Algorithmic Gauß-Manin Connection

Algorithms to Compute Hodge-theoretic Invariants of Isolated Hypersurface Singularities

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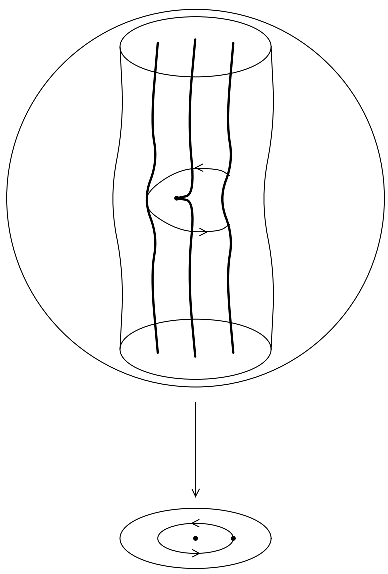

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Introduction

Human thinking is based on abstraction, the concept of objects and equality. Equal objects with respect to a notion of equality form a class. Such a class of objects is considered an object itself.

In mathematics, one can pass over from a given notion of equality to a coarser one by an equivalence relation on a class of objects. All objects equivalent to a given object are considered to be equal and form an (equivalence) class. This leads to a classification problem, that is the problem of describing all classes. One possible solution consists in associating to each class a normal form being an object in this class. The concept of invariants serves to approach classification problems. An invariant associates to each object an object of a possibly different type such that only one object is associated to all objects in a class. So it associates an object to each class. Of course, there is always the trivial invariant that associates to each object its class. But this does not help to solve a classification problem. A set of invariants forms an invariant and an invariant solves the classification problem if the associated object determines the class. This thesis is concerned with algorithms to compute certain invariants.

The objects in this thesis are a special class of singularities. Singularities occur in all fields of mathematics and the various mathematical definitions reflect the intuitive idea of a non-smooth point of a geometrical object. The singularities in this thesis are a special class of complex space germs. A complex space is locally the zero set

$$\mathrm{V}(\underline{f}) = \{\underline{x} \in \mathbb{C}^n | \underline{f}(\underline{x}) = \underline{0}\} \subset \mathbb{C}^n$$

of complex analytic functions $\underline{f} = f_1, \ldots, f_m$ and a hypersurface $\mathrm{V}(f)$ is a complex space defined by a single equation. It is equipped with a sheaf of local rings carrying the algebraic information of the defining equations. To consider complex spaces means to consider solution spaces of complex analytic equations as geometrical objects. A map of complex spaces is locally defined by complex analytic functions. The concept of germs allows one to consider complex spaces locally at a point. A germ of a complex space X

at a point $x \in X$ is the equivalence class (X, x) of neighbourhoods of x and a hypersurface germ is the germ (X, x) of a hypersurface X. Any complex space germ is the germ

$$\mathrm{V}(\underline{f}) = \big(\mathrm{V}(\underline{f}), \underline{0}\big) \subset (\mathbb{C}^n, 0)$$

of the zero-set of power series $\underline{f} = f_1, \ldots, f_m \in \mathbb{C}\{\underline{x}\}$ defined in a neighbourhood of $0 \in \mathbb{C}^n$ and any hypersurface germ is the germ

$$\mathrm{V}(f) = \big(\mathrm{V}(f), \underline{0}\big) \subset (\mathbb{C}^{n+1}, 0)$$

of the zero-set of a single power series $f \in \mathbb{C}\{\underline{x}\}$ defined in neighbourhood of $0 \in \mathbb{C}^{n+1}$. A map of complex space germs is the equivalence class of restrictions of a map of complex spaces. A singularity is a germ (X, x) of a complex space X at a non-smooth point $x \in X$. It is called a hypersurface singularity if X is a hypersurface and it is called isolated if there are only smooth points close to x. The objects in this thesis are isolated hypersurface singularities.

In chapter 1, we describe invariants of isolated hypersurface singularities. Let $\mathrm{V}(f) \subset (\mathbb{C}^{n+1}, 0)$ be an isolated hypersurface singularity. It is defined by a power series $f \in \mathbb{C}\{\underline{x}\}$ and one can consider f and its partial derivatives $\underline{\partial}(f) = \partial_{x_0}(f), \ldots, \partial_{x_n}(f)$ as maps

$$(\mathbb{C}^{n+1}, \underline{0}) \xrightarrow{f} (\mathbb{C}, 0)\ ,$$

$$(\mathbb{C}^{n+1}, \underline{0}) \xrightarrow{\underline{\partial}(f)} (\mathbb{C}^{n+1}, \underline{0})$$

of complex space germs. The fact that $\mathrm{V}(f) \subset (\mathbb{C}^{n+1}, 0)$ is an isolated singularity is equivalent to the fact that $0 \in \mathbb{C}^{n+1}$ is an isolated critical point of f, that is, $\mathrm{V}(\underline{\partial}(f)) = \{\underline{0}\}$. This implies that the Milnor number

$$\mu = \dim_{\mathbb{C}}\big(\mathbb{C}\{\underline{x}\}/\langle\underline{\partial}(f)\rangle\big)$$

is a finite number. Let X be the intersection of a closed ball in $\mathbb{C}^{n+1}$ centered at $\underline{0} \in \mathbb{C}^{n+1}$ with the preimage under f of an open disk T in $\mathbb{C}$ centered at $0 \in \mathbb{C}$. For appropriately chosen X and T, the restriction

$$X' = X \backslash f^{-1}(0) \xrightarrow{f} T \backslash \{0\} = T'$$

is a $\mathscr{C}^\infty$ fibre bundle, that is, locally at $t \in T'$, the fibres are $\mathscr{C}^\infty$-diffeomorphic to a product space of a neighbourhood of t and the smooth fibre

$$X_t = X \cap f^{-1}(t),$$

and the restriction

$$\delta X \xrightarrow{f} T$$

to the boundary δX of X is a trivial fibre bundle, that is, δX is $\mathscr{C}^\infty$-diffeomorphic to a product space of T and δX_t. The fibre bundle $f : X' \longrightarrow T$ is called a Milnor fibration. Different choices of X and T lead to diffeomorphic Milnor fibrations. The general fibre of the Milnor fibration is called the Milnor fibre. A parallel shift through the local product structures of the Milnor fibration over a counterclockwise loop around $0 \in T$ defines a diffeomorphism of the Milnor fibre which is trivial on the boundary. The relative isotopy class of this diffeomorphism is a topological invariant of the singularity and is called the geometrical monodromy. By J. Milnor [Mil68], the Milnor fibre is homotopy equivalent to a bouquet of μ n-spheres, that is, the space formed by μ spheres of dimension n glued at one point. This implies that the reduced (co)homology of the Milnor fibre is concentrated in dimension n and is a free Abelian group with μ generators, that is,

$$\widetilde{\mathrm{H}}_k(X_t) \cong \delta_{k,n}\mathbb{Z}^\mu \cong \widetilde{\mathrm{H}}^k(X_t)$$

where δ is the Kronecker symbol. The ((co)homological) monodromy M is the automorphism defined by the geometrical monodromy on the (co)homology of the Milnor fibre. By the monodromy theorem, the eigenvalues of M are roots on unity and the Jordan blocks have size at most $(n+1) \times (n+1)$ and at most $n \times n$ for eigenvalue 1. Let $\mathrm{M} = \mathrm{M}_s\mathrm{M}_u$ be the decomposition of M into semisimple and unipotent part and

$$\mathrm{N} = -\frac{\log \mathrm{M}_u}{2\pi\mathrm{i}}$$

the logarithm of the unipotent part. Then, by the monodromy theorem,

$$\mathrm{N}^{n+1} = 0$$

and even $\mathrm{N}^n = 0$ on the generalized 1-eigenspace. The (co)homology of the smooth fibres X_t, $t \in T'$, form the (co)homology bundle

