Folien zu dem Vortrag von

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A mapping of Clifford vector fields on regions of the complex plane with varying connectivity: applications to currents in biological tissues

CLIFFORD'S LEGACY
TO
MATHEMATICS

SATURDAY, 2 DECEMBER 1995

Introduction: Clifford vector fields on $\mathbb{R}^2$

name	grade	basis	number of elements
scalar	0	$1 \in \mathbb{R},\ 1^2 = 1$	$\binom{2}{0} = 1$
vector	1	$\vec{\sigma}_1, \vec{\sigma}_2,\quad \vec{\sigma}_1^2 = \vec{\sigma}_2^2 = 1,\quad \vec{\sigma}_1\vec{\sigma}_2 + \vec{\sigma}_2\vec{\sigma}_1 = 0$	$\binom{2}{1} = 2$
bivector	2	$\mathbb{i} = \vec{\sigma}_1\vec{\sigma}_2, \quad \mathbb{i}^2 = -1$	$\binom{2}{2} = 1$

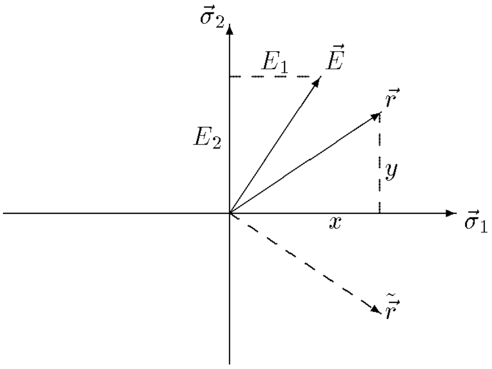
total number of elements $= 2^2 = 4$

Definition: A Clifford vector field is a sufficiently smooth mapping

$$\mathbb{R}^2 \overset{\vec{E}}{\to} \mathbb{R}^2, \qquad x, y, E_1, E_2 \in \mathbb{R},$$

$$\mathbb{R}^2 \ni \vec{r} = x\vec{\sigma}_1 + y\vec{\sigma}_2 \overset{\vec{E}}{\to} \vec{E} = E_1\vec{\sigma}_1 + E_2\vec{\sigma}_2 \in \mathbb{R}^2.$$

Isomorphism $\mathbb{R} \leftrightarrow \mathbb{C}(\mathbb{i})$: $\vec{r} = \vec{\sigma}_1(x + \mathbb{i}y) = \vec{\sigma}_1 z = \tilde{z}\vec{\sigma}_1$



complex conjugation

$$\tilde{z} = \vec{\sigma}_1 z \vec{\sigma}_1 = x - \mathbb{i}y$$

$$\tilde{\vec{r}} = \vec{\sigma}_1 \vec{r} \vec{\sigma}_1 = \vec{\sigma}_1 \tilde{z}$$

$$x = \tfrac{1}{2}(\tilde{z} + z) = \tfrac{1}{2}(\vec{\sigma}_1\vec{r} + \vec{r}\vec{\sigma}_1)$$

$$y = \tfrac{\mathbb{i}}{2}(\tilde{z} - z) = \tfrac{1}{2}(\vec{\sigma}_2\vec{r} + \vec{r}\vec{\sigma}_2)$$

Lift of Clifford vector fields on $\mathbb{R}^2$ to $\mathbb{C}(\mathrm{i})$-valued analytic functions on $\mathbb{C}(\mathrm{i}) \times \mathbb{C}(\mathrm{i}) \equiv \mathbb{C}^2(\mathrm{i})$:

$$\vec{E}(\vec{r}) = E(z,\tilde{z})\vec{\sigma}_1, \qquad E = E_1 - \mathrm{i}E_2,$$

$$\vec{\sigma}_1 E \vec{\sigma}_1 = E_1 + \mathrm{i}E_2 = \tilde{E}$$

Isomorphism:

$\vec{\partial} = \vec{\sigma}_1 \frac{\partial}{\partial x} + \vec{\sigma}_2 \frac{\partial}{\partial y} \quad \Leftrightarrow \quad \partial_{\tilde{z}}$

Dirac operator **Cauchy–Riemann operator**

$$\begin{aligned}
\vec{\sigma}_1 \vec{\partial} \vec{E} \vec{\sigma}_1 &= \left(\frac{\partial}{\partial x} + \mathrm{i}\frac{\partial}{\partial y}\right) E(z,\tilde{z}) \\
&= \left(\frac{\partial}{\partial x} + \mathrm{i}\frac{\partial}{\partial y}\right) E(x+\mathrm{i}y, x-\mathrm{i}y) \\
&= \lim_{\substack{\varepsilon\to 0 \\ \varepsilon\in\mathbb{R}}} \frac{\partial}{\partial \varepsilon}[E(\underbrace{x+\varepsilon+\mathrm{i}y}_{z+\varepsilon}, \underbrace{x+\varepsilon-\mathrm{i}y}_{\tilde{z}+\varepsilon}) \\
&\qquad + \mathrm{i}E(\underbrace{x+\mathrm{i}y+\mathrm{i}\varepsilon}_{z+\mathrm{i}\varepsilon}, \underbrace{x-\mathrm{i}y-\mathrm{i}\varepsilon}_{\tilde{z}-\mathrm{i}\varepsilon})]
\end{aligned}$$

Abbreviation: $\lim_{\substack{\varepsilon\to 0 \\ \varepsilon\in\mathbb{R}}} = \partial_\varepsilon|_0$

