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Properties of 3-dimensional line location models

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Abstract

We consider the problem of locating a line with respect to some existing facilities in 3-dimensional space, such that the sum of weighted distances between the line and the facilities is minimized. Measuring distance using the l_p norm is discussed, along with the special cases of Euclidean and rectangular norms. Heuristic solution procedures for finding a local minimum are outlined.

1 Introduction

The problem of locating a line in two-dimensional space was first considered by Wesolowsky [19] and further developed by Morris and Norback [13, 14, 15]. Many generalizations (like general distance measures, restrictions, or the location of hyperplanes) have been summarized in [17]. In computational geometry line and hyperplane location problems are also of interest [9, 10]. An overview about the more general case of locating any kind of dimensional facility has recently been given by [5].

In contrast to the two-dimensional line location problem, literature on finding a good line in $\mathbb{R}^3$ is rather rare. There are a few papers about finding the largest cylinder not containing a set of given points in $\mathbb{R}^3$ which can be considered as the location of an obnoxious line in three dimensions, see [6, 7]. This research has been motivated by problems in treating brain diseases, where radiation beams should not destroy the important organs within the brain. Finding the smallest

cylinder enclosing a set of given points, on the other hand, is equivalent to locating a line in $\mathbb{R}^3$ with center objective function and has been studied in [18]. The same problem has also been considered in the context of determining the width of a set by [8].
Another practical application for finding lines in $\mathbb{R}^3$ can be found in mining (see [2, 1]). Suppose that an area contains deposits of some mineral in various known locations underground. Instead of digging down separately to each deposit, it may be cheaper to construct a main shaft and reach the deposits by tunnels. The goal is to locate the shaft so as to minimize the annual transportation costs of moving the mineral through the tunnels (and up the shaft).

Line location problems in $\mathbb{R}^3$ can be formalized as follows. Given a set of existing facility locations, $\mathcal{A} = \{A_1, A_2, \ldots, A_M\}$, with $A_m = (a_{m1}, a_{m2}, a_{m3}) \in \mathbb{R}^3$ and non-negative weights w_m for $m \in \mathcal{M} = \{1, 2, \ldots, M\}$, along with a distance measure d, we want to find a straight line $L \subset \mathbb{R}^3$ so as to minimize

$$f(L) = \sum_{m \in \mathcal{M}} w_m d(A_m, L),$$

where the distance between a point $A \in \mathbb{R}^3$ and a line $L \subset \mathbb{R}^3$ is given by the shortest distance based on the distance measure d, i.e.

$$d(A, L) = \min_{X \in L} d(A, X).$$

In the mining example mentioned above the existing facilities represent the deposits and the line L models the mining shaft. The objective is to minimize the costs of the tunnel system which we assume to be related to the length of the tunnels. The length of a tunnel from a deposit located at A to the shaft L is given by $d(A, L)$ where d is mainly dependent on the properties of the tunnel system.

In the research carried out so far, see [2], it was assumed that the paths connecting the line to the existing facilities (the tunnels from the deposits to the shaft in the mining example) have to be horizontal. Within a horizontal plane, any p-norm may be considered as a distance measure. Such a distance measure is called horizontal l_p distance. Although it is by definition a three-dimensional distance, it reduces to the two-dimensional distance l_p in the horizontal plane through A, which significantly simplifies the line location problem. In [2] solution approaches for horizontal p-norm distances have been developed. For example, locating a line minimizing the sum of weighted horizontal Euclidean distances can be solved by a Weiszfeld type approach, which can be extended to l_p-distances with $1 < p < \infty$. For the rectangular distance l_1 special procedures have been suggested.

In this paper we relax the assumption that the tunnels connecting the shaft to the deposits have to be horizontal, i.e. we deal with p-norm distances instead of horizontal p-norm distances. For $1 \leq p < \infty$ the corresponding p-norm is given as $l_p(X) = l_p(x_1, x_2, x_3) = (\sum_{j=1}^{3} |x_j|^p)^{1/p}$. For the distance between a point $A \in \mathbb{R}^3$ and a line $L \subset \mathbb{R}^3$ the shortest distance based on the l_p norm is hence given by

$$\ell_p(A, L) = \min_{X \in L} l_p(A - X).$$

Given two parameters $\alpha, \beta \in \mathbb{R}^3$, we define an arbitrary line $L_{\alpha,\beta}$ by

$$L_{\alpha,\beta} = \{X \in \mathbb{R}^3 : X = \lambda\alpha + \beta, \lambda \in \mathbb{R}\}. \tag{1}$$

The following results of [2] are important for this research.