$$\mathscr{H} = \bigcup_{t \in T'} \mathrm{H}(X_t, \mathbb{C}).$$

It is a flat vector bundle, that is, locally at $t \in T'$, the fibres form a product space of a neighbourhood of t and the complex vectorspace $\mathrm{H}(X_t, \mathbb{C})$ such that on the intersection of such neighbourhoods the isomorphism of product spaces is independent of t. A (local) section of $\mathscr{H}$ is a (local) holomorphic section of the canonical projection

$$\mathscr{H} \longrightarrow T'\,.$$

By the flatness of the cohomology bundle, differentiation of coefficient functions in the local product structures defines a derivative

$$\mathscr{H} \xrightarrow{\partial_t} \mathscr{H}$$

on $\mathscr{H}$, that is, it fulfills the Leibniz rule

$$\partial_t(gv) = \partial_t(g)v + g\partial_t(v)$$

for (local) holomorphic functions g and (local) sections v of $\mathscr{H}$. In terms of a local basis of $\mathscr{H}$, $\partial_t = 0$ is a system of μ ordinary differential equations. The flat multivalued sections of $\mathscr{H}$ are the solutions of $\partial_t = 0$ and form a μ dimensional complex vectorspace. The monodromy is defined by shifting (co)homology classes along flat sections of $\mathscr{H}$ over a counterclockwise loop around in $0 \in T$. Let $i : T' \longrightarrow T$ by the canonical inclusion. Then a flat multivalued section of $\mathscr{H}$ in the generalized $\exp(-2\pi\mathrm{i}\alpha)$-eigenspace $\mathrm{H}^n(X_t, \mathbb{C})_{\exp(-2\pi\mathrm{i}\alpha)}$ of the monodromy corresponds to a local section of $\mathscr{H}$ at $0 \in T$ in the generalized α-eigenspace $C^\alpha = \ker(t\partial_t - \alpha)^{n+1} \subset (i_*\mathscr{H})_0$ of the operator $t\partial_t$ by an isomorphism

$$\mathrm{H}^n(X_t, \mathbb{C})_{\exp(-2\pi\mathrm{i}\alpha)} \xrightarrow[\sim]{\psi_\alpha} C^\alpha \tag{0.1}$$

of complex vectorspaces. The monodromy corresponds to the operator $t\partial_t$ by $(t\partial_t - \alpha) \circ \psi_\alpha = \psi_\alpha \circ \mathrm{N}$ and hence

$$\exp(-2\pi\mathrm{i}t\partial_t) \circ \psi_\alpha = \psi_\alpha \circ \mathrm{M}. \tag{0.2}$$

An elementary section is a section of $\mathscr{H}$ in a generalized eigenspace of $t\partial_t$. The (local) Gauß-Manin connection

$$G = \sum_{\alpha \in \mathbb{Q}} \mathbb{C}\{t\}[t^{-1}]C^\alpha$$

is the μ-dimensional $\mathbb{C}\{t\}[t^{-1}]$-subvectorspace of $(i_*\mathscr{H})_0$ generated by local elementary sections at $0 \in T$. It is a regular $\mathbb{C}\{t\}[\partial_t]$-module.

By the De Rham theorem, the cohomology bundle $\mathscr{H}$ can be described in terms of (relative) holomorphic differential forms. Following this idea, E. Brieskorn [Bri70] defined the Brieskorn lattice

$$H'' = \Omega^{n+1}_{X,0}/\mathrm{d}f \wedge \mathrm{d}\Omega^{n-1}_{X,0}.$$

It is the stalk of a locally free extension of the cohomology bundle $\mathscr{H}$ at $0 \in T$ and a $\mathbb{C}\{t\}$-lattice in G, that is a free $\mathbb{C}\{t\}$-submodule of G of rank μ. The Leray residue formula implies that

$$\partial_t[\mathrm{d}f \wedge \eta] = [\mathrm{d}\eta] \tag{0.3}$$

for $[\mathrm{d}f \wedge \eta] \in H''$.

E. Brieskorn [Bri70] found the first algorithm to compute compute the complex monodromy. It has been implemented in the computer algebra system Maple V by P.F.M. Nacken [Nac90] and in the computer algebra system Singular [GPS02] by the author [Sch99, Sch02b]. Brieskorn's algorithm and the algorithms in this thesis are based on (0.2) and (0.3).

The Gauß-Manin connection G has a rich structure. The generalized eigenspaces C^α of the operator $t\partial_t$ define a splitting of a decreasing filtration V of G by $\mathbb{C}\{t\}$-lattices

$$V^\alpha = \sum_{\alpha \leq \beta} \mathbb{C}\{t\} C^\beta.$$

This filtration is called the V-filtration and $tV^\alpha = V^{\alpha+1}$. For $\alpha > -1$, V^α is also a $\mathbb{C}\{\{s\}\}$-lattice, that is a free module of rank μ over a power series ring $\mathbb{C}\{\{s\}\}$ with $s = \partial_t^{-1}$ and $sV^\alpha = V^{\alpha+1}$. Since $[\partial_t, t] = 1$, t is a differential operator

$$t = s^2 \partial_s$$

on the $\mathbb{C}\{\{s\}\}$-module $V^{>-1}$. The regularity of the Gauß-Manin connection is reflected by the existence of the saturated, that is $t\partial_t$-invariant, $\mathbb{C}\{t\}$-lattices V^α. By a result of B. Malgrange [Mal74],

$$H'' \subset V^{>-1}$$

and, in particular, H'' is a $\mathbb{C}\{\{s\}\}$-lattice. In the sense of D. Barlet [Bar93, Bar00], the Brieskorn lattice H'' is a (t,s)-module, that is a free $\mathbb{C}\{\{s\}\}$-module with an operator t such that the commutator satisfies

$$[t, s] = s^2.$$

In contrast to Brieskorn's algorithm which is based on the $\mathbb{C}\{t\}$-structure, the algorithms in this thesis are based on the $\mathbb{C}\{\{s\}\}$-structure.

The nilpotent operator N commutes with t and s and defines a t- and s-invariant weight filtration W over $\mathbb{Q}$. Multiplying the Brieskorn lattice H'' by powers of t resp. s defines a filtration F by $\mathbb{C}\{t\}$-lattices

$$F_k = t^{-k} H''$$

on G resp. a filtration $\widetilde{F}$ by $\mathbb{C}\{\{s\}\}$-lattices

$$\widetilde{F}_k = s^{-k}H'' \cap V^{>-1}$$

on $V^{>-1}$. These filtrations are called Hodge filtrations. The Hodge filtration F was defined by A.N. Varchenko [Var82a] and the Hodge filtration $\widetilde{F}$ by J. Scherk and J.H.M. Steenbrink [SS85]. The weight resp. Hodge filtrations induce filtrations on the cohomology $\mathrm{H}^n(X_t, \mathbb{Q})$ resp. $\mathrm{H}^n(X_t, \mathbb{C})$ of the Milnor fibre. By J.H.M. Steenbrink [Ste76, SS85] and A.N. Varchenko [Var82a], the weight and Hodge filtrations define a mixed Hodge structure on the cohomology of the Milnor fibre, that is, on the graded parts of the weight filtration the induced Hodge filtrations are opposite to their complex conjugate shifted by the weight. Moreover, $\log \mathrm{M}_u$ is a morphism of mixed Hodge structures of type $(-1, -1)$, that is a morphism of type -2 resp. -1 with respect to the weight resp. Hodge filtrations. The Hodge numbers $h^{p,q}$ are the dimensions of the graded parts of the induced Hodge filtrations on the graded parts of the weight filtration, that is,

$$\dim_{\mathbb{C}} \mathrm{gr}_F^p \mathrm{gr}_{p+q}^W \mathrm{H}^n(X_t, \mathbb{C}) = h^{p,q} = \dim_{\mathbb{C}} \mathrm{gr}_{\widetilde{F}}^p \mathrm{gr}_{p+q}^W \mathrm{H}^n(X_t, \mathbb{C})$$

Properties of the mixed Hodge structure lead to certain symmetries of the Hodge numbers.

