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Five Essays on Overlapping-Generations Models

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DISSERTATION

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Paul Maximilian Ritschel

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Summary

This cumulative dissertation in theoretical macroeconomics presents five articles on deterministic, analytically tractable overlapping-generations models with two-period-lived agents and production. The microeconomic foundations are rooted in contract and game theory, while the analysis of macroeconomic dynamics builds on dynamical systems theory.

Article 1 establishes that financial intermediation which provides efficient risk sharing cannot generate endogenous fluctuations. Article 2 examines how competition in the banking sector affects economic growth and welfare in the real sector. Article 3 investigates the impact of market power and strategic interaction in the capital market on savings, capital accumulation, and the qualitative dynamics. Article 4 introduces a novel parameterization of the production-possibility frontier in terms of wage-rental ratios that can enhance both the generality and tractability of two-sector growth models and allows for a refinement of the Rybczynski theorem. Building on this parameterization, Article 5 develops a tractable two-sector model with intergenerational pollution externalities to study the role of fiscal policy in steering the green transformation of an economy.

The findings of this dissertation contribute to a deeper understanding of the determinants of economic growth, the sources of real business cycles, and normative welfare implications, while also providing recommendations for policymakers.

List of Publications

The following publications by the author, whether published in peer-reviewed academic journals or on preprint platforms, contain research that forms part of this cumulative dissertation and are presented below in order of their appearance within the dissertation:

1. Ritschel, P., & Wenzelburger, J. (2024). Financial Intermediation and Efficient Risk Sharing in Two-Period Lived OLG Models, *Economic Theory Bulletin, 12*(1), 57-78. https://doi.org/10.1007/s40505-024-00263-z

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2. Rauber, T., & Ritschel, P. (2024). Banking Competition and Capital Dependence of the Production Sector: Growth and Welfare Implications, *International Review of Economics and Finance, 89*(Part B), 676-698. https://doi.org/10.1016/j.iref.2023.10.011

3. Ritschel, P. (2023). Capital Market Power and Economic Growth in an Overlapping-Generations Model with Rational Expectations, *Theoretical Economics Letters, 13*(5), 1253-1265. https://doi.org/10.4236/tel.2023.135069

4. Ritschel, P., & Wenzelburger, J. (2025). Production-Possibility Frontiers, Rybczynski's Theorem, and Factor-Intensity Reversals, *SSRN Working Paper, ID 4973036*. https://dx.doi.org/10.2139/ssrn.4973036

5. Ritschel, P., & Wenzelburger, J. (2025). Green Transformation and Fiscal Policy in a Two-Sector OLG Model with Production Externalities, *SSRN Working Paper, ID 5169780*. https://dx.doi.org/10.2139/ssrn.5169780

Contents

Introduction

“We live in a world where new generations are always coming along.”
– Paul Samuelson (1958)

Background

Within macroeconomics, *neoclassical growth theory* provides a framework for understanding why economies grow over time and what factors shape cross-country differences in income levels and other economic characteristics. It also serves as a theoretical backbone for studying the driving forces underlying real business cycles. The development of neoclassical growth theory reflects a progressive refinement in how economists model intertemporal decision-making and capital accumulation. This has led from early descriptive models of growth to optimal growth models and, ultimately, to overlapping-generations models, cf. Azariadis (1993).

As a first step in this progression, *descriptive growth models* focus on characterizing the evolution of key aggregate variables such as capital, output, and consumption under stylized assumptions. These models are distinguished by their simplicity, analytical tractability, and strong ability to replicate empirical time series (Acemoglu, 2008). However, savings behavior is treated as exogenous and is not derived from optimizing behavior of rational households, which limits these models’ ability to provide insight into the determinants of capital accumulation. The most prominent model within this class is the one-sector growth model of Solow (1956) and Swan (1956), in which savings are assumed to be a constant fraction of income.

Building on the pioneering work of Ramsey (1928) on the theory of optimal savings, Cass (1965) and Koopmans (1963) developed the *optimal growth model*, also commonly referred to as *Ramsey-Cass-Koopmans model*. This class of models extends descriptive growth theory by providing a fully microfounded framework in which household behavior is explicitly derived from the intertemporal optimization problem of an infinitely-lived representative agent endowed with well-defined preferences. The incorporation of preferences not only allows one to comprehend why economies accumulate capital at particular rates, but also enables normative analysis. The notion of an *optimal* growth model stems from interpreting the representative agent as a central social planner who maximizes a given social welfare function. In this context, Spear and Young (2023, p. 74) note that

> *“the infinite ‘life’ of the central planner was an approximation of the fact that planning institutions, or more generally, governments, typically outlive the individual people who populate the society over which the government reigns. Treating the government or planner as being infinitely lived (particularly when the objective function is discounted) is then simply a mathematical shortcut rather than any kind of profound statement about the organization of society”.*

Optimal growth models are inherently characterized by an infinite planning horizon, a feature that is often regarded as limiting and unsatisfactory. To address this concern, two standard justifications are commonly invoked, cf. Acemoglu (2008). First, agents may be finitely lived but face stochastic mortality, which renders their *expected* planning horizon infinite and leads to an optimization problem that is formally equivalent to that of the optimal growth framework, see Yaari (1965). Second, and more prominently, the infinite planning horizon can be justified through dynastic altruism, whereby agents care about the welfare of their descendants by leaving bequests and therefore effectively behave as if they were infinitely lived, see Barro (1974). This interpretation, however, relies on strong assumptions about altruistic preferences that abstract from intergenerational conflicts of interest.

Overlapping-generations (OLG) models naturally overcome the limitation of an infinite planning horizon by introducing a sequence of finitely lived cohorts of agents born at different dates, which overlap and can trade with one another (Spear & Young, 2023). Such an explicit demographic structure allows for transfers across generations, which lie at the core of many of the most pressing economic challenges of our time, including climate change mitigation, social security, national debt, demographic change, public healthcare, and education (De La Croix & Michel, 2002; Tvede, 2017). By their nature, optimal growth models do not allow for intergenerational transfers, implying that many of these issues cannot be adequately addressed within a representative-agent framework but instead require an overlapping-generations structure. While the origin of overlapping generations in economic theory is commonly attributed to Samuelson's (1958) *"consumption-loan model"*, Malinvaud (1987) clarifies that their first appearance can in fact be traced back a decade earlier to a book by Allais (1947). Nonetheless, Samuelson's pure-exchange model sparked considerable interest among economists. Unlike optimal growth models, OLG models with an infinite time horizon feature an infinite number of goods and consumers, see Shell (1971). This so-called *double-infinity structure* implies incomplete markets and leads to a failure of the first welfare theorem. Samuelson (1958) has shown that competitive equilibria in such environments need not be Pareto-efficient; a finding that provides a rationale for government intervention even in the absence of the classical sources of market failure (Spear & Young, 2023). The role of government policy in addressing competitive inefficiency is further highlighted by the influential contribution of Diamond (1965), which extends Samuelson's framework by introducing production from the factors capital and labor. Diamond's model has since become a workhorse of modern macroeconomics, as it substantially broadens the scope of the overlapping-generations model to address questions for which capital accumulation is essential. Galor (1992) further extends the model by introducing a two-sector production structure that distinguishes between consumption and investment goods, highlighting the role of sectoral heterogeneity in shaping long-run economic development.[1]

Synopsis and Outline

This cumulative dissertation comprises five self-contained articles in macroeconomic theory. While each article addresses a distinct research question and can be read independently, all

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[1] Another important extension is Lucas (1972), who pioneered stochastic OLG models. A stochastic version of optimal growth models was first developed by Brock and Mirman (1972), which later gave rise to what became known as *real business cycle model*, see Kydland and Prescott (1982) and Long and Plosser (1983).

share a common methodological foundation in overlapping-generations growth models. These are employed in preference to optimal growth models because of their explicit incorporation of demographic structure and the scope for intergenerational transfers, as discussed above. As is standard in this literature, time is assumed to be discrete, a choice that naturally reflects the cohort structure inherent in OLG models. Moreover, economic agents typically make decisions at regular intervals rather than continuously, cf. Acemoglu (2008).

In line with Diamond (1965), the models include production from the inputs capital and labor, thereby rendering capital accumulation endogenous. The population consists of homogeneous, risk-averse agents who live for two periods. Young agents inelastically supply labor, while old agents are retired and supply capital. Since aggregate uncertainty is absent, the models are deterministic. The analysis is conducted in a real setting, that is, money is absent. A single final good serves for both consumption and investment. Finally, each model incorporates some form of market failure or imperfection, such as market power, production externalities, or idiosyncratic risk in the presence of incomplete markets.

Following Böhm and Wenzelburger (1999), a sequential modeling approach is pursued that not only enhances analytical tractability, but also permits the application of well-known results from dynamical systems theory. An economic law captures the fundamental mechanisms of capital accumulation, while expectations are formed on the basis of a forecasting rule in the sense of Grandmont (1985). The combination of an economic law and a forecasting rule yields a time-discrete dynamical system governing the growth paths of the economy. The analysis therefore focuses on the forward dynamics, which is naturally more meaningful than the backward dynamics, cf. Medio and Raines (2006). Expectations are assumed to be rational and homogeneous among agents. Nevertheless, the modeling approach is generic and can accommodate subjective beliefs. In this respect, the approach is behavioral and allows for bounded rationality: agents behave optimally given their expectations, but these expectations may, in principle, be erroneous.

As emphasized above, analytical tractability constitutes a central concern throughout the dissertation. Numerical investigations based on standard parameterizations for preferences and technology, as well as references to the empirical literature, complement the analysis.

A concise summary of each article follows, outlining the underlying models and highlighting the main findings and contributions.

Articles 1 and 2 are concerned with the finance–growth nexus and investigate how financial intermediation affects capital accumulation in an overlapping-generations model with idiosyncratic production risk. The analysis shows that financial intermediation promotes economic growth without altering the qualitative dynamics, thereby contradicting the claim of Banerji et al. (2004) that banking activity can expose an otherwise calm economy to the full variety of endogenous fluctuations.

Article 1, entitled *"Financial Intermediation and Business Cycles in Two-Period Lived OLG Models"*, is joint work with Jan Wenzelburger. The article circulates as

> Ritschel, P., & Wenzelburger, J. (2024). Financial Intermediation and Business Cycles in Two-Period Lived OLG Models, *SSRN ID 4286324*. http://dx.doi.org/10.2139/ssrn.4286324.

A shortened version, excluding Section 1.5 on business cycles, has been published as

> Ritschel, P., & Wenzelburger, J. (2024). Financial Intermediation and Efficient Risk Sharing in Two-Period Lived OLG Models, *Economic Theory Bulletin, 12*(1), 57-78. https://doi.org/10.1007/s40505-024-00263-z.

In the model, agents may finance their retirement by investing part of their wage income in risky production projects which expose them to idiosyncratic income shocks. Banks arise endogenously in such an environment, as providing insurance allows them to increase agents' welfare. We show that a collective bank offering agents welfare-maximizing loan and deposit contracts can implement the optimal allocation of a myopic social planner. Such *efficient contracts* provide full insurance without a premium for the bank, but must enlarge the disposable income of agents in order to be accepted. According to Banerji et al. (2004), this income effect is responsible for generating business cycles that cannot arise in the absence of banks. Our analysis contradicts this claim by establishing that endogenous fluctuations may only emerge if agents mistakenly accept inefficient contracts. Efficient contracts, by contrast, induce monotonic growth paths that are qualitatively indistinguishable from those of an economy without banks. The contribution of the article lies in clarifying that financial intermediation cannot be held responsible for instigating endogenous fluctuations, thereby advancing our understanding of the driving forces behind business cycles.

Article 2, entitled *"Banking Competition and Capital Dependence of the Production Sector: Growth and Welfare Implications"*, is coauthored by Tom Rauber. A shortened version of the article, excluding the discussion in Section 2.7 on financial stability, has appeared in

> Rauber, T., & Ritschel, P. (2024). Banking Competition and Capital Dependence of the Production Sector: Growth and Welfare Implications, *International Review of Economics and Finance, 89*(Part B), 676-698. https://doi.org/10.1016/j.iref.2023.10.011.

Building on the model considered in Article 1, we replace the benevolent collective bank with profit-maximizing commercial banks and investigate how competition in the banking sector affects welfare and economic growth in the real sector. Banks act as financial intermediaries by investing deposits, together with part of their equity, in production projects. The degree of competition among banks influences capital accumulation via the level of deposit rates, affecting both agents' saving incentives and banks' intermediation margins. Comparing a monopolistic with a perfectly competitive banking sector reveals that the impact of inter-bank competition on capital accumulation depends on the distribution of factor incomes and, therefore, on the underlying production technology. In labor-intensive economies, wage income of agents accounts for a large share of total income, making strong saving incentives essential for capital accumulation. In this case, intense competition among banks is favorable. In capital-intensive economies, however, saving incentives are of minor importance as wage income is relatively low. Instead, banks' abundant capital investments spur capital accumulation. In this case, less competition can be advantageous. The findings contribute to a deeper understanding of the finance-growth nexus and help explain an empirical puzzle concerning the correlation between banking competition and economic growth.

An important insight of this article is that, in a dynamic setting, increased competition need not be welfare improving. Since savings depend on preferences, the economy will generically not attain a golden-rule steady state, so that the capital accumulation path fails to maximize agents' consumption. Here, Lipsey and Lancaster's (1956) *general theory of the second best* comes into play. Imperfect competition in the banking sector may partially counteract inefficient capital accumulation and therefore lead to a superior overall outcome. In our setting, the consumption gains from a capital accumulation path closer to the golden rule may outweigh the welfare losses arising from monopolistic rent extraction. The main message of the article is that the debate on the optimal degree of interbank competition should take into account not only implications for financial stability, but also the effect on capital accumulation in the real sector. As an aside, the analysis complements Article 1 by establishing that financial intermediation has no bearing on the qualitative dynamics of the economy.

The collective bank considered in Article 1 is not a price taker, as it takes into account the impact of its loan contracts on the realized capital-rental rate. Adapting this approach, Article 3, entitled *"Capital Market Power and Economic Growth in an Overlapping-Generations Model with Rational Expectations"*, shifts the focus to market power on the side of agents. A previous version has been published as

> Ritschel, P. (2023). Capital Market Power and Economic Growth in an Overlapping-Generations Model with Rational Expectations, *Theoretical Economics Letters, 13*(5), 1253-1265. https://doi.org/10.4236/tel.2023.135069.

The article studies how market power and strategic interaction in the capital market affect savings, capital accumulation, and the qualitative dynamics in a canonical overlapping-generations model à la Diamond (1965). A finite set of agents chooses savings strategically, internalizing the effect on the realized capital-rental rate in their decision problems. We establish conditions that ensure the existence and uniqueness of a *Nash-equilibrium growth path*, defined as a trajectory of the economy generated by an infinite sequence of temporary Nash equilibria in pure strategies. It turns out that relative to price takers with perfect foresight, market power induces agents to reduce their savings, resulting in lower capital accumulation and long-term economic growth. Market power may also give rise to non-monotonic dynamics that render the Nash-equilibrium growth paths qualitatively distinct from the perfect-foresight growth paths under competitive markets. The article thus contributes to growth and business cycle theory by identifying imperfect competition in the capital market as both an impediment to economic growth and a source of endogenous fluctuations. The findings also carry clear implications for competition policy, highlighting the importance of preserving intense competition and eliminating barriers to entry in capital markets to foster strong and stable economic growth.

Article 4, entitled *"Production-Possibility Frontiers, Rybczynski's Theorem, and Factor-Intensity Reversals"*, is joint work with Jan Wenzelburger that circulates as

> Ritschel, P., & Wenzelburger, J. (2025). Production-Possibility Frontiers, Rybczynski's Theorem, and Factor-Intensity Reversals, *SSRN Working Paper ID 4973036*. https://dx.doi.org/10.2139/ssrn.4973036.

The article introduces a novel parameterization of the production-possibility frontier for two-factor, two-goods models with homogeneous production functions. Since the factor-price ratio serves as the curve parameter, the approach naturally accommodates factor-intensity reversals, which are often ruled out in the literature through restrictive assumptions, primarily for reasons of technical convenience. In contrast to the conventional parameterization of the frontier in terms of output levels, our parameterization is explicit for most standard production functions and is analytically more tractable. As a result, it has the potential to improve both the generality and tractability of two-sector growth models. Besides facilitating an elementary proof of the concavity of the production-possibility frontier, the proposed parameterization allows to state a generalized, global version of the Rybczynski (1955) theorem that incorporates factor-intensity reversals. It turns out that, under constant returns to scale, all profit-maximizing production plans with the same marginal rate of transformation trace out piecewise straight lines. Under decreasing returns to scale, these lines bend to curves along which violations of the classical Rybczynski theorem may be observed.

Article 5, entitled *"Green Transformation and Fiscal Policy in a Two-Sector OLG Model with Production Externalities"*, is a collaboration with Jan Wenzelburger that forms the centerpiece of this dissertation. The article circulates as

> Ritschel, P., & Wenzelburger, J. (2025). Green Transformation and Fiscal Policy in a Two-Sector OLG Model with Production Externalities, *SSRN Working Paper ID 5169780.* https://dx.doi.org/10.2139/ssrn.5169780.

The purpose of the article is to develop a tractable two-sector growth model that elucidates the role of fiscal policy in facilitating the transition toward a greener economy. In the spirit of Galor (1992), we extend the classical overlapping-generations framework of Diamond (1965) by introducing a polluting brown and a non-polluting green production sector, alongside environmental preferences of agents. While the outputs of both sectors are perfect substitutes for consumers, pollution deteriorates the quality of the environment inherited by future generations. To internalize this *intergenerational externality*, output in the brown sector must be reduced, requiring agents to sacrifice some consumption. The model thus captures an inherently intergenerational trade-off between consumption, pollution abatement, and capital accumulation. To investigate this trade-off, we adopt the perspective of a benevolent social planner who takes the welfare of future generations into account. We show that there exists an optimal fiscal policy that decentralizes the social planner's preferred allocation in a market economy. The optimal policy combines an emissions tax with intergenerational transfers that compensate agents for emissions-tax-induced reductions in factor incomes. In an overlapping-generations setting, these *climate dividends* serve a dual purpose, as they simultaneously function as a mechanism for implementing an efficient capital accumulation path. Building on the parameterization of the production-possibility frontier introduced in Article 4, the analysis reveals that the emissions tax rate effectively stipulates the wage-rental ratio, which in turn parameterizes temporary equilibria. Since the implementation of optimal allocations does not require bonds but instead maintains a zero primary deficit, the model highlights that effective climate policy can be implemented entirely without public debt. Moreover, given that the model may admit multiple modified golden-rule steady states with distinct pollution levels, it helps explain the coexistence of high- and low-carbon economies. The policy implication is clear: emissions tax revenues must be rebated to consumers to compensate them for the carbon price.

In summary, the findings of this dissertation advance our understanding of the determinants of economic growth and the forces shaping economic dynamics. They also provide insight into normative welfare aspects and the role of key instruments, such as fiscal policy and financial intermediation, in implementing social optima. The results carry clear and actionable policy implications and point to several promising avenues for further research.

The dissertation is organized as follows. Chapters 1 – 5 contain the five articles. Subsequently, a final chapter concludes by reflecting on limitations of the models and discussing potential directions for future research. All proofs are collected in Appendices A – E at the end of the dissertation, while references are provided on a chapter-by-chapter basis.

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Chapter 1

Financial Intermediation and Business Cycles in Two-Period Lived OLG Models*

with Jan Wenzelburger

Statement of Prior Publication

This chapter is based on the article

> Ritschel, P., & Wenzelburger, J. (2024). Financial Intermediation and Efficient Risk Sharing in Two-Period Lived OLG Models. *Economic Theory Bulletin, 12*(1), 57-78, https://doi.org/10.1007/s40505-024-00263-z,

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Abstract

This article investigates a two-period-lived overlapping-generations (OLG) model with production that incorporates financial intermediation. A risk-neutral bank offers loan and deposit contracts that insure risk-averse agents against idiosyncratic income shocks and implement the efficient allocation of a myopic social planner. Under such *efficient risk sharing*, agents prefer financial intermediation to capital markets. The analysis demonstrates that in any two-period-lived OLG model in which productive capital is increasing in investment levels, financial intermediation that implements efficient risk sharing cannot instigate business cycles or complex dynamics. The resulting dynamics is monotonic and qualitatively indistinguishable from the dynamics of the classical OLG model by Diamond (1965). Business cycles may only occur if the bank offers inefficient contracts. Efficient contracts will, in general, not induce dynamically efficient growth paths.

Keywords: Overlapping generations, financial intermediation, risk sharing, endogenous fluctuations, business cycles, loan contracts, deposit contracts.

JEL Classification: D53, E32, E44, G21, O41.

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1.1 INTRODUCTION

Understanding the pace and patterns of economic growth is one of the central topics in macroeconomics. The empirical evidence that a well-functioning financial system is vital for real economic development is plentiful, e.g., see Levine (1997) or the excellent reviews of the empirical literature by Aziakpono (2011) and Levine (2005). Over the past four decades, relatively few theoretical contributions have incorporated financial intermediation into growth models. These try to link the promotion of economic growth to the fundamental functions that financial intermediaries carry out in an economy, see Pagano (1993). The seminal contribution by Greenwood and Jovanovic (1990), for example, highlights how risk sharing and the informational advantage of financial intermediaries encourage high-yield investments and economic growth. Bencivenga and Smith (1991), to mention another important contribution, extends the Diamond and Dybvig (1983) view by showing that liquidity provision induces savings behavior that fosters capital accumulation.

The literature on business cycles in overlapping-generations models with financial intermediation is relatively scarce. Williamson (1987) demonstrates that indivisibilities in investment projects may be a cause of business cycles. Smith (1998) finds that monopolistic financial intermediaries can increase the severity of existing business cycles. Azariadis and Smith (1998) show that bank-loan financed capital investments may generate business cycles if there is an adverse selection problem regarding the ability of the borrowers to honor their debt. Banerji et al. (2004) consider an OLG model in which loan and deposit contracts allow risk-averse agents to fully insure against idiosyncratic income shocks. They argue that risk sharing may expose the economy to endogenous fluctuations in the form of real-sector business cycles and conclude that the promotion of economic growth by financial intermediaries comes at the cost of the full variety of complex dynamics. Finally, the stochastic OLG model with financial intermediation developed in Gersbach and Wenzelburger (2003, 2008, 2012) also exhibits persistent business cycles. Macroeconomic productivity shocks trigger the failure of individual production projects. This risk cannot be diversified away, so that the model, unlike that in Banerji et al. (2004), involves aggregate uncertainty.

This article examines the extent to which efficient risk sharing can induce endogenous business cycles in a two-period lived OLG model which, in the absence of financial intermediation, is known to admit only monotonic growth. Following on from Banerji et al. (2004) by allowing for the standard class of intertemporal preferences used in that literature, we find that a collective bank can implement the efficient allocation of a myopic social planner by offering suitable loan and deposit contracts. These contracts provide complete insurance and must enlarge the disposable income of young agents in order to be accepted. This income effect, which in Banerji et al. (2004) is deemed responsible for causing endogenous fluctuations, is a consequence of a mere incentive problem and it turns out that it does not alter the qualitative dynamics of the economy. The key feature of our model is that productive capital and thus capital income is increasing in investment levels. We demonstrate that financial intermediation that implements the efficient allocation and diversifies away idiosyncratic risk *does not* generate business cycles or even complex dynamics as the dynamics of the economy is always monotonic. Business cycles may only be triggered by contracts that implement a local welfare minimum and, therefore, would not be accepted by rational agents.

In our framework, agents' incentive compatibility and participation constraints are explicit. Our analysis reveals that despite the fact that the bank maximizes the welfare of agents, incentive problems remain, so that the acceptance of an efficient contract is not as straightforward as one would expect. Contrary to Banerji et al. (2004) who argue that financial intermediation may generate a complex *backward* dynamics, we will focus on the *forward* dynamics of the economy. The reason is that the usefulness of the backward dynamics for a forward-time interpretation is limited and that the analysis is often restricted to a limited range of model parameters in the neighborhood of a steady-state solution, e.g., see Grandmont (1989) and Medio and Raines (2006).

This article is organized as follows. The next section presents the model and states all essential assumptions. In Section 1.3, we formulate the decision problems of both agents and the bank. We then introduce our notion of an efficient contract and establish its existence and uniqueness. Section 1.4 is dedicated to the dynamics induced by efficient contracts and contains the main results. Section 1.5 demonstrates that inefficient contracts may induce business cycles by revisiting an example from Banerji et al. (2004). Section 1.6 concludes. All proofs are collected in Appendix A.

1.2 MODEL PREREQUISITES

We consider an overlapping-generations model with discrete time $t = 0, 1, \ldots, \infty$. There is a single perishable good that can be consumed and invested. At the beginning of each period $t \geq 0$, a new generation comprising a unit-mass continuum of homogeneous agents is born. Agents are risk-averse and live for two periods. Their intertemporal preferences are represented by a life-cycle utility function $U : \mathbb{R}^2_+ \to \mathbb{R}$, defined by

$$U(c^1, c^2) \coloneqq u(c^1) + v(c^2),$$

where $c^1, c^2 \geq 0$ denote youthful and old-age consumption, respectively. The following assumptions on the preferences are standard.

Assumption 1.1 (Preferences).
The utility functions $u, v : \mathbb{R}_+ \to \mathbb{R}$ are twice continuously differentiable, strictly increasing, strictly concave, and satisfy the Inada conditions

$$\lim_{c \to 0} u'(c) = \infty \quad \text{and} \quad \lim_{c \to 0} v'(c) = \infty.$$

A young agent may become an entrepreneur by undertaking a risky production project, which may either be successful or fail. The likelihood of success depends on the amount of capital invested and is determined by a *success function* $p : \mathbb{R}_+ \to (0, 1]$ that stipulates the success probability $p(I)$ of the capital investment $I \geq 0$. The uncertainty about the outcome of a project resolves one period after capital has been invested. A project generates a verifiable gross rate of return $\varrho > 0$ if successful and zero if it fails.[1] Invoking the law of large numbers,

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[1] As an alternative interpretation, one may think of the entrepreneur as an investor who invests capital in a risky firm. A failed project is then equivalent to a defaulting firm.

the productive capital stock of the economy is

$$\Omega(I) := p(I)I. \tag{1.1}$$

The properties of the success function p are of central importance to our analysis and are stated in terms of properties of Ω in the following assumption.

Assumption 1.2 (Productive capital).
The function $\Omega : \mathbb{R}_+ \to \mathbb{R}_+$, defined by (1.1), is twice continuously differentiable, strictly increasing, and concave.

The important assumption for our results is that productive capital $\Omega(I)$ is strictly increasing in the investment level I. The concavity of Ω implies that the success probability $p(I)$ is nonincreasing in I.[2] Such a choice features the economic intuition that large-scale projects are more likely to fail.

Example 1.1 (Success probability).
The function $\Omega(I) = \frac{\kappa I}{1+I}$, where $0 < \kappa \leq 1$ is some constant, satisfies Assumption 1.2. The corresponding success probability $p(I) = \frac{\kappa}{1+I}$ is strictly decreasing in I.

The production sector of the economy is perfectly competitive. A neoclassical technology transforms capital $K \geq 0$ and labor $N \geq 0$ with constant returns to scale into output. Inputs are paid their marginal products and capital depreciates fully during production. We denote by $k := K/N$ the capital-labor ratio and by $f : \mathbb{R}_+ \to \mathbb{R}_+$ the production function of the representative firm in intensive form. Our assumptions on the technology are the following.

Assumption 1.3 (Technology).
The production function $f : \mathbb{R}_+ \to \mathbb{R}_+$ is thrice continuously differentiable, strictly increasing, strictly concave, and satisfies the Inada conditions

$$\lim_{k\to 0} f'(k) = \infty \quad \textit{and} \quad \lim_{k\to\infty} f'(k) < 1.$$

Moreover, f satisfies

$$\frac{f''(k)k}{f'(k)} > -1 \quad \textit{and} \quad \frac{f'''(k)k}{f''(k)} > -2 \quad \textit{for all} \quad k \geq 0.$$

Assumption 1.3 is satisfied by many standard production functions from the literature, as for example Cobb-Douglas production functions and CES production functions with an elasticity of substitution greater than one. The last two properties imposed on f in Assumption 1.3 imply that capital income $f'(k)k$ is strictly increasing and strictly concave in the capital-labor ratio k. These properties facilitate the existence and uniqueness of an efficient loan contract.

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[2] The concavity of Ω implies that $\frac{\Omega'(I)I}{\Omega(I)} = \frac{p'(I)I}{p(I)} + 1 \leq 1$ for all $I > 0$, so that $p'(I) \leq 0$ for all $I > 0$.

The young generation constitutes the workforce of the economy. Each young agent supplies one unit of labor inelastically to a perfectly competitive labor market. The old generation is retired and receives capital income only. Given a capital investment I, the productive capital stock of the subsequent period is $k = \Omega(I)$ and is paid its marginal product $\varrho = f'(\Omega(I))$. The capital income of the old generation thus becomes

$$g(I) := f'(\Omega(I))\,\Omega(I). \tag{1.2}$$

The following properties of g are essential for our results.

Lemma 1.1 (Capital income).
Let Assumptions 1.2 and 1.3 be satisfied. Then the function $g : \mathbb{R}_+ \to \mathbb{R}_+$*, defined by (1.2), is strictly increasing, strictly concave, and satisfies* $g(0) = 0$.

In the proof of Lemma 1.1, it is shown that, under the hypotheses of Assumption 1.3, the concavity of productive capital Ω is sufficient for the strict concavity of capital income g.

1.3 Financial Intermediation

To transfer resources to the second period of their lives, young agents may invest part of their wage income in a production project which exposes them to the idiosyncratic risk of an old-age income shock. To share this risk, agents may form an endogenous coalition in the form of a risk-neutral collective bank.[3] In each period $t \geq 0$, the bank offers young agents a loan contract (B_t, I_t, R_t), where $B_t \geq 0$ is the size of the loan, $I_t \geq 0$ is the capital investment in the project, and $R_t \geq 0$ is the gross interest rate on loans. Agents are protected by limited liability, as they do not have to repay the loan in case their project fails.[4] To finance its loans, the bank raises deposits from young agents by offering them a risk-free gross rate $r_t \geq 0$ on deposits.

1.3.1 Decision Problems

Each young agent must decide whether to invest in a project by accepting a loan contract or to undertake the project without funding from the bank instead. Independently of his investment decision, however, a young agent is allowed to deposit part of his wage income at the bank. The decision problem of a typical young agent is thus the following. Suppose that, in period t, the bank offers the loan contract (B_t, I_t, R_t) and the deposit rate r_t on savings. Consider first the case in which the agent *accepts* the loan contract. Given his wage income $w_t \geq 0$, the agent must decide on how much to consume and how much to save for retirement. By accepting the loan contract, his disposable income becomes $w_t^d := w_t + B_t - I_t$, so that

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[3] For further details on financial-intermediary coalitions, we refer to the article by Boyd and Prescott (1986) and, for an overview, to the textbook by Freixas and Rochet (2008).

[4] We tacitly assume that the bank possesses a monitoring technology that enables it to observe agents' investment behavior and enforce the contract. A stipulated investment is a special form of monitoring. For details on the role of monitoring in settings with limited liability, we refer to Holmström and Tirole (1997).

youthful consumption in period t is

$$c_t^1 = w_t^d - D_t, \tag{1.3}$$

where $0 \leq D_t \leq w_t^d$ is the amount saved and deposited at the bank. Old-age consumption in period $t+1$ is $c_{t+1}^{2g} \geq 0$ if the project is successful, and $c_{t+1}^{2b} \geq 0$ if the project fails. Since agents have limited liability, the constraint for old-age consumption reads

$$\begin{cases} c_{t+1}^{2g} &= r_t D_t + \pi(I_t) - R_t B_t \\ c_{t+1}^{2b} &= r_t D_t \end{cases}, \tag{1.4}$$

where $r_t D_t$ are the proceeds from the deposits, $\pi(I_t) := f'(\Omega(I_t)) I_t$ is the revenue from a successful project, and $R_t B_t$ is the loan repayment obligation.

The objective of a young agent is to maximize his expected utility of lifetime consumption. Using the budget constraints (1.3) and (1.4), the agent's decision problem takes the form

$$\max_{0 \leq D \leq w_t^d} \; u(w_t^d - D) + p(I_t)\, v(r_t D + \pi(I_t) - R_t B_t) + (1 - p(I_t))\, v(r_t D). \tag{1.5}$$

Given a wage rate w_t, a deposit rate r_t, and a loan contract (B_t, I_t, R_t) with $w_t^d \geq 0$ and $r_t w_t^d + \pi(I_t) - R_t B_t \geq 0$, a solution to Problem (1.5) is given by the *savings function*

$$S(w_t, B_t, I_t, R_t, r_t) := \operatorname*{argmax}_{0 \leq D \leq w_t^d} \; u(w_t^d - D) + p(I_t)\, v(r_t D + \pi(I_t) - R_t B_t) + (1 - p(I_t))\, v(r_t D).$$

Inserting the savings function into the objective function establishes the value function for Problem (1.5), which is denoted by $V(w_t, B_t, I_t, R_t, r_t)$.

Consider now the case in which the agent *rejects* the loan contract and undertakes the project without funding from the bank. To do so, he will invest the amount I^{A} in the project and deposit the amount D^{A} at the bank in order to safeguard old-age consumption against the failure of the project. Formally, given the wage rate w_t and the deposit rate r_t, the corresponding decision problem reads

$$\begin{aligned} \max_{I^{\mathrm{A}}, D^{\mathrm{A}}} \; & u(w_t - D^{\mathrm{A}} - I^{\mathrm{A}}) + p(I^{\mathrm{A}})\, v(r_t D^{\mathrm{A}} + \pi(I^{\mathrm{A}})) + (1 - p(I^{\mathrm{A}}))\, v(r_t D^{\mathrm{A}}) \\ \text{s.t.} \quad & I^{\mathrm{A}}, D^{\mathrm{A}} \geq 0 \;\text{ and }\; I^{\mathrm{A}} + D^{\mathrm{A}} \leq w_t. \end{aligned} \tag{1.6}$$

The value function for Problem (1.6), which stipulates the agent's reservation utility, is well defined and denoted by $U_{\mathrm{res}}(w_t, r_t)$.

The decision problem of the bank is the following. Since the bank is collectively owned, it offers a loan contract (B_t, I_t, R_t) and a deposit rate r_t so as to maximize the expected utility $V(w_t, B_t, I_t, R_t, r_t)$ of a typical young agent. Using the law of large numbers, the bank correctly anticipates that the capital investment I_t results in the loan default rate $1 - p(I_t)$. Therefore, the two feasibility constraints of the bank are the *profit constraint*

$$p(I_t) R_t B_t - r_t D_t \geq 0, \tag{PrC}$$

stating that bank profits must be non-negative, and the *resource constraint*

$$D_t \geq B_t, \tag{RC}$$

noting that the bank has no equity. Moreover, both the loan and the deposit contract must be compatible with the agent's savings behavior. Since the amount saved is at the discretion of the agent, the bank has to fulfill the *incentive compatibility constraint*

$$D_t = S(w_t, B_t, I_t, R_t, r_t) \tag{IC}$$

in order to obtain the amount of deposits required in (RC). Finally, since the agent may decide to invest without funding from the bank, the loan contract must be designed in such a way that the agent prefers the loan contract to undertaking the project without the bank. Formally, this *participation constraint* reads

$$V(w_t, B_t, I_t, R_t, r_t) \geq U_{\text{res}}(w_t, r_t), \tag{PC}$$

stating that the expected utility of accepting both the loan and the deposit contract must be at least as high as the reservation utility.

Given the wage rate w_t, the decision problem of the bank thus takes the form

$$\begin{aligned} \max_{B,I,R,r} \quad & V(w_t, B, I, R, r) \\ \text{s.t.} \quad & p(I)RB - rS(w_t, B, I, R, r) \geq 0, \quad S(w_t, B, I, R, r) \geq B, \\ \text{and} \quad & V(w_t, B, I, R, r) \geq U_{\text{res}}(w_t, r). \end{aligned} \tag{1.7}$$

1.3.2 EFFICIENT ALLOCATIONS

We next establish the efficient allocation that a myopic social planner would implement. Given the wage rate w_t, the social planner's objective in period t is to maximize the welfare of the generation born in t.[5] By the law of large numbers, the mass of successful agents in the subsequent period is $p(I)$, while the mass of failed agents is $1 - p(I)$. The capital income in the subsequent period is $g(I)$, independently of the state of nature. The social planner's maximization problem thus takes the form

$$\begin{aligned} \max_{I,c^1,c^{2g},c^{2b}} \quad & u(c^1) + p(I)\, v(c^{2g}) + (1 - p(I))\, v(c^{2b}) \\ \text{s.t.} \quad & I, c^1, c^{2g}, c^{2b} \geq 0, \quad c^1 + I \leq w_t, \\ \text{and} \quad & p(I)\, c^{2g} + (1 - p(I))\, c^{2b} \leq g(I). \end{aligned} \tag{1.8}$$

The solution $(I_t^\star, c_t^{1\star}, c_{t+1}^{2g\star}, c_{t+1}^{2b\star})$ to Problem (1.8) will be referred to as *efficient allocation*. Its existence and uniqueness are established in the following proposition.

Proposition 1.1 (Efficient allocation).
Let Assumptions 1.1 – 1.3 be satisfied and $w_t > 0$ be given. Then Problem (1.8) admits a uniquely determined solution $(I_t^\star, c_t^{1\star}, c_{t+1}^{2g\star}, c_{t+1}^{2b\star})$, where the efficient consumption plan is

$$c_t^{1\star} = w_t - I_t^\star \quad \textit{and} \quad c_{t+1}^{2g\star} = c_{t+1}^{2b\star} = g(I_t^\star),$$

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[5] Since our research question is not concerned with intergenerational externalities as, for example, in Ennis and Keister (2003), a myopic social planner is justified.

and the efficient investment level $0 < I_t^\star < w_t$ *is*

$$I_t^\star = \underset{0 \leq I \leq w_t}{\operatorname{argmax}} \; u(w_t - I) + v(g(I)). \tag{1.9}$$

Proposition 1.1 states that if resources are allocated efficiently, the young generation consumes its wage income less the efficient investment level, while the old generation consumes aggregate capital income. Since agents are risk averse, old-age consumption with an efficient allocation is perfectly smoothed out across both possible states of nature. Observe that the uniqueness of the efficient allocation hinges on the strict concavity of the objective function in (1.9). This in turn is guaranteed, since the utility functions u and v are strictly concave by Assumption 1.1 and capital income g is concave by Lemma 1.1.

1.3.3 Efficient Contracts

The natural question now is whether financial intermediation that offers loan and deposit contracts in line with Problem (1.7) can implement the efficient allocation determined in Proposition 1.1. In situations in which agents' private actions are difficult to control, the arising incentive constraints make it questionable whether an efficient outcome can be achieved, see Myerson (1979). To address this problem, we will next define an *efficient contract* as a contract that implements the efficient allocation and is optimal for both agents and the bank.

Definition 1.1 (Efficient contract).
Given the wage rate w_t*, a loan contract* (B_t, I_t, R_t) *together with a deposit rate* r_t *is called an efficient contract (in period* t*) if the following holds true:*

(i) The quadruple (B_t, I_t, R_t, r_t) *solves the bank's problem (1.7).*

(ii) The allocation induced by (B_t, I_t, R_t, r_t) *is efficient in the sense of Proposition 1.1.*

With an efficient contract, agents are fully insured and old-age consumption is independent of the success of the project because the bank completely diversifies away idiosyncratic risk. Observe that, by definition, an efficient contract must be incentive compatible. Incentive compatibility of the efficient deposit rate r_t implies in particular that agents decide at their own discretion to save the amount of funds required for complete risk sharing.

Our next proposition establishes the existence of a unique efficient contract.[6]

Proposition 1.2 (Existence and uniqueness of the efficient contract).
Let Assumptions 1.1 – 1.3 be satisfied and $w_t > 0$ *be given. Then there exists a uniquely determined efficient contract* (B_t, I_t, R_t, r_t)*, which is given by the following equations:*

(i) The efficient investment level is

$$I_t = \underset{0 \leq I \leq w_t}{\operatorname{argmax}} \; u(w_t - I) + v(g(I)). \tag{1.10}$$

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[6] Notice again that the uniqueness of the efficient investment level hinges on the concavity of the objective function in Problem (1.10), which is implied by the (strict) concavity of u, v, and g.

(ii) The efficient deposit rate is

$$r_t = g'(I_t). \tag{1.11}$$

(iii) The efficient loan interest rate is

$$R_t = \frac{g'(I_t)}{p(I_t)}. \tag{1.12}$$

(iv) The efficient loan size is

$$B_t = \frac{g(I_t)}{g'(I_t)}. \tag{1.13}$$

(v) The profit constraint (PrC) and the resource constraint (RC) are binding with

$$D_t = B_t = S(w_t, B_t, I_t, R_t, r_t). \tag{1.14}$$

An immediate implication of Proposition 1.2 is that with an efficient contract, the bank extracts no rent as it seizes all the proceeds from the successful projects, $R_t B_t = \pi(I_t)$, and awards the whole surplus to the old consumers. A second implication of Proposition 1.2 is an *income effect*, already identified in Banerji et al. (2004). Since capital income g is strictly concave by Lemma 1.1, its elasticity is smaller than unity, so that (1.13) implies

$$B_t - I_t = \Big(\tfrac{g(I_t)}{g'(I_t) I_t} - 1\Big) I_t > 0. \tag{1.15}$$

Hence, the efficient contract enlarges the disposable income of the young agent, $w_t^d = w_t + B_t - I_t > w_t$.[7] By a slight abuse of notation, the savings function corresponding to an *efficient* loan contract in (1.14) thus takes the standard form

$$S(w_t^d, r_t) := \underset{0 \le D \le w_t^d}{\operatorname{argmax}} \; u(w_t^d - D) + v(r_t D). \tag{1.16}$$

Since Assumption 1.1 implies that both youthful and old-age consumption are normal goods, the young agent prefers the efficient contract to a pure deposit contract without investing. Note that if capital income g were not concave, then the income effect (1.15) may be negative, in which case the participation constraint (PC) is violated. In the proof of Proposition 1.2, we show that under the hypotheses of Assumptions 1.1 – 1.3, the participation constraint (PC) is satisfied by establishing that the expected utility of an efficient contract exceeds the expected utility of an investment without funding from the bank combined with precautionary savings.

Example 1.2 (Logarithmic utility).
For the logarithmic utility functions $u(c^1) = \ln(c^1)$ and $v(c^2) = \beta \ln(c^2)$, where $\beta > 0$ is a time-discount factor, the savings function (1.16) is independent of the deposit rate because

$$S(w_t^d, r_t) \equiv S(w_t^d) = \tfrac{\beta}{1+\beta}\, w_t^d,$$

that is, the agent saves a constant fraction out of his disposable income.

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[7] One may interpret an efficient contract as the agent selling his project to the bank for the amount $B_t - I_t > 0$ and then saving the amount $S(w_t^d, r_t)$.

We conclude this section with two important remarks.

Remark 1.1 (Tying contracts).
If capital income g is non-concave, w_t^d may fall below w_t because the income effect of the loan contract may be negative. In this case, the agent rejects the loan contract because he is better off saving out of wage income w_t, even without investing in a project. The bank can still implement the efficient allocation by tying the loan contract (B_t, I_t, R_t) to the deposit contract r_t and offering agents who want to save only a deposit rate $\tilde{r}_t \in (0, r_t)$ that makes them worse off. The existence of $\tilde{r}_t$ is seen as follows. The strict concavity of v implies that

$$U_{\text{res}}(w_t, r) < \max\Big\{u(w_t - D^{\text{A}} - I^{\text{A}}) + v(rD^{\text{A}} + g(I^{\text{A}})) \mid D^{\text{A}} \geq 0,\, I^{\text{A}} \geq 0,\, D^{\text{A}} + I^{\text{A}} \leq w_t\Big\}$$

for all $r \geq 0$. For $r = 0$, the r.h.s. attains its maximum at $(D_t^{\text{A}} = 0, I_t^{\text{A}} = I_t)$, yielding the utility level $u(w_t - I_t) + v(g(I_t))$. This shows that for $r = 0$, the agent is strictly better off accepting the tying contract. Since the reservation utility U_{res} is continuous and strictly increasing in r, there exists a largest positive deposit rate $\tilde{r}_t \in (0, r_t)$, defined by

$$U_{\text{res}}(w_t, \tilde{r}_t) = u(w_t - I_t) + v(g(I_t)),$$

such that the agent is indifferent between accepting the tying contract and investing in a project without funding from the bank.

Remark 1.2 (Efficient contract).
For each $I \geq 0$, define the two elasticities

$$\eta(I) := \frac{p'(I)I}{p(I)} \quad \text{and} \quad \epsilon(I) := \frac{f''(\Omega(I))\,\Omega(I)}{f'(\Omega(I))}.$$

Proposition 1.2 now implies that an efficient contract (B_t, I_t, R_t, r_t) satisfies

$$\begin{aligned} R_t &= \big[1 + \epsilon(I_t)\big]\big[1 + \eta(I_t)\big] f'(\Omega(I_t)) \\ r_t &= \big[1 + \epsilon(I_t)\big]\big[1 + \eta(I_t)\big] p(I_t) f'(\Omega(I_t)) \\ B_t &= \big(\big[1 + \epsilon(I_t)\big]\big[1 + \eta(I_t)\big]\big)^{-1} I_t, \end{aligned}$$

where I_t is the efficient investment level (1.10). Apart from the multiplier $\big[1 + \epsilon(I_t)\big]$, these equations coincide with those in Banerji et al. (2004, p. 2225). This multiplier accounts for the collective bank exploiting her knowledge of how the investment I_t affects the realized return on investment $f'(\Omega(I_t))$. For a Cobb-Douglas production function $f(k) = Ak^{\alpha}$, where $A > 0$ and $0 < \alpha < 1$, this multiplier is constant because $\epsilon(I) \equiv \alpha - 1$. In a model with perfect competition among banks, this multiplier vanishes as banks are price takers.

1.4 CAPITAL ACCUMULATION AND QUALITATIVE DYNAMICS

This section investigates the qualitative dynamics induced by efficient contracts. The aggregate capital stock of the economy is determined by the total capital endowment of the successful projects. The law of large numbers implies that, on aggregate, the share of successful production projects in period $t+1$ is $p(I_t)$, where I_t is the efficient investment level (1.10). Define the *investment function* $\mathcal{I} : \mathbb{R}_+ \to \mathbb{R}_+$ by setting

$$I_t = \mathcal{I}(w_t) := \underset{0 \le I \le w_t}{\operatorname{argmax}} \; u(w_t - I) + v(g(I)). \tag{1.17}$$

Given the wage rate w_t, the productive capital stock of the subsequent period $t+1$ is then[8]

$$k_{t+1} = \Omega(\mathcal{I}(w_t)).$$

Since competition in the labor market is perfect, the realized wage rate is determined by the marginal product of labor, so that $w_t = w(k_t) := f(k_t) - f'(k_t)k_t$. It follows that capital accumulation is driven by the time-one map $G : \mathbb{R}_+ \to \mathbb{R}_+$, defined by

$$k_{t+1} = G(k_t) := \Omega\big(\mathcal{I}(w(k_t))\big). \tag{1.18}$$

The dynamical system (1.18) governs the forward dynamics of the economy, in the sense that any growth path $\{k_t\}_{t=0}^{\infty}$ of the economy with initial capital $k_0 > 0$ is recursively defined by $k_{t+1} = G(k_t)$, $t = 0, 1, \ldots, \infty$.

We are now in the position to state our main result.

Theorem 1.1 (Monotonic dynamics)**.**
Let Assumptions 1.1 – 1.3 be satisfied and assume that the bank offers efficient contracts. Then $G' > 0$, so that the dynamics of the economy is monotonic.

Theorem 1.1 demonstrates that if financial intermediation implements efficient contracts, then the resulting dynamics is always monotonic. Depending on the initial condition $k_0 > 0$, all growth paths $\{k_t\}_{t=0}^{\infty}$ generated by (1.18) are either monotonically increasing or monotonically decreasing, as in the example portrayed in Figure 1.1.

Endogenous fluctuations, including complex dynamics, require the time-one map G to be decreasing at least in some neighborhood of a steady state $k_\star$ of G. This behavior is ruled out by Theorem 1.1. Since $\Omega' > 0$ by Assumption 1.2 and $w' > 0$ by Assumption 1.3, the chain rule applied to (1.18) implies that $G' > 0$ if and only if $\mathcal{I}' > 0$. In the proof of Theorem 1.1, we establish that the latter condition is ensured by the efficiency property of the investment level $\mathcal{I}(w)$ which determines a welfare *maximum* of agents.

The qualitative dynamics induced by efficient contracts may be classified by the stability properties of the steady states $k_\star = G(k_\star)$.

Proposition 1.3 (Existence of steady states)**.**
Let Assumptions 1.1 – 1.3 be satisfied and assume that the bank offers efficient contracts. Then the following holds true.

(i) The origin $k_\star = 0$ is a steady state of G if and only if $f(0) = 0$.

---

[8] Observe that since aggregate labor supply is unity at all times, k_{t+1} coincides with the capital-labor ratio in period $t+1$.

(ii) If either $f(0) > 0$ or $\lim_{k\to 0} G'(k) > 1$, then G has at least one positive steady state $k_\star > 0$. The largest one of these steady states is asymptotically stable.

Proposition 1.3 implies that the dynamics of our model is qualitatively equivalent to the dynamics of the standard two-period lived OLG model, e.g., see De La Croix and Michel (2002). The following example with a standard parameterization from the literature on OLG models is insightful.

Example 1.3 (Logarithmic utility and Cobb-Douglas technology). *Consider the success function $p(I) = \frac{1}{1+I}$ combined with the logarithmic utility functions $u(c^1) = \ln(c^1)$ and $v(c^2) = \beta \ln(c^2)$, where $\beta > 0$, and the Cobb-Douglas production function $f(k) = Ak^\alpha$, where $A > 0$ and $0 < \alpha < 1$. The investment function (1.17) takes the form*

$$\mathcal{I}(w_t) = \sqrt{\left(\tfrac{1+\alpha\beta}{2}\right)^2 + \alpha\beta w_t} - \tfrac{1+\alpha\beta}{2},$$

and the evolution of capital-labor ratios is governed by the dynamical system

$$k_{t+1} = G(k_t) = 1 - \left(\tfrac{1-\alpha\beta}{2} + \sqrt{\left(\tfrac{1+\alpha\beta}{2}\right)^2 + \alpha\beta(1-\alpha)Ak_t^\alpha}\right)^{-1}.$$

Since $G' > 0$, the dynamics is monotonic and all growth paths $\{k_t\}_{t=0}^\infty$ with $k_0 > 0$ converge to a unique positive steady state $k_\star > 0$ that is asymptotically stable, see Figure 1.1.

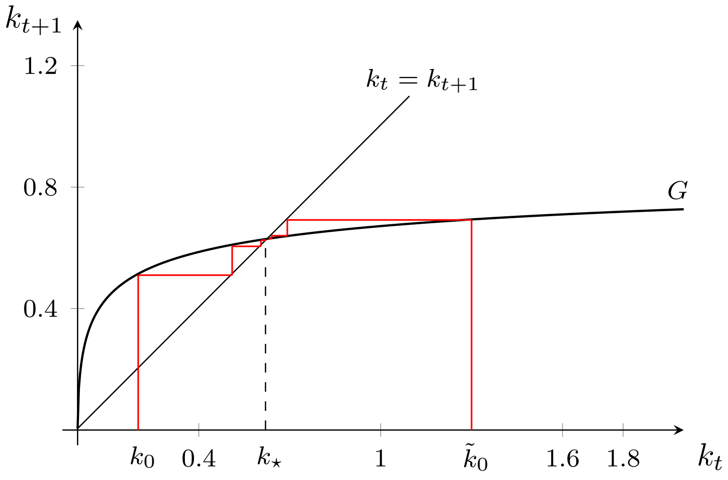

Figure 1.1. Monotonic convergence to a unique asymptotically stable steady state $k_\star > 0$ induced by efficient contracts ($A = 30$, $\alpha = 0.6$, $\beta = 1$).

The result that efficient contracts rule out business cycles and complex dynamics contradicts the findings in Banerji et al. (2004). They argue that efficient risk sharing through financial intermediation may generate endogenous fluctuations in an economy that otherwise would converge monotonically to a steady state. Our analysis shows that under the premise of efficient risk sharing, the qualitative dynamics of the model is monotonic, and that the enlarged disposable income is a mere byproduct of an incentive problem without any impact on the qualitative dynamics.

Remark 1.3 (Dynamics).
The evolution of the efficient investment levels is governed by a time-one map $H : \mathbb{R}_+ \to \mathbb{R}_+$ *of the form*

$$I_t = H(I_{t-1}) := \mathcal{I}\big(w(\Omega(I_{t-1}))\big), \tag{1.19}$$

noting that $w_t = w(\Omega(I_{t-1}))$. *The dynamics induced by* G *and* H *are qualitatively the same if* Ω *is invertible. Inserting (1.11) – (1.13) into (1.14) establishes an implicit difference equation that describes the same dynamics as (1.19). This difference equation corresponds to Equation (21) in Banerji et al. (2004).*

We proceed by demonstrating that the steady states of the economy will generically not coincide with the golden-rule steady state, so that the growth paths of the economy can be *dynamically inefficient* in the usual sense of macroeconomics. Given a wage rate w_t, youthful consumption in period t is

$$c_t^1 = w_t - I_t, \tag{1.20}$$

while old-age consumption in period t is

$$c_t^{2b} = c_t^{2g} = g(I_{t-1}) = c_t^2. \tag{1.21}$$

Adding (1.20) and (1.21), total consumption per capita in period t becomes

$$c_t := c_t^1 + c_t^2 = w_t - I_t + g(I_{t-1}) = f(k_t) - I_t. \tag{1.22}$$

Since $k_{t+1} = \Omega(I_t)$, it follows that stationary feasible allocations $(\bar{k}, \bar{c}_1, \bar{c}_2) \geq 0$ with $\bar{k} = \Omega(\bar{I})$ are defined by

$$\bar{c} := \bar{c}_1 + \bar{c}_2 = f(\Omega(\bar{I})) - \bar{I}.$$

Setting $\phi(\bar{I}) := f(\Omega(\bar{I})) - \bar{I}$, then stationary total consumption per capita $\bar{c}$ is maximal in a maximum of the function ϕ.

Lemma 1.2 (Maximal consumption).
Let Assumptions 1.2 and 1.3 be satisfied. Then the map ϕ *attains its global maximum at the golden-rule investment level* $I_G > 0$*, which is uniquely determined by*

$$f'(\Omega(I_G))\,\Omega'(I_G) = 1. \tag{1.23}$$

Lemma 1.2 resembles the golden rule of capital accumulation of the standard Solow (1956) growth model. Denote the capital-labor ratio corresponding to I_G by $k_G = \Omega(I_G)$. If capital depreciates fully and the population profile is stationary, then the golden-rule capital-labor ratio k_G^{S} in Solow model is determined by $f'(k_G^{\mathrm{S}}) = 1$. Since f is strictly concave and Assumption 1.2 implies $0 < \Omega' \leq 1$, it follows from (1.23) that $k_G \leq k_G^{\mathrm{S}}$. Thus, the positive failure rate of the production projects entails a lower golden-rule value in our model.

Observe also that k_G is solely determined by the production function f and the success probability function p. Since any steady state will, by construction, depend on agents' preferences,

k_G will generally not be attained as a steady state of the dynamical system (1.18). Indeed, k_G is a steady state of G if and only if $I_G = \mathcal{I}(w(k_G))$. Stated differently, the golden-rule consumption plan

$$c_G^1 = w(k_G) - I_G \quad \text{and} \quad c_G^2 = g(I_G)$$

is a steady-state consumption plan of the dynamical system (1.18) if and only if

$$\frac{u'(w(\Omega(I_G)) - I_G)}{v'(g(I_G))} = g'(I_G).$$

In this case, the economy attains the steady state with the highest possible welfare level. Finally, observe from (1.15) that

$$c_G^1 + \frac{c_G^2}{r_G} = w(k_G) + \Big(\tfrac{g(I_G)}{g'(I_G)I_G} - 1\Big) I_G =: w_G^d,$$

so that in a golden-rule steady state, disposable income is fully consumed. In general, however, the economy will not attain a golden-rule steady state, meaning that efficient contracts will, in general, *not* induce dynamically efficient growth paths.

1.5 Inefficient Contracts and Business Cycles

Banerji et al. (2004, Example 2, p. 2228) present an example in which the perfect-foresight dynamics displays the full range of complex behavior. This section revisits their example using our framework, showing that *efficient* contracts induce monotonic dynamics. In particular, we will demonstrate that endogenous fluctuations may only arise if the bank mistakenly offers *inefficient* contracts that should be rejected by rational agents.

Consider the production function $f(k) = Ak^\alpha$, where $A > 0$ and $0 < \alpha < 1$, combined with the utility functions $u(c) = v(c) = \ln(c)$ and the success probability function[9]

$$p(I) = \frac{0.32e^I}{1.2 + 1.901885e^I - I^3}, \quad I \geq 0. \tag{1.24}$$

The objective function in Problem (1.10) can be monotonically transformed into

$$V(I, w_t) := (w_t - I)\, g(I). \tag{1.25}$$

The problem now is that the productive capital Ω defined by (1.24) is neither strictly increasing nor concave. As a consequence, neither the capital income g nor the objective function $V(\cdot, w_t)$ in (1.25) is concave. Figure 1.2a portrays the objective function for varying levels of the wage rate. Since capital income g is not concave, the bank might be required to tie the loan and the deposit contract, see Remark 1.1.

The efficient investment level can be established as follows. For each $w_t > 0$, the objective function $I \mapsto V(I, w_t)$ is continuous on the compact interval $[0, w_t]$ and satisfies $V(0, w_t) =$

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[9] We slightly generalize the example from Banerji et al. (2004) by allowing the investment levels to become arbitrarily small. However, this change does not alter the main point of this section.

$V(w_t, w_t) = 0$. Thus, there exists at least one investment level $0 < I_t < w_t$ that maximizes $V(\cdot, w_t)$ on $[0, w_t]$. This maximizer is determined by the first-order condition

$$\frac{g'(I_t)I_t}{g(I_t)} = \frac{I_t}{w_t - I_t}. \tag{1.26}$$

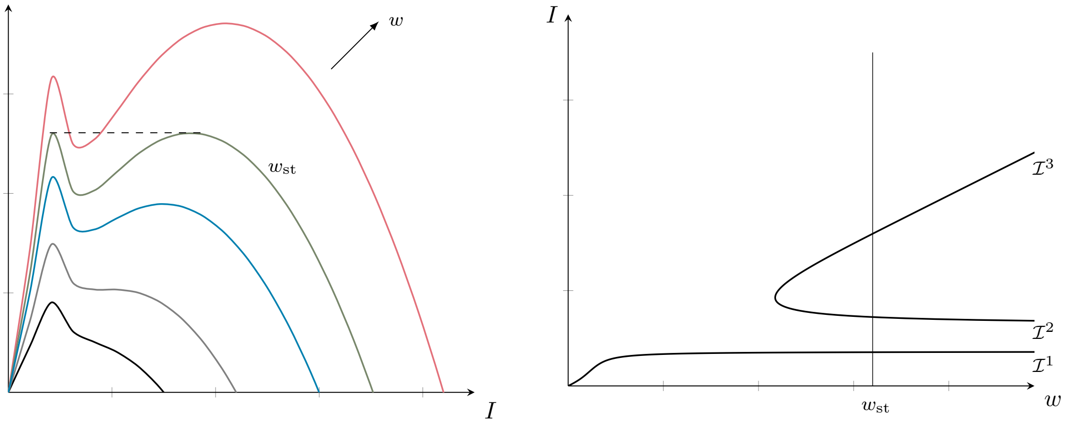
A numerical investigation reveals that depending on the level of w_t, the first-order condition (1.26) admits up to three distinct solutions, see Figure 1.2b. For all sufficiently low wage rates, (1.26) has a unique solution $0 < \mathcal{I}^1(w_t) < w_t$. For all sufficiently large wage rates, there exist three distinct solutions

$$0 < \mathcal{I}^1(w_t) < \mathcal{I}^2(w_t) < \mathcal{I}^3(w_t) < w_t.$$

From Figure 1.2a, we can infer that for all $w_t < w_{\text{st}}$, where $w_{\text{st}} > 0$ is defined by

$$V(\mathcal{I}^1(w_{\text{st}}), w_{\text{st}}) = V(\mathcal{I}^3(w_{\text{st}}), w_{\text{st}}),$$

the global maximum obtains at $\mathcal{I}^1(w_t)$. For all $w_t > w_{\text{st}}$, the global maximum obtains at $\mathcal{I}^3(w_t)$. Observe that $\mathcal{I}^2(w_t)$ is always a local minimum.



(a) Objective function $V(\cdot, w)$ for varying w.

(b) Solutions to (1.26).

Figure 1.2. Expected utility levels and critical points.

Since the *efficient* investment level must *maximize* the objective function (1.25), the investment function $\mathcal{I} : \mathbb{R}_+ \to \mathbb{R}_+$ may be defined by setting

$$I_t = \mathcal{I}(w_t) := \operatorname{argmax}\Big\{V(I^i, w_t) \;\Big|\; I^i = \mathcal{I}^i(w_t),\ i = 1, \dots, n(w_t)\Big\}, \tag{1.27}$$

where $\mathcal{I}^i(w_t)$ is a solution to (1.26) and $n(w_t)$ denotes the total number of solutions, given w_t. The dynamics in terms of efficient investment levels is then governed by the map

$$I_{t+1} = H(I_t) := \mathcal{I}\big(w(\Omega(I_t))\big), \tag{1.28}$$

where $I_0 = \mathcal{I}(w(k_0))$ is given. The map H is continuously differentiable, except for a discontinuity at $I_{\text{st}} := \mathcal{I}^1(w_{\text{st}})$. Its graph is the blue line depicted in Figures 1.3 and 1.4.

By standard arguments from dynamical systems theory, the dynamics generated by (1.28) is monotonic if H is strictly increasing. Applying the chain rule yields

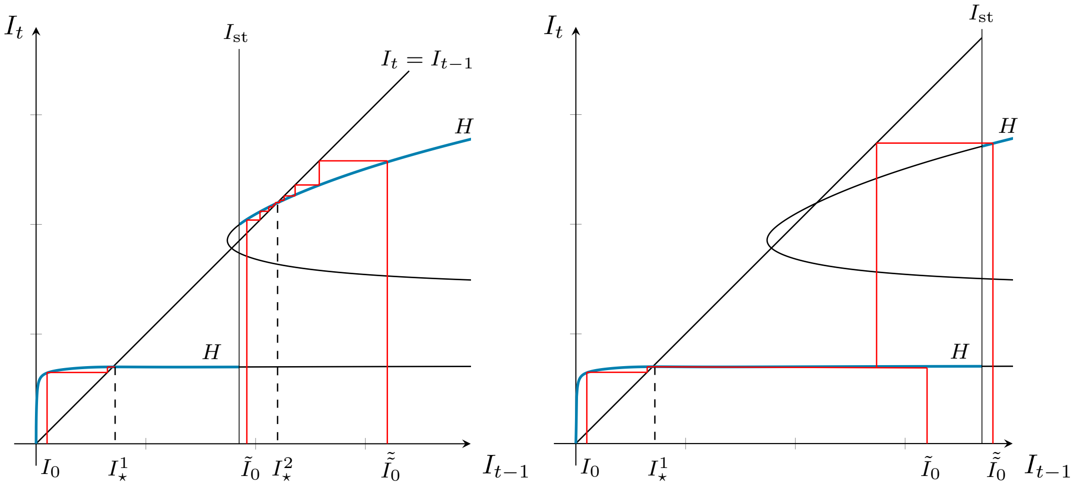
$$H'(I) = \mathcal{I}'\big(w(\Omega(I))\big)\, w'(\Omega(I))\, \Omega'(I) \quad \text{for all} \quad I_{\text{st}} \neq I \in \mathbb{R}_{++}. \tag{1.29}$$

Observe first that $w' > 0$ by Assumption 1.3. Since, for each $w > 0$, the investment level $I = \mathcal{I}(w)$ in (1.29) satisfies (1.26), it must fulfill $g'(I) > 0$ and, therefore, $\Omega'(I) > 0$.[10] Implicit differentiation of (1.26) now implies for each of its solutions $i = 1, 2, 3$ that

$$\mathcal{I}^{i\prime}(w) = -\frac{g'(\mathcal{I}^i(w))}{\frac{\partial^2 V}{\partial I^2}(\mathcal{I}^i(w), w)} \quad \text{for all} \quad w_{\text{st}} \neq w \in \mathbb{R}_{++}. \tag{1.30}$$

It follows from (1.29) and (1.30) that $H'(I) > 0$ whenever $I = \mathcal{I}(w)$ is an *efficient* investment level, given w. Stated differently, $H'(I) < 0$ if and only if $\frac{\partial^2 V}{\partial I^2}(I, w) > 0$, which implies that the investment level I determines a local welfare minimum and thus is *inefficient*.

The qualitative dynamics generated by the dynamical system (1.28) is portrayed in Figure 1.3. Since H is monotonically increasing, convergence is monotonic and endogenous fluctuations are ruled out. Depending on the magnitude of I_{st}, H has at least one and at most two positive steady states, denoted by $I^1_\star$ and $I^2_\star$, respectively. Depending on the initial condition I_0, the growth paths either converge to $I^1_\star$ or to $I^2_\star$.



(a) Two stable steady states.

(b) Unique stable steady state.

Figure 1.3. Dynamics induced by *efficient* contracts: dependence on initial conditions and technology.

This analysis demonstrates that efficient contracts cannot trigger endogenous fluctuations, even though productive capital Ω is not concave. This applies to both, the forward as well as the backward dynamics. However, the qualitative nature of the dynamics changes once we allow agents to mistakenly accept inefficient contracts. If the bank offers a contract characterized by the investment level $\mathcal{I}^2(w)$ that corresponds to a local welfare minimum, then, in view of (1.29), the dynamics is no longer governed by a strictly increasing time-one map. In this case, we find numerical evidence for three scenarios with endogenous fluctuations: First, non-monotonic convergence to an inefficient steady state $I^{\text{IE}}_\star$, as illustrated in Figure 1.4a. Second, a persistent period-2 cycle around an inefficient steady state $I^{\text{IE}}_\star$, as portrayed in Figure 1.4b. Third, initial fluctuations around an inefficient steady state $I^{\text{IE}}_\star$, which eventually converge to the stable efficient steady state $I^1_\star$, as illustrated in Figure 1.4c.

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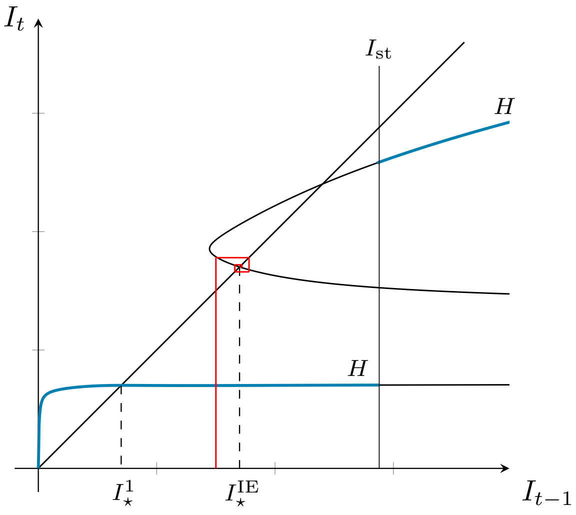
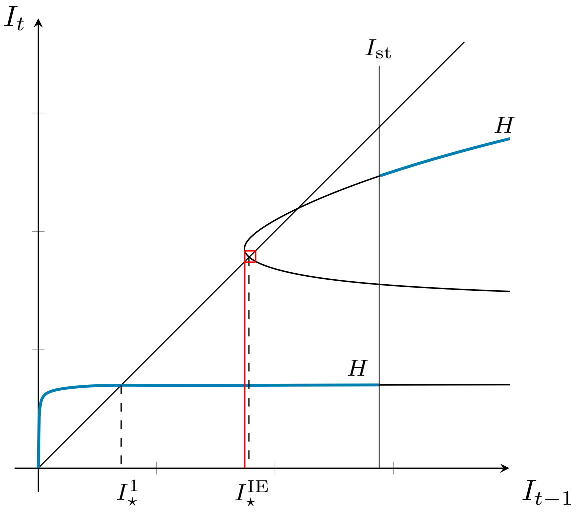
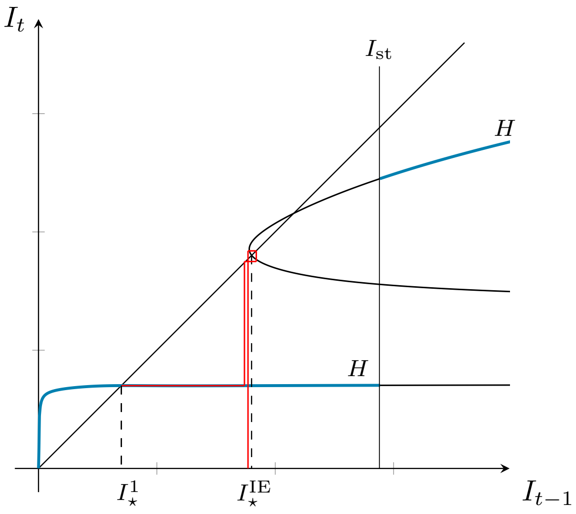
[10] The latter property can be read off Equation (A.1) in the proof of Lemma 1.1.



(a) Non-monotonic convergence to a stable inefficient steady state ($\alpha = 0.33$).



(b) Persistent period-2 cycle around an unstable inefficient steady state ($\alpha = 0.32$).



(c) Non-monotonic convergence to a stable efficient steady state ($\alpha = 0.315$).

Figure 1.4. Qualitative dynamics induced by *inefficient* contracts ($A = 60$).

1.6 Conclusion

This article examined an overlapping-generations model in which financial intermediation arises endogenously as a device to share idiosyncratic risk. Agents' welfare is maximized by efficient contracts that provide complete insurance and perfect consumption smoothing without a premium for the bank. In order to implement an efficient allocation, the bank must enlarge the disposable income of young agents because savings decisions are at their discretion. This implementation is only possible if capital income is concave in the investment level. Otherwise, the implementation of efficient allocations requires tying contracts. Our main contribution to the banking literature is the result that in any two-period lived OLG model with standard preferences and increasing productive capital, financial intermediation that implements efficient risk sharing can neither induce business cycles nor complex dynamics. The resulting dynamics is always monotonic and thus qualitatively indistinguishable from the dynamics of the classical two-period lived OLG model. This finding contradicts the results of Banerji et al. (2004), who claim the exact opposite, namely that financial intermediation can be responsible for endogenous business cycles. Our analysis reveals that business cycles may only be triggered by inefficient contracts that implement a local welfare minimum and thus would not be accepted by rational agents. To find conditions under which financial intermediation causes business cycles remains to be an open issue for future research.

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Chapter 2

Banking Competition and Capital Dependence of the Production Sector: Growth and Welfare Implications[⋆]

with Tom Rauber

Statement of Prior Publication

This chapter is based on the article

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In accordance with the terms of the publishing agreement, the authors retain the right to include the article (or a modified version of it) in a dissertation, whether in full or in part, provided it is not published commercially and appropriate credit is given. This chapter contains minor revisions aimed at improving clarity and exposition, and an additional discussion of financial stability in Section 2.7.

Abstract

This article examines how competition in the banking sector affects real economic growth and welfare in an overlapping-generations model with idiosyncratic production risk. We show that monopolistic banking can outperform its competitive counterpart in terms of both growth and welfare if the production sector is sufficiently capital-intensive. By contrast, perfect competition is favorable if production is sufficiently labor-intensive. Banking competition affects capital accumulation by determining both, agents' savings incentives and banks' intermediation margins. The distribution of factor incomes determines the decisive channel. The general theory of the second best explains the welfare-enhancing effect of a monopoly since capital accumulation is generally inefficient due to its dependence on preferences. We link these findings to empirical evidence.

Keywords: Overlapping generations, financial intermediation, deposit contracts, banking competition, financial dependence, economic growth, risk sharing.

JEL Classification: D53, E32, E44, G21, O41.

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2.1 Introduction

Understanding the determinants of economic growth is of central interest to both economic theorists and policymakers. Although discussed controversially in the past, it is by now virtually undisputed in macroeconomics that the financial sector also affects real economic development (Levine, 1997). As banks continue to play a pivotal role in the financial system, a sound understanding of their impact on economic growth is essential. In this context, the optimal degree of competition among banks has been subject to a contentious debate over the past three decades, both in theoretical and empirical literature (Coccorese, 2017). While some theoretical contributions advocate greater competition (e.g., see Guzman, 2000; Hamada et al., 2018; Smith, 1998), others emphasize the role of the monopolistic pole of the competitive continuum in promoting economic growth (e.g., see Cetorelli, 1997; Cetorelli and Peretto, 2000, 2012).

This article investigates the effect of interbank competition on real economic growth in an overlapping-generations (OLG) model with financial intermediation.[1] Banks offer risk-averse agents deposit contracts that insure against idiosyncratic production risk. We identify a simple condition under which a banking monopoly generates greater growth and higher levels of welfare than a perfectly competitive banking system and vice versa. It turns out that the distribution of factor incomes and thus the production technology is decisive. Monopolization can be beneficial if production is highly capital-intensive and the dividend payments of banks sufficiently constrained. However, if production is labor-intensive, then perfect competition maximizes growth and welfare. The underlying mechanism works as follows. In our model, banks contribute directly to capital accumulation via their equity, a channel that complements their intermediation of agents' deposits. By influencing deposit rates, competition among banks has a twofold impact on capital accumulation, as it determines agents' saving incentives on the one hand, and banks' equity through their intermediation margins on the other. Competition strengthens saving incentives but erodes intermediation margins. Which channel governs capital accumulation hinges on the factor-income distribution. If production is labor-intensive, wage income of agents accounts for a large share of total income in the economy, so that capital accumulation is primarily determined by the incentives to save. Since perfect competition yields high-powered saving incentives, it generates greater growth than its monopolistic counterpart. If production is capital-intensive, by contrast, wage income is low relative to capital income of banks, so that saving incentives play a minor role. Instead, economic growth is driven by the abundant capital investments of a monopolistic bank.

The welfare-improving effect of imperfect competition can be understood through the *general theory of the second best* due to Lipsey and Lancaster (1956). Since savings in overlapping-generations economies depend on preferences, agents will generally fail to implement an efficient capital accumulation path. Introducing an additional distortion, namely a banking monopoly, may therefore lead to a superior overall outcome, as the monopolist's investments contribute to capital accumulation and may shift the growth path closer to the golden rule.

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[1] Our overlapping-generations model builds on the frameworks used in Allen and Gale (1997), Banerji et al. (2004), and Ritschel and Wenzelburger (2024).

By taking a dynamic perspective on banking competition, this paper is, to the best of our knowledge, the first to show that imperfect competition in the banking sector can have positive effects on economic growth and welfare that arise purely from allocative effects.