Lemma 1 *Locating a vertical line in* $\mathbb{R}^3$ *with distance measure* l_p *is equivalent to a Weber problem with distance measure* l_p *in the plane.*

Lemma 2 *Locating a line in* $\mathbb{R}^3$ *with fixed origin* $(\beta_1, \beta_2, 0)$*, using the horizontal* l_p *distance, is equivalent to a Weber problem with distance measure* l_p *in the plane.*

The remainder of the paper is organized as follows. After discussing how to measure distances using l_p norms in the next section, we treat the theoretical aspects of the location models in Section 3. For some of the models, heuristic solution procedures are outlined in Section 4.
To clarify the various results found in this paper, we classify three-dimensional line location problems according to the following two properties:

The locational structure of the existing facilities :

- They all lie in the horizontal plane E,
- or in some given hyperplane H,
- or they are located arbitrarily in $\mathbb{R}^3$.

Restrictions on the line :

- The line must lie within the horizontal plane E,
- or within some given hyperplane H,
- or no restriction is given, i.e. we look for the best line in $\mathbb{R}^3$.

In the last section, a short summary according to this classification will be given.

2 Measuring Distances

We consider the problem of finding a line L such that

$$f(L) = \sum_{m \in \mathcal{M}} w_m \ell_p(A_m, L)$$

is minimized, where the A_m are given points in $\mathbb{R}^3$ with nonnegative weights w_m. Given two vectors, $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ and $\beta = (\beta_1, \beta_2, \beta_3)$, a line in $\mathbb{R}^3$ can be represented by

$$L_{\alpha,\beta} = \{X \in \mathbb{R}^3 : X = \lambda\alpha + \beta, \lambda \in \mathbb{R}\}.$$

In the following we assume that $\alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 1$. Before solving the problem we need to discuss the distance $\ell_p(A_m, L_{\alpha,\beta})$ from a point to a line in $\mathbb{R}^3$. We start with the Euclidean distance l_2.

2.1 The Euclidean distance

For any given point $A_m = (a_{m1}, a_{m2}, a_{m3}) \in \mathbb{R}^3$ the closest point on the line is found as the one with λ being the inner product $\lambda_m^* = \langle \alpha, A_m - \beta \rangle$, i.e., we get the following formula for calculating the distance between $A_m \in \mathbb{R}^3$ and $L = L_{\alpha,\beta}$, if α is normed to 1.

$$\begin{aligned} \ell_2(A_m, L) &= \sqrt{\sum_{j=1}^{3} (a_{mj} - \alpha_j \lambda_m^* - \beta_j)^2} \\ &= \sqrt{\langle A_m - \beta, A_m - \beta \rangle - \langle A_m - \beta, \alpha \rangle \langle A_m - \beta, \alpha \rangle} \end{aligned}$$

The objective function is hence given by

$$f(L_{\alpha,\beta}) = \sum_{m \in \mathcal{M}} w_m \sqrt{\langle A_m - \beta, A_m - \beta \rangle - \langle A_m - \beta, \alpha \rangle \langle A_m - \beta, \alpha \rangle}.$$

Unfortunately, this objective function is neither convex nor concave. The following property for the Euclidean distance will be helpful for developing an algorithm.

Lemma 3 *Let $L = L_{\alpha,\beta} \subset \mathbb{R}^3$ be a line and $A \in \mathbb{R}^3$ be a point. Then the shortest Euclidean path from A to L is a line segment orthogonal to L, i.e. it lies in a plane with normal vector α.*

2.2 p-norm distances

If we use a p-norm distance l_p instead of the Euclidean distance, the property of Lemma 3 is in general not true.
To determine the distance between a point A_m and a line $L = L_{\alpha,\beta}$ we have to find λ_m^* such that $P_m = \lambda_m^*\alpha + \beta$ is the closest point on the line (by solving a one-dimensional minimization problem). We get

$$\ell_p(A_m, L) = \min_{P \in L} l_p(A_m - P) = l_p(A_m - \lambda_m^*\alpha - \beta).$$

Again, the objective function

$$f(L_{\alpha,\beta}) = \sum_{m \in \mathcal{M}} w_m (\sum_{j=1}^{3} |a_{mj} - \alpha_j \lambda_m^* - \beta_j|^p)^{\frac{1}{p}}$$

is neither convex nor concave.