The spectral pairs reflect the embedding of the Brieskorn lattice H'' in the Gauß-Manin connection G with respect to the V- and weight filtration. They consist of μ pairs $(\alpha, l) \in \mathbb{Q} \times \mathbb{Z}$ with multiplicity

$$\dim_{\mathbb{C}} \mathrm{gr}_V^\alpha \mathrm{gr}_l^W (H''/tH'') = \dim_{\mathbb{C}} \mathrm{gr}_V^\alpha \mathrm{gr}_l^W (H''/sH'').$$

By the isomorphisms (0.1), they correspond to the Hodge numbers inheriting their symmetries and, by (0.2), they determine the Jordan data of the complex monodromy. The first components of the spectral pairs are the spectral numbers and form the (singularity) spectrum. They consist of μ rational numbers $\alpha \in \mathbb{Q}$ with multiplicity

$$\dim_{\mathbb{C}} \mathrm{gr}_V^\alpha (H''/tH'') = \dim_{\mathbb{C}} \mathrm{gr}_V^\alpha (H''/sH'')$$

and they determine the eigenvalues of the complex monodromy.

By M. Saito [Sai88], for Newton non-degenerate singularities, the V-filtration coincides with the Newton filtration which is defined by the Newton polyhedron of the power series $f \in \mathbb{C}\{\underline{x}\}$. Based on this result, S. Endrass [End02] implemented an algorithm to compute the singularity spectrum in SINGULAR.

By P. Deligne [Del72], there is a simultaneous splitting of the weight and Hodge filtration of a mixed Hodge structure. In particular, N is strict with respect to the weight and Hodge filtrations. By M. Saito [Sai89], this implies the existence of a $\mathbb{C}\{\{s\}\}$-basis of the Brieskorn lattice H'' such that the basis representation of t is

$$t \cong A_0 + sA_1 + s^2\partial_s.$$

The matrix $A = A_0 + sA_1$ of t determines the (t,s)-module structure of the Brieskorn lattice and, in particular, the spectral pairs and the complex monodromy.

In chapter 2, we describe algorithms to compute, for arbitrary isolated hypersurface singularities, the complex monodromy, the spectral pairs, and the (t,s)-module structure of the Brieskorn lattice in form of M. Saito's matrices A_0 and A_1. These algorithms are based on the following results and ideas.

In section 1.5, we show that the $\langle \underline{x} \rangle$-adic and $\langle s \rangle$-adic topologies on the Brieskorn lattice coincide and that there is a $\mathbb{C}[\![s]\!]$-isomorphism

$$\widehat{H}'' \cong \mathbb{C}[\![s,\underline{x}]\!]/\langle \underline{\partial}(f) - s\underline{\partial}\rangle \mathbb{C}[\![s,\underline{x}]\!] \tag{0.4}$$

where $\widehat{H}''$ is the completion of H'' with respect to this topology.

In section 1.10, we introduce standard bases with respect to splittings of refinements of the V-filtration. We show that the spectral pairs are the orders of a standard basis of the Brieskorn lattice with respect to a splitting of the weight refined V-filtration. We show that M. Saito's basis is a reduced standard basis with respect to a Hodge splitting of the V-filtration.

In section 2.1, we develop a theory of standard bases over formal power series rings based on the idea of monomial orderings [Buc65, Buc85, GP96]. We describe a normal form and standard basis algorithm based on Buchberger's algorithm [GP96] that converges with respect to the adic topology of the power series ring.

In section 2.2, we specialize this normal form algorithm to a normal form algorithm for the Brieskorn lattice using (0.4). In section 2.3, we show that it computes the matrix $A = \sum_{j\geq 0} s^j A_j$ of operator t with respect to a $\mathbb{C}\{\{s\}\}$-basis of the Brieskorn lattice such that

$$t \cong A + s^2\partial_s$$

is the basis representation of the operator t. Here, we use the finite determinacy theorem to assume that f is a polynomial. This assumption can be replaced by appropriate degree bounds. In the following sections 2.4, 2.5, 2.6, and 2.7, we describe a sequence of $\mathbb{C}\{\{s\}\}[s^{-1}]$-basis transformations and show that one compute the matrix A of t and a polynomial basis representation H of H'' with respect to the transformed bases. In section 2.4, the transformed basis is a $\mathbb{C}\{\{s\}\}$-basis of a saturated $\mathbb{C}\{\{s\}\}$-lattice and one can compute the eigenvalues of monodromy from A_1 by (0.2). Here, we use the regularity of the Gauß-Manin connection by computing the saturation of a lattice and the monodromy theorem by computing the eigenvalues of A_1 on the saturation. In section 2.4, the transformed basis is a $\mathbb{C}\{\{s\}\}$-basis of a direct sum of $\mathbb{C}\{\{s\}\}C^\alpha$ and one can compute the complex monodromy from A_1 by (0.2). In section 2.5, the transformed basis is a $\mathbb{C}\{\{s\}\}$-basis of a V^α and one can compute the spectral pairs from a standard basis of H by section 1.10. In section 2.5, the transformed basis is a $\mathbb{C}$-basis of a direct sum $\bigoplus_{\alpha\leq\beta<\alpha} C^\alpha$ and one can compute M. Saito's A_0 and A_1 from a finite jet of A and a finite jet of a reduced standard basis of H by section 1.10.

The algorithms in this thesis compute the complex monodromy, the singularity spectrum, the spectral pairs and M. Saito's matrices A_0 and A_1. Moreover, they compute a complex basis representation of the weight and Hodge filtrations on the cohomology of the Milnor fibre. But an algorithm to compute the underlying real structure and hence the mixed Hodge structure on the cohomology of the Milnor fibre is not yet known.

Appendix A contains the documentation of a Singular implementation by the author [Sch02a] of the algorithms in chapter 2. In chapter 3, we give examples and applications. We demonstrate the Singular implementation for a singularity with a 2×2 resp. 3×3 Jordan block of the monodromy. As an application, we verify Hertling's conjecture about the variance of the spectral numbers for singularities with Milnor number at most 16 using Arnold's classification [AGZV85].

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Chapter 1

Invariants of isolated hypersurface singularities

In this chapter, we describe the invariants of isolated hypersurface singularities which can be computed by the algorithms in chapter 2.

The local elementary sections of the cohomology bundle at the critical value form a regular $\mathbb{C}\{t\}[\partial_t]$-module, the Gauß-Manin connection. The complex monodromy can be identified with the operator $t\partial_t$. The description of complex cohomology in terms of differential forms by the De Rham isomorphism leads to an embedding of the Brieskorn lattice in the Gauß-Manin connection. The formal Brieskorn lattice is the completion of the Brieskorn lattice. We give an explicit description of the formal Brieskorn lattice leading to a normal form algorithm. The Gauß-Manin connection with the embedded Brieskorn lattice has a rich structure in form of the V-, weight, and Hodge filtration and a $\mathbb{C}\{t\}$- and $\mathbb{C}\{\{s\}\}$-module structure. The weight and Hodge filtration define a mixed Hodge structure on the cohomology of the Milnor fibre. This leads to symmetries of the filtrations and a close relation of the module structures in form of a certain simultaneous splitting of the V- and Hodge filtration. The relation of the filtrations is reflected by the spectral pairs corresponding to the Hodge numbers and the relation of the two module structures by M. Saito's matrices A_0 and A_1. These invariants can be computed by the algorithms in chapter 2. Standard bases are well known in computer algebra and occur naturally in the context of filtrations and splittings. We introduce standard bases with respect to the V-filtration and describe the invariants in terms of standard bases.

1.1 Milnor fibration

In this section, we introduce the Milnor fibration associated to an isolated hypersurface singularity. It is a fibre bundle formed by the smooth fibres near the singular fibre. Its general fibre is homotopy equivalent to a bouquet of spheres of dimension equal to the dimension of the singularity. The number of spheres is the Milnor number of the singularity. The construction and results in this section can be found in [Mil68] and [Loo84].