Our results constitute a novel theoretical foundation for an empirical puzzle concerning the correlation between banking competition and economic growth. In their empirical studies, both Cetorelli and Gambera (2001) and Hoxha (2013) find that banking concentration - which they associate with market power - promotes growth in financially dependent industries. By contrast, it has a growth-depressing effect in industries with low degrees of financial dependence. Maudos and Fernandez de Guevara (2006) show that, in industrialized economies, market power of banks is positively correlated with economic growth. On the other hand, Deidda and Fattouh (2005) find that the correlation between banking concentration and growth is negative only in low-income countries. Complementing this result, Beck et al. (2004) show that banking concentration implies financing obstacles for firms exclusively in developing countries. Taken together, this empirical evidence is consistent with our results, since industrialized countries tend to feature capital-intensive branches of industry, whereas labor-intensive manufacturing typically prevails in low-income countries.

Two further insights can be derived from our model. First, we demonstrate that the provision of risk sharing increases the long-run growth of an economy, independently of the degree of competition among banks. An economy with banks, even if endowed with significantly less initial capital, will eventually outgrow an economy without banks. This result helps explain why under-developed countries lacking a well-functioning banking system might experience development traps. Second, we establish that financial intermediation has no impact on the qualitative patterns of economic growth. In particular, it cannot adversely affect the dynamics of an economy that otherwise exhibits monotonic growth only.

The literature on the finance-growth nexus dates back to Schumpeter (1926), who held the view that financial services facilitate the initiation of entrepreneurial activity, cf. King and Levine (1993b, 1993c). However, it was not until the 1950s that the importance of finance for economic growth was explicitly noted by Gurley and Shaw (1955, p. 516), who suspected *"an inadvertent undervaluation by economists of the role that finance plays in determining the pace and pattern of growth"*. The presumed correlation between finance and growth was then indeed confirmed by the pioneering empirical studies of Goldsmith (1969), McKinnon (1973), and Shaw (1973).[2] Economic theory, on the other hand, attributes the promotion of economic growth to the fundamental functions that banks carry out in an economy (e.g., see Bencivenga and Smith, 1991; Greenwood and Jovanovic, 1990; Greenwood and Smith, 1997; Obstfeld, 1994; Pagano, 1993). Particularly important functions include the provision of liquidity (see Bencivenga and Smith, 1991; Diamond and Dybvig, 1983), risk sharing (see Bencivenga and Smith, 1991), and the informational advantage of financial intermediaries (see Hellwig, 1991).

Accordingly, both empirical and theoretical research in economics generally agrees that financial intermediation, and banking in particular, promotes real economic growth. However,

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[2] More recent studies finding a positive correlation between financial intermediation and economic growth are Beck et al. (2000), King and Levine (1993a), and Levine and Zervos (1998). For excellent overviews of the empirical literature, see Aziakpono (2011) and Levine (2005).

there is no clear-cut consensus on the optimal degree of competition among banks. Early contributions, such as Bencivenga and Smith (1991) and Greenwood and Jovanovic (1990), emphasize that market imperfections in the banking sector tend to inhibit growth, or, in the words of King and Levine (1993b, p. 515), *"financial sector distortions can reduce the rate of economic growth"*. Market power, above all, has attracted special attention ever since. Smith (1998) argues that imperfect interbank competition reduces economic growth by raising firms' financing costs. Moreover, he finds that monopolistic banking can amplify the severity of business cycles. Complementing Smith's results, Guzman (2000) shows that monopolistic banking impairs economic growth by exacerbating credit rationing. Finally, Hamada et al. (2018) find that lower banking competition reduces economic growth through lower deposit rates. Contrary findings were first brought forward by Cetorelli (1997) and Cetorelli and Peretto (2000, 2012), who argue that reduced competition among banks may benefit economic growth in environments with substantial idiosyncratic risk. The mechanism is that competition diminishes banks' incentives to engage in relationship lending, which can be detrimental to economic growth. However, this mechanism relies on informational asymmetries between banks and borrowers, as well as spill-over effects of financial intermediation on the performance of the production sector. Both of these features are absent in our model, as information is symmetric and the productivity of the real sector is unaffected by financial intermediation.

This article is organized as follows. The next section introduces the overlapping-generations model and states the assumptions on preferences and technology. Section 2.3 incorporates financial intermediation. In Section 2.4, we analyze capital accumulation and establish the perfect-foresight dynamics. In doing so, we adopt the sequential modeling approach of Böhm and Wenzelburger (1999): the basic mechanisms of capital accumulation are described by an economic law, while expectations are formed on the basis of a forecasting rule. Section 2.5 presents the main results concerning economic growth and welfare. Section 2.6 investigates whether financial intermediation may generate endogenous fluctuations, while Section 2.7 discusses implications for financial stability. Section 2.8 extends the model to include dividend payments, before the final section concludes. All proofs are collected in Appendix B. Throughout the article, we conduct a *ceteris-paribus* comparison of three economies: an economy without banks, an economy with perfect competition among a unit-mass continuum of homogeneous banks, and an economy with a monopolistic banking sector.

2.2 MODEL PREREQUISITES

We consider an overlapping-generations model with one production sector and markets for capital, labor, and a perishable good that can be consumed and invested. At the beginning of each period $t = 0, 1, \dots, \infty$, a new generation comprising a unit-mass continuum of homogeneous agents is born. Agents live for two periods, referred to as *young* and *old*. The initial old generation in $t = 0$ is endowed with capital $K_0 > 0$. Besides agents and firms, the economy accommodates financial intermediaries in the form of risk-neutral commercial banks, which are described below.

2.2.1 Preferences

Agents are risk averse and have intertemporal preferences over youthful consumption $c^1 \geq 0$ and old-age consumption $c^2 \geq 0$. The preferences are represented by a life-cycle utility function $U : \mathbb{R}_+^2 \to \mathbb{R}$ of the additive-separable form[3]

$$U(c^1, c^2) := u(c^1) + v(c^2)$$

that satisfies the following assumptions.

Assumption 2.1 (Preferences)**.**

(i) The utility functions $u, v : \mathbb{R}_+ \to \mathbb{R}$ *are twice continuously differentiable, strictly increasing, strictly concave, and satisfy the Inada conditions*

$$\lim_{c \to 0} u'(c) = \infty \quad \textit{and} \quad \lim_{c \to 0} v'(c) = \infty.$$

We normalize $v(0) = 0$*.*

(ii) The Arrow-Pratt coefficients of relative risk aversion

$$\alpha_u := -\frac{u''(c^1)c^1}{u'(c^1)} \quad \textit{and} \quad \alpha_v := -\frac{v''(c^2)c^2}{v'(c^2)}$$

are constants that satisfy $0 < \alpha_u \leq \alpha_v < 1$*.*

Assumption 2.1 is in particular satisfied by utility functions with a constant elasticity of intertemporal substitution, see Example 2.2. It implies that youthful and old-age consumption are normal goods and, since savings and investments are increasing in the corresponding rate of return, gross substitutes. Constant relative risk aversion is a widespread assumption in finance literature that helps to improve the analytical tractability of the model.

2.2.2 Production

The production sector is perfectly competitive. A representative firm transforms capital and labor with constant returns to scale into output. Capital and labor are paid their marginal products, while the output price is normalized to unity. The workforce consists of young agents only, who inelastically supply one unit of labor. Old agents are retired. Capital is provided to firms by old agents and banks and depreciates fully during production. Denote by $k \geq 0$ the capital-labor ratio and by $f : \mathbb{R}_+ \to \mathbb{R}_+$ the production function of the representative firm in intensive form, then $y = f(k)$ is the GDP per capita. For each $k \geq 0$, we define the three elasticities

$$\epsilon_f(k) := \frac{f'(k)k}{f(k)}, \quad \epsilon_{f'}(k) := \frac{f''(k)k}{f'(k)}, \quad \epsilon_{f''}(k) := \frac{f'''(k)k}{f''(k)},$$

and denote the marginal product of labor by

$$w(k) := f(k) - f'(k)k.$$

Our assumptions on the technology are as follows.

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[3] Note that a time discount factor $\beta > 0$ is included in the utility function v.

Assumption 2.2 (Technology).
The production function $f : \mathbb{R}_+ \to \mathbb{R}_+$ *is thrice continuously differentiable, strictly increasing, strictly concave, and satisfies the Inada conditions*

$$\lim_{k\to 0} f'(k) = \infty \quad \textit{and} \quad \lim_{k\to\infty} f'(k) < 1.$$

Moreover, f *satisfies*

$$-2 < \frac{f'''(k)k}{f''(k)} < -1 < \frac{f''(k)k}{f'(k)} < 0 \quad \textit{for all} \quad k \geq 0.$$

Assumption 2.2 is satisfied by many standard production functions from the OLG literature, including Cobb-Douglas production functions and CES productions functions with an elasticity of substitution greater than unity. It implies that total income $k \mapsto f(k)$, labor income $k \mapsto w(k)$, and capital income $k \mapsto f'(k)k$ are all strictly increasing and strictly concave in the capital-labor ratio k.

The elasticity ϵ_f is of central importance for our analysis, as it captures the share of capital income in total income of the economy and thus the capital dependence of the production sector. An increase in ϵ_f renders production more capital-intensive and less labor-intensive, raising the relative importance of the production factor capital. Observe that the strict concavity of f implies $0 < \epsilon_f(k) < 1$ for all $k \geq 0$.

Example 2.1 (Cobb-Douglas technology).
For the Cobb-Douglas production function $f(k) = Ak^\alpha$, *where* $A > 0$ *and* $0 < \alpha < 1$, *the elasticities* $\epsilon_f(k) = \alpha$, $\epsilon_{f'}(k) = \alpha - 1$, *and* $\epsilon_{f''}(k) = \alpha - 2$ *are constants. Thus, the parameter* α *fully determines the distribution of factor incomes.*

2.2.3 Projects and Expectations

Following Banerji et al. (2004), agents and banks can invest in risky projects (firms) in the production sector. A project has a stochastic binary outcome, that is, it can either be successful or fail. The probability of success is exogenously given by a constant success rate $p \in (0, 1)$. The uncertainty about the outcome of a project resolves one period after capital has been invested. A successful project yields a gross rate of return $\rho > 0$, whereas the gross return on a failed project is zero. In each period $t \geq 0$, agents form an expectation $\rho_t^e > 0$ with respect to the gross return on a successful project ρ_{t+1} realized in $t + 1$. Expectations are assumed to be homogeneous among all agents and banks.

2.3 Financial Intermediation

2.3.1 Autarky

In the absence of banks, a young agent may transfer resources to the second period of his life solely by investing part of his wage income in a risky production project, in which case he is fully exposed to the idiosyncratic risk of an old-age income shock.[4] The corresponding decision problem of a typical young agent in period t is the following. The agent's objective is to maximize his expected utility of lifetime consumption. Given the wage rate $w_t > 0$ and the anticipated gross return on a successful project ρ_t^e, the optimal investment $0 < I_t < w_t$ is uniquely determined by

$$I_t = I(w_t, \rho_t^e) := \operatorname*{argmax}_{0 \leq I \leq w_t} \; u(w_t - I) + p\, v(\rho_t^e I) + (1-p)\, v(0), \tag{2.1}$$

where $I : \mathbb{R}_+ \times \mathbb{R}_{++} \to \mathbb{R}_+$ is referred to as *investment function*. Since $v(0) = 0$, the first-order condition takes the form

$$\frac{u'(w_t - I_t)}{v'(\rho_t^e I_t)} = p\rho_t^e. \tag{2.2}$$

The value function associated with Problem (2.1) is defined by

$$\mathscr{U}_{\text{res}}(w_t, \rho_t^e) := u(w_t - I(w_t, \rho_t^e)) + p\, v(\rho_t^e I(w_t, \rho_t^e)) \tag{2.3}$$

and constitutes the agent's reservation utility in the presence of banks.

2.3.2 Deposit Contracts

In an environment with idiosyncratic risk, banks may be seen as coalitions of agents that arise endogenously as vehicles for risk sharing and rent extraction (Freixas & Rochet, 2008). Acting as a financial intermediary, the representative bank seeks to maximize its expected profit by collecting deposits from agents and investing them in production projects.[5] To this end, the bank offers agents in period t a risk-free gross deposit rate $r_t > 0$ on their savings. Deposits are at the discretion of the agents. Given the wage rate $w_t > 0$ and the deposit rate r_t, the optimal deposit supply $0 < D_t < w_t$ is uniquely determined by

$$D_t = D(w_t, r_t) := \operatorname*{argmax}_{0 \leq D \leq w_t} \; u(w_t - D) + v(r_t D), \tag{2.4}$$

where $D : \mathbb{R}_+ \times \mathbb{R}_{++} \to \mathbb{R}_+$ is referred to as *savings function*. The first-order condition reads

$$\frac{u'(w_t - D_t)}{v'(r_t D_t)} = r_t, \tag{2.5}$$

stating that the marginal rate of intertemporal substitution equals the gross return on savings. The value function for Problem (2.4) is denoted by

$$\mathscr{U}(w_t, r_t) := u(w_t - D(w_t, r_t)) + v(r_t D(w_t, r_t)).$$

Since the agent has the outside option of investing in a project himself, he will accept the deposit contract r_t offered by the bank if and only if it satisfies the *participation constraint*

$$\mathscr{U}(w_t, r_t) \geq \mathscr{U}_{\text{res}}(w_t, \rho_t^e), \tag{PC}$$

---

[4] Note that, unlike in Diamond and Dybvig (1983), our model does not feature a pure storage technology.

[5] Investing in production projects can be interpreted as the lending business of the bank. By the law of large numbers, the loan default rate is $1 - p$.

stating that the utility attained by saving and depositing the amount $D(w_t, r_t)$ at the bank must be at least as large as the expected utility of investing the amount $I(w_t, \rho_t^e)$ directly in a project while bearing the full idiosyncratic risk.[6]

It remains to formalize the objective function of the representative bank. Exploiting the law of large numbers, the bank correctly anticipates that there is no aggregate uncertainty, as the proportion p of projects will be successful. Therefore, the bank's portfolio of projects fully diversifies away idiosyncratic risk, and the *anticipated intermediation margin* per unit of deposits received in period t becomes $p\rho_t^e - r_t$. Under perfect foresight, the intermediation margin $p\rho_{t+1} - r_t$ realized in $t+1$ is non-negative at all times, so that the bank earns non-negative profits and accidental bankruptcies are ruled out. The bank's equity stock in period t is denoted by $e_t \geq 0$, normalizing that $e_0 = 0$.[7] Since there is no storage technology, bank equity is either reinvested in production projects or distributed to bank owners as dividend payments. Recall that bank equity is subject to strict regulatory requirements that impose an upper bound on divided payments, e.g., capital-adequacy ratios.[8] To simplify the analysis and keep the complexity manageable, we will first consider a stylized model in which bank equity is fully reinvested. Subsequently, in Section 2.8, we will argue that our main results are robust even when part of the banking surplus is distributed to the bank owners. Finally, since deposit management costs are irrelevant to the point of this paper, we neglect them. Given the wage rate w_t, the expectation ρ_t^e, and the equity stock e_t, the bank's anticipated profit then becomes

$$\pi_t^e(r) := D(w_t, r)\,[p\rho_t^e - r] + p\rho_t^e e_t.$$

We can now establish the optimal deposit contracts. Under perfect competition, banks compete in deposit rates until the anticipated intermediation margin is zero and agents are awarded the entire anticipated surplus, see Freixas and Rochet (2008). Deposit contracts are actuarially fair and provide complete insurance for agents without a risk premium for the bank, thereby offering a better risk-return profile than a direct investment in a project. Summarizing these results, we may state the following proposition.

Proposition 2.1 (Perfect competition).
Let Assumption 2.1 be satisfied and $w_t > 0$, $\rho_t^e > 0$, and $e_t \geq 0$ be given. Then the equilibrium deposit rate in case of perfect competition among banks is

$$r_t^{\mathrm{C}} = r^{\mathrm{C}}(w_t, \rho_t^e) := p\rho_t^e.$$

---

[6] Since the deposit contract is just a deposit rate r, we abstract from stochastic contracts, contracts specifying fixed fees, and from contracts that stipulate both the deposit rate *and* the level of deposits. Moreover, note that a young agent either saves in the form of deposits at the bank or invests with idiosyncratic risk instead. Under perfect competition, an agent has no incentive to invest in a project himself because the deposit contract awards him the entire anticipated surplus. A monopolistic bank, on the other hand, will naturally exploit her market power and design contracts that prohibit investments of agents in order to maximize the rent extracted.

[7] Allowing for erroneous expectations, then $e_t \in \mathbb{R}$ because the bank could be indebted owing to negative intermediation margins, see Section 2.7.

[8] See, for instance, Gersbach and Wenzelburger (2003, 2008, 2012) who investigate the need for and the effectiveness of banking regulation in an OLG model.

The opposite pole of the competitive continuum is considered next. Given w_t, ρ_t^e, and e_t, the profit-maximization problem of a monopolistic bank takes the form

$$\max_{0 \le r \le p\rho_t^e} \pi_t^e(r) \quad \text{s.t.} \quad \mathscr{U}(w_t, r) \ge \mathscr{U}_{\text{res}}(w_t, \rho_t^e). \tag{2.6}$$

Since youthful and old-age consumption are gross substitutes by Assumption 2.1, the price elasticity of the savings function

$$\eta(w_t, r_t) := \frac{\frac{\partial D}{\partial r}(w_t, r_t)\, r_t}{D(w_t, r_t)} \tag{2.7}$$

is strictly positive. Therefore, a solution $0 < r_t^{\mathrm{M}} < p\rho_t^e$ to the monopolist's problem (2.6) is either determined by the first-order condition for a profit maximum

$$\frac{p\rho_t^e - r_t^{\mathrm{M}}}{r_t^{\mathrm{M}}} = \frac{1}{\eta(w_t, r_t^{\mathrm{M}})} \tag{2.8}$$

corresponding to the relaxed problem without the participation constraint, or by the binding participation constraint

$$\mathscr{U}(w_t, r_t^{\mathrm{M}}) = \mathscr{U}_{\text{res}}(w_t, \rho_t^e). \tag{2.9}$$

Notice that (2.8) is the standard optimality condition for a monopoly, stating that the Lerner-index equals the inverse price elasticity.[9] On the other hand, (2.9) defines the minimum deposit rate that agents are willing to accept. Therefore, if (2.8) determines a sufficiently high deposit rate that is accepted by agents, then the monopolist optimally implements this contract as it maximizes the anticipated profit. Otherwise, the participation constraint (PC) is binding and the optimal deposit rate is determined by (2.9). In the latter case, the solution to (2.8) is not implementable because an agent would reject the bank's offer and invest himself instead. This leads to the following proposition.

Proposition 2.2 (Banking monopoly).
Let Assumption 2.1 be satisfied and $w_t > 0$, $\rho_t^e > 0$, and $e_t \ge 0$ be given. Then the monopolist's problem (2.6) admits a unique solution $r_t^{\mathrm{M}} = r^{\mathrm{M}}(w_t, \rho_t^e) \in (0, p\rho_t^e)$. It is given by

$$r^{\mathrm{M}}(w_t, \rho_t^e) := \max\{s(w_t, \rho_t^e), b(w_t, \rho_t^e)\}, \tag{2.10}$$

where the deposit rates $s(w_t, \rho_t^e)$ and $b(w_t, \rho_t^e)$ are defined by (2.8) and (2.9), respectively.

Proposition 2.2 shows that market power in the banking sector reduces deposit rates, $r_t^{\mathrm{M}} < r_t^{\mathrm{C}} = p\rho_t^e$. It is worth noting that whenever r_t^{M} is determined by the monopolist's first-order condition (2.8), then the utility attained with the deposit contract exceeds the agent's reservation utility because $r_t^{\mathrm{M}} = s(w_t, \rho_t^e) > b(w_t, \rho_t^e)$ implies that

$$\mathscr{U}(w_t, r_t^{\mathrm{M}}) > \mathscr{U}_{\text{res}}(w_t, \rho_t^e).$$

---

[9] Since $1/\eta$ is a measure of the bank's market power, the competitive limit $r_t^{\mathrm{M}} = r_t^{\mathrm{C}} = p\rho_t^e$ is obtained for $\eta \to \infty$. Moreover, it is worth noting that our model can be readily extended to cover symmetric banking oligopolies. The main difference is that the elasticity η in (2.8) must be multiplied by the number of competing banks. For a formal treatment of a related problem, we refer to Freixas and Rochet (2008).

In this case, the monopolist must pay agents an information rent in order to receive the desired amount of deposits. The following lemma demonstrates that this case arises in particular when the production projects are highly risky.

Lemma 2.1 (Success rate thresholds).
Let Assumption 2.1 be satisfied and $w_t > 0$, $\rho_t^e > 0$, *and* $e_t \geq 0$ *be given. Then there exist thresholds* $0 < \underline{p} \leq \overline{p} < 1$ *such that* $r_t^{\mathrm{M}} = s(w_t, \rho_t^e)$ *if* $p \leq \underline{p}$, *whereas* $r_t^{\mathrm{M}} = b(w_t, \rho_t^e)$ *if* $p \geq \overline{p}$.

The intuition underlying Lemma 2.1 is straightforward: if a direct investment in a project exposes agents to substantial idiosyncratic risk, the monopolist can exploit agents' risk aversion and implement the profit-maximizing deposit rate $r_t^{\mathrm{M}} = s(w_t, \rho_t^e)$. However, if the projects are relatively safe, then this contract is not accepted by agents.[10]

The case of agents who are endowed with CES preferences serves as the primary example throughout the article.

Example 2.2 (CES utility).
Consider the CES life-cycle utility function

$$U(c^1, c^2) = \tfrac{1}{\sigma}(c^1)^\sigma + \tfrac{\beta}{\sigma}(c^2)^\sigma,$$

where $\beta > 0$ *is a time-discount factor and the constant* $0 < \sigma < 1$ *stipulates the elasticity of intertemporal substitution* $\frac{1}{1-\sigma}$. *The Arrow-Pratt coefficients of relative risk aversion are constants because*

$$\alpha_u = \alpha_v = 1 - \sigma,$$

showing that Assumption 2.1 is satisfied. The investment function (2.1) takes the form

$$I(w_t, \rho_t^e) = \frac{1}{1 + (p\beta)^{\frac{1}{\sigma-1}}(\rho_t^e)^{\frac{\sigma}{\sigma-1}}} w_t$$

and the reservation utility (2.3) the form

$$\mathscr{U}_{\mathrm{res}}(w_t, \rho_t^e) = \left(\frac{(p\beta)^{\frac{1}{\sigma-1}}(\rho_t^e)^{\frac{\sigma}{\sigma-1}}}{1 + (p\beta)^{\frac{1}{\sigma-1}}(\rho_t^e)^{\frac{\sigma}{\sigma-1}}}\right)^{\sigma-1} \frac{w_t^\sigma}{\sigma}.$$

Moreover, the savings function (2.4) computes

$$D(w_t, r_t) = \frac{1}{1 + \beta^{\frac{1}{\sigma-1}} r_t^{\frac{\sigma}{\sigma-1}}} w_t.$$

For instance, let $\sigma = 0.5$ *and* $\beta = 1$, *then the monopolistic deposit rate (2.10) is*

$$r^{\mathrm{M}}(w_t, \rho_t^e) = \max\Big\{p^2\rho_t^e, \sqrt{p\rho_t^e + 1} - 1\Big\}.$$

In this case, the thresholds in Lemma 2.1 coincide, $\underline{p} = \overline{p}$.

---

[10] It is straightforward to verify that the reservation utility $\mathscr{U}_{\mathrm{res}}$ defined in (2.3) is strictly increasing in p, so that the participation constraint (PC) becomes more restrictive.

2.4 CAPITAL ACCUMULATION

The aggregate capital stock of the economy is determined by the total capital endowment of successful projects. Without banks, projects are financed exclusively from direct investments of agents. In the presence of banks, however, capital accumulation is driven by agents' deposits, which are channeled into the production sector through financial intermediaries, and by banks' equity investments. The degree of competition among banks has a twofold impact on these two funding sources. While intense competition generates high deposit rates and thus strong saving incentives for agents, it also reduces banks' equity investments by eroding their intermediation margins. The former effect is established in the following corollary.

Corollary 2.1.
Let Assumption 2.1 be satisfied. Then for each $w_t > 0$ *and* $\rho_t^e > 0$*, we have*

$$I(w_t, \rho_t^e) \leq D(w_t, r^{\mathrm{M}}(w_t, \rho_t^e)) < D(w_t, r^{\mathrm{C}}(w_t, \rho_t^e)),$$

where the first inequality is strict if and only if $s(w_t, \rho_t^e) > b(w_t, \rho_t^e)$*.*

Corollary 2.1 shows that risk sharing fosters capital accumulation by encouraging agents to allocate additional funds to the production sector. As we will see below, it is precisely this feature of financial intermediation that promotes long-run economic growth in our model.

In order to obtain dynamical systems that govern the evolution of the economies over time, we next introduce the respective economic laws of capital accumulation and then add forecasting rules that stipulate how expectations are formed. To this end, it is convenient to express the expectations in terms of the capital-labor ratio by setting $\rho_t^e = f'(k_t^e)$, where $k_t^e \geq 0$ is the expectation formed in period t with respect to the k_{t+1} realized in $t+1$.

Given a continuum of projects, the law of large numbers implies that, on aggregate, the proportion p of them will be successful. Without financial intermediation, the economic law of capital accumulation thus takes the form

$$k_{t+1} = G^{\mathrm{N}}(k_t, k_t^e) := pI(w(k_t), f'(k_t^e)). \tag{2.11}$$

In the presence of banks, capital accumulation obeys

$$k_{t+1} = p\big[D(w_t, r_t^i) + e_t^i\big], \tag{2.12}$$

where $i = \mathrm{C}, \mathrm{M}$ depending on the degree of competition. The bank's proceeds from the successful projects in period t are $f'(k_t)k_t$, while it has to pay the depositors the contractual amount $r_{t-1}^i D(w_{t-1}, r_{t-1}^i)$. Since the wage rate is determined by the marginal product of labor, it follows that bank equity in period t is given by the map[11]

$$e_t^i = e^i(k_t, k_{t-1}, k_{t-1}^e) := f'(k_t)k_t - r^i(w(k_{t-1}), f'(k_{t-1}^e))\, D\big(w(k_{t-1}), r^i(w(k_{t-1}), f'(k_{t-1}^e))\big). \tag{2.13}$$

---

[11] Note that the balance sheet of the bank satisfies

$$f(k_t) = w(k_t) + e^i(k_t, k_{t-1}, k_{t-1}^e) + r^i(w(k_{t-1}), f'(k_{t-1}^e))\, D\big(w(k_{t-1}), r^i(w(k_{t-1}), f'(k_{t-1}^e))\big),$$

showing that total output is split into wage payments to the young generation, contractual payments to the old generation, and bank equity.

Inserting (2.13) into (2.12) yields the economic law of capital accumulation

$$k_{t+1} = G^i(k_t, k_{t-1}, k_t^e, k_{t-1}^e) := p\big[D\big(w(k_t), r^i(w(k_t), f'(k_t^e))\big) + e^i(k_t, k_{t-1}, k_{t-1}^e)\big], \quad i = \mathrm{C}, \mathrm{M}.$$

Following Grandmont (1985), expectations are formed on the basis of a forecasting rule that depends solely on the current state of the economy. Agents have *perfect foresight* if expectations are correct at all times, that is, if $k_t^e = k_{t+1}$ for all $t \geq 0$. *Perfect forecasting rules* in the sense of Böhm and Wenzelburger (1999) generate perfect foresight along all possible growth paths of the economy. Formally, these may be defined as follows.[12]

Definition 2.1 (Perfect forecasting rule)**.**

(i) A forecasting rule

$$\psi^{\mathrm{N}} : \mathbb{R}_+ \to \mathbb{R}_+, \qquad k_t^e = \psi^{\mathrm{N}}(k_t),$$

is called a perfect forecasting rule *for the economic law* G^{N} *if it satisfies*

$$\psi^{\mathrm{N}}(k) = G^{\mathrm{N}}(k, \psi^{\mathrm{N}}(k)) \quad \textit{for all} \;\; k \in \mathbb{R}_+.$$

(ii) Let $i = \mathrm{C}, \mathrm{M}$. *A forecasting rule*

$$\psi^i : \mathcal{D}^i \to \mathbb{R}_+, \qquad k_t^e = \psi^i(k_t, k_{t-1}),$$

is called a perfect forecasting rule *for the economic law* G^i *if it satisfies*

$$\psi^i(k, \hat{k}) = G^i(k, \hat{k}, \psi^i(k, \hat{k}), k) \quad \textit{for all} \;\; (k, \hat{k}) \in \mathcal{D}^i := \big\{(k, \hat{k}) \in \mathbb{R}_+^2 \;\big|\; e^i(k, \hat{k}, k) \geq 0\big\}.$$

Existence and uniqueness of the perfect forecasting rules are established next.

Lemma 2.2 (Perfect foresight)**.**
Let Assumptions 2.1 and 2.2 be satisfied. Then the following holds true.

(i) There exists a uniquely determined perfect forecasting rule ψ^i, $i = \mathrm{N}, \mathrm{C}, \mathrm{M}$, *in the sense of Definition 2.1.*

(ii) Let $i = \mathrm{C}, \mathrm{M}$. *Then* $\mathcal{D}^i$ *is forward-invariant under* ψ^i*: for each* $(k_t, k_{t-1}) \in \mathcal{D}^i$, *we have* $(\psi^i(k_t, k_{t-1}), k_t) \in \mathcal{D}^i$.

Forward-invariance ensures that the economies with banks retain the possibility of perfect foresight over time. Under perfect foresight, banks never go bankrupt as their realized intermediation margins are non-negative.

Given the initial condition $k_0 > 0$, the perfect-foresight growth paths $\{k_t\}_{t=0}^{\infty}$ of the economy without banks are governed by the implicit difference equation

$$k_{t+1} = G^{\mathrm{N}}(k_t, k_{t+1}) = pI(w(k_t), f'(k_{t+1})), \quad t \geq 0, \qquad (\mathrm{PFD}^{\mathrm{N}})$$

---

[12] Observe that the functional form of a perfect forecasting rule depends on the economic law of capital accumulation and, therefore, on the prevailing form of financial intermediation.

or, equivalently, by the perfect forecasting rule

$$k_{t+1} = \psi^{\mathrm{N}}(k_t).$$

In the presence of banks, the perfect-foresight dynamics is governed by

$$k_{t+1} = G^i(k_t, k_{t-1}, k_{t+1}, k_t), \quad i = \mathrm{C}, \mathrm{M}, \quad t \geq 1. \tag{2.14}$$

The difference equation (2.14) deserves further attention. In a perfectly competitive banking sector, the realized intermediation margins under perfect foresight are zero at all times. Since $e_0 = 0$, it follows that bank equity along any perfect-foresight growth path is zero. In the competitive case, the difference equation (2.14) thus simplifies to

$$k_{t+1} = pD(w(k_t), pf'(k_{t+1})), \quad t \geq 0, \tag{PFD$^{\mathrm{C}}$}$$

or, equivalently,

$$k_{t+1} = \psi^{\mathrm{C}}(k_t).$$

In the monopolistic case, however, realized intermediation margins under perfect foresight are strictly positive, implying that the dynamics is governed by the second-order difference equation

$$k_{t+1} = G^{\mathrm{M}}(k_t, k_{t-1}, k_{t+1}, k_t) = p\big[D\big(w(k_t), r^{\mathrm{M}}(w(k_t), f'(k_{t+1}))\big) + e^{\mathrm{M}}(k_t, k_{t-1}, k_t)\big]. \tag{PFD$^{\mathrm{M}}$}$$

In view of the lagged variable in (PFD$^{\mathrm{M}}$), we will next set up a linearization and invoke the Hartman-Grobman theorem for further analyses, cf. Hartman (1960) and Grobman (1959). To do so, define the map

$$\Psi : \mathcal{D}^{\mathrm{M}} \to \mathcal{D}^{\mathrm{M}}, \qquad \begin{pmatrix} k_t \\ k_{t-1} \end{pmatrix} \mapsto \begin{pmatrix} \psi^{\mathrm{M}}(k_t, k_{t-1}) \\ k_t \end{pmatrix},$$

as well as the vector

$$\boldsymbol{k}_t := \begin{pmatrix} k_t \\ k_{t-1} \end{pmatrix}.$$

The second-order difference equation (PFD$^{\mathrm{M}}$) may then be transformed into the two-dimensional dynamical system

$$\boldsymbol{k}_{t+1} = \Psi(\boldsymbol{k}_t), \quad \boldsymbol{k}_t \in \mathcal{D}^{\mathrm{M}}, \quad t \geq 1.$$

Steady states $\boldsymbol{k}^{\mathrm{M}} = (k^{\mathrm{M}}, k^{\mathrm{M}}) \in \mathcal{D}^{\mathrm{M}}$ of Ψ are defined by $\boldsymbol{k}^{\mathrm{M}} = \Psi(\boldsymbol{k}^{\mathrm{M}})$.[13] Linearizing the map Ψ at $\boldsymbol{k}^{\mathrm{M}}$ yields

$$\boldsymbol{k}_{t+1} = \mathrm{D}_{\Psi}(\boldsymbol{k}^{\mathrm{M}})\boldsymbol{k}_t, \quad t \geq 1, \tag{2.15}$$

where the Jacobian matrix of Ψ takes the form

$$\mathrm{D}_{\Psi}(\boldsymbol{k}^{\mathrm{M}}) := \begin{pmatrix} \dfrac{\partial \psi^{\mathrm{M}}}{\partial k_t}(\boldsymbol{k}^{\mathrm{M}}) & \dfrac{\partial \psi^{\mathrm{M}}}{\partial k_{t-1}}(\boldsymbol{k}^{\mathrm{M}}) \\ 1 & 0 \end{pmatrix}. \tag{2.16}$$

---

[13] The existence of a hyperbolic steady state $\boldsymbol{k}^{\mathrm{M}}$ is established in Section 2.5.

The stability properties of the steady state $\boldsymbol{k}^{\mathrm{M}}$ can be deduced from the eigenvalues of the Jacobian matrix (2.16), which compute

$$\lambda_{1,2}(\boldsymbol{k}^{\mathrm{M}}) := \frac{1}{2}\frac{\partial \psi^{\mathrm{M}}}{\partial k_t}(\boldsymbol{k}^{\mathrm{M}}) \pm \sqrt{\frac{1}{4}\left(\frac{\partial \psi^{\mathrm{M}}}{\partial k_t}(\boldsymbol{k}^{\mathrm{M}})\right)^2 + \frac{\partial \psi^{\mathrm{M}}}{\partial k_{t-1}}(\boldsymbol{k}^{\mathrm{M}})}. \tag{2.17}$$

We complete this section with a technical lemma that will facilitate establishing our main results.

Lemma 2.3 (Eigenvalues).
Let Assumptions 2.1 and 2.2 be satisfied and $\boldsymbol{k}^{\mathrm{M}} \in \mathcal{D}^{\mathrm{M}}$ *be a steady state of* Ψ*. Then the eigenvalues (2.17) satisfy* $0 < \left|\lambda_2(\boldsymbol{k}^{\mathrm{M}})\right| \leq \left|\lambda_1(\boldsymbol{k}^{\mathrm{M}})\right|$.

2.5 ECONOMIC GROWTH AND WELFARE

Equipped with dynamical systems, we can now investigate how financial intermediation and in particular the degree of interbank competition affect economic growth and welfare.

The long-run development of the respective economies is determined by steady states $k^i \geq 0$ of the dynamical system (PFD^i), where $i = \mathrm{N}, \mathrm{C}, \mathrm{M}$. These are defined by solutions to

$$\frac{k}{p} \overset{!}{=} \begin{cases} I(w(k), f'(k)) & \text{if } i = \mathrm{N} \\ D(w(k), pf'(k)) & \text{if } i = \mathrm{C} \\ D\big(w(k), r^{\mathrm{M}}(w(k), f'(k))\big) + e^{\mathrm{M}}(k,k,k) & \text{if } i = \mathrm{M} \end{cases}, \tag{2.18}$$

respectively. As a first step, we analyse whether poverty traps may arise.[14]

Proposition 2.3 (Poverty traps).
Let Assumptions 2.1 and 2.2 be satisfied. Then the origin $k^i = 0$ *is a steady state of the dynamical system* (PFD^i), $i = \mathrm{N}, \mathrm{C}, \mathrm{M}$, *if and only if* $f(0) = 0$. *If the origin is a steady state, then it is unstable.*

Proposition 2.3 rules out poverty traps, regardless of the banking sector. Under perfect foresight, the economy never goes bankrupt as no growth path $\{k_t\}_{t=0}^{\infty}$ with $k_0 > 0$ converges to zero. Against this background, the question of the existence of positive steady states arises naturally. By characterizing the positive steady states, Theorem 2.1 now states the first main result of this article, namely the effect of financial intermediation on long-run economic growth.

Theorem 2.1 (Economic growth).
Let Assumptions 2.1 and 2.2 be satisfied and consider economies that are identical in all respects except for the banking sector. Then the following holds true.

---

[14] The term “poverty trap” is used ambiguously in the literature. We define a poverty trap as the origin $k^i = 0$ being an asymptotically stable steady state.

(i) Each dynamical system (PFD^i)*,* $i = \mathrm{N}, \mathrm{C}, \mathrm{M}$*, attains a uniquely determined positive steady state* $k^i > 0$*, which is asymptotically stable, globally on* $\mathbb{R}_{++}$*. These steady states satisfy* $0 < k^{\mathrm{N}} < k^{\mathrm{C}}, k^{\mathrm{M}}$*.*

(ii) If the elasticity ϵ_f *is sufficiently large, then* $k^{\mathrm{N}} < k^{\mathrm{C}} < k^{\mathrm{M}}$*. Conversely, if* ϵ_f *is sufficiently small, then* $k^{\mathrm{N}} < k^{\mathrm{M}} < k^{\mathrm{C}}$*.*

Theorem 2.1 (i) shows that financial intermediation promotes long-run growth, as the economies featuring banks attain a higher steady-state capital-labor ratio than the economy without banks. This result holds irrespective of the degree of interbank competition and the initial capital-labor ratio k_0. *Ceteris paribus*, an initially poor economy with financial intermediation will, in the long run, outgrow any prosperous economy without banks. As established in Corollary 2.1, the underlying mechanism is that risk sharing fosters capital accumulation by inducing agents to invest additional funds in the production sector. This finding helps explain why low-income countries lacking a well-functioning banking system might face obstacles to development.

Theorem 2.1 (ii) concerns the interplay between competition in the banking sector and the capital dependence of the production sector. It demonstrates that the effect of interbank competition on real long-run development depends on the distribution of factor incomes and, therefore, on the production technology. If production is highly labor-intensive, wage income of agents accounts for a large share of total income, rendering strong saving incentives crucial for capital accumulation. In this case, perfect competition among banks maximizes economic growth by generating high deposit rates. However, if production is highly capital-intensive, saving incentives play only a minor role, as agents' wage income is small relative to banks' capital income. Owing to its abundant capital income, a monopolistic bank is then best positioned to provide the production sector with sufficient capital and is therefore conducive to economic growth.

Since the pure growth perspective pursued so far abstracts from all normative considerations, the question of how banking competition affects welfare arises naturally. Adopting a utilitarian measure, welfare of agents is captured by their expected life-cycle utility. Given any capital-labor ratio $k_t \geq 0$ and expectation $k_t^e \geq 0$, the welfare of the generation born in period t thus becomes

$$\mathcal{W}^i(k_t, k_t^e) := \begin{cases} \mathscr{U}_{\mathrm{res}}(w(k_t), f'(k_t^e)) & \text{if } i = \mathrm{N} \\ \mathscr{U}\left(w(k_t), r^i(w(k_t), f'(k_t^e))\right) & \text{if } i = \mathrm{C}, \mathrm{M}. \end{cases}$$

From this definition, we can directly deduce the following corollary, which states the welfare effects of financial intermediation in a comparative-statics manner.

Corollary 2.2.
Let Assumptions 2.1 and 2.2 be satisfied. Then for each $k_t > 0$ *and* $k_t^e > 0$*, we have*

$$\mathcal{W}^{\mathrm{N}}(k_t, k_t^e) \leq \mathcal{W}^{\mathrm{M}}(k_t, k_t^e) < \mathcal{W}^{\mathrm{C}}(k_t, k_t^e),$$

where the first inequality is strict if and only if $s(w(k_t), f'(k_t^e)) > b(w(k_t), f'(k_t^e))$*.*

Corollary 2.2 is, in essence, a risk sharing result. Banks increase agents' welfare by insuring them against idiosyncratic production risk. From a static perspective, perfect competition in the banking sector is most beneficial for agents, as it eliminates rent extraction.

Turning to a dynamic perspective on welfare, we next establish the *golden rule of capital accumulation*. The resource constraint of the economy implies that, independently of financial intermediation, total consumption per capita in period t satisfies

$$c_t^1 + c_t^2 = f(k_t) - \frac{k_{t+1}}{p}. \tag{2.19}$$

It follows from (2.19) that a *stationary feasible allocation* $(\bar{k}, \bar{c}^1, \bar{c}^2) \geq 0$ must satisfy

$$\bar{c}^1 + \bar{c}^2 = \phi(\bar{k}) := f(\bar{k}) - \frac{\bar{k}}{p}, \tag{2.20}$$

where the hump-shaped function $\phi : \mathbb{R}_+ \to \mathbb{R}$ stipulates stationary total consumption per capita. The notion of a stationary feasible allocation now allows us to derive the following social-planner result, which is already found in Diamond (1965).

Lemma 2.4 (Golden-rule allocation).
Let Assumptions 2.1 and 2.2 be satisfied. Then the stationary feasible allocation with the highest possible level of welfare is the golden-rule allocation (k_G, c_G^1, c_G^2)*, where the steady-state capital-labor ratio is* $k_G = f'^{-1}\left(\frac{1}{p}\right) > 0$ *and the steady-state consumption plan* $(c_G^1, c_G^2) \gg 0$ *is uniquely determined by*

$$\frac{u'(c_G^1)}{v'(c_G^2)} = 1 \quad \text{and} \quad c_G^1 + c_G^2 = \phi(k_G).$$

Since steady states of an OLG economy define stationary feasible allocations, we may evaluate them using the golden-rule allocation (k_G, c_G^1, c_G^2) as an efficient benchmark. Observe that the golden-rule capital-labor ratio k_G is solely determined by the production technology and the success rate p, whereas the steady states of the dynamical systems (PFD^i), $i = \mathrm{N, C, M}$, depend on agents' preferences and the prevailing form of financial intermediation. For this reason, k_G will generically not be attained as a steady state, and the stationary feasible allocations associated with the respective steady states will generically deviate from the golden-rule allocation. Indeed, k_G is a steady state if and only if it solves (2.18). As a consequence, capital accumulation paths will generally be inefficient, and steady states will not be welfare-maximizing.

The preceding considerations reveal that, from a societal perspective, agents will generally fail to achieve investment efficiency. Since financial intermediation affects the steady states, the question arises to what extent banking can improve the steady-state welfare of agents. We are now in a position to present our second main result, namely the effect of financial intermediation on long-run welfare.

Theorem 2.2 (Welfare).
Let Assumptions 2.1 and 2.2 be satisfied and $k^i > 0$ *be the positive steady state of the dynamical system* (PFD^i)*,* $i = \mathrm{N, C, M}$*. Then the following holds true.*

(i) If $0 < k^{\mathrm{C}} \leq k_G$*, then the economy with a competitive banking sector experiences a higher level of steady-state welfare than the economy without banks, i.e.,*

$$\mathcal{W}^{\mathrm{N}}(k^{\mathrm{N}}, k^{\mathrm{N}}) < \mathcal{W}^{\mathrm{C}}(k^{\mathrm{C}}, k^{\mathrm{C}}).$$

(ii) If the elasticity ϵ_f *is sufficiently large, then the economy with a monopolistic banking sector experiences the highest level of steady-state welfare, i.e.,*

$$\mathcal{W}^{\mathrm{N}}(k^{\mathrm{N}}, k^{\mathrm{N}}) < \mathcal{W}^{\mathrm{C}}(k^{\mathrm{C}}, k^{\mathrm{C}}) < \mathcal{W}^{\mathrm{M}}(k^{\mathrm{M}}, k^{\mathrm{M}}).$$

Conversely, if ϵ_f *is sufficiently small, then*

$$\mathcal{W}^{\mathrm{M}}(k^{\mathrm{M}}, k^{\mathrm{M}}) < \mathcal{W}^{\mathrm{C}}(k^{\mathrm{C}}, k^{\mathrm{C}}).$$

Theorem 2.2 (i) implies that financial intermediation which extracts no rent increases the steady-state welfare of an economy if it alleviates under-accumulation of capital. By providing investment incentives for agents and propelling capital accumulation, financial intermediation moves the steady-state allocation closer to the golden-rule allocation, thereby increasing agents' consumption. Under over-accumulation, however, the effect of financial intermediation on steady-state welfare becomes generally ambiguous. While the provision of risk sharing always benefits agents, the resulting investment incentives exacerbate over-accumulation. Thus, a trade-off between insurance and reduced consumption arises whenever financial intermediation induces over-accumulation. The resolution of this trade-off, and whether financial intermediation ultimately increases or decreases steady-state welfare, depends on the technology and agents' preferences.

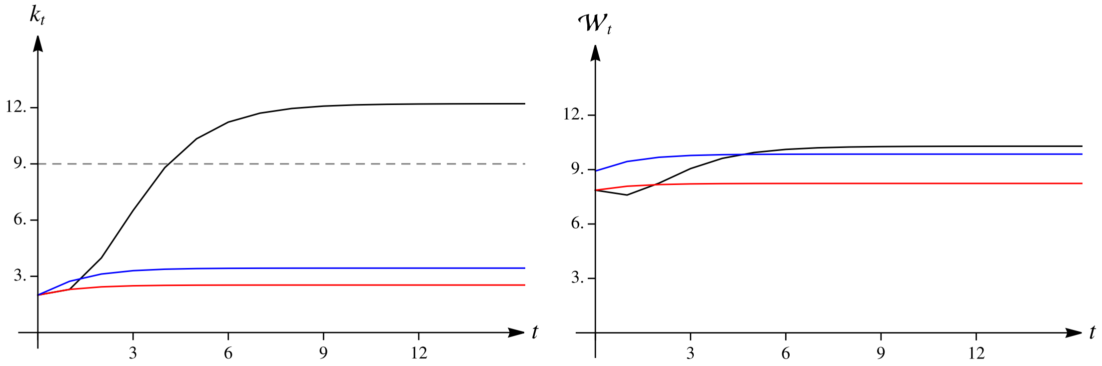
The second part of Theorem 2.2 reveals that the capital dependence of the production sector and the resulting distribution of factor incomes determine how interbank competition affects steady-state welfare. If production is highly capital-intensive, agents' savings alone may be insufficient to attain a steady state close to the golden rule. In this case, monopolistic capital investments can alleviate under-accumulation and thereby improve steady-state welfare. This mechanism can be explained by the *general theory of the second best*, see Lipsey and Lancaster (1956). Since each cohort of agents is only concerned with maximizing their own life-cycle utility and markets are incomplete, capital accumulation in OLG models is inherently inefficient. Introducing an additional distortion, namely a banking monopoly, may then partially counteract inefficient capital accumulation lead to a superior overall outcome, provided production is sufficiently capital-intensive. If production is labor-intensive, however, agents' savings are sufficient and the adverse effect of monopolistic rent extraction on agents' welfare outweighs. Overall, perfect competition is then favorable because it minimizes the rent extracted by banks.

At this point, it is worthwhile considering a numerical example with a standard parameterization from the literature in order to illustrate our central findings thus far.

Example 2.3 (CES utility and Cobb-Douglas technology).
Following on from Examples 2.1 and 2.2, consider the case of a CES utility function combined with a Cobb-Douglas production function. Figure 2.1 portrays the perfect-foresight growth paths $\{k_t\}_{t=0}^{\infty}$ *and corresponding welfare levels* $\{\mathcal{W}^i(k_t, k_{t+1})\}_{t=0}^{\infty}$ *generated by the dynamical*

systems (PFD^i), $i = \mathrm{N}, \mathrm{C}, \mathrm{M}$ *respectively, given a relatively capital-intensive technology. The dashed line marks the golden-rule value* k_G*. The following observations are immediate from Figure 2.1:*

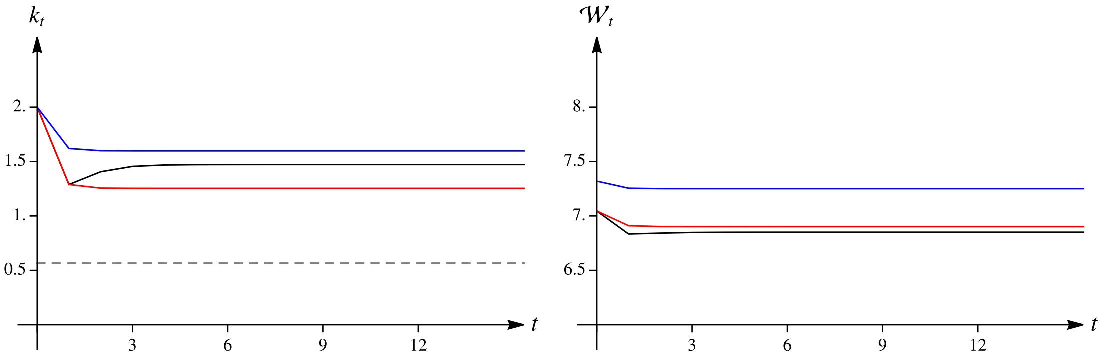
- *(i) The economies with banks display greater long-run growth than the economy without banks, cf. Theorem 2.1 (i).*
- *(ii) Since the economy lacking banks experiences under-accumulation, a competitive banking sector improves the steady-state welfare, cf. Theorem 2.2 (i).*
- *(iii) Since production is capital-intensive, the monopolistic economy experiences the highest level of long-run growth and welfare, cf. Theorems 2.1 (ii) and 2.2 (ii).*



(a) Perfect-foresight growth paths $\{k_t\}_{t=0}^{\infty}$ **(b)** Welfare levels $\{\mathcal{W}^i(k_t, k_{t+1})\}_{t=0}^{\infty}$

Figure 2.1. A banking monopoly maximizes long-run growth and welfare.
Color code: — (PFD$^{\mathrm{N}}$), — (PFD$^{\mathrm{C}}$), — (PFD$^{\mathrm{M}}$).
Parameters: $A = 10$, $\epsilon_f = \alpha = 0.5$, $p = 0.6$, $\sigma = 0.5$, $\beta = 1$

By contrast, Figure 2.2 portrays the perfect-foresight growth paths and welfare levels given a relatively labor-intensive technology. In this case, perfect competition among banks maximizes long-run growth and welfare, cf. Theorems 2.1 (ii) and 2.2 (ii). While, in the competitive case, the insurance-consumption trade-off is resolved in favor of insurance, reduced consumption outweighs in the monopolistic case, so that a reduction in steady-state welfare results.



(a) Perfect-foresight growth paths $\{k_t\}_{t=0}^{\infty}$ **(b)** Welfare levels $\{\mathcal{W}^i(k_t, k_{t+1})\}_{t=0}^{\infty}$

Figure 2.2. Perfect competition among banks maximizes long-run growth and welfare.
Color code: — (PFD$^{\mathrm{N}}$), — (PFD$^{\mathrm{C}}$), — (PFD$^{\mathrm{M}}$).
Parameters: $A = 10$, $\epsilon_f = \alpha = 0.1$, $p = 0.6$, $\sigma = 0.5$, $\beta = 1$.

2.6 Endogenous Fluctuations

The question of how financial intermediation affects the qualitative dynamics of the economy is addressed next. We investigate whether banking may induce endogenous fluctuations, as for example in the form of oscillations, periodic cycles, or chaos.

Recalling that youthful and old-age consumption are normal goods and gross substitutes by Assumption 2.1, we may state the following result.

Proposition 2.4 (Global monotonicity).
Let Assumptions 2.1 and 2.2 be satisfied. Then the perfect forecasting rules ψ^{N} and ψ^{C} defined by (PFD$^{\mathrm{N}}$) and (PFD$^{\mathrm{C}}$), respectively, are strictly increasing, i.e.,

$$\frac{\mathrm{d}\psi^{\mathrm{N}}}{\mathrm{d}k}(k)=\frac{\frac{\partial I}{\partial w}\big(w(k),f'(\psi^{\mathrm{N}}(k))\big)w'(k)}{\frac{1}{p}-\frac{\partial I}{\partial \rho^{e}}\big(w(k),f'(\psi^{\mathrm{N}}(k))\big)f''(\psi^{\mathrm{N}}(k))}>0$$

and

$$\frac{\mathrm{d}\psi^{\mathrm{C}}}{\mathrm{d}k}(k)=\frac{\frac{\partial D}{\partial w}\big(w(k),pf'(\psi^{\mathrm{C}}(k))\big)w'(k)}{\frac{1}{p}-\frac{\partial D}{\partial r}\big(w(k),pf'(\psi^{\mathrm{C}}(k))\big)pf''(\psi^{\mathrm{C}}(k))}>0$$

for all $k\geq 0$.

Proposition 2.4 rules out endogenous fluctuations. Depending on the initial condition k_0, the perfect-foresight growth paths $\{k_t\}_{t=0}^{\infty}$ generated by (PFD$^{\mathrm{N}}$) and (PFD$^{\mathrm{C}}$) are either monotonically increasing or monotonically decreasing, as in the example portrayed in Figure 2.2a. Under perfect foresight and perfect competition, financial intermediation cannot generate endogenous fluctuations in our model. This result stands in contrast with the findings of Banerji et al. (2004), who argue that risk sharing through financial intermediation may expose an economy to the full variety of complex dynamics.

The case of a banking monopoly is treated in the following proposition.

Proposition 2.5 (Local monotonicity).
Let Assumptions 2.1 and 2.2 be satisfied. Then, in a local neighborhood of the positive steady state $k^{\mathrm{M}}>0$, the perfect-foresight growth paths $\{k_t\}_{t=0}^{\infty}$ generated by (PFD$^{\mathrm{M}}$) are monotonic.

Recall that, by Theorem 2.1, any perfect-foresight growth path $\{k_t\}_{t=0}^{\infty}$ with $k_0>0$ generated by (PFD$^{\mathrm{M}}$) converges to the steady state $k^{\mathrm{M}}>0$ in the long run. Proposition 2.5 thus rules out periodic cycles and chaos, as it implies that initial fluctuations will, sooner or later, vanish and the dynamics become monotonic.

The bottom line of the preceding analysis is that, in our model, financial intermediation cannot affect the qualitative dynamics of the economy. The resulting growth paths are monotonic and thus qualitatively indistinguishable from those of the canonical overlapping-generations model without financial intermediation by Diamond (1965). In the monopolistic case, this result holds only locally in a neighborhood of a steady state, as the Hartman–Grobman theorem does not permit global inferences.

2.7 Financial Stability

In real-world economies, banking crises can significantly undermine economic growth and welfare. The well-known *competition-fragility hypothesis* posits that increased competition among banks can impair the stability of the financial system. As competition intensifies, profit margins shrink, prompting banks to engage in riskier lending and investment practices to maintain profitability, e.g., see Allen and Gale (2000), Hellmann et al. (2000), and Keeley (1990).[15] This section presents a short discussion on how banking competition affects financial stability in our model, showing that a competitive banking sector is more susceptible to forecast errors than a banking monopoly.

Allowing for subjective expectations, banks may accidentally go bankrupt when expectations are too optimistic, i.e., $\rho_t^e > \rho_{t+1}$. This is seen as follows. The law of large numbers implies that, under perfect competition, the maximum deposit rate a bank can offer, given the forecast ρ_t^e, is $r_t^{\mathrm{C}} = p\rho_t^e$. Any marginal forecast error $\rho_t^e - \rho_{t+1} > 0$ then results in a negative intermediation margin and, consequently, a loss for the bank. A bank defaults once it has insufficient equity to honor its deposit contracts. The intermediation margins of a monopolistic bank, however, remain positive as long as the forecast error is sufficiently small. Moreover, the monopolist holds a comparatively larger equity stock, which can buffer losses resulting from forecast errors. Indeed, for any given $(k_t, k_{t-1}, k_{t-1}^e)$, we have $e^{\mathrm{M}}(k_t, k_{t-1}, k_{t-1}^e) > e^{\mathrm{C}}(k_t, k_{t-1}, k_{t-1}^e)$. Under subjective expectations, competition among banks thus reduces the stability of the financial system by inducing banks to take more risk. Our model thus supports the competition-fragility hypothesis. The following numerical example illustrates this point.

Example 2.4 (Naive expectations)**.**
Consider a CES utility function combined with a Cobb-Douglas production function. Figure 2.3 portrays the growth paths generated by naive expectations, i.e., $k_t^e = k_t$ *for all* $t \geq 0$*. The growth paths of the monopolistic economy display significantly less volatility than the growth paths of the competitive economy. As shown in Figure 2.3b, the destabilizing effect of competition can become so severe that it triggers a collapse of the financial system, whereas the monopolistic economy converges to a stable steady state.*

2.8 Dividend Payments

Thus far, the model abstracts from dividend payments of banks. We next check the robustness of our main results by introducing dividend payments to the bank owners. Note that the case of perfect competition is unaffected by dividend payments since, under perfect foresight, banks earn zero profit.

Consider a risk-neutral monopolistic bank that is collectively owned by agents. The bank maximizes its anticipated profit by raising deposits from the public. Young agents become

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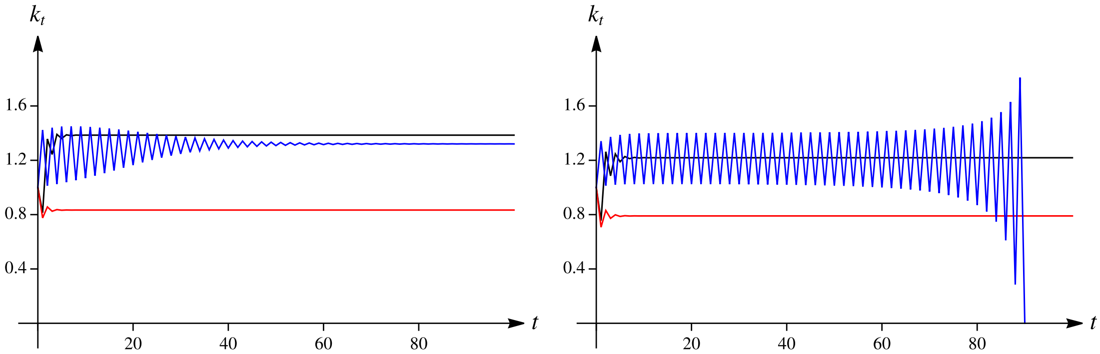
[15] However, the competition-fragility hypothesis is not undisputed among economists. For instance, Boyd and De Nicolo (2005) argue that market power of banks entails higher interest rates on loans, which in turn increases the risk of bankruptcy for borrowers.



(a) Damped oscillations ($\alpha = 0.2$) **(b)** Financial collapse ($\alpha = 0.169$)

Figure 2.3. Non-monotonic growth paths generated by naive expectations.
Color code: — No banks, — Perfect Competition, — Monopoly.
Parameters: $A = 10$, $p = 0.4$, $\sigma = 0.5$, $\beta = 1$.

shareholders of the bank by accepting a deposit contract. Dividend payments accrue to old agents. Let $\vartheta_t^e \geq 0$ denote a young agent's expectation formed in period t with respect to the dividend payment $\vartheta_{t+1} \geq 0$ realized in $t+1$. The bank's dividend policy is captured by the parameter $0 \leq \mu < 1$, which specifies the share of bank equity that is paid out to the shareholders. For simplicity, we assume that μ is exogenous and time-invariant.

Given the wage rate w_t, the deposit rate r_t, and the anticipated dividend payment ϑ_t^e, a young agent's optimal supply of deposits $0 \leq D_t < w_t$ is uniquely determined by

$$D_t = D(w_t, r_t, \vartheta_t^e) := \underset{0 \leq D \leq w_t}{\operatorname{argmax}} \; u(w_t - D) + v(r_t D + \vartheta_t^e), \tag{2.21}$$

noting that

$$D(w_t, r_t, \vartheta_t^e) > 0 \qquad \Longleftrightarrow \qquad r_t > \frac{u'(w_t)}{v'(\vartheta_t^e)}.$$

The participation constraint takes the form

$$u(w_t - D(w_t, r_t, \vartheta_t^e)) + v(r_t D(w_t, r_t, \vartheta_t^e) + \vartheta_t^e) \geq \mathscr{U}_{\text{res}}(w_t, \rho_t^e),$$

with the reservation utility $\mathscr{U}_{\text{res}}(w_t, \rho_t^e)$ as defined in (2.3). Observe that dividend payments relax the participation constraint relative to the baseline model. Since part of the banking surplus is distributed to shareholders, agents are now willing to accept deposit contracts that they would otherwise reject. Given w_t, e_t, and the expectations ρ_t^e and ϑ_t^e, the bank's profit-maximization problem reads

$$\begin{aligned} &\max_{0 \leq r \leq p\rho_t^e} \; D(w_t, r, \vartheta_t^e)[p\rho_t^e - r] + p\rho_t^e(1-\mu)e_t \\ &\qquad \text{s.t.} \quad u(w_t - D(w_t, r, \vartheta_t^e)) + v(r D(w_t, r, \vartheta_t^e) + \vartheta_t^e) \geq \mathscr{U}_{\text{res}}(w_t, \rho_t^e). \end{aligned} \tag{2.22}$$

Existence and uniqueness of a solution are established next.

Lemma 2.5 (Banking monopoly with dividend payments)**.**
Let Assumptions 2.1 and 2.2 be satisfied and $w_t > 0$, $e_t \geq 0$, $\rho_t^e > 0$, *and* $\vartheta_t^e \geq 0$ *with*

$p\rho_t^e > \frac{u'(w_t)}{v'(\vartheta_t^e)}$ *be given. Then the monopolistic bank's problem (2.22) admits a unique solution* $r_t^{\mathrm{M}} = r^{\mathrm{M}}(w_t, \rho_t^e, \vartheta_t^e) \in \big(\frac{u'(w_t)}{v'(\vartheta_t^e)}, p\rho_t^e\big)$.[16]

As before, the bank's equity stock in period t is determined by a map of the form

$$\begin{aligned} e_t^{\mathrm{M}} = e^{\mathrm{M}}(k_t, k_{t-1}, k_{t-1}^e, \vartheta_{t-1}^e) &:= f'(k_t)k_t \\ &\quad - r^{\mathrm{M}}(w(k_{t-1}), f'(k_{t-1}^e), \vartheta_{t-1}^e)\, D(w(k_{t-1}), r^{\mathrm{M}}(w(k_{t-1}), f'(k_{t-1}^e), \vartheta_{t-1}^e)\vartheta_{t-1}^e), \end{aligned}$$

Accordingly, the realized dividend payment to an old agent in period t is

$$\vartheta_t = \vartheta^{\mathrm{M}}(k_t, k_{t-1}, k_{t-1}^e, \vartheta_{t-1}^e) := \mu e^{\mathrm{M}}(k_t, k_{t-1}, k_{t-1}^e, \vartheta_{t-1}^e).$$

Since the remainder of equity is reinvested in production projects, the economic law of capital accumulation takes the form

$$\begin{aligned} k_{t+1} = G^{\mathrm{M}}(k_t, k_{t-1}, k_t^e, k_{t-1}^e, \vartheta_t^e, \vartheta_{t-1}^e) &:= pD(w(k_t), r^{\mathrm{M}}(w(k_t), f'(k_t^e), \vartheta_t^e), \vartheta_t^e) \\ &\quad + p(1-\mu)e^{\mathrm{M}}(k_t, k_{t-1}, k_{t-1}^e, \vartheta_{t-1}^e). \end{aligned}$$

Since agents must now form expectations regarding both, the future capital-labor ratio and the future dividend payment, the expectations-feedback effect becomes more involved. In particular, the forecasts k_t^e and ϑ_t^e must be consistent: if k_t^e is a correct forecast for k_{t+1}, then the forecast ϑ_t^e must satisfy

$$\vartheta_t^e = \vartheta^{\mathrm{M}}(k_t^e, k_t, k_t^e, \vartheta_t^e) \tag{2.23}$$

in order to be a correct forecast for ϑ_{t+1}.

Lemma 2.6 (Perfect foresight with dividend payments)**.**
Let Assumptions 2.1 and 2.2 be satisfied. Then there exists a function

$$\varphi : \mathcal{D}^{\mathrm{M}} \to \mathbb{R}_+, \qquad \vartheta_t^e = \varphi(k_t^e, k_t),$$

that satisfies

$$\varphi(k, \hat{k}) = \vartheta^{\mathrm{M}}\big(k, \hat{k}, k, \varphi(k, \hat{k})\big) \quad \textit{for all} \quad (k, \hat{k}) \in \mathcal{D}^{\mathrm{M}}.$$

For each $(k_t, k_{t-1}) \in \mathcal{D}^{\mathrm{M}}$*, there exists a pair of correct forecasts* (k_t^e, ϑ_t^e)*.*

As in the baseline model, the set $\mathcal{D}^{\mathrm{M}}$ is forward-invariant, implying that the possibility of perfect foresight is not lost over time. The perfect-foresight dynamics exists on $\mathcal{D}^{\mathrm{M}}$ and is governed by the implicit difference equation

$$k_{t+1} = G^{\mathrm{M}}\big(k_t, k_{t-1}, k_{t+1}, k_t, \varphi(k_{t+1}, k_t), \varphi(k_t, k_{t-1})\big), \quad t \geq 1. \tag{PFD$^\vartheta$}$$

An analytical investigation of the dynamics generated by (PFD$^\vartheta$) is beyond the scope of this article. The important point is that the model without dividend payments reobtains if the

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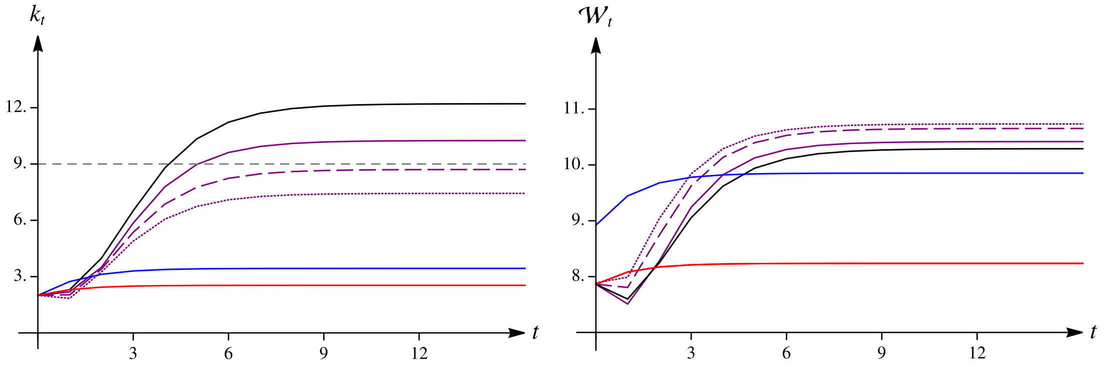
[16] A formal definition of the function $r^{\mathrm{M}}(w_t, \rho_t^e, \vartheta_t^e)$ is provided in the proof of Lemma 2.5.

parameter μ approaches zero. Under perfect foresight, agents correctly anticipate that no dividends will be paid out to them if $\mu \to 0$. Indeed, it follows from (2.23) that if $\mu \to 0$, then $\vartheta_t^e = \varphi(k_t^e, k_t) \to 0$ for all $t \geq 0$ such that the growth paths generated by (PFD$^{\mathrm{M}}$) and (PFD$^{\vartheta}$) coincide. Figure 2.4 provides an illustration using a numerical example with CES preferences and Cobb-Douglas production technology. We can thus conclude that the results in Theorems 2.1 and 2.2 also apply if the monopolistic bank distributes part of the surplus to agents, provided that the parameter μ is not too large. These considerations demonstrate that if a banking monopoly occurs in conjunction with a capital-intensive production sector and the dividend payments are sufficiently constrained (e.g., due to regulatory requirements), then it may outperform its perfectly competitive counterpart in terms of the induced growth and welfare levels. The model shows that from a dynamic viewpoint, more competition is not necessarily beneficial.



(a) Capital **(b)** Welfare

Figure 2.4. Effect of the dividend parameter μ on economic growth and welfare.
Color code: – (PFD$^{\mathrm{N}}$), – (PFD$^{\mathrm{C}}$), – (PFD$^{\mathrm{M}}$), – (PFD$^{\vartheta}$).
Parameters: $A = 10$, $\alpha = 0.5$, $p = 0.6$, $\sigma = 0.5$, $\beta = 1$, $\mu = 0.07$ (solid), 0.14 (dashed), 0.21 (dotted).

2.9 CONCLUSION

This article adopted a dynamical systems approach to explore how competition among banks affects real economic growth, the scope for endogenous fluctuations, and welfare of agents in an overlapping-generations model with idiosyncratic risk. Consistent with the empirical findings of Cetorelli and Gambera (2001), the analysis shows that financial intermediation promotes long-run economic growth by mobilizing additional funds for production through the provision of risk sharing. Financial intermediation improves welfare in economies that experience under-accumulation not only by providing insurance, but also by stimulating capital accumulation and thereby increasing investment efficiency. This result implies that establishing a well-functioning banking system is crucial in underdeveloped countries.

The main finding of this article is that a banking monopoly can generate higher levels of long-run growth and welfare than its competitive analog, provided the production sector is sufficiently capital-intensive and bank dividend payments sufficiently constrained. Otherwise, perfect competition among banks is favorable, particularly when production is labor-intensive. For this reason, a growth- and welfare-enhancing effect of reduced competition is solely conceivable in industrialized economies with capital-intensive industries such as automotive, pharmaceuticals, or telecommunications. Our theoretical findings are corroborated by substantial empirical evidence. Cetorelli and Gambera (2001) and Hoxha (2013) find that banking concentration has a growth-promoting effect in industries with high financial dependence. Maudos and Fernandez de Guevara (2006) argue that market power of banks promotes economic growth in industrialized countries. On the other hand, Beck et al. (2004) show that banking concentration creates financing obstacles for firms only in developing countries, i.e., countries that typically feature labor-intensive industries such as textiles, hospitality, or agriculture. For such countries, Deidda and Fattouh (2005) identify a negative impact of banking concentration on growth. Finally, Cetorelli and Gambera (2001) and Hoxha (2013) find that banking concentration curbs the growth of industries with low financial dependence.

Apart from these results, our analysis shows that, under perfect foresight, financial intermediation cannot adversely affect the qualitative dynamics of the economy. Allowing for erroneous expectations, it turns out that intense competition makes the banking system more susceptible to financial crises, thus providing support for the competition-fragility hypothesis. This issue warrants further investigation.

Like any other model, ours does not come without limitations. While most of our assumptions on preferences and technology are standard in this strand of the literature, a world with homogeneous two-period-lived agents is, of course, stylized. The approach of initially abstracting from dividend payments and incorporating them afterwards is limiting, but necessary to make the analytical complexity somewhat manageable. Nevertheless, the extension of the model suggests that our main findings are robust even when part of the banking surplus is distributed to agents. In view of the provoking result that a banking monopoly may outperform its competitive counterpart, we think that a fruitful avenue for future research is to investigate the optimal regulation of endogenous dividend payments and how these may affect economic growth, welfare, and the qualitative dynamics.

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Chapter 3

Capital Market Power and Economic Growth in an Overlapping-Generations Model with Rational Expectations[⋆]

Statement of Prior Publication

This chapter is based on the article

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Abstract

This article investigates how market power in the capital market affects savings, capital accumulation, and the qualitative dynamics in the canonical overlapping-generations model. Agents interact strategically by anticipating the influence of their savings decisions on the realized capital-rental rate. Under the assumption that capital income is strictly increasing and strictly concave in the capital-labor ratio, the model admits unique temporary Nash equilibria. The resulting growth paths may be non-monotonic and therefore qualitatively distinct from perfect-foresight growth paths under competitive markets. Relative to price takers with perfect foresight, market power induces agents to curtail savings, thereby reducing capital accumulation and long-run growth. Population growth restores the competitive outcomes and smooths away endogenous fluctuations. These findings underscore the importance of competitive capital markets for strong and stable economic growth.

Keywords: Overlapping generations, market power, imperfect competition, economic growth, endogenous fluctuations, temporary Nash equilibrium.

JEL Classification: D43, D50, D84, E32, O41.

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3.1 INTRODUCTION

A central objective of growth and business cycle theory is to elucidate the forces that drive and shape macroeconomic dynamics. In this context, overlapping-generations (OLG) models have become a standard tool of analysis owing to their explicit incorporation of demographic structure. In OLG models with production, aggregate dynamics are governed primarily by the intertemporal savings decisions of successive cohorts of agents and, when labor supply is endogenous, by their labor-leisure choices. Economic fluctuations therefore often originate from local properties of savings and labor supply functions, particularly when these decline near a steady-state equilibrium. In line with the original formulation by Diamond (1965), a widespread assumption in the OLG literature is perfect competition in all markets, so that agents take prices as given and make decisions based on rational expectations about the future, e.g., see Azariadis (1993) or De La Croix and Michel (2002). This assumption rules out strategic effects that may arise when market participation is concentrated among a small number of large agents such as institutional investors or labor unions, capable of influencing prices to their advantage. In such environments, agents may exercise market power by anticipating the impact of their savings and labor supply decisions on factor prices and adjust their behavior accordingly, creating a strategic channel that affects capital accumulation. This naturally raises the question of the extent to which the presence of market power in factor markets affects the dynamics of an OLG economy, both qualitatively and quantitatively.

Over the past four decades, a relatively small body of theoretical literature has emerged that relaxes the competitive-markets assumption and examines the effect of imperfect competition on economic growth and the occurrence of endogenous fluctuations within OLG models. Laitner (1982) shows that oligopolistic production structures can reduce capital accumulation and long-run output, while Chatterjee et al. (1993) identify them as a source of fluctuations. By contrast, Dos Santos Ferreira and Lloyd-Braga (2002) show that market power may facilitate sustained economic growth through intergenerational transfers from old to young cohorts. Dos Santos Ferreira and Lloyd-Braga (2005, 2008) demonstrate that imperfect competition in the intermediate-goods sector may generate business cycles, whereas Smith (1998) finds that market power in the banking sector can amplify them. Moreover, several contributions link business cycles to market power of labor unions, e.g., see Coimbra et al. (2005), Jacobsen (2000), and Kaas and Madden (2005). Further contributions addressing imperfect competition in OLG models include Aloi et al. (2000), Chatterjee and Cooper (1989), and Rivard (1994). Finally, Goenka et al. (1998) highlight the potential of strategic interaction among agents to expose a pure-exchange OLG economy to a wide range of complex dynamics.