2.3 The rectangular distance

In the special case of the rectangular distance l_1 we present the following formula for determining the distance between a point and a line in $\mathbb{R}^3$.

Lemma 4 *Let $A = (a_1, a_2, a_3) \in \mathbb{R}^3$ and let $L_{\alpha,\beta} \subset \mathbb{R}^3$ be a line defined by the parameters $\alpha, \beta \in \mathbb{R}^3$. Then*

$$\ell_1(A, L_{\alpha,\beta}) = \min \left\{ \sum_{j=1}^{3} \left| a_j - \frac{a_i - \beta_i}{\alpha_i}\alpha_j - \beta_j \right|, i = 1, 2, 3 \right\}$$

Proof:

$$\begin{aligned}
\ell_1(A, L_{\alpha,\beta}) &= \min_{X \in L} l_1(A - X) \\
&= \min_{\lambda \in \mathbb{R}} l_1(A - \lambda\alpha - \beta) \\
&= \min_{\lambda \in \mathbb{R}} (|a_1 - \lambda\alpha_1 - \beta_1| + |a_2 - \lambda\alpha_2 - \beta_2| + |a_3 - \lambda\alpha_3 - \beta_3|) \\
&= \min_{\lambda \in \mathbb{R}} \sum_{j=1}^{3} |\alpha_j| \left| \frac{a_j - \beta_j}{\alpha_j} - \lambda \right|,
\end{aligned}$$

assuming without loss of generalization that $\alpha_j \neq 0$, otherwise the j'th term is a constant and can hence be neglected. Since this is a weighted median problem there exists $i \in \{1, 2, 3\}$ such that

$$\lambda = \frac{a_i - \beta_i}{\alpha_i}$$

is optimal. Defining

$$P_i = \frac{a_i - \beta_i}{\alpha_i}\alpha + \beta \in \mathbb{R}^3, \; i = 1,2,3,$$

the distance between A and $L_{\alpha,\beta}$ is given by

$$\ell_1(A, L_{\alpha,\beta}) = \min\{l_1(A - P_1), l_1(A - P_2), l_1(A - P_3)\},$$

which proves the result. QED

Note that one shortest rectangular path from the point A to the line L in three-dimensional space is confined to a plane (since P_i and A share the same coordinate i). In particular, if the index i for the optimal λ in the proof of Lemma 4 is given by $i = 3$ then the path from A to l stays completely in the horizontal plane passing through A. Analogously, if $i = 1, 2$ the path lies completely in a plane perpendicular to respectively the x_1, x_2 axis. Unfortunately, the choice of the index i for λ is not only dependent on the parameters of the line (as in the two-dimensional case), but also on the position of the point A, so the property of Lemma 3 does not hold for the rectangular distance l_1.

3 Theoretical results

For the line location problem in the plane it has been shown by several authors (the earliest proof is in [19]) that with Euclidean distance there always exists an optimal line passing through two of the existing facilities. In [9] this statement was sharpened: For the Euclidean distance, *all* optimal lines pass through two of the existing facilities. Generalizations of this incidence property to other distances than the Euclidean can be found in [17]. With this background one might suspect that such an incidence property is also true for locating a line in three-dimensional space. But in the following counterexample *no* optimal line passes through two existing facilities, so the two-dimensional incidence property cannot be generalized.
Assume $M = 8$ existing facilities as the vertices of a cuboid, given by the following coordinates.