We denote row vectors by a lower bar and column vectors by an upper bar. Lower indices are row indices and upper indices are column indices. Let $\mathrm{V}(f) \subset (\mathbb{C}^{n+1}, 0)$ be an isolated hypersurface singularity. Then

$$(\mathbb{C}^{n+1}, \underline{0}) \xrightarrow{f} (\mathbb{C}, 0)$$

is the germ of a holomorphic function with isolated critical point $\underline{0} \in \mathbb{C}^{n+1}$. Let $\underline{x} = (x_0, \ldots, x_n)$ be a local coordinate system at $\underline{0} \in \mathbb{C}^{n+1}$ and $\underline{\partial} := (\partial_0, \ldots, \partial_n)$ where $\partial_j := \partial_{x_j}$. Let t be a local coordinate at $0 \in \mathbb{C}$. Then the fact that f has an isolated critical point is equivalent to

$$\mathrm{V}(\underline{\partial}(f)) = \{\underline{0}\} = \mathrm{V}(\underline{x})$$

and hence to $\sqrt{\langle \underline{\partial}(f) \rangle} = \langle \underline{x} \rangle$ by the analytic Nullstellensatz [dJP00, Thm. 2.4.4]. This implies that there is a $k \geq 1$ such that $\langle \underline{x} \rangle^k \subset \langle \underline{\partial}(f) \rangle$.

Definition 1.1.1. The dimension

$$1 \leq \mu := \dim_{\mathbb{C}} \big(\mathbb{C}\{\underline{x}\} / \langle \underline{\partial}(f) \rangle \big) < \infty$$

of the **Jacobian algebra** $\mathbb{C}\{\underline{x}\}/\langle \underline{\partial}(f) \rangle$ is called the **Milnor number**.

An isolated hypersurface singularity is determined by a finite jet of the defining power series.

Theorem 1.1.2 (Finite determinacy theorem). *Let $f \in \langle \underline{x} \rangle \subset \mathbb{C}\{\underline{x}\}$ with $\langle \underline{x} \rangle^{k+1} \subset \langle \underline{x} \rangle^2 \langle \underline{\partial}(f) \rangle$. Then f is k-**determined**, that is, for any $g \in \mathbb{C}\{\underline{x}\}$ with $f - g \in \langle \underline{x} \rangle^{k+1}$, there is an automorphism $\phi \in \operatorname{Aut} \mathbb{C}\{\underline{x}\}$ such that $\phi(f) = g$. In particular, if $k \geq \mu + 1$ then f is k-determined.*

Proof: [dJP00, Thm. 9.1.4] □

Let D^{2n+2} be the open unit ball and S^{2n+1} the unit sphere in $\mathbb{C}^{n+1}$. By the curve selection lemma [Mil68, Lem. 3.1], we can choose $\delta > 0$ such that f is defined on δD^{2n+2} with only critical point 0 and the singular fibre $f^{-1}(0)$ intersects $\delta' S^{2n+1}$, $\delta' < \delta$, transversely. Since S^{2n+1} is compact, we can choose $\epsilon > 0$ such that the fibre $f^{-1}(t)$, $t \in \epsilon D^2$, intersects δS^{2n+1} transversely. Finally, we set

$$T := \epsilon D^2, \qquad X := \delta D^{2n+2} \cap f^{-1}(T),$$
$$T' := T \backslash \{0\}, \quad X' := X \backslash f^{-1}(0),$$

and denote by $i : T' \hookrightarrow T$ the inclusion. Then

$$X \xrightarrow{f} T$$

is a good representative of f in the sense of [Loo84, 2.7]. We denote by

$$X_t := X \cap f^{-1}(t)$$

the fibre of f over $t \in T$ and by

$$X_U := X \cap f^{-1}(U)$$

the preimage of an open subset $U \subset T$ under f. By the Ehresmann fibration theorem,

$$X' \xrightarrow{f} T'$$

is a $\mathscr{C}^\infty$ fibre bundle and, by [Loo84, Prop. 2.9], it is independent of the choices of ϵ and δ.

Definition 1.1.3. The $\mathscr{C}^\infty$ fibre bundle $f : X' \longrightarrow T'$ is called the **Milnor fibration** and the general fibre X_t, $t \in T'$, is called the **Milnor fibre**.

Figure 1.1 shows the Milnor fibration.

J. Milnor studied the Milnor fibration and proved the following theorem.

Theorem 1.1.4. *The Milnor fibre is homotopy equivalent to the bouquet of* μ *n-spheres.*

Proof: [Mil68, Thm. 7.2] □

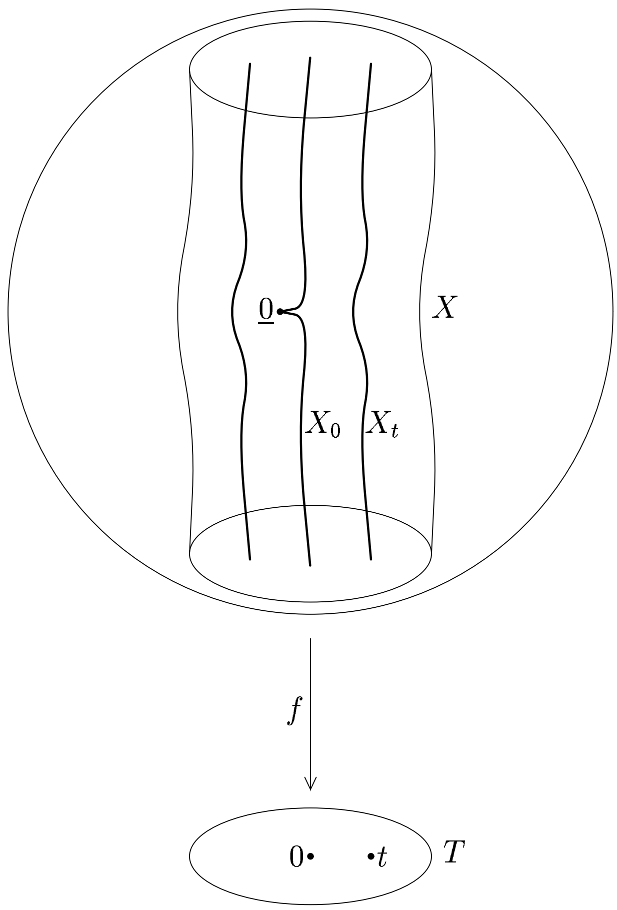
Recall that the bouquet of a set of pointed topological spaces is the topological space obtained by gluing these spaces at their base points.

Corollary 1.1.5. *The reduced (co)homology of the Milnor fibre is concentrated in the dimension of the singularity, that is,*

$$\widetilde{\mathrm{H}}_k(X_t) \cong \delta_{k,n}\mathbb{Z}^\mu \cong \widetilde{\mathrm{H}}^k(X_t)$$

where δ *is the Kronecker symbol.*

Figure 1.1: The Milnor fibration



1.2 Cohomology bundle

In this section, we introduce the (co)homology bundle and the (co)homological monodromy. The (co)homology of the fibres of the Milnor fibration form a local system with a natural flat connection on the sheaf of holomorphic sections. Shifting (co)homology classes along flat sections around the singular fibre defines an automorphism of the (co)homology of the Milnor fibre, the (co)homological monodromy. By the monodromy theorem, its eigenvalues are roots of unity and the size of its Jordan blocks is bounded by the dimension of the singularity added by one. An introduction to local systems, (flat) connections, and monodromy can be found in [Del70].