Despite the central role of the savings-and-investment channel in OLG models, the implications of market power in the capital market have, to the best of our knowledge, not yet been addressed in this literature. The purpose of this article is to fill this gap. We incorporate capital market power into the canonical OLG model of Diamond (1965). A finite set of two-period-lived agents chooses savings strategically, internalizing the effect of their decisions on the realized capital-rental rate. We show that this approach constitutes a natural extension of perfect foresight, as it requires agents to possess the same information and to account for the same feedback mechanisms. Adopting methods from game theory, we establish that under standard assumptions on preferences, technology, and capital income that is strictly increasing and strictly concave in the capital-labor ratio, the economy admits unique temporary Nash equilibria in pure strategies. A comparison between the resulting *Nash-equilibrium growth paths* and the competitive perfect-foresight growth paths yields two main insights. First, market power curbs capital accumulation, as agents strategically reduce savings in order to raise the realized return on savings. In the long run, this engenders a reduction in steady-state output. Second, it turns out that the transition dynamics may differ qualitatively, as the Nash-equilibrium growth paths under a non-stationary population profile may display transient fluctuations. In essence, this non-monotonicity arises from strategic interaction in the capital market, since temporary Nash equilibria may depend sensitively on the total number of agents supplying savings. If population growth is unbounded, these fluctuations vanish over time as agents' market power is gradually eliminated and the competitive limit is approached, which permits only monotonic growth. Summarizing, this article contributes to the overlapping-generations literature by identifying capital market power as both a source of non-monotonic dynamics and an impediment to economic growth.

The article proceeds as follows. The next section introduces the model and states the assumptions on preferences and technology. Section 3.3 revisits the case of competitive markets and derives the perfect-foresight dynamics. Section 3.4 introduces capital market power and contains the main results. Section 3.5 presents a model extension that incorporates market power in the labor market. Section 3.6 concludes. Throughout the article, standard examples are used to illustrate the main results. All proofs are provided in Appendix C.

3.2 MODEL PREREQUISITES

We consider a one-sector overlapping-generations model with production in the spirit of Diamond (1965). At the beginning of each period $t = 0, 1, \dots, \infty$, a new generation consisting of $N_t \in \mathbb{N}_1 := \mathbb{N} \setminus \{0\}$ identical, two-period-lived agents is born.[1] In principle, the population profile may be given by any sequence $\{N_t\}_{t=0}^{\infty}$ generated by some map $\mathcal{N} : \mathbb{N}_1 \to \mathbb{N}_1$. For simplicity, however, we assume that the population grows at a constant rate $n \in \mathbb{N}_0$, so that[2]

$$N_{t+1} = \mathcal{N}(N_t) := (1+n)N_t.$$

The intertemporal preferences of an agent are represented by a life-cycle utility function $U : \mathbb{R}_+^2 \to \mathbb{R}$ of the additive-separable form[3]

$$U(c^1, c^2) := u(c^1) + v(c^2),$$

where $c^1, c^2 \geq 0$ denote youthful and old-age consumption, respectively.

Our assumptions on the preferences are as follows.

Assumption 3.1 (Preferences).
The utility functions $u, v : \mathbb{R}_+ \to \mathbb{R}$ *are twice continuously differentiable, strictly increasing,*

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1. If convenient, we may interpret each agent as a coalition of multiple households.
2. Since $n \in \mathbb{N}_0$, the number of agents stays integer at all times.
3. A time-discount factor $\beta > 0$ is included in the utility function v.

strictly concave, and satisfy the Inada conditions

$$\lim_{c \to 0} u'(c) = \infty \quad \textit{and} \quad \lim_{c \to 0} v'(c) = \infty.$$

Moreover, the elasticity of v' satisfies

$$-\frac{v''(c)c}{v'(c)} \leq 1 \quad \textit{for all} \ \ c \geq 0.$$

Assumption 3.1 implies that youthful and old-age consumption are normal goods and, since savings are non-decreasing in the anticipated return on savings, weak gross substitutes.

The production sector comprises a large number of price-taking firms that transform capital $K \geq 0$ and labor $L \geq 0$ into a single consumption-and-investment good, which serves as the numéraire. Capital and labor are paid their marginal products. Each young agent inelastically supplies one unit of labor. Old agents are retired and consume their capital income. The technology of the representative firm has constant returns to scale. Denote by $f : \mathbb{R}_+ \to \mathbb{R}_+$ the production function of the representative firm in intensive form and by $k := \frac{K}{L}$ the capital-labor ratio, then $y = f(k)$ is the GDP per capita.

We impose the following assumptions on the technology.

Assumption 3.2 (Technology).
The production function $f : \mathbb{R}_+ \to \mathbb{R}_+$ is thrice continuously differentiable, strictly increasing, strictly concave, and satisfies

$$\epsilon_{f'}(k) := \frac{f''(k)k}{f'(k)} > -1 \quad \textit{and} \quad \frac{f'''(k)k}{f''(k)} > -2 \quad \textit{for all} \ \ k \geq 0. \tag{3.1}$$

Assumption 3.2 is relatively general and is satisfied by many standard production functions from the OLG literature, including CES production functions with an elasticity of factor substitution greater than one. It follows from (3.1) that the capital income $k \mapsto f'(k)k$ of the old generation is strictly increasing and strictly concave in the capital-labor ratio. As will become clear below, these two properties help ensure that the decision problem of an agent with market power admits a *unique* solution.

3.3 COMPETITIVE MARKETS

To establish a benchmark for assessing the effects of market power, this section briefly revisits the case of competitive markets, adopting the sequential modeling approach presented in Wenzelburger (2025). In each period $t \geq 0$, agents form an expectation $R_t^e > 0$ with respect to the gross return on savings R_{t+1} realized in $t + 1$. Given the expectation R_t^e and the wage rate $w_t \geq 0$, the objective of a young agent is to choose a consumption plan that maximizes his life-cycle utility.[4] The resulting *savings function* $s : \mathbb{R}_+ \times \mathbb{R}_{++} \to \mathbb{R}_+$ is

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[4] Clearly, the price-taking assumption is only reasonable when the number of agents N_t is sufficiently large.

uniquely determined by

$$s_t = s(w_t, R_t^e) := \underset{0 \le s \le w_t}{\operatorname{argmax}} \; u(w_t - s) + v(R_t^e s). \tag{3.2}$$

The Inada conditions imposed on the utility functions u and v in Assumption 3.1 imply that the optimal consumption plan is interior and thus satisfies the first-order condition

$$\frac{u'(w_t - s_t)}{v'(R_t^e s_t)} = R_t^e, \tag{3.3}$$

stating that the marginal rate of intertemporal substitution is equal to the anticipated gross return on savings. The capital-labor ratio of the subsequent period $t+1$ then becomes

$$k_{t+1} = \frac{K_{t+1}}{L_{t+1}} = \frac{N_t s_t}{N_{t+1}} = \tfrac{1}{1+n} s(w_t, R_t^e). \tag{3.4}$$

Since capital is paid its marginal product, the realized gross return on savings is

$$R_{t+1} = R(k_{t+1}) := 1 - \delta + f'(k_{t+1}), \tag{3.5}$$

where $\delta \in (0,1]$ denotes the capital depreciation rate. Inserting (3.4) into (3.5), it follows that

$$R_{t+1} = R\big(\tfrac{1}{1+n} s(w_t, R_t^e)\big), \tag{3.6}$$

showing that the expectation R_t^e generally feeds back into the realization R_{t+1}. The formation of *correct* expectations thus necessitates knowledge of the production function f because, otherwise, agents cannot take the expectations-feedback effect into account. Indeed, a correct expectation R_t^e, given the wage rate w_t, is determined by

$$R_t^e = R\big(\tfrac{1}{1+n} s(w_t, R_t^e)\big). \tag{3.7}$$

Under the hypotheses of Assumptions 3.1 and 3.2, (3.7) defines a unique *perfect forecasting rule* $\psi : \mathbb{R}_+ \to \mathbb{R}_{++}$ in the sense of Böhm and Wenzelburger (1999) that generates correct expectations for all possible wage rates, that is,

$$\psi(w) = R\big(\tfrac{1}{1+n} s(w, \psi(w))\big) \quad \text{for all} \quad w \ge 0. \tag{3.8}$$

By construction, the function ψ depends on the fundamentals of the economy. However, if an agent knows the production function f, then, in principle, he can internalize the feedback effect in his decision problem and anticipate how his savings behavior affects the realized return on savings. Put differently, the agent may exert market power in the capital market. This observation motivates our analysis of strategic interaction in the capital market.

Before introducing market power, however, it remains to establish a dynamical system. The realized wage rate is determined by the marginal product of labor, so that

$$w_t = w(k_t) := f(k_t) - f'(k_t) k_t. \tag{3.9}$$

It now follows from (3.4), (3.8), and (3.9) that the evolution of capital-labor ratios under perfect foresight is governed by a time-one map $G : \mathbb{R}_+ \to \mathbb{R}_+$, defined by[5]

$$k_{t+1} = G(k_t) := \tfrac{1}{1+n} s\big(w(k_t), \psi(w(k_t))\big). \tag{3.10}$$

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[5] Observe that due to the constant population growth rate, G is independent of the population size N.

Given any initial condition $k_0 > 0$, the map G recursively defines a *perfect-foresight growth path* $\{k_t\}_{t=0}^{\infty}$. The following lemma restates the result that when youthful and old-age consumption are normal goods and weak gross substitutes, the perfect-foresight growth paths are either monotonically increasing or monotonically decreasing, see Galor and Ryder (1989).

Lemma 3.1 (Monotonic dynamics)**.**
Let Assumptions 3.1 and 3.2 be satisfied. Then any perfect-foresight growth path $\{k_t\}_{t=0}^{\infty}$ *is monotonic and bounded.*

Lemma 3.1 rules out endogenous fluctuations, as for example oscillations, periodic cycles, or topological chaos. Since any perfect-foresight growth path $\{k_t\}_{t=0}^{\infty}$ generated by G is monotonic and bounded, it must attain a limit $k_{\mathrm{PF}} := \lim_{t\to\infty} k_t$. In general, G may admit multiple steady states with distinct basins of attraction, so that k_{PF} may depend on the initial condition k_0. To rule out poverty traps, we assume for the remainder of this article that either $G(0) > 0$ or $G'(0) > 1$, so that the origin is either not a steady state or, if it is a steady state, then it is unstable. The perfect-foresight growth path then attains a positive but finite limit $0 < k_{\mathrm{PF}} < \infty$. In particular, there exists at least one asymptotically stable positive steady state of G.[6]

3.4 Market Power and Strategic Interaction

Since capital accumulation is driven by agents' savings behavior, the question of to which extent market power in the capital market affects the dynamics of the economy arises naturally. We next develop the game-theoretic foundation required to model imperfect competition.

Let period t be arbitrary but fixed, and consider a typical young agent indexed by $i \in \{1, \ldots, N_t\}$. Since there is strategic interaction among young agents, we consider a simultaneous-move game with $N_t \in \mathbb{N}_1$ symmetric players and complete but imperfect information. The appropriate solution concept is Nash equilibrium in pure strategies. Given the wage rate $w_t > 0$, a *pure strategy* of agent i is an amount of savings $s^i \in [0, w_t]$. Denote by $\boldsymbol{s}_t^{-i} \in [0, w_t]^{N_t - 1}$ the strategy profile of all other young agents except i, resulting in aggregate savings[7]

$$S_t^{-i} := \|\boldsymbol{s}_t^{-i}\|_1 = s_t^1 + s_t^2 + \ldots + s_t^{i-1} + s_t^{i+1} + \ldots + s_t^{N_t}.$$

As the capital-rental rate is determined by the marginal product of capital, agent i exploits his knowledge of the production function and anticipates that if he saves the amount s^i, then the gross return on savings realized in period $t+1$ is

$$R_{t+1} = R(k_{t+1}) = R\big(\tfrac{1}{1+n}\tfrac{s^i + S_t^{-i}}{N_t}\big) = 1 - \delta + f'\big(\tfrac{1}{1+n}\tfrac{s^i + S_t^{-i}}{N_t}\big).$$

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[6] Sufficient conditions that rule out poverty traps are found in De La Croix and Michel (2002). Moreover, Galor and Ryder (1989) and Li and Lin (2012) provide sufficient conditions for the uniqueness of the positive steady state.

[7] In the case of a monopoly, $N_t = 1$, there is no strategic interaction and we have $S_t^{-i} = 0$.

Note that by Assumption 3.2, the agent's capital income is strictly increasing in his savings because

$$\frac{\partial}{\partial s^i}\Big[R\big(\tfrac{1}{1+n}\tfrac{s^i+S_t^{-i}}{N_t}\big)s^i\Big] = 1-\delta+f'\big(\tfrac{1}{1+n}\tfrac{s^i+S_t^{-i}}{N_t}\big)\Big[1+\tfrac{s^i}{s^i+S_t^{-i}}\,\epsilon_{f'}\big(\tfrac{1}{1+n}\tfrac{s^i+S_t^{-i}}{N_t}\big)\Big] > 0$$

for all $s^i \geq 0$. Setting

$$\mathscr{U}(s^i, \boldsymbol{s}_t^{-i}; w_t, N_t) \coloneqq u(w_t - s^i) + v\big(R\big(\tfrac{1}{1+n}\tfrac{s^i+S_t^{-i}}{N_t}\big)s^i\big),$$

then a *best response* of agent i to the strategy profile $\boldsymbol{s}_t^{-i}$, given w_t and N_t, is determined by a solution to the life-cycle utility maximization problem

$$\max_{0\leq s^i\leq w_t} \mathscr{U}(s^i, \boldsymbol{s}_t^{-i}; w_t, N_t). \tag{3.11}$$

Existence and uniqueness of the best response are established in the following lemma.

Lemma 3.2 (Existence and uniqueness of the best response).
Let Assumptions 3.1 and 3.2 be satisfied. Then for each wage rate $w_t > 0$, population size $N_t \in \mathbb{N}_1$, and strategy profile $\boldsymbol{s}_t^{-i} \in [0, w_t]^{N_t-1}$, Problem (3.11) admits a unique solution $s_t^i \in (0, w_t)$, which satisfies the first-order condition

$$\frac{u'(w_t - s_t^i)}{v'\big(R\big(\tfrac{1}{1+n}\tfrac{s_t^i+S_t^{-i}}{N_t}\big)s_t^i\big)} = 1-\delta+f'\big(\tfrac{1}{1+n}\tfrac{s_t^i+S_t^{-i}}{N_t}\big)\Big[1+\tfrac{s_t^i}{s_t^i+S_t^{-i}}\,\epsilon_{f'}\big(\tfrac{1}{1+n}\tfrac{s_t^i+S_t^{-i}}{N_t}\big)\Big]. \tag{3.12}$$

Lemma 3.2 states that for any given strategy set $[0, w_t]$ and strategy profile $\boldsymbol{s}_t^{-i}$ chosen by his $N_t - 1$ competitors, agent i has a unique best response. In a *temporary Nash equilibrium*, the savings decisions of all young agents must then constitute mutual best responses. A more formal definition is the following.

Definition 3.1 (Temporary Nash equilibrium).
Given the wage rate $w_t > 0$ and the population size $N_t \in \mathbb{N}_1$, a temporary Nash equilibrium *in period t is a strategy profile $\boldsymbol{s}_t^* = (s_t^{i*}, \boldsymbol{s}_t^{-i*}) \in [0, w_t]^{N_t}$ that satisfies*

$$\mathscr{U}(s_t^{i*}, \boldsymbol{s}_t^{-i*}; w_t, N_t) \geq \mathscr{U}(s, \boldsymbol{s}_t^{-i*}; w_t, N_t) \quad \textit{for all} \;\; s \in [0, w_t],\;\; i \in \{1, \ldots, N_t\}.$$

Exploiting the symmetry of young agents, the following proposition now establishes the existence and uniqueness of temporary Nash equilibria.

Proposition 3.1 (Existence and uniqueness of temporary Nash equilibria).
Let Assumptions 3.1 and 3.2 be satisfied. Then for each wage rate $w_t > 0$ and population size $N_t \in \mathbb{N}_1$, there exists a uniquely determined, symmetric temporary Nash equilibrium $\boldsymbol{s}_t^* = (s_t^*, \ldots, s_t^*)$ in period t. The equilibrium savings level $s_t^* \in (0, w_t)$ satisfies

$$\frac{u'(w_t - s_t^*)}{v'\big(R\big(\tfrac{1}{1+n}s_t^*\big)s_t^*\big)} = 1-\delta+f'\big(\tfrac{1}{1+n}s_t^*\big)\Big[1+\frac{\epsilon_{f'}\big(\tfrac{1}{1+n}s_t^*\big)}{N_t}\Big]. \tag{3.13}$$

Equation (3.13) is of central importance. Given any wage rate and population size, it stipulates the savings of a typical young agent in a temporary Nash equilibrium. Formally, it defines the *strategic savings function*

$$\mathfrak{s} : \mathbb{R}_{++} \times \mathbb{N}_1 \to \mathbb{R}_{++}, \qquad (w_t, N_t) \mapsto s_t^*,$$

where s_t^* is determined by (3.13). The left-hand side of (3.13) is the marginal rate of intertemporal substitution between youthful and old-age consumption. The term $\epsilon_{f'}\big(\frac{1}{1+n}s_t^*\big)/N_t$ on the right-hand side captures the sensitivity of the realized capital-rental rate $f'\big(\frac{1}{1+n}s_t^*\big)$ to changes in individual savings and, therefore, serves as an index of the capital market power of a typical young agent in period t. The degree of market power depends on the elasticity of the marginal product of capital $\epsilon_{f'}$, and on the total number of agents N_t contributing savings to the capital market. If the marginal product of capital is relatively inelastic, i.e. $\epsilon_{f'}$ is close to zero, agents possess little market power, as strategic adjustments in savings have only a negligible effect on the realized capital-rental rate. Moreover, the competitive limit is obtained as $N_t \to \infty$, as the weight of each agent then becomes infinitely small. In this case, (3.13) simplifies to

$$\frac{u'(w_t - s_t^*)}{v'\big(R\big(\frac{1}{1+n}s_t^*\big)s_t^*\big)} = R\big(\tfrac{1}{1+n}s_t^*\big)$$

and coincides with a price taker's first-order condition (3.3) under perfect foresight. This observation leads to the following proposition.

Proposition 3.2 (Strategic savings supply).
Let Assumptions 3.1 and 3.2 be satisfied. Then savings in a temporary Nash equilibrium are strictly smaller than savings in a competitive temporary equilibrium under perfect foresight,

$$\mathfrak{s}(w, N) < s(w, \psi(w)) \quad \textit{for all} \ \ (w, N) \in \mathbb{R}_{++} \times \mathbb{N}_1.$$

In particular, for each $w > 0$, $\lim_{N\to\infty} \mathfrak{s}(w, N) = s(w, \psi(w))$.

Proposition 3.2 reveals that market power induces strategic savings behavior, in the sense that agents withhold funds from the capital market in order to increase the realized return on savings. An agent with market power therefore saves less than a price taker who has perfect foresight. In the competitive limit, the strategic savings function $\mathfrak{s}$ coincides with the standard savings function s defined in (3.2), and the temporary Nash equilibrium becomes a competitive temporary equilibrium with perfect foresight in the usual sense, e.g., see De La Croix and Michel (2002). Observe, however, that Proposition 3.2 does generally not apply when expectations are subjective. Under subjective expectations, a price taker may erroneously underestimate the return on savings and, consequently, save *less* than an agent who enjoys market power.

At this point, it is worthwhile considering a standard parameterization from the literature.

Example 3.1 (Log-linear utility and Cobb-Douglas technology).
Consider the log-linear life-cycle utility function $U(c^1, c^2) = \ln(c^1) + \beta \ln(c^2)$, *where* $\beta > 0$ *is a time-discount factor, combined with the Cobb-Douglas production function* $f(k) = Ak^{\alpha}$,

where $A > 0$ scales the total factor productivity and the parameter $0 < \alpha < 1$ determines the distribution of factor incomes. It is straightforward to verify that Assumptions 3.1 and 3.2 are satisfied. Since the marginal product of capital is isoelastic,

$$\epsilon_{f'}(k) \equiv \alpha - 1,$$

market power is independent of the savings level. Let capital depreciate fully, $\delta = 1$, then the savings functions take the form

$$s(w, R^e) \equiv s(w) = \tfrac{\beta}{1+\beta}\, w \quad \textit{and} \quad \mathfrak{s}(w, N) = \tfrac{\beta+\beta(\alpha-1)N^{-1}}{1+\beta+\beta(\alpha-1)N^{-1}}\, w.$$

Clearly, $\lim\limits_{N\to\infty} \mathfrak{s}(w, N) = s(w) = \tfrac{\beta}{1+\beta}\, w.$

Intuitively, market power raises agents' life-cycle utility, as it allows them to influence the capital-rental rate to their advantage. Since capital income $s \mapsto R\big(\tfrac{1}{1+n}s\big)s$ is strictly increasing in savings by Assumption 3.2, we can infer from Proposition 3.2 that an agent with market power consumes more when young, but *less* when old. Indeed, for each $(w, N) \in \mathbb{R}_{++} \times \mathbb{N}_1$, youthful consumption satisfies

$$w - \mathfrak{s}(w, N) > w - s(w, \psi(w)),$$

while old-age consumption satisfies

$$R\big(\tfrac{1}{1+n}\mathfrak{s}(w, N)\big)\mathfrak{s}(w, N) < R\big(\tfrac{1}{1+n}s(w, \psi(w))\big)s(w, \psi(w)).$$

In general, however, nothing can be said about the impact on total lifetime consumption.

We now turn to capital accumulation. Since labor is paid its marginal product, it follows from Proposition 3.1 that the dynamics induced by the sequence of all temporary Nash equilibria is governed by the two-dimensional system

$$\begin{cases} k_{t+1} &= \mathcal{G}(k_t, N_t) := \tfrac{1}{1+n}\mathfrak{s}(w(k_t), N_t) \\ N_{t+1} &= \mathcal{N}(N_t) \end{cases}. \tag{3.14}$$

The capital accumulation law $\mathcal{G} : \mathbb{R}_{++} \times \mathbb{N}_1 \to \mathbb{R}_{++}$ inherits its dependence on the population size from the dependence of the temporary Nash equilibrium on the number of players, see Definition 3.1. Given any initial condition $(k_0, N_0) \in \mathbb{R}_{++} \times \mathbb{N}_1$, the dynamical system (3.14) recursively defines a *Nash-equilibrium growth path* in the following sense.

Definition 3.2 (Nash-equilibrium growth path).
Given the initial condition $(k_0, N_0) \in \mathbb{R}_{++} \times \mathbb{N}_1$ and corresponding population profile $\{N_t\}_{t=0}^{\infty}$ generated by $N_{t+1} = \mathcal{N}(N_t)$, a Nash-equilibrium growth path *is a sequence $\{k_t\}_{t=0}^{\infty}$ that satisfies $k_{t+1} = \mathcal{G}(k_t, N_t)$ for all $t \geq 0$.*

An immediate consequence of Proposition 3.2 is the following corollary, showing that the strategic savings behavior of agents curbs capital accumulation.

Corollary 3.1 (Capital accumulation).
The capital accumulation laws satisfy

$$\mathcal{G}(k,N) < G(k) \quad \textit{for all} \quad (k,N) \in \mathbb{R}_{++} \times \mathbb{N}_1.$$

In particular, for each $k > 0$, $\lim_{N\to\infty} \mathcal{G}(k,N) = G(k)$.

In view of the impact on capital accumulation, the central question now is how market power affects the qualitative dynamics and the long-run development of the economy. In this context, our next lemma implies that, in the absence of population growth, $n = 0$, the Nash-equilibrium growth paths are always monotonic.

Lemma 3.3 (Monotonicity of $\mathcal{G}$).
Let Assumptions 3.1 and 3.2 be satisfied. Then for each $N \in \mathbb{N}_1$, *the map* $k \mapsto \mathcal{G}(k,N)$ *is strictly increasing.*

Under a stationary population profile, imperfect competition in the capital market has no bearing on the qualitative dynamics, as the Nash-equilibrium growth paths are qualitatively indistinguishable from the competitive perfect-foresight growth paths. Any form of fluctuations is ruled out. By contrast, a straightforward numerical investigation reveals that non-monotonic dynamics may emerge in the presence of population growth, see Example 3.2. In essence, these fluctuations arise from strategic interaction among agents in the capital market, as the temporary Nash equilibrium may depend sensitively on the total number of agents supplying savings.

The following theorem now establishes the effect of market power on long-run growth.[8]

Theorem 3.1 (Economic growth).
Let Assumptions 3.1 and 3.2 be satisfied. Then for each initial condition $(k_0, N_0) \in \mathbb{R}_{++} \times \mathbb{N}_1$, *the limit* $k_* := \lim_{t\to\infty} k_t$ *attained by the Nash-equilibrium growth path* $\{k_t\}_{t=0}^{\infty}$ *satisfies* $k_* \leq k_{\mathrm{PF}}$, *where the inequality is strict if and only if* $\lim_{t\to\infty} N_t < \infty$.

Theorem 3.1 shows that market power in the capital market reduces the economy's long-run GDP. The underlying mechanism is the strategic savings behavior of agents, which leads to a lower long-run capital stock. However, the long-term development of the economy is unaffected whenever the population grows infinitely large. The reason is straightforward: unbounded population growth gradually eliminates agents' ability to influence prices, so that the competitive limit is restored in the long run.

The following example illustrates our main results.

Example 3.2 (Log-linear utility and Cobb-Douglas technology).
Following on from Example 3.1, the capital accumulation laws take the form

$$k_{t+1} = G(k_t) = \tfrac{1}{1+n}\tfrac{\beta}{1+\beta}A(1-\alpha)k_t^{\alpha}$$

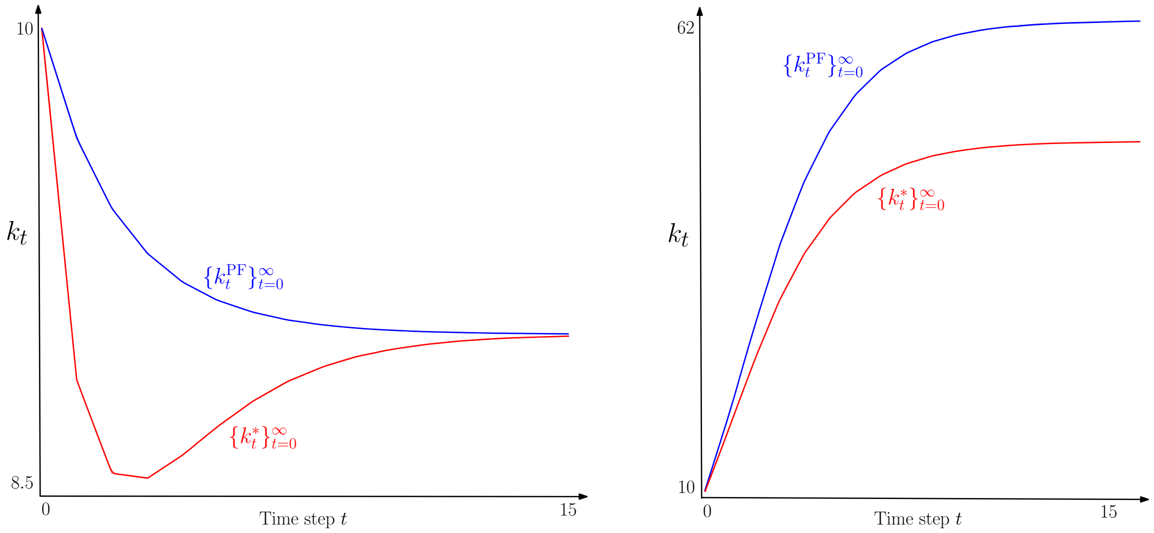
---

[8] Recall that $0 < k_{\mathrm{PF}} < \infty$ is the limit attained by the perfect-foresight growth path.

and

$$k_{t+1} = \mathcal{G}(k_t, N_t) = \tfrac{1}{1+n}\tfrac{\beta+\beta(\alpha-1)N_t^{-1}}{1+\beta+\beta(\alpha-1)N_t^{-1}}A(1-\alpha)k_t^{\alpha}.$$

Figure 3.1 portrays the resulting perfect-foresight and Nash-equilibrium growth paths in blue and red color, respectively. Figure 3.1a indicates that if $n > 0$*, market power may induce non-monotonic growth paths. Figure 3.1b, on the other hand, shows that if* $n = 0$*, then it engenders a reduction in economic growth without altering the qualitative dynamics.*



(a) Distinct qualitative dynamics, $n = 1$.

(b) Reduced economic growth, $n = 0$.

Figure 3.1. Influence of capital market power on qualitative dynamics and economic growth.
Color code: — Nash-equilibrium growth path, — Perfect-foresight growth path.
Parameter set: $A = 37$, $\alpha = 0.65$, $\beta = 0.5$, $N_0 = 3$, $\delta = 1$, $k_0 = 10$.

The perfect-foresight growth path converges monotonically to the limit

$$k_{\mathrm{PF}} = \left(\tfrac{(1+n)(1+\beta)}{\beta(1-\alpha)A}\right)^{\frac{1}{\alpha-1}},$$

which is the unique positive steady state of G*. Since* $G'(k_{\mathrm{PF}}) = \alpha \in (0,1)$*, it is asymptotically stable, globally on* $\mathbb{R}_{++}$*. The limit attained by the Nash-equilibrium growth path is*

$$k_* = \left(\tfrac{(1+n)(1+\beta)}{\beta(1-\alpha)A}\,\tfrac{\frac{1}{\alpha-1}+\frac{\beta}{1+\beta}N_\infty^{-1}}{\frac{1}{\alpha-1}+N_\infty^{-1}}\right)^{\frac{1}{\alpha-1}},$$

where

$$N_\infty := \lim_{t\to\infty} N_t = \begin{cases} N_0 & \text{if } \ n = 0 \\ \infty & \text{if } \ n > 0 \end{cases}.$$

It is readily seen that $k_* < k_{\mathrm{PF}}$ *if* $n = 0$*, whereas* $k_* = k_{\mathrm{PF}}$ *if* $n > 0$*.*

To conclude this section, it is worth noting that while market power and strategic interaction in the capital market may trigger non-monotonic dynamics in the form of transient fluctuations, the Nash-equilibrium growth paths in our model can never display persistent business cycles, let alone chaos. The reason is that population growth, while necessary for fluctuations, progressively eradicates agents' market power and thereby removes the very source of such fluctuations. In other words, population growth smooths away endogenous fluctuations by implementing the competitive limit, which permits only monotonic growth. An analysis of the qualitative dynamics under bounded population growth remains an open issue for future research. Perhaps, market power can induce more intricate dynamics in such a setting.

3.5 ELASTIC LABOR SUPPLY

Since the labor supply of young agents is totally inelastic, the model so far abstracts from market power in the labor market. This section discusses how the game-theoretic framework developed in Section 3.4 can be extended to incorporate market power in *both* factor markets.

Assume, without loss of generality, that each young agent is endowed with one unit of time, which can be allocated between leisure and labor. The preferences over youthful consumption, old-age consumption, and leisure $z \in [0,1]$ are represented by a life-cycle utility function

$$U(c^1, c^2, z) := u(c^1) + v(c^2) + \mu(z),$$

where $\mu : [0,1] \to \mathbb{R}$ is assumed to be twice continuously differentiable, strictly increasing, strictly concave, and to satisfy the Inada condition

$$\lim_{z \to 0} \mu'(z) = \infty.$$

Fix an arbitrary period t. A pure strategy of a typical young agent $i \in \{1, \ldots, N_t\}$ is a pair (l_t^i, s_t^i) consisting of labor supply and savings. The labor-supply choices of all other young agents are contained in the vector $\boldsymbol{l}_t^{-i} \in [0,1]^{N_t - 1}$ and give rise to the aggregate labor supply

$$L_t^{-i} := \|\boldsymbol{l}_t^{-i}\|_1 = l_t^1 + l_t^2 + \ldots + l_t^{i-1} + l_t^{i+1} + \ldots + l_t^{N_t}.$$

Suppose now that the old generation owns the aggregate capital stock $K_t > 0$. Since the wage rate is determined by the marginal product of labor, agent i anticipates that supplying l_t^i units of labor yields a realized labor income of

$$w_t l_t^i = w(k_t) l_t^i = w\big(\tfrac{K_t}{l_t^i + L_t^{-i}}\big) l_t^i.$$

Since savings cannot exceed labor income, the strategy set of agent i is given by

$$\mathcal{S}_i(\boldsymbol{l}_t^{-i}, K_t) := \Big\{ (l^i, s^i) \in [0,1] \times \mathbb{R}_+ \;\Big|\; s^i \le w\big(\tfrac{K_t}{l^i + L_t^{-i}}\big) l^i \Big\}. \tag{3.15}$$

Assuming that, in addition to Assumption 3.2, the production function f satisfies

$$\frac{f'''(k)k}{f''(k)} < -1 \quad \text{for all} \quad k \ge 0,$$

then the marginal product of labor w is strictly concave and its elasticity satisfies

$$\epsilon_w(k) := -\frac{w'(k)k}{w(k)} \in (-1, 0) \quad \text{for all} \quad k \ge 0.$$

These properties are sufficient to ensure that the agent's labor income $l^i \mapsto w\big(\frac{K_t}{l^i + L_t^{-i}}\big) l^i$ is strictly increasing and strictly concave in his labor supply, which in turn implies that each strategy set (3.15) is convex and compact.

A temporary Nash equilibrium can now be characterized as follows. Note first that the realized capital-labor ratio of the subsequent period $t+1$ depends on the aggregate labor supply of the young generation born in $t+1$. For this reason, agents living in period t must form an expectation $L_t^e > 0$ with respect to the aggregate labor supply realized in $t+1$. A best response (l_t^i, s_t^i) to the strategy profile $(\boldsymbol{l}_t^{-i}, \boldsymbol{s}_t^{-i})$, given K_t and L_t^e, is then determined by[9]

$$(l_t^i, s_t^i) = \underset{(l^i, s^i) \in \mathcal{S}_i(\boldsymbol{l}_t^{-i}, K_t)}{\operatorname{argmax}} \quad u\Big(w\big(\tfrac{K_t}{l^i + L_t^{-i}}\big) l^i - s^i\Big) + v\Big(R\big(\tfrac{s^i + S_t^{-i}}{L_t^e}\big) s^i\Big) + \mu(1 - l^i). \tag{3.16}$$

The Inada conditions imposed on the utility functions u, v, and μ imply that a best response (l_t^i, s_t^i) satisfies $0 < l_t^i < 1$ and $0 < s_t^i < w\big(\frac{K_t}{l_t^i + L_t^{-i}}\big) l_t^i$. Therefore, it satisfies the two first-order conditions

$$\frac{u'\Big(w\big(\frac{K_t}{l_t^i + L_t^{-i}}\big) l_t^i - s_t^i\Big)}{v'\Big(R\big(\frac{s_t^i + S_t^{-i}}{L_t^e}\big) s_t^i\Big)} = 1 - \delta + f'\big(\tfrac{s_t^i + S_t^{-i}}{L_t^e}\big) \Big[1 + \frac{s_t^i}{s_t^i + S_t^{-i}}\, \epsilon_{f'}\big(\tfrac{s_t^i + S_t^{-i}}{L_t^e}\big)\Big] \tag{3.17}$$

and

$$\frac{\mu'(1 - l_t^i)}{u'\Big(w\big(\frac{K_t}{l_t^i + L_t^{-i}}\big) l_t^i - s_t^i\Big)} = w\big(\tfrac{K_t}{l_t^i + L_t^{-i}}\big) \Big[1 + \frac{l_t^i}{l_t^i + L_t^{-i}}\, \epsilon_w\big(\tfrac{K_t}{l_t^i + L_t^{-i}}\big)\Big]. \tag{3.18}$$

Taking advantage of the symmetry of young agents, it now follows from (3.17) and (3.18) that a *temporary Nash equilibrium in period* t is a strategy profile $(\boldsymbol{l}_t^*, \boldsymbol{s}_t^*) = (l_t^*, \ldots, l_t^*, s_t^*, \ldots, s_t^*)$ that satisfies

$$\frac{u'\Big(w\big(\frac{\hat{k}_t}{l_t^*}\big) l_t^* - s_t^*\Big)}{v'\Big(R\big(\frac{1}{1+n} \frac{s_t^*}{l_t^e}\big) s_t^*\Big)} = 1 - \delta + f'\big(\tfrac{1}{1+n} \tfrac{s_t^*}{l_t^e}\big) \left[1 + \frac{\epsilon_{f'}\big(\frac{1}{1+n} \frac{s_t^*}{l_t^e}\big)}{N_t}\right] \tag{3.19}$$

and

$$\frac{\mu'(1 - l_t^*)}{u'\Big(w\big(\frac{\hat{k}_t}{l_t^*}\big) l_t^* - s_t^*\Big)} = w\big(\tfrac{\hat{k}_t}{l_t^*}\big) \left[1 + \frac{\epsilon_w\big(\frac{\hat{k}_t}{l_t^*}\big)}{N_t}\right], \tag{3.20}$$

where $\hat{k}_t := \frac{K_t}{N_t}$ denotes the *capital stock per capita* and $l_t^e := \frac{L_t^e}{N_{t+1}}$ is the expectation for the future labor supply expressed in per capita terms.

Notice that (3.20) is the counterpart to (3.19) for the labor market: the left-hand side is the marginal rate of substitution between leisure and youthful consumption, while the term $\epsilon_w\big(\frac{\hat{k}_t}{l_t^*}\big)/N_t$ on the right-hand side is an index of labor market power. The more elastic the marginal product of labor, the greater an agent's market power. As one would expect, the competitive limit is obtained for $N_t \to \infty$.

The following example presents a standard parameterization for which temporary Nash equilibria are uniquely determined and, conveniently, independent of expectations.

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[9] By the extreme value theorem, a best response exists because (3.16) is a maximization problem with a continuous objective function on a convex and compact set of constraints.

Example 3.3 (Log-linear utility and Cobb-Douglas technology).
Consider the log-linear life-cycle utility function

$$U(c^1, c^2, z) = \ln(c^1) + \beta \ln(c^2) + \ln(z)$$

combined with a Cobb-Douglas production function and full depreciation of capital. In this case, there exists a uniquely determined temporary Nash equilibrium, defined by (3.19) and (3.20). The equilibrium strategies are

$$l_t^* = \frac{N_t(1+\beta) - \alpha - \beta - \alpha\beta(\alpha-1)N_t^{-1}}{N_t(2+\beta) - \alpha - \beta - \alpha\beta(\alpha-1)N_t^{-1}}$$

and

$$s_t^* = \mathfrak{s}(\hat{k}_t, N_t) := \frac{A(1-\alpha)\hat{k}_t^{\alpha}\,\Omega(N_t)}{N_t(2+\beta) - \alpha - \beta - \alpha\beta(\alpha-1)N_t^{-1}} \left[\frac{2N_t - \alpha + \Omega(N_t)}{N_t - \alpha + \Omega(N_t)}\right]^{\alpha},$$

where

$$\Omega(N_t) := \beta(N_t - \alpha)\Big[1 + \tfrac{\alpha-1}{N_t}\Big].$$

Hence, capital accumulation is independent of agents' expectations, so that given any initial condition $(\hat{k}_0, N_0) \in \mathbb{R}_{++} \times \mathbb{N}_1$, *the Nash-equilibrium growth paths are recursively generated by*

$$\begin{cases} \hat{k}_{t+1} &= \frac{1}{1+n}\,\mathfrak{s}(\hat{k}_t, N_t) \\ N_{t+1} &= (1+n)N_t \end{cases}.$$

In the competitive limit $N_t \to \infty$, *we obtain*

$$l_t^* = \tfrac{1+\beta}{2+\beta} \quad \textit{and} \quad s_t^* = A(1-\alpha)\tfrac{\beta}{1+\beta}\left(\tfrac{1+\beta}{2+\beta}\right)^{1-\alpha}\hat{k}_t^{\alpha}.$$

A comprehensive analysis of the effects of labor market power is outside the scope of this article and is deferred to future research.

3.6 CONCLUSION

This article extended the classical overlapping-generations model of Diamond (1965) by introducing strategic interaction and market power in the capital market. The analysis shows that, under relatively general assumptions on preferences and technology, the solution concept *temporary Nash equilibrium* generates well-defined growth paths. The competitive perfect-foresight growth paths emerge as a limiting case. Since agents strategically cut back on their savings to raise the realized capital-rental rate, market power reduces capital accumulation and long-term growth. It may also alter the qualitative dynamics of the economy, as the resulting growth paths are not necessarily monotonic. These findings carry clear policy implications, indicating that maintaining competition and removing entry barriers in capital markets is essential for robust and stable economic growth. We assumed exponential population growth for technical convenience; yet this assumption is restrictive: unbounded population growth inevitably implements the competitive limit and thus precludes an analysis of the dynamics near a steady state in which agents retain market power. Future research should therefore examine how imperfect competition in the capital market shapes the dynamics of a model with bounded population growth. Moreover, the extent to which regulatory intervention can address the potentially adverse effects of capital market power remains an open question. In this context, a welfare analysis may be particularly informative, as capital accumulation in OLG models is generally inefficient owing to incomplete markets. The *general theory of the second best*, see Lipsey and Lancaster (1956), suggests that market power may actually *enhance* welfare if it alleviates over-accumulation of capital and moves the growth path of the economy closer to the golden rule. Finally, an empirical investigation of the correlation between market power and economic growth in real-world economies could provide a valuable complement to our theoretical analysis.

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Chapter 4

Production-Possibility Frontiers, Rybczynski's Theorem, and Factor-Intensity Reversals[⋆]

with Jan Wenzelburger

Abstract

This article introduces an analytically tractable parameterization of the production-possibility frontier for two-factor, two-goods models with homogeneous production functions in terms of the factor-price ratio. The natural incorporation of factor-intensity reversals allows for a generalized version of the Rybczynski theorem. Under constant returns to scale, all production plans that are profit maximizing at fixed output prices trace out piecewise straight lines, along which the factor-price ratio and the output-substitution rate remain constant as long as both sectors produce. Under non-constant returns to scale, this effect is violated because the marginal rate of transformation is no longer independent of the economy-wide factor-intensity. The findings build on an elementary and concise proof of the concavity of the production-possibility frontier and are illustrated with both CES and VES production functions.

Keywords: Efficient factor allocations, production-possibility frontiers, homogeneous production functions, factor-intensity reversals, Rybczynski theorem.

JEL Classification: D24, D61, E23, F20.

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4.1 INTRODUCTION

The production-possibility set is a fundamental concept in production theory and encountered in numerous areas of economics. The concavity of the *production-possibility frontier*, also referred to as transformation curve, is of central importance to the existence and uniqueness of equilibria in neoclassical welfare economics and the theory of international trade, see Mas-Colell et al. (1995) or Krugman et al. (2018). The standard approach in the literature on two-sector models is to define the production-possibility frontier as the value function of a constrained output-maximization problem, subject to feasibility of the factor allocation; see, for example, Boldrin (1989, 1991), Boldrin and Deneckere (1990), Dalal (2006), and Venditti (2005). The frontier is then parameterized in the output level of one sector, and its key properties are derived from the envelope theorem.

A widespread assumption in the literature on two-sector growth models is the so-called *'no-factor-intensity-reversals' assumption*, which requires that the factor intensity of one sector exceeds that of the other sector for all possible factor-price ratios, e.g., see Cremers (2006) or Galor (1992). This assumption, however, is rather restrictive, as it rules out a wide range of standard production functions with both constant and variable elasticities of factor substitution. Already Uzawa (1963, p. 109) notes that, in most cases, the exclusion of factor-intensity reversals is *"required mainly for reasons of a mathematical nature and [...] it seems to be difficult to give any economic justification"*.

This article introduces a novel and analytically tractable parameterization of the production-possibility frontier for two-factor, two-goods models with homogeneous production functions, expressed in terms of the factor-price ratio. Since our approach naturally accommodates factor-intensity reversals, it has the potential to improve the tractability and generality of two-sector growth and overlapping-generations models, e.g., see Ritschel and Wenzelburger (2025) or Schmitz and Wenzelburger (2024).

The relationship between the curvature of the production-possibility frontier and the returns to scale of the underlying production functions has been studied extensively in the literature; see, for example, Herberg (1969, 1973), Herberg and Kemp (1969), Khang (1971), Lancaster (1968), Mayer (1974), and Samuelson (1949). Technical proofs establishing the concavity of the frontier in models with constant returns to scale and convex isoquants are provided in Herberg (1969), Khang (1971), and Quirk and Saposnik (1966). However, these proofs are quite complicated and hard to grasp, which might explain why textbooks on international trade devote relatively little attention to formally establish this result, e.g., see Krugman et al. (2018) or Rübel (2008).

The generalization of the Rybczynski theorem to models with variable returns to scale was first addressed by Jones (1968) and Panagariya (1980). Following on from these contributions, Hansson and Lundahl (1983) and Ylönen (1987) show that, under decreasing returns to scale, an increase in the economy-wide factor intensity at constant output prices may cause the output levels in *both* sectors to increase. The classical output-substitution effect in Rybczynski (1955), where the output in one sector is raised while the output in the other sector is lowered, arises only when the difference in sector-specific factor intensities is sufficiently large. Our parameterization of the production-possibility frontier demonstrates that, under non-constant returns to scale, such violations of the Rybczynski effect may be observed because the marginal rate of transformation is no longer independent of the economy-wide factor intensity.

In contrast to the conventional value-function approach, our parameterization of the production-possibility frontier is explicit for most homogeneous standard production functions, including the entire class of CES production functions and certain VES production functions. A major advantage is that all essential properties of the frontier can be derived from elementary calculus instead of the envelope theorem. It turns out that the factor-price ratio is the more natural curve parameter than the output level because it also parameterizes all Pareto-efficient factor allocations. We establish that, under constant returns to scale, the marginal rate of transformation depends solely on the factor-price ratio. This observation allows for a generalization of the classical Rybczynski (1955) theorem that incorporates factor-intensity reversals, showing that all production plans that are profit-maximizing at fixed output prices trace out piecewise straight lines. The factor-price ratio is the sole determinant of Rybczynski's output-substitution rate. To the best of our knowledge, these findings are novel. As an aside, our parameterization allows for a concise and elementary proof that the frontier is concave whenever both sectors have non-increasing returns to scale, whereas concavity is violated if one sector exhibits increasing returns to scale.

This article is organized as follows. The technical prerequisites are introduced in Section 4.2. Section 4.3 presents a characterization of Pareto-efficient factor allocations. In Section 4.4, we establish our parameterization of the production-possibility frontier, focusing on the case of constant returns to scale. Section 4.5 develops a refined version of the Rybczynski theorem. The generalization of our approach to technologies with variable returns to scale is addressed in Section 4.6, before Section 4.7 concludes. Examples with both CES and VES production functions are discussed in Section 4.8. All proofs are collected in Appendix D.

4.2 MODEL PREREQUISITES

We consider two production sectors that transform two factors, e.g., capital and labor, into output. The production factors are perfectly mobile between the sectors and are paid their marginal products. The aggregate production function of the representative firm in sector $j = 1, 2$ is of the form

$$F_j : \mathbb{R}_+^2 \to \mathbb{R}_+, \qquad Y_j = F_j(K_j, L_j),$$

where $Y_j \geq 0$ is the sector-specific output and $K_j, L_j \geq 0$ are the sector-specific inputs of capital and labor, respectively. The corresponding intensive form $f_j : \mathbb{R}_+ \to \mathbb{R}_+$ is defined by

$$f_j(k_j) := F_j(k_j, 1), \tag{4.1}$$

where $k_j := \frac{K_j}{L_j}$ is the sector-specific capital-labor ratio. Assuming that F_j is homogeneous of degree $n_j > 0$, the marginal products of labor and capital take the form

$$\begin{aligned}\frac{\partial F_j}{\partial L}(K_j, L_j) &= L_j^{n_j - 1}\big[n_j f_j(k_j) - f_j'(k_j) k_j\big]\\ \frac{\partial F_j}{\partial K}(K_j, L_j) &= L_j^{n_j - 1} f_j'(k_j),\end{aligned}$$

respectively. As a consequence, the marginal rate of technical substitution (MRTS) at any $(K_j, L_j) \in \mathbb{R}^2_{++}$ satisfies

$$\text{MRTS}_j(K_j, L_j) := \frac{\frac{\partial F_j}{\partial L}(K_j, L_j)}{\frac{\partial F_j}{\partial K}(K_j, L_j)} = \frac{n_j f_j(k_j)}{f_j'(k_j)} - k_j$$

and thus depends solely on the sector-specific capital-labor ratio k_j.

Our exposition rests on the following technological assumptions.

Assumption 4.1 (Technologies)**.**
The two production sectors $j = 1, 2$ are characterized by the following properties:

(i) Each production function $F_j : \mathbb{R}^2_+ \to \mathbb{R}_+$ is homogeneous of degree $n_j > 0$, twice continuously differentiable, strictly increasing in both arguments, and all isoquants are downward sloping and strictly convex curves.

(ii) Each MRTS *function*

$$\Omega_j : \mathbb{R}_+ \to \mathbb{R}_+, \qquad k \mapsto \frac{n_j f_j(k)}{f_j'(k)} - k, \tag{4.2}$$

satisfies the boundary conditions

$$\lim_{k \to 0} \Omega_j(k) = 0 \quad \textit{and} \quad \lim_{k \to \infty} \Omega_j(k) = \infty.$$

Assumption 4.1 dates back to Drandakis (1963) and is satisfied by most standard production functions that are quasi-concave, as for example those with a constant elasticity of substitution (CES).[1] Assumption 4.1 (i) implies that each MRTS function Ω_j is continuously differentiable and strictly increasing. Assumption 4.1 (ii) then ensures the existence of continuously differentiable inverse functions $\kappa_j : \mathbb{R}_+ \to \mathbb{R}_+$, $j = 1, 2$ such that

$$\Omega_j(\kappa_j(\omega)) = \omega \quad \text{for all} \quad \omega \geq 0. \tag{4.3}$$

The functions κ_j stipulate the relative factor demands, so that given a wage-rental ratio ω, the capital-labor ratio in sector $j = 1, 2$ is $k_j = \kappa_j(\omega)$. Our approach naturally incorporates *factor-intensity reversals* in the following sense. Denote by $k := \frac{K}{L}$ the economy-wide capital-labor ratio. If $\Omega_i(k) < \Omega_j(k)$, where $i \neq j$, then sector i is more capital-intensive than sector j for all wage-rental ratios in $\big(\Omega_i(k), \Omega_j(k)\big)$ because

$$\kappa_j(\omega) < k < \kappa_i(\omega) \quad \text{for all} \quad \omega \in \big(\Omega_i(k), \Omega_j(k)\big). \tag{4.4}$$

The marginal rates of technical substitution thus determine which of the two sectors is more capital-intensive, so that a restrictive assumption on factor intensities is not needed.[2] Figure 4.1 portrays factor-intensity reversals in an example with CES production functions. The details of the parameterization are found in Section 4.8.

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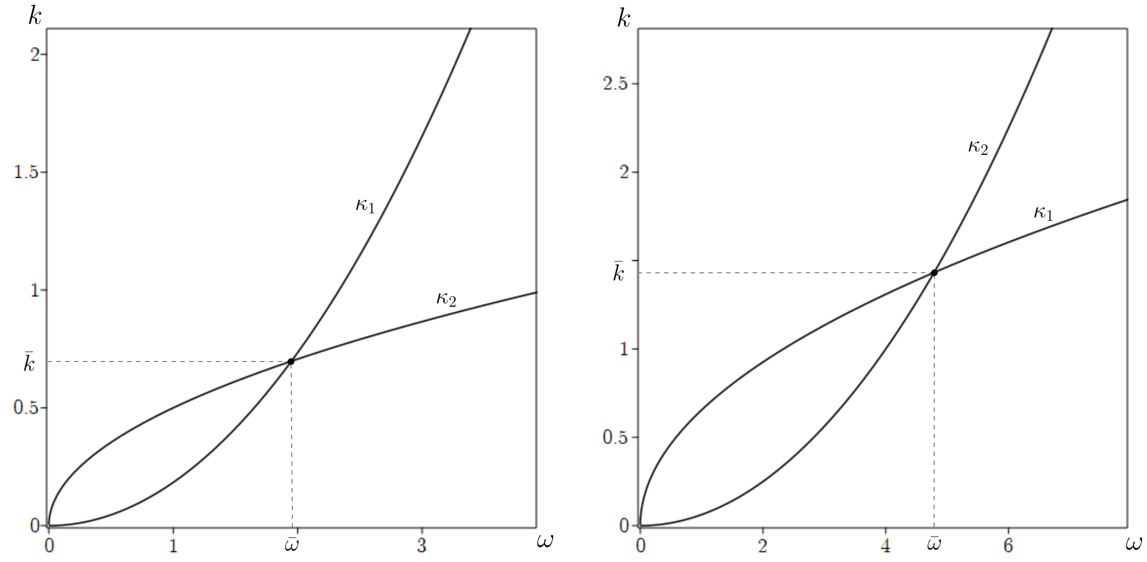
[1] An important exception are Leontief production functions, which violate Assumption 4.1 (i).

[2] It is straightforward to verify that $\Omega_1(k) \gtreqless \Omega_2(k) \iff \frac{\varepsilon_1(k)}{n_1} \lesseqgtr \frac{\varepsilon_2(k)}{n_2}$, where ε_j denotes the elasticity of the production function f_j, $j = 1, 2$.



(a) Parameter set A.

(b) Parameter set B.

Figure 4.1. Factor-intensity reversals for CES production functions.

4.3 Pareto-Efficient Factor Allocations

Given an aggregate supply of capital and labor $(K, L) \in \mathbb{R}^2_+$, a *feasible factor allocation* is a list $(K_1, K_2, L_1, L_2) \geq 0$ that satisfies the feasibility constraints

$$(i) \quad K = K_1 + K_2 \quad \text{and} \quad (ii) \quad L = L_1 + L_2.$$

A *Pareto-efficient factor allocation* is a feasible factor allocation for which the output of a sector cannot be raised without lowering the output of the other sector. Except for the two boundary factor allocations $(K, 0, L, 0)$ and $(0, K, 0, L)$, the marginal rates of technical substitution in both sectors must then coincide, so that the corresponding isoquants in the factor box intersect tangentially. In intensive form, a Pareto-efficient factor allocation may be defined as follows.

Definition 4.1 (Pareto-efficient factor allocation)**.**
Given an economy-wide capital-labor ratio $k \geq 0$*, a* Pareto-efficient factor allocation *(in intensive form) is a list* $(k_1, k_2, l_1, l_2) \geq 0$ *that satisfies*

$$\begin{aligned} (i) &\quad k = l_1 k_1 + l_2 k_2 \\ (ii) &\quad 1 = l_1 + l_2 \\ (iii) &\quad \Omega_1(k_1) = \Omega_2(k_2). \end{aligned}$$

We next characterize the set of all Pareto-efficient factor allocations. For each $k \geq 0$, set

$$\Omega_{\min}(k) := \min\big\{\Omega_1(k), \Omega_2(k)\big\} \quad \text{and} \quad \Omega_{\max}(k) := \max\big\{\Omega_1(k), \Omega_2(k)\big\}.$$

For each $k > 0$ with $\Omega_1(k) \neq \Omega_2(k)$, define the *labor-share functions* by setting

$$\ell_1(k,\cdot) : \big[\Omega_{\min}(k), \Omega_{\max}(k)\big] \to [0,1], \qquad \omega \mapsto \frac{\kappa_2(\omega) - k}{\kappa_2(\omega) - \kappa_1(\omega)},$$

and (4.5)

$$\ell_2(k,\cdot) : \big[\Omega_{\min}(k), \Omega_{\max}(k)\big] \to [0,1], \qquad \omega \mapsto \frac{k - \kappa_1(\omega)}{\kappa_2(\omega) - \kappa_1(\omega)}.$$

Notice that the labor-share functions (4.5) are well defined because of (4.4). Moreover, they add up to unity and satisfy

$$\ell_1(k, \Omega_2(k)) = \ell_2(k, \Omega_1(k)) = 0 \quad \text{and} \quad \ell_1(k, \Omega_1(k)) = \ell_2(k, \Omega_2(k)) = 1. \tag{4.6}$$

The following lemma states that except for the non-generic case $\Omega_1(k) = \Omega_2(k)$ in which both sectors have the same capital-labor ratio, Pareto-efficient factor allocations are parameterized by the wage-rental ratios $\omega \in \big[\Omega_{\min}(k), \Omega_{\max}(k)\big]$.[3]

Lemma 4.1 (Pareto-efficient factor allocations)**.**
Let Assumption 4.1 be satisfied and $k \geq 0$ be given. Then the following holds true.

(i) If $\Omega_1(k) \neq \Omega_2(k)$, then all Pareto-efficient factor allocations in the sense of Definition 4.1 are given by

$$\big(\kappa_1(\omega), \kappa_2(\omega), \ell_1(k,\omega), \ell_2(k,\omega)\big), \quad \omega \in \big[\Omega_{\min}(k), \Omega_{\max}(k)\big]. \tag{4.7}$$

(ii) If $\Omega_1(k) = \Omega_2(k)$, then all Pareto-efficient factor allocations in the sense of Definition 4.1 are given by

$$\big(k, k, l_1, 1 - l_1\big), \quad l_1 \in [0,1].$$

4.4 THE CONCAVITY OF THE PRODUCTION-POSSIBILITY FRONTIER REVISITED

We assume throughout this and the following section that, in addition to Assumption 4.1, both production functions F_1 and F_2 are linear-homogeneous, i.e., $n_1 = n_2 = 1$. In Section 4.6, we will then extend our results to technologies with variable returns to scale.

In view of Lemma 4.1, the production-possibility frontier may be introduced as follows. In the generic case, for each $k > 0$ with $\Omega_1(k) \neq \Omega_2(k)$, the output of sector $j = 1,2$ per unit of aggregate labor supply produced with the factor allocation (4.7) is given by the *output function*

$$\mathsf{y}_j(k,\cdot) : \big[\Omega_{\min}(k), \Omega_{\max}(k)\big] \to \big[0, f_j(k)\big], \qquad \omega \mapsto \ell_j(k,\omega) f_j(\kappa_j(\omega)). \tag{4.8}$$

The production-possibility frontier is then described by the curve

$$\mathcal{T}_F(k,\cdot) : \big[\Omega_{\min}(k), \Omega_{\max}(k)\big] \to \mathbb{R}^2_+, \qquad \omega \mapsto \big(\mathsf{y}_1(k,\omega), \mathsf{y}_2(k,\omega)\big), \tag{4.9}$$

---

[3] Observe that for the two boundary factor allocations $(k_1 = k, k_2, 1, 0)$ and $(k_1, k_2 = k, 0, 1)$, the capital-labor ratio of the non-producing sector is stipulated by Condition (iii) in Definition 4.1. Indeed, if $l_j = 0$, then $L_j = l_j \cdot L = 0$ and $K_j = k_j L_j = 0$ for all $k_j \geq 0$.

which traces out all *Pareto-efficient* production plans in the (y_1, y_2)-plane. The novelty of the representation (4.9) is that the wage-rental ratio serves as the 'natural' curve parameter. The *marginal rate of transformation* (MRT) pertaining to the production plan $\mathcal{T}_F(k,\omega)$ is the slope of the curve (4.9) at that point, so that

$$\mathrm{MRT}\big(\mathcal{T}_F(k,\omega)\big) = -\frac{\frac{\partial \mathsf{y}_2}{\partial \omega}(k,\omega)}{\frac{\partial \mathsf{y}_1}{\partial \omega}(k,\omega)}.$$

In the non-generic case, for each $k > 0$ with $\Omega_1(k) = \Omega_2(k)$, Lemma 4.1 (ii) implies that the production-possibility frontier is the straight line[4]

$$\tilde{\mathcal{T}}_F(k,\cdot) : [0,1] \to \mathbb{R}^2_+, \qquad l_1 \mapsto \big(l_1 f_1(k), (1-l_1) f_2(k)\big), \tag{4.10}$$

so that the marginal rate of transformation simply is

$$\mathrm{MRT}\big(\tilde{\mathcal{T}}_F(k,l_1)\big) = \frac{f_2(k)}{f_1(k)}.$$

The following theorem now establishes the concavity of the production-possibility frontier and reveals that the marginal rate of transformation $\mathrm{MRT}(y_1, y_2)$ at a Pareto-efficient production plan (y_1, y_2) is, except for the non-generic case $\Omega_1(k) = \Omega_2(k)$, independent of the economy-wide capital-labor ratio k.

Theorem 4.1 (Concavity of the production-possibility frontier - linear-homogeneous case)**.** *Let Assumption 4.1 be satisfied and* $n_1 = n_2 = 1$*. Then the following holds true.*

(i) For each $k > 0$ *with* $\Omega_1(k) \neq \Omega_2(k)$*, the production-possibility frontier (4.9) is strictly concave. The marginal rate of transformation at* $\mathcal{T}_F(k,\omega)$*,* $\omega \in \big[\Omega_{\min}(k), \Omega_{\max}(k)\big]$*, satisfies*

$$\mathrm{MRT}\big(\mathcal{T}_F(k,\omega)\big) = \frac{f_2'(\kappa_2(\omega))}{f_1'(\kappa_1(\omega))}. \tag{4.11}$$

(ii) For each $k > 0$ *with* $\Omega_1(k) = \Omega_2(k)$*, the production-possibility frontier (4.10) is linear with a constant marginal rate of transformation*

$$\mathrm{MRT}\big(\tilde{\mathcal{T}}_F(k,l_1)\big) = \frac{f_2'(k)}{f_1'(k)} \quad \text{for all} \ \ l_1 \in [0,1]. \tag{4.12}$$

Theorem 4.1 shows that with our parameterization, the marginal rate of transformation coincides with the ratio of the marginal products of capital. The generic case of Theorem 4.1 is illustrated with CES production functions in Figure 4.2.

The more familiar parameterization of the frontier in terms of the output of sector 1 obtains as follows. In the proof of Theorem 4.1, we show that the output functions (4.8) are monotonic with respect to ω, so that for each $k > 0$ with $\Omega_1(k) \neq \Omega_2(k)$, the inverse function

$$\mathsf{y}_1^{-1}(k,\cdot) : \big[0, f_1(k)\big] \to \big[\Omega_{\min}(k), \Omega_{\max}(k)\big], \qquad y_1 \mapsto \mathsf{y}_1^{-1}(k, y_1),$$

---

[4] This observation is already found in Johnson (1966).

defined by[5]

$$\omega = \mathrm{y}_1^{-1}(k, \mathrm{y}_1(k,\omega)) \quad \text{for all} \quad \omega \in \big[\Omega_{\min}(k), \Omega_{\max}(k)\big],$$

exists. Rewriting the expressions (4.9) and (4.10) as functions of y_1 now yields the curve

$$T_F(k,\cdot) : \big[0, f_1(k)\big] \to \big[0, f_2(k)\big], \qquad y_1 \mapsto T_F(k, y_1), \tag{4.13}$$

where

$$y_2 = T_F(k, y_1) := \begin{cases} \mathrm{y}_2(k, \mathrm{y}_1^{-1}(k, y_1)) & \text{if } \Omega_1(k) \neq \Omega_2(k) \\ f_2(k) - \frac{f_2(k)}{f_1(k)} y_1 & \text{if } \Omega_1(k) = \Omega_2(k) \end{cases}. \tag{4.14}$$

Each pair $\big(y_1, T_F(k, y_1)\big)$ with $y_1 \in \big[0, f_1(k)\big]$ is a Pareto-efficient production plan on the production-possibility frontier pertaining to the economy-wide capital-labor ratio $k > 0$.

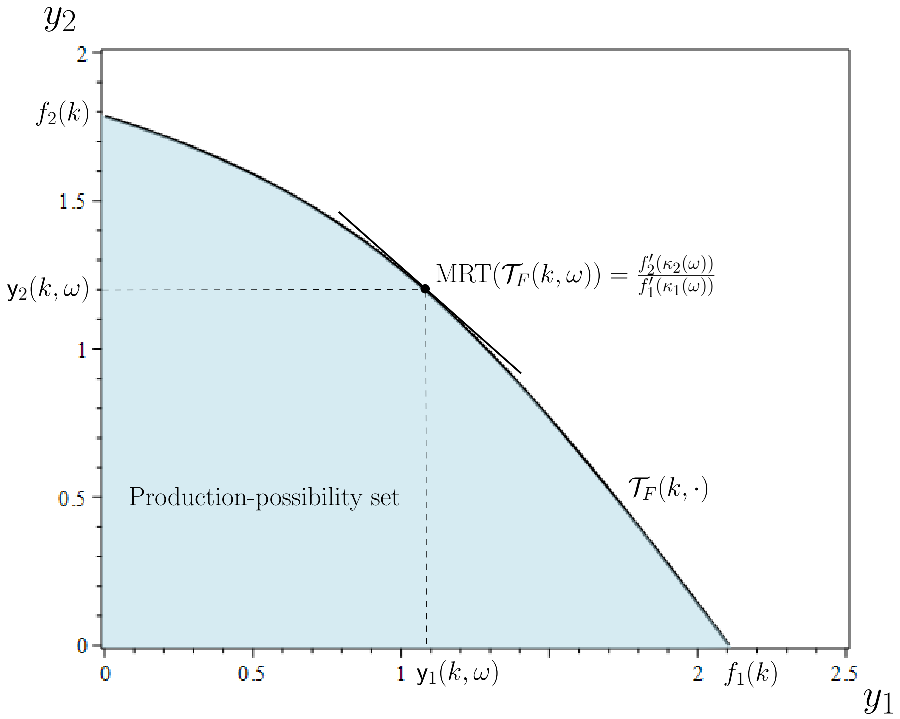


Figure 4.2. Strictly concave production-possibility frontier for linear-homogeneous CES production functions. Parameter set A; $k = 0.1$; $\Omega_2(k) < \omega < \Omega_1(k)$.

For all two-sector growths models, the concavity of the whole map T_F is often useful, e.g., see the dynamic programming problem in Ritschel and Wenzelburger (2025). This property is established next.

Lemma 4.2 (Concavity of T_F).
Let Assumption 4.1 be satisfied and $n_1 = n_2 = 1$*. Then the map*

$$T_F : \mathcal{D} \to \mathbb{R}_+, \qquad (k, y_1) \mapsto T_F(k, y_1),$$

---

[5] This result is a consequence of the well-known *Stolper-Samuelson theorem*.

where

$$\mathcal{D} := \Big\{(k, y_1) \in \mathbb{R}_{++} \times \mathbb{R}_+ \;\Big|\; y_1 \leq f_1(k)\Big\},$$

is concave. For each $(k, y_1) \in \mathcal{D}$*, its partial derivatives are*

$$(i) \quad \frac{\partial T_F}{\partial k}(k, y_1) = f_2'(\kappa_2(\omega)) \quad \textit{and} \quad (ii) \quad \frac{\partial T_F}{\partial y_1}(k, y_1) = -\frac{f_2'(\kappa_2(\omega))}{f_1'(\kappa_1(\omega))},$$

where

$$\omega = \begin{cases} \mathsf{y}_1^{-1}(k, y_1) & \textit{if } \Omega_1(k) \neq \Omega_2(k) \\ \Omega_1(k) & \textit{if } \Omega_1(k) = \Omega_2(k) \end{cases}.$$

4.5 Rybczynski Theorem with Factor-Intensity Reversals

Theorem 4.1 states that for linear-homogeneous production functions, the marginal rate of transformation is equal to the ratio of the two marginal products of capital and thus depends solely on the wage-rental ratio ω. In this section, we will exploit this observation and define the function

$$\varrho : \mathbb{R}_{++} \to \mathbb{R}_{++}, \qquad \omega \mapsto \frac{f_2'(\kappa_2(\omega))}{f_1'(\kappa_1(\omega))}, \tag{4.15}$$

which assigns to each wage-rental ratio $\omega > 0$ a marginal rate of transformation $\varrho(\omega)$. The following corollary establishes that the marginal rate of transformation is increasing in the wage-rental ratio if and only if sector 2 is more capital-intensive than sector 1.

Corollary 4.1 (Factor-intensity reversals)**.**
Let Assumption 4.1 be satisfied and $n_1 = n_2 = 1$*. Then for each* $\omega > 0$*,*

$$\varrho'(\omega) \geq 0 \iff \kappa_2(\omega) \geq \kappa_1(\omega).$$

Corollary 4.1 implies that factor-intensity reversals occur at wage-rental ratios $\bar{\omega}$ for which $\varrho'(\bar{\omega}) = 0$ and, therefore, $\kappa_1(\bar{\omega}) = \kappa_2(\bar{\omega}) = \bar{k}$. Figure 4.3 illustrates this result using CES production functions, for which the marginal rate of transformation ϱ is either hump-shaped or U-shaped due to a single factor-intensity reversal.

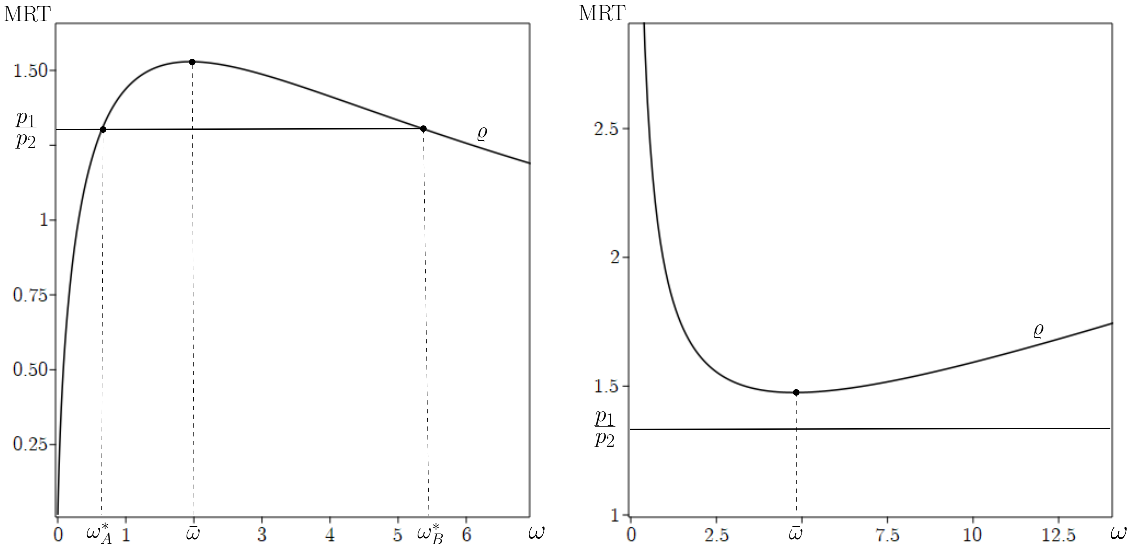
For technologies with constant returns to scale, the marginal products of labor are

$$w_j(k_j) := f_j(k_j) - f_j'(k_j)k_j, \quad j = 1, 2.$$

It follows from the very definition of the relative factor demand functions (4.3) that

$$\frac{f_2'(\kappa_2(\omega))}{f_1'(\kappa_1(\omega))} = \frac{w_2(\kappa_2(\omega))}{w_1(\kappa_1(\omega))} \quad \text{for all} \quad \omega > 0.$$

Thus, the marginal rate of transformation (4.15) is also equal to the ratio of the two marginal products of labor. The intuition behind this result is, of course, the classical one. Given an



(a) Maximum at $\bar{\omega}$. Parameter set A.

(b) Minimum at $\bar{\omega}$. Parameter set B.

Figure 4.3. The shape of ϱ for linear-homogeneous CES production functions. $\frac{p_1}{p_2} = 1.3$.

output-price vector $(p_1, p_2) \gg 0$, perfect factor mobility between the sectors implies that factor prices must be equal whenever both sectors produce, so that

$$p_1 w_1(\kappa_1(\omega)) = p_2 w_2(\kappa_2(\omega)) \quad \text{and} \quad p_1 f_1'(\kappa_1(\omega)) = p_2 f_2'(\kappa_2(\omega)). \tag{4.16}$$

On the other hand, a production plan $(y_1, y_2) \gg 0$ is profit maximizing for output prices $(p_1, p_2) \gg 0$ if and only if

$$\text{MRT}(y_1, y_2) = \frac{p_1}{p_2}. \tag{4.17}$$

Inserting (4.16) into (4.17), Theorem 4.1 implies that a profit-maximizing production plan $(y_1, y_2) \gg 0$ is determined by a wage-rental ratio ω that solves

$$\varrho(\omega) = \frac{p_1}{p_2}.$$

The capital-labor ratio k stipulates the marginal rates of transformation $\varrho(\Omega_1(k))$ and $\varrho(\Omega_2(k))$ at the two corner points of the production-possibility frontier. The concavity established in Theorem 4.1 implies that

$$\varrho(\Omega_2(k)) \leq \varrho(\Omega_1(k)) \quad \text{for all} \quad k > 0,$$

where the inequality is strict whenever $\Omega_1(k) \neq \Omega_2(k)$. The following lemma on the existence and uniqueness of profit-maximizing production plans is well known.

Lemma 4.3 (Profit-maximizing production plans - linear-homogeneous case)**.**
Let Assumption 4.1 be satisfied, $n_1 = n_2 = 1$, and $k > 0$ and $(p_1, p_2) \gg 0$ be arbitrary but fixed. Then the following holds true.

(i) If $\Omega_1(k) \neq \Omega_2(k)$, then profits of both sectors at output prices (p_1, p_2) are maximized by the production plan $\mathcal{T}_F(k, \omega)$ that is determined by a unique wage-rental ratio $\omega \in$

$\big[\Omega_{\min}(k), \Omega_{\max}(k)\big]$, *given by*

$$\omega = \begin{cases} \Omega_2(k) & \textit{if } \frac{p_1}{p_2} \leq \varrho(\Omega_2(k)) \\ \varrho^{-1}\big(\frac{p_1}{p_2}\big) & \textit{if } \varrho(\Omega_2(k)) < \frac{p_1}{p_2} < \varrho(\Omega_1(k)) \\ \Omega_1(k) & \textit{if } \varrho(\Omega_1(k)) \leq \frac{p_1}{p_2} \end{cases} .$$

(ii) If $\Omega_1(k) = \Omega_2(k)$, then profits of both sectors at output prices (p_1, p_2) are maximized by the production plan $\tilde{\mathcal{T}}_F(k, l_1)$ that is determined by the labor share

$$l_1 = \begin{cases} 0 & \textit{if } \frac{p_1}{p_2} < \varrho(\Omega_2(k)) \\ [0,1] & \textit{if } \frac{p_1}{p_2} = \varrho(\Omega_1(k)) \\ 1 & \textit{if } \varrho(\Omega_1(k)) < \frac{p_1}{p_2} \end{cases} .$$

Expect for the case $\frac{p_1}{p_2} = \varrho(\Omega_1(k)) = \varrho(\Omega_2(k))$, all profit-maximizing production plans are uniquely determined.[6]

The well-known *Rybczynski theorem* states that for any fixed output-price ratio $\frac{p_1}{p_2}$, an increase in the economy-wide capital-labor ratio k causes a disproportional change in the profit-maximizing production plans, whereby the output level in the more capital-intensive sector is raised and the output level in the less capital-intensive sector is lowered, see Rybczynski (1955). Our parameterization allows for a refinement of this theorem by incorporating factor-intensity reversals, which Rybczynski has ruled out. This is seen as follows. For each $\omega \geq 0$, the minimum and the maximum factor intensity are

$$\kappa_{\min}(\omega) := \min\big\{\kappa_1(\omega), \kappa_2(\omega)\big\} \quad \text{and} \quad \kappa_{\max}(\omega) := \max\big\{\kappa_1(\omega), \kappa_2(\omega)\big\},$$

respectively. Since the marginal rate of transformation is independent of the economy-wide capital-labor ratio, Theorem 4.1 implies that, for any fixed wage-rental ratio $\omega^* > 0$ with $\kappa_1(\omega^*) \neq \kappa_2(\omega^*)$, all production plans determined by the map

$$\mathcal{T}_F(\cdot, \omega^*) : \big[\kappa_{\min}(\omega^*), \kappa_{\max}(\omega^*)\big] \to \mathbb{R}^2_+, \qquad k \mapsto \big(\mathsf{y}_1(k, \omega^*), \mathsf{y}_2(k, \omega^*)\big) \tag{4.18}$$

have the same marginal rate of transformation $\varrho(\omega^*)$. Therefore, all production plans determined by (4.18) are profit maximizing at the output-price ratio $\frac{p_1}{p_2} = \varrho(\omega^*)$. In view of the output functions (4.8), it is clear that the change in output levels $\mathsf{y}_j(k, \omega^*)$, $j = 1, 2$ due to an increase in k is solely caused by a change in the labor shares $\ell_j(k, \omega^*)$. In particular, the sector-specific capital-labor ratios $\kappa_j(\omega^*)$, $j = 1, 2$ along with the respective unit outputs $f_j(\kappa_j(\omega^*))$ remain constant. The following refinement of the Rybczynski theorem states that the curves (4.18) are straight lines along which the output-substitution rate is independent of the economy-wide capital-labor ratio and, therefore, constant.