$$A_1 = (0,0,-1), A_2 = (0,0,1), A_3 = (0,2,1), A_4 = (0,2,-1),$$

$$A_5 = (e,0,-1), A_6 = (e,0,1), A_7 = (e,2,1), A_8 = (e,2,-1),$$

where $e > 0$.
Consider the line L_1 passing through the points $(0,1,0)$ and $(e,1,0)$. We get that $\ell_2(A_m, L_1) = \sqrt{2}$ for all $m = 1,\ldots,8$, such that

$$f(L_1) = 8\sqrt{2},$$

independent of e, when all weights are one.
We want to show that for large e the line L_1 is better than any line passing through two of the existing facilities. For the line $L_2 = L_{\alpha,\beta}$ with $\alpha = \frac{1}{\sqrt{e^2+8}}(e,2,2)$ and $\beta = (0,0,-1)$, passing through A_1 and A_7 we get

$$\begin{aligned}
\ell_2(A_1, L_2) &= \ell_2(A_7, L_2) = 0,\\
\ell_2(A_2, L_2) &= \ell_2(A_4, L_2) = \ell_2(A_6, L_2) = \ell_2(A_8, L_2) = 2\sqrt{\frac{e^2+4}{e^2+8}},\\
\ell_2(A_3, L_2) &= \ell_2(A_5, L_2) = 2\sqrt{\frac{2e^2}{e^2+8}},\\
\Longrightarrow f(L_2) &= \frac{4}{\sqrt{e^2+8}}(2\sqrt{e^2+4} + \sqrt{2e^2}).
\end{aligned}$$

For $e \to \infty$ we get $f(L_2) \to 8 + 4\sqrt{2} > 8\sqrt{2} = f(L_1)$. The vertical and horizontal lines passing through two of the facilities are even worse, and the lines which are diagonals in one of the faces (as the line through A_2 and A_7) are also worse than L_2. This means that, for large enough e, the line L_1 is better than all lines passing through two of the existing facilities, so no such line is optimal.

We now deal with some special configurations of the existing facilities and the new line. Assume, e.g., that *all* existing facilities lie in one common plane H. In the following lemma we consider the special case that this plane is

$$E := \mathbb{R}^2 \times \{0\} = \{(x_1, x_2, x_3) : x_3 = 0\},$$

meaning that $a_{m3} = 0$ for all $m \in \mathcal{M}$. In this case, all optimal lines with respect to any l_p-norm, $p < \infty$, are also contained in E.

Lemma 5 *Let $a_{m3} = 0$ for all $m \in \mathcal{M}$. Then there exists an optimal line $L_{\alpha,\beta}$ with respect to l_p which satisfies*

$$\alpha_3 = 0 \text{ and } \beta_3 = 0.$$

Moreover, if $p < \infty$ all optimal lines satisfy the condition.

Proof: For $X = (x_1, x_2, x_3) \in \mathbb{R}^3$ let $P(X) = (x_1, x_2, 0)$ be its projection onto $\mathbb{R}^2 \times \{0\}$. Note that for all l_p-norms it holds that

$$l_p(X - Y) \geq l_p(P(X) - P(Y))$$

and moreover, for $p < \infty$, $X \neq P(X)$, and $Y = P(Y)$ it can easily be verified that

$$l_p(X - Y) > l_p(P(X) - Y).$$

Now let $L = L_{\alpha,\beta}$ be any line in $\mathbb{R}^3$, and let

$$P(L) = L_{P(\alpha),P(\beta)}$$

be its projection onto $\mathbb{R}^2 \times \{0\}$. Using the above inequalities, the distance between a point $A = (a_1, a_2, 0) = P(A)$ and the line l satisfies

$$\begin{aligned} \ell_p(A, L) &= \min_{X \in L} l_p(A - X) \\ &\geq \min_{X \in L} l_p(A - P(X)) \\ &= \min_{X \in P(L)} l_p(A - X) \\ &= \ell_p(A, P(L)). \end{aligned}$$

where the above inequality holds strictly if $p < \infty$ and $L \neq P(L)$.

QED

Corollary 1 *Let all existing facilities be contained in $E = \mathbb{R}^2 \times \{0\}$. For all l_p-norms, $1 \leq p \leq \infty$ there exists an optimal line passing through two of the existing facilities.*

Corollary 2 *Let H be a plane in $\mathbb{R}^3$ containing all existing facilities. Then all optimal lines with respect to the Euclidean distance l_2 are contained in H.*

Proof: Since l_2 is rotation invariant, we can assume without loss of generality that $H = E = \{(x_1, x_2, x_3) : x_3 = 0\}$ and the result follows directly from Lemma 5.