In the following, we denote the n-th homology resp. cohomology by H_n

resp. H^n and omit the index n if the statement applies to homology and cohomology. Since $f: X' \longrightarrow T'$ is a $\mathscr{C}^\infty$ fibre bundle,

$$\mathrm{H}(U) := \mathrm{H}(X_U)$$

defines a $\mathbb{Z}$-sheaf H. Note that $\mathrm{H}^n \otimes_{\mathbb{Z}} \mathbb{C} = \mathrm{R}^n f_* \mathbb{C}_{X'}$. The $\mathscr{O}_{T'}$-sheaf

$$\mathscr{H} := \mathscr{O}_{T'} \otimes_{\mathbb{Z}_{T'}} \mathrm{H}$$

is the sheaf of holomorphic sections of H. By corollary 1.1.5, H is a locally free $\mathbb{Z}$-sheaf of rank μ and hence $\mathscr{H}$ is a locally free $\mathscr{O}_{T'}$-sheaf of rank μ. Note that $\mathrm{H} \otimes_{\mathbb{Z}} \mathbb{C}$ is a local system in the sense of [Del70].

Definition 1.2.1. The $\mathbb{Z}$-sheaf H is called the **(co)homology sheaf** and the $\mathscr{O}_{T'}$-sheaf $\mathscr{H}$ is called the **(co)homology bundle**.

We denote by $(\Omega^\bullet, \mathrm{d})$ the complex of sheaves of holomorphic differential forms and by $\mathscr{T}$ the sheaf of holomorphic vector fields. A **connection** on $\mathscr{H}$ is a map

$$\mathscr{H} \xrightarrow{\nabla} \Omega^1_{T'} \otimes_{\mathscr{O}_{T'}} \mathscr{H}$$

which fulfills the **Leibniz rule**

$$\nabla(gv) = \mathrm{d}(g) \otimes v + g \otimes \nabla(v)$$

for (local) sections $g \in \Gamma(U, \mathscr{O}_{T'})$ and $v \in \Gamma(U, \mathscr{H})$. A (local) section $v \in \Gamma(U, \mathscr{H})$ is called a **flat section** if $\nabla(v) = 0$ and the sheaf $\ker(\nabla)$ of flat sections is a locally constant $\mathbb{C}$-sheaf. The **covariant derivative**

$$\mathscr{H} \xrightarrow{\nabla_X} \mathscr{H}$$

of ∇ along a (local) vector field $X \in \Gamma(U, \mathscr{T}_{T'})$ is defined by

$$\nabla_X(v) = \langle \nabla(v), X \rangle.$$

It fulfills the **Leibniz rule**

$$\nabla_X(g \cdot v) = X(g) \cdot v + g \cdot \nabla_X(v)$$

for (local) sections $g \in \Gamma(U, \mathscr{O}_{T'})$ and $v \in \Gamma(U, \mathscr{H})$. A connection ∇ on $\mathscr{H}$ extends to a map of complexes

$$\Omega^\bullet_{T'} \otimes_{\mathscr{O}_{T'}} \mathscr{H} \longrightarrow \Omega^\bullet_{T'}[1] \otimes_{\mathscr{O}_{T'}} \mathscr{H}$$

by

$$\nabla(\omega \otimes v) = \mathrm{d}\omega \otimes v + (-1)^k \nabla(v)$$

for (local) sections $\omega \in \Gamma(U, \Omega^k_{T'})$ and $v \in \Gamma(U, \mathscr{H})$. If $\nabla^2 = 0$ or equivalently

$$\nabla_{[X,Y]} = [\nabla_X, \nabla_Y]$$

then ∇ is called a **flat connection**.

Definition 1.2.2. The flat connection ∇ on $\mathscr{H}$ with $\ker \nabla = \mathrm{H}$ is called the **Gauß-Manin connection**.

Note that the Gauß-Manin connection

$$\mathscr{H} \xrightarrow{\nabla} \Omega^1_{T'} \otimes_{\mathscr{O}_{T'}} \mathscr{H}$$

is defined by

$$\nabla(gv) := \mathrm{d}g \otimes v$$

for (local) sections $g \in \Gamma(U, \mathscr{O}_{T'})$ and $v \in \Gamma(U, \mathrm{H})$. A connection ∇ on $\mathscr{H}$ defines a **dual connection** ∇^* on $\mathscr{H}^*$ by

$$\nabla^*_X(\phi)(v) = X(\phi(v)) - \phi(\nabla_X(v))$$

for (local) sections $\phi \in \Gamma(U, \mathscr{H}^*)$ and $v \in \Gamma(U, \mathscr{H})$ and (local) vector fields $X \in \Gamma(U, \mathscr{T}_{T'})$. Note that the homological and cohomological Gauß-Manin connections are dual connections.

We denote the covariant derivative of ∇ along ∂_t on $\mathscr{H}$ and $i_*\mathscr{H}$ by

$$\partial_t := \nabla_{\partial_t}.$$

Let

$$\begin{aligned} T^\infty &\xrightarrow{\quad u \quad} T \\ \tau &\longmapsto \exp(2\pi \mathrm{i}\tau) \end{aligned}$$

be the universal covering of T' where τ is a coordinate on $T^\infty \in \mathbb{C}$.

Definition 1.2.3. The pullback

$$X^\infty := X' \times_{T'} T^\infty$$

is called the **canonical Milnor fibre**.

Then

$$X^\infty \longrightarrow T^\infty$$

is a $\mathscr{C}^\infty$ fibre bundle with $X^\infty_\tau = X_{u(\tau)}$. Since T^∞ is contractible,

$$H(U) := \mathrm{H}(X^\infty_U)$$

defines a free $\mathbb{Z}$-sheaf H of rank μ on T^∞ and u_*H is the sheaf of **flat multivalued sections** of $\mathscr{H}$. For a field extension $\mathbb{Q} \subset K$, we denote by

$$H_K := H \otimes_{\mathbb{Z}} K$$

the corresponding extension of scalars. We consider $A \in H$ as a global flat multivalued section $A(t)$ of $\mathscr{H}$. Note that

$$\partial_t A(t) = 0$$

for $A \in H$.

Lifting closed paths in T' along section in H defines the **monodromy representation**

$$\begin{aligned} \pi_1(T', t) &\longrightarrow \mathrm{Aut}(\mathrm{H}_t) \\ \circlearrowleft &\longmapsto \mathrm{M}_t \end{aligned}$$

on H_t where $\circlearrowleft$ is the counterclockwise generator. The monodromy representations on the H_t induce the **monodromy representation**

$$\begin{aligned} \pi_1(T') &\longrightarrow \mathrm{Aut}(H) \\ \circlearrowleft &\longmapsto \mathrm{M} \end{aligned}$$

on H where $\circlearrowleft$ is the counterclockwise generator such that

$$(\mathrm{M}s)(\tau) = s(\tau + 1)$$

for $s \in H$. The sheaf H is determined by the monodromy representation up to isomorphism.

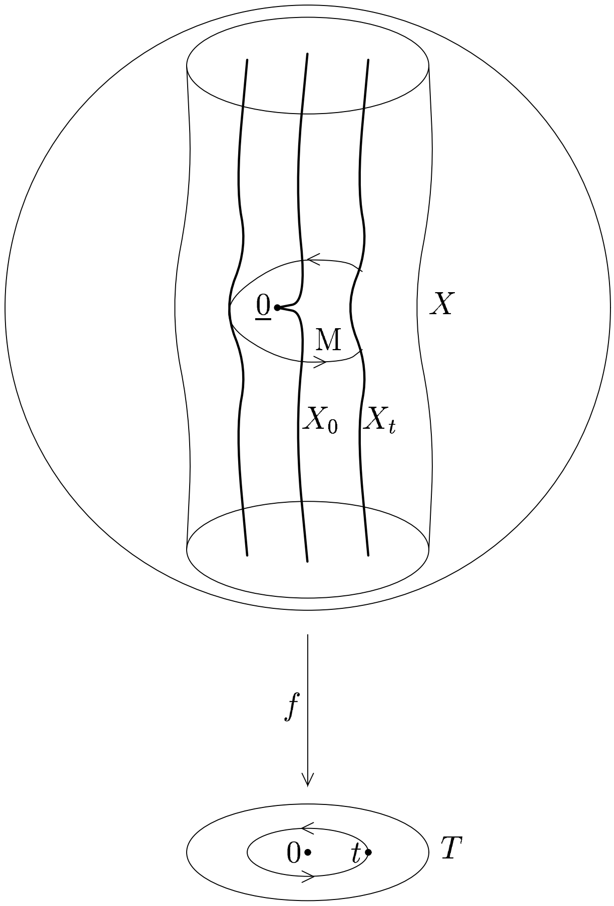
Definition 1.2.4. The automorphism

$$\mathrm{M} \in \mathrm{Aut}(H)$$

is called the **((co)homological) monodromy**.