Definition:

$$\partial_z E(z,\tilde z) = \partial_\varepsilon|_0\, E(z+\varepsilon,\tilde z),\ \partial_{\tilde z} E(z,\tilde z) = \partial_\varepsilon|_0\, E(z,\tilde z+\varepsilon)$$

$$\Rightarrow \partial_\varepsilon|_0\, E(z+a\varepsilon,\tilde z) = a\,\partial_\varepsilon|_0\, E(z+\varepsilon,\tilde z) = a\partial_z E,\ a\in\mathbb{C}(\mathbb{i})$$

$$\Rightarrow \partial_\varepsilon|_0\, E(z,\tilde z+b\varepsilon) = b\,\partial_\varepsilon|_0\, E(z,\tilde z+\varepsilon) = b\partial_z E,\ b\in\mathbb{C}(\mathbb{i})$$

$$\Rightarrow \vec\sigma_1\vec\partial\vec E\vec\sigma_1 = 2\partial_{\tilde z}E(z,\tilde z) \quad \Rightarrow \quad \vec\partial\vec E = 2\vec\sigma_1\partial_{\tilde z}\vec E \Rightarrow$$

Dirac operator $\boxed{\vec\partial = 2\vec\sigma_1\partial_{\tilde z}}$ Cauchy–Riemann operator

Corollary $\boxed{\vec\sigma_1\vec\partial\vec\sigma_1 = 2\vec\sigma_1\partial_z = 2\partial_{\tilde z}\vec\sigma_1}$

A new inversion of the Dirac operator on $\mathbb{R}^2$

$$\vec\partial\vec E = 2\varrho(x,y), \qquad \varrho\in\mathbb{R}$$

Atiyah–Kähler relation:

$$\vec\partial\vec E = \vec\partial\cdot\vec E + \vec\partial\wedge\vec E = 2\varrho \iff \vec\partial\cdot\vec E = 2\varrho, \qquad \vec\partial\wedge\vec E = 0$$

$$\partial_{\tilde z}E(z,\tilde z) = \varrho(x,y) = \omega(z,\tilde z)\in\mathbb{R}$$

$$\boxed{E(z,\tilde z) = f(z) + \tilde z\int_0^1 d\alpha\ \omega(z,\alpha\tilde z) = \vec E(\vec r)\vec\sigma_1}$$

Example: A **Dirac δ–source of intensity** $I > 0$ at the point $\vec{p} = \vec{\sigma}_1 p$

$$\begin{aligned} \vec{\partial}\vec{E} &= \pi I \delta(\vec{r} - \vec{p}) = \pi I \lim_{\substack{\eta\to\infty \\ \eta\in\mathbb{R}}} \eta^2 e^{-\pi\eta^2 (z-p)(\tilde{z}-\tilde{p})} \\ &= 2\vec{\sigma}_1 \partial_{\tilde{z}} E(z,\tilde{z})\vec{\sigma}_1 \end{aligned}$$

Solution:

$$E(z,\tilde{z}) = f(z) + \lim_{\substack{\eta\to\infty \\ \eta\in\mathbb{R}}} \frac{I}{2(z-p)} \left[1 - e^{-\pi\eta^2 (z-p)(\tilde{z}-\tilde{p})}\right] =$$

$$E(z,\tilde{z}) = f(z) + \frac{I}{2(z-p)}$$

for

$$|z-p| =_{+} \sqrt{(z-p)(\tilde{z}-\tilde{p})} > 0.$$

E is holomorphic except on a microlocal region at $z = p$.

Stationary currents $\vec{J} = \kappa\vec{E}$ on the plane $\mathbb{R}^2$

$\kappa = \kappa(\vec{r}) =$ scalar conductivity field
$\vec{E} = \vec{E}(\vec{r}) =$ electric Clifford vector field
$\vec{J} = \kappa\vec{E} =$ Ohm's law

Maxwell equations: **pointsource** **pointsink**

$$\vec{\partial} \wedge \vec{E} = 0, \quad \vec{\partial} \cdot (\kappa\vec{E}) = \pi\kappa I[\delta(\vec{r} - \vec{p}) - \delta(\vec{r} - \vec{q})], \quad I > 0$$

$$2\vec{\partial} \wedge \vec{E} = \vec{\partial}\vec{E} - (\vec{E}\vec{\partial}), \quad 2\vec{\partial} \cdot \vec{E} = \vec{\partial}\vec{E} + (\vec{E}\vec{\partial})$$

Atiyah–Kähler relation $\vec{\partial}\vec{E} = \vec{\partial} \cdot \vec{E} + \vec{\partial} \wedge \vec{E} \Rightarrow$
Unification of all Maxwell equations

$$\boxed{\kappa\vec{\partial}\vec{E} + \vec{E} \cdot (\vec{\partial}\kappa) = \kappa\pi I[\delta(\vec{r} - \vec{p}) - \delta(\vec{r} - \vec{q})]}$$

Lift to $\mathbb{C}^2(\mathbb{i})$

$$\boxed{\kappa\partial_{\tilde{z}}E + \frac{1}{2}(E\partial_{\tilde{z}}\kappa + \tilde{E}\partial_z\kappa) = \frac{\pi}{2}\kappa I[\delta(\vec{r} - \vec{p}) - \delta(\vec{r} - \vec{q})]}$$

A pointsource at $\vec{p} = \vec{\sigma}_1 p$ and a pointsink at $\vec{q} = \vec{\sigma}_1 q$ of intensity $I > 0$ in a region $\subset \mathbb{R}^2$ of <u>constant</u> conductivity $\kappa = const. \in \mathbb{R}$:

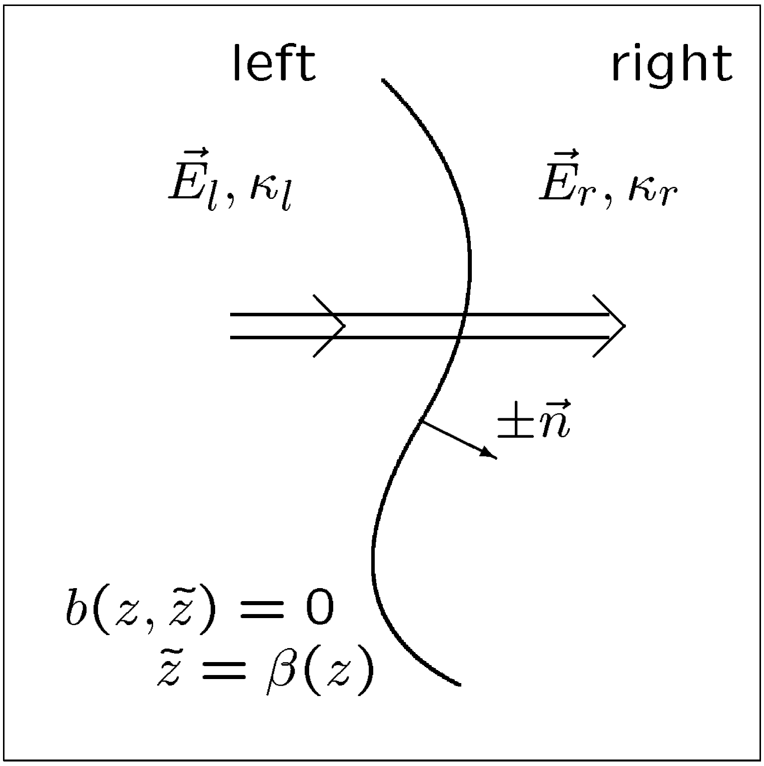
$$2\partial_{\tilde{z}}E = \pi I[\delta(\vec{r} - \vec{p}) - \delta(\vec{r} - \vec{q})]$$

$$\boxed{\Rightarrow E = \frac{I}{2}\left(\frac{1}{z-p} - \frac{1}{z-q}\right) + f(z) = E(z) \text{ for } z \neq p, q}$$

Transition across a smooth curve

$$c(x,y) = 0 = b(z,\tilde{z}) = \vec{\sigma}_1 b(z,\tilde{z})\vec{\sigma}_1$$

with unit normal $\vec{n} = \frac{\vec{\partial}c}{|\vec{\partial}c|} = n\vec{\sigma}_1$, separating two regions of different conductivities



no sources, no sinks on the boundary!

$$\vec{\partial} \wedge \vec{E} = 0 \Rightarrow \vec{n} \wedge (\vec{E}_r - \vec{E}_l)\Big|_{c=0} = 0$$

$$\vec{\partial} \cdot (\kappa\vec{E}) = 0 \Rightarrow \vec{n} \cdot (\kappa_r\vec{E}_r - \kappa_l\vec{E}_l)\Big|_{c=0} = 0$$

Atiyah–Kähler unification in terms of the Clifford product

$$(1+\eta)\vec{E}_r = \vec{E}_l - \eta\vec{n}\vec{E}_l\vec{n}, \qquad \eta = \frac{\kappa_r - \kappa_l}{\kappa_r + \kappa_l}$$

Lift to $\mathbb{C}^2(\mathbb{i})$

$$\boxed{(1+\eta)E_r = E_l - \eta n^2 \vec{\sigma}_1 E_l \vec{\sigma}_1,\ b(z,\tilde{z}) = 0 = \vec{\sigma}_1 b(z,\tilde{z})\vec{\sigma}_1}$$

Proposition $b(z,\tilde{z}) = 0 \Rightarrow \tilde{z} = \beta(z)$

$$\Rightarrow b(z,\beta(z)) = 0 \Rightarrow \partial_z b + \beta' \partial_{\tilde{z}} b = 0$$

$$\vec{n} = \frac{\vec{\partial} c}{|\vec{\partial} c|} = n\vec{\sigma}_1 \ \Rightarrow\ n = \frac{\partial_z b}{|\partial_z b|} \ \Rightarrow\ \underline{n^2 = \frac{\partial_z b}{\partial_{\tilde{z}} b} = -\beta'(z)}$$

$$\boxed{(1+\eta)E_r = E_l + \eta\beta' \underbrace{\vec{\sigma}_1 E_l \vec{\sigma}_1}_{[E_l]^\sim},\ \tilde{z} = \beta(z),\ \eta = \frac{\kappa_r - \kappa_l}{\kappa_r + \kappa_l}}$$

Neumann–condition for $\vec{E}_l$: $\kappa_r = 0 \Rightarrow \vec{J}_r = \kappa_r \vec{E}_r = 0$
insulator, provided $|\vec{E}_r| < \infty$! $\Downarrow \eta = -1$

$$\vec{E}_l = -\vec{n}\vec{E}_l\vec{n} = \vec{E}_l + 2\vec{n}(\vec{n}\cdot\vec{E}_l) \qquad \Rightarrow$$

$$\vec{n}\cdot\vec{E}_l = 0, \quad \vec{E}_l = \textbf{tangential}$$

Lift to $\mathbb{C}(\mathbb{i})$

$$\boxed{E_l = \beta'\vec{\sigma}_1 E_l \vec{\sigma}_1 = -[E_l]^\sim \frac{\partial_z b}{\partial_{\tilde{z}} b}}$$

In regions of constant conductivity κ, outside of sources and sinks, the functions $E = E_{l,r}$ are holomorphic, i.e.,

$$\partial_{\tilde{z}} E = 0 \iff E = E(z)$$

The Schwarz–conjugate $\hat{E}$ of the holomorphic function E:

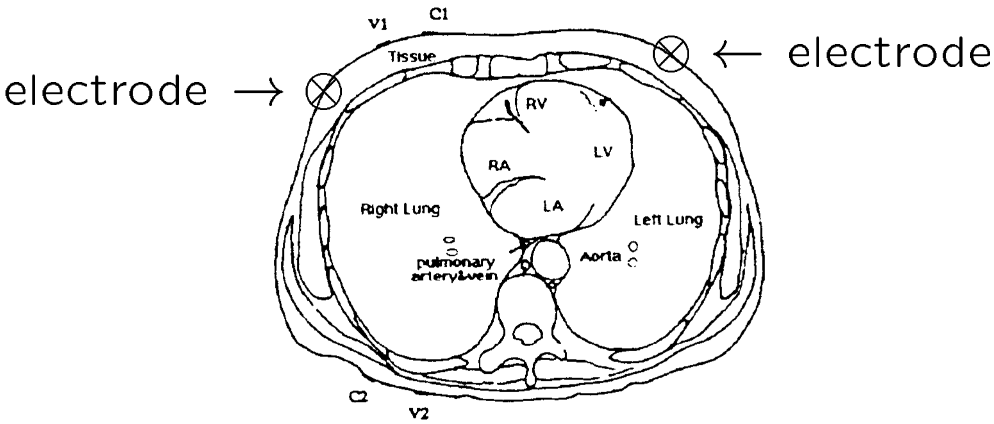
$$\hat{E}(z) = \vec{\sigma}_1 E(\vec{\sigma}_1 z \vec{\sigma}_1)\vec{\sigma}_1 = [E(\tilde{z})]^{\sim} \Rightarrow \hat{E}(\tilde{z}) = [E(z)]^{\sim} \Rightarrow$$

Transition across the boundary curve $\tilde{z} = \beta(z)$ separating two regions of different **constant** conductivities

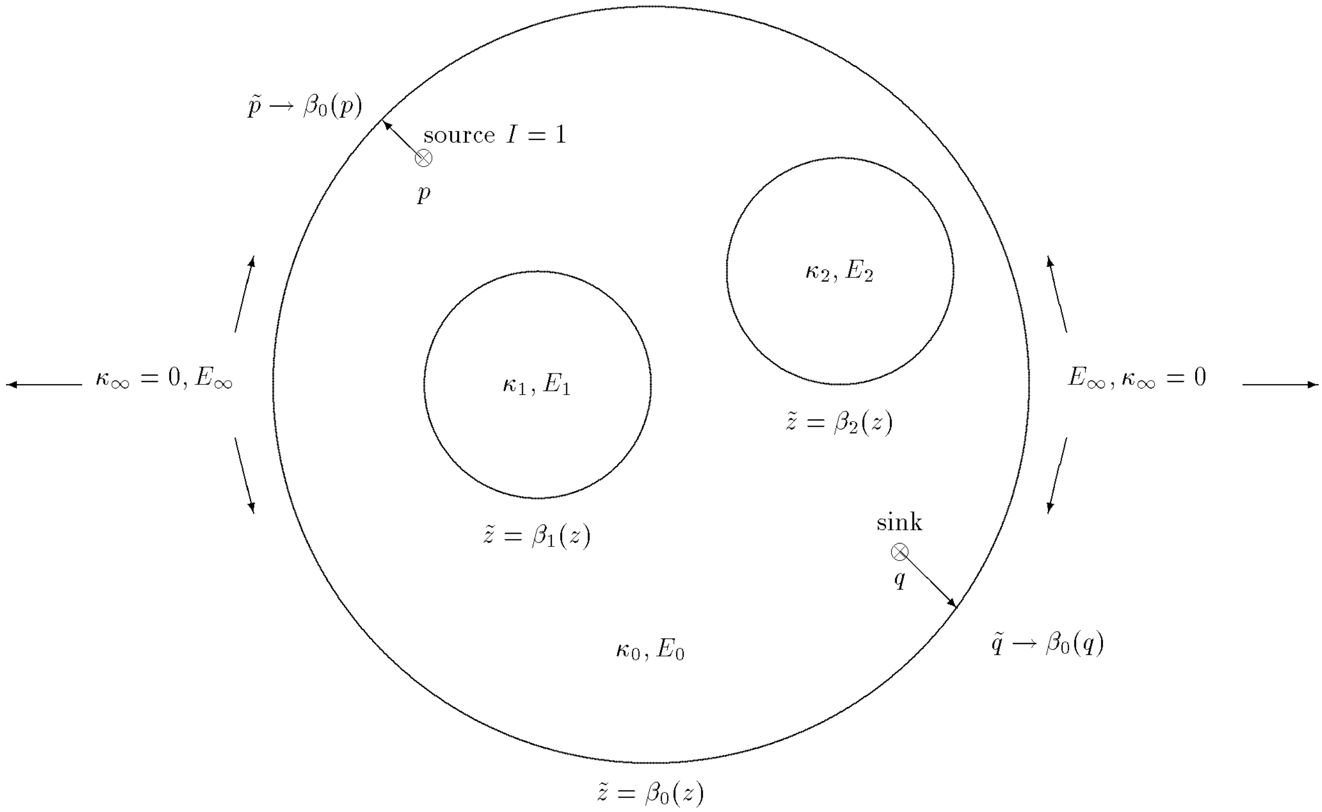
$$\boxed{(1+\eta)E_r(z) = \underbrace{E_l(z) + \eta\beta'(z)\hat{E}_l \circ \beta(z)}_{\text{transformation law of a 1–form}},\ \eta = \frac{\kappa_r - \kappa_l}{\kappa_r + \kappa_l}}$$

A model for currents in plane biological tissues:

Cross section of the human thorax



Currents on an insulated disk with circular anomalies external to one another of constant conductivities



μ**–th circle of radius** $r_\mu > 0$**, center** $\vec{m}_\mu = \vec{\sigma}_1 m_\mu$**:**

$$(z - m_\mu)(\tilde{z} - \tilde{m}_\mu) = r_\mu^2$$

$$\Rightarrow \; \tilde{z} = \beta_\mu(z) = \frac{\tilde{m}_\mu z + r_\mu^2 - (m_\mu)^2}{z - m_\mu}, \quad \widehat{\beta}_\mu(z) = [\beta_\mu(\tilde{z})]^\sim$$

$$\widehat{\beta}_\mu \circ \beta_\mu(z) = \mathbb{1}(z) = z, \quad \widehat{\beta}_\mu = \beta_\mu^{-1} \quad \text{inverse function}$$