QED

Lemma 5 deals with a plane containing all existing facilities. In the next property, however, arbitrary existing facilities are allowed, but only lines within $E = \mathbb{R}^2 \times \{0\}$ will be considered.

Lemma 6 *Let L be a line in the plane $E = \mathbb{R}^2 \times \{0\} \subset \mathbb{R}^3$, $A = (a_1, a_2, a_3) \in \mathbb{R}^3$, and $P(A) = (a_1, a_2, 0)$ the projection of A onto E. Then*

$$\ell_p(A, L) = l_p[\ell_p(P(A), L), a_3].$$

Note that the symbol ℓ_p refers to a distance measure in $\mathbb{R}^3$ while l_p on the right-hand-side is the p-norm in only two dimensions. Moreover, $\ell_p(P(A), L)$ can be replaced by a 2-dimensional distance, since both $P(A)$ and L lie in $E = \mathbb{R}^2 \times \{0\}$.
Proof: Given two points $X = (x_1, x_2, x_3)$ and $Y = (y_1, y_2, y_3)$ we will use the following property for the l_p-norms:

$$l_p(X - Y) = l_p[l_p(P(X) - P(Y)), x_3 - y_3].$$

Noting that all $X \in L$ satisfy $x_3 = 0$, yields

$$\begin{aligned}\ell_p(A, L) &= \min_{X\in L} l_p(A - X) \\ &= \min_{X\in L}\{l_p[l_p(P(A) - P(X)), a_3 - 0]\} \\ &= l_p[\min_{X\in L}\{l_p(P(A) - P(X))\}, a_3] \\ &= l_p[\ell_p(P(A), L), a_3].\end{aligned}$$

QED

Lemma 6 motivates the definition of the following *planarly restricted line location problem* (PRL):

(PRL) Given a set $\mathcal{A}$ of existing facilities in $\mathbb{R}^3$, find a line L within the plane $E = \mathbb{R}^2 \times \{0\}$ that minimizes $f(L) = \sum_{m\in\mathcal{M}} w_m \ell_p(A_m, L)$.

Using the constraint $L \subset E$ and Lemma 6 the objective function of (PRL) can be reformulated as

$$f(L) = \sum_{m\in\mathcal{M}} w_m l_p[\ell_p(P(A_m), L), a_{m3}].$$

Theorem 1 *Problem (PRL) with rectangular distance l_1 is equivalent to a line location problem in the plane, where the existing facilities are given by the projections of the given points A_m onto E, and the distance function is given by the two-dimensional distance l_1.*

Proof: Using Lemma 6 we get for l_1:

$$\begin{aligned}f(L) &= \sum_{m\in\mathcal{M}} w_m l_1[\ell_1(P(A_m), L), a_3] \\ &= \sum_{m\in\mathcal{M}} w_m \ell_1(P(A_m), L) + \sum_{m\in\mathcal{M}} w_m |a_{m3}|,\end{aligned}$$

meaning that for l_1 (PRL) is equivalent to

$$\min \sum_{m\in\mathcal{M}} w_m \ell_1[(a_{m1}, a_{m2}), L].$$

The latter problem is a planar line location problem, with the two-dimensional projections of the A_m as existing facilities.

QED

This result leads to the following consequences.

Corollary 3 *For problem (PRL) with distance l_1 there exists an optimal line passing through at least two of the projection points $P(A_m)$.*

Proof: The Corollary follows directly from the properties of planar line location problems, see, e.g. [14, 17]. QED

Note that since separation of the coordinates does not hold for $1 < p < \infty$, the results of Theorem 1 and Corollary 3 are not extendable to general p-norms. As counterexample (for the Euclidean distance l_2), consider the problem instance given earlier on page 6. If we are looking for an optimal line within E, the line L_1 with objective value $f(L_1) = 8\sqrt{2}$ (which is not passing through any of the projection points of $\mathcal{A}$) is better than the best line passing through two of the projection points (with objective value $f(L) \to 4(1+\sqrt{5})$ for $e \to \infty$).

From the symmetry properties of the rectangular distance we also get the following.