Note that the homological monodromy is dual to the inverse of the cohomological monodromy. Figure 1.2 shows the monodromy. The most important result on the monodromy is the following theorem.

Figure 1.2: The (geometrical) monodromy



Theorem 1.2.5 (Monodromy theorem). *The eigenvalues of* M *are roots of unity, the Jordan blocks have size at most* $(n+1)\times(n+1)$*, and the Jordan blocks with eigenvalue* 1 *have size at most* $n\times n$*.*

Proof: [Bri70] and others.

From now on, we denote the homology resp. cohomology sheaf by H_n resp. H^n and the homology resp. cohomology bundle by $\mathscr{H}_n$ resp. $\mathscr{H}^n$.

1.3 Gauß-Manin connection

In this section, we introduce the (local) Gauß-Manin connection. The local elementary sections of the cohomology bundle at the critical value generate a regular $\mathbb{C}\{t\}[\partial_t]$-module, the (local) Gauß-Manin connection. It has a $\mathbb{C}\{t\}$- and $\mathbb{C}\{\{s\}\}$-module structure and the operator $t\partial_t$ corresponds to the complex monodromy. The construction in this section can be found in [AGZV88] and [Her93, Her00].

Let

$$\mathrm{M} = \mathrm{M}_s\mathrm{M}_u = \mathrm{M}_u\mathrm{M}_s$$

be the decomposition of M into semisimple part M_s and unipotent part M_u and

$$\mathrm{N} := -\frac{\log \mathrm{M}_u}{2\pi \mathrm{i}} \in \mathrm{End}_{\mathbb{C}}(H_{\mathbb{C}}).$$

By theorem 1.2.5, the eigenvalues of M_s are roots of unity. Note that $-2\pi\mathrm{i}\mathrm{N} \in \mathrm{End}_{\mathbb{Q}}(H_{\mathbb{Q}})$ is defined over $\mathbb{Q}$. Let

$$H_{\mathbb{C}} = \bigoplus_{\lambda} H_{\mathbb{C}}^{\lambda}$$

be the decomposition of $H_{\mathbb{C}}$ into generalized eigenspaces

$$H_{\mathbb{C}}^{\lambda} := \ker(\mathrm{M}_s - \lambda)$$

of M and $\mathrm{M}^{\lambda} := \mathrm{M}|_{H_{\mathbb{C}}^{\lambda}}$. Let $H_{\mathbb{C}}^{\neq 1} := \bigoplus_{\lambda\neq 1} H_{\mathbb{C}}^{\lambda}$ and $H_{\mathbb{Q}}^{\neq 1} := H_{\mathbb{C}}^{\neq 1} \cap H_{\mathbb{Q}}$. Note that the decomposition $H_{\mathbb{C}} = H_{\mathbb{C}}^{1} \oplus H_{\mathbb{C}}^{\neq 1}$ is defined over $\mathbb{Q}$, that is,

$$H_{\mathbb{Q}} = H_{\mathbb{Q}}^{1} \oplus H_{\mathbb{Q}}^{\neq 1}.$$

By theorem 1.2.5,

$$\mathrm{N}^{n+1} = 0,$$
$$\left(\mathrm{N}|_{H_{\mathbb{C}}^{1}}\right)^{n} = 0.$$

For $\alpha \in \mathbb{Q}$, we denote

$$\lambda_\alpha := \exp(-2\pi\mathrm{i}\alpha).$$

Lemma 1.3.1. *For $A \in H_{\mathbb{C}}^{\lambda_\alpha}$, the multivalued section $s_\alpha(A)$ of $\mathscr{H}^n$ defined by*

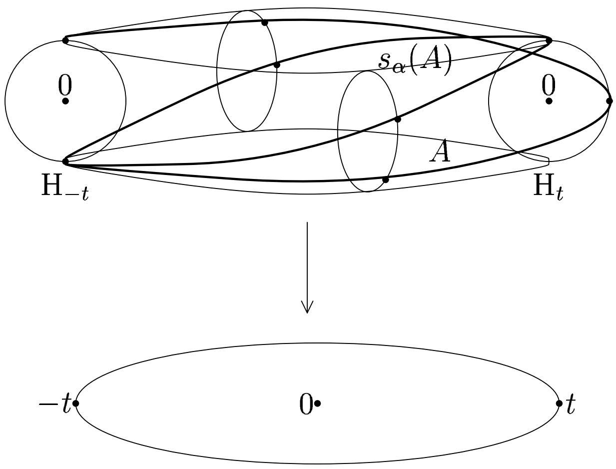
$$s_\alpha(A)(t) := t^\alpha \exp(\mathrm{N}\log t)A(t)$$

is single-valued.

Proof: Since

$$\begin{aligned}\mathrm{M}(s_\alpha(A))(t) &= t^\alpha \exp(2\pi\mathrm{i}\alpha)\exp(\mathrm{N}\log t)\exp(2\pi\mathrm{i}\mathrm{N})\mathrm{M}A(t)\\ &= t^\alpha\lambda_\alpha^{-1}\exp(\mathrm{N}\log t)\mathrm{M}_u^{-1}\mathrm{M}_s\mathrm{M}_u A(t)\\ &= t^\alpha\exp(\mathrm{N}\log t)A(t)\\ &= s_\alpha(A)(t)\end{aligned}$$

Figure 1.3: An elementary section



the section $s_\alpha(A)$ is M-invariant and hence single-valued. Note that the twist $t^\alpha \exp(\mathrm{N}\log t)$ is inverse to the action of the monodromy on $H_{\mathbb{C}}^{\lambda_\alpha}$. □

Definition 1.3.2. A section $s_\alpha(A)$ is called an **elementary section**.

Figure 1.3 shows an elementary section. The flat multivalued section $s_\alpha(A)$ defined by $A \in H_{\mathbb{C}}^{\lambda_\alpha}$ is twisted by the factor $t^\alpha \exp(\mathrm{N}\log t)$ depending on t such that it glues over -1 to a global single-valued section.

Let $\mathscr{D}$ be the sheaf of holomorphic differential operators. Note that $\mathscr{D}_{T,0} = \mathbb{C}\{t\}[\partial_t]$ with $[\partial_t, t] = 1$. The elementary sections generate a $\mathscr{D}_T$-module

$$\mathscr{G} := \big\langle i_* s_\alpha\big(H_{\mathbb{C}}^{\lambda_\alpha}\big)\big|\alpha \in \mathbb{Q}\big\rangle \mathscr{O}_T \subset i_*\mathscr{H}^n.$$

Note that $\mathscr{O}_T s_\alpha\big(H_{\mathbb{C}}^{\lambda_\alpha}\big)$ and hence $\mathscr{G}$ is independent of the coordinate t.