Matching of the fields across the boundaries

$$E_\infty(z) = f_0(z) + 2f(z), \qquad \underbrace{|f(z)| < \infty}_{\text{on the complement of the anomalies}}$$

$$f_0(z) = \frac{1}{z-p} - \frac{1}{z-q}, \; \tilde{p} = \beta_0(p), \; \tilde{q} = \beta_0(q)$$

$$\Rightarrow \boxed{\beta_0' \hat{f}_0 \circ \beta_0 = f_0}$$

Matching $E_\infty \to E_0$:

$$E_0 = \underbrace{\frac{1}{2}(E_\infty + \beta_0' \hat{E}_\infty \circ \beta_0) = f_0 + f + \beta_0' \hat{f} \circ \beta_0}_{\substack{\text{satisfies the Neumann–condition} \\ \text{for arbitrary } f=f(z)}}$$

Partition of f into separate contributions from each anomaly

$\boxed{f = f_1 + f_2}$, $\quad f_1 \to$ poles in 1, $\quad f_2 \to$ poles in 2, ...

Matching $E_0 \to E_1$: $\quad \eta_1 = \dfrac{\kappa_1 - \kappa_0}{\kappa_1 + \kappa_0}$

$$(1 + \eta_1)E_1 = E_0 + \eta_1 \beta_1' \hat{E}_0 \circ \beta_1 =$$

$$= f_0 + \beta_0'(\hat{f} \circ \beta_0) + f_2 + \eta_1 \beta_1' \hat{f}_1 \circ \beta_1 +$$

$$\left.\begin{matrix} f_1 + \eta_1 (\hat{\beta}_0 \circ \beta_1)'(f \circ \hat{\beta}_0 \circ \beta_1) \\ \eta_1 \beta_1' \hat{f}_2 \circ \beta_1 + \eta_1 \beta_1' \hat{f}_0 \circ \beta_1 \end{matrix}\right\} \text{poles in 1} \;\; \Rightarrow \stackrel{!}{=} 0$$

symmetry $1 \leftrightarrow 2$

Coupled functional equations:

$$\eta_k = \frac{\kappa_k - \kappa_0}{\kappa_k + \kappa_0},\ k = 1, 2$$

$$f_{\substack{1\\2}} + \eta_{\substack{1\\2}} \widehat{\beta}_0 \circ \beta'_{\substack{1\\2}} f \circ \widehat{\beta}_0 \circ \beta_{\substack{1\\2}} = -\eta_{\substack{1\\2}} \beta'_{\substack{1\\2}} (\widehat{f}_0 \circ \beta_{\substack{1\\2}} + f_{\substack{2\\1}} \circ \beta_{\substack{1\\2}})$$

$$(1 + \eta_{\substack{1\\2}}) E_{\substack{1\\2}} = f_0 + \beta'_0 \widehat{f} \circ \beta_0 + f_{\substack{2\\1}} + \eta_{\substack{1\\2}} \beta'_{\substack{1\\2}} \widehat{f}_{\substack{1\\2}} \circ \beta_{\substack{1\\2}}$$

$$E_{\substack{1\\2}} = (1 - \eta_{\substack{1\\2}})(f_0 + \beta'_0 \widehat{f} \circ \beta_0 + f_{\substack{2\\1}})$$

$$E_0 = \underbrace{f_0 + f + \beta'_0 \widehat{f} \circ \beta_0}$$

defines E_0 on a triply–connected region, and, satisfies the Neumann–condition on the outer circle

Summary: Provided the anomalies are external to one another and have no contact with the outer boundary of the disk, the coupled functional equations can be solved by iteration.

Start of iteration: $f_k0 = -\eta_k \beta'_k \widehat{f}_0 \circ \beta_k, \quad k = 1, 2$

Noting, that for any non–singular linear fractional transformation $T(z) = \dfrac{az + b}{cz + d},\ ad - bc \neq 0 \Rightarrow$

$$\boxed{\frac{T'(z)}{T(z) - p} = \frac{1}{z - T^{-1}(p)} - \frac{c}{cz + d}}$$

$$\Rightarrow \beta'_k \widehat{f}_0 \circ \beta_k = \frac{1}{z - \widehat{\beta}_k \circ \beta_0(p)} - \frac{1}{z - \widehat{\beta}_k \circ \beta_0(q)}$$

Repeating this procedure, one finds for f_1 and f_2 convergent series of partial fractions (Mittag–Leffler, Lord Kelvin, Poincaré, Burnside), which represent **generalized Theta–fuchsian, pseudo–automorphic functions.**

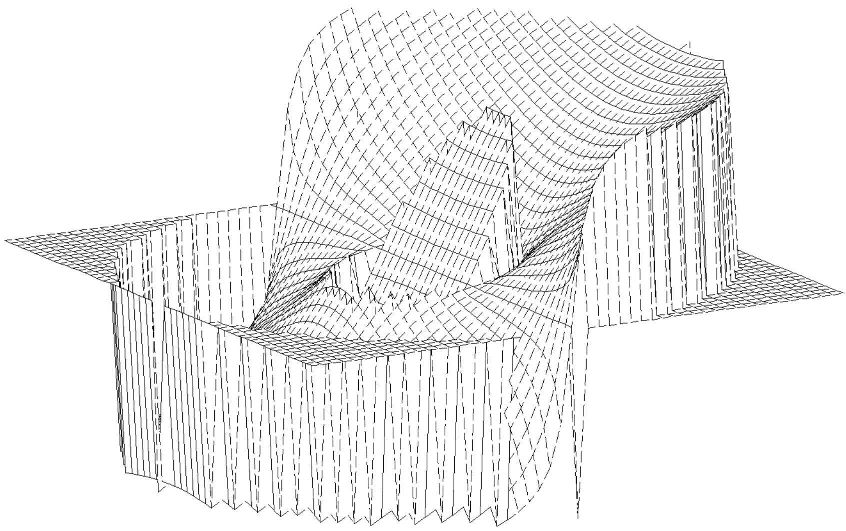
These generalized Theta–series have been summed on a PC in order to visualize the various Clifford vector fields.