Corollary 4 *Let H be a given hyperplane with normal vector n, satisfying*

$$n \in \{(1,0,0),(0,1,0),(0,0,1)\}.$$

Then, for the rectangular distance l_1, the problem of finding an optimal line within H is equivalent to a line location problem in the plane. Moreover, all optimal lines are passing through at least two of the projection points of the existing facilities A_m onto H.

Another special case occurs, when also all the existing facilities are contained in some given hyperplane H. For the rectangular distance and the set of hyperplanes mentioned in Corollary 4 we directly know that there exists an optimal line passing through two of the existing facilities. Corollary 2 shows the same result for the Euclidean distance l_2, independent of the hyperplane H, and Corollary 1 yields this incidence property for one special hyperplane E, but allows all distances l_p. In the last result of this section we generalize this property to all distances l_p, **and** to arbitrary hyperplanes H. We first formulate the *line location problem within a hyperplane (LH)*:

(LH) Given a set $\mathcal{A}$ of existing facilities within some hyperplane $H \subset \mathbb{R}^3$, find a line L within H that minimizes $f(L) = \sum_{m\in\mathcal{M}} w_m \ell_p(A_m, L)$.

Lemma 7 *If a line $L = L_{\alpha*,\beta*}$ is a solution to the problem of minimizing $f(L) = \sum_{m\in\mathcal{M}} w_m \ell_p(A_m, L)$, then the translated line $L = L_{\alpha*,\beta*+\Delta}$ is a solution to the translated problem of minimizing $g(L) = \sum_{m\in\mathcal{M}} w_m \ell_p(A_m + \Delta, L)$.*

Proof: First we note that similar translation of a point and a line does not change the distance: $\ell_p(A+\Delta, L_{\alpha,\beta+\Delta}) = \min_\lambda l_p(A+\Delta-\alpha\lambda-\beta-\Delta) =$
$\min_\lambda l_p(A-\alpha\lambda-\beta) = \ell_p(A, L_{\alpha,\beta})$.
For all lines $L = L_{\alpha,\beta}$ we have $g(L_{\alpha*,\beta*+\Delta}) = \sum_{m\in\mathcal{M}} w_m \ell_p(A_m+\Delta, L_{\alpha*,\beta*+\Delta}) =$

$\sum_{m\in\mathcal{M}} w_m \ell_p(A_m, L_{\alpha*,\beta*}) \leq \sum_{m\in\mathcal{M}} w_m \ell_p(A_m, L_{\alpha,\beta-\Delta}) =$
$\sum_{m\in\mathcal{M}} w_m \ell_p(A_m + \Delta, L_{\alpha,\beta}) = g(L_{\alpha,\beta})$.

QED

Theorem 2 *Given a hyperplane $H \subset \mathbb{R}^3$. For all l_p-norms, $1 \leq p \leq \infty$ there exists an optimal line for (LH) passing through two of the existing facilities.*

Proof: Let $H = H_{n,b} = \{X \in \mathbb{R}^3 : n_1x_1 + n_2x_2 + n_3x_3 = b\}$ be a hyperplane with normal vector $n \neq 0$. According to Lemma 7 we can assume without loss of generality that the origin is contained in H, i.e. $b = 0$. We denote the unit ball of the l_p-norm by B_p, i.e.

$$B_p = \{X \in \mathbb{R}^3 : l_p(X) \leq 1\}.$$

As unit ball of a norm, B_p is a convex set, containing the origin in its interior, and symmetric with respect to the origin. Note that $\tilde{B} := B_p \cap H$ is a two-dimensional set, which still is convex, contains the origin in its (relative) interior and is symmetric with respect to the origin. Consequently (see [12]), $\tilde{B}$ defines a norm on H, given by

$$\tilde{\gamma}(X) := \inf\{\lambda \geq 0 : X \in \lambda\tilde{B}\}, X \in H.$$

This means that (LH) is equivalent to a planar line location problem, with respect to some norm $\tilde{\gamma}$, (not necessarily l_p) and for such problems it is known ([16, 17]) that there exists an optimal line passing through two of the existing facilities.