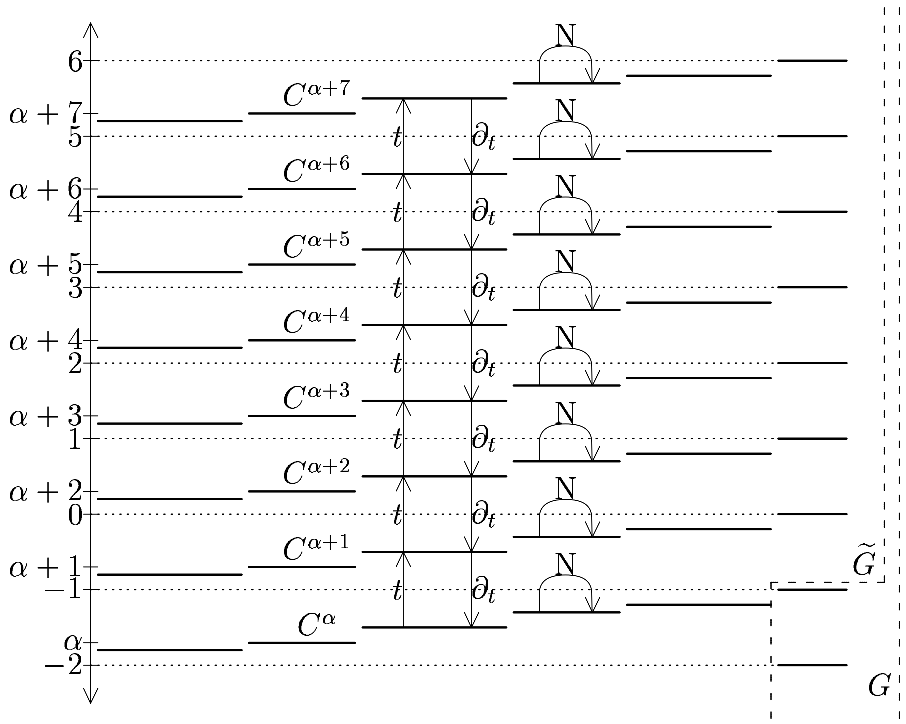
Definition 1.3.3. We call the $\mathscr{D}_{T,0}$-module

$$G := \mathscr{G}_0$$

the **(local) Gauß-Manin connection**.

Figure 1.4 shows the (local) Gauß-Manin connection. The eigenvalue level of the operator $t\partial_t$ is on the vertical axis $\mathbb{Q}$. We denote the finite dimensional $\mathbb{C}$-vectorspaces C^α by horizontal line segments and consider their lengths as the corresponding dimensions. We denote operators by arrows and submodules by dashed lines.

Figure 1.4: The (local) Gauß-Manin connection



Since $t^\alpha \exp(\mathrm{N}\log t)$ is invertible,

$$\begin{aligned} H_{\mathbb{C}}^{\lambda_\alpha} &\overset{\psi_\alpha}{\hookrightarrow} G \\ A &\longmapsto (i_* s_\alpha(A))_0 \end{aligned}$$

is an inclusion with image

$$C^\alpha := \operatorname{im} \psi_\alpha.$$

Note that, for an N-invariant subspace $F \subset C^\alpha$, $\mathbb{C}\{t\}F$ is independent of the coordinate t.

Lemma 1.3.4.

1. $t \circ \psi_\alpha = \psi_{\alpha+1}$
2. $\partial_t \circ \psi_\alpha = \psi_{\alpha-1} \circ (\alpha + \mathrm{N})$
3. $(t\partial_t - \alpha) \circ \psi_\alpha = \psi_\alpha \circ \mathrm{N}$

4. $\exp(-2\pi i t\partial_t) \circ \psi_\alpha = \psi_\alpha \circ \mathrm{M}^{\lambda_\alpha}$

Proof: Equality 1 and 2 follow from the definition of ψ_α. Combining equalities 1 and 2 gives equality 3. Applying the exponential to equality 3 gives equality 4. □

Note that, by lemma 1.3.4.3,

$$C^\alpha = \ker(t\partial_t - \alpha)^{n+1}$$

is the generalized α-eigenspace of $t\partial_t$ on $(i_*\mathscr{H})_0$. From now on, we identify the operator N on $H_{\mathbb{C}}^{\lambda_\alpha}$ with the operator

$$\mathrm{N} : C^\alpha \xrightarrow[\sim]{\psi_\alpha} H_{\mathbb{C}}^{\lambda_\alpha} \xrightarrow{\mathrm{N}} H_{\mathbb{C}}^{\lambda_\alpha} \xleftarrow[\sim]{\psi_\alpha} C^\alpha$$

on C^α such that

$$t\partial_t - \alpha = \mathrm{N}.$$

Corollary 1.3.5.

1. *The operator*

$$C^\alpha \xrightarrow[\sim]{t} C^{\alpha+1}$$

is bijective and $[t, \mathrm{N}] = 0$.

2. *The operator*

$$C^\alpha \xrightarrow[\sim]{\partial_t} C^{\alpha+1}$$

is bijective for $\alpha \neq 0$ *and* $[\partial_t, \mathrm{N}] = 0$.

3.

$$C^\alpha = \operatorname{im}\psi_\alpha = \ker(t\partial_t - \alpha)^{n+1}$$

4. $\mathscr{G}$ *is a free* $\mathscr{O}_T[t^{-1}]$*-module of rank* μ.

Proof: This follows from corollary 1.1.5 and lemma 1.3.4. □

By lemma 1.3.4.2, the operators t and ∂_t are related by $\partial_t = t^{-1}(\alpha + \mathrm{N})$. This relation generalizes to a relation of their powers.

Lemma 1.3.6. *As* $\mathbb{C}$*-homomorphisms* $C^{\alpha+k} \longrightarrow C^\alpha$,

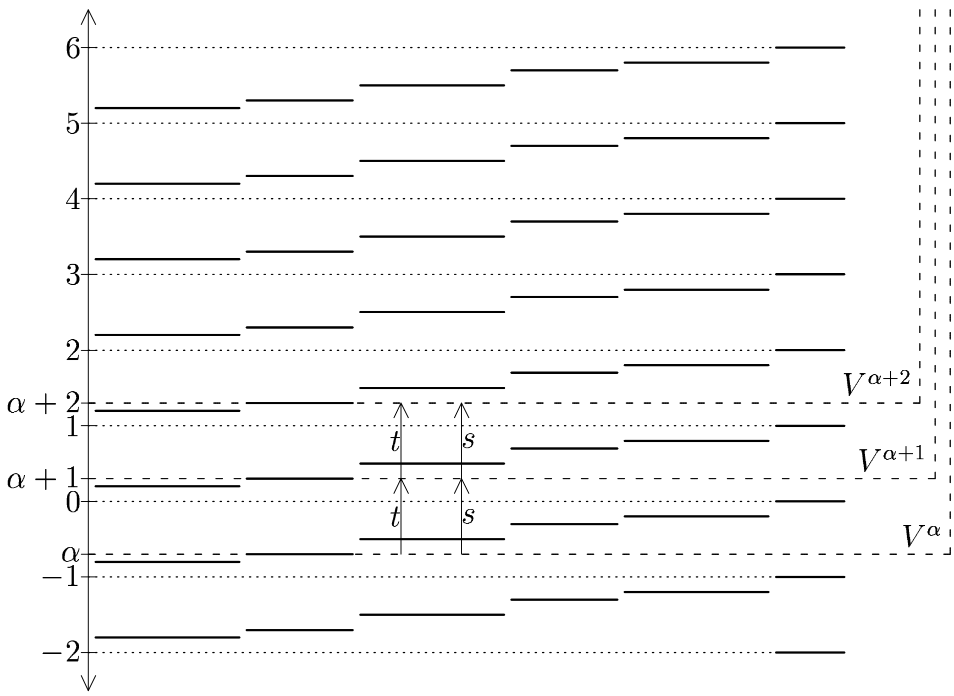
$$\partial_t^k = t^{-k} \prod_{j=1}^{k} (\alpha + j + \mathrm{N}).$$

Proof: By lemma 1.3.4.3 and corollary 1.3.5.1,

$$\begin{aligned}\partial_t^k &= \prod_{j=1}^{k} t^{-1} t\partial_t \\ &= \prod_{j=1}^{k} t^{-1}(\alpha + j + \mathrm{N}) \\ &= t^{-k} \prod_{j=1}^{k} (\alpha + j + \mathrm{N}).\end{aligned}$$