---

[6] In the generic case, the uniqueness of a profit-maximizing production plan follows from the fact that, by Corollary 4.1, the map $\omega \mapsto \varrho(\omega)$ is either strictly increasing or strictly decreasing on the interval $\big[\Omega_{\min}(k), \Omega_{\max}(k)\big]$. In the non-generic case, the production plan is indeterminate if the slope of the linear production-possibility frontier coincides with the output-price ratio.

Proposition 4.1 (Rybczynski lines).
Let Assumption 4.1 be satisfied and $n_1 = n_2 = 1$. *Then all production plans* $(y_1, y_2) \in \mathbb{R}^2_+$ *determined by (4.18) are located on the straight line*

$$y_2 = f_2(\kappa_2(\omega^*)) - \frac{f_2(\kappa_2(\omega^*))}{f_1(\kappa_1(\omega^*))} y_1, \quad y_1 \in \big[0, f_1(\kappa_1(\omega^*)\big], \tag{4.19}$$

and are profit maximizing at the output-price ratio $\frac{p_1}{p_2} = \varrho(\omega^*)$. *The output-substitution rate satisfies*

$$\frac{f_2(\kappa_2(\omega^*))}{f_1(\kappa_1(\omega^*))} \geq \varrho(\omega^*) \iff \kappa_2(\omega^*) \geq \kappa_1(\omega^*).$$

Proposition 4.1 implies that all production plans produced at a fixed wage-rental ratio ω^* are located on a downward-sloping straight line. The output-substitution rate $\frac{f_2(\kappa_2(\omega^*))}{f_1(\kappa_1(\omega^*))}$ quantifies the classical Rybczynski effect. Our parameterization reveals that this rate is independent of the economy-wide capital-labor ratio k and exceeds the marginal rate of transformation $\varrho(\omega^*)$ if and only if sector 2 is more capital-intensive than sector 1.[7]

From Corollary 4.1, we know that, due to factor-intensity reversals, there may exist multiple wage-rental ratios that yield the same marginal rate of transformation. As a consequence, the production plans that maximize profits at a given output-price ratio $\frac{p_1}{p_2}$ may trace out *several* line segments of the form (4.19) in the (y_1, y_2)-plane. In the CES example portrayed in Figure 4.4, the production plans (y_1, y_2) located on the two lime-green lines have the same marginal rate of transformation $\varrho(\omega_A^*) = \varrho(\omega_B^*) = \frac{p_1}{p_2}$, where $\omega_A^* < \omega_B^*$. The red line marks the linear production-possibility frontier pertaining to the capital-labor ratio $\bar{k} = \kappa_1(\bar{\omega}) = \kappa_2(\bar{\omega})$ at which the only factor-intensity reversal occurs. For the chosen parameterization, sector 2 is more capital-intensive on $(0, \bar{\omega})$, while sector 1 is more capital-intensive on $(\bar{\omega}, \infty)$, see also Figure 4.1. It follows that

$$\kappa_1(\omega_A^*) < \kappa_2(\omega_A^*) < \bar{k} < \kappa_2(\omega_B^*) < \kappa_1(\omega_B^*).$$

Figure 4.4 depicts five line segments colored in green representing the production plans that are profit-maximizing at the output-price ratio $\frac{p_1}{p_2}$. These are functions of the economy-wide capital-labor ratio k:

1. For any $k \in \big[0, \kappa_1(\omega_A^*)\big]$, only sector 1 produces, even though sector 2 is more capital-intensive. Production in sector 2 is not profitable as $\varrho(\Omega_1(k)) \leq \frac{p_1}{p_2}$. The output of sector 1 increases with k while sector 2 remains inactive. The wage-rental ratio is $\Omega_1(k)$ and increases with k as well.
2. For any $k \in \big(\kappa_1(\omega_A^*), \kappa_2(\omega_A^*)\big)$, both sectors produce at the wage-rental ratio ω_A^*. In line with the classical Rybczynski theorem, output increases in the more capital-intensive sector 2 and decreases in the less capital-intensive sector 1 as k increases. Since the wage-rental ratio remains constant at ω_A^*, this change in output levels is driven solely by an increase in the labor share of sector 2. In particular, the sector-specific capital-labor ratios remain constant.

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[7] In the non-generic case $\kappa_1(\omega^*) = \kappa_2(\omega^*)$, the line (4.19) is the production-possibility frontier.

3. For any $k \in \big[\kappa_2(\omega_A^*), \kappa_2(\omega_B^*)\big]$, production in sector 1 is not profitable as $\varrho(\Omega_2(k)) \geq \frac{p_1}{p_2}$. The output of sector 2 increase with k, independently of which sector is more capital-intensive. The wage-rental ratio is $\Omega_2(k)$ and increases with k.
4. For any $k \in \big(\kappa_2(\omega_B^*), \kappa_1(\omega_B^*)\big)$, both sectors produce again because both firms can afford to pay the higher wage-rental ratio $\omega_B^* > \omega_A^*$. Output increases in the more capital-intensive sector 1 and decreases in the less capital-intensive sector 2 as k increases. Since the wage-rental ratio remains constant at ω_B^*, this change in output levels is caused by an increase in the labor share of sector 1. Again, the sector-specific capital-labor ratios remain constant.
5. For any $k \in \big[\kappa_1(\omega_B^*), \infty\big)$, sector 1 is more capital intensive. Since $\varrho(\Omega_1(k)) \leq \frac{p_1}{p_2}$, only sector 1 produces. The wage-rental ratio is $\Omega_1(k)$ and increases with k.

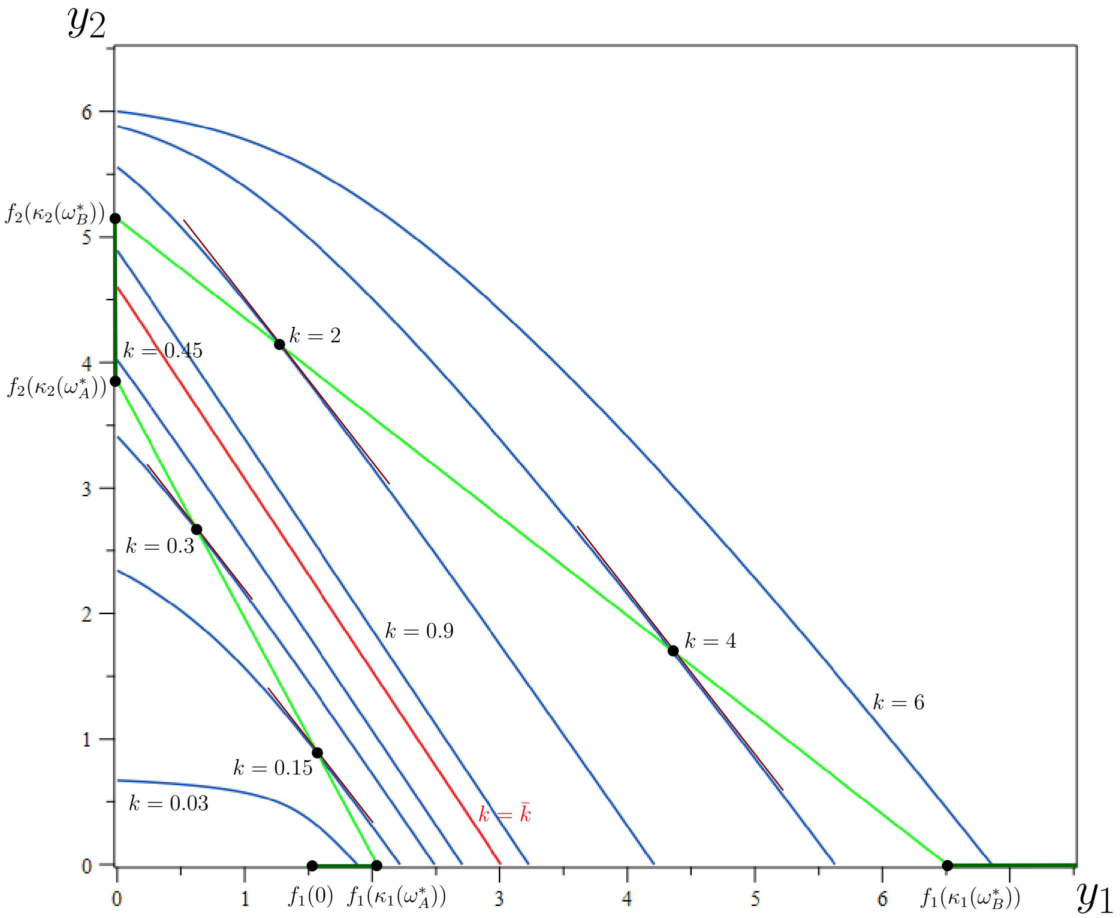

Figure 4.4. Rybczynski theorem with factor-intensity reversals for linear-homogeneous CES production functions. Parameter set A; $\frac{p_1}{p_2} = 1.3$, $\omega_A^* = 0.644$, $\omega_B^* = 5.43$, $\bar{k} = 0.698$.

The preceding discussion of the CES example leads to the following generalization of the Rybczynski theorem.

Theorem 4.2 (Rybczynski theorem with factor-intensity reversals).
Let Assumption 4.1 be satisfied, $n_1 = n_2 = 1$*, and* $(p_1, p_2) \gg 0$ *be arbitrary but fixed.*

(i) If sector 2 is more capital-intensive for all $\omega \in (\bar{\omega}_A, \bar{\omega}_B)$ *and* $\frac{p_1}{p_2} \in \big(\varrho(\bar{\omega}_A), \varrho(\bar{\omega}_B)\big)$*, then all production plans*

$$\mathcal{R}_F(k) := \begin{cases} (f_1(k), 0) & \text{if} \quad k \in \big[\kappa_1(\bar{\omega}_A), \kappa_1(\omega^*)\big] \\ \mathcal{T}_F(k, \omega^*) & \text{if} \quad k \in \big(\kappa_1(\omega^*), \kappa_2(\omega^*)\big) \\ (0, f_2(k)) & \text{if} \quad k \in \big[\kappa_2(\omega^*), \kappa_2(\bar{\omega}_B)\big] \end{cases},$$

where $\omega^* = \varrho^{-1}\left(\frac{p_1}{p_2}\right) \in (\bar{\omega}_A, \bar{\omega}_B)$*, are profit-maximizing at output prices* (p_1, p_2)*.*

(ii) If sector 1 is more capital-intensive for all $\omega \in (\bar{\omega}_A, \bar{\omega}_B)$ *and* $\frac{p_1}{p_2} \in \big(\varrho(\bar{\omega}_B), \varrho(\bar{\omega}_A)\big)$*, then all production plans*

$$\mathcal{R}_F(k) := \begin{cases} (0, f_2(k)) & \text{if} \quad k \in \big[\kappa_2(\bar{\omega}_A), \kappa_2(\omega^*)\big] \\ \mathcal{T}_F(k, \omega^*) & \text{if} \quad k \in \big(\kappa_2(\omega^*), \kappa_1(\omega^*)\big) \\ (f_1(k), 0) & \text{if} \quad k \in \big[\kappa_1(\omega^*), \kappa_1(\bar{\omega}_B)\big] \end{cases},$$

where $\omega^* = \varrho^{-1}\left(\frac{p_1}{p_2}\right) \in (\bar{\omega}_A, \bar{\omega}_B)$*, are profit maximizing at output prices* (p_1, p_2)*.*

Notice that the *Rybczynski-curve* $\mathcal{R}_F$ defined in Theorem 4.2 is continuous since

$$\mathcal{T}_F(\kappa_1(\omega^*), \omega^*) = \big(f(\kappa_1(\omega^*)), 0\big) \quad \text{and} \quad \mathcal{T}_F(\kappa_2(\omega^*), \omega^*) = \big(0, f(\kappa_2(\omega^*))\big).$$

Theorem 4.2 shows that the classical Rybczynski effect arises only when both sectors are active, so that the prevailing wage-rental ratio is constant. The increase in output of the more (less) capital-intensive sector requires a proportional increase (decrease) in labor relative to the change in capital, so that the sector-specific capital-labor ratios remain constant. Allowing for factor-intensity reversals, Theorem 4.2 also reveals that the question of whether or not a sector produces is independent of its use of capital.

4.6 GENERALIZATION TO VARIABLE RETURNS TO SCALE

Adapting the approach of Herberg (1969), we now extend our results to production functions $G_j : \mathbb{R}^2_+ \to \mathbb{R}_+$, $j = 1, 2$ that satisfy Assumption 4.1 and are homogeneous of some arbitrary degree $n_j > 0$. Setting

$$F_j(K_j, L_j) := G_j(K_j, L_j)^{\frac{1}{n_j}}, \qquad j = 1, 2, \tag{4.20}$$

the production functions (F_1, F_2) satisfy Assumption 4.1 as well but are linear-homogeneous. In terms of the respective intensive forms, we have

$$g_j(k_j) := G_j(k_j, 1) = F_j(k_j, 1)^{n_j} = f_j(k_j)^{n_j}, \qquad j = 1, 2, \tag{4.21}$$

where, as before, $k_j = \frac{K_j}{L_j}$ is the sector-specific capital-labor ratio. Observe that the marginal rates of technical substitution for the production functions (G_1, G_2) coincide with those for the production functions (F_1, F_2) because (4.21) implies that the corresponding MRTS functions satisfy[8]

$$\Omega_j(k) = \frac{n_j g_j(k)}{g_j'(k)} - k = \frac{f_j(k)}{f_j'(k)} - k, \qquad j = 1, 2.$$

As a consequence, the two pairs of production functions (G_1, G_2) and (F_1, F_2) have the same Pareto-efficient factor allocations, see Lemma 4.1. In particular, the relative factor demand functions (4.3) and the labor-share functions (4.5) are the very same functions.

Using (4.21), the output per unit of aggregate labor supply corresponding to G_j is

$$x_j = \frac{G_j(K_j, L_j)}{L} = \frac{L_j^{n_j} g_j(k_j)}{L} = L^{n_j - 1}\big(l_j f_j(k_j)\big)^{n_j}, \qquad j = 1, 2.$$

For any fixed aggregate labor supply $L > 0$, define the one-to-one mapping

$$h_j : \mathbb{R}_+ \to \mathbb{R}_+, \qquad y \mapsto L^{n_j - 1} y^{n_j}, \qquad j = 1, 2. \tag{4.22}$$

For each economy-wide capital-labor ratio $k > 0$ with $\Omega_1(k) \neq \Omega_2(k)$, the Pareto-efficient production plans corresponding to (G_1, G_2) are then determined by the functions

$$\mathsf{x}_j(k, \cdot) : \big[\Omega_{\min}(k), \Omega_{\max}(k)\big] \to \mathbb{R}_+, \qquad \omega \mapsto h_j\big(\mathsf{y}_j(k, \omega)\big), \qquad j = 1, 2, \tag{4.23}$$

where y_j is the output function for F_j defined in (4.8). The production-possibility frontier is given by the curve

$$\mathcal{T}_G(k, \cdot) : \big[\Omega_{\min}(k), \Omega_{\max}(k)\big] \to \mathbb{R}^2_+, \qquad \omega \mapsto \big(\mathsf{x}_1(k, \omega), \mathsf{x}_2(k, \omega)\big), \tag{4.24}$$

and the marginal rate of transformation is

$$\mathrm{MRT}\big(\mathcal{T}_G(k, \omega)\big) = -\frac{\frac{\partial \mathsf{x}_2}{\partial \omega}(k, \omega)}{\frac{\partial \mathsf{x}_1}{\partial \omega}(k, \omega)}.$$

In the non-generic case, for each $k > 0$ with $\Omega_1(k) = \Omega_2(k)$, Lemma 4.1 (ii) implies that the production-possibility frontier is given by the curve

$$\tilde{\mathcal{T}}_G(k, \cdot) : [0, 1] \to \mathbb{R}^2_+, \qquad l_1 \mapsto \big(h_1(l_1 f_1(k)), h_2((1 - l_1) f_2(k))\big). \tag{4.25}$$

With this notation, it is now straightforward to show that for technologies without constant returns to scale, the marginal rates of transformation are no longer independent of k.

Lemma 4.4 (Marginal rate of transformation - general case)**.**
Let Assumption 4.1 be satisfied. Then the following holds true.

(i) For each $k > 0$ with $\Omega_1(k) \neq \Omega_2(k)$, the marginal rate of transformation at $\mathcal{T}_G(k, \omega)$, $\omega \in \big(\Omega_{\min}(k), \Omega_{\max}(k)\big)$, is

$$\mathrm{MRT}\big(\mathcal{T}_G(k, \omega)\big) = L^{n_2 - n_1} \frac{\ell_2(k, \omega)^{n_2 - 1}}{\ell_1(k, \omega)^{n_1 - 1}} \frac{g_2'(\kappa_2(\omega))}{g_1'(\kappa_1(\omega))}. \tag{4.26}$$

---

[8] The isoquants of G_j and F_j form the same curves in $\mathbb{R}^2_+$ because

$$\Big\{(K_j, L_j) \in \mathbb{R}^2_+ \;\Big|\; G_j(K_j, L_j) = Y\Big\} = \Big\{(K_j, L_j) \in \mathbb{R}^2_+ \;\Big|\; F_j(K_j, L_j) = Y^{\frac{1}{n_j}}\Big\} \quad \text{for all} \quad Y \geq 0.$$

(ii) *For each* $k > 0$ *with* $\Omega_1(k) = \Omega_2(k)$, *the marginal rate of transformation at* $\tilde{\mathcal{T}}_G(k, l_1)$, $l_1 \in (0, 1)$, *is*

$$\mathrm{MRT}\big(\tilde{\mathcal{T}}_G(k, l_1)\big) = L^{n_2 - n_1} \frac{(1 - l_1)^{n_2 - 1}}{l_1^{n_1 - 1}} \frac{g_2'(k)}{g_1'(k)}. \tag{4.27}$$

Lemma 4.4 generalizes Theorem 4.1. Since

$$\frac{\partial G_j}{\partial K}(K_j, L_j) = L_j^{n_j - 1} g_j'(k_j), \qquad j = 1, 2,$$

the marginal rate of transformation again coincides with the ratio of the two marginal products of capital. In the absence of constant returns to scale, however, the marginal rate of transformation at the two corner points of the production-possibility frontier is either zero or infinity.

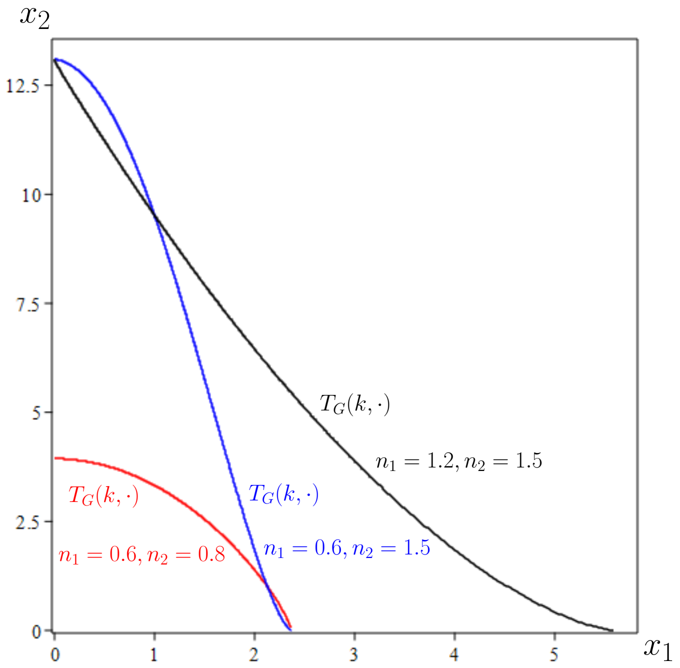

Figure 4.5. Production-possibility frontiers for CES production functions with non-constant returns to scale. Parameter sets C – E; $k = 2$, $L = 1$.

For non-increasing returns to scale, the concavity of the production-possibility frontiers $\mathcal{T}_G(k, \cdot)$ and $\tilde{\mathcal{T}}_G(k, \cdot)$ now follows from the concavity of the corresponding frontiers $\mathcal{T}_F(k, \cdot)$ and $\tilde{\mathcal{T}}_F(k, \cdot)$ established in Theorem 4.1. In this regard, the following theorem summarizes a collection of results found in Herberg (1969).

Theorem 4.3 (Concavity of the production-possibility frontier - general case)**.**
Let Assumption 4.1 be satisfied. Then the following hold true.

(i) *If* $n_1, n_2 \leq 1$, *then for each* $k > 0$, *the production-possibility frontier* $\mathcal{T}_G(k, \cdot)$, *respectively* $\tilde{\mathcal{T}}_G(k, \cdot)$, *is concave, where the concavity is strict if* $n_j < 1$ *for some* $j = 1, 2$.

(ii) *If* $n_j > 1$ *for some* $j = 1, 2$, *then for each* $k > 0$, *the production-possibility frontier* $\mathcal{T}_G(k, \cdot)$, *respectively* $\tilde{\mathcal{T}}_G(k, \cdot)$, *is* not *concave. In particular, if* $n_1 > 1$, *then the frontier is strictly convex in a neighborhood of* $\big(0, \frac{G_2(K,L)}{L}\big)$. *By contrast, if* $n_2 > 1$, *then the frontier is strictly convex in a neighborhood of* $\big(\frac{G_1(K,L)}{L}, 0\big)$.

Theorem 4.3 is illustrated in Figure 4.5, which portrays a family of production-possibility frontiers for CES production functions with varying degrees of homogeneity.

Our final result characterizes profit-maximizing production plans in the case where one sector has non-increasing returns to scale, while the other has decreasing returns to scale.[9]

Lemma 4.5 (Profit-maximizing production plans).
Let Assumption 4.1 be satisfied, $n_1, n_2 \leq 1$ *with* $n_j < 1$ *for some* $j = 1, 2$, *and* $(p_1, p_2) \gg 0$ *be arbitrary but fixed. Then the following holds true.*

(i) *For each* $k > 0$ *with* $\Omega_1(k) \neq \Omega_2(k)$, *there exist a uniquely determined wage-rental ratio* $\Omega_G(k) \in \big(\Omega_{\min}(k), \Omega_{\max}(k)\big)$ *such that*

$$\mathrm{MRT}\big(\mathcal{T}_G(k, \Omega_G(k))\big) = \frac{p_1}{p_2}. \tag{4.28}$$

The production plan $\mathcal{T}_G(k, \Omega_G(k)) \gg 0$ *is profit-maximizing at output prices* (p_1, p_2). *The wage-rental ratio* $\Omega_G(k)$ *is increasing for all* $k > 0$ *with* $\Omega_1(k) \neq \Omega_2(k)$.

(ii) *For each* $k > 0$ *with* $\Omega_1(k) = \Omega_2(k)$, *there exists a uniquely determined labor share* $l_G(k) \in (0, 1)$ *such that*

$$\mathrm{MRT}\big(\tilde{\mathcal{T}}_G(k, l_G(k))\big) = \frac{p_1}{p_2}. \tag{4.29}$$

The production plan $\tilde{\mathcal{T}}_G(k, l_G(k)) \gg 0$ *is profit-maximizing at output prices* (p_1, p_2).

Lemma 4.5 demonstrates that, under decreasing returns to scale, production always takes place in both sectors, regardless of the magnitude of (non-zero) output prices. Since the marginal rate of transformation in (4.28) is no longer independent of the economy-wide capital-labor ratio k, all production plans that are profit-maximizing at output prices $(p_1, p_2) \gg 0$ are now located on the *Rybczynski curve*[10]

$$k \mapsto \mathcal{R}_G(k) \coloneqq \mathcal{T}_G(k, \Omega_G(k)) \quad \text{for all} \quad k > 0 \text{ with } \Omega_1(k) \neq \Omega_2(k).$$

Since the wage-rental ratio $\Omega_G(k)$ is strictly increasing by Lemma 4.5, the unit outputs $g_j\big(\kappa_j(\Omega_G(k))\big)$ of both sectors $j = 1, 2$ are strictly increasing in k. This is the reason why Rybczynski's output-substitution effect need not hold in models with decreasing returns to scale, first observed by Hansson and Lundahl (1983) and Ylönen (1987). The increase in unit-output $g_1\big(\kappa_1(\Omega_G(k))\big)$, for example, may overcompensate a decrease in the labor share $\ell_1(k, \Omega_G(k))$, in which case both outputs levels $\mathsf{x}_1(k, \Omega_G(k))$ and $\mathsf{x}_2(k, \Omega_G(k))$ increase with k.

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[9] The case of increasing returns to scale is omitted since, by Theorem 4.3, the production-possibility frontier is not concave.

[10] From a mathematical perspective, a closer look at the expression (4.26) reveals that the wage-rental ratio would be the more natural curve parameter here as well.

Such violations of the Rybczynski-effect are illustrated in Figure 4.6 using CES production functions with decreasing returns to scale. All production plans on the lime-green curve $k \mapsto \mathcal{R}_G(k)$ are profit-maximizing at output prices $(p_1, p_2) \gg 0$ and therefore have the same marginal rate of transformation $\frac{p_1}{p_2}$. In a neighborhood of $\bar{k}$ at which the factor-intensity reversal occurs, the Rybczynski effect is not observed because the substitution of output in sector 1 to output in sector 2 is reversed to substitution of output in sector 2 to output in sector 1.

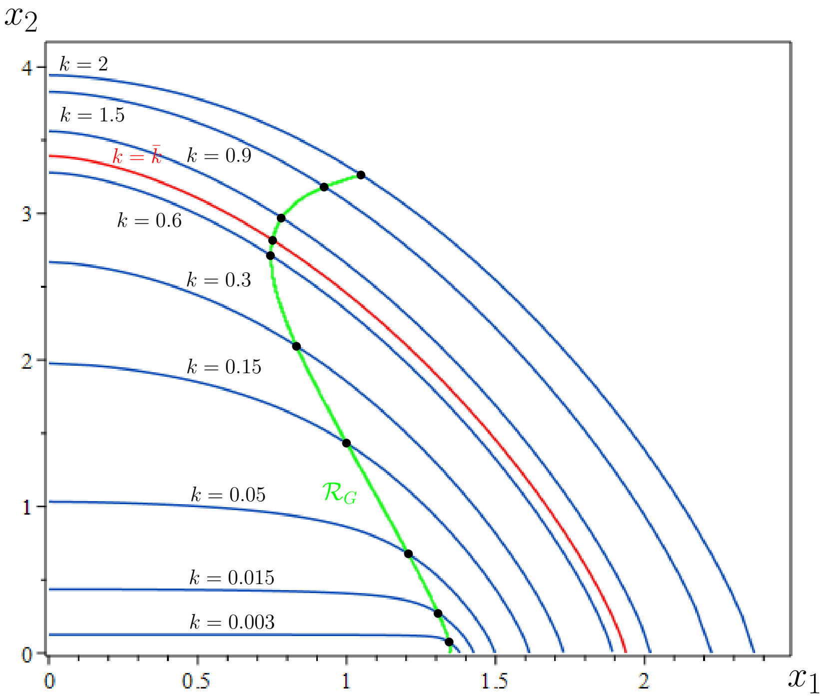

Figure 4.6. Rybczynski curve for CES production functions with decreasing returns to scale. Parameter set C; $\frac{p_1}{p_2} = 1.3$, $L = 1$.

Example 4.1 (Rybczynski curves).

Consider the case $n_1 = n_2 = n \in (0, 1)$. The first-order condition (4.28) then reads

$$\phi(\omega) = \frac{k - \kappa_1(\omega)}{\kappa_2(\omega) - k}, \tag{4.30}$$

where

$$\phi(\omega) := \frac{f_1(\kappa_1(\omega))}{f_2(\kappa_2(\omega))} \left(\frac{p_2}{p_1} \varrho(\omega) \right)^{\frac{1}{1-n}}. \tag{4.31}$$

Solving (4.30) for k yields the inverse of Ω_G, which takes the form

$$k = \Omega_G^{-1}(\omega) := \frac{\kappa_1(\omega) + \phi(\omega)\kappa_2(\omega)}{1 + \phi(\omega)}, \quad \omega \in \mathbb{R}_{++}.$$

The profit-maximizing labor shares are

$$\ell_1(k, \Omega_G(k)) = \frac{1}{1 + \phi(\Omega_G(k))} \quad \textit{and} \quad \ell_2(k, \Omega_G(k)) = \frac{\phi(\Omega_G(k))}{1 + \phi(\Omega_G(k))}$$

and may be non-monotonic in k, while the profit-maximizing capital-labor ratios $\kappa_1(\Omega_G(k))$ and $\kappa_2(\Omega_G(k))$ are strictly increasing in k. The profit-maximizing production plans are

$$\mathsf{x}_1(k,\Omega_G(k)) = L^{n-1}\left(\frac{1}{1+\phi(\Omega_G(k))}\, f_1(\kappa_1(\Omega_G(k)))\right)^n \tag{4.32}$$

$$\mathsf{x}_2(k,\Omega_G(k)) = L^{n-1}\left(\frac{\phi(\Omega_G(k))}{1+\phi(\Omega_G(k))}\, f_2(\kappa_2(\Omega_G(k)))\right)^n. \tag{4.33}$$

Figures 4.7 portrays Rybczynski curves $\mathcal{R}_G$ for CES production functions with varying degrees of homogeneity $n \in (0,1)$. Notice that as $n \to 1$, $\mathcal{R}_G$ approaches the piecewise straight lines displayed in Figure 4.4.

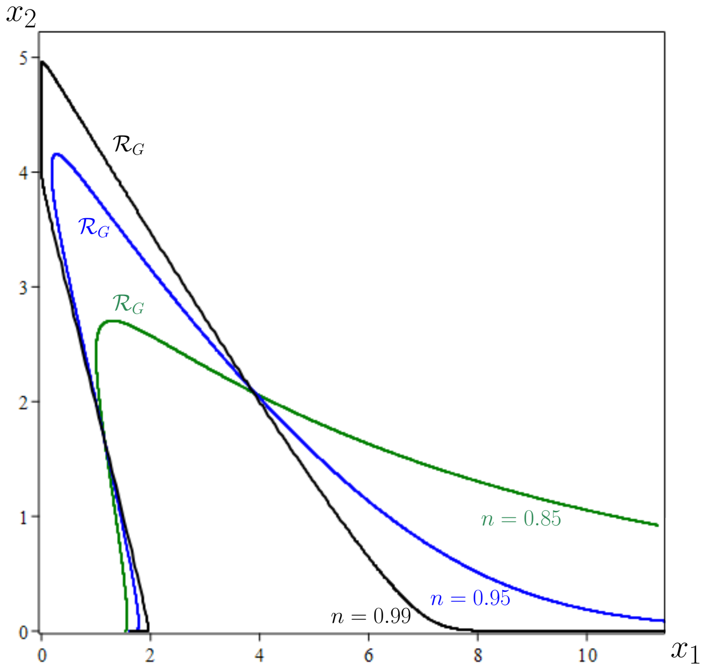

Figure 4.7. A family of Rybczynski curves for CES production functions with decreasing returns to scale. Parameter set C; $\frac{p_1}{p_2} = 1.3$, $L = 1$.

For the case of Cobb-Douglas production functions introduced in (4.36), the function (4.31) takes the form

$$\phi(\omega) = C\omega^{\frac{n}{1-n}(\alpha_2-\alpha_1)}, \tag{4.34}$$

where

$$C := \left(\frac{p_2}{p_1}\right)^{\frac{1}{1-n}} \left(\frac{A_2}{A_1}\right)^{\frac{n}{1-n}} \left(\frac{1-\alpha_1}{\alpha_1}\right)^{\frac{n\alpha_1}{1-n}} \left(\frac{\alpha_2}{1-\alpha_2}\right)^{\frac{n\alpha_2}{1-n}} \left(\frac{1-\alpha_2}{1-\alpha_1}\right)^{\frac{1}{1-n}} > 0.$$

Recall that Ω_G is strictly increasing. Without loss of generality, let $\alpha_2 > \alpha_1$. Then the map $\omega \mapsto \phi(\omega)$ in (4.34) is strictly increasing, so that output in sector 2 is strictly increasing for all $k > 0$ by (4.33). Differentiating (4.32), it is straightforward to verify that output in sector 1 is strictly increasing in k if and only if

$$[n\alpha_2 - \alpha_1]\,\phi(\Omega_G(k)) < (1-n)\alpha_1. \tag{4.35}$$

If $n \leq \frac{\alpha_1}{\alpha_2}$*, then Condition (4.35) is satisfied for all* $k > 0$ *because the term in the brackets is non-positive. By contrast, if* $\frac{\alpha_1}{\alpha_2} < n < 1$*, then there exists a uniquely determined* $k_{\mathsf{Ryb}} > 0$ *that solves (4.35) with equality. In this case, output in sector 1 is strictly increasing for all* $k < k_{\mathsf{Ryb}}$*, and strictly decreasing for all* $k > k_{\mathsf{Ryb}}$*. Therefore, the Rybczynski effect is violated for all sufficiently small capital-labor ratios. If the returns to scale are sufficiently decreasing, then the effect is violated for* all *capital-labor ratios, see Figure 4.8.*

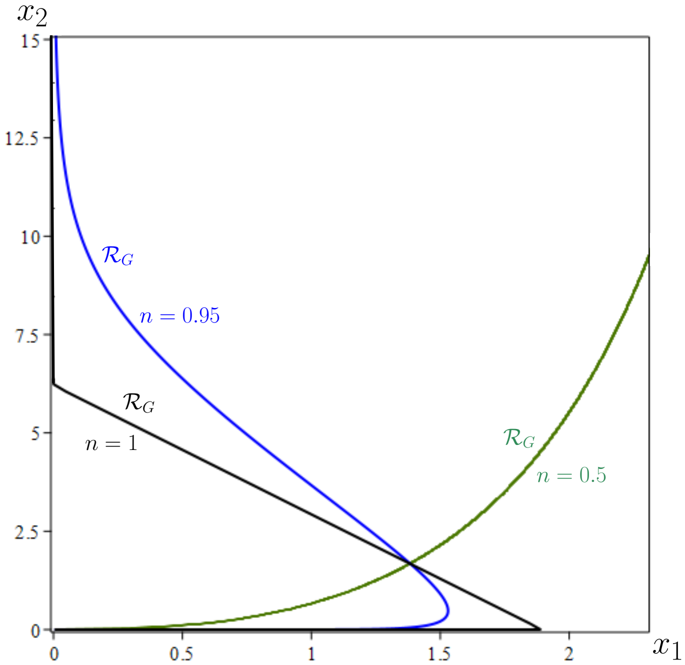

Figure 4.8. A family of Rybczynski curves for Cobb-Douglas production functions with non-increasing returns to scale. Parameter set F; $\frac{p_1}{p_2} = 1.3$, $L = 1$.

4.7 Conclusion

The necessity of the *'no-factor-intensity-reversals'* assumption is a nuisance in neoclassical two-sector models. This article introduced a novel and analytically tractable parameterization of the production-possibility frontier in terms of the wage-rental ratio that naturally incorporates factor-intensity reversals. Apart from an elementary proof of the frontier's concavity, our parameterization allows for a refined version of the Rybczynski theorem that holds for the whole range of possible capital-labor ratios. Exploiting the observation that, for linear-homogeneous production functions, the marginal rate of transformation is independent of the economy-wide capital-labor ratio, the findings of this article allow for a comprehensive analysis of the validity of Rybczynski's output-substitution effect. It turns out that the effect does not hold if the economy-wide capital-labor ratio is either too small or near a state at which a factor-intensity reversal occurs. Our approach is directly applicable to all CES production functions, as well as VES production functions that are homogeneous. We hope that the parameterization will enhance the tractability of dynamic models with two production sectors. Future research should extend our approach to multiple sectors and factors of production.

4.8 EXAMPLES

CES PRODUCTION FUNCTIONS

In this section, we provide a more detailed discussion of the CES example used in the previous sections to illustrate our results. The class of CES production functions is defined by

$$F_j(K_j, L_j) := \begin{cases} A_j\left[K_j^{\alpha_j} L_j^{1-\alpha_j}\right]^{n_j} & \text{if} \quad \rho_j = 0 \\ A_j\left[\alpha_j K_j^{-\rho_j} + (1-\alpha_j)L_j^{-\rho_j}\right]^{-\frac{n_j}{\rho_j}} & \text{if} \quad \rho_j > -1 \end{cases}, \quad (K_j, L_j) \in \mathbb{R}_+^2, \tag{4.36}$$

where $n_j > 0$ is the degree of homogeneity, $A_j > 0$ is the total factor productivity, $0 < \alpha_j < 1$ scales the distribution of factor incomes, and the constant $\rho_j > -1$ stipulates the elasticity of factor substitution $\sigma_j = \frac{1}{1+\rho_j}$. Assumption 4.1 (i) is satisfied. In the limiting case $\rho_j \to 0$, the CES production function becomes a Cobb-Douglas production function with substitution elasticity $\sigma_j = 1$. However, Cobb-Douglas production functions do not allow for factor-intensity reversals, as sector 2 is more capital-intensive than sector 1 if and only if $\alpha_2 > \alpha_1$. The parameter sets for our examples are listed in Table 4.1.

Parameter set	A_1	A_2	α_1	α_2	ρ_1	ρ_2	n_1	n_2
A	3.33	5	0.3	0.2	-0.5	1	1	1
B	3.33	5	0.3	0.2	1	-0.5	1	1
C	3.33	5	0.3	0.2	-0.5	1	0.6	0.8
D	3.33	5	0.3	0.2	-0.5	1	0.6	1.5
E	3.33	5	0.3	0.2	-0.5	1	1.2	1.5
F	3.33	5	0.5	0.8	0	0	$0.6, 0.95, 1$	$0.6, 0.95, 1$

Table 4.1. Parameter sets for CES production functions.

The MRTS functions Ω_j defined in (4.2) are independent of n_j and take the form

$$\Omega_j(k) = \tfrac{1-\alpha_j}{\alpha_j} k^{1+\rho_j},$$

showing that Assumption 4.1 (ii) is also satisfied. The relative factor demand functions κ_j defined in (4.3) become

$$\kappa_j(\omega) = \left(\tfrac{\alpha_j}{1-\alpha_j}\omega\right)^{\frac{1}{1+\rho_j}},$$

As portrayed in Figure 4.1, factor-intensity reversals occur unless $\rho_1 = \rho_2$. Specifically, for $\rho_1 > \rho_2$, exactly one factor-intensity reversal occurs at the wage-rental ratio $\bar{\omega}$ because

$$\kappa_2(\omega) \gtreqless \kappa_1(\omega) \iff \omega \gtreqless \bar{\omega} := \left(\tfrac{\alpha_1}{1-\alpha_1}\right)^{\frac{1+\rho_2}{\rho_1-\rho_2}} \left(\tfrac{1-\alpha_2}{\alpha_2}\right)^{\frac{1+\rho_1}{\rho_1-\rho_2}}.$$

The labor-share functions ℓ_j defined in (4.5) take the form

$$\ell_1(k,\omega) = \frac{\left(\frac{\alpha_2}{1-\alpha_2}\omega\right)^{\frac{1}{1+\rho_2}} - k}{\left(\frac{\alpha_2}{1-\alpha_2}\omega\right)^{\frac{1}{1+\rho_2}} - \left(\frac{\alpha_1}{1-\alpha_1}\omega\right)^{\frac{1}{1+\rho_1}}} \quad \text{and} \quad \ell_2(k,\omega) = \frac{k - \left(\frac{\alpha_1}{1-\alpha_1}\omega\right)^{\frac{1}{1+\rho_1}}}{\left(\frac{\alpha_2}{1-\alpha_2}\omega\right)^{\frac{1}{1+\rho_2}} - \left(\frac{\alpha_1}{1-\alpha_1}\omega\right)^{\frac{1}{1+\rho_1}}}$$

and are well defined, except at $\bar{\omega}$. Figures 4.2 and 4.5 portray the resulting production-possibility frontiers for the different parameter sets. The frontier is non-concave whenever there are increasing returns to scale in one sector. The function ϱ defined in (4.15) becomes

$$\varrho(\omega) = \frac{n_2 A_2 \Big(\alpha_1^{\frac{1}{1+\rho_1}} + (1-\alpha_1)^{\frac{1}{1+\rho_1}}\,\omega^{\frac{\rho_1}{1+\rho_1}}\Big)^{\frac{1+\rho_1}{\rho_1}} \Big(\alpha_2\big(\tfrac{\alpha_2}{1-\alpha_2}\omega\big)^{\frac{-\rho_2}{1+\rho_2}} + (1-\alpha_2)\Big)^{\frac{1-n_2}{\rho_2}}}{n_1 A_1 \Big(\alpha_2^{\frac{1}{1+\rho_2}} + (1-\alpha_2)^{\frac{1}{1+\rho_2}}\,\omega^{\frac{\rho_2}{1+\rho_2}}\Big)^{\frac{1+\rho_2}{\rho_2}} \Big(\alpha_1\big(\tfrac{\alpha_1}{1-\alpha_1}\omega\big)^{\frac{-\rho_1}{1+\rho_1}} + (1-\alpha_1)\Big)^{\frac{1-n_1}{\rho_1}}}. \tag{4.37}$$

In the linear-homogeneous case, i.e., $n_1 = n_2 = 1$, (4.37) simplifies to

$$\varrho(\omega) = \frac{A_2 \Big(\alpha_1^{\frac{1}{1+\rho_1}} + (1-\alpha_1)^{\frac{1}{1+\rho_1}}\,\omega^{\frac{\rho_1}{1+\rho_1}}\Big)^{\frac{1+\rho_1}{\rho_1}}}{A_1 \Big(\alpha_2^{\frac{1}{1+\rho_2}} + (1-\alpha_2)^{\frac{1}{1+\rho_2}}\,\omega^{\frac{\rho_2}{1+\rho_2}}\Big)^{\frac{1+\rho_2}{\rho_2}}} \tag{4.38}$$

and stipulates the marginal rate of transformation corresponding to any wage-rental ratio $\omega > 0$. Figure 4.3 shows that the marginal rate of transformation (4.38) is either hump or U-shaped. The relative size of the parameters ρ_1 and ρ_2 determines whether ϱ has a global maximum or a global minimum at $\bar{\omega}$.

VES Production Functions

We next apply our approach to a production function with a variable elasticity of substitution (VES). Consider the production function

$$F_j(K_j, L_j) := A_j \Big[a_j K_j^{-\rho_j} + b_j L_j^{-\rho_j} + \big(K_j^{-\rho_j} L_j^{-\rho_j}\big)^{\frac{1}{2}}\Big]^{-\frac{n_j}{\rho_j}}, \quad (K_j, L_j) \in \mathbb{R}_+^2,$$

where $A_j, a_j, b_j > 0$ determine the factor productivities, $n_j > 0$ is the degree of homogeneity, and $0 \neq \rho_j \geq -1$ is some curvature parameter. The MRTS functions Ω_j in (4.2) are independent of n_j and take the form

$$\Omega_j(k) = \frac{2b_j k^{1+\rho_j} + k^{1+\frac{\rho_j}{2}}}{2a_j + k^{\frac{\rho_j}{2}}}.$$

Since Ω_j is strictly positive and strictly increasing, the isoquants of F_j are downward-sloping and strictly convex curves, so that Assumption 4.1 (i) is satisfied. It is straightforward to verify that Assumption 4.1 (ii) is also satisfied. Since the elasticity of factor substitution

$$\sigma_j(k) := \frac{\Omega_j(k)}{\Omega_j'(k)k}$$

is *not* constant, F_j belongs to the class of VES production functions, see Revankar (1971) and Chilarescu (2019) for details. For $\rho_1 = \rho_2 = -1$, the MRTS functions simplify to

$$\Omega_j(k) = \frac{2b_j + k^{\frac{1}{2}}}{2a_j + k^{-\frac{1}{2}}},$$

and the relative factor demand functions (4.3) take the explicit form

$$\kappa_j(\omega) = \left(a_j\,\omega - b_j + \sqrt{(a_j\,\omega - b_j)^2 + \omega}\right)^2.$$

If $a_1 = a_2$, then sector 2 is always more capital-intensive than sector 1 if and only if $b_1 > b_2$. Likewise, if $b_1 = b_2$, then sector 2 is always more capital-intensive than sector 1 if and only if $a_2 > a_1$. Otherwise, a factor-intensity reversal occurs at the wage-rental ratio $\bar{\omega} \coloneqq \frac{b_1 - b_2}{a_1 - a_2}$ whenever $\bar{\omega} > 0$.

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Chapter 5

Green Transformation and Fiscal Policy in a Two-Sector OLG Model with Production Externalities[⋆]

with Jan Wenzelburger

Abstract

This article studies the decarbonization of a two-sector overlapping-generations economy with a non-polluting green and a polluting brown sector that produce the same good. Agents derive utility from consumption and disutility from pollution. We construct a balanced-budget fiscal policy consisting of an emissions tax and intergenerational transfers that implements an optimal allocation of a social planner. Emissions tax receipts must be redistributed as climate dividends to compensate agents for policy-induced reductions in factor incomes. The optimal emissions tax rate is determined by a marginal rate of substitution between current consumption and discounted future pollution damages. Depending on the fundamentals, multiple modified golden-rule steady states with distinct consumption and pollution levels may coexist. The sector-specific capital intensities determine the stability properties.

Keywords: Overlapping generations, two-sector growth models, production externalities, green transformation, climate policy, golden rules.

JEL Classification: D61, D62, E62, H23, O41, P28.

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5.1 Introduction

The decarbonization and green transformation of large economies have become central priorities on policy agendas worldwide, driven by the increasingly severe environmental and economic consequences of anthropogenic climate change, e.g., see the Stern (2007) review. The European Green Deal stands as a prominent example, committing the European Union to climate neutrality by 2050 and underscoring the magnitude of the structural, technological, and policy changes required to accomplish this transition. Given that the industrial and energy sectors together account for more than half of global greenhouse gas emissions (Friedlingstein et al., 2022; Lee et al., 2023), policymakers face the key challenge of designing incentives that induce firms to adopt less emission-intensive processes and technologies. Green technologies are widely available across most industries and are becoming increasingly cost-effective (Lee et al., 2023). Beyond environmental benefits, emissions reduction can strengthen competitiveness of firms both by mitigating the financial burden of carbon pricing and by responding to the growing environmental awareness of the general public. At the same time, climate change involves an inherently complex intergenerational challenge. While today's emissions negatively affect the welfare of all future generations, the costs of decarbonization are borne by the current generation alone. Balancing these abatement costs against the benefits accruing to future generations significantly complicates the internalization of the climate change externality.

Against this background, numerous contributions such as Howarth (2000), Schneider et al. (2012), and Stephan et al. (1997) have highlighted the suitability of overlapping-generations (OLG) models for addressing intergenerational aspects of climate economics. OLG models complement the more commonly used optimal growth models that feature an infinitely-lived representative agent (ILA). While it is well known that certain assumptions on altruism can generate an *'observational equivalence'* between OLG and ILA approaches (e.g., see Barro, 1974), the extent to which such equivalence applies when intergenerational externalities such as climate change are present remains unclear; see, for example, Jaimes (2023), Schneider et al. (2012), and Stephan et al. (1997). Moreover, the ongoing public debate on climate change mitigation suggests that consumption and production decisions are largely driven by self-interest rather than altruistic concern for future generations. The goal of this article is to develop a tractable two-sector OLG model with intergenerational pollution externalities to investigate the role of fiscal policy for the decarbonization of an economy.

The literature on OLG models with pollution externalities dates back to the seminal contributions of Howarth and Norgaard (1992) and John and Pecchenino (1994). Within this literature, both the origins and effects of pollution externalities are modeled in various ways. On the production side, pollution may arise from output generation (Marini and Scaramozzino, 1995; Howarth, 1998), labor supply (Andersen et al., 2020), energy use (Howarth and Norgaard, 1992), or the extraction of natural resources (Mourmouras, 1991). Alternatively, pollution may be caused by consumption (John and Pecchenino, 1994; John et al., 1995; Ono, 1996). The consequences of pollution also differ across models. In Andersen et al. (2020), Goussebaile (2024), and Howarth and Norgaard (1992), pollution reduces future production, whereas in Gutiérrez (2008) it lowers future consumption. John and Pecchenino (1994) and

John et al. (1995), on the other hand, assume that environmental quality enters agents' preferences, such that pollution reduces welfare directly.

Despite the abundance of overlapping-generations models in the literature on environmental economics, analytically tractable models with more than one sector are scarce. Rausch and Yonezawa (2023) compare the welfare effects of technology policy and carbon pricing in an OLG model with a green and a brown intermediate-goods sector, showing that green technology policy - unlike carbon pricing - may act as an implicit capital subsidy that disproportionately benefits the current generation at the expense of future generations. Green and brown intermediate-goods sectors also feature in Dao and Edenhofer (2018), whose model gives rise to poverty-environment traps, that is, steady states characterized by low environmental quality and a low capital stock per capita. Dao and Edenhofer (2018) further show that the optimal allocation chosen by a social planner can be decentralized through a combination of emissions and capital-income taxes. Similar poverty-environment traps are found in Ikefuji and Horii (2007). Karp and Rezai (2014) analyze whether environmental policies can generate Pareto improvements by altering asset prices in an OLG model with a resource sector and a production sector. Nakabayashi (2010) derives optimal second-best tax rules in an OLG model with a green and a brown production sector. Finally, in a one-sector OLG framework, Dao and Davila (2014) examine the implementation of golden-rule steady states through alternative tax-and-transfer schemes. More elaborate models with multiple sectors require a numerical analysis, e.g., see Kotlikoff et al. (2021).

This article extends the classical overlapping-generations framework of Diamond (1965) by introducing a polluting *brown* and a non-polluting *green* production sector as well as environmental preferences of agents.[1] In the spirit of Galor (1992), both sectors compete for capital and labor but produce the same good. Pollution degrades environmental quality inherited by future generations, thereby causing an intergenerational externality. In the absence of carbon pricing, the competitive equilibrium fails to internalize this externality. Achieving the social optimum requires a reduction in brown output, so that agents must reduce consumption. Adopting the perspective of a social planner, we investigate the trade-off between consumption, capital accumulation, and emissions abatement. We establish the existence of a unique optimal allocation and at least one welfare-maximizing steady state, that is, a *modified golden-rule steady state*. Depending on technologies, discounting, and agents' subjective evaluation of environmental damages, this steady state may involve production exclusively in the green sector, exclusively in the brown sector, or in both sectors. Optimal pollution levels are determined by the relationship between the marginal rate of transformation and the marginal rate of substitution between consumption and discounted future pollution damages. Since all three types of steady states may coexist, the model provides a theoretical foundation for the observed heterogeneity in pollution levels across real-world economies.

The main contribution of this article concerns the decentralization of the optimal allocation, which has important policy implications. We show that the social planner's allocation can be implemented in a competitive market economy through a balanced-budget fiscal policy, consisting of an emissions tax combined with intergenerational transfers. While the emissions

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[1] We consider more general consumption preferences than John and Pecchenino (1994) and John et al. (1995) by allowing agents to consume when they are young.

tax internalizes the pollution externality, it distorts factor prices and thereby reduces factor incomes. To offset the resulting welfare losses, the government must redistribute the emissions tax receipts to agents in the form of *climate dividends* via the intergenerational transfers. Beyond compensation, these transfers enable the government to control the intergenerational income distribution and thus to implement an efficient capital accumulation path, resolving the problem of dynamic inefficiency inherent in OLG models.

Our approach is relatively general in that it imposes no a-priori restrictions on the relative productivities of green and brown technologies. In particular, it accommodates factor-intensity reversals, which may naturally arise in the course of the green transformation. In contrast to most existing multi-sector models (e.g., see Rausch and Yonezawa, 2023), our approach therefore does not rely on restrictive factor-intensity assumptions. Instead, the relative capital intensities play a central role in determining the stability properties of modified golden-rule steady states.

This article is organized as follows. The next section introduces the model and states the assumptions on technologies and preferences. Section 5.3 investigates temporary equilibria of the market economy and establishes the perfect-foresight dynamics. Section 5.4 derives the optimal allocation, while Section 5.5 addresses its implementation. Section 5.6 characterizes modified golden-rule steady states, before Section 5.7 concludes. Proofs and technical details on dynamic programming are collected in Appendix E.

5.2 MODEL PREREQUISITES

We consider an overlapping-generations model with production. At the beginning of each period $t = 0, 1, \dots, \infty$, a new generation consisting of a continuum of homogeneous, two-period-lived agents is born. The population grows at a constant rate $n \geq 0$. Assuming that each young agent inelastically supplies one unit of labor, then aggregate labor supply in period t becomes $L_t = (1+n)^t L_0$, where $L_0 > 0$ is given. Old agents are retired and consume the income of inelastically renting capital to firms.

5.2.1 TECHNOLOGIES

The economy comprises two production sectors, a non-polluting *'green'* and a polluting *'brown'* sector, indexed by $j = g, b$, respectively. Both sectors transform capital and labor into a single *'all-purpose'* good that can be consumed and invested. The aggregate production function of the representative firm in sector $j = g, b$ is

$$F_j : \mathbb{R}_+^2 \to \mathbb{R}_+, \qquad Y^j = F_j(K^j, L^j),$$

where $Y^j \geq 0$ denotes the sector-specific output and $K^j, L^j \geq 0$ are the sector-specific inputs of capital and labor, respectively. Following Galor (1992), capital and labor are perfectly mobile between the sectors and are paid their marginal products. The output good serves as the numéraire, so that its price is normalized to unity. Since both sectors produce the same good, the total output of economy is $Y = Y^g + Y^b$. During production, capital depreciates at the rate $\delta \in (0, 1]$.

The technology in either sector has constant returns to scale, meaning that both production functions F_g and F_b are linear-homogeneous. Their respective intensive forms $f_j : \mathbb{R}_+ \to \mathbb{R}_+$ are defined by

$$f_j(k^j) := F_j(k^j, 1), \quad j = g, b,$$

where $k^j := \frac{K^j}{L^j}$ is the sector-specific capital-labor ratio. The marginal products of labor $w_j : \mathbb{R}_+ \to \mathbb{R}_+$ then take the form

$$w_j(k^j) := f_j(k^j) - f_j'(k^j)k^j, \quad j = g, b,$$

and the marginal rates of technical substitution (MRTS) $\Omega_j : \mathbb{R}_+ \to \mathbb{R}_+$ are defined by

$$\Omega_j(k^j) := \frac{w_j(k^j)}{f_j'(k^j)}, \quad j = g, b.$$

We impose the following assumptions on the technologies.

Assumption 5.1 (Technologies).
The technologies of the two sectors $j = g, b$ are characterized by the following properties:

(i) Each production function $f_j : \mathbb{R}_+ \to \mathbb{R}_+$ is twice continuously differentiable, strictly increasing, $f_j' > 0$, strictly concave, $f_j'' < 0$, and satisfies $f_j(0) = 0$.

(ii) Each MRTS function $\Omega_j : \mathbb{R}_+ \to \mathbb{R}_+$ satisfies the boundary conditions

$$\lim_{k^j \to 0} \Omega_j(k^j) = 0 \quad \textit{and} \quad \lim_{k^j \to \infty} \Omega_j(k^j) = \infty.$$

Since Assumption 5.1 (i) implies that each MRTS function Ω_j is strictly increasing, Assumption 5.1 (ii) ensures the existence of inverse functions $\kappa_j : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\Omega_j(\kappa_j(\omega)) = \omega \quad \text{for all} \quad \omega \geq 0. \tag{5.1}$$

These so-called *relative factor demand functions* stipulate the sector-specific capital-labor ratios $\kappa_j(\omega)$, given some wage-rental ratio ω. It is worth noting that our approach naturally incorporates *factor-intensity reversals* in the following sense. If $k := \frac{K}{L}$ is the economy-wide capital-labor ratio and $\Omega_i(k) < \Omega_j(k)$, where $i \neq j$, then

$$\kappa_i(\omega) < k < \kappa_j(\omega) \quad \text{for all} \quad \omega \in \big(\Omega_i(k), \Omega_j(k)\big).$$

The marginal rates of technical substitution thus determine whether the green or the brown sector is more capital-intensive, so that a restrictive assumption on factor intensities is not needed. The following example, which is adapted from Schmitz and Wenzelburger (2024), demonstrates that factor-intensity reversals occur in particular for production functions with a constant elasticity of substitution (CES).

Example 5.1 (CES technologies).
The CES production function $f_j : \mathbb{R}_+ \to \mathbb{R}_+$ is defined by

$$f_j(k) = \begin{cases} A_j k^{\alpha_j} & \textit{if } \sigma_j = 1 \\ A_j \Big[\alpha_j k^{\frac{\sigma_j - 1}{\sigma_j}} + (1 - \alpha_j)\Big]^{\frac{\sigma_j}{\sigma_j - 1}} & \textit{if } \sigma_j \neq 1 \end{cases}, \tag{5.2}$$

where $A_j > 0$ *is the total factor productivity,* $0 < \alpha_j < 1$ *scales the factor-income distribution, and the constant* $\sigma_j > 0$ *is the elasticity of factor substitution. In the limiting case* $\sigma_j = 1$*, the CES production function becomes a Cobb-Douglas production function. The relative factor demand functions* $\kappa_j : \mathbb{R}_+ \to \mathbb{R}_+$ *defined in (5.1) take the form*

$$\kappa_j(\omega) = \left(\tfrac{\alpha_j}{1-\alpha_j}\omega\right)^{\sigma_j}, \quad j = g, b.$$

Unless $\sigma_g = \sigma_b$*, then exactly one factor-intensity reversal occurs at the wage-rental ratio* $\overline{\omega}$*. In particular, for* $\sigma_g > \sigma_b$*, we have*

$$\kappa_g(\omega) \gtreqless \kappa_b(\omega) \iff \omega \gtreqless \overline{\omega} := \left(\tfrac{1-\alpha_g}{\alpha_g}\right)^{\frac{\sigma_g}{\sigma_g-\sigma_b}} \left(\tfrac{\alpha_b}{1-\alpha_b}\right)^{\frac{\sigma_b}{\sigma_g-\sigma_b}}.$$

5.2.2 Pollution

To produce one unit of output, the brown sector emits $\epsilon_b > 0$ units of pollutants into the environment, as for example in the form of greenhouse gases. The green sector operates emissions neutrally, so that $\epsilon_g = 0$. The pollution stock per capita of the young generation *at the beginning* of period t is denoted by $\chi_t \geq 0$. Assuming that pollution decays at a constant rate $\zeta \in (0, 1]$, the evolution of χ_t over time is governed by the *pollution accumulation law* $E : \mathbb{R}^2_+ \to \mathbb{R}_+$, defined by

$$\chi_{t+1} = E(\chi_t, y^b_t) := \tfrac{1}{1+n}\big[(1-\zeta)\chi_t + \epsilon_b y^b_t\big], \tag{5.3}$$

where $\epsilon_b y^b_t \geq 0$ are the per-capita emissions of the brown sector in period t.

5.2.3 Preferences

Agents have intertemporal preferences over consumption and the quality of the environment. The preferences are represented by a life-cycle utility function $U : \mathbb{R}^4_+ \to \mathbb{R}$ of the additive-separable form

$$U(c^1_t, c^2_{t+1}, z_t, z_{t+1}) := u(c^1_t) - \mu(z_t) + \beta\big[u(c^2_{t+1}) - \mu(z_{t+1})\big], \tag{5.4}$$

where $c^1_t, c^2_{t+1} \geq 0$ denote youthful and old-age consumption, respectively, and

$$z_t := (1-\zeta)\chi_t + \epsilon_b y^b_t = (1+n)\chi_{t+1} \tag{5.5}$$

is an environmental pollution index, measured *at the end* of period t. The factor $\beta > 0$ is the agent-specific time-discount factor. Our assumptions on the preferences are as follows.

Assumption 5.2 (Preferences).

(i) The utility function $u : \mathbb{R}_+ \to \mathbb{R}$ *is twice continuously differentiable, strictly increasing,* $u' > 0$*, strictly concave,* $u'' < 0$*, and satisfies the Inada condition*

$$\lim_{c \to 0} u'(c) = \infty.$$

(ii) The damage function $\mu : \mathbb{R}_+ \to \mathbb{R}_+$ *is twice continuously differentiable, strictly increasing,* $\mu' > 0$*, and convex,* $\mu'' \geq 0$*.*

Assumption 5.2 implies that youthful and old-age consumption are normal goods, whereas environmental pollution is a *bad*. It follows from (5.3) that

$$\chi_{t+1} = \left(\tfrac{1-\zeta}{1+n}\right)^{t+1}\chi_0 + \tfrac{1}{1+n}\sum_{i=0}^{t}\left(\tfrac{1-\zeta}{1+n}\right)^{t-i}\epsilon_b y_i^b,$$

showing that pollution reduces the welfare of all future generations of agents, unless $\zeta = 1$. The brown sector therefore has both, an *intra-* and *intergenerational* production externality.[2]

5.3 The Market Economy

We first investigate a market economy subject to fiscal policy intervention. Suppose the government taxes pollution emissions in period t at the rate $\tau_t \in \mathbb{R}$. Moreover, it pays each young agent a lump-sum transfer $a_t^1 \in \mathbb{R}$, and each old agent a lump-sum transfer $a_t^2 \in \mathbb{R}$.[3] As usual, negative transfers imply lump-sum taxes. Since the government does neither issue nor hold bonds, its budget constraint in per-capita terms takes the form

$$a_t^1 + \tfrac{a_t^2}{1+n} = \tau_t\epsilon_b y_t^b, \tag{5.6}$$

implying that the emissions tax receipts $\tau_t\epsilon_b y_t^b$ are fully redistributed to agents. Note that our approach also covers the case $\tau_t < 0$, in which the brown sector is subsidized.

5.3.1 Profit Maximization

Given the wage rate $w_t > 0$, the capital-rental rate $r_t > 0$, and the emissions tax rate $\tau_t \in \mathbb{R}$, the profit-maximization problem of the representative firm in sector $j = g, b$ reads

$$\max_{K^j,L^j\geq 0} \quad F_j(K^j,L^j) - r_tK^j - w_tL^j - \tau_t\epsilon_jF_j(K^j,L^j). \tag{5.7}$$

The first-order conditions for the profit-maximizing factor inputs (K_t^j, L_t^j) take the form

$$\begin{array}{llcl} (i) & \big[1-\epsilon_j\tau_t\big]\dfrac{\partial F_j}{\partial K}(K_t^j,L_t^j) & = & \big[1-\epsilon_j\tau_t\big]f_j'(k_t^j) \overset{!}{=} r_t \\ (ii) & \big[1-\epsilon_j\tau_t\big]\dfrac{\partial F_j}{\partial L}(K_t^j,L_t^j) & = & \big[1-\epsilon_j\tau_t\big]w_j(k_t^j) \overset{!}{=} w_t \end{array}, \tag{5.8}$$

showing that due to linear-homogeneity of the objective function in Problem (5.7), only the profit-maximizing capital-labor ratio k_t^j is well defined by the first-order conditions (5.8). Given any wage-rental ratio $\omega_t := \frac{w_t}{r_t} > 0$, these reduce to

$$\Omega_j(k_t^j) \overset{!}{=} \omega_t, \quad j = g, b. \tag{5.9}$$

Since capital and labor are perfectly mobile, it follows from (5.8) that factor prices in both sectors must be equal whenever both produce. Since $\epsilon_g = 0$, we have

$$\big[1-\epsilon_b\tau_t\big]w_b(k_t^b) = w_g(k_t^g) = w_t \quad \text{and} \quad \big[1-\epsilon_b\tau_t\big]f_b'(k_t^b) = f_g'(k_t^g) = r_t. \tag{5.10}$$

---

[2] Observe that our model abstracts from a direct effect of pollution on future production.

[3] Since labor supply is totally inelastic, we may likewise consider a non-distortionary, proportional tax (subsidy) on the labor income of young agents.

5.3.2 UTILITY MAXIMIZATION

The utility-maximization problem of a typical young agent in period t is considered next. The agent forms an expectation $R_t^e > 0$ with respect to the gross return on savings $R_{t+1} := 1 - \delta + r_{t+1}$ realized in $t+1$, and an expectation $a_t^{2,e} \in \mathbb{R}$ with respect to his old-age transfer a_{t+1}^2. Given the expectations $(a_t^{2,e}, R_t^e)$ and the disposable income $w_t^d := w_t + a_t^1$, the agent's savings decision problem takes the form

$$\begin{aligned} &\max_s \; u(w_t^d - s) - \mu(z_t) + \beta\big[u(R_t^e s + a_t^{2,e}) - \mu(z_{t+1})\big] \\ &\text{s.t.} \quad \max\Big\{\tfrac{-a_t^{2,e}}{R_t^e}, 0\Big\} \le s \le w_t^d. \end{aligned} \tag{5.11}$$

The two constraints in Problem (5.11) ensure that both youthful and old-age consumption are non-negative, and that the agent does not borrow against his anticipated future income.[4] Naturally, since the agent has mass zero, he does not take into account how his savings behavior affects the environmental pollution index z_{t+1}.

Given that both the disposable youthful income and the discounted lifetime income of the agent are non-negative, i.e., $w_t^d \ge 0$ and $w_t^d + \frac{a_t^{2,e}}{R_t^e} \ge 0$, Problem (5.11) admits a uniquely determined solution $s_t = s(w_t^d, a_t^{2,e}, R_t^e)$. The economy-wide capital-labor ratio of the subsequent period $t+1$ then becomes

$$k_{t+1} = \tfrac{K_{t+1}}{L_{t+1}} = \tfrac{1}{1+n} s(w_t^d, a_t^{2,e}, R_t^e).$$

The realized levels of youthful and old-age consumption in period t are

$$c_t^1 = w_t + a_t^1 - s(w_t + a_t^1, a_t^{2,e}, R_t^e) \quad \text{and} \quad c_t^2 = (1+n)R_t k_t + a_t^2. \tag{5.12}$$

Observe that the fiscal policy parameters (τ_t, a_t^1, a_t^2) are only feasible if the induced disposable incomes of both generations in period t are non-negative. Since, by (5.6), $a_t^1 = \tau_t \epsilon_b y_t^b - \frac{a_t^2}{1+n}$, we can read off (5.12) that this is the case if and only if

$$-R_t k_t \le \tfrac{a_t^2}{1+n} \le w_t + \tau_t \epsilon_b y_t^b. \tag{5.13}$$

The feasibility condition (5.13) implies that the disposable income of the old generation must neither be negative nor exceed the total income of the economy.

5.3.3 TEMPORARY EQUILIBRIA

Given an economy-wide capital-labor ratio $k_t \ge 0$, a *feasible factor allocation* is a list $(k_t^g, k_t^b, l_t^g, l_t^b) \ge 0$ that satisfies the market-clearing conditions

$$(i) \quad k_t = l_t^g k_t^g + l_t^b k_t^b \quad \text{and} \quad (ii) \quad 1 = l_t^g + l_t^b, \tag{5.14}$$

where $l_t^g := \frac{L_t^g}{L_t}$ and $l_t^b := \frac{L_t^b}{L_t}$ denote the labor shares of the green and the brown sector, respectively. The corresponding per-capita outputs are

$$y_t^j := \tfrac{Y_t^j}{L_t} = l_t^j f_j(k_t^j), \quad j = g, b.$$

---

[4] Recall that capital accumulation in OLG models requires positive savings.

In a *temporary equilibrium*, the markets for capital, labor, and goods clear simultaneously and, given expectations, individual decisions of agents and firms are optimal.

Definition 5.1 (Temporary equilibrium).
Given an economy-wide capital-labor ratio $k_t \geq 0$*, the expectations* $(a_t^{2,e}, R_t^e) \in \mathbb{R} \times \mathbb{R}_{++}$*, and the fiscal policy parameters* $(\tau_t, a_t^2) \in \mathbb{R}^2$*, a* temporary equilibrium *is an allocation* $(k_t^g, k_t^b, l_t^g, l_t^b, c_t^1, c_t^2) \geq 0$ *with prices* $(w_t, r_t) \geq 0$ *that satisfies (5.6), (5.10), (5.12), and (5.14).*

Definition 5.1 ensures that factor prices in both sectors are equal, even when only one sector is active.[5] Observe that the non-negativity constraints on the consumption plan (c_t^1, c_t^2) imply that the fiscal policy parameters (a_t^1, a_t^2, τ_t) must satisfy the feasibility condition (5.13), and that the savings function $s(w_t + a_t^1, a_t^{2,e}, R_t^e)$ in (5.12) is only well defined for expectations that satisfy $w_t + a_t^1 + \frac{a_t^{2,e}}{R_t^e} \geq 0$.

The following result can be deduced directly from Definition 5.1.

Lemma 5.1 (Equilibrium factor incomes).
The factor incomes in a temporary equilibrium satisfy

$$w_t + [1 - \delta + r_t]k_t + \tau_t \epsilon_b y_t^b = y_t^g + y_t^b + (1 - \delta)k_t. \tag{5.15}$$

The accounting identity (5.15) shows that emissions taxation reduces the factor incomes of both generations by distorting the factor prices. The government may compensate agents for this reduction by redistributing the emissions tax receipts via the lump-sum transfers (a_t^1, a_t^2). The transfers can thus be interpreted as *climate dividends* that offset the emissions-tax-induced decline in factor incomes.

A further implication of (5.15) is that the goods market clears whenever the factor markets are in equilibrium. Inserting (5.6) and (5.12), the accounting identity (5.15) takes the form

$$c_t^1 + \frac{c_t^2}{1+n} + s(w_t + a_t^1, a_t^{2,e}, R_t^e) = y_t^g + y_t^b + (1 - \delta)k_t,$$

showing that the excess demand for goods is zero. This result is, of course, a direct consequence of Walras' law.

5.3.4 CHARACTERIZATION OF TEMPORARY EQUILIBRIA

The temporary equilibria of the economy may now be characterized by parameterizing the factor allocation, factor prices, and production plan in the wage-rental ratio ω_t. The firms' reduced first-order conditions (5.9) imply that profits in both sectors are maximized if and only if the sector-specific capital-labor ratios are

$$k_t^g = \kappa_g(\omega_t) \quad \text{and} \quad k_t^b = \kappa_b(\omega_t), \tag{5.16}$$

---

[5] Since $l_t^j = 0$ implies $L_t^j = l_t^j L_t = 0$ and thus $K_t^j = l_t^j k_t^j L_t = 0$, one has $y_t^j = l_t^j f_j(k_t^j) = 0$ for all $k_t^j \geq 0$. Hence, the capital-labor ratio of the inactive sector may be chosen such that factor prices are equal.

where $\kappa_g, \kappa_b : \mathbb{R}_+ \to \mathbb{R}_+$ are the relative factor demand functions introduced in (5.1).

The corresponding labor shares obtain as follows. Given $k_t \geq 0$, there are two cases: the *'generic'* case $\Omega_g(k_t) \neq \Omega_b(k_t)$, and the *'non-generic'* case $\Omega_g(k_t) = \Omega_b(k_t)$. Set

$$\Omega_{\min}(k_t) := \min\{\Omega_g(k_t), \Omega_b(k_t)\} \quad \text{and} \quad \Omega_{\max}(k_t) := \max\{\Omega_g(k_t), \Omega_b(k_t)\}.$$

We may then define the *labor-share functions*

$$\ell_g(k_t, \cdot), \ell_b(k_t, \cdot) : \big[\Omega_{\min}(k_t), \Omega_{\max}(k_t)\big] \to [0, 1]$$

by setting

$$l_t^g = \ell_g(k_t, \omega_t) := \begin{cases} \dfrac{\kappa_b(\omega_t) - k_t}{\kappa_b(\omega_t) - \kappa_g(\omega_t)} & \text{if } \ \Omega_g(k_t) \neq \Omega_b(k_t) \\ \hat{l}_t^g & \text{if } \ \Omega_g(k_t) = \Omega_b(k_t) \end{cases} \tag{5.17}$$

and

$$l_t^b = \ell_b(k_t, \omega_t) := 1 - \ell_g(k_t, \omega_t),$$

respectively, where $\hat{l}_t^g \in [0, 1]$ is some arbitrary feasible labor share.[6] It is now straightforward to verify that the factor allocation

$$(k_t^g, k_t^b, l_t^g, l_t^b) = \big(\kappa_g(\omega_t), \kappa_b(\omega_t), \ell_g(k_t, \omega_t), \ell_b(k_t, \omega_t)\big) \tag{5.18}$$

with $\omega_t \in \big[\Omega_{\min}(k_t), \Omega_{\max}(k_t)\big]$ is *feasible* as it satisfies the market-clearing conditions (5.14). Since $\Omega_g(k_t^g) = \Omega_b(k_t^b)$ is also satisfied because of (5.1), the factor allocation (5.18) is *Pareto-efficient* in the sense that the output of a sector cannot be raised without lowering the output of the other sector. The per-capita outputs produced with the factor allocation (5.18) are given by the *output functions*

$$\mathsf{y}_j(k_t, \cdot) : \big[\Omega_{\min}(k_t), \Omega_{\max}(k_t)\big] \to \big[0, f_j(k_t)\big],$$

defined by

$$y_t^j = \mathsf{y}_j(k_t, \omega_t) := \ell_j(k_t, \omega_t) f_j(\kappa_j(\omega_t)), \quad j = g, b. \tag{5.19}$$

Note that, generically, the output functions (5.19) satisfy

$$\mathsf{y}_j(k_t, \Omega_i(k_t)) = 0 \quad \text{and} \quad \mathsf{y}_j(k_t, \Omega_j(k_t)) = f_j(k_t) \qquad i, j \in \{g, b\}, \ i \neq j,$$

so that boundary production plans that allocate all factors to one sector only are obtained for a wage-rental ratio ω_t at either boundary of the interval $\big[\Omega_{\min}(k_t), \Omega_{\max}(k_t)\big]$.

It follows from (5.10) that the factor allocation (5.18) is an equilibrium factor allocation if and only if

$$\big[1 - \epsilon_b \tau_t\big] w_b(\kappa_b(\omega_t)) = w_g(\kappa_g(\omega_t)) \quad \text{and} \quad \big[1 - \epsilon_b \tau_t\big] f_b'(\kappa_b(\omega_t)) = f_g'(\kappa_g(\omega_t)). \tag{5.20}$$

---

[6] In the non-generic case $\Omega_g(k_t) = \Omega_b(k_t) = \omega_t$, the sector-specific capital-labor ratios (5.16) coincide, so that $k_t^g = k_t^b = k_t$ and the market-clearing conditions (5.14) hold with *any* feasible allocation of labor. Later, we will then set $\hat{l}_t^g \in [0, 1]$ such that the economy is in a welfare maximum.