The pertinent software has been developed by my doctoral fellow

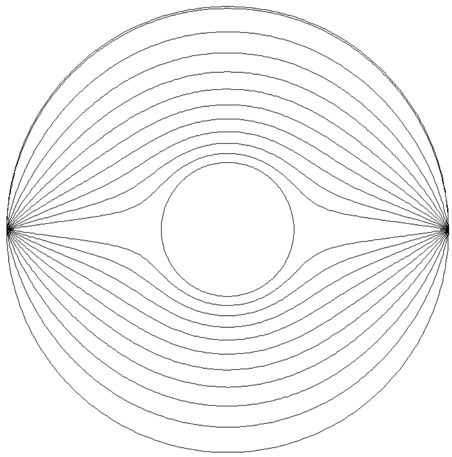
Martin Menzel

3–d plot of streamfunction $= \mathrm{Im}\Omega(z) = \Psi(z)$

$$E(z) = -\partial_z \Omega(z)$$



lines $\Psi(z) =$const. = fieldlines of $\vec{E} = E(z)\vec{\sigma}_1$



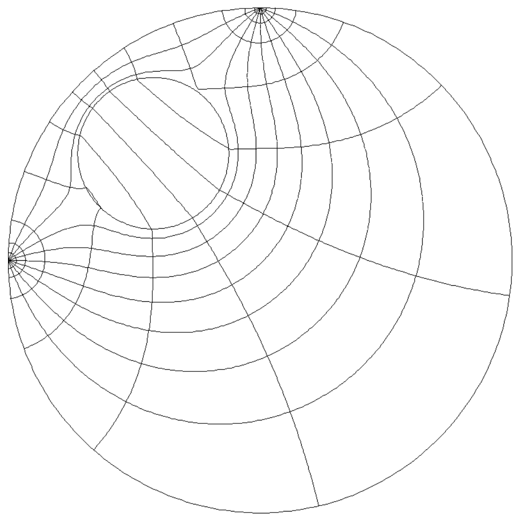
$$\vec{E}\vec{\sigma}_1 = E(z) = -\partial_z \Omega(z)$$

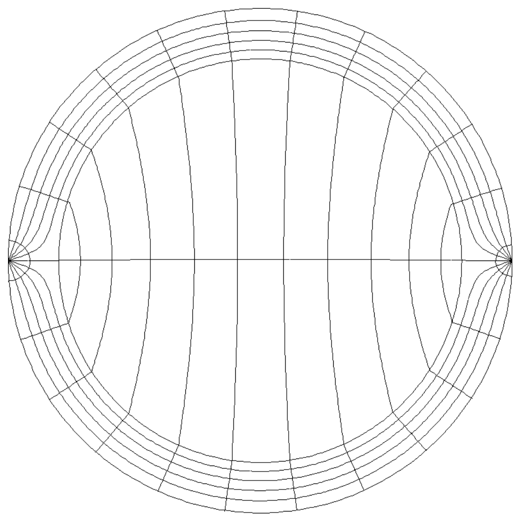
complex potential $\Omega(z) = \Phi(z) + \mathrm{i}\Psi(z), \quad \Phi, \Psi \in \mathbb{R}$