QED

Note that an equivalent planar problem in the previous proof can be defined analytically as follows: Let $H = H_{n,b}$ be the hyperplane and assume that $n_3 \neq 0$, and $b = 0$. Define $P : H \to E$ as the (bijective) projection from H onto the xy-plane, and let P^{-1} be its inverse mapping. Let $\tilde{\mathcal{A}} := \{P(A) : A \in \mathcal{A}\}$ and define

$$\tilde{\gamma}(X) = l_p(P^{-1}(X)).$$

From the previous proof it follows that $\tilde{\gamma}$ is a norm on H (which can also be easily checked by calculation). Moreover, since $l_p(X - Y) = \tilde{\gamma}(P(X) - P(Y))$ for all $X, Y \in H$ we obtain

$$l_p(A, L) = \tilde{\gamma}(P(A), P(L)),$$

yielding that (LH) is, in fact, equivalent to the planar line location problem.

4 Local search procedures

As mentioned before, the objective function of the three-dimensional line location problems introduced in the previous sections are neither convex nor concave. Hence, without extensive search we can only expect a local minimum. Again, we first consider the Euclidean distance.

4.1 The Euclidean case

Lemma 3 states that if the slope of the line $L_{\alpha,\beta}$ is fixed (i.e. the vector α is given) then the distances from the existing facilities to the line can be calculated according to the Euclidean distance within the plane orthogonal to $L_{\alpha,\beta}$. Thus, our problem may be approximated as the location of an arbitrary line with respect to the horizontal Euclidean distance after applying a rotation $r : \mathbb{R}^3 \to \mathbb{R}^3$, such that L initially becomes a vertical line. By applying Algorithm 2 in [2], an improved solution L_r is found for horizontal Euclidean distances which then retransforms into an even better solution for shortest Euclidean distances. The details are outlined in the following heuristic.

Algorithm 1 (for locating a line with shortest Euclidean distance)

Step 1. Choose an initial solution L^0, $g = 0$.

Step 2. Find a rotation r which maps L^g to a vertical line. Determine $\mathcal{A}^r = \{r(A) : A \in \mathcal{A}\}$.

Step 3. Determine L_r by solving the problem with respect to $\mathcal{A}^r$ using the horizontal Euclidean distance using Algorithm 2 of [2]

Calculate $L^{g+1} = r^{-1}(L_r)$ by retransforming L_r.

Step 4. If $f(L^g) - f(L^{g+1}) < \delta$, STOP;
else set $g = g + 1$ and return to step 2.

For a quicker solution, step 3 in Algorithm 1 may be replaced by

Step 3a. For the rotated axes, fix $\alpha^g = (0, 0, 1)$ and find the best starting point $\beta^{g+1} = (\beta_1^{g+1}, \beta_2^{g+1}, 0)$ for the vertical line L^g by using Lemma 1.

Step 3b. Fix β^{g+1} and optimize for α^{g+1} with respect to the horizontal Euclidean distance by using Lemma 2. Retransform $L^{g+1} = r^{-1}(L_{\alpha^{g+1},\beta^{g+1}})$.

4.2 The p-norm case

If we use a p-norm distance instead of the Euclidean distance, the objective function (developed in Section 2.2)

$$f(L_{\alpha,\beta}) = \sum_{m\in\mathcal{M}} w_m(\sum_{j=1}^{3} |a_{mj} - \alpha_j\lambda_m^* - \beta_j|^p)^{\frac{1}{p}}$$

is neither convex nor concave, but a local minimum may be found by the following scheme.

Algorithm 2 (for locating a line with shortest l_p distance)

Step 1. Choose an initial solution (α^0, β^0), compute the λ_m^* values and the objective function value $f(L^0_{\alpha^0,\beta^0})$, and set counter $g = 0$.

Step 2a. Holding α^g and the λ_m^* values fixed find the best starting point $\beta^{g+1} = (\beta_1^{g+1}, \beta_2^{g+1}, \beta_3^{g+1})$ for the line by a generalized Weiszfeld algorithm (e.g., see [3]) for the Weber problem.

Step 2b. Holding β^{g+1} and the λ_m^* values fixed, perform Weiszfeld-type iterations on α until a stopping criterion is reached.

Denote the current solution by $(\alpha^{g+1}, \beta^{g+1})$.