Define the function $\varrho : \mathbb{R}_{++} \to \mathbb{R}_{++}$ by setting

$$\varrho(\omega_t) := \frac{f_g'(\kappa_g(\omega_t))}{f_b'(\kappa_b(\omega_t))}, \tag{5.21}$$

then (5.20) holds if and only if the wage-rental ratio ω_t satisfies[7]

$$\varrho(\omega_t) = 1 - \epsilon_b \tau_t. \tag{5.22}$$

In Ritschel and Wenzelburger (2025), we show that $\varrho(\omega_t)$ is the *marginal rate of transformation* corresponding to the factor allocation (5.18). The concavity of the production-possibility frontier implies that, for each $k_t > 0$, ϱ is either monotonically increasing or decreasing on the interval $\big[\Omega_{\min}(k_t), \Omega_{\max}(k_t)\big]$ and satisfies $\varrho(\Omega_g(k_t)) \leq \varrho(\Omega_b(k_t))$. Thus, (5.22) is nothing but the standard condition for two-sector models stating that the marginal rate of transformation must coincide with the output-price ratio, which in our case is stipulated by the emissions tax rate τ_t since both sectors produce the same good. To account for the fact that a wage-rental ratio $\omega_t \in \big[\Omega_{\min}(k_t), \Omega_{\max}(k_t)\big]$ that satisfies (5.22) does *not* exist when the emissions tax rate is either too high or too low, we define the function $\Omega_{\text{eq}}(k_t, \cdot) : \mathbb{R} \to \big[\Omega_{\min}(k_t), \Omega_{\max}(k_t)\big]$ by setting

$$\omega_t = \Omega_{\text{eq}}(k_t, \tau_t) := \begin{cases} \Omega_g(k_t) & \text{if } 1 - \epsilon_b\tau_t \leq \varrho(\Omega_g(k_t)) \\ \varrho^{-1}(1 - \epsilon_b\tau_t) & \text{if } \varrho(\Omega_g(k_t)) < 1 - \epsilon_b\tau_t < \varrho(\Omega_b(k_t)) \\ \Omega_b(k_t) & \text{if } \varrho(\Omega_b(k_t)) \leq 1 - \epsilon_b\tau_t \end{cases}. \tag{5.23}$$

5.3.5 Feasible Fiscal Policies

We now turn to the question of which fiscal policies are feasible. Since, in a temporary equilibrium, factor prices are determined by the marginal products of the green sector, the feasibility condition (5.13) takes the form

$$-\big[1 - \delta + f_g'(\kappa_g(\omega_t))\big]k_t \;\leq\; \tfrac{a_t^2}{1+n} \;\leq\; w_g(\kappa_g(\omega_t)) + \tau_t\epsilon_b\mathsf{y}_b(k_t, \omega_t), \tag{5.24}$$

where $\omega_t = \Omega_{\text{eq}}(k_t, \tau_t)$. Assume now that the emissions tax rate τ_t is stipulated by a state-dependent, continuous *emissions-tax-policy rule* of the form

$$\tau(k_t, \cdot) : \mathbb{R}_+ \to \Big[\tfrac{1-\varrho(\Omega_b(k_t))}{\epsilon_b}, \tfrac{1-\varrho(\Omega_g(k_t))}{\epsilon_b}\Big], \qquad \tau_t = \tau(k_t, \chi_t). \tag{5.25}$$

By construction, emissions-tax-policy rules of the form (5.25) ensure that for each $(k_t, \chi_t) \in \mathbb{R}_+^2$, the *'output-price ratio'* $1 - \epsilon_b\tau(k_t, \chi_t)$ lies within the range $\big[\varrho(\Omega_g(k_t)), \varrho(\Omega_b(k_t))\big]$ of the marginal rate of transformation. Hence, there exists a unique wage-rental ratio $\omega_t \in \big[\Omega_{\min}(k_t), \Omega_{\max}(k_t)\big]$ satisfying (5.22), given by $\omega_t = \Omega_{\text{eq}}(k_t, \tau(k_t, \chi_t))$.

Assuming that the old-age transfer a_t^2 is determined by a continuous *transfer-policy rule*

$$a^2 : \mathbb{R}_+^2 \to \mathbb{R}, \qquad a_t^2 = a^2(k_t, \chi_t), \tag{5.26}$$

then a *feasible fiscal policy* may be formally defined as follows.

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[7] This follows from the very definition of the relative factor demand functions κ_g and κ_b in (5.1).

Definition 5.2 (Feasible fiscal policy).
An emissions-tax-policy rule of the form (5.25) together with a transfer-policy rule of the form (5.26) is called a feasible fiscal policy *if these satisfy (5.24) for all* $(k_t, \chi_t) \in \mathbb{R}^2_+$.

Under a feasible fiscal policy, transfers are chosen such that the disposable incomes of both generations are non-negative in all possible states of the economy.

5.3.6 EXISTENCE AND UNIQUENESS OF TEMPORARY EQUILIBRIA

Having introduced the notion of a feasible fiscal policy, we can now establish the existence and uniqueness of temporary equilibria.

Proposition 5.1 (Existence and uniqueness of temporary equilibria).
Let Assumption 5.1 be satisfied and the expectations $(a_t^{2,e}, R_t^e) \in \mathbb{R} \times \mathbb{R}_{++}$, *the state* $(k_t, \chi_t) \in \mathbb{R}^2_+$, *and the fiscal policy parameters* $\tau_t = \tau(k_t, \chi_t)$ *and* $a_t^2 = a^2(k_t, \chi_t)$ *be given, where* $\tau, a^2 : \mathbb{R}^2_+ \to \mathbb{R}$ *is some feasible fiscal policy. Then the allocation* $(k_t^g, k_t^b, l_t^g, l_t^b, c_t^1, c_t^2)$ *with corresponding prices* (w_t, r_t), *determined by (5.6), (5.10), (5.12), (5.18), (5.19), and (5.23), is a temporary equilibrium in the sense of Definition 5.1 if and only if* $w_t + a_t^1 + \frac{a_t^{2,e}}{R_t^e} \geq 0$. *In particular, it is the* only *temporary equilibrium whenever* $\Omega_g(k_t) \neq \Omega_b(k_t)$.[8]

Proposition 5.1 shows that the emissions tax rate τ_t stipulates the equilibrium wage-rental ratio $\omega_t = \Omega_{\text{eq}}(k_t, \tau_t)$ and thereby the factor allocation, factor prices, and production plan $(\mathsf{y}_b(k_t, \omega_t), \mathsf{y}_g(k_t, \omega_t))$. Since the two boundary productions plans $(0, f_g(k_t))$ and $(f_b(k_t), 0)$ obtain for sufficiently high and sufficiently low emissions tax rates, respectively, the government can implement any feasible production plan and thus fully control the pollution levels.

The well-known *Stolper-Samuelson theorem* takes the following form.

Lemma 5.2 (Equilibrium factor prices).
Let Assumption 5.1 be satisfied and $\omega_t = \Omega_{\text{eq}}(k_t, \tau_t)$ *be the equilibrium wage-rental ratio. Then for each* $\varrho(\Omega_g(k_t)) < 1 - \epsilon_b \tau_t < \varrho(\Omega_b(k_t))$, *the equilibrium factor prices satisfy*

$$\frac{\mathrm{d}}{\mathrm{d}\tau} w_g(\kappa_g(\omega_t)) \lesseqgtr 0 \quad \textit{and} \quad \frac{\mathrm{d}}{\mathrm{d}\tau} f_g'(\kappa_g(\omega_t)) \gtreqless 0 \qquad \Longleftrightarrow \qquad \kappa_g(\omega_t) \gtreqless \kappa_b(\omega_t).$$

Lemma 5.2 states that, depending on which of the two sectors is more capital-intensive, an increase in the emissions tax rate may raise or lower the equilibrium factor prices.

5.3.7 PERFECT-FORESIGHT ALLOCATIONS

A *perfect-foresight allocation* is a sequence of temporary equilibria generated by expectations that are correct at all times, that is, $R_t^e = R_{t+1}$ and $a_t^{2,e} = a_{t+1}^2$ for all $t \geq 0$. More formally, we can state the following definition.

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[8] In the non-generic case $\Omega_g(k_t) = \Omega_b(k_t)$, the labor shares are indeterminate.

Definition 5.3 (Perfect-foresight allocation).
Given the initial condition $(k_0, \chi_0) \in \mathbb{R}^2_+$ *and a feasible fiscal policy* $\tau, a^2 : \mathbb{R}^2_+ \to \mathbb{R}$*, a* perfect-foresight allocation *is a sequence* $\{(k_t, \chi_t, y^b_t, c^1_t, c^2_t)\}_{t=0}^{\infty}$ *with corresponding prices* $\{(w_t, R_t)\}_{t=0}^{\infty}$ *that satisfies*

$$\begin{aligned}
k_{t+1} &= \tfrac{1}{1+n} s(w_t + a^1_t, a^2_{t+1}, R_{t+1}), & \chi_{t+1} &= \tfrac{1}{1+n}\left[(1-\zeta)\chi_t + \epsilon_b y^b_t\right], & y^b_t &= \mathsf{y}_b(k_t, \omega_t),\\
\tau_t &= \tau(k_t, \chi_t), & a^2_t &= a^2(k_t, \chi_t), & a^1_t &= \tau_t \epsilon_b y^b_t - \tfrac{a^2_t}{1+n},\\
c^1_t &= w_t + a^1_t - (1+n)k_{t+1}, & c^2_t &= (1+n)R_t k_t + a^2_t, & &\\
w_t &= w_g(\kappa_g(\omega_t)), & R_t &= 1 - \delta + f'_g(\kappa_g(\omega_t)), & \omega_t &= \Omega_{\mathrm{eq}}(k_t, \tau_t),
\end{aligned}$$

for all times $t \geq 0$.

We next address the existence and uniqueness of perfect-foresight allocations. Given a feasible fiscal policy $\tau, a^2 : \mathbb{R}^2_+ \to \mathbb{R}$, let the state of the economy be $(k_t, \chi_t) \in \mathbb{R}^2_+$ and set $\omega_t = \Omega_{\mathrm{eq}}(k_t, \tau(k_t, \chi_t))$. Then the non-negative disposable income of a young agent is

$$w^d_t = w^d(k_t, \chi_t) := w_g(\kappa_g(\omega_t)) + \tau(k_t, \chi_t)\epsilon_b \mathsf{y}_b(k_t, \omega_t) - \tfrac{a^2(k_t, \chi_t)}{1+n},$$

and the pollution stock per capita of the subsequent period $t+1$ is

$$\chi_{t+1} = \vartheta(k_t, \chi_t) := E(\chi_t, \mathsf{y}_b(k_t, \omega_t)). \tag{5.27}$$

It follows that, under perfect foresight, the capital-labor ratio k_{t+1} must be a solution to

$$k_{t+1} \overset{!}{=} \tfrac{1}{1+n} s\big(w^d(k_t, \chi_t), a^2(k_{t+1}, \chi_{t+1}), 1 - \delta + f'_g(\kappa_g(\omega_{t+1}))\big), \tag{5.28}$$

where $\omega_{t+1} = \Omega_{\mathrm{eq}}(k_{t+1}, \tau(k_{t+1}, \chi_{t+1}))$. Condition (5.28) defines a *perfect forecasting rule* in the sense of Böhm and Wenzelburger (1999) that generates correct expectations along all possible growth paths of the economy.

Definition 5.4 (Perfect forecasting rule).
Given a feasible fiscal policy $\tau, a^2 : \mathbb{R}^2_+ \to \mathbb{R}$*, a* perfect forecasting rule *is a function*

$$\varphi : \mathbb{R}^2_+ \to \mathbb{R}_+, \qquad k_{t+1} = \varphi(k_t, \chi_t),$$

that satisfies (5.28) for all $(k_t, \chi_t) \in \mathbb{R}^2_+$.

The following theorem establishes the existence and uniqueness of a perfect forecasting rule.

Theorem 5.1 (Existence and uniqueness of the perfect forecasting rule).
Let Assumptions 5.1 and 5.2 be satisfied. Then the following holds true.

(i) For any given feasible fiscal policy $\tau, a^2 : \mathbb{R}^2_+ \to \mathbb{R}$*, there exists a perfect forecasting rule* φ *in the sense of Definition 5.4.*

(ii) Set $\omega = \Omega_{\mathrm{eq}}(k, \tau(k, \chi))$*. Then* φ *is uniquely determined if, in addition, youthful and old-age consumption are weak gross substitutes and, the tax-policy rules satisfy*

$$\frac{\partial a^2}{\partial k}(k, \chi) \geq 0 \quad \textit{and} \quad \frac{\partial \tau}{\partial k}(k, \chi) \begin{cases} \leq 0 & \textit{if } \kappa_g(\omega) \geq \kappa_b(\omega) \\ \geq 0 & \textit{if } \kappa_g(\omega) < \kappa_b(\omega) \end{cases}, \quad (k, \chi) \in \mathbb{R}^2_+. \tag{5.29}$$

By Theorem 5.1, a perfect forecasting rule exists. Its uniqueness requires additional restrictions on the preferences and the behavior of the tax-policy rules. In particular, the emissions-tax-policy rule $\tau(k,\chi)$ must be non-decreasing in the economy-wide capital-labor ratio k whenever the brown sector is more capital-intensive than the green sector. This property is conducive for fiscal policy, since brown output $\mathsf{y}_b(k,\omega)$ is increasing in k whenever the brown sector is more capital-intensive.

The perfect-foresight dynamics of the economy is governed by the perfect forecasting rule φ together with the map ϑ defined in (5.27), in the sense that given any feasible fiscal policy $\tau, a^2 : \mathbb{R}^2_+ \to \mathbb{R}$ and initial condition $(k_0,\chi_0) \in \mathbb{R}^2_+$, the *perfect-foresight growth paths* $\{(k_t,\chi_t)\}_{t=0}^{\infty}$ are recursively generated by the dynamical system

$$\begin{cases} k_{t+1} &= \varphi(k_t,\chi_t) \\ \chi_{t+1} &= \vartheta(k_t,\chi_t) \end{cases} . \tag{5.30}$$

Each perfect-foresight growth path $\{(k_t,\chi_t)\}_{t=0}^{\infty}$, in turn, defines a perfect-foresight allocation $\{(k_t,\chi_t,y_t^b,c_t^1,c_t^2)\}_{t=0}^{\infty}$ in the sense of Definition 5.3.

A steady state $(k_\star,\chi_\star) \in \mathbb{R}^2_+$ of the dynamical system (5.30) is determined by

$$\begin{aligned} k_\star &= \tfrac{1}{1+n} s\big(w^d(k_\star,\chi_\star), a^2(k_\star,\chi_\star), 1-\delta+f_g'(\kappa_g(\omega_\star))\big) \\ \chi_\star &= \tfrac{\epsilon_b}{n+\zeta}\, \mathsf{y}_b(k_\star,\omega_\star), \end{aligned}$$

where $\omega_\star = \Omega_{\text{eq}}(k_\star,\tau(k_\star,\chi_\star))$, and defines a stationary perfect-foresight allocation denoted by $\{(k_\star,\chi_\star,y_\star^b,c_\star^1,c_\star^2)\}$. The existence of such *perfect-foresight steady states* will be addressed below together with the implementability of the optimal allocation.

5.4 WELFARE ANALYSIS

This section derives the optimal allocation of the economy by taking the perspective of a benevolent social planner who maximizes the sum of the discounted welfare levels of all generations. In doing so, the social planner must take the intergenerational effects of the pollution externality into account and faces the challenge of reconciling economic prosperity with environmental sustainability.

Our welfare analysis requires an additional assumption on the technologies.

Assumption 5.3 (Technologies).
The marginal product of capital in sector $j=g,b$ satisfies

$$(i) \quad \lim_{k^j\to 0} f_j'(k^j) > \tfrac{1+n}{\gamma} - 1 + \delta \quad \textit{and} \quad (ii) \quad \lim_{k^j\to\infty} f_j'(k^j) = 0, \tag{5.31}$$

where $\gamma \in (0,1)$ is the social discount factor.

Condition (5.31) (ii) implies that any feasible growth path $\{(k_t,\chi_t)\}_{t=0}^{\infty}$ is bounded, so that sustained economic growth is ruled out. Condition (5.31) (i), which is weaker than the Inada condition $\lim_{k^j\to 0} f_j'(k^j) = \infty$ and thus allows for CES production functions, is needed to ensure the existence of a modified golden-rule steady state, see Section 5.6.

5.4.1 PRODUCTION-POSSIBILITY FRONTIER

In each period $t \geq 0$, the social planner chooses a production plan (y_t^b, y_t^g), a consumption plan (c_t^1, c_t^2), and a capital investment k_{t+1}. Observe that the planner will only implement Pareto-efficient production plans on the production-possibility frontier because if a production plan were inefficient, then the output in the green sector could always be raised and the social welfare be increased without lowering the output in the brown sector.

Following Ritschel and Wenzelburger (2025), a tractable parameterization of the production-possibility frontier that allows for factor-intensity reversals obtains as follows. Define the set

$$\mathcal{D} := \Big\{ (k, y^b) \in \mathbb{R}_+^2 \;\big|\; y^b \leq f_b(k) \Big\}$$

and the map $T : \mathcal{D} \to \mathbb{R}_+$ by setting

$$y_t^g = T(k_t, y_t^b) := \begin{cases} \mathsf{y}_g(k_t, \mathsf{y}_b^{-1}(k_t, y_t^b)) & \text{if } \Omega_g(k_t) \neq \Omega_b(k_t) \\ f_g(k_t) - \frac{f_g(k_t)}{f_b(k_t)} y_t^b & \text{if } \Omega_g(k_t) = \Omega_b(k_t) \end{cases}, \tag{5.32}$$

where y_g and y_b are the output functions introduced in (5.19).[9] The production-possibility frontier pertaining to any economy-wide capital-labor ratio $k_t \geq 0$ is then given by the curve

$$T(k_t, \cdot) : \big[0, f_b(k_t)\big] \to \big[0, f_g(k_t)\big], \qquad y^b \mapsto T(k_t, y^b). \tag{5.33}$$

As portrayed in Figure 5.1, each pair $(y_t^b, T(k_t, y_t^b))$ with $(k_t, y_t^b) \in \mathcal{D}$ is a Pareto-efficient production plan on the production-possibility frontier.

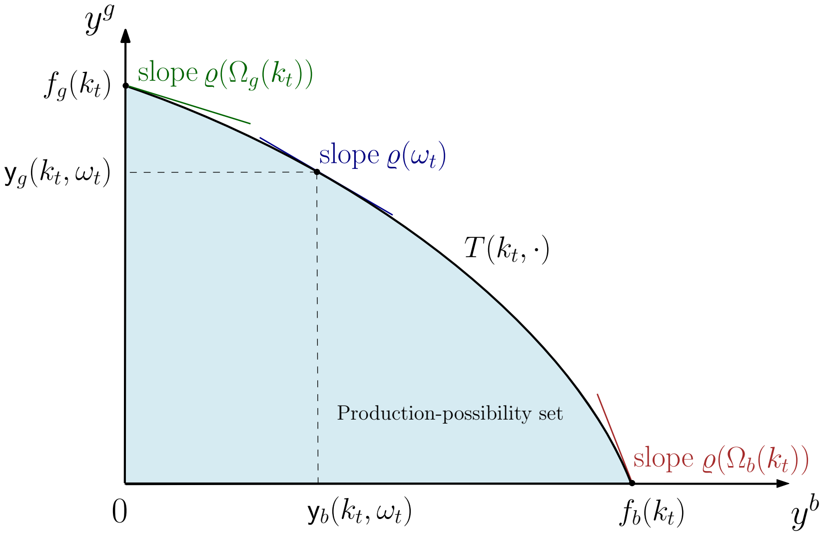

Figure 5.1. Strictly concave production-possibility frontier; $\Omega_g(k_t) \neq \Omega_b(k_t)$.

Our next lemma states properties of the map T that are essential for our analysis.

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[9] Assumption 5.1 implies that the output functions (5.19) are monotonic w.r.t. ω, so that for each $k_t \geq 0$ with $\Omega_g(k_t) \neq \Omega_b(k_t)$, the inverse functions $\mathsf{y}_j^{-1}(k_t, \cdot) : \big[0, f_j(k_t)\big] \to \big[\Omega_{\min}(k_t), \Omega_{\max}(k_t)\big]$, $j = g, b$ are well defined.

Lemma 5.3 (Concavity of T, Ritschel and Wenzelburger, 2025)**.**
Let Assumption 5.1 be satisfied. Then the map $T : \mathcal{D} \to \mathbb{R}_+$ *is concave and its partial derivatives are*

$$(i) \quad \frac{\partial T}{\partial y^b}(k_t, y_t^b) = -\varrho(\omega_t) \quad \textit{and} \quad (ii) \quad \frac{\partial T}{\partial k}(k_t, y_t^b) = f_g'(\kappa_g(\omega_t)),$$

where

$$\omega_t = \Omega(k_t, y_t^b) := \begin{cases} \mathsf{y}_b^{-1}(k_t, y_t^b) & \textit{if } \ \Omega_g(k_t) \neq \Omega_b(k_t) \\ \Omega_g(k_t) & \textit{if } \ \Omega_g(k_t) = \Omega_b(k_t) \end{cases}.$$

5.4.2 THE SOCIAL PLANNER'S PROBLEM

We next formalize the decision problem of the social planner. Adopting a utilitarian measure, the welfare of the generation born in period $t \geq 0$ is $U(c_t^1, c_{t+1}^2, z_t, z_{t+1})$ as defined in (5.4). The welfare of the initial old generation in $t = 0$ is given by $\beta[u(c_0^2) - \mu(z_0)] \in \mathbb{R}$. The *social welfare function* thus takes the form

$$W\big(\{(k_t, \chi_t, y_t^b, c_t^1, c_t^2)\}_{t=0}^{\infty}\big) := \sum_{t=0}^{\infty} \gamma^t g(k_t, \chi_t, y_t^b, c_t^1, c_t^2), \tag{5.34}$$

where $\gamma \in (0, 1)$ is the social discount factor and

$$g(k_t, \chi_t, y_t^b, c_t^1, c_t^2) := u(c_t^1) + \tfrac{\beta}{\gamma} u(c_t^2) - \big(1 + \tfrac{\beta}{\gamma}\big)\mu\big((1-\zeta)\chi_t + \epsilon_b y_t^b\big) \tag{5.35}$$

is the one-period return function.

The constraints faced by the social planner are the following. The gross total output per capita produced with the production plan $(y_t^b, T(k_t, y_t^b))$ is given by the *social production function*

$$f : \mathcal{D} \to \mathbb{R}_+, \qquad (k_t, y_t^b) \mapsto y_t^b + T(k_t, y_t^b) + (1-\delta)k_t. \tag{5.36}$$

Given an economy-wide capital-labor ratio $k_t \geq 0$, the *set of feasible policies* is

$$Q(k_t) := \Big\{(y^b, c^1, c^2) \in \mathbb{R}_+^3 \ \big| \ c^1 + \tfrac{c^2}{1+n} \leq f(k_t, y^b) \text{ and } y^b \leq f_b(k_t)\Big\}, \tag{5.37}$$

where $c^1 + \frac{c^2}{1+n}$ is total consumption per capita. The concavity of T established in Lemma 5.3 implies that f is concave, so that each set $Q(k_t)$ is compact and convex. It now follows from the resource constraint of the economy that the capital investment k_{t+1} is determined by the *capital accumulation law* $A(k_t, \cdot) : Q(k_t) \to \mathbb{R}_+$, defined by

$$k_{t+1} = A(k_t, y_t^b, c_t^1, c_t^2) := \tfrac{1}{1+n}\Big[f(k_t, y_t^b) - \big(c_t^1 + \tfrac{c_t^2}{1+n}\big)\Big]. \tag{5.38}$$

The pollution accumulation law E was already defined in (5.3). Given any initial condition $(k_0, \chi_0) \in \mathbb{R}_+^2$, the *set of feasible allocations* is therefore[10]

$$\Pi(k_0, \chi_0) := \Big\{\{(k_t, \chi_t, y_t^b, c_t^1, c_t^2)\}_{t=0}^{\infty} \ \Big| \ (y_t^b, c_t^1, c_t^2) \in Q(k_t), \\ k_{t+1} = A(k_t, y_t^b, c_t^1, c_t^2), \text{ and } \chi_{t+1} = E(\chi_t, y_t^b) \ \text{ for all } t \geq 0\Big\}.$$

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[10] More precisely, $\Pi(k_0, \chi_0)$ is the set of all feasible allocations with Pareto-efficient factor allocations in each period $t \geq 0$.

The social planner's decision problem now takes the form

$$\max_{\{(k_t,\chi_t,y_t^b,c_t^1,c_t^2)\}_{t=0}^{\infty}} W\big(\{(k_t,\chi_t,y_t^b,c_t^1,c_t^2)\}_{t=0}^{\infty}\big) \quad \text{s.t.} \quad \{(k_t,\chi_t,y_t^b,c_t^1,c_t^2)\}_{t=0}^{\infty} \in \Pi(k_0,\chi_0). \tag{5.39}$$

In Appendix E, we demonstrate that Problem (5.39) is well posed by establishing the corresponding value function and the Bellman equation, see Stokey and Lucas (1989) for details on dynamic programming.

5.4.3 OPTIMAL ALLOCATIONS

A solution $\{(k_t^*,\chi_t^*,y_t^{b*},c_t^{1*},c_t^{2*})\}_{t=0}^{\infty}$ to Problem (5.39) is referred to as an *optimal allocation.* To characterize the optimal allocation, define for each $t \geq 0$ the *marginal rate of substitution between consumption and pollution*

$$\psi_t^* := \sum_{i=0}^{\infty} \left[\tfrac{\gamma(1-\zeta)}{1+n}\right]^i \left[\frac{1}{u'(c_t^{1*})} + \tfrac{1}{1+n}\frac{1}{u'(c_t^{2*})}\right] \mu'\big((1+n)\chi_{t+1+i}^*\big), \tag{5.40}$$

which reflects the discounted marginal damages borne by all future generations. Setting

$$\omega_t^* := \Omega(k_t^*,y_t^{b*}) \quad \text{and} \quad r_t^* := \max\Big\{f_g'(\kappa_g(\omega_t^*)), \big[1-\epsilon_b\psi_t^*\big] f_b'(\kappa_b(\omega_t^*))\Big\}, \tag{5.41}$$

the following theorem establishes the existence and uniqueness of the optimal allocation and provides its characterization in terms of first-order conditions.

Theorem 5.2 (Existence and uniqueness of the optimal allocation)**.**
Let Assumptions 5.1 – 5.3 be satisfied. Then the following holds true.

(i) For any given initial condition $(k_0,\chi_0) \in \mathbb{R}_{++} \times \mathbb{R}_+$*, there exists a uniquely determined optimal allocation* $\{(k_t^*,\chi_t^*,y_t^{b*},c_t^{1*},c_t^{2*})\}_{t=0}^{\infty} \in \Pi(k_0,\chi_0)$*. The optimal allocation is generated by a triple of continuous policy functions* $y_*^b : \mathbb{R}_{++} \times \mathbb{R}_+ \to \mathbb{R}_+$ *and* $c_*^1, c_*^2 : \mathbb{R}_{++} \times \mathbb{R}_+ \to \mathbb{R}_{++}$ *in the sense that for each* $t \geq 0$*,*

$$k_{t+1}^* = A(k_t^*,y_t^{b*},c_t^{1*},c_t^{2*}) > 0 \tag{5.42}$$

$$\chi_{t+1}^* = E(\chi_t^*,y_t^{b*}) \geq 0 \tag{5.43}$$

$$(y_t^{b*},c_t^{1*},c_t^{2*}) = \big(y_*^b(k_t^*,\chi_t^*), c_*^1(k_t^*,\chi_t^*), c_*^2(k_t^*,\chi_t^*)\big) \in Q(k_t^*). \tag{5.44}$$

(ii) For each $t \geq 0$*, the optimal allocation satisfies the first-order conditions*

$$\frac{u'(c_t^{1*})}{\beta u'(c_t^{2*})} = \tfrac{1+n}{\gamma} \tag{5.45}$$

$$\frac{u'(c_t^{1*})}{\beta u'(c_{t+1}^{2*})} = 1-\delta+r_{t+1}^* \tag{5.46}$$

$$y_t^{b*} = \operatorname*{argmax}_{0\leq y^b\leq f_b(k_t^*)} \; T(k_t^*,y^b) + \big[1-\epsilon_b\psi_t^*\big]y^b. \tag{5.47}$$

The first-order conditions (5.45) – (5.47) carry a clear intuition. Equation (5.45) is a standard condition for OLG models determining the optimal allocation of consumption between any two coexisting generations. Since (5.46) prescribes the marginal rate of intertemporal substitution between youthful and old-age consumption, r_{t+1}^* may be interpreted as the *optimal capital-rental rate*. Equation (5.47) is a novel condition arising from the two-sector structure and the pollution externality. It states that the optimal production plan $(y_t^{b*}, T(k_t^*, y_t^{b*}))$ is the solution to an economy-wide revenue-maximization problem. In particular, it follows from (5.47) and Lemma 5.3 that whenever $0 < y_t^{b*} < f_b(k_t^*)$, the optimal production plan $(y_t^{b*}, T(k_t^*, y_t^{b*}))$ satisfies

$$\varrho(\Omega(k_t^*, y_t^{b*})) = 1 - \epsilon_b \psi_t^*.$$

Hence, $\omega_t^* = \Omega(k_t^*, y_t^{b*})$ is nothing but the *optimal wage-rental ratio*, while $1 - \epsilon_b\psi_t^*$ is the *optimal output-price ratio* that fully internalizes the brown sector's pollution externality. The latter observation follows from the following result.

Corollary 5.1 (Equilibrium production plan).
Under the hypotheses of Proposition 5.1, the brown output in a temporary equilibrium satisfies

$$\mathsf{y}_b(k_t, \Omega_{\text{eq}}(k_t, \tau_t)) = \operatorname*{argmax}_{0 \le y^b \le f_b(k_t)} \; T(k_t, y^b) + \big[1 - \epsilon_b \tau_t\big] y^b. \tag{5.48}$$

Unfortunately, closed-form solutions for the policy functions are not available. However, there are two special cases in which the first-order condition (5.47) allows for a more tractable characterization of the optimal production plan. Define the function

$$\psi(k, \chi, y^b) := \left[\frac{1}{u'(c_*^1(k,\chi))} + \frac{1}{1+n}\frac{1}{u'(c_*^2(k,\chi))}\right]\mu'\big((1+n)E(\chi, y^b)\big), \tag{5.49}$$

then these case are presented in the following corollary.

Corollary 5.2 (Full decay of pollution).
For each $(k_t^*, \chi_t^*) \in \mathbb{R}_{++} \times \mathbb{R}_+$, the optimal brown output level $y_t^{b*} = y_*^b(k_t^*, \chi_t^*)$ satisfies:

(i) If $\zeta = 1$, then

$$y_t^{b*} = \begin{cases} 0 & \textit{if} \quad 1 - \epsilon_b \psi(k_t^*, \chi_t^*, 0) \le \varrho(\Omega_g(k_t^*)) \\ f_b(k_t^*) & \textit{if} \quad \varrho(\Omega_b(k_t^*)) \le 1 - \epsilon_b \psi(k_t^*, \chi_t^*, f_b(k_t^*)) \\ \textit{solves} & \varrho(\Omega(k_t^*, y_t^{b*})) = 1 - \epsilon_b \psi(k_t^*, \chi_t^*, y_t^{b*}) \quad \textit{otherwise.} \end{cases}$$

(ii) If $\mu' \equiv 0$, then $y_t^{b*} = \operatorname*{argmax}_{0 \le y^b \le f_b(k_t^*)} \; f(k_t^*, y^b).$

Corollary 5.2 (i) characterizes the optimal production plan $(y_t^{b*}, T(k_t^*, y_t^{b*}))$ in case the pollution stock decays fully between any two periods, so that only the instantaneous emissions $\epsilon_b y_t^{b*}$ affect the welfare of the two generations living in period t. The optimal brown output level y_t^{b*} then depends on how the marginal rate of transformation $\varrho(\Omega(k_t^*, y_t^{b*}))$ relates to the marginal rate of substitution $\psi(k_t^*, \chi_t^*, y_t^{b*})$. Corollary 5.2 (ii), on the other hand, implies that if pollution plays no role, then the optimal production plan $(y_t^{b*}, T(k_t^*, y_t^{b*}))$ maximizes the total output of the economy, given the capital-labor ratio k_t^*.

5.5 IMPLEMENTATION

We now investigate the extent to which fiscal policy can implement the optimal allocation of the social planner in a market economy. To this end, we first present a more general result, which is adapted from De La Croix and Michel (2002).

Theorem 5.3 (Decentralization of feasible allocations)**.**
Let Assumptions 5.1 – 5.3 be satisfied and $\{(\hat{k}_t, \hat{\chi}_t, \hat{y}_t^b, \hat{c}_t^1, \hat{c}_t^2)\}_{t=0}^{\infty} \in \Pi(\hat{k}_0, \hat{\chi}_0)$ *be a feasible allocation with corresponding* $\hat{\omega}_t := \Omega(\hat{k}_t, \hat{y}_t^b)$ *that satisfies*

$$\frac{u'(\hat{c}_{t-1}^1)}{\beta u'(\hat{c}_t^2)} = 1 - \delta + f_g'(\kappa_g(\hat{\omega}_t)) \tag{5.50}$$

for all $t \geq 1$. *Then there exists a sequence of fiscal policy parameters* $\{(\hat{\tau}_t, \hat{a}_t^1, \hat{a}_t^2)\}_{t=0}^{\infty}$ *such that the allocation* $\{(\hat{k}_t, \hat{\chi}_t, \hat{y}_t^b, \hat{c}_t^1, \hat{c}_t^2)\}_{t=0}^{\infty}$ *together with prices* $\{(\hat{w}_t, \hat{R}_t)\}_{t=0}^{\infty}$, *determined by*

$$\hat{w}_t = w_g(\kappa_g(\hat{\omega}_t)) \quad \text{and} \quad \hat{R}_t = 1 - \delta + f_g'(\kappa_g(\hat{\omega}_t)),$$

is a perfect-foresight allocation in the sense of Definition 5.3. For each $t \geq 0$, *the sequence* $\{(\hat{\tau}_t, \hat{a}_t^1, \hat{a}_t^2)\}_{t=0}^{\infty}$ *satisfies the feasibility condition*

$$-\big[1 - \delta + f_g'(\kappa_g(\hat{\omega}_t))\big]\hat{k}_t \leq \tfrac{\hat{a}_t^2}{1+n} \leq w_g(\kappa_g(\hat{\omega}_t)) + \hat{\tau}_t \epsilon_b \hat{y}_t^b. \tag{5.51}$$

Theorem 5.3 is, in essence, a consequence of the second welfare theorem. Condition (5.50) is the optimality condition for an individual savings decision of a young agent who correctly anticipates the gross return on savings $1-\delta+f_g'(\kappa_g(\hat{\omega}_{t+1}))$ and, therefore, ensures compatibility with perfect foresight.

In view of the social planner's first-order condition (5.46), we may now state the following corollary to Theorem 5.3.

Corollary 5.3 (Decentralization of optimal allocations)**.**
Let Assumptions 5.1 – 5.3 be satisfied and $\{(k_t^*, \chi_t^*, y_t^{b*}, c_t^{1*}, c_t^{2*})\}_{t=0}^{\infty} \in \Pi(k_0, \chi_0)$ *be an optimal allocation that satisfies* $0 \leq y_t^{b*} < f_b(k_t^*)$ *for all* $t \geq 0$. *Then there exists a sequence of fiscal policy parameters* $\{(\tau_t^*, a_t^{1*}, a_t^{2*})\}_{t=0}^{\infty}$ *such that* $\{(k_t^*, \chi_t^*, y_t^{b*}, c_t^{1*}, c_t^{2*})\}_{t=0}^{\infty}$ *is a perfect-foresight allocation in the sense of Definition 5.3. The sequence* $\{(\tau_t^*, a_t^{1*}, a_t^{2*})\}_{t=0}^{\infty}$ *is generated by a feasible fiscal policy* $\tau_*, a_*^2 : \mathbb{R}_{++} \times \mathbb{R}_+ \to \mathbb{R}$ *in the sense of Definition 5.2, meaning that for each* $t \geq 0$,

$$\tau_t^* = \tau_*(k_t^*, \chi_t^*), \quad a_t^{2*} = a_*^2(k_t^*, \chi_t^*), \quad \text{and} \quad a_t^{1*} = \tau_t^* \epsilon_b y_t^{b*} - \tfrac{a_t^{2*}}{1+n}.$$

If $0 < y_t^{b*} < f_b(k_t^*)$, *then the optimal emissions tax rate satisfies*

$$\tau_t^* = \psi_t^* = \sum_{i=0}^{\infty} \left[\tfrac{\gamma(1-\zeta)}{1+n}\right]^i \left[\frac{1}{u'(c_t^{1*})} + \tfrac{1}{1+n}\frac{1}{u'(c_t^{2*})}\right] \mu'\big((1+n)\chi_{t+1+i}^*\big).$$

Otherwise, if $y_t^{b*} = 0$, *then* $\tau_t^* \leq \psi_t^*$.

Corollary 5.3 states that there exists an *optimal* (feasible) fiscal policy $\tau_*, a_*^2 : \mathbb{R}_{++} \times \mathbb{R}_+ \to \mathbb{R}$ that decentralizes the optimal allocation in a market economy with perfect foresight. The purpose of the emissions tax rate τ_t^* is to implement the optimal production plan $(y_t^{b*}, T(k_t^*, y_t^{b*}))$. To accomplish this, it must implement the optimal wage-rental ratio $\omega_t^* = \Omega(k_t^*, y_t^{b*})$ as an equilibrium wage-rental ratio by stipulating the output-price ratio $\varrho(\omega_t^*)$. Thus, the emissions-tax-policy rule τ_* is formally defined by

$$\varrho\big(\Omega(k, y_*^b(k,\chi))\big) = 1 - \epsilon_b \tau_*(k,\chi), \quad (k,\chi) \in \mathbb{R}_{++} \times \mathbb{R}_+. \tag{5.52}$$

Since $\Omega(k, y_*^b(k,\chi)) \in \big[\Omega_{\min}(k), \Omega_{\max}(k)\big]$, the function τ_* obeys the form (5.25) and, therefore, generates equal factor prices in both sectors. Notably, if production takes place in both sectors, then the optimal emissions tax rate τ_t^* equals the marginal rate of substitution between consumption and pollution ψ_t^*. Since $\tau_t^* \geq 0$ whenever $y_t^{b*} > 0$, it is never optimal to subsidize the brown sector.

The optimal transfer system (a_t^{1*}, a_t^{2*}) induces disposable incomes of the two generations such that, under perfect foresight, a young agent decides to save the optimal amount $(1+n)k_{t+1}^*$. As a consequence, the economy attains an optimal capital accumulation path. Formally, if φ_* is the perfect forecasting rule pertaining to the *optimal* fiscal policy, then

$$\varphi_*(k,\chi) = A\big(k, y_*^b(k,\chi), c_*^1(k,\chi), c_*^2(k,\chi)\big), \quad (k,\chi) \in \mathbb{R}_{++} \times \mathbb{R}_+.$$

Remark 5.1 (Implementation of optimal allocations).
The implementation result in Corollary 5.3 can be extended to optimal allocations that include brown production plans, $y_t^{b*} = f_b(k_t^*)$, in which case the optimal emissions tax rate is always $\tau_t^* = \psi_t^*$. However, since generically, $\varrho(\Omega_b(k_t^*)) < 1 - \epsilon_b \psi_t^*$ when $y_t^{b*} = f_b(k_t^*)$, this implementation requires a more general emissions-tax-policy rule than (5.25). In particular, Definition 5.1 must be adapted so that the factor-price-equalization condition (5.10) may hold with inequality when only the brown sector is active.

5.6 THE MODIFIED GOLDEN RULE

A stationary allocation $\{(\bar{k}_\gamma, \bar{\chi}_\gamma, \bar{y}_\gamma^b, \bar{c}_\gamma^1, \bar{c}_\gamma^2)\}$ that satisfies the optimality conditions in Theorem 5.2 is referred to as a *modified golden-rule steady state* (MGRSS). The pollution accumulation law (5.43) implies that the steady-state pollution stock per capita is

$$\bar{\chi}_\gamma = \tfrac{\epsilon_b}{n+\zeta}\, \bar{y}_\gamma^b. \tag{5.53}$$

Moreover, the resource constraint of the economy implies that total consumption per capita in any stationary feasible allocation is determined by the function

$$\phi : \mathcal{D} \to \mathbb{R}, \qquad (k, y^b) \mapsto f(k, y^b) - (1+n)k.$$

It now follows from the capital accumulation law (5.42) and the first-order condition (5.45) that given any pair $(\bar{k}_\gamma, \bar{y}_\gamma^b) \in \mathcal{D}$ with $\phi(\bar{k}_\gamma, \bar{y}_\gamma^b) > 0$, the steady-state consumption plan

$$\bar{c}_\gamma^1 = c_*^1\big(\bar{k}_\gamma, \tfrac{\epsilon_b}{n+\zeta}\, \bar{y}_\gamma^b\big) \quad \text{and} \quad \bar{c}_\gamma^2 = c_*^2\big(\bar{k}_\gamma, \tfrac{\epsilon_b}{n+\zeta}\, \bar{y}_\gamma^b\big) \tag{5.54}$$

is uniquely determined by

$$\bar{c}^1_\gamma + \tfrac{\bar{c}^2_\gamma}{1+n} = \phi(\bar{k}_\gamma, \bar{y}^b_\gamma) \quad \text{and} \quad \frac{u'(\bar{c}^1_\gamma)}{\beta u'(\bar{c}^2_\gamma)} = \tfrac{1+n}{\gamma}. \tag{5.55}$$

This shows that any MGRSS $\{(\bar{k}_\gamma, \bar{\chi}_\gamma, \bar{y}^b_\gamma, \bar{c}^1_\gamma, \bar{c}^2_\gamma)\}$ is determined by the pair $(\bar{k}_\gamma, \bar{y}^b_\gamma)$. Since the output function y_b defined in (5.19) is invertible, we have $\bar{y}^b_\gamma = \mathsf{y}_b(\bar{k}_\gamma, \bar{\omega}_\gamma)$ and can conclude that any MGRSS is determined by an economy-wide capital-labor ratio $\bar{k}_\gamma$ together with a wage-rental ratio $\bar{\omega}_\gamma$.

The central question now is how $(\bar{k}_\gamma, \bar{\omega}_\gamma)$ must be chosen such that the social welfare attains its maximum. For each sector $j = g, b$, define the capital-labor ratio $\bar{k}^j > 0$ by

$$f_j'(\bar{k}^j) = \tfrac{1+n}{\gamma} - 1 + \delta. \tag{5.56}$$

Denote the corresponding wage-rental ratios by $\bar{\omega}^j := \Omega_j(\bar{k}^j)$. To avoid a tedious non-generic case, assume that $\kappa_g(\bar{\omega}^g) \neq \kappa_b(\bar{\omega}^g)$. The wage-rental ratio $\bar{\omega}^g$ then defines a non-empty interval $\big[\bar{k}_{\min}, \bar{k}_{\max}\big]$, where

$$\bar{k}_{\min} := \min\big\{\kappa_g(\bar{\omega}^g), \kappa_b(\bar{\omega}^g)\big\} \quad \text{and} \quad \bar{k}_{\max} := \max\big\{\kappa_g(\bar{\omega}^g), \kappa_b(\bar{\omega}^g)\big\}.$$

It follows from (5.40) that the marginal rate of substitution between steady-state consumption and pollution is given by the function[11]

$$\bar{\psi}(k, \omega) := \frac{1}{1 - \frac{\gamma(1-\zeta)}{1+n}} \, \psi\big(k, \tfrac{\epsilon_b}{n+\zeta} \, \mathsf{y}_b(k, \omega), \mathsf{y}_b(k, \omega)\big), \tag{5.57}$$

where the map ψ was defined in (5.49). The following theorem now establishes the existence of modified golden-rule steady states.

Theorem 5.4 (Existence of MGRSS)**.**
Let Assumptions 5.1 – 5.3 be satisfied. Then there exists at least one modified golden-rule steady state $\{(\bar{k}_\gamma, \bar{\chi}_\gamma, \bar{y}^b_\gamma, \bar{c}^1_\gamma, \bar{c}^2_\gamma)\}$, which is determined by the pair $(\bar{k}_\gamma, \bar{\omega}_\gamma)$.

(i) If $1 - \epsilon_b \bar{\psi}(\kappa_g(\bar{\omega}^g), \bar{\omega}^g) \leq \varrho(\bar{\omega}^g)$, then $(\bar{k}^g, \bar{\omega}^g)$ is a MGRSS. Since $\bar{y}^b_\gamma = \mathsf{y}_b(\bar{k}^g, \bar{\omega}^g) = 0$, this steady state is called 'green'.

(ii) If $\varrho(\bar{\omega}^g) \leq 1 - \epsilon_b \bar{\psi}(\kappa_b(\bar{\omega}^g), \bar{\omega}^g)$, then $(\bar{k}_\gamma, \Omega_b(\bar{k}_\gamma))$ is a MGRSS, where the capital-labor ratio $\bar{k}_\gamma \in \big[\kappa_b(\bar{\omega}^g), \bar{k}^b\big)$ is determined by

$$f_b'(\bar{k}_\gamma)\big[1 - \epsilon_b \bar{\psi}(\bar{k}_\gamma, \Omega_b(\bar{k}_\gamma))\big] = \tfrac{1+n}{\gamma} - 1 + \delta. \tag{5.58}$$

Since $\bar{y}^b_\gamma = \mathsf{y}_b(\bar{k}_\gamma, \Omega_b(\bar{k}_\gamma)) = f_b(\bar{k}_\gamma)$, this steady state is called 'brown'.

(iii) If $1 - \epsilon_b \bar{\psi}(\kappa_b(\bar{\omega}^g), \bar{\omega}^g) < \varrho(\bar{\omega}^g) < 1 - \epsilon_b \bar{\psi}(\kappa_g(\bar{\omega}^g), \bar{\omega}^g)$, then $(\bar{k}_\gamma, \bar{\omega}^g)$ is a MGRSS, where the capital-labor ratio $\bar{k}_\gamma \in \big(\bar{k}_{\min}, \bar{k}_{\max}\big)$ is determined by

$$\varrho(\bar{\omega}^g) = 1 - \epsilon_b \bar{\psi}(\bar{k}_\gamma, \bar{\omega}^g). \tag{5.59}$$

Since $\bar{y}^b_\gamma = \mathsf{y}_b(\bar{k}_\gamma, \bar{\omega}^g) \in \big(0, f_b(\bar{k}_\gamma)\big)$, this steady state is called 'mixed'.

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[11] Observe that $k \mapsto \bar{\psi}(k, \bar{\omega}^g)$ is only well defined on $\big[\bar{k}_{\min}, \bar{k}_{\max}\big]$.

(iv) If $1 - \epsilon_b \bar{\psi}(\kappa_g(\bar{\omega}^g), \bar{\omega}^g) < \varrho(\bar{\omega}^g) < 1 - \epsilon_b \bar{\psi}(\kappa_b(\bar{\omega}^g), \bar{\omega}^g)$*, then there exist at least three distinct MGRSS: a green one in the sense of (i), a brown one in the sense of (ii), and a mixed one in the sense of (iii).*

Which type of MGRSS exists depends on the technologies, the discount factors β and γ, and agents' preferences, especially their evaluation of the externality via the damage function μ. The resulting trade-off is described by the relationship between the marginal rate of transformation ϱ and the marginal rate of substitution $\bar{\psi}$. A complete decarbonization of the economy requires that $1 - \epsilon_b \bar{\psi}(\kappa_g(\bar{\omega}^g), \bar{\omega}^g) \leq \varrho(\bar{\omega}^g)$ because the green technology must be sufficiently productive. In particular, since $\bar{\psi} \geq 0$, there exists a uniquely determined *green* MGRSS whenever $1 \leq \varrho(\bar{\omega}^g)$, that is, if the green technology is more productive than the brown technology.

Remark 5.2 (MGRSS without pollution externalities)**.**
The characterization of the MGRSS of a model without pollution externalities is included in Theorem 5.4 by the case $\bar{\psi} \equiv 0$*. In this case, it follows from Theorem 5.4 that:*

(i) If $1 \leq \varrho(\bar{\omega}^g)$*, then* $(\bar{k}^g, \bar{\omega}^g)$ *is a MGRSS.*

(ii) If $\varrho(\bar{\omega}^g) \leq 1$*, then* $(\bar{k}^b, \bar{\omega}^b)$ *is a MGRSS.*

Thus, if pollution is irrelevant, then, generically, only the more productive sector is active.

The intuition of Theorem 5.4 relates to the well-known Rybczynski theorem, see Rybczynski (1955). In a MGRSS, the capital-labor ratio of the green sector must always be $\bar{k}^g$. As a consequence, the factor prices in any *mixed* MGRSS are determined by the marginal products $w_g(\bar{k}^g)$ and $f_g'(\bar{k}^g)$, so that the wage-rental ratio becomes $\Omega_g(\bar{k}^g) = \bar{\omega}^g$. Since, by Lemma 5.3, the marginal rate of transformation depends solely on the wage-rental ratio, it is fixed at $\varrho(\bar{\omega}^g)$, independently of the economy-wide capital-labor ratio. The marginal rate of transformation, in turn, fixes the output-price ratio, so that the Rybczynski theorem applies. In Ritschel and Wenzelburger (2025), we show that in the (y^b, y^g)-plane, all Pareto-efficient production plans with the same marginal rate of transformation $\varrho(\bar{\omega}^g)$ are located on the straight line

$$y^g = f_g(\kappa_g(\bar{\omega}^g)) - \frac{f_g(\kappa_g(\bar{\omega}^g))}{f_b(\kappa_b(\bar{\omega}^g))} y^b, \qquad y^b \in \big[0, f_b(\kappa_b(\bar{\omega}^g))\big]. \tag{5.60}$$

In the example portrayed in Figure 5.2, this is the dashed red line. The details of the parameterization are presented in Section 5.8. The output-substitution rate along the line (5.60) is $\frac{f_g(\kappa_g(\bar{\omega}^g))}{f_b(\kappa_b(\bar{\omega}^g))}$. All production-possibility frontiers $T(\bar{k}, \cdot)$, where $\bar{k} \in \big[\bar{k}_{\min}, \bar{k}_{\max}\big]$, intersect the line (5.60).

The existence of a mixed MGRSS depends on the existence of an economy-wide capital-labor ratio $\bar{k}_\gamma \in \big(\bar{k}_{\min}, \bar{k}_{\max}\big)$ that satisfies (5.59). The marginal rates of substitution $\bar{\psi}(\kappa_g(\bar{\omega}^g), \bar{\omega}^g)$ and $\bar{\psi}(\kappa_b(\bar{\omega}^g), \bar{\omega}^g)$ at the two end points of the line (5.60) are essential. Theorem 5.4 states that if

$$1 - \epsilon_b \bar{\psi}(\kappa_b(\bar{\omega}^g), \bar{\omega}^g) < \varrho(\bar{\omega}^g) < 1 - \epsilon_b \bar{\psi}(\kappa_g(\bar{\omega}^g), \bar{\omega}^g),$$

then there coexist at least three MGRSS with distinct consumption and pollution levels. In contrast to the classical OLG framework by Diamond (1965), our model thus allows for multiple MGRSS. Whether a steady state can be implemented will depend on its stability properties and the initial conditions of the economy.

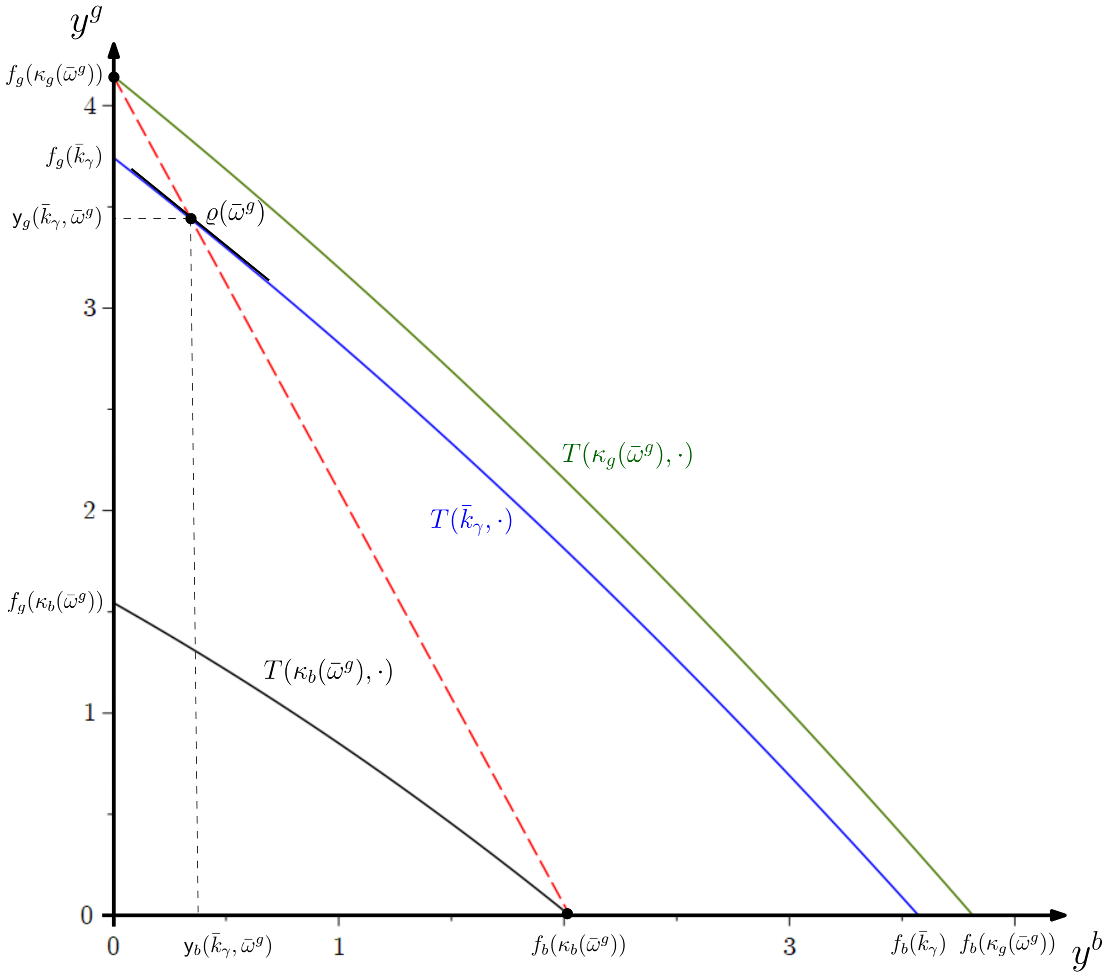

Figure 5.2. Existence of a mixed MGRSS; Parameter Set A.

An immediate implication of Corollary 5.3 is that the government may implement both green and mixed modified golden-rule steady states as perfect-foresight steady states. It follows from (5.52) and Theorem 5.4 that the optimal steady-state emissions tax rate is

$$\bar{\tau}_\gamma = \tau_*(\bar{k}_\gamma, \bar{\chi}_\gamma) = \frac{1 - \varrho(\bar{\varpi}^g)}{\epsilon_b}$$

and thus solely depends on the technologies of the two sectors and the pollution intensity ϵ_b. Surprisingly, it is independent of agents' preferences. This feature arises because the optimal steady-state output-price ratio is already fixed by the marginal rate of transformation $\varrho(\bar{\varpi}^g)$, which is independent of the economy-wide capital-labor ratio.[12] However, we can infer from (5.59) that whenever the steady state is *mixed*, $0 < \bar{y}^b_\gamma < f_b(\bar{k}_\gamma)$, then $\bar{\tau}_\gamma$ coincides with the

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[12] Notice that $\bar{\tau}_\gamma$ is *decreasing* in ϵ_b as the government taxes emissions rather than the brown output.

marginal rate of substitution between steady-state consumption and pollution because[13]

$$\bar{\tau}_\gamma = \frac{1-\varrho(\bar{\omega}^g)}{\epsilon_b} = \bar{\psi}(\bar{k}_\gamma, \bar{\omega}^g).$$

In the special case of a constant marginal damage of pollution, Theorem 5.2 allows to construct a simple difference equation that governs the dynamics of the optimal wage-rental ratios $\omega_t^* = \Omega(k_t^*, y_t^{b*})$ in a local neighborhood of a mixed MGRSS. Since the wage-rental ratio parameterizes the factor allocation, factor prices, and production plans, the stability of this dynamics is pivotal.

Proposition 5.2 (Stability of mixed MGRSS).
Let Assumptions 5.1 – 5.3 be satisfied, the damage function μ be linear, and assume there exists a mixed MGRSS in the sense of Theorem 5.4 (iii). Then, in a local neighborhood of $\bar{\omega}^g$, the sequence of optimal wage-rental ratios $\{\omega_t^*\}_{t=0}^\infty$ satisfies the difference equation

$$\frac{1-\varrho(\omega_{t+1}^*)}{1-\varrho(\omega_t^*)} = \tfrac{\gamma}{1+n}\Big[1-\delta+f_g'(\kappa_g(\omega_{t+1}^*))\Big]. \tag{5.61}$$

The steady state $\bar{\omega}^g$ is asymptotically stable if

$$\kappa_g(\bar{\omega}^g) - \kappa_b(\bar{\omega}^g) < \tfrac{1}{2}\tfrac{\gamma}{1+n}[1-\varrho(\bar{\omega}^g)]f_b(\kappa_b(\bar{\omega}^g)) \tag{5.62}$$

and unstable otherwise. In particular, $\bar{\omega}^g$ is asymptotically stable if $\kappa_b(\bar{\omega}^g) > \kappa_g(\bar{\omega}^g)$.

Proposition 5.2 suggests that the stability properties of mixed MGRSS depend on the sector-specific capital-labor ratios. If the brown sector is more capital-intensive than the green sector, $\kappa_b(\bar{\omega}^g) > \kappa_g(\bar{\omega}^g)$, then the dynamics (5.61) is stable. However, if the green sector is more capital-intensive than the brown sector, $\kappa_g(\bar{\omega}^g) > \kappa_b(\bar{\omega}^g)$, then instability may arise. In Section 5.8, we provide a tractable example with a standard parameterization from the literature on OLG models. A comprehensive stability analysis is beyond the scope of this article and remains an open question for future research.

5.7 CONCLUSION

This article extended the classical overlapping-generations model of Diamond (1965) by incorporating a polluting production sector and environmental preferences of agents. The result is a tractable two-sector growth model that provides a framework for studying the role of fiscal policy in steering the green transformation of an economy. Our assumptions are very general and allow for factor-intensity reversals, which may arise naturally in the course of the green transformation.

The analysis demonstrates that the implemented degree of pollution reduction depends on the technologies, discount factors, and agents' subjective evaluation of environmental damages. The optimal allocation, including modified golden-rule steady states, can be decentralized through an emissions tax in combination with intergenerational transfers. These transfers

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[13] Otherwise, if the steady state is green, $\bar{y}_\gamma^b = 0$, then $\bar{\tau}_\gamma \le \psi(\bar{k}_\gamma, \bar{\omega}^g)$.

allow the government to control the intergenerational income distribution and thus serve a dual purpose. First, they act as climate dividends that compensate agents for the decline in factor incomes caused by the emissions tax. Such climate dividends have become a central element in current public debates on how to deal with climate change. Second, the transfers serve to implement efficient savings and thereby prevent both over- and under-accumulation of capital. The ability of the government to implement the optimal allocation while maintaining a zero primary deficit highlights that, in theory, effective climate policy need not rely on public borrowing. Since the optimal allocation is derived from the perspective of a social planner who takes agents' preferences into account, this notion of *optimality* need not comply with the objectives of climate science. However, we have shown that fiscal policy can implement a wide range of feasible allocations, including those proposed by climate scientists.

In contrast to the classical OLG framework, our model allows for multiple modified golden-rule steady states with distinct consumption and pollution levels and may therefore help explain the coexistence of high and low-carbon economies. This finding further suggests that optimal fiscal policies may give rise to complex dynamics. An investigation of the qualitative dynamics and the stability properties of modified golden-rule steady states is outside the scope of this article and, in view of implementing optimal allocations, should be addressed by future research. Our results so far indicate that the stability of mixed steady states hinges on the relative capital intensities of the two sectors. Instability may emerge if the green sector is too capital-intensive. Since agents in our model are homogeneous, another interesting avenue for further research is to introduce wealth and income heterogeneity. This would allow for an analysis of the extent to which wealthier individuals contribute disproportionately to climate change and how compensation for carbon pricing can be designed in a socially just manner. The assumption of additive separable consumption and environmental preferences is restrictive, as the utility of consumption may depend on environmental quality. Addressing this interdependence by adopting a more general class of preferences could yield interesting insights. Finally, it is worth emphasizing that our model may be readily adapted to incorporate carbon capture technologies. Given the positive intergenerational externality associated with such technologies, analyzing their optimal subsidization could provide valuable insights into effective climate change mitigation.

5.8 A STANDARD EXAMPLE

The following example demonstrates that, in general, uniqueness and stability of a mixed MGRSS will depend on the relative capital intensities.

Consider the logarithmic utility function $u(c) = \ln(c)$ together with the linear damage function $\mu(z) = dz$, where the constant $d > 0$ is the marginal damage of pollution. It follows from (5.55) that the steady-state consumption plan is

$$\bar{c}^1_\gamma = \frac{\phi(\bar{k}_\gamma, \bar{y}^b_\gamma)}{1+\frac{\beta}{\gamma}} \quad \text{and} \quad \bar{c}^2_\gamma = (1+n)\frac{\beta}{\gamma}\frac{\phi(\bar{k}_\gamma, \bar{y}^b_\gamma)}{1+\frac{\gamma}{\beta}},$$

where the map

$$\phi(k, \mathsf{y}_b(k,\omega)) = \mathsf{y}_g(k,\omega) + \mathsf{y}_b(k,\omega) - (n+\delta)k$$

stipulates the stationary total consumption per capita. Using (5.49), the steady-state marginal rate of substitution $\bar{\psi}$ defined in (5.57) takes the form

$$\bar{\psi}(k,\omega) = \frac{d}{1-\frac{\gamma(1-\zeta)}{1+n}}\,\phi(k,\mathsf{y}_b(k,\omega)). \tag{5.63}$$

This observation leads to the following result.

Lemma 5.4 (Uniqueness of MGRSS).
Let Assumptions 5.1 – 5.3 be satisfied, the utility function u be logarithmic, and the damage function μ be linear. Then there exists at most *one green, one brown, and one mixed MGRSS. If, in addition, $\kappa_b(\bar{\omega}^g) > \kappa_g(\bar{\omega}^g)$, then there exists a unique MGRSS.*

By Lemma 5.4, the coexistence of multiple MGRSS in this example requires that the green sector is more capital-intensive than the brown sector, i.e., $\kappa_g(\bar{\omega}^g) > \kappa_b(\bar{\omega}^g)$. Consider the special case with Cobb-Douglas production functions

$$f_g(k) = A_g\,k^{\alpha_g} \quad \text{and} \quad f_b(k) = A_b\,k^{\alpha_b},$$

where $A_g, A_b > 0$ and $0 < \alpha_g, \alpha_b < 1$. The marginal rate of transformation (5.21) is

$$\varrho(\omega) = \left(\frac{\alpha_b^{\alpha_b}(1-\alpha_b)^{1-\alpha_b}A_b}{\alpha_g^{\alpha_g}(1-\alpha_g)^{1-\alpha_g}A_g}\right)\omega^{\alpha_b-\alpha_g}.$$

For each sector $j = g, b$,

$$\kappa_j(\omega) = \tfrac{\alpha_j}{1-\alpha_j}\,\omega, \quad \bar{k}^j = \left(\frac{A_j\alpha_j}{\frac{1+n}{\gamma}-1+\delta}\right)^{\frac{1}{1-\alpha_j}}, \quad \text{and} \quad \bar{\omega}^j = \tfrac{1-\alpha_j}{\alpha_j}\,\bar{k}^j.$$

The green sector is always more capital-intensive than the brown sector if $\alpha_g > \alpha_b$, while the converse holds true if $\alpha_b > \alpha_g$.

Parameter set	ϵ_b	ζ	d	γ	β	n	δ	A_b	A_g	α_b	α_g
Set A	1	0.9	0.04	0.5	1	0	1	3	2.85	0.5	0.78
Set B	1	0.9	0.04	0.5	1	0	1	5	4.2	0.5	0.45

Table 5.1. Parameter sets for the example.

For Parameter Set A in Table 5.1, we obtain

$$\bar{\omega}^g = 0.456, \quad \kappa_b(\bar{\omega}^g) = 0.456, \quad \text{and} \quad \kappa_g(\bar{\omega}^g) = 1.617.$$

Since

$$1-\epsilon_b\bar{\psi}(\kappa_g(\bar{\omega}^g),\bar{\omega}^g) = 0.894 < \varrho(\bar{\omega}^g) = 0.900 < 1-\epsilon_b\bar{\psi}(\kappa_b(\bar{\omega}^g),\bar{\omega}^g) = 0.934,$$

Theorem 5.4 (iv) implies that there exists one green, one mixed, and one brown MGRSS. These are characterized by the economy-wide capital-labor ratios $\bar{k}_\gamma^1 = 1.617$, $\bar{k}_\gamma^2 = 1.419$, and

$\bar{k}_\gamma^3 = 0.489$, respectively. Since the stability condition (5.62) is violated, the mixed MGRSS $\bar{k}_\gamma^2$ is unstable.

For Parameter Set B in Table 5.1, we get

$$\bar{\omega}^g = 1.103, \quad \kappa_g(\bar{\omega}^g) = 0.902, \quad \text{and} \quad \kappa_b(\bar{\omega}^g) = 1.103.$$

Since

$$1 - \epsilon_b \bar{\psi}(\kappa_b(\bar{\omega}^g), \bar{\omega}^g) = 0.825 < \varrho(\bar{\omega}^g) = 0.840 < 1 - \epsilon_b \bar{\psi}(\kappa_g(\bar{\omega}^g), \bar{\omega}^g) = 0.869,$$

it follows from Theorem 5.4 (iii) that there exists a uniquely determined mixed MGRSS, characterized by the economy-wide capital-labor ratio $\bar{k}_\gamma = 1.035$. Since the brown sector is more capital-intensive than the green sector, $\bar{k}_\gamma$ is stable by Lemma 5.4.

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Conclusions and Outlook

To conclude this dissertation, we discuss the limitations of the models treated in the five articles and, building on these limitations, outline potential directions for future research.

First of all, while the overlapping-generations models developed in this dissertation include financial intermediation, idiosyncratic risk, market power, production externalities, and multi-sector structures, they remain confined to agents with two-period lifespans. Therefore, a natural next step is to introduce multi-period lifespans, thereby allowing agents to borrow against future income. Such an extension, however, is likely to come at the cost of analytical tractability. It is well known that multi-period OLG models typically permit perfect foresight only locally in the neighborhood of steady states, e.g., see Böhm and Wenzelburger (2004). Similar challenges may arise when incorporating elastic labor supply, e.g., see Nourry (2001).

Article 1, "*Financial Intermediation and Business Cycles in Two-Period Lived OLG Models*", contributes to the banking literature by showing that, when agents are unable to diversify away idiosyncratic risk individually, financial intermediation can serve as an incentive-compatible mechanism for implementing efficient allocations. The article further contributes to economic growth theory by establishing that, under the premise of *efficient* contracts, financial intermediation cannot generate endogenous fluctuations. Instead, any non-monotonic dynamics arise from inefficient contracts that rational agents would reject.

Promising directions for future research include the following. First, our notion of an efficient contract is derived from the problem of a *myopic* social planner who is solely concerned with the welfare of the current generation. This naturally raises the question of whether a bank can also implement the allocation chosen by a social planner with an infinite horizon, as for example in Ennis and Keister (2003). Efficient contracts, in this sense, would need to account for how capital accumulation affects the welfare of future generations and therefore be expected to induce dynamically efficient growth paths that guide the economy toward a modified golden-rule steady state. This, in turn, has implications for financial stability, as modified golden-rule steady states are typically saddle points.

Second, the dynamics of capital accumulation are driven by investments in risky projects. Following on from Banerji et al. (2004), these investments were assumed to be contractible and enforceable. An important open question is to what extent endogenous fluctuations may emerge once agents are allowed to choose investment levels at their own discretion, giving rise to moral hazard. Addressing this issue necessitates the introduction of an additional incentive-compatibility constraint on the side of the bank that disciplines agents' investment behavior, preventing both underinvestment and excessive risk-taking. Agents would then be required to optimally allocate their resources among consumption, deposits, and risky investments. In the presence of moral hazard, it is no longer clear whether the efficient allocation remains implementable through financial intermediation.

Third, aggregate uncertainty is absent in the model, as production risk is purely idiosyncratic. However, systemic risk is a key driver of business cycles in OLG models, e.g., see Gersbach and Wenzelburger (2003, 2008, 2012). Following Smith (1998), one could introduce aggregate shocks and examine whether financial intermediation amplifies or dampens output volatility.

Article 2, "*Banking Competition and Capital Dependence of the Production Sector: Growth and Welfare Implications*", contributes to the literature on the finance-growth nexus by demonstrating that the impact of banking competition on real economic growth depends on the production technology and the distribution of factor incomes. It turns out that when production is highly capital-intensive, intense competition among banks may be detrimental to capital accumulation, as it reduces productive investment by eroding intermediation margins. The analysis further shows that the risk-sharing function of financial intermediation is a key driver of long-term economic growth. By highlighting these mechanisms, the article challenges the conventional wisdom that increased competition is always beneficial.

The theoretical findings are supported by ample empirical evidence. Nevertheless, the model has several limitations that should be addressed by future research. First, the monopolistic bank's market power is one-sided, as it cannot influence the capital-rental rate paid by firms. It remains unclear whether the results would change if the bank also exerted market power in the capital market. Second, incorporating a storage technology, as in Diamond and Dybvig (1983), would allow agents to self-insure against idiosyncratic risk and enable banks to store equity. Such an extension, however, would complicate the decision problems, raising doubts about whether the model would retain its tractability. Third, endogenizing dividend payments and introducing regulatory requirements, such as capital-adequacy ratios, would constitute natural extensions of the analysis. Finally, the model suggests that intense competition among banks may impair financial stability. It would thus be intriguing to quantify the risk of a banking crisis using a probability of default.

Article 3, "*Capital Market Power and Economic Growth in an Overlapping-Generations Model with Rational Expectations*", highlights the importance of competitive capital markets for economic growth by showing that market power and strategic interaction curb capital accumulation and may induce non-monotonic dynamics. The article contributes to the overlapping-generations literature by introducing the notion of a *Nash-equilibrium growth path* and establishing its existence and uniqueness. The assumption of unbounded population growth is restrictive, as it rules out steady states with market power and thereby excludes potentially richer dynamic behavior. Since the analysis abstracts from normative considerations, examining the implications of market power for welfare and dynamic efficiency constitutes a promising avenue for future research. Moreover, it remains an open question whether a capital gains tax amplifies or mitigates the effects of capital market power. Finally, the existence and uniqueness of temporary Nash equilibria with elastic labor supply warrant further investigation.

Article 4, "*Production-Possibility Frontiers, Rybczynski's Theorem, and Factor-Intensity Reversals*", contributes to production theory and the literature on international trade by proposing a novel parameterization of the production-possibility frontier for two-factor, two-goods economies. The approach can enhance both the generality and tractability of two-sector growth models and provides insight into the validity of the Rybczynski theorem under decreasing returns to scale. Future research should consider extensions to more than two sectors and factors of production. Allowing for multiple inputs would enable an analysis of the efficient allocation of energy between sectors, which appears to be particularly relevant in times of anthropogenic climate change.

Article 5, "*Green Transformation and Fiscal Policy in a Two-Sector OLG Model with Production Externalities*", contributes to the literature on the growth-environment nexus and the debate on climate change mitigation. The article elucidates how the benefits and burdens of emissions reduction must be distributed across generations so as to reconcile the green transformation with economic prosperity. The policy implications of our findings are clear and readily implementable: an emissions tax is required to internalize the pollution externality, but the tax revenues must be redistributed to compensate agents for the policy-induced reduction in factor incomes. The finding that optimal fiscal policy maintains a zero primary deficit underscores that effective climate policy can be implemented without generating new public debt and shifting financial burdens onto future generations.

Building on this article, several directions for future research emerge naturally. First and foremost, a comprehensive analysis of the stability properties of modified golden-rule steady states, as well as of the qualitative dynamics arising from optimal fiscal policies, lies beyond the scope of the article. The results obtained so far suggest that stability hinges on the relative capital intensities of the two sectors. Numerical methods such as those proposed by Laffargue (1990) and Boucekkine (1995) may provide further insight in this regard.

Second, pollution in the model is generated exclusively by production. Following John et al. (1995), incorporating emissions arising from consumption would be an important next step. Such an extension could prove particularly informative when combined with wealth and income heterogeneity, allowing for an analysis of the extent to which wealthier individuals contribute disproportionately to global warming relative to poorer individuals – a central issue in current societal debates. On the other hand, the model abstracts from any impact of pollution on future production. A more general framework could introduce endogenous total factor productivity that depends on the pollution levels, as done in Howarth and Norgaard (1992).

Finally, the introduction of an emissions tax may incentivize carbon-intensive firms to relocate to jurisdictions with lower carbon-pricing pressures. Thus, extending the model to a two-country setting could prove fruitful, as it would allow for an analysis of international trade in the spirit of Heckscher-Ohlin-Samuelson, as well as an examination of issues related to carbon leakage and border carbon adjustments.

In conclusion, it is worth emphasizing that many of the most pressing economic challenges of our time are inherently intergenerational in nature, including climate change mitigation, rising public debt, or the sustainable financing of social security and public healthcare systems. Overlapping-generations models provide a particularly well-suited framework for analyzing such complex issues and for deriving actionable policy recommendations. It is the author's hope that the renewed interest in overlapping-generations models within economic theory will endure.