$$\Phi(z) = \text{const} \qquad \Psi(z) = \text{const}$$

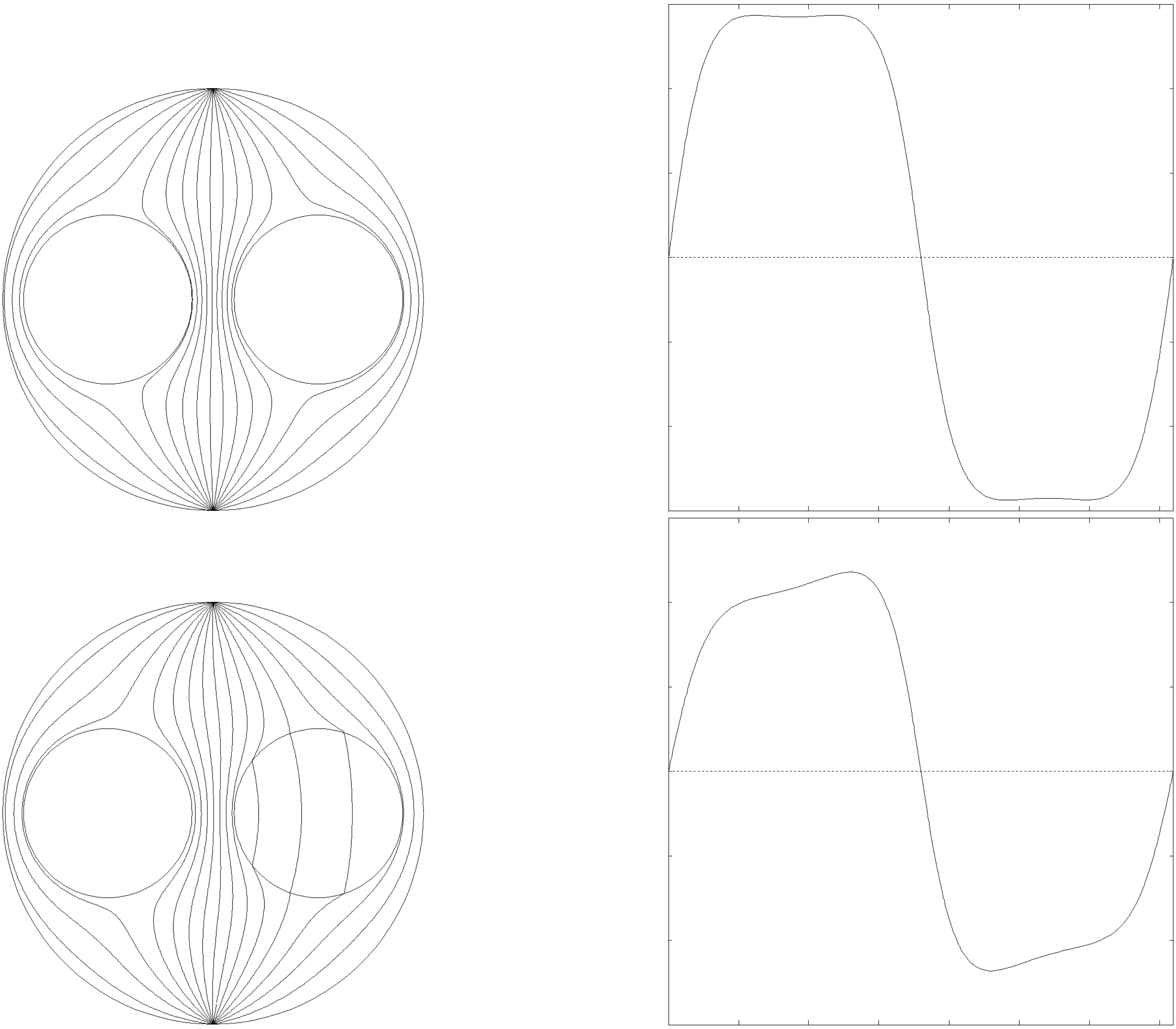
$$\downarrow \qquad\qquad \downarrow$$

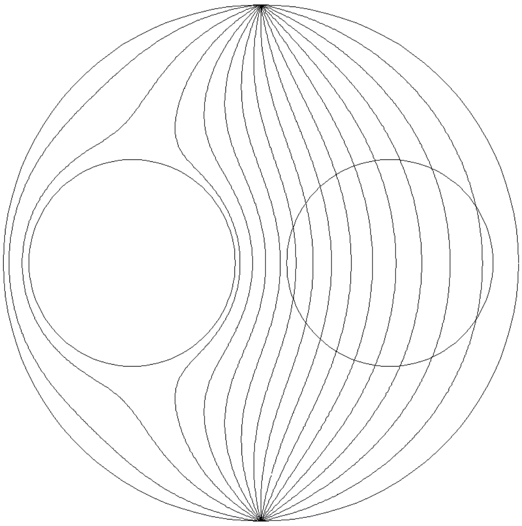
Equipotential lines and fieldlines

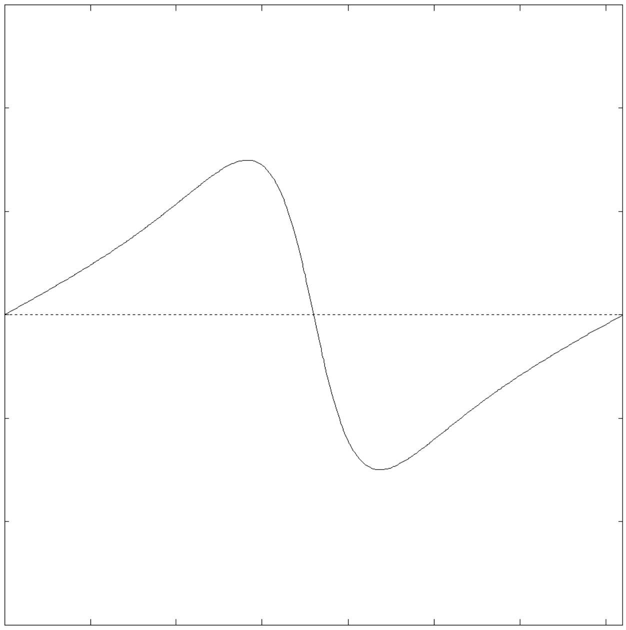


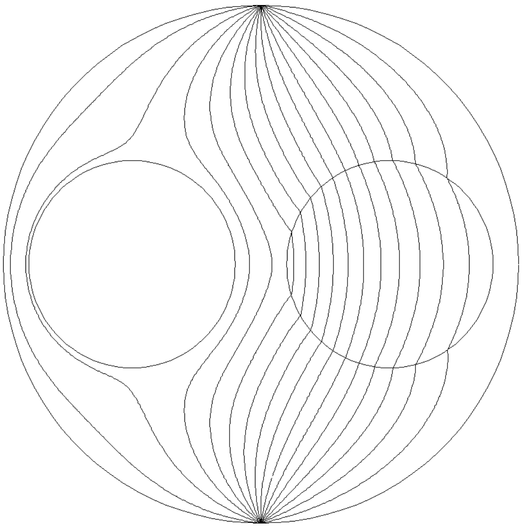


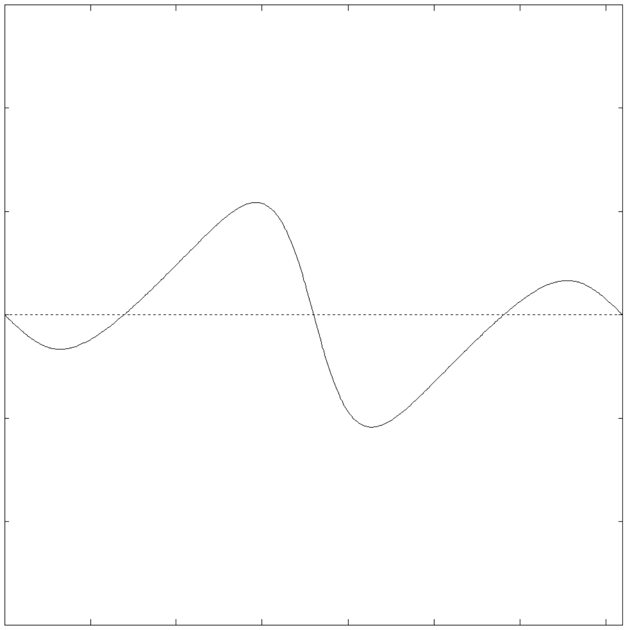
Potential with anomalies **minus** potential without anomalies along the outer boundary of the disk: Variation of the connectivity rank