Step 3. Compute $\lambda_m^*, m \in \mathcal{M}$ for the current solution.
If $f(L^g_{\alpha,\beta}) - f(L^{g+1}_{\alpha,\beta}) < \delta$, STOP;
else set $g = g + 1$ and return to step 2a.

In step 2a it turns out that the problem to find β^{g+1} reduces to a classical single facility problem in $\mathbb{R}^3$ with l_p distance where the existing facilities are given by

$$A'_m = (a_{m1} - \lambda_m^*\alpha_1, a_{m2} - \lambda_m^*\alpha_2, a_{m3} - \lambda_m^*\alpha_3) \text{ for all } m \in \mathcal{M}.$$

The Weiszfeld iterations in both parts of step 2 result in a sequence of descent moves for the fixed values of $\lambda_m^*, m \in \mathcal{M}$. By updating the λ_m^* values in step 3 for the new line $L^{g+1}_{\alpha,\beta}$, we are replacing distances to the line by shortest distances, thereby providing a further improvement of the objective function. The iteration scheme thus converges to a stationary point. A multi-start version of Algorithm 2 with random initial solutions may be used to improve the likelihood of finding the global optimum.

4.3 The rectangular case

To solve line location problems in $\mathbb{R}^3$ with respect to the rectangular distance one may again use a local search to find a local minimum as for the p-norm case, but steps 2a and 2b of Algorithm 2 can be combined to run in linear time, as the following approach shows.

Algorithm 3 (for locating a line with shortest rectangular distance)
Step 1. Choose an initial solution (α^0, β^0), compute the λ_m^* values and the objective function value $f(L^0_{\alpha^0,\beta^0})$, and set counter $g = 0$.
Step 2. Holding the λ_m^* values fixed optimize for α and β. Denote the solution by $L^{g+1} = L_{\alpha^{g+1},\beta^{g+1}}$.
Step 3. Compute $\lambda_m^*, m \in \mathcal{M}$ for L^{g+1} If $f(L^g_{\alpha,\beta}) - f(L^{g+1}_{\alpha,\beta}) < \delta$, STOP; else set $g = g + 1$ and return to step 2.

The minimization problem of step 2 is given by

$$\min_{\alpha,\beta} \sum_{m\in\mathcal{M}} w_m \left(|a_{m1} - \lambda_m^*\alpha_1 - \beta_1| + |a_{m2} - \lambda_m^*\alpha_2 - \beta_2| + |a_{m3} - \lambda_m^*\alpha_3 - \beta_3|\right).$$

It can be separated into three independent subproblems $P_k, k = 1, 2, 3$, each being a line location problem in the plane with respect to the vertical distance. The existing facilities in subproblem P_k are given by

$$A'_m = (\lambda_m^*, a_{mk}) \text{ for all } m \in \mathcal{M},$$

the weights are given by the original weights w_m, and the optimal solution yields a line with slope α_k^* and intercept β_k^*. All three subproblems can be solved in linear time by linear programming [20].

5 Summary

Line	Existing facilities		
	in E	in H	in $\mathbb{R}^3$
in E	planar line location	for l_1 equivalent to planar line location with l_1 (PRL, Theorem 1)	for l_1 equivalent to planar line location with l_1 (PRL, Theorem 1)
in H		for all l_p equivalent to planar line location with some norm (LH, Theorem 2)	Corollary 4 for l_1
in $\mathbb{R}^3$	an optimal line in E for all l_p (Lemma 5)	an optimal line in H for l_2 (Corollary 2); open for l_p	Algorithm 1 for l_2 Algorithm 3 for l_1 Algorithm 2 for l_p

Concerning the incidence property *there exists an optimal line passing through two of the existing facilities* our results can be summarized as follows.

Line	Existing facilities		
	in E	in H	in $\mathbb{R}^3$
in E	for all l_p through two of the ex. fac.s	for l_1 through two of the projections of the ex. fac.s	for l_1 through two of the projections of the ex. fac.s
in H	needs not contain any of the ex. fac.s	for all l_p through two of the ex. fac.s	needs not contain any of the ex. fac.s
in $\mathbb{R}^3$	for all l_p through two of the ex. fac.s	for l_2 through two of the ex. fac.s	needs not contain any of the ex. fac.s

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