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Appendix A

Proofs of Chapter 1

Proof of Lemma 1.1. Differentiating $g(I) = f'(\Omega(I))\,\Omega(I)$ yields

$$g'(I) = \Omega'(I)f'(\Omega(I))\left[1 + \frac{f''(\Omega(I))\,\Omega(I)}{f'(\Omega(I))}\right]. \tag{A.1}$$

By Assumption 1.2, $\Omega' > 0$. Moreover, $f' > 0$, $f'' < 0$, and $f''(k)k/f'(k) > -1$ for all $k \geq 0$ by Assumption 1.3. Hence, (A.1) is strictly positive, so that $g' > 0$. Since $\Omega(0) = 0$ and $\lim_{k\to 0} f'(k)k = 0$, it follows that $g(0) = 0$.[1] Differentiating (A.1) gives

$$g''(I) = \Omega''(I)f'(\Omega(I))\left[1 + \frac{f''(\Omega(I))\,\Omega(I)}{f'(\Omega(I))}\right] + \Omega'(I)^2 f''(\Omega(I))\left[2 + \frac{f'''(\Omega(I))\,\Omega(I)}{f''(\Omega(I))}\right]. \tag{A.2}$$

Since $\Omega'' \leq 0$ by Assumption 1.2 and $f'''(k)k/f''(k) > -2$ for all $k \geq 0$ by Assumption 1.2, it follows that (A.2) is strictly negative, so that $g'' < 0$. ☐

Proof of Proposition 1.1. Let $w_t > 0$ be arbitrary but fixed. By Lemma 1.1, g is strictly increasing and strictly concave with $g(0) = 0$. Since $u' > 0$, it follows that, in an optimum, $c^1 = w_t - I$. Therefore, the first-order conditions are

$$-u'(w_t - I) + p'(I)\big[v(c^{2g}) - v(c^{2b})\big] + \lambda\big[g'(I) - p'(I)(c^{2g} - c^{2b})\big] \overset{!}{=} 0 \tag{A.3}$$

$$p(I)\,v'(c^{2g}) - \lambda p(I) \overset{!}{=} 0 \tag{A.4}$$

$$(1 - p(I))\,v'(c^{2b}) - \lambda(1 - p(I)) \overset{!}{=} 0 \tag{A.5}$$

and the complementary slackness condition is

$$\lambda\big[g(I) - p(I)\,c^{2g} - (1 - p(I))\,c^{2b}\big] \overset{!}{=} 0. \tag{A.6}$$

Assumption 1.1 implies that, in an optimum, the constraint on the consumption plan must hold with equality. Since $g(0) = 0$, the Inada condition on v implies that $I > 0$ because, otherwise, $c^{2g} = c^{2b} = 0$ by (A.6). Therefore, $0 < p(I) < 1$. Conditions (A.4) and (A.5) then imply $v'(c^{2g}) = v'(c^{2b}) = \lambda > 0$. Since $v'' < 0$, it follows from (A.6) that a social optimum requires

$$c^{2g} = c^{2b} = g(I). \tag{A.7}$$

Moreover, (A.3) reduces to

$$-u'(w_t - I) + v'(g(I))\,g'(I) = 0. \tag{A.8}$$

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[1] A proof that Assumption 1.3 implies $\lim_{k\to 0} f'(k)k = 0$ is found in De La Croix and Michel (2002, p. 308).

Observe that (A.8) is the first-order condition for the maximization problem

$$\max_{0 \le I \le w_t} \; u(w_t - I) + v(g(I)). \tag{A.9}$$

The objective function in (A.9) is either already a continuous function or can be transformed into a continuous function on the compact interval $[0, w_t]$ using the exponential function. Hence, a solution $I_t^\star$ to (A.9) exists. It follows that any solution $(I_t^\star, c_{t+1}^{2g\star}, c_{t+1}^{2b\star})$ to the first-order conditions (A.3) – (A.5) must satisfy (A.7) with $I_t^\star$ being a maximizer of (A.9). In other words, the social planner's problem (1.8) reduces to (A.9). Any maximizer $I_t^\star$ of (A.9) together with (A.7) and $c_t^{1\star} = w_t - I_t^\star$ is a maximizer of (1.8). The concavity of g implies that the objective function in (A.9) is strictly concave, so that the social optimum is uniquely determined. □

Proof of Proposition 1.2. The proof comprises four steps.

Step 1 (Relaxed problem). We establish the existence and uniqueness of a solution to Problem (1.7) without the participation constraint. Let $w_t > 0$ be arbitrary but fixed. Setting $D = S(w_t, B, I, R, r)$, the Lagrangian for Problem (1.7) without the participation constraint is

$$\begin{aligned} \mathcal{L}(B, I, R, r, \lambda_1, \lambda_2) \coloneqq \; & u(w_t + B - I - D) + p(I)\, v(rD + \pi(I) - RB) \\ & + (1 - p(I))\, v(rD) + \lambda_1(p(I)RB - rD) + \lambda_2(D - B), \end{aligned} \tag{A.10}$$

where $\lambda_1, \lambda_2 \ge 0$ are the Lagrange multipliers. The four first-order conditions for a solution (B_t, I_t, R_t, r_t) are:

$$\begin{aligned} 0 &= p(I_t)R_t\Big[v'(r_tD_t + \pi(I_t) - R_tB_t) - \lambda_1\Big] + \lambda_2 - u'(w_t + B_t - I_t - D_t) \\ &\quad + (\lambda_1 r_t - \lambda_2)\frac{\partial S}{\partial B}(w_t, B_t, I_t, R_t, r_t) && \text{(A.11)} \\ 0 &= h'(I_t)p(I_t)v'(r_tD_t + \pi(I_t) - R_tB_t) + p'(I_t)\Big[v(r_tD_t + \pi(I_t) - R_tB_t) - v(r_tD_t)\Big] \\ &\quad + \lambda_1 p'(I_t)R_tB_t - u'(w_t + B_t - I_t - D_t) - (\lambda_1 r_t - \lambda_2)\frac{\partial S}{\partial I}(w_t, B_t, I_t, R_t, r_t) && \text{(A.12)} \\ 0 &= \Big[\lambda_1 - v'\big(r_tD_t + \pi(I_t) - R_tB_t\big)\Big]p(I_t)B_t - (\lambda_1 r_t - \lambda_2)\frac{\partial S}{\partial R}(w_t, B_t, I_t, R_t, r_t) && \text{(A.13)} \\ 0 &= \Big[p(I_t)\, v'\big(r_tD_t + \pi(I_t) - R_tB_t\big) + (1 - p(I_t))\, v'(r_tD_t) - \lambda_1\Big]D_t \\ &\quad - (\lambda_1 r_t - \lambda_2)\frac{\partial S}{\partial r}(w_t, B_t, I_t, R_t, r_t), && \text{(A.14)} \end{aligned}$$

where $D_t = S(w_t, B_t, I_t, R_t, r_t)$. The two complementary slackness conditions are

$$\lambda_1(p(I_t)R_tB_t - r_tD_t) = 0 \tag{A.15}$$

$$\lambda_2(D_t - B_t) = 0. \tag{A.16}$$

Assume that $\lambda_1 r_t - \lambda_2 = 0$. We will show below with (A.35) that in an optimum, this identity must hold. As a consequence, all terms involving derivatives of S in the first-order conditions

(A.11) – (A.14) are zero.[2] Since $p > 0$, (A.13) is equivalent to

$$\Big[\lambda_1 - v'(r_tD_t + \pi(I_t) - R_tB_t)\Big]B_t = 0. \tag{A.17}$$

Since $v' > 0$ by Assumption 1.1, two cases can occur in (A.17). First,

$$B_t > 0 \quad \text{and} \quad \lambda_1 = v'(r_tD_t + \pi(I_t) - R_tB_t) > 0. \tag{A.18}$$

Second, $B_t = 0$.

Case 1. Since $B_t > 0$ and $\lambda_1 > 0$, (A.15) requires

$$p(I_t)R_tB_t = r_tD_t, \tag{A.19}$$

so that the profit constraint is binding. Inserting (A.18), we see that (A.11) holds with

$$\lambda_2 = u'(w_t + B_t - I_t - D_t) > 0. \tag{A.20}$$

Since $\lambda_2 > 0$, (A.16) implies that the resource constraint is binding,

$$B_t = D_t > 0. \tag{A.21}$$

Using (A.21), it follows from (A.19) that

$$r_t = p(I_t)R_t. \tag{A.22}$$

Inserting (A.18) into (A.14) yields

$$(1 - p(I_t))D_t\Big[v'(r_tD_t) - v'(r_tD_t + \pi(I_t) - R_tB_t)\Big] = 0. \tag{A.23}$$

Equation (A.23) has two possible solutions. First, since $0 < p(I) < 1$ for all $I > 0$, $I_t = 0$ is a solution whenever $p(0) = 1$. In this case, (A.22) implies $R_t = r_t$ and, therefore, $R_tB_t = r_tD_t$. The attained utility level is $u(w_t) + v(0)$. By Assumption 1.1, this level cannot be optimal. Since $v'' < 0$, the second solution to (A.23) is

$$R_tB_t = \pi(I_t). \tag{A.24}$$

It follows from (A.18) and (A.20) that

$$\lambda_1 = v'(r_tD_t) > 0 \quad \text{and} \quad \lambda_2 = u'(w_t - I_t) > 0. \tag{A.25}$$

Combining (A.22) with (A.24) yields

$$B_t = \frac{g(I_t)}{r_t}. \tag{A.26}$$

Since $D_t = B_t$, (A.26) implies

$$r_tD_t = g(I_t) \tag{A.27}$$

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[2] In essence, this result is a consequence of the envelope theorem.

and, therefore,

$$\lambda_1 = v'(g(I_t)). \tag{A.28}$$

Inserting (A.21), (A.24), (A.27), and (A.28), Condition (A.12) reduces to

$$-u'(w_t - I_t) + v'(g(I_t))\, g'(I_t) = 0. \tag{A.29}$$

Condition (A.29) determines the optimal investment level I_t. Observe that (A.29) is the first-order condition for the maximization problem

$$\max_{0 \le I \le w_t} \; u(w_t - I) + v(g(I)). \tag{A.30}$$

Equations (A.24) and (A.27) imply that any utility-maximizing consumption plan of the relaxed problem (A.10) has to satisfy

$$c_t^1 = w_t - I_t \quad \text{and} \quad c_{t+1}^{2g} = c_{t+1}^{2b} = r_t D_t = g(I_t). \tag{A.31}$$

Hence, any solution to (A.10) is already determined by a solution I_t to (A.30). Since Problem (A.30) coincides with Problem (1.10), existence and uniqueness of $0 < I_t < w_t$ obtain from the same arguments as presented in the proof of Proposition 1.1.

Case 2. If $B_t = 0$, then (PrC) implies that $r_t D_t = 0$. The strict concavity of v yields

$$u(w_t - I - D) + p(I)\, v(\pi(I)) + (1 - p(I))\, v(0) < u(w_t - I) + v(g(I)) \tag{A.32}$$

for all $I, D > 0$ with $I + D \le w_t$. Note that the r.h.s. of (A.32) is the objective function of Problem (A.30), which attains its maximum in $0 < I_t < w_t$ with I_t being determined by (A.29). Hence, $B_t = 0$ cannot be optimal.

It follows that $B_t > 0$ is optimal and that the optimal solution to the relaxed problem (A.10) is uniquely determined by (A.29) together with (A.31).

Step 2 (Incentive compatibility). In Step 1, the optimal deposit rate r_t has not yet been determined. Given the loan contract (B_t, I_t, R_t) determined in Step 1, the incentive constraint (IC) implies that the deposits $D_t = S(w_t, B_t, I_t, R_t, r_t)$ must satisfy the first-order condition

$$u'(w_t + B_t - I_t - D_t) = \Big[p(I_t)\, v'(r_t D_t + \pi(I_t) - R_t B_t) + (1 - p(I_t))\, v'(r_t D_t)\Big] r_t.$$

Inserting (A.21), (A.24), and (A.27), this condition takes the form

$$-u'(w_t - I_t) + v'(g(I_t)) r_t = 0. \tag{A.33}$$

A comparison of (A.29) with (A.33) shows that the optimal deposit rate is

$$r_t = g'(I_t). \tag{A.34}$$

Using (A.25), (A.28), and (A.33), it follows that r_t satisfies

$$\lambda_1 r_t - \lambda_2 = v'(g(I_t)) r_t - u'(w_t - I_t) = 0, \tag{A.35}$$

thus justifying the assumption made at the outset of the proof. Inserting (A.34) into (A.26) and (A.22) yields

$$B_t = \frac{g(I_t)}{g'(I_t)} \quad \text{and} \quad R_t = \frac{g'(I_t)}{p(I_t)}.$$

Step 3 (Efficiency). To see that the contract (B_t, I_t, R_t, r_t) established above implements the efficient allocation, recall that the first-order conditions (A.8) and (A.29) coincide, so that $I_t = I_t^\star$ is the efficient investment level. It follows from (A.24) and (A.27) that

$$c_{t+1}^{2g} = c_{t+1}^{2b} = r_t S(w_t, I_t, B_t, R_t, r_t) = g(I_t) = g(I_t^\star) = c_{t+1}^{2g\star} = c_{t+1}^{2b\star}.$$

Thus, (B_t, I_t, R_t, r_t) implements the efficient allocation.

Step 4 (Participation constraint). We prove that the relaxed problem (A.10) without the participation constraint leads to the same solution as Problem (1.7) by showing that agents will accept the efficient contract (B_t, I_t, R_t, r_t) established above.

An agent who rejects the efficient loan contract may save and invest with idiosyncratic risk, solving the autarky problem (1.6). The strict concavity of v implies that the objective function in (1.6) satisfies

$$\begin{aligned} u(w_t - I^{\mathrm{A}} - D^{\mathrm{A}}) + p(I^{\mathrm{A}})\, v(r_t D^{\mathrm{A}} + \pi(I^{\mathrm{A}})) + (1 - p(I^{\mathrm{A}}))\, v(r_t D^{\mathrm{A}}) \\ \leq u(w_t - I^{\mathrm{A}} - D^{\mathrm{A}}) + v(r_t D^{\mathrm{A}} + g(I^{\mathrm{A}})) \end{aligned} \tag{A.36}$$

for all $I^{\mathrm{A}}, D^{\mathrm{A}} \geq 0$ with $I^{\mathrm{A}} + D^{\mathrm{A}} \leq w_t$. Replacing the objective function in (1.6) with the r.h.s. of (A.36) yields an auxiliary problem. We next establish that the unique maximizer of this auxiliary problem is $(I_t^{\mathrm{A}} = I_t, D_t^{\mathrm{A}} = 0)$, where I_t is the efficient investment level, and show that agents are worse off rejecting the efficient contract.

Observe first that the auxiliary objective function is strictly concave if g is strictly concave. The Inada conditions on u and v imply that any solution $(I_t^{\mathrm{A}}, D_t^{\mathrm{A}})$ to Problem (1.6) must satisfy $0 < I_t^{\mathrm{A}} + D_t^{\mathrm{A}} < w_t$. Thus, there are three possible cases.

Case 1: $I_t^{\mathrm{A}} = 0$, $D_t^{\mathrm{A}} > 0$. The resulting first-order conditions in this case read

$$-u'(w_t - D^{\mathrm{A}}) + v'(r_t D^{\mathrm{A}}) g'(0) + \lambda_1 = 0 \tag{A.37}$$

$$-u'(w_t - D^{\mathrm{A}}) + v'(r_t D^{\mathrm{A}}) r_t = 0. \tag{A.38}$$

The Inada conditions imply that a solution $0 < D_t^{\mathrm{A}} < w_t$ to (A.38) exists. Inserting (A.38) into (A.37), we see that $(I_t^{\mathrm{A}} = 0, D_t^{\mathrm{A}})$ is a possible maximum if

$$\lambda_1 = v'(r_t D_t^{\mathrm{A}})[r_t - g'(0)] \geq 0.$$

However, since $r_t = g'(I_t)$ and $g'' < 0$, it follows that λ_1 must be negative. Hence, $(I_t^{\mathrm{A}} = 0,\ D_t^{\mathrm{A}})$ does not solve the first-order conditions.

Case 2: $I_t^{\mathrm{A}} > 0$, $D_t^{\mathrm{A}} = 0$. The corresponding first-order conditions are

$$-u'(w_t - I^{\mathrm{A}}) + v'(g(I^{\mathrm{A}})) g'(I^{\mathrm{A}}) = 0 \tag{A.39}$$

$$-u'(w_t - I^{\mathrm{A}}) + v'(g(I^{\mathrm{A}})) r_t + \lambda_2 = 0. \tag{A.40}$$

As shown in the proof of Proposition 1.1, the unique solution to (A.39) is the efficient investment level $I_t^{\mathrm{A}} = I_t$. Since $r_t = g'(I_t)$, it follows that $(I_t^{\mathrm{A}} = I_t, D_t^{\mathrm{A}} = 0)$ together with $\lambda_2 = 0$ solves the first-order conditions.

Case 3: $I_t^{\mathrm{A}} > 0$, $D_t^{\mathrm{A}} > 0$. The resulting first-order conditions are

$$-u'(w_t - I^{\mathrm{A}} - D^{\mathrm{A}}) + v'(r_t D^{\mathrm{A}} + g(I^{\mathrm{A}}))g'(I^{\mathrm{A}}) = 0 \tag{A.41}$$

$$-u'(w_t - I^{\mathrm{A}} - D^{\mathrm{A}}) + v'(r_t D^{\mathrm{A}} + g(I^{\mathrm{A}}))r_t = 0. \tag{A.42}$$

A comparison of (A.41) with (A.42) shows that any solution I_t^{A} requires $r_t = g'(I_t^{\mathrm{A}})$. Since $r_t = g'(I_t)$ and $g'' < 0$, it follows that $I_t^{\mathrm{A}} = I_t$. A comparison with Case 2 shows that $(I_t^{\mathrm{A}} = I_t, D_t^{\mathrm{A}} = 0)$ solves (A.41) and (A.42). Since both equations are strictly decreasing in D, no solution $(I_t^{\mathrm{A}} = I_t, D_t^{\mathrm{A}} > 0)$ exists.

These considerations show that the uniquely determined maximizer of the auxiliary problem is $(I_t^{\mathrm{A}} = I_t, D_t^{\mathrm{A}} = 0)$, yielding the utility level $u(w_t - I_t) + v(g(I_t))$. Inequality (A.36) now implies

$$U_{\mathrm{res}}(w_t, r_t) \leq u(w_t - I_t) + v(g(I_t)) = V(w_t, B_t, I_t, R_t, r_t),$$

showing that agents are indeed willing to accept the efficient contract (B_t, I_t, R_t, r_t). □

Proof of Theorem 1.1. Endogenous fluctuations are ruled out if $G' > 0$. Differentiating (1.18) using the chain rule yields

$$G'(k) = \Omega'\big(\mathcal{I}(w(k))\big)\mathcal{I}'(w(k))w'(k). \tag{A.43}$$

Observe first that $\Omega' > 0$ by Assumption 1.2, and $w' > 0$ by Assumption 1.3. We next show that $\mathcal{I}' > 0$. For each $w > 0$, the investment function $\mathcal{I}(w)$ satisfies the first-order condition

$$-u'(w - \mathcal{I}(w)) + v'\big(g(\mathcal{I}(w))\big)g'(\mathcal{I}(w)) = 0. \tag{A.44}$$

Implicit differentiation of (A.44) yields

$$\mathcal{I}'(w) = \frac{u''(w - \mathcal{I}(w))}{u''(w - \mathcal{I}(w)) + v''\big(g(\mathcal{I}(w))\big)g'(\mathcal{I}(w))^2 + v'\big(g(\mathcal{I}(w))\big)g''(\mathcal{I}(w))}. \tag{A.45}$$

Since $u'' < 0$ by Assumption 1.1, the numerator in (A.45) is strictly negative. Since $0 < \mathcal{I}(w) < w$ is a maximizer, the second-order condition for a maximum of Problem (A.30) is satisfied, which implies that the denominator in (A.45) is strictly negative. Thus, we can conclude that (A.45) is strictly positive. Hence, $G' > 0$. □

Proof of Proposition 1.3. *(i).* Steady states of G are determined by solutions $k_\star \geq 0$ to

$$k \overset{!}{=} \Omega\big(\mathcal{I}(w(k))\big). \tag{A.46}$$

Observe that $f(0) = 0$ if and only if $w(0) = 0$. Thus, if $f(0) = 0$, then $\mathcal{I}(w(0)) = 0$. Since $\Omega(0) = 0$, it follows that $k_\star = 0$ solves (A.46) if $f(0) = 0$. By contrast, if $f(0) > 0$, then the Inada conditions stated in Assumption 1.1 imply that $0 < \mathcal{I}(w(0)) < w(0)$. Since $\Omega' > 0$, it follows that $\Omega\big(\mathcal{I}(w(0))\big) > 0$, so that $k_\star = 0$ cannot solve (A.46). Therefore, $k_\star = 0$ solves (A.46) if and only if $f(0) = 0$.

(ii). If $f(0) > 0$, then $G(0) > 0$, so that $G(k) > k$ for all $k \in (0, \epsilon)$ with $\epsilon > 0$ sufficiently small. On the other hand, if $f(0) = 0$, then $G(0) = 0$. In this case, $\lim_{k \to 0} G'(k) > 1$ ensures that $G(k) > k$ for all $k \in (0, \epsilon)$. It follows from the definition of Ω and Proposition 1.2 that

$$0 \leq G(k) = \Omega\big(\mathcal{I}(w(k))\big) \leq f(k) \quad \text{for all} \quad k \geq 0, \tag{A.47}$$

showing that G is bounded from above by the strictly concave production function f. The condition $\lim_{k \to \infty} f'(k) < 1$ implies that there exists $\bar{k} > 0$ such that $f(\bar{k}) = \bar{k}$. As a consequence, there exists at least one solution $k_\star \in \big(0, \bar{k}\big]$ to (A.46). The largest one of these solutions must satisfy $0 < G'(k_\star) < 1$ and, hence, be asymptotically stable. □

Proof of Lemma 1.2. Assumptions 1.2 and 1.3 imply that $I \mapsto f(\Omega(I))$ is strictly increasing and strictly concave. Hence, a solution I_G to

$$\max_{I \geq 0} f(\Omega(I)) - I \tag{A.48}$$

is unique, if it exists. Observe that $f(\Omega(I)) - I \leq f(I) - I$ for all $I \geq 0$. It follows from Assumption 1.3 that $I \mapsto f(I) - I$ has a unique maximum. Hence, Problem (A.48) admits a unique solution $I_G > 0$, which is determined by the first-order condition (1.23). □

Appendix B

Proofs of Chapter 2

We first establish three technical lemmas that facilitate the proofs of the main results.

Lemma B.1 (Marginal product of labor)**.**
Under the hypotheses of Assumption 2.2, the marginal product of labor $w(k)$ *is strictly increasing, strictly concave, and its elasticity satisfies*

$$0 < \frac{w'(k)k}{w(k)} < 1 \quad \textit{for all} \quad k \geq 0.$$

Moreover, $\lim\limits_{k\to 0} w'(k) = \infty$.

Proof of Lemma B.1. Differentiating $w(k) = f(k) - f'(k)k$ using Assumption 2.2 yields

$$w'(k) = -f''(k)k > 0 \quad \text{and} \quad w''(k) = -f''(k)\big[1 + \epsilon_{f''}(k)\big] < 0$$

for all $k \geq 0$. The elasticity of a strictly increasing, strictly concave function lies within $(0,1)$. Since $\frac{f''(k)k}{f'(k)} < 0$ for all $k \geq 0$ by Assumption 2.2, we have

$$\frac{w'(k)}{f'(k)} > 0 \quad \text{for all} \quad k \geq 0. \tag{B.1}$$

It now follows from (B.1) and the Inada condition $\lim\limits_{k\to 0} f'(k) = \infty$ that $\lim\limits_{k\to 0} w'(k) = \infty$. □

Lemma B.2 (Comparative statics)**.**
Let Assumption 2.1 be satisfied. Then the following holds true.

(i) $I(w, \rho^e)$ *defined in (2.1) is strictly increasing in both* w *and* ρ^e.
(ii) $D(w, r)$ *defined in (2.4) is strictly increasing in both* w *and* r.
(iii) $\eta(w, r)$ *defined in (2.7) is non-decreasing in* w *and strictly decreasing in* r.
(iv) $r^{\mathrm{M}}(w, \rho^e)$ *defined in (2.10) is strictly increasing in* ρ^e *and non-decreasing in* w.
(v) $\eta(w, r, \vartheta^e) := \frac{\partial D}{\partial r}(w, r, \vartheta^e) r / D(w, r, \vartheta^e)$ *defined by (2.21) is strictly decreasing in* r.

Proof of Lemma B.2. *(i).* Since, by Assumption 2.1 (ii), the Arrow-Pratt coefficients of relative risk aversion are constants $0 < \alpha_u \leq \alpha_v < 1$, implicit differentiation of (2.2) yields

$$\frac{\partial I}{\partial w}(w, \rho^e) = \frac{\alpha_u}{\alpha_u + \alpha_v \frac{w - I(w,\rho^e)}{I(w,\rho^e)}} > 0 \quad \text{and} \quad \frac{\partial I}{\partial \rho^e}(w, \rho^e) = \frac{(1-\alpha_v) I(w, \rho^e)}{\rho^e\big(\alpha_u \frac{I(w,\rho^e)}{w - I(w,\rho^e)} + \alpha_v\big)} > 0.$$

(ii). Since $0 < \alpha_u \leq \alpha_v < 1$, implicit differentiation of (2.5) yields

$$\frac{\partial D}{\partial w}(w, r) = \frac{\alpha_u}{\alpha_u + \alpha_v \frac{w - D(w,r)}{D(w,r)}} > 0 \quad \text{and} \quad \frac{\partial D}{\partial r}(w, r) = \frac{(1-\alpha_v) D(w, r)}{r\big(\alpha_u \frac{D(w,r)}{w - D(w,r)} + \alpha_v\big)} > 0. \tag{B.2}$$

Part (iii). Using (B.2), the elasticity η may be written as

$$\eta(w,r) = \frac{\frac{\partial D}{\partial r}(w,r)r}{D(w,r)} = \frac{1-\alpha_v}{\alpha_u \frac{D(w,r)}{w-D(w,r)} + \alpha_v} > 0. \tag{B.3}$$

Since $0 < \alpha_u \leq \alpha_v < 1$, differentiation of (B.3) gives

$$\frac{\partial \eta}{\partial r}(w,r) = -\alpha_u \Big(\frac{\eta(w,r)}{w-D(w,r)} \Big)^2 \frac{wD(w,r)}{r\big(\alpha_u \frac{D(w,r)}{w-D(w,r)} + \alpha_v\big)} < 0. \tag{B.4}$$

Setting

$$\xi(w,r) := \frac{\frac{\partial D}{\partial w}(w,r)w}{D(w,r)},$$

then Assumption 2.1 implies $0 < \xi(w,r) \leq 1$ for all $(w,r) \in \mathbb{R}^2_{++}$. As a consequence,

$$\frac{\partial \eta}{\partial w}(w,r) = \frac{\big[1-\xi(w,r)\big]D(w,r)(1-\alpha_v)\alpha_u}{\Big(\big[\alpha_u \frac{D(w,r)}{w-D(w,r)} + \alpha_v\big]\big[w-D(w,r)\big]\Big)^2} \geq 0. \tag{B.5}$$

(iv). Since $v' > 0$ by Assumption 2.1 and $I(w,\rho^e) \equiv D(w,b(w,\rho^e))$ by Corollary 2.1, implicit differentiation of (2.9) yields

$$\frac{\partial b}{\partial \rho^e}(w,\rho^e) = \frac{v'(\rho^e I(w,\rho^e))pI(w,\rho^e)}{v'\big(b(w,\rho^e)D(w,b(w,\rho^e))\big)D(w,b(w,\rho^e))} > 0$$

and

$$\frac{\partial b}{\partial w}(w,\rho^e) = \frac{u'(w-I(w,\rho^e)) - u'\big(w-D(w,b(w,\rho^e))\big)}{v'\big(b(w,\rho^e)D(w,b(w,\rho^e))\big)D(w,b(w,\rho^e))} = 0.$$

Using (B.3) – (B.5), implicit differentiation of (2.8) yields

$$\frac{\partial s}{\partial \rho^e}(w,\rho^e) = p\Big(\frac{p\rho^e}{s(w,\rho^e)} - \frac{s(w,\rho^e)}{\eta(w,s(w,\rho^e))}\frac{\partial \eta}{\partial r}(w,s(w,\rho^e))\Big)^{-1} > 0$$

and

$$\frac{\partial s}{\partial w}(w,\rho^e) = \frac{\partial \eta}{\partial w}(w,s(w,\rho^e))\Big(\frac{p\rho^e\eta(w,s(w,\rho^e))^2}{s(w,\rho^e)^2} - \frac{\partial \eta}{\partial r}(w,s(w,\rho^e))\Big)^{-1} \geq 0.$$

This establishes that $r^{\mathrm{M}}(w,\rho^e) = \max\big\{b(w,\rho^e), s(w,\rho^e)\big\}$ is strictly increasing in ρ^e and non-decreasing in w.

(v). The elasticity $\eta(w,r,\vartheta^e)$ may be written as

$$\eta(w,r,\vartheta^e) = \frac{\frac{\partial D}{\partial r}(w,r,\vartheta^e)r}{D(w,r,\vartheta^e)} = \frac{1-\alpha_v + \frac{\vartheta^e}{rD(w,r,\vartheta^e)}}{\alpha_u \frac{D(w,r,\vartheta^e)+\frac{\vartheta^e}{r}}{w-D(w,r,\vartheta^e)} + \alpha_v} > 0. \tag{B.6}$$

Setting

$$T(w,r,\vartheta^e) := \alpha_u \frac{D(w,r,\vartheta^e)+\frac{\vartheta^e}{r}}{w-D(w,r,\vartheta^e)} + \alpha_v > 0,$$

then differentiation of (B.6) yields

$$\frac{\partial \eta}{\partial r}(w,r,\vartheta^e) = -\frac{\big[1+\eta(w,r,\vartheta^e)\big]\vartheta^e}{r^2 D(w,r,\vartheta^e)T(w,r,\vartheta^e)} - \frac{\eta(w,r,\vartheta^e)\frac{\partial T}{\partial r}(w,r,\vartheta^e)}{T(w,r,\vartheta^e)}. \tag{B.7}$$

Since $\eta(w,r,\vartheta^e) > 0$ and $T(w,r,\vartheta^e) > 0$, it follows that (B.7) is strictly negative if

$$\frac{\partial T}{\partial r}(w,r,\vartheta^e) > 0 \iff \frac{\eta(w,r,\vartheta^e)}{\vartheta^e} > \frac{w - D(w,r,\vartheta^e)}{D(w,r,\vartheta^e)[\vartheta^e + wr]}. \tag{B.8}$$

Using (B.6), it is readily seen that (B.8) holds if

$$1 > \frac{rw\alpha_v + \vartheta^e \alpha_u}{rw + \vartheta^e}. \tag{B.9}$$

Since $0 < \alpha_u \leq \alpha_v < 1$, (B.9) and hence (B.8) is satisfied, implying that the derivative (B.7) is strictly negative. ☐

Lemma B.3 (Over-accumulation)**.**
Let Assumptions 2.1 and 2.2 be satisfied. Then the positive steady state $k^{\mathrm{M}} > 0$ *of the dynamical system (PFD*$^{\mathrm{M}}$*) established in Theorem 2.1 satisfies* $k^{\mathrm{M}} > f'^{-1}\big(\frac{1}{p}\big)$.

Proof of Lemma B.3. Because of perfect foresight, the monopolistic bank's equity stock in any steady state $k^{\mathrm{M}} > 0$ of (PFD$^{\mathrm{M}}$) must be strictly positive. Since, by Proposition 2.2,

$$pf'(k^{\mathrm{M}}) - r^{\mathrm{M}}(w(k^{\mathrm{M}}), f'(k^{\mathrm{M}})) > 0,$$

it follows that

$$e^{\mathrm{M}}(k^{\mathrm{M}}, k^{\mathrm{M}}, k^{\mathrm{M}}) = \frac{pf'(k^{\mathrm{M}}) - r^{\mathrm{M}}(w(k^{\mathrm{M}}), f'(k^{\mathrm{M}}))}{1 - pf'(k^{\mathrm{M}})} D\big(w(k^{\mathrm{M}}), r^{\mathrm{M}}(w(k^{\mathrm{M}}), f'(k^{\mathrm{M}}))\big) > 0$$

if and only if $1 - pf'(k^{\mathrm{M}}) > 0$. Assumption 2.2 now implies that $k^{\mathrm{M}} > f'^{-1}\big(\frac{1}{p}\big)$. ☐

Proof of Proposition 2.1. Given any expectation $\rho_t^e > 0$, the anticipated intermediation margin is zero if and only if $r_t^{\mathrm{C}} = p\rho_t^e$. By the strict concavity of v, the deposit rate $r_t^{\mathrm{C}} = p\rho_t^e$ satisfies the participation constraint (PC) because

$$\begin{aligned}\mathscr{U}_{\mathrm{res}}(w_t, \rho_t^e) &< u(w_t - I(w_t,\rho_t^e)) + v(p\rho_t^e I(w_t,\rho_t^e)) \\ &= u(w_t - I(w_t,\rho_t^e)) + v(r_t^{\mathrm{C}} I(w_t,\rho_t^e)) \\ &\leq u(w_t - D(w_t, r_t^{\mathrm{C}})) + v(r_t^{\mathrm{C}} D(w_t, r_t^{\mathrm{C}})) = \mathscr{U}(w_t, r_t^{\mathrm{C}}),\end{aligned}$$

where the last inequality follows from (2.4). ☐

Proof of Proposition 2.2. The proof comprises two steps. Step 1 establishes the existence and uniqueness of a solution to Problem (2.6) using the intermediate-value theorem. Step 2 verifies that the sufficient conditions for a maximum are satisfied.

Step 1. Let $w_t > 0$, $\rho_t^e > 0$, and $e_t \geq 0$ be arbitrary but fixed. It follows from (2.8) that the deposit rate $s(w_t, \rho_t^e)$ is defined by a solution to

$$J_t(r) := \frac{p\rho_t^e}{r} - 1 - \frac{1}{\eta(w_t, r)} \overset{!}{=} 0. \tag{B.10}$$

Since $0 < \alpha_v < 1$, (B.3) implies that $\eta(w_t, 0) = \frac{1-\alpha_v}{\alpha_v} > 0$. Since $\eta(w_t, p\rho_t^e) > 0$, we see that the l.h.s. in (B.10) satisfies

$$\lim_{r \to 0} J_t(r) = \infty - \frac{1}{\eta(w_t, 0)} > 0 \quad \text{and} \quad \lim_{r \to p\rho_t^e} J_t(r) = \frac{-1}{\eta(w_t, p\rho_t^e)} < 0.$$

Lemma B.2 (iii) implies that

$$J_t'(r) = -\frac{p\rho_t^e}{r^2} + \frac{1}{\eta(w_t, r)^2} \frac{\partial \eta}{\partial r}(w_t, r) < 0$$

for all $r \in (0, p\rho_t^e]$. Hence, (B.10) admits a unique solution $s(w_t, \rho_t^e) \in (0, p\rho_t^e)$.

It follows from (2.9) that the deposit rate $b(w_t, \rho_t^e)$ is defined by a solution to

$$\mathscr{U}(w_t, r) \overset{!}{=} \mathscr{U}_{\text{res}}(w_t, \rho_t^e). \tag{B.11}$$

By Assumption 2.1, $v(0) = 0$. Moreover, weak gross substitutability in consumption ensures that $\lim\limits_{r \to 0} D(w_t, r) = 0$. Hence,

$$\lim_{r \to 0} \mathscr{U}(w_t, r) = u(w_t - D(w_t, 0)) + v(0) = u(w_t) < \mathscr{U}_{\text{res}}(w_t, \rho_t^e).$$

On the other hand, it follows from (2.4) and the strict concavity of v that

$$\begin{aligned}\lim_{r \to p\rho_t^e} \mathscr{U}(w_t, r) &= u(w_t - D(w_t, p\rho_t^e)) + v(p\rho_t^e D(w_t, p\rho_t^e)) \\ &\geq u(w_t - I(w_t, \rho_t^e)) + v(p\rho_t^e I(w_t, \rho_t^e)) \\ &> u(w_t - I(w_t, \rho_t^e)) + p\, v(\rho_t^e I(w_t, \rho_t^e)) \\ &= \mathscr{U}_{\text{res}}(w_t, \rho_t^e).\end{aligned}$$

Since $v' > 0$, the envelope theorem implies that

$$\frac{\partial \mathscr{U}}{\partial r}(w_t, r) = v'(rD(w_t, r))D(w_t, r) > 0 \tag{B.12}$$

for all $r \in (0, p\rho_t^e]$. Hence, (B.11) admits a unique solution $b(w_t, \rho_t^e) \in (0, p\rho_t^e)$.

Since the l.h.s. of the participation constraint (PC) is strictly increasing in r by (B.12), it follows that the maximizer of Problem (2.6) must be of the form (2.10).

Step 2. The objective function $r \mapsto \pi_t^e(r)$ in Problem (2.6) satisfies

$$\lim_{r \to 0} \pi_t^e(r) = p\rho_t^e e_t \quad \text{and} \quad \lim_{r \to p\rho_t^e} \pi_t^e(r) = p\rho_t^e e_t$$

because either deposits are zero or the intermediation margin is zero. Since the first-order condition (2.8) has a unique solution on the interval $[0, p\rho_t^e]$, the objective function $r \mapsto \pi_t^e(r)$ has a unique stationary point on $[0, p\rho_t^e]$ at $s(w_t, \rho_t^e)$. Since $\pi_t^e(s(w_t, \rho_t^e)) > p\rho_t^e e_t$, we can conclude that the objective function is strictly quasi-concave on $[0, p\rho_t^e]$. Moreover, it follows from (B.12) that the participation constraint (PC) defines a convex set of deposit rates. Hence, (2.10) is the unique maximizer of Problem (2.6). □

Proof of Lemma 2.1. Let $w_t > 0$, $\rho_t^e > 0$, and $e_t \geq 0$ be arbitrary but fixed. By a slight abuse of notation, denote by $\mathcal{P}_t$ the set of solutions $0 < p < 1$ to

$$b(w_t, \rho_t^e, p) \overset{!}{=} s(w_t, \rho_t^e, p). \tag{B.13}$$

We first show that $\mathcal{P}_t$ is non-empty and, subsequently, define the thresholds for p. As illustrated in Figure B.1, the proof exploits the intermediate-value theorem.

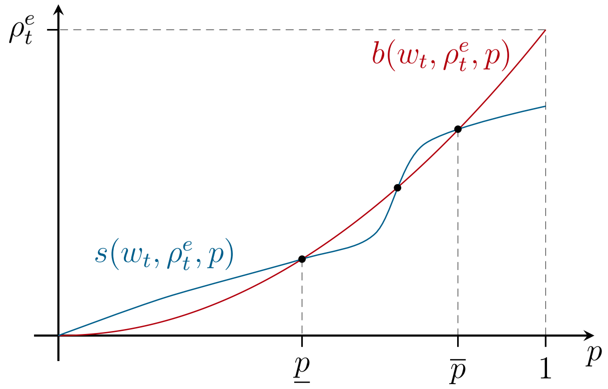

Figure B.1. The minimum and the maximum of $\mathcal{P}_t$ define the thresholds for p.

Step 1. Since $0 < s(w_t, \rho_t^e, p), b(w_t, \rho_t^e, p) < p\rho_t^e$, it follows that

$$\lim_{p\to 0} b(w_t, \rho_t^e, p) = 0 \quad \text{and} \quad \lim_{p\to 0} s(w_t, \rho_t^e, p) = 0.$$

Moreover, we can deduce from (2.9) that $\lim_{p\to 1} b(w_t, \rho_t^e, p) = \rho_t^e$, and from (2.8) that $\lim_{p\to 1} s(w_t, \rho_t^e, p) < \rho_t^e$. We next show that

$$\lim_{p\to 0} \frac{\partial s}{\partial p}(w_t, \rho_t^e, p) > \lim_{p\to 0} \frac{\partial b}{\partial p}(w_t, \rho_t^e, p),$$

so that the existence of a solution $0 < p < 1$ to (B.13) follows from the intermediate-value theorem. Implicit differentiation of (2.8) yields

$$\frac{\partial s}{\partial p}(w_t, \rho_t^e, p) = \frac{(1-\alpha_v)\rho_t^e}{1 + \alpha_u \dfrac{\big[1 + \eta(w_t, s(w_t, \rho_t^e, p)\big] D(w_t, s(w_t, \rho_t^e, p)) w_t - D(w_t, s(w_t, \rho_t^e, p))^2}{\big[w_t - D(w_t, s(w_t, \rho_t^e, p))\big]^2}}.$$

Since $\lim_{p\to 0} D(w, s(w, \rho^e, p)) = 0$ and $0 < \alpha_v < 1$, it follows that

$$\lim_{p\to 0} \frac{\partial s}{\partial p}(w, \rho^e, p) = (1-\alpha_v)\rho_t^e > 0. \tag{B.14}$$

By the envelope theorem, the r.h.s. of (2.9) satisfies

$$\lim_{p\to 0} \frac{\partial \mathscr{U}_{\text{res}}}{\partial p}(w_t, \rho_t^e, p) = v(0) = 0.$$

By contrast, the l.h.s. of (2.9) is independent of p. Hence,

$$\lim_{p\to 0} \frac{\partial b}{\partial p}(w, \rho^e, p) = 0. \tag{B.15}$$

It now follows from (B.14) and (B.15) that

$$\lim_{p\to 0} \frac{\partial s}{\partial p}(w_t, \rho_t^e, p) > \lim_{p\to 0} \frac{\partial b}{\partial p}(w_t, \rho_t^e, p).$$

Hence, the set $\mathcal{P}_t$ is non-empty because there exists at least one solution $0 < p < 1$ to (B.13).

Step 2. From the limits of $s(w_t, \rho_t^e, p)$ and $b(w_t, \rho_t^e, p)$ established in Step 1, it follows that

$$s(w_t, \rho_t^e, p) > b(w_t, \rho_t^e, p) \quad \text{for all} \;\; p < \min \mathcal{P}_t$$

and

$$s(w_t, \rho_t^e, p) < b(w_t, \rho_t^e, p) \quad \text{for all} \;\; p > \max \mathcal{P}_t.$$

Proposition 2.2 now implies that

$$r_t^{\text{M}} = \max\bigl\{s(w_t, \rho_t^e, p), b(w_t, \rho_t^e, p)\bigr\} = s(w_t, \rho_t^e, p) \quad \text{for all} \;\; p < \min \mathcal{P}_t,$$

whereas

$$r_t^{\text{M}} = \max\bigl\{s(w_t, \rho_t^e, p), b(w_t, \rho_t^e, p)\bigr\} = b(w_t, \rho_t^e, p) \quad \text{for all} \;\; p > \max \mathcal{P}_t.$$

Setting $\underline{p} \coloneqq \min \mathcal{P}_t$ and $\overline{p} \coloneqq \max \mathcal{P}_t$ completes the proof. □

Proof of Corollary 2.1. Let $w_t > 0$, $\rho_t^e > 0$, and $e_t \geq 0$ be arbitrary but fixed. Since $r \mapsto D(w_t, r)$ is strictly increasing by Lemma B.2 (ii), the second inequality follows from the fact that $r^{\text{M}}(w_t, \rho_t^e) < r^{\text{C}}(w_t, \rho_t^e)$ by Proposition 2.2. We next establish the first inequality. Suppose that $r^{\text{M}}(w_t, \rho_t^e) = b(w_t, \rho_t^e) \geq s(w_t, \rho_t^e)$. Then the agent gets his reservation utility and the monopolist's rent is determined by the risk premium $RP > 0$, which is defined by

$$\begin{aligned}
\mathscr{U}_{\text{res}}(w_t, \rho_t^e) &\overset{!}{=} u(w_t - I(w_t, \rho_t^e)) + v(p\rho_t^e I(w_t, \rho_t^e) - RP) \\
&= u(w_t - I(w_t, \rho_t^e)) + v\bigl(p\rho_t^e\bigl[I(w_t, \rho_t^e) - D(w_t, b(w_t, \rho_t^e))\bigr] \\
&\qquad + b(w_t, \rho_t^e) D(w_t, b(w_t, \rho_t^e))\bigr).
\end{aligned}$$

Substituting $\mathscr{U}_{\text{res}}(w_t, \rho_t^e)$ with (2.9) yields

$$\begin{aligned}
&u\bigl(w_t - D(w_t, b(w_t, \rho_t^e))\bigr) + v\bigl(b(w_t, \rho_t^e) D(w_t, b(w_t, \rho_t^e))\bigr) \\
&\quad \overset{!}{=} u(w_t - I(w_t, \rho_t^e)) + v\bigl(p\rho_t^e \left[I(w_t, \rho_t^e) - D(w_t, b(w_t, \rho_t^e))\right] \\
&\qquad + b(w_t, \rho_t^e) D(w_t, b(w_t, \rho_t^e))\bigr).
\end{aligned} \tag{B.16}$$

Since $u', v' > 0$, (B.16) is satisfied with $D(w_t, b(w_t, \rho_t^e)) = I(w_t, \rho_t^e)$. Since $r \mapsto D(w_t, r)$ is strictly increasing, it follows that $D(w_t, s(w_t, \rho_t^e)) > D(w_t, b(w_t, \rho_t^e)) = I(w_t, \rho_t^e)$ whenever $s(w_t, \rho_t^e) > b(w_t, \rho_t^e)$. This establishes that $D(w_t, r^{\mathrm{M}}(w_t, \rho_t^e)) \geq I(w_t, \rho_t^e)$, where the inequality is strict if and only if $s(w_t, \rho_t^e) > b(w_t, \rho_t^e)$. □

Proof of Lemma 2.2. *Part (i).* We first establish the existence and uniqueness of ψ^{N}. Let $k_t \geq 0$ be arbitrary but fixed. A correct forecast k_t^e is determined by a solution to

$$k^e - pI(w(k_t), f'(k^e)) \overset{!}{=} 0. \tag{B.17}$$

For $k^e \to 0$, the l.h.s. in (B.17) is non-positive. Since the investment function defined in (2.1) satisfies $pI(w(k_t), f'(k^e)) \leq w(k_t)$ for all $k^e \geq 0$, the l.h.s. in (B.17) grows to infinity for $k^e \to \infty$. It follows from Lemma B.2 (i) and Assumption 2.2 that the l.h.s. in (B.17) is strictly increasing because

$$1 - p\frac{\partial I}{\partial \rho^e}(w(k_t), f'(k^e))f''(k^e) > 0$$

for all $k^e \geq 0$. Hence, (B.17) admits a unique solution $0 \leq k_t^e < \infty$. Since $k_t \geq 0$ is arbitrary, it follows that there exists a uniquely determined forecasting rule ψ^{N} that is perfect in the sense of Definition 2.1 (i).

We next establish the existence and uniqueness of ψ^i, $i = \mathrm{C}, \mathrm{M}$. Let $(k_t, k_{t-1}) \in \mathcal{D}^i$ be arbitrary but fixed. A correct forecast k_t^e is determined by a solution to

$$\frac{k^e}{p} - D\big(w(k_t), r^i(w(k_t), f'(k^e))\big) \overset{!}{=} e^i(k_t, k_{t-1}, k_t). \tag{B.18}$$

For $k^e \to 0$, the l.h.s. in (B.18) is non-positive, while for $k^e \to \infty$, it grows to infinity. By construction of the set $\mathcal{D}^i$, we have

$$0 \leq e^i(k_t, k_{t-1}, k_t) < \infty,$$

so that the existence of a solution $0 \leq k_t^e < \infty$ to (B.18) follows from the intermediate-value theorem. It follows from Assumption 2.2 and Lemma B.2 that the l.h.s. in (B.18) is strictly increasing because

$$\frac{1}{p} - \frac{\partial D}{\partial r}\big(w(k_t), r^i(w(k_t), f'(k^e))\big)\frac{\partial r^i}{\partial \rho^e}(w(k_t), f'(k^e))f''(k^e) > 0 \tag{B.19}$$

for all $k^e \geq 0$. Hence, (B.18) admits a unique solution k_t^e. Since $(k_t, k_{t-1}) \in \mathcal{D}^i$ is arbitrary, it follows that there exists a uniquely determined forecasting rule ψ^i, $i = \mathrm{C}, \mathrm{M}$, that is perfect in the sense of Definition 2.1 (ii).

It remains to prove that $\mathcal{D}^i$ is forward-invariant under ψ^i, $i = \mathrm{C}, \mathrm{M}$. Consider some arbitrary period $t \geq 1$. Let $(k_t, k_{t-1}) \in \mathcal{D}^i$, so that $e^i(k_t, k_{t-1}, k_t) \geq 0$. It follows from Step 1 that there exists a uniquely determined correct forecast $k_t^e = k_{t+1} = \psi^i(k_t, k_{t-1}) \geq 0$. Since, under perfect foresight, the realized intermediation margin is non-negative, it must be the case that $e^i(k_{t+1}, k_t, k_{t+1}) \geq 0$ and, therefore, $(k_{t+1}, k_t) \in \mathcal{D}^i$. Since t was arbitrary, it follows that $\mathcal{D}^i$ must be forward-invariant under ψ^i. □

Proof of Lemma 2.3. We establish the signs of the partial derivatives of the perfect forecasting rule ψ^{M}. For better readability, set $r^{\mathrm{M}} = r^{\mathrm{M}}(w(k^{\mathrm{M}}), f'(k^{\mathrm{M}}))$. Differentiating (PFD$^{\mathrm{M}}$) and evaluating in a steady state $\boldsymbol{k}^{\mathrm{M}} = (k^{\mathrm{M}}, k^{\mathrm{M}}) \in \mathcal{D}^{\mathrm{M}}$ yields

$$\begin{aligned}\frac{\partial \psi^{\mathrm{M}}}{\partial k_t}(\boldsymbol{k}^{\mathrm{M}}) = {}& \frac{w'(k^{\mathrm{M}})\frac{\mathrm{d}}{\mathrm{d}w}D(w(k^{\mathrm{M}}), r^{\mathrm{M}}) + f'(k^{\mathrm{M}})\big[1+\epsilon_{f'}(k^{\mathrm{M}})\big]}{\frac{1}{p} - \frac{\partial D}{\partial r}(w(k^{\mathrm{M}}), r^{\mathrm{M}})\frac{\partial r^{\mathrm{M}}}{\partial \rho^e} f''(k^{\mathrm{M}})} \\ & - \frac{f''(k^{\mathrm{M}})\frac{\partial r^{\mathrm{M}}}{\partial \rho^e} D(w(k^{\mathrm{M}}), r^{\mathrm{M}})\big[1+\eta(w(k^{\mathrm{M}}), r^{\mathrm{M}})\big]}{\frac{1}{p} - \frac{\partial D}{\partial r}(w(k^{\mathrm{M}}), r^{\mathrm{M}})\frac{\partial r^{\mathrm{M}}}{\partial \rho^e} f''(k^{\mathrm{M}})}\end{aligned} \tag{B.20}$$

as well as

$$\frac{\partial \psi^{\mathrm{M}}}{\partial k_{t-1}}(\boldsymbol{k}^{\mathrm{M}}) = -\frac{w'(k^{\mathrm{M}})\frac{\partial r^{\mathrm{M}}}{\partial w} D(w(k^{\mathrm{M}}), r^{\mathrm{M}})\big[1+\eta(w(k^{\mathrm{M}}), r^{\mathrm{M}})\big] + w'(k^{\mathrm{M}}) r^{\mathrm{M}} \frac{\partial D}{\partial w}(w(k^{\mathrm{M}}), r^{\mathrm{M}})}{\frac{1}{p} - \frac{\partial D}{\partial r}(w(k^{\mathrm{M}}), r^{\mathrm{M}})\frac{\partial r^{\mathrm{M}}}{\partial \rho^e} f''(k^{\mathrm{M}})}. \tag{B.21}$$

By Assumption 2.2, we have $f' > 0$, $f'' < 0$, $w' > 0$, and $\epsilon_{f'} > -1$. Using Lemma B.2 (ii) and (iv), we can read off (B.20) and (B.21) that

$$\frac{\partial \psi^{\mathrm{M}}}{\partial k_t}(\boldsymbol{k}^{\mathrm{M}}) > 0 \quad \text{and} \quad \frac{\partial \psi^{\mathrm{M}}}{\partial k_{t-1}}(\boldsymbol{k}^{\mathrm{M}}) < 0. \tag{B.22}$$

It follows from (B.22) that the eigenvalues in (2.17) satisfy $0 < \big|\lambda_2(\boldsymbol{k}^{\mathrm{M}})\big| \leq \big|\lambda_1(\boldsymbol{k}^{\mathrm{M}})\big|$. ☐

Proof of Proposition 2.3. The proof comprises two steps. Step 1 shows that the origin is a steady state if and only if $f(0) = 0$. Step 2 establishes its instability.

Step 1. The origin $k^i = 0$ is a steady state of (PFD^i), $i = \mathrm{N}, \mathrm{C}, \mathrm{M}$, if and only if it solves (2.18). Observe that $f(0) > 0$ if and only if $w(0) > 0$. For $k = 0$, the l.h.s. in (2.18) is zero. Since the respective r.h.s. in (2.18) are bounded from above by $f(k)$, they are zero if $k = 0$ and $f(0) = 0$. Hence, $k = 0$ solves (2.18) if $f(0) = 0$. On the other hand, the Inada condition imposed on v in Assumption 2.1 implies that the respective r.h.s. in (2.18) are strictly positive if $k = 0$ and $f(0) > 0$ because either savings or investments are strictly positive. Hence, $k = 0$ does *not* solve (2.18) if $f(0) > 0$. This shows that $k^i = 0$ is a steady state of (PFD^i), $i = \mathrm{N}, \mathrm{C}, \mathrm{M}$, if and only if $f(0) = 0$.

Step 2. We analyze each case $i = \mathrm{N}, \mathrm{C}, \mathrm{M}$ separately. Suppose that $k^{\mathrm{N}} = 0$ is a steady state of (PFD$^{\mathrm{N}}$). Then $k^{\mathrm{N}} = 0$ is unstable if

$$\lim_{k \to 0} \frac{\mathrm{d}}{\mathrm{d}k}\psi^{\mathrm{N}}(k) > 1 \quad \Longleftrightarrow \quad \lim_{k \to 0} \frac{\mathrm{d}}{\mathrm{d}k} I(w(k), f'(k)) > \frac{1}{p}. \tag{B.23}$$

In a steady state of (PFD$^{\mathrm{N}}$), the first-order condition (2.2) takes the form

$$\frac{u'\big(w(k) - I(w(k), f'(k))\big)}{v'\big(f'(k) I(w(k), f'(k))\big)} = p f'(k). \tag{B.24}$$

It follows from (B.24) and the Inada conditions imposed on u, v, and f that

$$I(w(k), f'(k)) \to w(k)$$

as $k \to 0$. As a consequence,

$$\lim_{k\to 0} \frac{\mathrm{d}}{\mathrm{d}k} I(w(k), f'(k)) = \lim_{k\to 0} w'(k) = \infty,$$

where the last equation follows from Lemma B.1. It now follows from (B.23) that $k^{\mathrm{N}} = 0$ is unstable.

Next, suppose that $k^{\mathrm{C}} = 0$ is a steady state of ($\mathrm{PFD}^{\mathrm{C}}$). By analogous arguments, we have

$$\lim_{k\to 0} \frac{\mathrm{d}}{\mathrm{d}k} D(w(k), pf'(k)) = \lim_{k\to 0} w'(k) = \infty > \frac{1}{p}, \tag{B.25}$$

so that $k^{\mathrm{C}} = 0$ is unstable.

Finally, suppose that $k^{\mathrm{M}} = 0$ is a steady state of ($\mathrm{PFD}^{\mathrm{M}}$). Then $k^{\mathrm{M}} = 0$ is unstable if the eigenvalue λ_1 given in (2.17) satisfies

$$\lim_{k\to 0} \big|\lambda_1(\boldsymbol{k})\big| > 1. \tag{B.26}$$

Observe that (B.26) holds in particular if $\lim\limits_{k\to 0} \frac{\partial \psi^{\mathrm{M}}}{\partial k_t}(\boldsymbol{k}) = \infty$. Since the denominator in (B.20) is strictly positive, we must distinguish two cases.

Case 1. Suppose that for $k \to 0$, the denominator in (B.20) is finite. Since all summands in the numerator of (B.20) are non-negative, we have

$$\lim_{k\to 0} \frac{\partial \psi^{\mathrm{M}}}{\partial k_t}(\boldsymbol{k}) \geq \lim_{k\to 0} \frac{f'(k)\big[1+\epsilon_{f'}(k)\big]}{\frac{1}{p} - \frac{\partial D}{\partial r}\big(w(k), r^{\mathrm{M}}(w(k), f'(k))\big)\frac{\partial r^{\mathrm{M}}}{\partial \rho^e}(w(k), f'(k))f''(k)}. \tag{B.27}$$

Since, by Assumption 2.2, $\lim\limits_{k\to 0} f'(k)\big[1+\epsilon_{f'}(k)\big] = \infty$, (B.27) implies that $\lim\limits_{k\to 0} \frac{\partial \psi^{\mathrm{M}}}{\partial k_t}(\boldsymbol{k}) = \infty$.

Case 2. Suppose that for $k \to 0$, the denominator in (B.20) is infinity. Since all summands in the numerator of (B.20) are non-negative, we have

$$\begin{aligned}
\lim_{k\to 0} \frac{\partial \psi^{\mathrm{M}}}{\partial k_t}(\boldsymbol{k}) &\geq \lim_{k\to 0} \frac{-\frac{\partial D}{\partial r}\big(w(k), r^{\mathrm{M}}(w(k), f'(k))\big) r^{\mathrm{M}}(w(k), f'(k))\frac{\partial r^{\mathrm{M}}}{\partial \rho^e}(w(k), f'(k))f''(k)}{\frac{1}{p} - \frac{\partial D}{\partial r}\big(w(k), r^{\mathrm{M}}(w(k), f'(k))\big)\frac{\partial r^{\mathrm{M}}}{\partial \rho^e}(w(k), f'(k))f''(k)} \\
&= \lim_{k\to 0} \frac{r^{\mathrm{M}}(w(k), f'(k))}{1 - \Big[p\frac{\partial D}{\partial r}\big(w(k), r^{\mathrm{M}}(w(k), f'(k))\big)\frac{\partial r^{\mathrm{M}}}{\partial \rho^e}(w(k), f'(k))f''(k)\Big]^{-1}} \\
&= \lim_{k\to 0} r^{\mathrm{M}}(w(k), f'(k)).
\end{aligned} \tag{B.28}$$

Since $\lim\limits_{k\to 0} f'(k) = \infty$, it follows from (2.8) that $\lim\limits_{k\to 0} s(w(k), f'(k)) = \infty$. Since, by (2.10), the monopolistic deposit rate satisfies $r^{\mathrm{M}}(w(k), f'(k)) \geq s(w(k), f'(k))$ for all $k \geq 0$, we can conclude that $\lim\limits_{k\to 0} r^{\mathrm{M}}(w(k), f'(k)) = \infty$. It now follows from (B.28) that $\lim\limits_{k\to 0} \frac{\partial \psi^{\mathrm{M}}}{\partial k_t}(\boldsymbol{k}) = \infty$.

Since (B.26) holds in both cases, we can conclude that $k^{\mathrm{M}} = 0$ is indeed unstable. □

Proof of Theorem 2.1. *(i).* We analyze each case $i = \mathrm{N}, \mathrm{C}, \mathrm{M}$ separately. Let $i = \mathrm{N}$. It follows from (B.23) that in a sufficiently small neighborhood of $k = 0$, we have

$$I(w(k), f'(k)) > \frac{k}{p}.$$

Since $I(w(k), f'(k)) \leq f(k)$ for all $k \geq 0$, it follows from the strict concavity of f and the property $\lim\limits_{k \to 0} f'(k) < 1$ that the steady-state condition (2.18) admits a positive solution $k^{\mathrm{N}} > 0$. This establishes the existence. We next show that, for any $k^{\mathrm{N}} > 0$,

$$\frac{\mathrm{d}}{\mathrm{d}k}\psi^{\mathrm{N}}(k^{\mathrm{N}}) < 1 \qquad \Longleftrightarrow \qquad \frac{\mathrm{d}}{\mathrm{d}k} I(w(k^{\mathrm{N}}), f'(k^{\mathrm{N}})) < \frac{1}{p}.$$

This property ensures that $I(w(k), f'(k))$ and $\frac{k}{p}$ have a unique positive intersection, so that the positive steady state $k^{\mathrm{N}} > 0$ is uniquely determined. Since ψ^{N} is monotonically increasing, see Lemma 2.4, it also implies asymptotic stability of the steady state $k^{\mathrm{N}} > 0$. Using (2.18), total differentiation shows that

$$\begin{aligned} & \frac{\mathrm{d}}{\mathrm{d}k} I(w(k^{\mathrm{N}}), f'(k^{\mathrm{N}})) < \frac{1}{p} \\ \Longleftrightarrow \quad & \alpha_u \frac{f'(k^{\mathrm{N}})}{w'(k^{\mathrm{N}})} \left(\frac{p w'(k^{\mathrm{N}}) k^{\mathrm{N}} - k^{\mathrm{N}}}{p w(k^{\mathrm{N}}) - k^{\mathrm{N}}} \right) + \alpha_v \left(1 - \frac{f'(k^{\mathrm{N}})}{w'(k^{\mathrm{N}})} \right) < 1. \end{aligned} \tag{B.29}$$

Since $p w(k^{\mathrm{N}}) > k^{\mathrm{N}}$ and, by Lemma B.1, $0 < w'(k^{\mathrm{N}}) k^{\mathrm{N}} < w(k^{\mathrm{N}})$, Condition (B.29) is in particular satisfied if

$$\frac{f'(k^{\mathrm{N}})}{w'(k^{\mathrm{N}})} (\alpha_u - \alpha_v) < 1 - \alpha_v. \tag{B.30}$$

Since $f' > 0$, $w' > 0$, and $0 < \alpha_u \leq \alpha_v < 1$, we can conclude that (B.30) holds for each $k^{\mathrm{N}} > 0$. Hence, $k^{\mathrm{N}} > 0$ is uniquely determined and asymptotically stable, globally on $\mathbb{R}_{++}$.

Next, let $i = \mathrm{C}$. It follows from (B.25) that in a sufficiently small neighborhood of $k = 0$,

$$D(w(k), p f'(k)) > \frac{k}{p}.$$

Since $D(w(k), p f'(k)) \leq f(k)$ for all $k \geq 0$, it follows from the strict concavity of f together with the property $\lim\limits_{k \to \infty} f'(k) < 1$ that the steady-state condition (2.18) admits a positive solution $k^{\mathrm{C}} > 0$. We next prove that $k^{\mathrm{C}} > k^{\mathrm{N}}$ by contradiction. Assume, for a moment, that $k^{\mathrm{C}} \leq k^{\mathrm{N}}$. By the uniqueness of k^{N}, we then have

$$p I(w(k^{\mathrm{C}}), f'(k^{\mathrm{C}})) \geq p D(w(k^{\mathrm{C}}), p f'(k^{\mathrm{C}})). \tag{B.31}$$

Since (B.31) contradicts Corollary 2.1, it follows that $k^{\mathrm{C}} > k^{\mathrm{N}} > 0$ must hold. Uniqueness and asymptotic stability of the steady state $k^{\mathrm{C}} > 0$ follow from the same arguments as presented above: the assumption $0 < \alpha_u \leq \alpha_v < 1$ implies that, for any $k^{\mathrm{C}} > 0$,

$$\frac{\mathrm{d}D}{\mathrm{d}k}(w(k^{\mathrm{C}}), p f'(k^{\mathrm{C}})) < \frac{1}{p}.$$

Finally, let $i = \mathrm{M}$. The instability of the origin established in Proposition 2.3 implies the existence of a positive steady state $k^{\mathrm{M}} > 0$. This is seen as follows. It follows from (B.20), (B.21), and (2.17) that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}k}\Big(D(w(k), r^{\mathrm{M}}(w(k), f'(k))) + e^{\mathrm{M}}(k,k,k)\Big) &\gtreqless \frac{1}{p} \\ \iff \qquad \frac{\partial \psi^{\mathrm{M}}}{\partial k_t}(\boldsymbol{k}) + \frac{\partial \psi^{\mathrm{M}}}{\partial k_{t-1}}(\boldsymbol{k}) &\gtreqless 1 \\ \iff \qquad \big|\lambda_1(\boldsymbol{k})\big| &\gtreqless 1. \end{aligned} \tag{B.32}$$

In the proof of Proposition 2.3, we have shown that $\lim\limits_{k\to 0} \big|\lambda_1(\boldsymbol{k})\big| = \infty$. Hence, by (B.32),

$$\frac{\mathrm{d}}{\mathrm{d}k}\Big(D\big(w(k), r^{\mathrm{M}}(w(k), f'(k))\big) + e^{\mathrm{M}}(k,k,k)\Big) > \frac{1}{p}.$$

It follows that in a sufficiently small neighborhood of $k = 0$, we have

$$D\big(w(k), r^{\mathrm{M}}(w(k), f'(k))\big) + e^{\mathrm{M}}(k,k,k) > \frac{k}{p}.$$

Since $0 \le D\big(w(k), r^{\mathrm{M}}(w(k), f'(k))\big) + e^{\mathrm{M}}(k,k,k) \le f(k)$ for all $k \ge 0$, the steady-state condition (2.18) has a positive solution $k^{\mathrm{M}} > 0$.

We next show that $k^{\mathrm{M}} > k^{\mathrm{N}} > 0$. Suppose, for a moment, that $0 < k^{\mathrm{M}} \le k^{\mathrm{N}}$. By Corollary 2.1 and the fact that the monopolist's equity in a perfect-foresight steady state must be strictly positive, we obtain a contradiction:

$$\begin{aligned} &p\, I(w(k^{\mathrm{M}}), f'(k^{\mathrm{M}})) \ge p\Big(D\big(w(k^{\mathrm{M}}), r^{\mathrm{M}}(w(k^{\mathrm{M}}), f'(k^{\mathrm{M}}))\big) + e^{\mathrm{M}}(k^{\mathrm{M}}, k^{\mathrm{M}}, k^{\mathrm{M}})\Big) > pD(w(k^{\mathrm{M}}), r^{\mathrm{M}}(\cdot)) \\ \iff \quad & 0 \ge I(w(k^{\mathrm{M}}), f'(k^{\mathrm{M}})) - D\big(w(k^{\mathrm{M}}), r^{\mathrm{M}}(w(k^{\mathrm{M}}), f'(k^{\mathrm{M}}))\big) \ge e^{\mathrm{M}}(k^{\mathrm{M}}, k^{\mathrm{M}}, k^{\mathrm{M}}) > 0. \end{aligned}$$

Hence, $k^{\mathrm{M}} > k^{\mathrm{N}} > 0$ must hold. Again, it is straightforward to verify that the assumption $0 < \alpha_u \le \alpha_v < 1$ implies that, for any $k^{\mathrm{M}} > 0$, we have

$$\frac{\mathrm{d}}{\mathrm{d}k}\Big(D\big(w(k^{\mathrm{M}}), r^{\mathrm{M}}(w(k^{\mathrm{M}}), f'(k^{\mathrm{M}}))\big) + e^{\mathrm{M}}(k^{\mathrm{M}}, k^{\mathrm{M}}, k^{\mathrm{M}})\Big) < \frac{1}{p}. \tag{B.33}$$

Hence, the positive steady state $k^{\mathrm{M}} > 0$ is uniquely determined. Moreover, it follows from (B.33) and (B.32) that $\big|\lambda_1(\boldsymbol{k}^{\mathrm{M}})\big| < 1$. In view of Lemma 2.3, it follows that

$$0 < \big|\lambda_2(\boldsymbol{k}^{\mathrm{M}})\big| \le \big|\lambda_1(\boldsymbol{k}^{\mathrm{M}})\big| < 1,$$

showing that the steady state $k^{\mathrm{M}} > 0$ is asymptotically stable.

(ii). Observe first that $\epsilon_f(k) = \frac{f'(k)k}{f(k)} \to 1$ implies $w(k) = f(k) - f'(k)k \to 0$ and $f'(k)k \to f(k)$. Since the savings and investment functions satisfy $I(w(k), f'(k)) \le D(w(k), pf'(k)) \le w(k)$ for all $k \ge 0$, we can infer from the steady-state condition (2.18) that if $\epsilon_f \to 1$, then $k^{\mathrm{N}} \to 0$ and $k^{\mathrm{C}} \to 0$, whereas $k^{\mathrm{M}} > f'^{-1}(\frac{1}{p}) > 0$ by Lemma B.3. By Theorem 2.1 (i), the positive steady states satisfy $0 < k^{\mathrm{N}} < k^{\mathrm{C}}$. This shows that if the elasticity ϵ_f is sufficiently small, then $0 < k^{\mathrm{N}} < k^{\mathrm{C}} < k^{\mathrm{M}}$.

On the other hand, $\epsilon_f(k) \to 0$ implies $w(k) \to f(k)$ and $f'(k)k \to 0$, in which case the positive steady state $k^{\mathrm{M}} > 0$ of ($\mathrm{PFD}^{\mathrm{M}}$) satisfies

$$\frac{k^{\mathrm{M}}}{p} = D\big(w(k^{\mathrm{M}}), r^{\mathrm{M}}(w(k^{\mathrm{M}}), f'(k^{\mathrm{M}}))\big)\big[1 - r^{\mathrm{M}}(w(k^{\mathrm{M}}), f'(k^{\mathrm{M}}))\big]. \tag{B.34}$$

It follows from Corollary 2.1 and (B.34) that

$$\frac{k^{\mathrm{M}}}{p} = D\big(w(k^{\mathrm{M}}), r^{\mathrm{M}}(w(k^{\mathrm{M}}), f'(k^{\mathrm{M}}))\big)\big[1 - r^{\mathrm{M}}(w(k^{\mathrm{M}}), f'(k^{\mathrm{M}}))\big] < D\big(w(k^{\mathrm{M}}), pf'(k^{\mathrm{M}})\big),$$

showing that ($\mathrm{PFD}^{\mathrm{C}}$) attains a positive steady state $k^{\mathrm{C}} > k^{\mathrm{M}} > 0$. By Theorem 2.1 (i), we have $0 < k^{\mathrm{N}} < k^{\mathrm{M}}$. This shows that if the elasticity ϵ_f is sufficiently small, then $0 < k^{\mathrm{N}} < k^{\mathrm{M}} < k^{\mathrm{C}}$. ☐

Proof of Corollary 2.2. Let $k_t > 0$ and $k_t^e > 0$ be arbitrary but fixed. Since, by (B.12), welfare of agents is strictly increasing in the deposit rate, the second inequality follows from the fact that $r^{\mathrm{M}}(w(k_t), f'(k_t^e)) < r^{\mathrm{C}}(w(k_t), f'(k_t^e))$ by Proposition 2.2. If $b(w(k_t), f'(k_t^e)) \geq s(w(k_t), f'(k_t^e))$, then the participation constraint (PC) is binding, so that the first inequality must hold with equality. It now follows from (B.12) that the first inequality is strict whenever $s(w(k_t), f'(k_t^e)) > b(w(k_t), f'(k_t^e))$. ☐

Proof of Lemma 2.4. Consider the maximization problem

$$\max_{\bar{c}^1, \bar{c}^2, \bar{k} \geq 0} \quad u(\bar{c}^1) + v(\bar{c}^2) \quad \text{s.t.} \quad \bar{c}^1 + \bar{c}^2 = \phi(\bar{k}). \tag{B.35}$$

A solution $(c_G^1, c_G^2, k_G) \gg 0$ to Problem (B.35) must satisfy

$$u'(\phi(k_G) - c_G^2) \overset{!}{=} v'(c_G^2) \tag{B.36}$$

$$u'(\phi(k_G) - c_G^2)\phi'(k_G) \overset{!}{=} 0 \tag{B.37}$$

$$c_G^1 + c_G^2 \overset{!}{=} \phi(k_G). \tag{B.38}$$

Since $u' > 0$, (B.37) implies that

$$\phi'(k_G) \overset{!}{=} 0 \iff f'(k_G) \overset{!}{=} \tfrac{1}{p}. \tag{B.39}$$

Assumption 2.2 ensures that (B.39) admits a unique solution $k_G = f'^{-1}\big(\frac{1}{p}\big) > 0$. Since $\phi(k_G) > 0$, it follows from Assumption 2.1 that $(c_G^1, c_G^2) \gg 0$ is uniquely determined by (B.36) together with (B.38). ☐

Proof of Theorem 2.2. *(i).* Welfare in a steady state $k^{\mathrm{C}} > 0$ of ($\mathrm{PFD}^{\mathrm{C}}$) is

$$\mathcal{W}^{\mathrm{C}}(k^{\mathrm{C}}, k^{\mathrm{C}}) = u\big(w(k^{\mathrm{C}}) - D(w(k^{\mathrm{C}}), pf'(k^{\mathrm{C}}))\big) + v\big(pf'(k^{\mathrm{C}})D(w(k^{\mathrm{C}}), pf'(k^{\mathrm{C}}))\big). \tag{B.40}$$

Using the envelope theorem and the steady-state condition (2.18), it follows that (B.40) attains its maximum if and only if k^{C} satisfies

$$\frac{u'\big(w(k^{\mathrm{C}}) - D(w(k^{\mathrm{C}}), pf'(k^{\mathrm{C}}))\big)}{v'\big(pf'(k^{\mathrm{C}})\, D(w(k^{\mathrm{C}}), pf'(k^{\mathrm{C}}))\big)} \overset{!}{=} 1. \tag{B.41}$$

The first-order condition (2.5) implies that $k^{\mathrm{C}} > 0$ satisfies

$$\frac{u'\big(w(k^{\mathrm{C}}) - D(w(k^{\mathrm{C}}), pf'(k^{\mathrm{C}}))\big)}{v'\big(pf'(k^{\mathrm{C}}) D(w(k^{\mathrm{C}}), pf'(k^{\mathrm{C}}))\big)} = pf'(k^{\mathrm{C}}). \tag{B.42}$$

Inserting (B.42) into (B.41), we see that (B.40) attains its maximum if and only if $pf'(k^{\mathrm{C}}) \overset{!}{=} 1$, that is, $k^{\mathrm{C}} \overset{!}{=} k_G$.

Given any steady states $0 < k^{\mathrm{N}}$ and $0 < k^{\mathrm{C}} \leq k_G$, Theorem 2.1 implies that $0 < k^{\mathrm{N}} < k^{\mathrm{C}} \leq k_G$. By Corollary 2.2, we have

$$\mathcal{W}^{\mathrm{N}}(k^{\mathrm{N}}, k^{\mathrm{N}}) < \mathcal{W}^{\mathrm{C}}(k^{\mathrm{N}}, k^{\mathrm{N}}). \tag{B.43}$$

Since $k \mapsto \mathcal{W}^{\mathrm{C}}(k, k)$ is strictly increasing on $[0, k_G]$, it follows from (B.43) that

$$\mathcal{W}^{\mathrm{N}}(k^{\mathrm{N}}, k^{\mathrm{N}}) < \mathcal{W}^{\mathrm{C}}(k^{\mathrm{N}}, k^{\mathrm{N}}) < \mathcal{W}^{\mathrm{C}}(k^{\mathrm{C}}, k^{\mathrm{C}}) \leq \mathcal{W}^{\mathrm{C}}(k_G, k_G).$$

Hence, $\mathcal{W}^{\mathrm{N}}(k^{\mathrm{N}}, k^{\mathrm{N}}) < \mathcal{W}^{\mathrm{C}}(k^{\mathrm{C}}, k^{\mathrm{C}})$ whenever $0 < k^{\mathrm{C}} \leq k_G$.