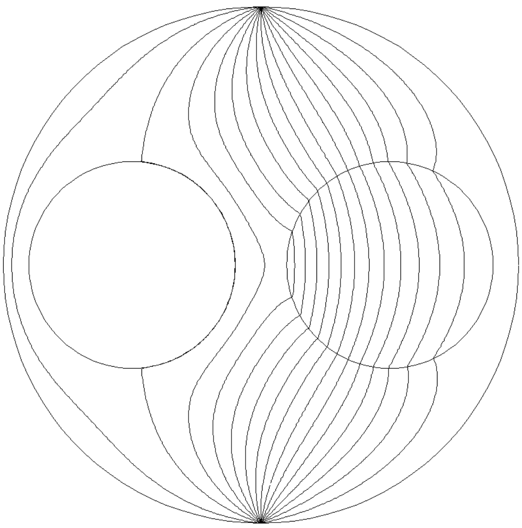


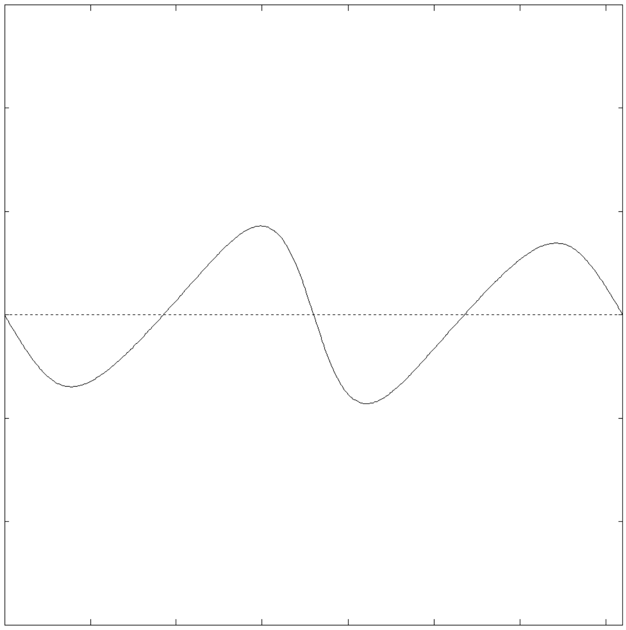




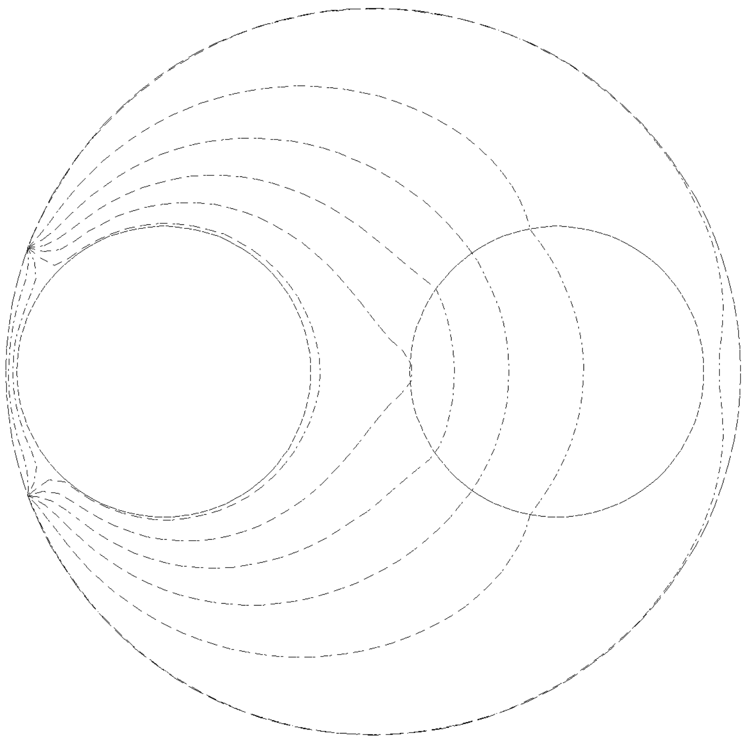


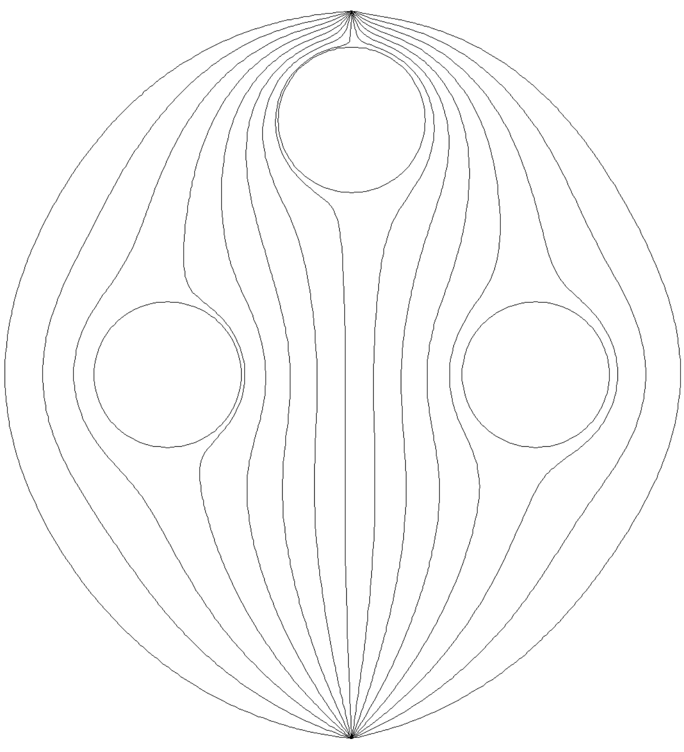






“a current needs not to take the shortest path”





a vector field of connectivity rank four boundary potential difference: connectivity rank 4