(ii). We first establish the first claim. In the proof of Theorem 2.1 (ii), we have shown that the positive steady states $k^{\mathrm{N}}, k^{\mathrm{C}} > 0$ of ($\mathrm{PFD}^{\mathrm{N}}$) and ($\mathrm{PFD}^{\mathrm{C}}$), respectively, become arbitrarily small as $\epsilon_f \to 1$, so that

$$\mathcal{W}^{\mathrm{N}}(k^{\mathrm{N}}, k^{\mathrm{N}}) \;\to\; u(0) \quad \text{and} \quad \mathcal{W}^{\mathrm{C}}(k^{\mathrm{C}}, k^{\mathrm{C}}) \;\to\; u(0).$$

On the other hand, Lemma B.3 implies that $k^{\mathrm{M}} > k_G > 0$, even as $\epsilon_f \to 1$. It follows that if ϵ_f is sufficiently close to one, then $0 < k^{\mathrm{N}} < k^{\mathrm{C}} << k_G < k^{\mathrm{M}}$ and Theorem 2.2 (i) implies that the steady-state welfare levels satisfy

$$\mathcal{W}^{\mathrm{N}}(k^{\mathrm{N}}, k^{\mathrm{N}}) < \mathcal{W}^{\mathrm{C}}(k^{\mathrm{C}}, k^{\mathrm{C}}) < \mathcal{W}^{\mathrm{M}}(k^{\mathrm{M}}, k^{\mathrm{M}}).$$

We next establish the second claim. Let $i = \mathrm{C}, \mathrm{M}$. If $\epsilon_f \to 0$, then $r^i(w(k^i), f'(k^i))\, D\big(w(k^i), r^i(w(k^i), f'(k^i))\big) \to 0$ because $f'(k^i)k^i \to 0$, $w(k^i) \to f(k^i)$, and

$$0 \leq r^i(w(k^i), f'(k^i))\, D\big(w(k^i), r^i(w(k^i), f'(k^i))\big) \leq f'(k^i)k^i.$$

This, in turn, implies that if $\epsilon_f \to 0$, then

$$\mathcal{W}^{\mathrm{C}}(k^{\mathrm{C}}, k^{\mathrm{C}}) \to u(w(k^{\mathrm{C}})) \quad \text{and} \quad \mathcal{W}^{\mathrm{M}}(k^{\mathrm{M}}, k^{\mathrm{M}}) \to u(w(k^{\mathrm{M}})). \tag{B.44}$$

Since, by Theorem 2.1 (ii), $\epsilon_f \to 0$ implies $k^{\mathrm{C}} > k^{\mathrm{M}}$ and, therefore, $w(k^{\mathrm{C}}) > w(k^{\mathrm{M}})$, it follows that the steady-state welfare levels in (B.44) satisfy $u(w(k^{\mathrm{C}})) > u(w(k^{\mathrm{M}}))$ because $u' > 0$. This shows that if ϵ_f is sufficiently close to zero, then

$$\mathcal{W}^{\mathrm{C}}(k^{\mathrm{C}}, k^{\mathrm{C}}) > \mathcal{W}^{\mathrm{M}}(k^{\mathrm{M}}, k^{\mathrm{M}}).$$

The proof is now complete. ☐

Proof of Proposition 2.4. The derivatives of the perfect forecasting rules ψ^{N} and ψ^{C} stated in Proposition 2.4 obtain from implicit differentiation of ($\mathrm{PFD}^{\mathrm{N}}$) and ($\mathrm{PFD}^{\mathrm{C}}$), respectively. Since $w' > 0$ by Lemma B.1 and $f'' < 0$ by Assumption 2.2, it follows from Lemma B.2 (i) and (ii) that both derivatives are strictly positive. ☐

Proof of Proposition 2.5. By Theorem 2.1 (i), the steady state $k^{\mathrm{M}} > 0$ of ($\mathrm{PFD}^{\mathrm{M}}$) is asymptotically stable and thus hyperbolic. The Hartman-Grobman theorem implies that, in a local neighborhood of $k^{\mathrm{M}} > 0$, the dynamics of the linearized system (2.15) is qualitatively equivalent to the dynamics of the non-linear system ($\mathrm{PFD}^{\mathrm{M}}$). We will show that both eigenvalues (2.17) are positive real numbers. Let us first demonstrate that the eigenvalues are real. The eigenvalues (2.17) are real if and only if

$$\frac{1}{4}\Big(\frac{\partial\psi^{\mathrm{M}}}{\partial k_t}(\boldsymbol{k}^{\mathrm{M}})\Big)^2 + \frac{\partial\psi^{\mathrm{M}}}{\partial k_{t-1}}(\boldsymbol{k}^{\mathrm{M}}) \geq 0. \tag{B.45}$$

Using (B.20) and (B.21), it is readily seen that (B.45) is satisfied if

$$\frac{\mathrm{d}}{\mathrm{d}k}\Big(D\big(w(k^{\mathrm{M}}), r^{\mathrm{M}}(w(k^{\mathrm{M}}), f'(k^{\mathrm{M}}))\big) + e^{\mathrm{M}}(k^{\mathrm{M}}, k^{\mathrm{M}}, k^{\mathrm{M}})\Big) > 0. \tag{B.46}$$

Calculating the total derivative in (B.46) yields

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}k}\Big(D\big(w(k^{\mathrm{M}}), r^{\mathrm{M}}(w(k^{\mathrm{M}}), f'(k^{\mathrm{M}}))\big) + e^{\mathrm{M}}(k^{\mathrm{M}}, k^{\mathrm{M}}, k^{\mathrm{M}})\Big)\\ &= f'(k^{\mathrm{M}})\big[1+\epsilon_{f'}(k^{\mathrm{M}})\big] + \frac{\mathrm{d}}{\mathrm{d}k}\Big(D\big(w(k^{\mathrm{M}}), r^{\mathrm{M}}(w(k^{\mathrm{M}}), f'(k^{\mathrm{M}}))\big)\big[1 - r^{\mathrm{M}}(w(k^{\mathrm{M}}), f'(k^{\mathrm{M}}))\big]\Big).\end{aligned}$$

Assumption 2.2 implies that $f'(k^{\mathrm{M}})\big[1+\epsilon_{f'}(k^{\mathrm{M}})\big] > 0$, so that the first summand is strictly positive. By Lemma B.3, the positive steady state $k^{\mathrm{M}} > 0$ satisfies $k^{\mathrm{M}} > k_G$. Since stationary total consumption per capita ϕ is a hump-shaped function that attains its global maximum at k_G, it must be the case that $\phi'(k^{\mathrm{M}}) < 0$. From the identity

$$\phi(k) \equiv w(k) - D\big(w(k), r^{\mathrm{M}}(w(k), f'(k))\big)\big[1 - r^{\mathrm{M}}(w(k), f'(k))\big]$$

and the observation that $\phi'(k^{\mathrm{M}}) < 0$, we can now conclude that

$$0 < w'(k^{\mathrm{M}}) < \frac{\mathrm{d}}{\mathrm{d}k}\Big(D\big(w(k^{\mathrm{M}}), r^{\mathrm{M}}(w(k^{\mathrm{M}}), f'(k^{\mathrm{M}}))\big)\big[1 - r^{\mathrm{M}}(w(k^{\mathrm{M}}), f'(k^{\mathrm{M}}))\big]\Big).$$

Hence, the second summand is also strictly positive, implying that (B.46) and hence (B.45) holds. Hence, the eigenvalues (2.17) are real. The fact that the eigenvalues (2.17) are positive follows from (B.22). Positive real eigenvalues rule out that the linearized dynamics (2.15) fluctuates in a local neighborhood of the steady state $k^{\mathrm{M}} > 0$. Hence, in a local neighborhood of $k^{\mathrm{M}} > 0$, the dynamics ($\mathrm{PFD}^{\mathrm{M}}$) must be monotonic. ☐

Proof of Lemma 2.5. Let $w_t > 0$, $\rho^e_t > 0$, $\vartheta^e_t \geq 0$, and $e_t \geq 0$ with $p\rho^e_t > \frac{u'(w_t)}{v'(\vartheta^e_t)}$ be arbitrary but fixed. The price elasticity

$$\eta(w_t, r, \vartheta^e_t) := \frac{\frac{\partial D}{\partial r}(w_t, r, \vartheta^e_t) r}{D(w_t, r, \vartheta^e_t)}$$

is well defined and strictly positive for all $r > \frac{u'(w_t)}{v'(\vartheta_t^e)}$. A solution to Problem (2.22) is either determined by a solution $s(w_t, \rho_t^e, \vartheta_t^e)$ to the first-order condition for a profit maximum

$$\Gamma_t(r) := \frac{p\rho_t^e - r}{r} - \frac{1}{\eta(w_t, r, \vartheta_t^e)} \overset{!}{=} 0, \tag{B.47}$$

or by a solution $b(w_t, \rho_t^e, \vartheta_t^e)$ to the binding participation constraint

$$u(w_t - D(w_t, r, \vartheta_t^e)) + v(rD(w_t, r, \vartheta_t^e) + \vartheta_t^e) \overset{!}{=} \mathscr{U}_{\text{res}}(w_t, \rho_t^e). \tag{B.48}$$

It follows from (B.6) that

$$\lim_{r \to \frac{u'(w_t)}{v'(\vartheta_t^e)}} \eta(w_t, r, \vartheta_t^e) = \infty,$$

so that

$$\lim_{r \to \frac{u'(w_t)}{v'(\vartheta_t^e)}} \Gamma_t(r) = \frac{p\rho_t^e - \frac{u'(w_t)}{v'(\vartheta_t^e)}}{\frac{u'(w_t)}{v'(\vartheta_t^e)}} > 0.$$

On the other hand,

$$\lim_{r \to p\rho_t^e} \Gamma_t(r) = -\frac{1}{\eta(w_t, p\rho_t^e, \vartheta_t^e)} < 0.$$

Since, by Lemma B.2 (v),

$$\Gamma_t'(r) = -\frac{p\rho_t^e}{r^2} + \frac{1}{\eta(w_t, r, \vartheta_t^e)^2} \frac{\partial \eta}{\partial r}(w_t, r, \vartheta_t^e) < 0$$

for all $r \in \left(\frac{u'(w_t)}{v'(\vartheta_t^e)}, p\rho_t^e\right)$, it follows that (B.47) admits a unique solution $s(w_t, \rho_t^e, \vartheta_t^e) \in \left(\frac{u'(w_t)}{v'(\vartheta_t^e)}, p\rho_t^e\right)$. For $r \to \frac{u'(w_t)}{v'(\vartheta_t^e)}$, the l.h.s. in (B.48) becomes $u(w_t) + v(\vartheta_t^e)$. By contrast, for $r = p\rho_t^e$, it is

$$\begin{aligned}
& u(w_t - D(w_t, p\rho_t^e, \vartheta_t^e)) + v(p\rho_t^e D(w_t, p\rho_t^e, \vartheta_t^e) + \vartheta_t^e) \\
\geq\ & u(w_t - I(w_t, \rho_t^e)) + v(p\rho_t^e I(w_t, \rho_t^e) + \vartheta_t^e) \\
\geq\ & u(w_t - I(w_t, \rho_t^e)) + v(p\rho_t^e I(w_t, \rho_t^e)) \\
>\ & \mathscr{U}_{\text{res}}(w_t, \rho_t^e),
\end{aligned}$$

where the first inequality follows from (2.21), the second inequality from the fact that $v' > 0$ and $\vartheta_t^e \geq 0$, and the last inequality from the strict concavity of v. Since the l.h.s. in (B.48) is strictly increasing in r, two cases must be distinguished:

Case 1. If $u(w_t) + v(\vartheta_t^e) < \mathscr{U}_{\text{res}}(w_t, \rho_t^e)$, then the existence of a unique solution $b(w_t, \rho_t^e, \vartheta_t^e) \in \left(\frac{u'(w_t)}{v'(\vartheta_t^e)}, p\rho_t^e\right)$ to (B.48) follows from the intermediate-value theorem.

Case 2. If $u(w_t) + v(\vartheta_t^e) \geq \mathscr{U}_{\text{res}}(w_t, \rho_t^e)$, then no solution to (B.48) exists. Instead, the participation constraint is satisfied for *any* positive deposit rate. In particular, it is satisfied for the deposit rate $s(w_t, \rho_t^e, \vartheta_t^e)$ that satisfies (B.47).

This establishes that Problem (2.22) admits a unique solution, given by

$$r_t^{\mathrm{M}} = r^{\mathrm{M}}(w_t, \rho_t^e, \vartheta_t^e) := \begin{cases} \max\bigl\{b(w_t, \rho_t^e, \vartheta_t^e), s(w_t, \rho_t^e, \vartheta_t^e)\bigr\} & \text{if } u(w_t) + v(\vartheta_t^e) < \mathscr{U}_{\mathrm{res}}(w_t, \rho_t^e), \\ s(w_t, \rho_t^e, \vartheta_t^e) & \text{otherwise.} \end{cases}$$

□

Proof of Lemma 2.6. We first establish the existence of φ. Given k_t and a correct forecast k_t^e for k_{t+1}, a correct forecast ϑ_t^e for ϑ_{t+1} is determined by a solution to

$$\vartheta^e \overset{!}{=} \mu e^{\mathrm{M}}(k_t^e, k_t, k_t^e, \vartheta^e). \tag{B.49}$$

For $\vartheta^e = 0$, the l.h.s. in (B.49) is zero, while the r.h.s. takes the value $\mu e^{\mathrm{M}}(k_t^e, k_t, k_t^e, 0)$. For $\vartheta^e \to \infty$, the l.h.s. in (B.49) grows infinitely large, whereas the r.h.s. in (B.49) remains finite because it is bounded from above by $f(k_t)$. Therefore, the intermediate-value theorem ensures that a solution $0 \le \vartheta_t^e < \infty$ to (B.49) exists if $(k_t^e, k_t) \in \mathcal{D}^{\mathrm{M}}$, where, as before,

$$\mathcal{D}^{\mathrm{M}} := \Bigl\{(k, \hat{k}) \in \mathbb{R}_+^2 \mid e^{\mathrm{M}}(k, \hat{k}, k, 0) \ge 0\Bigr\}.$$

This solution defines a map

$$\varphi : \mathcal{D}^{\mathrm{M}} \to \mathbb{R}_+, \qquad \vartheta_t^e = \varphi(k_t^e, k_t),$$

that satisfies

$$\varphi(k, \hat{k}) = \mu e^{\mathrm{M}}\bigl(k, \hat{k}, k, \varphi(k, \hat{k})\bigr) \ge 0 \quad \text{for all} \quad (k, \hat{k}) \in \mathcal{D}^{\mathrm{M}}. \tag{B.50}$$

Observe that $\mathcal{D}^{\mathrm{M}}$ is non-empty. We next show that there indeed exists a correct forecast k_t^e. Given k_t and k_{t-1}, it follows from the economic law of capital accumulation that a correct forecast k_t^e for k_{t+1} is determined by a solution to

$$\begin{aligned} \frac{k^e}{p} - D\bigl(w(k_t), r^{\mathrm{M}}(w(k_t), f'(k^e), \varphi(k^e, k_t)), \varphi(k^e, k_t)\bigr) \\ \overset{!}{=} (1-\mu) e^{\mathrm{M}}(k_t, k_{t-1}, k_t, \varphi(k_t, k_{t-1})). \end{aligned} \tag{B.51}$$

Define the set

$$\mathcal{C}^{\mathrm{M}} := \Bigl\{(k, \hat{k}) \in \mathbb{R}_+^2 \mid e^{\mathrm{M}}\bigl(k, \hat{k}, k, \varphi(k, \hat{k})\bigr) \ge 0\Bigr\},$$

then (B.50) implies $\mathcal{D}^{\mathrm{M}} \subseteq \mathcal{C}^{\mathrm{M}}$. Invoking the intermediate-value theorem, we see that a solution k_t^e to (B.51) exists whenever $(k_t, k_{t-1}) \in \mathcal{D}^{\mathrm{M}}$ because both $D(w(k_t), r^{\mathrm{M}}(w(k_t), f'(k^e), \varphi(k^e, k_t)), \varphi(k^e, k_t))$ and $e^{\mathrm{M}}(k_t, k_{t-1}, k_t, \varphi(k_t, k_{t-1}))$ are bounded from above by $f(k_t)$. Since this solution satisfies $(k_t^e, k_t) \in \mathcal{D}^{\mathrm{M}}$, it follows that for each $(k_t, k_{t-1}) \in \mathcal{D}^{\mathrm{M}}$, there exists a pair of correct forecasts (k_t^e, ϑ_t^e). □

Appendix C

Proofs of Chapter 3

Proof of Lemma 3.1. The time-one map $G : \mathbb{R}_+ \to \mathbb{R}_+$ is defined by

$$G(k) = \tfrac{1}{1+n} s\big(w(k), R(G(k))\big), \quad k \in \mathbb{R}_+. \tag{C.1}$$

Observe first that $G(k) \leq w(k)$ for all $k \geq 0$. Since Assumption 3.2 implies $\lim\limits_{k\to\infty} \frac{w(k)}{k} = 0$, it follows that $\lim\limits_{k\to\infty} \frac{G(k)}{k} = 0$, so that $G(k) < k$ for all sufficiently large k. Hence, any perfect-foresight growth path is bounded. Implicit differentiation of (C.1) yields

$$G'(k) = \frac{\frac{\partial s}{\partial w}\big(w(k), R(G(k))\big) w'(k)}{1 + n - \frac{\partial s}{\partial R^e}\big(w(k), R(G(k))\big) f''(G(k))}. \tag{C.2}$$

Since youthful and old-age consumption are normal goods and weak gross substitutes by Assumption 3.1, it is straightforward to verify that

$$\frac{\partial s(w, R^e)}{\partial w} \in (0,1) \quad \text{and} \quad \frac{\partial s(w, R^e)}{\partial R^e} \geq 0 \quad \text{for all} \quad (w, R^e) \in \mathbb{R}_+ \times \mathbb{R}_{++}.$$

Since $w' > 0$ and $f'' < 0$ by Assumption 3.2, it follows that (C.2) is strictly positive, so that $G' > 0$. Hence, any perfect-foresight growth path is monotonic. □

Proof of Lemma 3.2. Let $w_t \in \mathbb{R}_{++}$, $N_{t+1} = (1+n)N_t \in \mathbb{N}_1$, and $\boldsymbol{s}_t^{-i} \in [0, w_t]^{N_t - 1}$ with corresponding $S_t^{-i} = \|\boldsymbol{s}_t^{-i}\|_1 \in \mathbb{R}_+$ be arbitrary but fixed. Assumptions 3.1 and 3.2 imply that the objective function $s^i \mapsto \mathscr{U}(s^i, \boldsymbol{s}_t^{-i}; w_t, N_t)$ in Problem (3.11) is either already continuous or can be monotonically transformed into a continuous function. Since Problem (3.11) is a maximization problem on the compact interval $[0, w_t]$ with a continuous objective function, the extreme-value theorem ensures the existence of a maximizer s_t^i. Since Assumption 3.2 implies that the capital income $s^i \mapsto R\big(\frac{s^i + S_t^{-i}}{N_{t+1}}\big) s^i$ is zero if $s^i = 0$, it follows from the Inada conditions imposed on u and v in Assumption 3.1 that $s_t^i \in (0, w_t)$. Thus, s_t^i is determined by the first-order condition (3.12). Setting $\epsilon_{f''}(k) = f'''(k)k/f''(k)$, then Assumption 3.2 implies that $\epsilon_{f''}(k) > -2$ for all $k \geq 0$. Since $u', v' > 0$, $u'', v'' < 0$, and $R' < 0$, it follows that the objective function in Problem (3.11) is strictly concave because

$$\begin{aligned} \frac{\partial^2 \mathscr{U}}{\partial (s^i)^2}(s^i, \boldsymbol{s}_t^{-i}; w_t, N_t) = u''(w_t - s^i) + v''\big(R\big(\tfrac{s^i + S_t^{-i}}{N_{t+1}}\big) s^i\big) \left[R\big(\tfrac{s^i + S_t^{-i}}{N_{t+1}}\big) + R'\big(\tfrac{s^i + S_t^{-i}}{N_{t+1}}\big) \tfrac{s^i}{N_{t+1}} \right]^2 \\ + \tfrac{1}{N_{t+1}} v'\big(R\big(\tfrac{s^i + S_t^{-i}}{N_{t+1}}\big) s^i\big) R'\big(\tfrac{s^i + S_t^{-i}}{N_{t+1}}\big) \left[2 + \tfrac{s^i}{s^i + S_t^{-i}} \epsilon_{f''}\big(\tfrac{s^i + S_t^{-i}}{N_{t+1}}\big) \right] < 0 \end{aligned}$$

for all $s^i \in (0, w_t)$. Hence, (3.12) admits a unique solution $s_t^i \in (0, w_t)$. □

Proof of Proposition 3.1. Let $w_t \in \mathbb{R}_{++}$ and $N_t \in \mathbb{N}_1$ be arbitrary but fixed. The symmetry of young agents implies symmetry of best responses, so that

$$S_t^{-i} = (N_t - 1)s_t^i, \quad \frac{s_t^i + S_t^{-i}}{N_{t+1}} = \frac{N_t s_t^i}{N_{t+1}} = \tfrac{1}{1+n} s_t^i, \quad s_t^i = s_t \quad \text{for all} \quad i \in \{1, \dots, N_t\}. \tag{C.3}$$

Substituting (C.3) into the first-order condition (3.12) yields (3.13). We establish the existence and uniqueness of a solution to (3.13). Assumption 3.2 implies that $\lim\limits_{k\to 0} R(k)k = 0$, see De La Croix and Michel (2002, p. 308). It now follows from the Inada conditions on u and v that the l.h.s. in (3.13) is zero if $s_t = 0$, and infinity if $s_t = w_t$. Since $f' > 0$ and $\epsilon_{f'} > -1$, the r.h.s. in (3.13) is strictly positive. Since all expressions are continuous, the intermediate-value theorem ensures the existence of a solution $s_t^* \in (0, w_t)$ to (3.13). Uniqueness is seen as follows. Differentiating the r.h.s. in (3.13) w.r.t. s yields

$$\tfrac{1}{(1+n)N_t} f''\big(\tfrac{1}{1+n}s\big)\Big[1 + N_t + \epsilon_{f''}\big(\tfrac{1}{1+n}s\big)\Big],$$

which is strictly negative for all $s \in [0, w_t]$ because $f'' < 0$, $N_t \geq 1$, and $\epsilon_{f''} > -2$. Differentiating the l.h.s. in (3.13) w.r.t. s yields

$$-\frac{u''(w_t - s)v'\big(R\big(\tfrac{1}{1+n}s\big)s\big) + u'(w_t - s)v''\big(R\big(\tfrac{1}{1+n}s\big)s\big)R\big(\tfrac{1}{1+n}s\big)\big[1 + \epsilon_{f'}\big(\tfrac{1}{1+n}s\big)\big]}{v'\big(R\big(\tfrac{1}{1+n}s\big)s\big)^2},$$

which is strictly positive for all $s \in [0, w_t]$ since $u', v' > 0$, $u'', v'' < 0$, and $\epsilon_{f'} > -1$. Hence, (3.13) admits a unique solution $s_t^* \in (0, w_t)$. □

Proof of Proposition 3.2. The perfect forecasting rule $\psi : \mathbb{R}_+ \to \mathbb{R}_{++}$ is defined by

$$\psi(w) = 1 - \delta + f'\big(\tfrac{1}{1+n}s(w, \psi(w))\big), \quad w \in \mathbb{R}_+. \tag{C.4}$$

It follows from (3.3) and (C.4) that, for each $w > 0$, the savings level $s = s(w, \psi(w))$ satisfies

$$\frac{u'(w - s)}{v'\big(R\big(\tfrac{1}{1+n}s\big)s\big)} = 1 - \delta + f'\big(\tfrac{1}{1+n}s\big). \tag{C.5}$$

On the other hand, it follows from (3.13) that, for each $w > 0$ and each $N \in \mathbb{N}_1$, the savings level $s = \mathfrak{s}(w, N)$ satisfies

$$\frac{u'(w - s)}{v'\big(R\big(\tfrac{1}{1+n}s\big)s\big)} = 1 - \delta + f'\big(\tfrac{1}{1+n}s\big)\left[1 + \frac{\epsilon_{f'}\big(\tfrac{1}{1+n}s\big)}{N}\right]. \tag{C.6}$$

Assumptions 3.1 and 3.2 imply that the marginal rate of substitution on the left-hand sides of (C.5) and (C.6) is strictly increasing in s, whereas the respective right-hand sides are strictly decreasing in s. Since the term in brackets in (C.6) lies between zero and unity, we can conclude that the savings levels determined by (C.5) and (C.6) satisfy $\mathfrak{s}(w, N) < s(w, \psi(w))$ for all $(w, N) \in \mathbb{R}_{++} \times \mathbb{N}_1$. The claim that $\lim\limits_{N\to\infty} \mathfrak{s}(w, N) = s(w, \psi(w))$ follows from the fact that for $N \to \infty$, the first-order conditions (C.5) and (C.6) coincide. □

Proof of Corollary 3.1. The claim follows directly from Proposition 3.2 and the definition of the capital accumulation laws G and $\mathcal{G}$. ☐

Proof of Lemma 3.3. Setting $s = \mathfrak{s}(w, N)$, then implicit differentiation of (3.13) yields

$$\begin{aligned}\frac{\partial \mathfrak{s}(w,N)}{\partial w} &= u''(w-s)\Big[u''(w-s) + v''\big(R\big(\tfrac{1}{1+n}s\big)s\big) \\ &\quad\cdot \Big(1-\delta+f'\big(\tfrac{1}{1+n}s\big)\big[1+\epsilon_{f'}\big(\tfrac{1}{1+n}s\big)N^{-1}\big]\Big)\Big(1-\delta+f'\big(\tfrac{1}{1+n}s\big)\big[1+\epsilon_{f'}\big(\tfrac{1}{1+n}s\big)\big]\Big) \\ &\quad + v'\big(R\big(\tfrac{1}{1+n}s\big)s\big)f''\big(\tfrac{1}{1+n}s\big)\tfrac{1}{1+n}N^{-1}\big[1+N+\epsilon_{f''}\big(\tfrac{1}{1+n}s\big)\big]\Big]^{-1}.\end{aligned} \tag{C.7}$$

By Assumptions 3.1 and 3.2, we have $u', v' > 0$, $u'', v'' < 0$, $f' > 0$, $f'' < 0$, $\epsilon_{f'} > -1$, $\epsilon_{f''} > -2$. Since $N \geq 1$, we can read off (C.7) that $\frac{\partial \mathfrak{s}}{\partial w}(w, N) \in (0, 1)$ for all $(w, N) \in \mathbb{R}_{++} \times \mathbb{N}_1$. Since $w' > 0$, straightforward differentiation of (3.14) establishes that

$$\frac{\partial \mathcal{G}(k,N)}{\partial k} = \tfrac{1}{1+n}\frac{\partial \mathfrak{s}(w(k),N)}{\partial w}w'(k) > 0 \quad \text{for all} \quad (k,N) \in \mathbb{R}_{++} \times \mathbb{N}_1.$$

Hence, $k \mapsto \mathcal{G}(k, N)$ is strictly increasing for each $N \in \mathbb{N}_1$. ☐

Proof of Theorem 3.1. Let the initial condition $(k_0, N_0) \in \mathbb{R}_{++} \times \mathbb{N}_1$ be arbitrary but fixed, and the corresponding perfect-foresight growth path $\{k_t^{\mathrm{PF}}\}_{t=0}^{\infty}$ and Nash-equilibrium growth path $\{k_t^*\}_{t=0}^{\infty}$ be given. Assumption 3.2 rules out unbounded growth, so that both sequences $\{k_t^{\mathrm{PF}}\}_{t=0}^{\infty}$ and $\{k_t^*\}_{t=0}^{\infty}$ are bounded. Since poverty traps are ruled out, $\{k_t^{\mathrm{PF}}\}_{t=0}^{\infty}$ attains a positive but finite limit $0 < k_{\mathrm{PF}} < \infty$. We first show that $\{k_t^*\}_{t=0}^{\infty}$ also attains a finite limit. There are two cases, depending on the population growth rate n.

(i) If $n > 0$, then $\lim\limits_{t\to\infty} N_t = \infty$ and Corollary 3.1 implies that $k_* = \lim\limits_{t\to\infty} k_t^* = k_{\mathrm{PF}} < \infty$.
(ii) If $n = 0$, then $N_t = N_0$ for all $t \geq 0$. Since, by Lemma 3.3, $k \mapsto \mathcal{G}(k, N_0)$ is strictly increasing, it follows that $\{k_t^*\}_{t=0}^{\infty}$ is monotonic and bounded and, therefore, attains a finite limit $0 \leq k_* < \infty$.

We next demonstrate that $k_* < k_{\mathrm{PF}}$ whenever $n = 0$. Since, generically, k_* is an asymptotically stable steady state of $\mathcal{G}(\cdot, N_0)$, it satisfies

$$k_* = \mathcal{G}(k_*, N_0) \quad \text{and} \quad \frac{\partial \mathcal{G}(k_*, N_0)}{\partial k} \in (0,1).$$

Analogously, k_{PF} is generically an asymptotically stable steady state of G and thus satisfies

$$k_{\mathrm{PF}} = G(k_{\mathrm{PF}}) \quad \text{and} \quad G'(k_{\mathrm{PF}}) \in (0,1).$$

Since both capital accumulation laws $k \mapsto G(k)$ and $k \mapsto \mathcal{G}(k, N_0)$ are strictly increasing and, by Corollary 3.1, satisfy

$$\mathcal{G}(k, N_0) < G(k) \quad \text{for all} \quad k > 0,$$

it follows from the phase portrait in the (k_t, k_{t+1})-plane that for any given initial capital-labor ratio $k_0 > 0$, the growth path $\{k_t^*\}_{t=0}^{\infty}$ recursively defined by $k_{t+1}^* = \mathcal{G}(k_t^*, N_0)$ with $k_0^* = k_0$ converges monotonically to a strictly smaller limit than the path $\{k_t^{\mathrm{PF}}\}_{t=0}^{\infty}$ recursively defined by $k_{t+1}^{\mathrm{PF}} = G(k_t^{\mathrm{PF}})$ with $k_0^{\mathrm{PF}} = k_0$. This result holds true independently of the uniqueness of the positive steady states of $\mathcal{G}(\cdot, N_0)$ and G. Moreover, it also holds true in the non-generic case where k_0 coincides with an unstable steady state of either $\mathcal{G}(\cdot, N_0)$ or G. $\square$

Appendix D

Proofs of Chapter 4

Proof of Lemma 4.1. It follows from (4.3) that the Pareto-efficiency condition (iii) in Definition 4.1 is by construction satisfied for all $\omega \geq 0$. Moreover, for each $\omega \notin \big[\Omega_{\min}(k), \Omega_{\max}(k)\big]$, either $\kappa_1(\omega), \kappa_2(\omega) > k$ or $\kappa_1(\omega), \kappa_2(\omega) < k$. In both cases, no labor shares $l_1, l_2 \in [0,1]$ exist that satisfy the feasibility constraints (i) and (ii) in Definition 4.1. Thus, it remains to be shown that the factor allocations given in Lemma 4.1 (i) and (ii) are feasible.

(i). If $\Omega_1(k) \neq \Omega_2(k)$, then it is straightforward to verify that $k_j = \kappa_j(\omega)$, $j = 1, 2$ together with $l_j = \ell_j(k, \omega)$, $j = 1, 2$ satisfy the feasibility constraints (i) and (ii) in Definition 4.1 for all $\omega \in \big[\Omega_{\min}(k), \Omega_{\max}(k)\big]$.

(ii). If $\Omega_1(k) = \Omega_2(k)$, then $\kappa_1(\Omega_1(k)) = \kappa_2(\Omega_2(k)) = k$, so that the feasibility constraint (i) holds whenever (ii) holds. Hence, any labor shares $l_1 \in [0,1]$ and $l_2 = 1 - 1_1$ are feasible. ☐

Proof of Theorem 4.1. *Step 1 (Monotonicity of the output functions).* Since the output functions y_1 and y_2 defined in (4.8) are symmetric, establishing monotonicity for y_1 suffices. For each $k > 0$ with $\Omega_1(k) \neq \Omega_2(k)$ and each $\omega \in \big[\Omega_{\min}(k), \Omega_{\max}(k)\big]$, the derivative of the labor-share function ℓ_1 defined in (4.5) takes the form

$$\begin{aligned}\frac{\partial \ell_1}{\partial \omega}(k, \omega) &= \frac{\big[\kappa_2(\omega) - k\big]\kappa_1'(\omega) + \big[k - \kappa_1(\omega)\big]\kappa_2'(\omega)}{\big[\kappa_2(\omega) - \kappa_1(\omega)\big]^2} \\ &= \frac{\ell_1(k,\omega)\kappa_1'(\omega) + \ell_2(k,\omega)\kappa_2'(\omega)}{\kappa_2(\omega) - \kappa_1(\omega)}. \end{aligned} \tag{D.1}$$

Using (D.1), differentiation of (4.8) yields

$$\begin{aligned}\frac{\partial \mathsf{y}_1}{\partial \omega}(k, \omega) = \frac{f_1(\kappa_1(\omega))}{\kappa_2(\omega) - \kappa_1(\omega)} \bigg(&\ell_1(k,\omega)\kappa_1'(\omega) \left[1 + \frac{f_1'(\kappa_1(\omega))}{f_1(\kappa_1(\omega))}\big[\kappa_2(\omega) - \kappa_1(\omega)\big]\right] \\ &+ \ell_2(k,\omega)\kappa_2'(\omega)\bigg). \end{aligned} \tag{D.2}$$

Since $\Omega_1 \geq 0$ and $n_1 = 1$, it follows from (4.2) that the elasticity of f_1 is smaller than unity. Thus, the term in large brackets in (D.2) is strictly positive. Since $\kappa_j' > 0$ and $\ell_j \geq 0$, $j = 1, 2$, it follows that the term in parentheses in (D.2) is strictly positive. The factor-intensity reversal condition (4.4) together with the symmetry of y_1 and y_2 now implies that for each $k > 0$ with $\Omega_1(k) \neq \Omega_2(k)$, either

$$\begin{aligned} &\frac{\partial \mathsf{y}_1}{\partial \omega}(k,\omega) < 0 < \frac{\partial \mathsf{y}_2}{\partial \omega}(k,\omega) \quad \text{for all} \quad \omega \in \big[\Omega_1(k), \Omega_2(k)\big] \\ \text{or} \\ &\frac{\partial \mathsf{y}_1}{\partial \omega}(k,\omega) > 0 > \frac{\partial \mathsf{y}_2}{\partial \omega}(k,\omega) \quad \text{for all} \quad \omega \in \big[\Omega_2(k), \Omega_1(k)\big]. \end{aligned} \tag{D.3}$$

Step 2 (MRT). Let $k > 0$ with $\Omega_1(k) \neq \Omega_2(k)$. The marginal rate of transformation is

$$\mathrm{MRT}\big(\mathcal{T}_F(k,\omega)\big) = -\frac{\frac{\partial \mathsf{y}_2}{\partial \omega}(k,\omega)}{\frac{\partial \mathsf{y}_1}{\partial \omega}(k,\omega)}, \quad \omega \in \big[\Omega_{\min}(k), \Omega_{\max}(k)\big]. \tag{D.4}$$

Since $n_j = 1$, it follows from (4.2) and (4.3) that for each $\omega \geq 0$,

$$f_j(\kappa_j(\omega)) = \big[\omega + \kappa_j(\omega)\big] f_j'(\kappa_j(\omega)), \qquad j = 1, 2. \tag{D.5}$$

Using (D.5), differentiation of (4.8) yields

$$\begin{aligned}\frac{\frac{\partial \mathsf{y}_2}{\partial \omega}(k,\omega)}{\frac{\partial \mathsf{y}_1}{\partial \omega}(k,\omega)} &= \frac{\frac{\partial \ell_2}{\partial \omega}(k,\omega)\big[\omega + \kappa_2(\omega)\big] f_2'(\kappa_2(\omega)) + \ell_2(k,\omega) f_2'(\kappa_2(\omega))\kappa_2'(\omega)}{\frac{\partial \ell_1}{\partial \omega}(k,\omega)\big[\omega + \kappa_1(\omega)\big] f_1'(\kappa_1(\omega)) + \ell_1(k,\omega) f_1'(\kappa_1(\omega))\kappa_1'(\omega)} \\ &= \frac{\frac{\partial \ell_2}{\partial \omega}(k,\omega)\omega + \frac{\mathrm{d}}{\mathrm{d}\omega}\big(\ell_2(k,\omega)\kappa_2(\omega)\big)}{\frac{\partial \ell_1}{\partial \omega}(k,\omega)\omega + \frac{\mathrm{d}}{\mathrm{d}\omega}\big(\ell_1(k,\omega)\kappa_1(\omega)\big)} \cdot \frac{f_2'(\kappa_2(\omega))}{f_1'(\kappa_1(\omega))}.\end{aligned} \tag{D.6}$$

By Lemma 4.1,

$$\tfrac{\mathrm{d}}{\mathrm{d}\omega}\big(\ell_1(k,\omega)\kappa_1(\omega) + \ell_2(k,\omega)\kappa_2(\omega)\big) = 0 \quad \text{and} \quad \frac{\partial \ell_1}{\partial \omega}(k,\omega) + \frac{\partial \ell_2}{\partial \omega}(k,\omega) = 0.$$

Hence, the first ratio in (D.6) satisfies

$$\frac{\frac{\partial \ell_2}{\partial \omega}(k,\omega)\omega + \frac{\mathrm{d}}{\mathrm{d}\omega}\big(\ell_2(k,\omega)\kappa_2(\omega)\big)}{\frac{\partial \ell_1}{\partial \omega}(k,\omega)\omega + \frac{\mathrm{d}}{\mathrm{d}\omega}\big(\ell_1(k,\omega)\kappa_1(\omega)\big)} = -1.$$

As a consequence, the marginal rate of transformation (D.4) takes the form

$$\mathrm{MRT}\big(\mathcal{T}_F(k,\omega)\big) = -\frac{\frac{\partial \mathsf{y}_2}{\partial \omega}(k,\omega)}{\frac{\partial \mathsf{y}_1}{\partial \omega}(k,\omega)} = \frac{f_2'(\kappa_2(\omega))}{f_1'(\kappa_1(\omega))}, \quad \omega \in \big[\Omega_{\min}(k), \Omega_{\max}(k)\big]. \tag{D.7}$$

This establishes (4.11).

The non-generic case $\Omega_1(k) = \Omega_2(k) = \omega$ implies $\kappa_1(\omega) = \kappa_2(\omega) = k$ and $\frac{f_2(k)}{f_1(k)} = \frac{f_2'(k)}{f_1'(k)}$, so that the marginal rate of transformation takes the form

$$\mathrm{MRT}\big(\tilde{\mathcal{T}}_F(k,l_1)\big) = \frac{f_2(k)}{f_1(k)} = \frac{f_2'(k)}{f_1'(k)}. \tag{D.8}$$

This establishes (4.12).

Step 3 (Concavity). Let $k > 0$ with $\Omega_1(k) \neq \Omega_2(k)$. Using the function ϱ defined in (4.15), the marginal rate of transformation (D.7) takes the form

$$\mathrm{MRT}\big(\mathcal{T}_F(k,\omega)\big) = \varrho(\omega), \quad \omega \in \big[\Omega_{\min}(k), \Omega_{\max}(k)\big]. \tag{D.9}$$

Since the output function $\mathsf{y}_1(k,\cdot)$ is invertible, we may interchange the curve parameter in (D.9) by setting $\omega = \mathsf{y}_1^{-1}(k,y_1)$ and write

$$\mathrm{MRT}\big(\mathcal{T}_F(k,\mathsf{y}_1^{-1}(k,y_1))\big) = \varrho\big(\mathsf{y}_1^{-1}(k,y_1)\big), \quad y_1 \in \big[0, f_1(k)\big]. \tag{D.10}$$

Differentiating (D.10) w.r.t. y_1 using the inverse function rule yields

$$\frac{\mathrm{d}}{\mathrm{d}y_1}\mathrm{MRT}\big(\mathcal{T}_F(k,\mathrm{y}_1^{-1}(k,y_1))\big)=\frac{\varrho'\big(\mathrm{y}_1^{-1}(k,y_1)\big)}{\frac{\partial \mathrm{y}_1}{\partial\omega}\big(k,\mathrm{y}_1^{-1}(k,y_1)\big)},\tag{D.11}$$

where

$$\varrho'(\omega)=\varrho(\omega)\left[\frac{f_2''(\kappa_2(\omega))\kappa_2'(\omega)}{f_2'(\kappa_2(\omega))}-\frac{f_1''(\kappa_1(\omega))\kappa_1'(\omega)}{f_1'(\kappa_1(\omega))}\right].\tag{D.12}$$

Using (D.5), the derivatives of the functions κ_j defined in (4.3) can be written as

$$\kappa_j'(\omega)=\frac{1}{\Omega_j'(\kappa_j(\omega))}=-\frac{f_j'(\kappa_j(\omega))}{f_j''(\kappa_j(\omega))}\cdot\frac{1}{\omega+\kappa_j(\omega)},\qquad j=1,2.\tag{D.13}$$

Inserting (D.13) into (D.12) yields

$$\varrho'(\omega)=\frac{\varrho(\omega)\big[\kappa_2(\omega)-\kappa_1(\omega)\big]}{\big[\omega+\kappa_2(\omega)\big]\big[\omega+\kappa_1(\omega)\big]}.\tag{D.14}$$

It now follows from (D.14), (D.3), and the factor-intensity reversal condition (4.4) that the derivative (D.11) is strictly positive. Hence, the production-possibility frontier (4.9) is strictly concave for each $k>0$ with $\Omega_1(k)\neq\Omega_2(k)$.

In the non-generic case $\Omega_1(k)=\Omega_2(k)=\omega$, the production-possibility frontier (4.10) is linear and, therefore, concave. □

Proof of Lemma 4.2. *Step 1 (Derivatives).* Let $(k,y_1)\in\mathcal{D}$ with $\Omega_1(k)\neq\Omega_2(k)$. Since $T_F(k,y_1)=\mathcal{T}_F(k,\mathrm{y}_1^{-1}(k,y_1))$, it follows from (D.10) that

$$\frac{\partial T_F}{\partial y_1}(k,y_1)=-\varrho\big(\mathrm{y}_1^{-1}(k,y_1)\big).\tag{D.15}$$

By the implicit function theorem, we have

$$\frac{\partial \mathrm{y}_1^{-1}}{\partial k}(k,y_1)=-\frac{\frac{\partial \mathrm{y}_1}{\partial k}(k,\mathrm{y}_1^{-1}(k,y_1))}{\frac{\partial \mathrm{y}_1}{\partial\omega}(k,\mathrm{y}_1^{-1}(k,y_1))}.\tag{D.16}$$

Differentiation of (4.13) w.r.t. k using (D.5), (D.7), (D.16) and setting $\omega=\mathrm{y}_1^{-1}(k,y_1)$ yields

$$\begin{aligned}\frac{\partial T_F}{\partial k}(k,y_1)&=\frac{\partial \mathrm{y}_2}{\partial k}(k,\omega)+\frac{\partial \mathrm{y}_2}{\partial\omega}(k,\omega)\,\frac{\partial \mathrm{y}_1^{-1}}{\partial k}(k,y_1)\\&\overset{(D.16)}{=}\frac{\partial \mathrm{y}_2}{\partial k}(k,\omega)-\frac{\frac{\partial \mathrm{y}_2}{\partial\omega}(k,\omega)}{\frac{\partial \mathrm{y}_1}{\partial\omega}(k,\omega)}\,\frac{\partial \mathrm{y}_1}{\partial k}(k,\omega)\\&\overset{(D.7)}{=}\frac{f_2'(\kappa_2(\omega))}{f_1'(\kappa_1(\omega))}\,\frac{\partial \mathrm{y}_1}{\partial k}(k,\omega)+\frac{\partial \mathrm{y}_2}{\partial k}(k,\omega)\\&=\frac{\frac{f_2'(\kappa_2(\omega))}{f_1'(\kappa_1(\omega))}f_1(\kappa_1(\omega))-f_2(\kappa_2(\omega))}{\kappa_1(\omega)-\kappa_2(\omega)}\\&\overset{(D.5)}{=}\frac{\frac{f_2'(\kappa_2(\omega))}{f_1'(\kappa_1(\omega))}\,f_1'(\kappa_1(\omega))\big[\omega+\kappa_1(\omega)\big]-f_2'(\kappa_2(\omega))\big[\omega+\kappa_2(\omega)\big]}{\kappa_1(\omega)-\kappa_2(\omega)}\\&=f_2'(\kappa_2(\omega)).\end{aligned}\tag{D.17}$$

In the non-generic case, differentiation of (4.13) w.r.t. k yields

$$\begin{aligned}\frac{\partial T_F}{\partial k}(k,y_1) &= f_2'(k) - y_1\,\frac{f_2'(k)f_1(k)-f_2(k)f_1'(k)}{f_1(k)^2}\\ &= f_2'(k) = f_2'(\kappa_2(\omega)).\end{aligned}\tag{D.18}$$

Equations (D.17) and (D.18) establish the desired partial derivative w.r.t. k. The partial derivative w.r.t. y_1 follows from (D.15) and (D.8).

Step 2 (Concavity). The map $T_F : \mathcal{D} \to \mathbb{R}_+$ is concave if

$$\frac{\partial^2 T_F}{\partial k^2}(k,y_1) \le 0 \quad \text{and} \quad \det \mathrm{H}_{T_F}(k,y_1) \ge 0 \quad \text{for all} \quad (k,y_1)\in\mathcal{D},$$

where H_{T_F} is Hessian matrix of T_F. This is established next.

Let $(k,y_1)\in\mathcal{D}$ with $\Omega_1(k)\neq\Omega_2(k)$ and set $\omega = \mathrm{y}_1^{-1}(k,y_1)$. Differentiation of (D.15) w.r.t. y_1 using the inverse function rule yields

$$\frac{\partial^2 T_F}{(\partial y_1)^2}(k,y_1) = -\frac{\varrho'\big(\mathrm{y}_1^{-1}(k,y_1)\big)}{\frac{\partial \mathrm{y}_1}{\partial\omega}\big(k,\mathrm{y}_1^{-1}(k,y_1)\big)}.\tag{D.19}$$

Differentiation of (D.15) w.r.t. k using (D.16) yields

$$\frac{\partial^2 T_F}{\partial k\partial y_1}(k,y_1) = \varrho'(\omega)\,\frac{\frac{\partial \mathrm{y}_1}{\partial k}(k,\omega)}{\frac{\partial \mathrm{y}_1}{\partial \omega}(k,\omega)}.\tag{D.20}$$

Moreover, differentiation of (D.17) w.r.t. y_1 and k yields

$$\frac{\partial^2 T_F}{\partial y_1\partial k}(k,y_1) = \frac{f_2''(\kappa_2(\omega))\,\kappa_2'(\omega)}{\frac{\partial \mathrm{y}_1}{\partial \omega}(k,\omega)}\tag{D.21}$$

and

$$\frac{\partial^2 T_F}{\partial k^2}(k,y_1) = -f_2''(\kappa_2(\omega))\,\kappa_2'(\omega)\,\frac{\frac{\partial \mathrm{y}_1}{\partial k}(k,\omega)}{\frac{\partial \mathrm{y}_1}{\partial \omega}(k,\omega)},\tag{D.22}$$

respectively. Since $f_2''<0$, $\kappa_2'>0$, and, by (D.3),

$$\frac{\partial \mathrm{y}_1}{\partial k}(k,\omega)\cdot\frac{\partial \mathrm{y}_1}{\partial \omega}(k,\omega) < 0 \quad \text{for all} \quad \omega\in\big[\Omega_{\min}(k),\Omega_{\max}(k)\big],$$

it follows that (D.22) is strictly negative. Inserting (D.19) – (D.22), the determinant of the Hessian matrix becomes

$$\begin{aligned}\det \mathrm{H}_{T_F}(k,y_1) &= \frac{\partial^2 T_F}{\partial k^2}(k,y_1)\cdot\frac{\partial^2 T_F}{(\partial y_1)^2}(k,y_1) - \frac{\partial^2 T_F}{\partial k\partial y_1}(k,y_1)\cdot\frac{\partial^2 T_F}{\partial y_1\partial k}(k,y_1)\\ &= \frac{f_2''(\kappa_2(\omega))\kappa_2'(\omega)\varrho'(\omega)\frac{\partial \mathrm{y}_1}{\partial k}(k,\omega)}{\frac{\partial \mathrm{y}_1}{\partial \omega}(k,\omega)\frac{\partial \mathrm{y}_1}{\partial \omega}(k,\omega)} - \frac{f_2''(\kappa_2(\omega))\,\kappa_2'(\omega)\varrho'(\omega)\frac{\partial \mathrm{y}_1}{\partial k}(k,\omega)}{\frac{\partial \mathrm{y}_1}{\partial \omega}(k,\omega)\frac{\partial \mathrm{y}_1}{\partial \omega}(k,\omega)}\\ &= 0.\end{aligned}$$

Hence, T_F is concave for all $(k,y_1)\in\mathcal{D}$ with $\Omega_1(k)\neq\Omega_2(k)$.

In the non-generic case, differentiation of (D.18) and (4.14) yields

$$\frac{\partial^2 T_F}{\partial k^2}(k, y_1) = f_2''(k) \quad \text{and} \quad \frac{\partial^2 T_F}{(\partial y_1)^2}(k, y_1) = \frac{\partial^2 T_F}{\partial y_1 \partial k}(k, y_1) = \frac{\partial^2 T_F}{\partial k \partial y_1}(k, y_1) = 0. \tag{D.23}$$

Since $f_2'' < 0$, it follows from (D.23) that

$$\frac{\partial^2 T_F}{\partial k^2}(k, y_1) < 0 \quad \text{and} \quad \det \mathrm{H}_{T_F}(k, y_1) = 0.$$

Hence, T_F is concave for all $(k, y_1) \in \mathcal{D}$ with $\Omega_1(k) = \Omega_2(k)$. ☐

Proof of Corollary 4.1. Since $\varrho(\omega), \kappa_1(\omega), \kappa_2(\omega) > 0$ for all $\omega > 0$, the claim follows directly from (D.14). ☐

Proof of Lemma 4.3. *(i).* Let $k > 0$ with $\Omega_1(k) \neq \Omega_2(k)$. A profit-maximizing interior production plan $\big(\mathsf{y}_1(k, \omega), \mathsf{y}_2(k, \omega)\big) \gg 0$ obtains for an $\omega \in \big(\Omega_{\min}(k), \Omega_{\max}(k)\big)$ such that

$$\varrho(\omega) \overset{!}{=} \tfrac{p_1}{p_2}. \tag{D.24}$$

Recall that $\varrho(\Omega_2(k)) < \varrho(\Omega_1(k))$ owing to the strict concavity of the production-possibility frontier. Since, by Corollary 4.1, $\omega \mapsto \varrho(\omega)$ is either strictly decreasing or strictly increasing on the interval in question, a unique solution $\omega = \varrho^{-1}\big(\tfrac{p_1}{p_2}\big) \in \big(\Omega_{\min}(k), \Omega_{\max}(k)\big)$ to (D.24) exists if and only if $\varrho(\Omega_2(k)) < \tfrac{p_1}{p_2} < \varrho(\Omega_1(k))$. Otherwise, a boundary production plan obtains. If $\tfrac{p_1}{p_2} < \varrho(\Omega_2(k))$, then

$$p_2 f_2'(\kappa_2(\omega)) > p_1 f_1'(\kappa_1(\omega)) \quad \text{and} \quad p_2 w_2(\kappa_2(\omega)) > p_1 w_1(\kappa_1(\omega))$$

for all $\omega \in \big[\Omega_{\min}(k), \Omega_{\max}(k)\big]$. In this case, sector 2 receives all factors inputs since $\tfrac{p_1}{p_2}$ is too low. The production plan is $\big(\mathsf{y}_1(k, \Omega_2(k)), \mathsf{y}_2(k, \Omega_2(k))\big) = (0, f_2(k))$. If $\varrho(\Omega_1(k)) < \tfrac{p_1}{p_2}$, then

$$p_2 f_2'(\kappa_2(\omega)) < p_1 f_1'(\kappa_1(\omega)) \quad \text{and} \quad p_2 w_2(\kappa_2(\omega)) < p_1 w_1(\kappa_1(\omega))$$

for all $\omega \in \big[\Omega_{\min}(k), \Omega_{\max}(k)\big]$. In this case, sector 1 receives all factors inputs because $\tfrac{p_1}{p_2}$ is too high. The production plan is $\big(\mathsf{y}_1(k, \Omega_1(k)), \mathsf{y}_2(k, \Omega_1(k))\big) = (f_1(k), 0)$.

(ii). Let $k > 0$ with $\Omega_1(k) = \Omega_2(k)$, implying that $\tfrac{w_2(k)}{w_1(k)} = \tfrac{f_2'(k)}{f_1'(k)}$. If $\tfrac{p_1}{p_2} < \varrho(\Omega_2(k))$, then

$$p_2 f_2'(k) > p_1 f_1'(k) \quad \text{and} \quad p_2 w_2(k) > p_1 w_1(k),$$

so that the production plan is $(0, f_2(k))$ because sector 2 receives all factor inputs. If $\varrho(\Omega_1(k)) < \tfrac{p_1}{p_2}$, then

$$p_2 f_2'(k) < p_1 f_1'(k) \quad \text{and} \quad p_2 w_2(k) < p_1 w_1(k),$$

so that the production plan is $(f_1(k), 0)$ because sector 1 receives all factor inputs. The marginal products in both sectors coincide if and only if $\tfrac{p_1}{p_2} = \varrho(\Omega_1(k)) = \varrho(\Omega_2(k))$, in which case any production plan $\big(l_1 f_1(k), (1 - l_1) f_2(k)\big)$ with $l_1 \in [0, 1]$ is profit maximizing. ☐

Proof of Proposition 4.1. Let $\omega^* > 0$ with $\kappa_1(\omega^*) \neq \kappa_2(\omega^*)$ be arbitrary but fixed. Solving the output function y_1 defined in (4.8) for k yields

$$k = \kappa_2(\omega^*) - \frac{\mathsf{y}_1(k,\omega^*)}{f_1(\kappa_1(\omega^*))}\big[\kappa_2(\omega^*) - \kappa_1(\omega^*)\big]. \tag{D.25}$$

Substituting (D.25) into the output function y_2 defined in (4.8) yields

$$\mathsf{y}_2(k,\omega^*) = f_2(\kappa_2(\omega^*)) - \frac{f_2(\kappa_2(\omega^*))}{f_1(\kappa_1(\omega^*))}\,\mathsf{y}_1(k,\omega^*). \tag{D.26}$$

Since, by Lemma 4.3, a production plan $\mathcal{T}_F(k,\omega^*) = \big(\mathsf{y}_1(k,\omega^*), \mathsf{y}_2(k,\omega^*)\big) \gg 0$ is profit maximizing at output prices (p_1, p_2) if and only if $\varrho(\omega^*) = \frac{p_1}{p_2}$, it follows from (D.26) that all production plans that are profit maximizing at the output-price ratio $\frac{p_1}{p_2} = \varrho(\omega^*)$ are located on the straight line (4.19). It follows from (D.5) that

$$\frac{f_2(\kappa_2(\omega^*))}{f_1(\kappa_1(\omega^*))} \geq \varrho(\omega^*) = \frac{f_2'(\kappa_2(\omega^*))}{f_1'(\kappa_1(\omega^*))}$$

if and only if $\kappa_2(\omega^*) \geq \kappa_1(\omega^*)$. □

Proof of Theorem 4.2. *(i)*. Since sector 2 is more capital-intensive on $(\bar{\omega}_A, \bar{\omega}_B)$, Corollary 4.1 implies that ϱ is strictly increasing on $(\bar{\omega}_A, \bar{\omega}_B)$. Hence, for each $\frac{p_1}{p_2} \in \big(\varrho(\bar{\omega}_A), \varrho(\bar{\omega}_B)\big)$, there exist a unique $\omega^* \in (\bar{\omega}_A, \bar{\omega}_B)$ such that $\varrho(\omega^*) = \frac{p_1}{p_2}$. The qualifiers in claim (i) of the theorem distinguish three cases. Since ϱ is strictly increasing on $(\bar{\omega}_A, \bar{\omega}_B)$ and $\Omega_1', \Omega_2' > 0$, we have

$$\begin{array}{lll} (a) & \kappa_1(\bar{\omega}_A) \leq k \leq \kappa_1(\omega^*) \iff & \varrho(\bar{\omega}_A) \leq \varrho(\Omega_1(k)) \leq \frac{p_1}{p_2} \\ (b) & \kappa_1(\omega^*) < k < \kappa_2(\omega^*) \iff & \varrho(\Omega_2(k)) < \frac{p_1}{p_2} < \varrho(\Omega_1(k)) \\ (c) & \kappa_2(\omega_*) \leq k \leq \kappa_2(\bar{\omega}_B) \iff & \frac{p_1}{p_2} \leq \varrho(\Omega_2(k)) \leq \varrho(\bar{\omega}_B). \end{array}$$

It follows from Lemma 4.3 that in (a), the profit-maximizing production plan is $(f_1(k), 0)$, in (b), it is $\mathcal{T}_F(k,\omega^*)$, and in (c), it is $(0, f_2(k))$.

(ii). Since sector 1 is more capital-intensive on $(\bar{\omega}_A, \bar{\omega}_B)$, Corollary 4.1 implies that ϱ is strictly decreasing on $(\bar{\omega}_A, \bar{\omega}_B)$. Hence, for each $\frac{p_1}{p_2} \in \big(\varrho(\bar{\omega}_B), \varrho(\bar{\omega}_A)\big)$, there exist a unique $\omega^* \in (\bar{\omega}_A, \bar{\omega}_B)$ such that $\varrho(\omega^*) = \frac{p_1}{p_2}$. Again, the qualifiers in claim (ii) of the theorem distinguish three cases. Since ϱ is strictly decreasing on $(\bar{\omega}_A, \bar{\omega}_B)$, we have

$$\begin{array}{lll} (a) & \kappa_2(\bar{\omega}_A) \leq k \leq \kappa_2(\omega^\star) \iff & \varrho(\bar{\omega}_A) \geq \varrho(\Omega_2(k)) \geq \frac{p_1}{p_2} \\ (b) & \kappa_1(\omega^*) < k < \kappa_2(\omega^*) \iff & \varrho(\Omega_2(k)) < \frac{p_1}{p_2} < \varrho(\Omega_1(k)) \\ (c) & \kappa_1(\omega_*) \leq k \leq \kappa_1(\bar{\omega}_B) \iff & \frac{p_1}{p_2} \geq \varrho(\Omega_1(k)) \geq \varrho(\bar{\omega}_B). \end{array}$$

It follows from Lemma 4.3 that in (a), the profit-maximizing production plan is $(f_2(k), 0)$, in (b), it is $\mathcal{T}_F(k,\omega^*)$, and in (c), it is $(f_1(k), 0)$. □

Proof of Lemma 4.4. Let $L > 0$ be arbitrary but fixed.

(i). Let $k > 0$ with $\Omega_1(k) \neq \Omega_2(k)$. By (4.24), the marginal rate of transformation is

$$\mathrm{MRT}\big(\mathcal{T}_G(k,\omega)\big) = -\frac{\frac{\partial \mathsf{x}_2}{\partial \omega}(k,\omega)}{\frac{\partial \mathsf{x}_1}{\partial \omega}(k,\omega)}, \quad \omega \in \big(\Omega_{\min}(k), \Omega_{\max}(k)\big), \tag{D.27}$$

where the output functions $\mathsf{x}_1, \mathsf{x}_2$ were defined in (4.23). Differentiating (4.23) using the chain rule and Theorem 4.1 yields

$$\begin{aligned} -\frac{\frac{\partial \mathsf{x}_2}{\partial \omega}(k,\omega)}{\frac{\partial \mathsf{x}_1}{\partial \omega}(k,\omega)} &= -\frac{h_2'(\mathsf{y}_2(k,\omega))}{h_1'(\mathsf{y}_1(k,\omega))}\frac{\frac{\partial \mathsf{y}_2}{\partial \omega}(k,\omega)}{\frac{\partial \mathsf{y}_1}{\partial \omega}(k,\omega)} \\ &= L^{n_2-n_1}\frac{n_2}{n_1}\frac{\mathsf{y}_2(k,\omega)^{n_2-1}}{\mathsf{y}_1(k,\omega)^{n_1-1}}\frac{f_2'(\kappa_2(\omega))}{f_1'(\kappa_1(\omega))} \\ &= L^{n_2-n_1}\frac{n_2}{n_1}\frac{\ell_2(k,\omega)^{n_2-1}}{\ell_1(k,\omega)^{n_1-1}}\frac{f_2(\kappa_2(\omega))^{n_2-1}}{f_1(\kappa_1(\omega))^{n_1-1}}\frac{f_2'(\kappa_2(\omega))}{f_1'(\kappa_1(\omega))} \\ &= L^{n_2-n_1}\frac{\ell_2(k,\omega)^{n_2-1}}{\ell_1(k,\omega)^{n_1-1}}\frac{g_2'(\kappa_2(\omega))}{g_1'(\kappa_1(\omega))}, \end{aligned} \tag{D.28}$$

where the last equation follows from (4.21). Inserting (D.28) into (D.27) now yields

$$\mathrm{MRT}\big(\mathcal{T}_G(k,\omega)\big) = L^{n_2-n_1}\frac{\ell_2(k,\omega)^{n_2-1}}{\ell_1(k,\omega)^{n_1-1}}\frac{g_2'(\kappa_2(\omega))}{g_1'(\kappa_1(\omega))}, \quad \omega \in \big(\Omega_{\min}(k), \Omega_{\max}(k)\big),$$

and establishes (4.26).

(ii). Let $k > 0$ with $\Omega_1(k) = \Omega_2(k)$. By (4.25), the marginal rate of transformation is

$$\begin{aligned} \mathrm{MRT}\big(\tilde{\mathcal{T}}_G(k,l_1)\big) &= \mathrm{MRT}\big(h_1(l_1 f_1(k)), h_2((1-l_1)f_2(k))\big) \\ &= \frac{h_2'\big((1-l_1)f_2(k)\big)f_2(k)}{h_1'\big(l_1 f_1(k)\big)f_1(k)} \\ &= L^{n_2-n_1}\frac{n_2}{n_1}\frac{f_2(k)^{n_2}}{f_1(k)^{n_1}}\frac{(1-l_1)^{n_2-1}}{l_1^{n_1-1}} \\ &= L^{n_2-n_1}\frac{(1-l_1)^{n_2-1}}{l_1^{n_1-1}}\frac{g_2'(k)}{g_1'(k)}, \end{aligned} \tag{D.29}$$

where the last equation follows from (4.21). This establishes (4.27). □

Proof of Theorem 4.3. Let $L > 0$ and $k > 0$ be arbitrary but fixed. The maximum attainable output of sector $j = 1, 2$ then is

$$\overline{x}_j := h_j\big(f_j(k)\big).$$

Using the production-possibility frontier T_F corresponding to (F_1, F_2) defined in (4.14), the frontier corresponding to (G_1, G_2) is now given by the curve

$$T_G(k,\cdot) : \big[0, \overline{x}_1\big] \to \big[0, \overline{x}_2\big], \qquad x_1 \mapsto T_G(k, x_1), \tag{D.30}$$

defined by

$$x_2 = T_G(k, x_1) := h_2\big(T_F(k, h_1^{-1}(x_1))\big).$$

Differentiation of (D.30) using the chain rule yields

$$\frac{\partial T_G}{\partial x_1}(k, x_1) = h_2'\big(T_F(k, h_1^{-1}(x_1))\big)\, \frac{\partial T_F}{\partial y_1}(k, h_1^{-1}(x_1))\, (h_1^{-1})'(x_1). \tag{D.31}$$

Since $h_1', h_2' > 0$ and $\frac{\partial T_F}{\partial y_1}(k, h_1^{-1}(x_1)) < 0$ for all $x_1 \in [0, \bar{x}_1]$ by Theorem 4.1, it follows that (D.31) is strictly negative. Thus, (D.30) is strictly decreasing. Differentiating (D.31) w.r.t. x_1, it is readily seen that

$$\frac{\partial^2 T_G}{\partial x_1^2}(k, x_1) \leq 0$$

if and only if

$$\begin{aligned}\frac{h_2''\big(T_F(k, h_1^{-1}(x_1))\big) T_F(k, h_1^{-1}(x_1))}{h_2'\big(T_F(k, h_1^{-1}(x_1))\big)} \cdot \frac{\frac{\partial T_F}{\partial y_1}(k, h_1^{-1}(x_1))(h_1^{-1})'(x_1)x_1}{T_F(k, h_1^{-1}(x_1))}& \\ + \frac{\frac{\partial^2 T_F}{\partial y_1^2}(k, h_1^{-1}(x_1))(h_1^{-1})'(x_1)x_1}{\frac{\partial T_F}{\partial y_1}(k, h_1^{-1}(x_1))} + \frac{(h_1^{-1})''(x_1)x_1}{(h_1^{-1})'(x_1)} &\geq 0.\end{aligned} \tag{D.32}$$

Since

$$\frac{(h_1^{-1})'(x)x}{h_1^{-1}(x)} = \frac{1}{n_1}, \quad \frac{(h_1^{-1})''(x)x}{(h_1^{-1})'(x)} = \frac{1-n_1}{n_1}, \quad \text{and} \quad \frac{h_2''(y)y}{h_2'(y)} = n_2 - 1,$$

Condition (D.32) takes the form

$$(n_2 - 1)\, \frac{\frac{\partial T_F}{\partial y_1}(k, h_1^{-1}(x_1))h_1^{-1}(x_1)}{T_F(k, h_1^{-1}(x_1))} + \frac{\frac{\partial^2 T_F}{\partial y_1^2}(k, h_1^{-1}(x_1))h_1^{-1}(x_1)}{\frac{\partial T_F}{\partial y_1}(k, h_1^{-1}(x_1))} \geq n_1 - 1. \tag{D.33}$$

Since $y_1 \mapsto T_F(k, y_1)$ is strictly decreasing and concave by Theorem 4.1, (D.33) holds for all $x_1 \in [0, \bar{x}_1]$ if $n_1, n_2 \in (0, 1]$. In this case, (D.30) is concave. If either $n_1 < 1$ or $n_2 < 1$, then (D.33) holds with strict inequality, so that (D.30) is strictly concave.

If $n_2 > 1$, then the first term on the l.h.s. of (D.33) is negative, while the second term is positive and finite. Since

$$T_F(k, h_1^{-1}(\overline{x}_1)) = T_F(k, f_1(k)) = 0 \quad \text{and} \quad \frac{\partial T_F}{\partial y_1}(k, h_1^{-1}(\overline{x}_1)) = -\varrho(\Omega_1(k)) < 0,$$

it follows that (D.33) is violated for all x_1 sufficiently close to $\overline{x}_1$, in which case (D.30) is strictly convex. The same argument applies for the case $n_1 > 1$ by interchanging the role of the variables y_1 and y_2 in the parameterization of T_F. ☐

Proof of Lemma 4.5. Let $L > 0$ be arbitrary but fixed.

(i). Let $k > 0$ with $\Omega_1(k) \neq \Omega_2(k)$. Since $n_1, n_2 \leq 1$ and $n_j < 1$ for some $j = 1, 2$, the marginal rate of transformation $\mathrm{MRT}\big(\mathcal{T}_G(k,\omega)\big)$ given in (4.26) is zero for $\omega = \Omega_1(k)$ and becomes infinitely large for $\omega \to \Omega_2(k)$. Hence, there exists $\omega^* = \Omega_G(k) \in \big(\Omega_{\min}(k), \Omega_{\max}(k)\big)$ such that

$$\mathrm{MRT}\big(\mathcal{T}_G(k,\omega^*)\big) = L^{n_2 - n_1} \frac{n_2}{n_1} \frac{\mathsf{y}_1(k,\omega^*)^{1-n_1}}{\mathsf{y}_2(k,\omega^*)^{1-n_2}} \varrho(\omega^*) = \frac{p_1}{p_2}. \tag{D.34}$$

Since, by Theorem 4.3, the production-possibility frontier is strictly concave whenever $n_1, n_2 \leq 1$ and $n_j < 1$ for some $j = 1, 2$, ω^* must be uniquely determined.

The result that $\Omega_G(k)$ is increasing for all $k > 0$ with $\Omega_1(k) \neq \Omega_2(k)$ follows from the implicit function theorem applied to (D.34), Corollary 4.1, the inequalities (D.3), and the fact that for each $k > 0$ with $\Omega_1(k) \neq \Omega_2(k)$, either

$$\frac{\partial \mathsf{y}_1}{\partial k}(k,\omega) > 0 > \frac{\partial \mathsf{y}_2}{\partial k}(k,\omega) \quad \text{for all} \quad \omega \in \big[\Omega_1(k), \Omega_2(k)\big]$$

or

$$\frac{\partial \mathsf{y}_1}{\partial k}(k,\omega) < 0 < \frac{\partial \mathsf{y}_2}{\partial k}(k,\omega) \quad \text{for all} \quad \omega \in \big[\Omega_2(k), \Omega_1(k)\big].$$

(ii). Let $k > 0$ with $\Omega_1(k) = \Omega_2(k)$. Since $n_1, n_2 \leq 1$ and $n_j < 1$ for some $j = 1, 2$, the marginal rate of transformation $\mathrm{MRT}\big(\tilde{\mathcal{T}}_G(k,\omega)\big)$ given in (4.27) is zero for $l_1 = 1$ and becomes infinitely large for $l_1 \to 0$. Hence, there exists $l_1^* = l_G(k) \in (0,1)$ such that

$$\mathrm{MRT}\big(\tilde{\mathcal{T}}_G(k,l_1^*)\big) = L^{n_2 - n_1} \frac{(1 - l_1^*)^{n_2 - 1}}{(l_1^*)^{n_1 - 1}} \frac{g_2'(k)}{g_1'(k)} = \frac{p_1}{p_2}. \tag{D.35}$$

The strict concavity of the frontier implies that l_1^* is uniquely determined. □

Appendix E

Proofs of Chapter 5

To solve the social planner's problem (5.39) using dynamic programming methods, a number of technical results found in De La Croix and Michel (2002) must be adapted.

The following lemma ensures that Problem (5.39) is well posed.

Lemma E.1 (Existence of the value function)**.**
Let Assumptions 5.1 – 5.3 be satisfied. Then for each $(k_0, \chi_0) \in \mathbb{R}_{++} \times \mathbb{R}_+$*, the value function*

$$\mathcal{V}(k_0, \chi_0) := \sup\Big\{ W\big(\{(k_t, \chi_t, y_t^b, c_t^1, c_t^2)\}_{t=0}^{\infty}\big) \;\Big|\; \{(k_t, \chi_t, y_t^b, c_t^1, c_t^2)\}_{t=0}^{\infty} \in \Pi(k_0, \chi_0) \Big\}$$

is well defined and finite.

Proof of Lemma E.1. Let $(k_0, \chi_0) \in \mathbb{R}_{++} \times \mathbb{R}_+$ be arbitrary but fixed.

Step 1 (Upper bound). We show that for every feasible allocation $\{(k_t, \chi_t, y_t^b, c_t^1, c_t^2)\}_{t=0}^{\infty} \in \Pi(k_0, \chi_0)$, the social welfare $W\big(\{(k_t, \chi_t, y_t^b, c_t^1, c_t^2)\}_{t=0}^{\infty}\big)$ is bounded from above. For each $k \geq 0$ and each $(y^b, c^1, c^2) \in Q(k)$, the capital accumulation law A defined in (5.38) satisfies

$$A(k, y^b, c^1, c^2) \leq \tfrac{1}{1+n}\Big[f_b(k) + f_g(k) + (1-\delta)k\Big]. \tag{E.1}$$

It follows from Assumptions 5.1 and 5.3 and (E.1) that every sequence $\{k_t\}_{t=0}^{\infty}$ recursively generated by A is bounded. This, in turn, implies that every sequence $\{(k_t, \chi_t, y_t^b, c_t^1, c_t^2)\}_{t=0}^{\infty} \in \Pi(k_0, \chi_0)$ is bounded. As a consequence, the one-period return function (5.35) is bounded from above, that is,

$$g(k_t, \chi_t, y_t^b, c_t^1, c_t^2) < \infty \quad \text{for all } t \geq 0.$$

Since $\gamma \in (0,1)$, it follows from (5.34) that

$$W\big(\{(k_t, \chi_t, y_t^b, c_t^1, c_t^2)\}_{t=0}^{\infty}\big) < \infty.$$

Step 2 (Lower bound). We construct a feasible allocation $\{(k_t, \chi_t, y_t^b, c_t^1, c_t^2)\}_{t=0}^{\infty} \in \Pi(k_0, \chi_0)$ for which the social welfare attains a value larger than $-\infty$. For each $t \geq 0$, choose

$$y_t^b = 0 \quad \text{and} \quad c_t^1 = \tfrac{c_t^2}{1+n} = \tfrac{1}{2}\Big[f_g(k_t) - (n+\delta)k_t\Big].$$

For this choice, only the green sector produces, so that

$$\begin{aligned} k_{t+1} &= A(k_t, 0, c_t^1, c_t^2) = \tfrac{1}{1+n}\Big[f_g(k_t) + (1-\delta)k_t - c_t^1 - \tfrac{c_t^2}{1+n}\Big] = k_t \\ \chi_{t+1} &= E(\chi_t, 0) = \tfrac{1-\zeta}{1+n}\,\chi_t \end{aligned}$$

for all $t \geq 0$. Clearly, the social welfare $W\big(\{(k_0, \chi_0, 0, c_0^1, c_0^2)\}_{t=0}^{\infty}\big)$ attained with the resulting feasible allocation $\{(k_0, \chi_0, 0, c_0^1, c_0^2)\}_{t=0}^{\infty} \in \Pi(k_0, \chi_0)$ is finite.

Step 3 (Value function). Step 1 and Step 2 imply that the supremum

$$\mathcal{V}(k_0, \chi_0) = \sup\Big\{W\big(\{(k_t, \chi_t, y_t^b, c_t^1, c_t^2)\}_{t=0}^{\infty}\big) \;\Big|\; \{(k_t, \chi_t, y_t^b, c_t^1, c_t^2)\}_{t=0}^{\infty} \in \Pi(k_0, \chi_0)\Big\}$$

exists and is finite. Since $(k_0, \chi_0) \in \mathbb{R}_{++} \times \mathbb{R}_+$ was arbitrary, we can conclude that there exists a finite value function $\mathcal{V} : \mathbb{R}_{++} \times \mathbb{R}_+ \to \mathbb{R}$. □

Our next lemma summarizes technical properties of the value function $\mathcal{V}$.

Lemma E.2 (Properties of the value function).
Let Assumptions 5.1 – 5.3 be satisfied. Then for each $(k_0, \chi_0) \in \mathbb{R}_{++} \times \mathbb{R}_+$, the value function $\mathcal{V}$ satisfies the Bellman equation

$$\mathcal{V}(k_0, \chi_0) = \sup\Big\{g(k_0, \chi_0, y^b, c^1, c^2) + \gamma \mathcal{V}\big(A(k_0, y^b, c^1, c^2), E(\chi_0, y^b)\big) \;\Big|\; (y^b, c^1, c^2) \in Q(k_0)\Big\}.$$

Moreover, $\mathcal{V}$ is concave, continuous, non-decreasing in the first argument, and non-increasing in the second argument.

Proof of Lemma E.2. Let $(k_0, \chi_0) \in \mathbb{R}_{++} \times \mathbb{R}_+$ be arbitrary but fixed.

Step 1 (Bellman equation). For every feasible allocation $\{(k_t, \chi_t, y_t^b, c_t^1, c_t^2)\}_{t=0}^{\infty} \in \Pi(k_0, \chi_0)$ generated by the accumulation laws A and E, we have

$$\begin{aligned} &W\big(\{(k_t, \chi_t, y_t^b, c_t^1, c_t^2)\}_{t=0}^{\infty}\big) = \sum_{t=0}^{\infty} \gamma^t g(k_t, \chi_t, y_t^b, c_t^1, c_t^2) \qquad \text{(E.2)} \\ &= g(k_0, \chi_0, y_0^b, c_0^1, c_0^2) + \gamma \sum_{t=0}^{\infty} \gamma^t g(k_{t+1}, \chi_{t+1}, y_{t+1}^b, c_{t+1}^1, c_{t+1}^2) \\ &\leq g(k_0, \chi_0, y_0^b, c_0^1, c_0^2) + \gamma \mathcal{V}(k_1, \chi_1) \\ &\leq \sup\Big\{g(k_0, \chi_0, y^b, c^1, c^2) + \gamma \mathcal{V}\big(A(k_0, y^b, c^1, c^2), E(\chi_0, y^b)\big) \;\Big|\; (y^b, c^1, c^2) \in Q(k_0)\Big\}. \end{aligned}$$

The claim now follows from the definition of the supremum: $\mathcal{V}(k_0, \chi_0)$ is the smallest upper bound for the social welfare (E.2), while $\mathcal{V}(k_1, \chi_1)$ is the smallest upper bound for the social welfare of all feasible allocations contained in $\Pi(k_1, \chi_1)$, where

$$k_1 = A(k_0, y_0^b, c_0^1, c_0^2) \quad \text{and} \quad \chi_1 = E(\chi_0, y_0^b).$$

Step 2 (Concavity and continuity of $\mathcal{V}$). We first demonstrate that any convex combination of any two growth paths $\{(k_t^j, \chi_t^j)\}_{t=0}^{\infty}$, $j = 1, 2$, associated with two feasible allocations may be generated by another feasible allocation. Let $\lambda \in [0, 1]$ be arbitrary but fixed and, for each $t \geq 0$, set

$$k_t^{\lambda} = \lambda k_t^1 + (1 - \lambda) k_t^2 \quad \text{and} \quad \chi_t^{\lambda} = \lambda \chi_t^1 + (1 - \lambda) \chi_t^2.$$

We will show that, for each $t \geq 0$, there exists $q_t^\lambda = (y_t^{b\lambda}, c_t^{1\lambda}, c_t^{2\lambda}) \in Q(k_t^\lambda)$ such that

$$\begin{aligned} k_{t+1}^\lambda &= A(k_t^\lambda, q_t^\lambda) \\ \chi_{t+1}^\lambda &= E(\chi_t^\lambda, y_t^{b\lambda}) = \tfrac{1}{1+n}\big[(1-\zeta)\chi_t^\lambda + \epsilon_b y_t^{b\lambda}\big]. \end{aligned} \tag{E.3}$$

Lemma 5.3 implies that the capital accumulation law A is concave. Hence,

$$k_{t+1}^\lambda \leq A(k_t^\lambda, \lambda q_t^1 + (1-\lambda) q_t^2), \tag{E.4}$$

where $q_t^j = (y_t^{bj}, c_t^{1j}, c_t^{2j}) \in Q(k_t^j)$, $j = 1, 2$, are the policies belonging to the respective feasible allocations. The convexity of the set $Q(k_t^\lambda)$ implies that

$$\lambda q_t^1 + (1-\lambda) q_t^2 \in Q(k_t^\lambda).$$

Since the pollution accumulation law E is linear, we have $y_t^{b\lambda} = \lambda y_t^{b1} + (1-\lambda) y_t^{b2}$. Given $y_t^{b\lambda}$, it follows from (E.4) that there exists a consumption bundle $(c_t^{1\lambda}, c_t^{2\lambda})$ such that $q_t^\lambda = (y_t^{b\lambda}, c_t^{1\lambda}, c_t^{2\lambda}) \in Q(k_t^\lambda)$ and (E.3) holds. As a consequence, $\{(k_t^\lambda, \chi_t^\lambda, q_t^\lambda)\}_{t=0}^\infty \in \Pi(k_0^\lambda, \chi_0^\lambda)$. Since

$$c_t^{1\lambda} + \tfrac{c_t^{2\lambda}}{1+n} > \lambda \left(c_t^{11} + \tfrac{c_t^{21}}{1+n} \right) + (1-\lambda) \left(c_t^{12} + \tfrac{c_t^{22}}{1+n} \right),$$

the consumption bundle $(c_t^{1\lambda}, c_t^{2\lambda})$ may be chosen such that

$$g(k_t^\lambda, \chi_t^\lambda, q_t^\lambda) > \lambda g(k_t^1, \chi_t^1, q_t^1) + (1-\lambda) g(k_t^2, \chi_t^2, q_t^2). \tag{E.5}$$

The concavity of the value function $\mathcal{V}$ is now established as follows. By definition of $\mathcal{V}$, for any $\varepsilon > 0$, there exists a feasible allocation $\{(k_t^j, \chi_t^j, q_t^j)\}_{t=0}^\infty \in \Pi(k_0^j, \chi_0^j)$ such that

$$\sum_{t=0}^\infty \gamma^t g(k_t^j, \chi_t^j, q_t^j) > \mathcal{V}(k_0^j, \chi_0^j) - \varepsilon. \tag{E.6}$$

Since $\{(k_t^\lambda, \chi_t^\lambda, q_t^\lambda)\}_{t=0}^\infty \in \Pi(k_0^\lambda, \chi_0^\lambda)$, it follows from (E.5) that

$$\mathcal{V}(k_0^\lambda, \chi_0^\lambda) \geq \sum_{t=0}^\infty \gamma^t g(k_t^\lambda, \chi_t^\lambda, q_t^\lambda) \geq \lambda \sum_{t=0}^\infty \gamma^t g(k_t^1, \chi_t^1, q_t^1) + (1-\lambda) \sum_{t=0}^\infty \gamma^t g(k_t^2, \chi_t^2, q_t^2).$$

Taking suprema, (E.6) implies that

$$\mathcal{V}(k_0^\lambda, \chi_0^\lambda) \geq \lambda \mathcal{V}(k_0^1, \chi_0^1) + (1-\lambda) \mathcal{V}(k_0^2, \chi_0^2) - \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, we can conclude that $\mathcal{V}$ is concave. Concave functions defined on open subsets of $\mathbb{R}^2$ are continuous, e.g., see Rockafellar (1970).

Step 3 (Monotonicity of $\mathcal{V}$). Observe first that $Q(k) \subseteq Q(k')$ for all $k' \geq k$. Let $(k_0, \chi_0) \in \mathbb{R}_{++} \times \mathbb{R}_+$ be arbitrary but fixed. By definition of $\mathcal{V}$, for any $\varepsilon > 0$, there exists $\{(k_t, \chi_t, q_t)\}_{t=0}^\infty \in \Pi(k_0, \chi_0)$ such that

$$\sum_{t=0}^\infty \gamma^t g(k_t, \chi_t, q_t) \geq \mathcal{V}(k_0, \chi_0) - \varepsilon.$$

Choose $k_0' \geq k_0$ and $\chi_0' \leq \chi_0$. Since $\frac{\partial A}{\partial k}(k_t, q_t) > 0$ for all $q_t \in Q(k_t)$, $k_t \in \mathbb{R}_+$, we may recursively define

$$k_{t+1}' = A(k_t', q_t) \geq A(k_t, q_t) = k_{t+1} \quad \text{and} \quad \chi_{t+1}' = E(\chi_t', y_t^b) \leq \chi_{t+1},$$

where $q_t \in Q(k_t) \subseteq Q(k_t')$ for all $t \geq 0$. It follows from the properties of the return function g that

$$\mathcal{V}(k_0', \chi_0') \geq \sum_{t=0}^{\infty} \gamma^t g(k_t', \chi_t', q_t) \geq \sum_{t=0}^{\infty} \gamma^t g(k_t, \chi_t, q_t) \geq \mathcal{V}(k_0, \chi_0) - \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, $\mathcal{V}(k_0', \chi_0') \geq \mathcal{V}(k_0, \chi_0)$. ☐

Proof of Lemma 5.1. For each sector $j = g, b$, we have $y_t^j = l_t^j f_j(k_t^j)$ and $f_j(k_t^j) = w_j(k_t^j) + f_j'(k_t^j) k_t^j$. Thus, the accounting identity (5.15) may be rewritten as

$$w_t + r_t k_t = l_t^g \big[w_g(k_t^g) + f_g'(k_t^g) k_t^g\big] + \big[1 - \epsilon_b \tau_t\big] l_t^b \big[w_b(k_t^b) + f_b'(k_t^b) k_t^b\big]. \tag{E.7}$$

Using (5.10) and (5.14), (E.7) takes the form

$$w_g(k_t^g) + f_g'(k_t^g)\big[l_t^g k_t^g + l_t^b k_t^b\big] = l_t^g \big[w_g(k_t^g) + f_g'(k_t^g) k_t^g\big] + l_t^b w_t^g(k_t^g) + l_t^b f_g'(k_t^g) k_t^b. \tag{E.8}$$

Since $l_t^g + l_t^b = 1$, we see that (E.8) and hence (5.15) is satisfied. ☐

Proof of Proposition 5.1. Given a feasible fiscal policy $\tau, a^2 : \mathbb{R}_+^2 \to \mathbb{R}$ in the sense of Definition 5.2, let the state $(k_t, \chi_t) \in \mathbb{R}_+^2$ be arbitrary but fixed. The corresponding emissions tax rate is $\tau(k_t, \chi_t)$ and the old-age transfer is $a^2(k_t, \chi_t)$. The firms' first-order conditions (5.8) imply that profits in both sectors are maximized if and only if $k_t^j = \kappa_j(\omega_t)$, $j = g, b$. Hence, the market-clearing conditions (5.14) are satisfied if and only if $l_t^j = \ell_j(k_t, \omega_t)$, $j = g, b$. The factor prices satisfy (5.10) if and only if

$$\varrho(\omega_t) = 1 - \epsilon_b \tau(k_t, \chi_t). \tag{E.9}$$

Since the emissions-tax-policy rule τ is of the form (5.25), it follows that (E.9) is satisfied if and only if $\omega_t = \Omega_{\text{eq}}(k_t, \tau(k_t, \chi_t))$, where the function Ω_{eq} was defined in (5.23).

Given $\omega_t = \Omega_{\text{eq}}(k_t, \tau(k_t, \chi_t))$, the labor shares $\ell_g(k_t, \omega_t)$ and $\ell_g(k_t, \omega_t)$ are feasible because $\omega_t \in \big[\Omega_{\min}(k_t), \Omega_{\max}(k_t)\big]$. Since the brown output level is $y_t^b = \mathsf{y}_b(k_t, \omega_t) \in \big[0, f_b(k_t)\big]$, it follows from the government's budget constraint (5.6) that the youthful transfer must be

$$a_t^1 = \tau(k_t, \chi_t) \epsilon_b \mathsf{y}_b(k_t, \omega_t) - \tfrac{a_t^2}{1+n}.$$

The question of whether the list given in Proposition 5.1 is a temporary equilibrium now reduces to the question of whether the savings function $s(w_t + a_t^1, a_t^{2,e}, R_t^e)$ is well defined. Since the fiscal policy is feasible, Definition 5.2 implies that

$$w_t = w_g(\kappa_g(\omega_t)) \geq \tfrac{a_t^2}{1+n} - \tau(k_t, \chi_t) \epsilon_b \mathsf{y}_b(k_t, \omega_t) = -a_t^1,$$

so that the disposable youthful income is non-negative, $w_t + a_t^1 \geq 0$. As a consequence, $s(w_t + a_t^1, a_t^{2,e}, R_t^e)$ is well defined if and only if $w_t + a_t^1 + \frac{a_t^{2,e}}{R_t^e} \geq 0$, in which case the list given in Proposition 5.1 is a temporary equilibrium. Since, by (5.17), the labor shares $l_t^j = \ell_j(k_t, \omega_t)$, $j = g, b$, are uniquely determined whenever $\Omega_g(k_t) \neq \Omega_b(k_t)$, we can conclude that this list is the *only* temporary equilibrium if $\Omega_g(k_t) \neq \Omega_b(k_t)$. ☐

Proof of Lemma 5.2. Let $p_t^j > 0$ denote the output price in sector $j = g, b$, then we may set $\frac{p_t^b}{p_t^g} = 1 - \epsilon_b \tau_t$. The proof of the claim is then found in Ritschel and Wenzelburger (2025). ☐

Proof of Theorem 5.1. *(i).* The proof follows De La Croix and Michel (2002, p. 20-22). Given a feasible fiscal policy $\tau, a^2 : \mathbb{R}_+^2 \to \mathbb{R}$ in the sense of Definition 5.2, let the state $(k_t, \chi_t) \in \mathbb{R}_+^2$ be arbitrary but fixed. Set

$$w_t^d = w^d(k_t, \chi_t) \quad \text{and} \quad \chi_{t+1} = \vartheta(k_t, \chi_t).$$

The feasibility of the fiscal policy implies that $w_t^d \geq 0$. Set

$$R_{t+1}(k) := 1 - \delta + f_g'\big(\kappa_g\big(\Omega_{\mathrm{eq}}(k, \tau(k, \chi_{t+1}))\big)\big) \tag{E.10}$$

and

$$\mathcal{E}_{t+1}(k) := \tfrac{1}{1+n} s\big(w_t^d, a^2(k, \chi_{t+1}), R_{t+1}(k)\big) - k.$$

The map $\mathcal{E}_{t+1}$ is only well defined for a non-negative discounted lifetime income, i.e., for

$$\frac{a^2(k, \chi_{t+1})}{R_{t+1}(k)} \geq -w_t^d.$$

Since the fiscal policy is feasible, it follows from the first inequality in (5.24) that

$$\frac{a^2(k, \chi_{t+1})}{R_{t+1}(k)} \geq -(1+n)k \quad \text{for all} \quad k \geq 0.$$

Hence, $\mathcal{E}_{t+1}$ is well defined on the interval $\big[0, \frac{1}{1+n} w_t^d\big]$. Using the intermediate-value theorem, we establish the existence of a solution $k_{t+1} \in \big[0, \frac{1}{1+n} w_t^d\big]$ to

$$\mathcal{E}_{t+1}(k) \overset{!}{=} 0. \tag{E.11}$$

There are two cases. First, if $w_t^d = 0$, then $k_{t+1} = 0$ is the unique solution to (E.11). Second, $w_t^d > 0$. Since savings are bounded from above by the disposable income w_t^d, we have

$$\mathcal{E}_{t+1}\big(\tfrac{w_t^d}{1+n}\big) < 0.$$

Since old-age consumption is an ordinary good, the map $R \mapsto R s(w_t^d, a^2, R) + a^2$ is strictly increasing for each $a^2 \in \mathbb{R}$ with $w_t^d + \frac{a^2}{R} \geq 0$. Since Assumption 5.1 implies $\Omega_{\mathrm{eq}}(0, \cdot) =$

0, $\kappa_g(0) = 0$, $\lim_{k\to 0} f_g'(k)k = 0$, $w_g(0) = 0$, and $\mathsf{y}_b(0,\cdot) = 0$, it follows from (E.10) that $\lim_{k\to 0} R_{t+1}(k)k = 0$, and from (5.24) that $\lim_{k\to 0} a^2(k,\chi_{t+1}) = 0$. As a consequence,

$$\begin{aligned}&\lim_{k\to 0}\Big[R_{t+1}(k)s(w_t^d, a^2(k,\chi_{t+1}), R_{t+1}(k)) + a^2(k,\chi_{t+1})\Big]\\ &\quad = \lim_{k\to 0} R_{t+1}(k)s(w_t^d, 0, R_{t+1}(k)) \geq \bar{c} \ > 0.\end{aligned}$$

It now follows that

$$\lim_{k\to 0}\frac{s(w_t^d, a^2(k,\chi_{t+1}), R_{t+1}(k))}{k} = \lim_{k\to 0}\frac{R_{t+1}(k)s(w_t^d, 0, R_{t+1}(k))}{R_{t+1}(k)k} \geq \lim_{k\to 0}\frac{\bar{c}}{R_{t+1}(k)k} = \infty.$$

Hence, for $\varepsilon > 0$ sufficiently small, we have

$$\mathcal{E}_{t+1}(k) > 0 \quad \text{for all} \quad k \in (0,\varepsilon).$$

Since the function $\mathcal{E}_{t+1}$ is continuous on the interval in question, the intermediate-value theorem ensures the existence of a solution $k_{t+1} \in \big(0, \frac{1}{1+n}w_t^d\big)$ to (E.11). Since $(k_t,\chi_t) \in \mathbb{R}_+^2$ was arbitrary, we can conclude that there exists a forecasting rule $\varphi : \mathbb{R}_+^2 \to \mathbb{R}_+$ that is perfect in the sense of Definition 5.4. In particular, since the feasible fiscal policy $\tau, a^2 : \mathbb{R}_+^2 \to \mathbb{R}$ was arbitrary, a perfect forecasting rule exists for *any* feasible fiscal policy in the sense of Definition 5.2.

(ii). Uniqueness is addressed next. Under the hypotheses of Assumption 5.2 (i) and weak gross substitutability between youthful and old-age consumption, the savings function $s(w_t^d, a^2, R)$ is non-decreasing in R and non-increasing in a^2. The transfer-policy-rule $a^2(k,\chi_{t+1})$ is non-decreasing in k by (5.29). Thus, $k \mapsto \mathcal{E}_{t+1}(k)$ is strictly decreasing on $\big[0, \frac{1}{1+n}w_t^d\big]$ if

$$\begin{aligned}R_{t+1}'(k) = f_g''\big(\kappa_g\big(\Omega_{\text{eq}}(k,\tau(k,\chi_{t+1}))\big)\big)\,\kappa_g'\big(\Omega_{\text{eq}}(k,\tau(k,\chi_{t+1}))\big)&\\ \left[\frac{\partial\Omega_{\text{eq}}}{\partial k}(k,\tau(k,\chi_{t+1})) + \frac{\partial\Omega_{\text{eq}}}{\partial\tau}(k,\tau(k,\chi_{t+1}))\frac{\partial\tau}{\partial k}(k,\chi_{t+1})\right] &\leq 0\end{aligned} \tag{E.12}$$

for all $k \in \big[0, \frac{1}{1+n}w_t^d\big]$. By Assumption 5.1, we have $f_g'' < 0$ and $\kappa_g' > 0$. Moreover, (5.23) implies that $\frac{\partial\Omega_{\text{eq}}}{\partial k}(k,\tau(k,\chi_{t+1})) \geq 0$ for all $k \geq 0$. Therefore, (E.12) is satisfied if

$$\frac{\partial\Omega_{\text{eq}}}{\partial\tau}(k,\tau(k,\chi_{t+1}))\frac{\partial\tau}{\partial k}(k,\chi_{t+1}) \geq 0. \tag{E.13}$$

There are three cases in (E.13). First, if $1-\epsilon_b\tau(k,\chi_{t+1}) \leq \varrho(\Omega_g(k))$, then $\Omega_{\text{eq}}(k,\tau(k,\chi_{t+1})) = \Omega_g(k)$. Second, if $\varrho(\Omega_b(k)) \leq 1-\epsilon_b\tau(k,\chi_{t+1})$, then $\Omega_{\text{eq}}(k,\tau(k,\chi_{t+1})) = \Omega_b(k)$. In both cases, (E.13) is satisfied because $\frac{\partial\Omega_{\text{eq}}}{\partial\tau}(k,\tau(k,\chi_{t+1})) = 0$. The third case is $\varrho(\Omega_g(k)) < 1-\epsilon_b\tau(k,\chi_{t+1}) < \varrho(\Omega_b(k))$. In this case,

$$\Omega_{\text{eq}}(k,\tau(k,\chi_{t+1})) = \varrho^{-1}(1-\epsilon_b\tau(k,\chi_{t+1})),$$

so that the inverse function rule yields

$$\frac{\partial\Omega_{\text{eq}}}{\partial\tau}(k,\tau(k,\chi_{t+1})) = -\frac{\epsilon_b}{\varrho'\big(\Omega_{\text{eq}}(k,\tau(k,\chi_{t+1}))\big)}.$$

In Ritschel and Wenzelburger (2025), we show that for each $\omega > 0$, $\varrho'(\omega) \geq 0$ if and only if $\kappa_g(\omega) \geq \kappa_b(\omega)$. It follows from (5.29) that (E.13) holds. This shows that under the hypotheses of Theorem 5.1 (ii), the solution k_{t+1} to (E.11) is unique. As a consequence, the perfect forecasting rule φ is uniquely determined. $\square$

Proof of Lemma 5.3. See Ritschel and Wenzelburger (2025, Lemma 2). $\square$

Proof of Theorem 5.2. *Step 1 (Attainability of a maximum).* Without loss of generality, we may consider period $t = 0$. Given an arbitrary but fixed state $(k_0, \chi_0) \in \mathbb{R}_{++} \times \mathbb{R}_+$, consider the maximization problem

$$\max_{(y_0^b, c_0^1, c_0^2) \in Q(k_0)} V(k_0, \chi_0, y_0^b, c_0^1, c_0^2), \tag{E.14}$$

where

$$V(k_0, \chi_0, y_0^b, c_0^1, c_0^2) := g(k_0, \chi_0, y_0^b, c_0^1, c_0^2) + \gamma \mathcal{V}\big(A(k_0, y_0^b, c_0^1, c_0^2), E(\chi_0, y_0^b)\big).$$

We will show that Problem (E.14) admits a unique solution $(y_0^{b*}, c_0^{1*}, c_0^{2*}) \in Q(k_0)$ with $c_0^{1*} > 0$ and $c_0^{2*} > 0$, so that the Bellman equation takes the form

$$\mathcal{V}(k_0, \chi_0) = V(k_0, \chi_0, y_0^{b*}, c_0^{1*}, c_0^{2*}). \tag{E.15}$$

Recall that the set $Q(k_0)$ is non-empty, convex, and compact. Assumption 5.2 implies that $(y_0^b, c_0^1, c_0^2) \mapsto g(k_0, \chi_0, y_0^b, c_0^1, c_0^2)$ is strictly concave. Hence, objective function $V(k_0, \chi_0, \cdot)$ in (E.14) is strictly concave on $Q(k_0)$ if we can show that

$$(y_0^b, c_0^1, c_0^2) \mapsto \mathcal{V}\big(A(k_0, y_0^b, c_0^1, c_0^2), E(\chi_0, y_0^b)\big) \tag{E.16}$$

is concave on $Q(k_0)$. Choose any policies $q_0^1, q_0^2 \in Q(k_0)$ and set

$$q_0^\lambda := \lambda q_0^1 + (1 - \lambda) q_0^2,$$

where $\lambda \in [0, 1]$ is arbitrary. Since $Q(k_0)$ is convex, $q_0^\lambda \in Q(k_0)$. Setting

$$k_1^i := A(k_0, q_0^i) \quad \text{and} \quad e_1^i = E(\chi_0, q_0^i), \quad i = 1, 2, \lambda,$$

we have $k_1^\lambda \geq \lambda k_1^1 + (1 - \lambda) k_1^2$ because $A(k_0, \cdot)$ is concave and $e_1^\lambda = \lambda e_1^1 + (1 - \lambda) e_1^2$ because $E(\chi_0, \cdot)$ is linear. By Lemma E.2, the value function $\mathcal{V}$ is concave, non-decreasing in its first, and non-increasing in its second argument. As a consequence,

$$\mathcal{V}(k_1^\lambda, e_1^\lambda) \geq \lambda \mathcal{V}(k_1^1, e_1^1) + (1 - \lambda) \mathcal{V}(k_1^2, e_1^2). \tag{E.17}$$

Since $q_0^1, q_0^2 \in Q(k_0)$ and $\lambda \in [0, 1]$ were arbitrary, we conclude that (E.16) is concave on $Q(k_0)$.

We next show that the maximum (E.14) cannot be attained at any $(y_0^b, c_0^1, c_0^2) \in Q(k_0)$ with $c_0^1 = 0$, $c_0^2 = 0$, or $A(k_0, y_0^b, c_0^1, c_0^2) = 0$. Applying the Bellman equation twice, Lemma E.2 implies that

$$\begin{aligned}\mathcal{V}(k_0, \chi_0) = \sup\Big\{ & g(k_0, \chi_0, y_0^b, c_0^1, c_0^2) + \gamma g(k_1, \chi_1, y_1^b, c_1^1, c_1^2) + \gamma^2 \mathcal{V}(k_2, \chi_2) \;\Big|\; \\ & (y_0^b, c_0^1, c_0^2) \in Q(k_0),\ k_1 = A(k_0, y_0^b, c_0^1, c_0^2),\ \chi_1 = E(\chi_0, y_0^b), \\ & \text{and } (y_1^b, c_1^1, c_1^2) \in Q(k_1),\ k_2 = A(k_1, y_1^b, c_1^1, c_1^2),\ \chi_2 = E(\chi_1, y_1^b) \Big\}.\end{aligned} \tag{E.18}$$

Set

$$k_2 = A(k_1, y_1^b, c_1^1, c_1^2) \quad \text{and} \quad \hat{k}_2 = A(\hat{k}_1, \hat{y}_1^b, \hat{c}_1^1, \hat{c}_1^2).$$

Observe that $k_2 = \hat{k}_2$ if and only if

$$f(k_1, y_1^b) - f(\hat{k}_1, \hat{y}_1^b) = \big(c_1^1 + \tfrac{c_1^2}{1+n}\big) - \big(\hat{c}_1^1 + \tfrac{\hat{c}_1^2}{1+n}\big). \tag{E.19}$$

Consider any $(y_0^b, c_0^1, c_0^2) \in Q(k_0)$ with $c_0^1 = 0$. Choose $\varepsilon_0 > 0$ and $(\hat{y}_0^b, \hat{c}_0^1, \hat{c}_0^2) = (y_0^b, \varepsilon_0, c_0^2) \in Q(k_0)$ so that

$$\hat{k}_1 = A(k_0, \hat{y}_0^b, \hat{c}_0^1, \hat{c}_0^2) = k_1 - \tfrac{\varepsilon_0}{1+n} \quad \text{and} \quad \hat{\chi}_1 = E(\hat{\chi}_0, y_0^b) = \chi_1.$$

Next, choose $(y_1^b, c_1^1, c_1^2) \in Q(k_1)$ and $(\hat{y}_1^b, \hat{c}_1^1, \hat{c}_1^2) \in Q(\hat{k}_1)$ with $\hat{y}_1^b \leq y_1^b$, $\hat{c}_1^1 < c_1^1$, and $\hat{c}_1^2 = c_1^2$ such that (E.19) holds. Then $k_2 = \hat{k}_2$ and $\hat{\chi}_2 \leq \chi_2$ so that, by Lemma E.2, $\mathcal{V}(\hat{k}_2, \hat{\chi}_2) \geq \mathcal{V}(k_2, \chi_2)$. If $\varepsilon_0 > 0$ is sufficiently small, the Inada condition on u implies that

$$g(k_0, \chi_0, y_0^b, \varepsilon_0, c_0^2) + \gamma g(\hat{k}_1, \chi_1, \hat{y}_1^b, \hat{c}_1^1, c_1^2) > g(k_0, \chi_0, y_0^b, 0, c_0^2) + \gamma g(k_1, \chi_1, y_1^b, c_1^1, c_1^2).$$

Thus, the maximum (E.14) cannot be attained with $c_0^1 = 0$. An analogous argument applies for any $(y_0^b, c_0^1, c_0^2) \in Q(k_0)$ with $c_0^2 = 0$.

Consider now any $(y_0^b, c_0^1, c_0^2) \in Q(k_0)$ with $c_0^1, c_0^2 > 0$ and arbitrarily small $k_1 > 0$. Choose $\varepsilon_0, \varepsilon_1 > 0$ such that

$$(\hat{y}_0^b, \hat{c}_0^1, \hat{c}_0^2) = (y_0^b, c_0^1 - \varepsilon_0, c_0^2) \in Q(k_0) \quad \text{and} \quad (\hat{y}_1^b, \hat{c}_1^1, \hat{c}_1^2) = (y_1^b, c_1^1 + \varepsilon_1, c_1^2) \in Q(\hat{k}_1)$$

so that $\hat{k}_1 > k_1$ and $\hat{k}_2 = k_2$ holds via (E.19). Since $\hat{y}_j^b = y_j^b$, $j = 0, 1$, it follows that $\hat{\chi}_1 = \chi_1$ and $\hat{\chi}_2 = \chi_2$. Since $f(0,0) = 0$, c_1^1 will become arbitrarily small if k_1 is arbitrarily small. The Inada condition on u then implies that

$$g(k_0, \chi_0, y_0^b, c_0^1 - \varepsilon_0, c_0^2) + \gamma g(\hat{k}_1, \chi_1, y_1^b, \hat{c}_1^1 + \varepsilon_1, c_1^2) > g(k_0, \chi_0, y_0^b, c_0^1, c_0^2) + \gamma g(k_1, \chi_1, y_1^b, c_1^1, c_1^2).$$

Hence, the maximum (E.14) cannot be attained with arbitrarily small $k_1 = A(k_0, y_0^b, c_0^1, c_0^2)$.

The objective function $V(k_0, \chi_0, \cdot)$ in the maximization problem (E.14) is strictly concave and continuous on any compact subset of the convex set $Q(k_0)$. Therefore, it attains a unique maximum at $(y_0^{b*}, c_0^{1*}, c_0^{2*}) \in Q(k_0)$ with $c_0^{1*} > 0$, $c_0^{2*} > 0$, and $k_1^* = A(k_0, y_0^{b*}, c_0^{1*}, c_0^{2*}) > 0$.

Step 2 (Existence of the optimal allocation). By the nature of the maximization problem (E.14), the maximizer depends solely on (k_0, χ_0). Since $(k_0, \chi_0) \in \mathbb{R}_{++} \times \mathbb{R}_+$ was arbitrary, there exist continuous policy functions of the form $y^b_* : \mathbb{R}_{++} \times \mathbb{R}_+ \to \mathbb{R}_+$ and $c^1_*, c^2_* : \mathbb{R}_{++} \times \mathbb{R}_+ \to \mathbb{R}_{++}$ such that $y^{b*}_0 = y^b_*(k_0, \chi_0) \in [0, f_b(k_0)]$, $c^{1*}_0 = c^1_*(k_0, \chi_0)$, and $c^{2*}_0 = c^2_*(k_0, \chi_0)$. Setting $(k^*_0, \chi^*_0) = (k_0, k_0)$, the recursively defined allocation

$$\begin{gathered} k^*_{t+1} = A\big(k^*_t, y^{b*}_t, c^{1*}_t, c^{2*}_t\big), \quad \chi^*_{t+1} = E\big(\chi^*_t, y^{b*}_t\big), \\ y^{b*}_t = y^b_*(k^*_t, \chi^*_t), \quad c^{1*}_t = c^1_*(k^*_t, \chi^*_t), \quad c^{2*}_t = c^2_*(k^*_t, \chi^*_t), \end{gathered}$$

$t \geq 0$, is feasible by construction, so that $\{(k^*_t, \chi^*_t, y^{b*}_t, c^{1*}_t, c^{2*}_t)\}^\infty_{t=0} \in \Pi(k_0, \chi_0)$ and

$$\mathcal{V}(k^*_t, \chi^*_t) = g(k^*_t, \chi^*_t, y^{b*}_t, c^{1*}_t, c^{2*}_t) + \gamma \mathcal{V}(k^*_{t+1}, \chi^*_{t+1}) \quad \text{for all} \quad t \geq 0. \tag{E.20}$$

Repeated application of (E.20) shows that

$$\mathcal{V}(k_0, \chi_0) = \sum_{t=0}^{\tau-1} \gamma^t g(k^*_t, \chi^*_t, y^{b*}_t, c^{1*}_t, c^{2*}_t) + \gamma^\tau \mathcal{V}(k^*_\tau, \chi^*_\tau) \quad \text{for all} \quad \tau \geq 1. \tag{E.21}$$

Since the sequence $\{(k^*_t, \chi^*_t)\}^\infty_{t=0}$ is bounded, $\{\mathcal{V}(k^*_t, \chi^*_t)\}^\infty_{t=0}$ is bounded as well. Hence,

$$\mathcal{V}(k_0, \chi_0) = \sum_{t=0}^{\infty} \gamma^t g(k^*_t, \chi^*_t, y^{b*}_t, c^{1*}_t, c^{2*}_t) = W(\{(k^*_t, \chi^*_t, y^{b*}_t, c^{1*}_t, c^{2*}_t)\}^\infty_{t=0}).$$

Step 3 (Differentiability of $\mathcal{V}$). We establish the differentiability of the value function $\mathcal{V}$, which in turn implies differentiability of the objective function V in Problem (E.14). Let $(k_0, \chi_0) \in \mathbb{R}_{++} \times \mathbb{R}_+$ be arbitrary but fixed. As shown in Step 1, there exists $(y^b_0, c^1_0, c^2_0) \in Q(k_0)$ such that

$$\mathcal{V}(k_0, \chi_0) = g(k_0, \chi_0, y^b_0, c^1_0, c^2_0) + \gamma \mathcal{V}\big(A(k_0, y^b_0, c^1_0, c^2_0), E(\chi_0, y^b_0)\big). \tag{E.22}$$

Given $\chi_1 = E(\chi_0, y^b_0)$, define the function

$$\Phi(\chi) := \tfrac{1+n}{\epsilon_b} \chi_1 - \tfrac{1-\zeta}{\epsilon_b} \chi.$$

By construction, the function Φ satisfies

$$E(\chi, \Phi(\chi)) = E(\chi_0, y^b_0) \quad \text{for all} \quad \chi \in \mathbb{R}_+. \tag{E.23}$$

Now, choose an open neighborhood $\mathscr{U}(k_0, \chi_0)$ of (k_0, χ_0) and set

$$c_1(k, \chi) := f(k, \Phi(\chi)) - f(k_0, y^b_0) + c^1_0 \quad \text{and} \quad c_2(k, \chi) := c^2_0,$$

so that

$$A(k, \Phi(\chi), c_1(k, \chi), c^2_0) = A(k_0, y^b_0, c^1_0, c^2_0) = k_1$$

for all $(k, \chi) \in \mathscr{U}(k_0, \chi_0)$. Observe that

$$y^b_0 = \Phi(\chi_0), \quad c^1_0 = c_1(k_0, \chi_0), \quad \text{and} \quad c^2_0 = c_2(k_0, \chi_0).$$

From (E.22), we can infer that for each $(k, \chi) \in \mathscr{U}(k_0, \chi_0)$,

$$\mathcal{V}(k, \chi) \geq g(k, \chi, \Phi(\chi), c_1(k, \chi), c_2(k, \chi)) + \gamma \mathcal{V}(k_1, \chi_1)$$

and, therefore,

$$\mathcal{V}(k,\chi)-\mathcal{V}(k_0,\chi_0)\geq g(k,\chi,\Phi(\chi),c_1(k,\chi),c_2(k,\chi))-g(k_0,\chi_0,y_0^b,c_0^1,c_0^2).$$

This shows that the concave function $\mathcal{V}$ is bounded from below on $\mathscr{U}(k_0,\chi_0)$ by a differentiable function. Hence, $\mathcal{V}$ itself must be differentiable at (k_0,χ_0). Since $(k_0,\chi_0)\in\mathbb{R}_{++}\times\mathbb{R}_+$ was arbitrary, we can conclude that $\mathcal{V}$ is differentiable on its entire domain.

Step 4 (First-order conditions). We establish the first-order conditions (5.45) – (5.47). Given some arbitrary but fixed state $(k_t,\chi_t)\in\mathbb{R}_{++}\times\mathbb{R}_+$, the Lagrangian for the maximization problem (E.14) in period $t\geq 0$ reads

$$\mathcal{L}_t(y^b,c^1,c^2,\lambda_t^0,\lambda_t^1,\lambda_t^2)=V(k_t,\chi_t,y^b,c^1,c^2)+\lambda_t^0\big[f(k_t,y^b)-c^1-\tfrac{c^2}{1+n}\big]+\lambda_t^1y^b+\lambda_t^2\big[f_b(k_t)-y^b\big].$$

It follows from Step 3 that $\mathcal{L}_t$ is differentiable. Setting

$$k_{t+1}=A(k_t,y_t^b,c_t^1,c_t^2)\quad\text{and}\quad\chi_{t+1}=E(\chi_t,y_t^b)=\tfrac{z_t}{1+n},$$

then differentiation of $\mathcal{L}_t$ shows that the optimal policy $(y_t^b,c_t^1,c_t^2)\in Q(k_t)$ satisfies the first-order conditions

$$-(1+\tfrac{\beta}{\gamma})\epsilon_b\mu'(z_t)+\tfrac{\gamma}{1+n}\frac{\partial\mathcal{V}}{\partial k}(k_{t+1},\chi_{t+1})\frac{\partial f}{\partial y^b}(k_t,y_t^b)+\tfrac{\gamma}{1+n}\frac{\partial\mathcal{V}}{\partial\chi}(k_{t+1},\chi_{t+1})\epsilon_b\overset{!}{=}\lambda_t^2-\lambda_t^1 \tag{E.24}$$

$$u'(c_t^1)-\tfrac{\gamma}{1+n}\frac{\partial\mathcal{V}}{\partial k}(k_{t+1},\chi_{t+1})\overset{!}{=}\lambda_t^0 \tag{E.25}$$

$$\tfrac{\beta}{\gamma}u'(c_t^2)-\tfrac{\gamma}{(1+n)^2}\frac{\partial\mathcal{V}}{\partial k}(k_{t+1},\chi_{t+1})\overset{!}{=}\tfrac{\lambda_t^0}{1+n} \tag{E.26}$$

together with the complementary slackness conditions

$$\lambda_t^0\big[f(k_t,y_t^b)-c_t^1-\tfrac{c_t^2}{1+n}\big]\overset{!}{=}0 \tag{E.27}$$

$$\lambda_t^1y_t^b\overset{!}{=}0 \tag{E.28}$$

$$\lambda_t^2\big[f_b(k_t)-y_t^b\big]\overset{!}{=}0, \tag{E.29}$$

where $\lambda_t^0,\lambda_t^1,\lambda_t^2\geq 0$. As shown in Step 1, the optimal policy (y_t^b,c_t^1,c_t^2) satisfies $k_{t+1}=A(k_t,y_t^b,c_t^1,c_t^2)>0$, implying that (E.27) holds with $\lambda_t^0=0$. Conditions (E.25) and (E.26) now take the form

$$\frac{u'(c_t^1)}{\beta u'(c_t^2)}\overset{!}{=}\tfrac{1+n}{\gamma} \tag{E.30}$$

$$u'(c_t^1)\overset{!}{=}\tfrac{\gamma}{1+n}\frac{\partial\mathcal{V}}{\partial k}(k_{t+1},\chi_{t+1}). \tag{E.31}$$

Equation (E.30) establishes (5.45).

We next establish (5.47). Applying the envelope theorem (e.g., see Mas-Colell et al., 1995, p. 965) to the maximization problem (E.14) and using the Bellman equation (E.20), differentiation of $\mathcal{V}$ yields

$$\frac{\partial\mathcal{V}}{\partial\chi}(k_t,\chi_t)=-(1-\zeta)\big(1+\tfrac{\beta}{\gamma}\big)\mu'(z_t)+\tfrac{\gamma(1-\zeta)}{1+n}\frac{\partial\mathcal{V}}{\partial\chi}(k_{t+1},\chi_{t+1}). \tag{E.32}$$

Repeated application of (E.32) yields

$$\frac{\partial\mathcal{V}}{\partial\chi}(k_t,\chi_t) = -(1-\zeta)\big(1+\tfrac{\beta}{\gamma}\big)\sum_{i=0}^{\tau-1}\Big[\tfrac{\gamma(1-\zeta)}{1+n}\Big]^i \mu'(z_{t+i}) + \Big[\tfrac{\gamma(1-\zeta)}{1+n}\Big]^\tau \frac{\partial\mathcal{V}}{\partial\chi}(k_{t+\tau},\chi_{t+\tau}) \tag{E.33}$$

for all $\tau \geq 1$. Since $\{(k_t,\chi_t)\}_{t=0}^{\infty}$ is bounded, so is the sequence $\big\{\frac{\partial\mathcal{V}}{\partial\chi}(k_t,\chi_t)\big\}_{t=0}^{\infty}$. Hence,

$$\frac{\partial\mathcal{V}}{\partial\chi}(k_t,\chi_t) = -(1-\zeta)\big(1+\tfrac{\beta}{\gamma}\big)\sum_{i=0}^{\infty}\Big[\tfrac{\gamma(1-\zeta)}{1+n}\Big]^i \mu'(z_{t+i}). \tag{E.34}$$

Inserting (E.34) and (E.31) into (E.24) yields, after re-normalizing the Lagrange multipliers,

$$\frac{\epsilon_b\big(1+\frac{\beta}{\gamma}\big)}{u'(c_t^1)}\sum_{i=0}^{\infty}\Big[\tfrac{\gamma(1-\zeta)}{1+n}\Big]^i \mu'(z_{t+i}) \overset{!}{=} \frac{\partial f}{\partial y^b}(k_t,y_t^b) + \lambda_t^1 - \lambda_t^2. \tag{E.35}$$

Inserting (E.30), the l.h.s. of (E.35) becomes the marginal rate of substitution ψ_t^* defined in (5.40). Using Lemma 5.3, (E.35) takes the form

$$1 - \epsilon_b\psi_t^* \overset{!}{=} \varrho(\Omega(k_t,y_t^b)) + \lambda_t^2 - \lambda_t^1. \tag{E.36}$$

Equations (E.28), (E.29), and (E.36) are precisely the optimality conditions for

$$\max_{0\leq y^b\leq f_b(k_t)} \; T(k_t,y^b) + \big[1-\epsilon_b\psi_t^*\big]y^b.$$

This establishes (5.47).

It remains to establish (5.46). Applying the envelope theorem to the maximization problem (E.14) and exploiting the Bellman equation (E.20), differentiation of $\mathcal{V}$ yields

$$\frac{\partial\mathcal{V}}{\partial k}(k_t,\chi_t) = \tfrac{\gamma}{1+n}\frac{\partial\mathcal{V}}{\partial k}(k_{t+1},\chi_{t+1})\frac{\partial f}{\partial k}(k_t,y_t^b) + \lambda_t^2 f_b'(k_t). \tag{E.37}$$

Inserting (E.30) and (E.31) into (E.37) and re-normalizing the Lagrange multiplier, we obtain

$$\frac{u'(c_{t-1}^1)}{\beta u'(c_t^2)} = \frac{\partial f}{\partial k}(k_t,y_t^b) + \lambda_t^2 f_b'(k_t). \tag{E.38}$$

Applying Lemma 5.3, (E.38) takes the form

$$\frac{u'(c_{t-1}^1)}{\beta u'(c_t^2)} = 1 - \delta + f_g'(\kappa_g(\omega_t)) + \lambda_t^2 f_b'(k_t), \tag{E.39}$$

where $\omega_t = \Omega(k_t,y_t^b)$. Since (E.29) implies that $k_t = \kappa_b(\omega_t)$ whenever $\lambda_t^2 > 0$, (5.46) holds whenever

$$f_g'(\kappa_g(\omega_t)) + \lambda_t^2 f_b'(\kappa_b(\omega_t)) = r_t^* := \max\Big\{f_g'(\kappa_g(\omega_t)),\big[1-\epsilon_b\psi_t^*\big]f_b'(\kappa_b(\omega_t))\Big\}. \tag{E.40}$$

Using the function ϱ and (E.36), Condition (E.40) is equivalent to

$$\varrho(\omega_t) + \lambda_t^2 = \max\big\{\varrho(\omega_t),\varrho(\omega_t) + \lambda_t^2 - \lambda_1^1\big\}. \tag{E.41}$$

Since (E.28) and (E.29) imply that either $\lambda_t^1 = 0$ or $\lambda_t^2 = 0$, (E.41) holds. This establishes (5.46) and completes the proof. □

Proof of Corollary 5.1. We know from (5.23) that, for each $k_t > 0$ and $\tau_t \in \mathbb{R}$, the equilibrium brown output level $y_t^b = \mathsf{y}_b(k_t, \Omega_{\text{eq}}(k_t, \tau_t))$ satisfies

$$y_t^b = \begin{cases} 0 & \text{if } \ 1 - \epsilon_b \tau_t \leq \varrho(\Omega_g(k_t)) \\ \mathsf{y}_b(k_t, \varrho^{-1}(1 - \epsilon_b \tau_t)) & \text{if } \ \varrho(\Omega_g(k_t)) < 1 - \epsilon_b \tau_t < \varrho(\Omega_b(k_t)) \, . \\ f_b(k_t) & \text{if } \ \varrho(\Omega_b(k_t)) \leq 1 - \epsilon_b \tau_t \end{cases} \tag{E.42}$$

On the other hand, Lemma 5.3 implies that a solution y_t^b to the maximization problem (5.48) satisfies the first-order condition

$$1 - \epsilon_b \tau_t = \varrho(\Omega(k_t, y_t^b)) + \lambda_t^2 - \lambda_t^1 \tag{E.43}$$

together with the complementary slackness conditions (E.28) and (E.29). The claim now follows from the fact that (E.42) satisfies (E.43), (E.28), and (E.29). ☐

Proof of Corollary 5.2. *(i).* Let $(k_t^*, \chi_t^*) \in \mathbb{R}_{++} \times \mathbb{R}_+$ be arbitrary but fixed. If $\zeta = 1$, then the first-order condition (E.36) for the optimal brown output level $y_t^{b*} \in \big[0, f_b(k_t^*)\big]$ takes the form

$$1 - \epsilon_b \psi(k_t^*, \chi_t^*, y_t^{b^*}) = \varrho(\Omega(k_t^*, y_t^{b^*})) + \lambda_t^2 - \lambda_t^1, \tag{E.44}$$

where the function ψ was defined in (5.49). In addition, $y_t^{b*} \in \big[0, f_b(k_t^*)\big]$ must satisfy the complementary slackness conditions (E.28) and (E.29). Noting that $\Omega(k_t^*, 0) = \Omega_g(k_t^*)$ and $\Omega(k_t^*, f_b(k_t^*)) = \Omega_b(k_t^*)$, solving the optimality conditions (E.44), (E.28), and (E.29) establishes the claim.

(ii). If $\mu' \equiv 0$, then $\psi_t^* \equiv 0$, so that (5.47) takes the form

$$y_t^{b*} = \underset{0 \leq y^b \leq f_b(k_t^*)}{\operatorname{argmax}} \quad T(k_t^*, y^b) + y^b.$$

The claim now follows from the definition of the function f in (5.36). ☐

Proof of Theorem 5.3. Let a feasible allocation $\{(\hat{k}_t, \hat{\chi}_t, \hat{y}_t^b, \hat{c}_t^1, \hat{c}_t^2)\}_{t=0}^{\infty} \in \Pi(\hat{k}_0, \hat{\chi}_0)$ that satisfies (5.50) be given. The proof proceeds by induction. Let $(\hat{k}_t, \hat{\chi}_t) \in \mathbb{R}_{++} \times \mathbb{R}_+$ be arbitrary but fixed.

Step 1 (Emissions tax rate). We establish the emissions tax rate. Setting

$$\hat{\omega}_t := \Omega(\hat{k}_t, \hat{y}_t^b) \quad \text{and} \quad \hat{\tau}_t := \tfrac{1 - \varrho(\hat{\omega}_t)}{\epsilon_b}, \tag{E.45}$$

then (5.23) implies that the equilibrium wage-rental ratio in period t is

$$\Omega_{\text{eq}}\big(\hat{k}_t, \tfrac{1 - \varrho(\hat{\omega}_t)}{\epsilon_b}\big) = \hat{\omega}_t. \tag{E.46}$$

It follows from (E.45) and (E.46) that the equilibrium brown output level in period t is[1]

$$\mathsf{y}_b(\hat{k}_t, \hat{\omega}_t) = \mathsf{y}_b(\hat{k}_t, \Omega(\hat{k}_t, \hat{y}_t^b)) = \hat{y}_t^b.$$

Step 2 (Intergenerational transfers). We establish the intergenerational transfers. The feasibility property of the allocation $\{(\hat{k}_t, \hat{\chi}_t, \hat{y}_t^b, \hat{c}_t^1, \hat{c}_t^2)\}_{t=0}^{\infty}$ implies that

$$\hat{k}_{t+1} = A(\hat{k}_t, \hat{y}_t^b, \hat{c}_t^1, \hat{c}_t^2) = \tfrac{1}{1+n}\Big[f(\hat{k}_t, \hat{y}_t^b) - \hat{c}_t^1 - \tfrac{\hat{c}_t^2}{1+n}\Big] \geq 0. \tag{E.47}$$

Since the emissions tax rate $\hat{\tau}_t$ implements $\hat{\omega}_t$ as an equilibrium wage-rental ratio and $\hat{y}_t^b$ as an equilibrium brown output level, the accounting identity (5.15) implies that

$$f(\hat{k}_t, \hat{y}_t^b) = w_g(\kappa_g(\hat{\omega}_t)) + \big[1 - \delta + f_g'(\kappa_g(\hat{\omega}_t))\big]\hat{k}_t + \hat{\tau}_t \epsilon_b \hat{y}_t^b. \tag{E.48}$$

Inserting (E.48) into (E.47) yields

$$\hat{k}_{t+1} = \tfrac{1}{1+n}\Big[w_g(\kappa_g(\hat{\omega}_t)) + \big[1 - \delta + f_g'(\kappa_g(\hat{\omega}_t))\big]\hat{k}_t + \hat{\tau}_t \epsilon_b \hat{y}_t^b - \hat{c}_t^1 - \tfrac{\hat{c}_t^2}{1+n}\Big]. \tag{E.49}$$

Now, let the lump-sum transfer to an old agent be

$$\hat{a}_t^2 := \hat{c}_t^2 - \big[1 - \delta + f_g'(\kappa_g(\hat{\omega}_t))\big](1+n)\hat{k}_t. \tag{E.50}$$

The budget constraint (5.6) implies that the lump-sum transfer to a young agent is

$$\hat{a}_t^1 = \hat{\tau}_t \epsilon_b \hat{y}_t^b - \tfrac{\hat{a}_t^2}{1+n}. \tag{E.51}$$

Inserting (E.50) and (E.51) into (E.49) yields

$$\hat{c}_t^1 = w_g(\kappa_g(\hat{\omega}_t)) + \hat{\tau}_t \epsilon_b \hat{y}_t^b - \tfrac{\hat{a}_t^2}{1+n} - (1+n)\hat{k}_{t+1} = w_g(\kappa_g(\hat{\omega}_t)) + \hat{a}_t^1 - (1+n)\hat{k}_{t+1}. \tag{E.52}$$

Inserting (E.52) and (E.50) into Condition (5.50) gives

$$\frac{u'\big(w_g(\kappa_g(\hat{\omega}_t)) + \hat{a}_t^1 - (1+n)\hat{k}_{t+1}\big)}{\beta u'\big(\hat{a}_{t+1}^2 + \big[1 - \delta + f_g'(\kappa_g(\hat{\omega}_{t+1}))\big](1+n)\hat{k}_{t+1}\big)} = 1 - \delta + f_g'(\kappa_g(\hat{\omega}_{t+1})). \tag{E.53}$$

Condition (E.53) is the optimality condition for an individual savings decision of a young agent who has the disposable income $w_g(\kappa_g(\hat{\omega}_t)) + \hat{a}_t^1$ and correctly anticipates the gross return on capital $1 - \delta + f_g'(\kappa_g(\hat{\omega}_{t+1}))$ as well as the old-age transfer $\hat{a}_{t+1}^2$. Hence,

$$\hat{k}_{t+1} = \tfrac{1}{1+n} s\big(w_g(\kappa_g(\hat{\omega}_t)) + \hat{a}_t^1, \hat{a}_{t+1}^2, 1 - \delta + f_g'(\kappa_g(\hat{\omega}_{t+1}))\big). \tag{E.54}$$

Since t was arbitrary, we can conclude that there exists a sequence $\{(\hat{\tau}_t, \hat{a}_t^2, \hat{a}_t^1)\}_{t=0}^{\infty}$ such that $\{(\hat{k}_t, \hat{\chi}_t, \hat{y}_t^b, \hat{c}_t^1, \hat{c}_t^2)\}_{t=0}^{\infty}$ is a perfect-foresight allocation in the sense of Definition 5.3.

Step 3 (Feasibility of the transfers). The feasibility of the allocation $\{(\hat{k}_t, \hat{\chi}_t, \hat{y}_t^b, \hat{c}_t^1, \hat{c}_t^2)\}_{t=0}^{\infty}$ implies that $\hat{c}_t^1 \geq 0$ and $\hat{k}_{t+1} \geq 0$. Hence, it follows from (E.52) that

$$\tfrac{\hat{a}_t^2}{1+n} \leq w_g(\kappa_g(\hat{\omega}_t)) + \hat{\tau}_t \epsilon_b \hat{y}_t^b - (1+n)\hat{k}_{t+1} \leq w_g(\kappa_g(\hat{\omega}_t)) + \hat{\tau}_t \epsilon_b \hat{y}_t^b,$$

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[1] In the non-generic case $\Omega_g(\hat{k}_t) = \Omega_b(\hat{k}_t)$, there exists a unique labor share $\hat{l}_t^g \in [0,1]$ such that $\mathsf{y}_b(\hat{k}_t, \hat{\omega}_t) = (1 - \hat{l}_t^g) f_b(\hat{k}_t) = \hat{y}_t^b$ and the market economy is in a temporary equilibrium.

showing that the second inequality in (5.51) is satisfied. Moreover, since $\hat c_t^2 \geq 0$, it follows from (E.50) that

$$\tfrac{\hat a_t^2}{1+n} \geq -\big[1-\delta+f_g'(\kappa_g(\hat\omega_t))\big]\hat k_t,$$

showing that the first inequality in (5.51) is also satisfied. □

Proof of Corollary 5.3. For each $t \geq 0$, set $\omega_t^* := \Omega(k_t^*, y_t^{b*})$. By construction, the optimal allocation $\{(k_t^*, \chi_t^*, y_t^{b*}, c_t^{1*}, c_t^{2*})\}_{t=0}^\infty$ with $(k_0^*, \chi_0^*) = (k_0, \chi_0)$ is feasible, so that $\{(k_t^*, \chi_t^*, y_t^{b*}, c_t^{1*}, c_t^{2*})\}_{t=0}^\infty \in \Pi(k_0, \chi_0)$. If $0 \leq y_t^{b*} < f_b(k_t^*)$ for all $t \geq 0$, then (E.29) implies that $\lambda_t^2 = 0$ for all $t \geq 0$. The social planner's first-order condition (E.39) thus reads

$$\frac{u'(c_{t-1}^{1*})}{\beta u'(c_t^{2*})} = 1-\delta+f_g'(\kappa_g(\omega_t^*))$$

and fulfills the form required by (5.50). It now follows from Theorem 5.3 that there exists a sequence $\{(\tau_t^*, a_t^{1*}, a_t^{2*})\}_{t=0}^\infty$ such that the optimal allocation $\{(k_t^*, \chi_t^*, y_t^{b*}, c_t^{1*}, c_t^{2*})\}_{t=0}^\infty$ is a perfect-foresight allocation. By (E.45), the optimal emissions tax rate is

$$\tau_t^* = \frac{1-\varrho(\omega_t^*)}{\epsilon_b}. \tag{E.55}$$

Since $\lambda_t^1 \geq [=]\ 0$ if $y_t^{b*} = [>]\ 0$, it follows from (E.36) and (E.55) that $\tau_t^* \leq [=]\ \psi_t^*$ if $y_t^{b*} = [>]\ 0$. In view of (E.50) and (E.51), the optimal transfers are

$$a_t^{2*} = c_t^{2*} - \big[1-\delta+f_g'(\kappa_g(\omega_t^*))\big](1+n)k_t^* \quad \text{and} \quad a_t^{1*} = \tau_t^*\epsilon_b y_t^{b*} - \tfrac{a_t^{2*}}{1+n}. \tag{E.56}$$

Since, by Theorem 5.2 (i),

$$(y_t^{b*}, c_t^{1*}, c_t^{2*}) = \big(y_*^b(k_t^*, \chi_t^*), c_*^1(k_t^*, \chi_t^*), c_*^2(k_t^*, \chi_t^*)\big),$$

the fiscal policy parameters $(\tau_t^*, a_t^{2*}, a_t^{1*})$ given in (E.55) and (E.56) depend solely on the arbitrary current state $(k_t^*, \chi_t^*) \in \mathbb{R}_{++} \times \mathbb{R}_+$. Hence, there must exist *tax-policy rules* of the form $\tau_*, a_*^2 : \mathbb{R}_{++} \times \mathbb{R}_+ \to \mathbb{R}$ such that

$$\tau_t^* = \tau_*(k_t^*, \chi_t^*) \quad \text{and} \quad a_t^{2*} = a_*^2(k_t^*, \chi_t^*) \quad \text{for all} \quad t \geq 0. \tag{E.57}$$

Since the policy functions y_*^b, c_*^1, c_*^2 are continuous, the tax-policy rules τ_*, a_*^2 are continuous. Theorem 5.3 implies that, for each $(k_t^*, \chi_t^*) \in \mathbb{R}_{++} \times \mathbb{R}_+$, the fiscal policy parameters (E.57) satisfy the feasibility condition (5.51). Thus, the tax-policy rules τ_*, a_*^2 constitute a feasible fiscal policy in the sense of Definition 5.2. □

Proof of Theorem 5.4. Using the function $\bar\psi$ defined in (5.57), it follows from (E.30), (E.36), and (E.39) that the pair $(\bar k_\gamma, \bar y_\gamma^b)$ must satisfy the first-order conditions

$$1-\delta+f_g'\big(\kappa_g(\Omega(\bar k, \bar y^b))\big) \overset{!}{=} \tfrac{1+n}{\gamma} - \bar\lambda^2 f_b'(\bar k) \tag{E.58}$$

$$1-\varrho(\Omega(\bar k, \bar y^b)) \overset{!}{=} \epsilon_b \bar\psi(\bar k, \Omega(\bar k, \bar y^b)) - \bar\lambda^1 + \bar\lambda^2 \tag{E.59}$$

together with the complementary slackness conditions

$$\bar\lambda^1 \bar{y}^b \overset{!}{=} 0 \quad \text{and} \quad \bar\lambda^2\big[f_b(\bar k) - \bar{y}^b\big] \overset{!}{=} 0, \qquad \bar\lambda^1, \bar\lambda^2 \geq 0. \tag{E.60}$$

There are three possible cases.

Green Case. $\bar\lambda^1 > 0$ and $\bar\lambda^2 = 0$. Then $\bar{y}^b_\gamma = 0$ due to (E.60). Since $\Omega(\bar k, 0) = \Omega_g(\bar k)$, the first-order condition (E.58) takes the form

$$f_g'(\bar k) \overset{!}{=} \tfrac{1+n}{\gamma} - 1 + \delta. \tag{E.61}$$

Assumptions 5.1 and 5.3 imply that (E.61) admits a unique solution $0 < \bar k^g < \infty$. Setting $\bar\omega^g = \Omega_g(\bar k^g)$, then it follows from (E.59) that

$$\bar\lambda^1 \geq 0 \iff 1 - \epsilon_b \bar\psi(\bar k^g, \bar\omega^g) \leq \varrho(\bar\omega^g).$$

Therefore, the pair $(\bar k^g, \bar\omega^g)$ determines a MGRSS if $\varrho(\bar\omega^g) \geq 1 - \epsilon_b \bar\psi(\bar k^g, \bar\omega^g)$. This steady state is green because $y^b_\gamma = \mathsf{y}_b(\bar k^g, \bar\omega^g) = 0$.

Mixed Case. $\bar\lambda^1 = \bar\lambda^2 = 0$. Then the first-order condition (E.59) takes the form

$$\varrho(\Omega(\bar k, \bar{y}^b)) \overset{!}{=} 1 - \epsilon_b \bar\psi(\bar k, \Omega(\bar k, \bar{y}^b)) \tag{E.62}$$

and (E.58) the form

$$f_g'\big(\kappa_g(\Omega(\bar k, \bar{y}^b))\big) \overset{!}{=} \tfrac{1+n}{\gamma} - 1 + \delta. \tag{E.63}$$

It follows from (E.63) that

$$\Omega(\bar k, \bar{y}^b) \overset{!}{=} \bar\omega^g. \tag{E.64}$$

Inserting (E.64) into (E.62), it follows that the capital-labor ratio is determined by

$$\varrho(\bar\omega^g) \overset{!}{=} 1 - \epsilon_b \bar\psi(\bar k, \bar\omega^g). \tag{E.65}$$

By the intermediate-value theorem, a solution $\bar k_\gamma \in \big(\bar k_{\min}, \bar k_{\max}\big)$ to (E.65) exists if either the condition stated in Theorem 5.4 (iii) or the condition stated in Theorem 5.4 (iv) is satisfied. Given $\bar k_\gamma$, (E.64) implies that $\bar{y}^b_\gamma = \mathsf{y}_b(\bar k_\gamma, \bar\omega^g) \in \big(0, f_b(\bar k_\gamma)\big)$, so that the steady state is mixed.

Brown Case. $\bar\lambda^1 = 0$ and $\bar\lambda^2 > 0$. Then $\bar{y}^b_\gamma = f_b(\bar k_\gamma)$ due to (E.60). Since $\Omega(\bar k, f_b(\bar k)) = \Omega_b(\bar k)$, the first-order condition (E.58) takes the form

$$\bar\lambda^2 \overset{!}{=} \frac{\tfrac{1+n}{\gamma} - 1 + \delta - f_g'\big(\kappa_g(\Omega_b(\bar k))\big)}{f_b'(\bar k)} \tag{E.66}$$

and (E.59) the form

$$\bar\lambda^2 \overset{!}{=} 1 - \epsilon_b \bar\psi(\bar k, \Omega_b(\bar k)) - \varrho(\Omega_b(\bar k)). \tag{E.67}$$

We can read off (E.66) that

$$\bar\lambda^2 \geq 0 \iff \Omega_b(\bar k) \geq \bar\omega^g \iff \bar k \geq \kappa_b(\bar\omega^g).$$

Inserting (E.66) into (E.67), it follows that the capital-labor ratio is determined by

$$f_b'(\bar{k})\big[1-\epsilon_b\bar{\psi}(\bar{k},\Omega_b(\bar{k}))\big] \overset{!}{=} \tfrac{1+n}{\gamma}-1+\delta. \tag{E.68}$$

The existence of a solution $\bar{k}_\gamma$ to (E.68) follows from the intermediate-value theorem. Setting $\bar{k}^b = f_b'^{-1}(\frac{1+n}{\gamma}-1+\delta)$, we have

$$f_b'(\bar{k}^b)\big[1-\epsilon_b\bar{\psi}(\bar{k}^b,\Omega_b(\bar{k}^b))\big] < \tfrac{1+n}{\gamma}-1+\delta.$$

Since $f_j'' < 0$ and $\kappa_j' > 0$, $j = g, b$, we also have

$$\varrho(\bar{\omega}^g) \;=\; \frac{\frac{1+n}{\gamma}-1+\delta}{f_b'(\kappa_b(\bar{\omega}^g))} = \frac{f_b'\big(\kappa_b(\Omega_b(\bar{k}^b))\big)}{f_b'\big(\kappa_b(\bar{\omega}^g)\big)} < 1 \iff \bar{\omega}^g < \Omega_b(\bar{k}^b) \iff \kappa_b(\bar{\omega}^g) < \bar{k}^b.$$

On the other hand,

$$f_b'(\kappa_b(\bar{\omega}^g))\big[1-\epsilon_b\bar{\psi}(\kappa_b(\bar{\omega}^g),\bar{\omega}^g)\big] \ge \tfrac{1+n}{\gamma}-1+\delta$$

if and only if

$$1-\epsilon_b\bar{\psi}(\kappa_b(\bar{\omega}^g),\omega^g) \ge \varrho(\bar{\omega}^g). \tag{E.69}$$

Therefore, if (E.69) holds, then there exists a solution $\bar{k}_\gamma \in \big[\kappa_b(\bar{\omega}^g),\bar{k}^b\big)$ to (E.68) such that the pair $(\bar{k}_\gamma,\Omega_b(\bar{k}_\gamma))$ determines a MGRSS. Since $\bar{y}^b_\gamma = \mathsf{y}_b(\bar{k}_\gamma,\Omega_b(\bar{k}_\gamma)) = f_b(\bar{k}_\gamma)$, this steady state is brown. ☐

Proof of Proposition 5.2. In a sufficiently small neighborhood of a mixed MGRSS, we have $0 < y_t^{b*} < f_b(k_t^*)$, so that $\lambda_t^1 = 0$ and $\lambda_t^2 = 0$ due to (E.28) and (E.29). If the marginal damage is a constant $d > 0$, then (E.35) takes the form

$$u'(c_t^{1*}) = \frac{\epsilon_b(1+\frac{\beta}{\gamma})}{1-\frac{\gamma(1-\zeta)}{1+n}}\,\frac{d}{\frac{\partial f}{\partial y^b}(k_t^*,y_t^{b*})}. \tag{E.70}$$

Moreover, (E.38) and (E.30) imply that

$$\frac{u'(c_t^{1*})}{u'(c_{t+1}^{1*})} = \frac{\gamma}{1+n}\frac{\partial f}{\partial k}(k_{t+1}^*,y_{t+1}^{b*}). \tag{E.71}$$

Substituting (E.70) into (E.71) yields

$$\frac{\frac{\partial f}{\partial y^b}(k_{t+1}^*,y_{t+1}^{b*})}{\frac{\partial f}{\partial y^b}(k_t^*,y_t^{b*})} = \frac{\gamma}{1+n}\frac{\partial f}{\partial k}(k_{t+1}^*,y_{t+1}^{b*}). \tag{E.72}$$

Using Lemma 5.3, the derivatives of f in (E.72) take the form

$$\frac{1-\varrho(\omega_{t+1}^*)}{1-\varrho(\omega_t^*)} = \frac{\gamma}{1+n}\Big[1-\delta+f_g'(\kappa_g(\omega_{t+1}^*))\Big]. \tag{E.73}$$

This establishes the difference equation (5.61). The stability condition obtains as follows. Setting $\omega^*_{t+1} = \Gamma(\omega^*_t)$, then implicit differentiation of (E.73) yields

$$\Gamma'(\omega^*_t) = \frac{\frac{\gamma}{1+n}\big[1-\delta+f_g'(\kappa_g(\omega^*_{t+1}))\big]\varrho'(\omega^*_t)}{\varrho'(\Gamma(\omega^*_t)) + \frac{\gamma}{1+n}f_g''\big(\kappa_g(\Gamma(\omega^*_t))\big)\kappa_g'(\Gamma(\omega^*_t))\big[1-\varrho(\omega^*_t)\big]}. \tag{E.74}$$

The unique steady state of (E.73) is $\bar\omega^g$. Hence, $\bar\omega^g = \Gamma(\bar\omega^g)$ and, in a steady state, the derivative (E.74) reads[2]

$$\Gamma'(\bar\omega^g) = \frac{1}{1+\frac{\gamma}{1+n}f_g''(\kappa_g(\bar\omega^g))\kappa_g'(\bar\omega^g)\frac{1-\varrho(\bar\omega^g)}{\varrho'(\bar\omega^g)}}. \tag{E.75}$$

Since

$$\kappa_j'(\bar\omega^g) = \frac{f_j'(\kappa_j(\bar\omega^g))^2}{-f_j''(\kappa_j(\bar\omega^g))f_j(\kappa_j(\bar\omega^g))}, \quad j = g, b,$$

the derivative (E.75) may be rewritten as

$$\Gamma'(\bar\omega^g) = \frac{1}{1+\frac{\gamma}{1+n}\frac{[1-\varrho(\bar\omega^g)]f_b(\kappa_b(\bar\omega^g))}{\kappa_b(\bar\omega^g)-\kappa_g(\bar\omega^g)}}. \tag{E.76}$$

The existence of a mixed MGRSS implies that $\varrho(\bar\omega^g) < 1$. It now follows from (E.76) that $\Gamma'(\bar\omega^g) \in (-1, 1)$ if and only if (5.62) holds, in which case the steady state is asymptotically stable. In particular, (E.76) implies that $\Gamma'(\bar\omega^g) \in (0, 1)$ whenever $\kappa_b(\bar\omega^g) > \kappa_g(\bar\omega^g)$. ☐

Proof of Lemma 5.4. *Step 1*. Since $f_g'' < 0$, (5.56) has a unique solution, implying that there exists at most one green MGRSS. Inserting (5.63), (5.58) takes the form

$$f_b'(k)\left[1-\epsilon_b\frac{d}{1-\frac{\gamma(1-\zeta)}{1+n}}\big(f_b(k)-(n+\delta)k\big)\right] = \tfrac{1+n}{\gamma} - 1 + \delta. \tag{E.77}$$

Since $f_b'' < 0$, the l.h.s. in (E.77) is strictly decreasing if $k \mapsto f_b(k) - (n+\delta)k$ is strictly increasing, that is, whenever $0 < k < f_b'^{-1}(n+\delta)$. Since $0 < \bar k^b < f_b'^{-1}(n+\delta)$ by (5.56), it follows that any solution $\bar k_\gamma \in (0, \bar k^b)$ to (E.77) is unique, if it exists. Hence, there exists at most one brown MGRSS. Using Lemma 5.3, differentiation of ϕ yields

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}k}\Big[\phi(k, \mathsf{y}_b(k, \bar\omega^g))\Big] &= 1-\delta+f_g'(\kappa_g(\bar\omega^g)) - (1+n) + \frac{\big[1-\varrho(\bar\omega^g)\big]f_b(\kappa_b(\bar\omega^g)}{\kappa_b(\bar\omega^g)-\kappa_g(\bar\omega^g)} \\ &= (1+n)\tfrac{1-\gamma}{\gamma} + \frac{\big[1-\varrho(\bar\omega^g)\big]f_b(\kappa_b(\bar\omega^g))}{\kappa_b(\bar\omega^g)-\kappa_g(\bar\omega^g)}\end{aligned} \tag{E.78}$$

for all $k \in \big[\bar k_{\min}, \bar k_{\max}\big]$. It follows from (E.78) that the map $k \mapsto \bar\psi(k, \bar\omega^g)$ in (5.63) is either monotonically increasing or monotonically decreasing on $\big[\bar k_{\min}, \bar k_{\max}\big]$. This implies that any solution $\bar k_\gamma \in \big(\bar k_{\min}, \bar k_{\max}\big)$ to (5.59) is unique, if it exists. Hence, there exists at most one mixed MGRSS.

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[2] The assumption that $\kappa_g(\bar\omega^g) \neq \kappa_b(\bar\omega^g)$ implies $\varrho'(\bar\omega^g) \neq 0$.

Step 2. There are two cases. First, $\varrho(\bar{\omega}^g) \geq 1$. In this case, there exists a unique green MGRSS by Theorem 5.4. Second, $\varrho(\bar{\omega}^g) < 1$. As shown in Step 1, there exists at most one green, one brown, and one mixed MGRSS. If $\kappa_b(\bar{\omega}^g) > \kappa_g(\bar{\omega}^g)$, then (E.78) implies that the map $k \mapsto \bar{\psi}(k, \bar{\omega}^g)$ in (5.63) is strictly increasing on $\left[\bar{k}_{\min}, \bar{k}_{\max}\right] = \left[\kappa_g(\bar{\omega}^g), \kappa_b(\bar{\omega}^g)\right]$, which rules out Case (iv) in Theorem 5.4. As a consequence, there exists a uniquely determined MGRSS, which is either green, brown, or mixed. $\square$

Authorship Contribution Statement

The following document outlines the authors' contributions to the respective articles.

Article 1: *"Financial Intermediation and Business Cycles in Two-Period Lived OLG Models"*

Paul Ritschel has: developed the research question; reviewed the related literature; formulated the decision problems of the agent, the bank, and the myopic social planner; developed the concept of an efficient contract; analyzed the qualitative dynamics of the model and established the monotonicity; uncovered the relationship between inefficient contracts and business cycles; created the proofs of the results; written the manuscript; visualized the results and created the figures; developed the numerical examples; reviewed the manuscript.

Jan Wenzelburger has: applied the dynamical systems approach to the topic; analyzed the qualitative dynamics of the model and established the monotonicity; created the proofs of the results; written the manuscript; reviewed the manuscript.

Article 2: "Banking Competition and Capital Dependence of the Production Sector: Growth and Welfare Implications"

Paul Ritschel has: developed the research question; reviewed the related literature; applied the overlapping-generations and dynamical systems approach to the topic; formulated the decision problems of the agent and the monopolistic bank; analyzed the qualitative dynamics of the model; characterized the steady states; created the proofs of the results; written the manuscript; developed the numerical examples; reviewed the manuscript; revised the manuscript after publication.

Tom Rauber has: reviewed the related literature; developed the contract theoretical foundation; formulated the decision problems of the agent and the monopolistic bank; analyzed the qualitative dynamics of the model; characterized the steady states; created the proofs of the results; written the manuscript; visualized the results and created the figures; developed the numerical examples; reviewed the manuscript.

Article 4: "Production-Possibility Frontiers, Rybczynski's Theorem, and Factor-Intensity Reversals"

Paul Ritschel has: developed the research question; reviewed the related literature; proven the concavity of the production-possibility frontier; developed a generalized Rybczynski theorem that incorporates factor-intensity reversals; extended the results to production functions without constant returns to scale; created the proofs of the results; written the manuscript; visualized the results and created the figures; reviewed the manuscript.

Jan Wenzelburger has: developed the research question; reviewed the related literature; developed the parameterization of efficient factor allocations; developed a generalized Rybczynski theorem that incorporates factor-intensity reversals; created the proofs of the results; written the manuscript; developed the examples with CES production functions; reviewed the manuscript.

Article 5: "*Green Transformation and Fiscal Policy in a Two-Sector OLG Model with Production Externalities*"

Paul Ritschel has: developed the research question; reviewed the related literature; developed the model including the preferences and the production externality; characterized the temporary equilibria; formulated the social planner's problem; applied dynamic programming methods to the topic; characterized the optimal allocation; characterized the modified golden-rule steady states; established the implementation of the optimal allocation through optimal fiscal policy; created the proofs of the results; written the manuscript; visualized the results and created the figures; developed the numerical examples; reviewed the manuscript.

Jan Wenzelburger has: developed the research question; adapted the two-sector framework to the topic; developed the parameterization of efficient factor allocations; applied dynamic programming methods to the topic; established the properties of the value function; characterized the modified golden-rule steady states; created the proofs of the results; written the manuscript; reviewed the manuscript.

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Nairobi, Kenya: Meeting of the Econometric Society

PEER-REVIEWED PUBLICATIONS

- Rauber, T., & Ritschel, P. (2024). Banking Competition and Capital Dependence of the Production Sector: Growth and Welfare Implications, *International Review of Economics and Finance, 89*(Part B), 676-698.
- Ritschel, P., & Wenzelburger, J. (2024). Financial Intermediation and Efficient Risk Sharing in Two-Period Lived OLG Models, *Economic Theory Bulletin, 12*(1), 57-78.
- Ritschel, P. (2023). Capital Market Power and Economic Growth in an Overlapping-Generations Model with Rational Expectations, *Theoretical Economics Letters, 13*(5), 1253-1265.