□

Figure 1.5: The V-filtration on the Gauß-Manin connection



Definition 1.3.7. The **V-filtration** $V = (V^\alpha)_{\alpha\in\mathbb{Q}}$ on G is the decreasing

filtration by $\mathbb{C}\{t\}$-modules

$$V^\alpha := \sum_{\alpha\le\beta} \mathbb{C}\{t\}C^\beta = \bigoplus_{\alpha\le\beta<\alpha+1} \mathbb{C}\{t\}C^\beta,$$
$$V^{>\alpha} := \sum_{\alpha<\beta} \mathbb{C}\{t\}C^\beta = \bigoplus_{\alpha<\beta\le\alpha+1} \mathbb{C}\{t\}C^\beta.$$

Figure 1.5 shows the V-filtration on the Gauß-Manin connection. Note that the V^α and $V^{>\alpha}$ are independent of the coordinate t. The generalized eigenspaces C^α define a canonical splitting

$$C^\alpha \cong V^\alpha/V^{>\alpha}$$

of the V-filtration and one can consider C^α as a subspace and a subquotient of G. For a filtration F on G, we denote by FC^α the induced filtration on C^α considered as a subquotient of G.

Proposition 1.3.8. *The V^α and $V^{>\alpha}$ are free $\mathbb{C}\{t\}$-modules of rank μ.*

Proof: This follows from corollary 1.1.5 and 1.3.5. □

By corollary 1.3.5.2, the inverse of the operator ∂_t is defined on $V^{>-1}$. It extends to a module structure over a power series ring.

Definition 1.3.9. The **ring of microdifferential operators** with constant coefficients is defined by

$$\mathbb{C}\{\{s\}\} := \Big\{\sum_{k=0}^{\infty} a_k s^k \in \mathbb{C}[\![s]\!] \Big| \sum_{k=0}^{\infty} \frac{a_k}{k!} t^k \in \mathbb{C}\{t\}\Big\}.$$

Note that $\mathbb{C}\{\{s\}\}$ is a discrete valuation ring.

Lemma 1.3.10. *For $\alpha > -1$ or $\alpha \notin \mathbb{Z}$, $\mathbb{C}\{t\}t^\alpha$ is a free $\mathbb{C}\{\{s\}\}$-module of rank 1 with $s := \partial_t^{-1} := \int_0^t \mathrm{d}t$.*

Proof: Since

$$\sum_{k=0}^{\infty} a_k t^k t^\alpha = \sum_{k=0}^{\infty} \prod_{j=1}^{k} (\alpha+j) a_k \partial_t^{-k} t^\alpha,$$

the claim is equivalent to the fact that $\sum_{k=0}^\infty a_k t^k \in \mathbb{C}\{t\}$ if and only if $\sum_{k=0}^\infty \frac{\prod_{j=1}^k(\alpha+j)}{k!} a_k t^k \in \mathbb{C}\{t\}$. □

Proposition 1.3.11. *For $\alpha > -1$ or $\alpha \notin \mathbb{Z}$, $\mathbb{C}\{t\}C^\alpha$ is a free $\mathbb{C}\{\{s\}\}$-module of rank $\dim_{\mathbb{C}} C^\alpha$ with $s := \partial_t^{-1}$. In particular, for $\alpha > -1$ resp. $\alpha \ge -1$, V^α resp. $V^{>\alpha}$ is a free $\mathbb{C}\{\{s\}\}$-module of rank μ.*

Proof: Since N is nilpotent on C^α,

$$\dim_{\mathbb{C}} \mathrm{N}(C^\alpha) < \dim_{\mathbb{C}} C^\alpha.$$

By induction on $\dim_{\mathbb{C}} C^\alpha$ and by corollary 1.3.5.1 and 1.3.5.2,

$$\mathbb{C}\{t\}\mathrm{N}(C^\alpha) = \mathrm{N}(\mathbb{C}\{t\}C^\alpha)$$

is a free $\mathbb{C}\{\{s\}\}$-module of rank $\dim_{\mathbb{C}} C^\alpha$. By lemma 1.3.4.3 and 1.3.10,

$$\begin{aligned}(\mathbb{C}\{t\}t^\alpha)^{\dim_{\mathbb{C}}(C^\alpha/\mathrm{N}(C^\alpha))} &\cong_{\mathbb{C}\{\{s\}\}} \mathbb{C}\{t\}(C^\alpha/\mathrm{N}(C^\alpha))\\ &\cong_{\mathbb{C}\{\{s\}\}} (\mathbb{C}\{t\}C^\alpha)/(\mathbb{C}\{t\}\mathrm{N}(C^\alpha))\end{aligned}$$

is a free $\mathbb{C}\{\{s\}\}$-module of rank $\dim_{\mathbb{C}} C^\alpha - \dim_{\mathbb{C}} \mathrm{N}(C^\alpha)$ and hence $\mathbb{C}\{t\}C^\alpha$ is a free $\mathbb{C}\{\{s\}\}$-module of rank $\dim_{\mathbb{C}} C^\alpha$. Then the claim follows from corollary 1.3.5.4. □

Definition 1.3.12. We denote

$$s := \partial_t^{-1}$$

Definition 1.3.13. We call the maximal $\mathbb{C}\{\{s\}\}$-module

$$\widetilde{G} := \mathbb{C}\{t\}C^0 \oplus \bigoplus_{-1<\alpha<0} \mathbb{C}\{t\}[t^{-1}]C^\alpha$$

in G the **reduced (local) Gauß-Manin connection**. The μ-dimensional $\mathbb{C}\{\{s\}\}[s^{-1}]$-vector space $\widetilde{G} \otimes_{\mathbb{C}\{\{s\}\}} \mathbb{C}\{\{s\}\}[s^{-1}]$ is called the **Gauß-Manin system** [Pha79, SS85].

Note that, for $\alpha > -1$,

$$V^\alpha = V^\alpha \widetilde{G}.$$

The following lemma, summarizes the basic properties of the V-filtration.

Lemma 1.3.14.

1. *The generalized eigenspaces C^α define a canonical splitting*

$$C^\alpha \cong V^\alpha / V^{>\alpha}$$

of the V-filtration.

2. *The operators*

$$V^\alpha \xrightarrow[\sim]{t} V^{\alpha+1},$$

$$\text{and } V^{>\alpha} \xrightarrow[\sim]{t} V^{>\alpha+1}$$

are bijective.

3. The operator

$$V^\alpha \xrightarrow[\sim]{\partial_t} V^{\alpha-1}$$

$$\textit{resp.}\quad V^{>\alpha} \xrightarrow[\sim]{\partial_t} V^{>\alpha-1}$$

is bijective for $\alpha > 0$ resp. $\alpha \geq 0$.

4. For $\alpha > -1$ resp. $\alpha \geq -1$, the operator

$$V^\alpha \xrightarrow[\sim]{s} V^{\alpha+1}$$

$$\textit{resp.}\quad V^{>\alpha} \xrightarrow[\sim]{s} V^{>\alpha+1}$$

is bijective.

Proof: This follows from corollary 1.3.5.1 and 1.3.5.2. □

The operator t is a differential operator with respect to the $\mathbb{C}\{\{s\}\}$-structure.

Definition 1.3.15. We denote

$$\partial_s := \partial_t^2 t$$

Note that $\partial_t t = s\partial_s$.

Lemma 1.3.16.

$$[\partial_s, s] = 1$$

In particular, $t = s^2\partial_s$ is a differential operator with respect to the $\mathbb{C}\{\{s\}\}$-structure.

Proof: Since $[\partial_t, t] = \partial_t t - t\partial_t = 1$,

$$[t, s] = ts - st = t\partial_t^{-1} - \partial_t^{-1}t = \partial_t^{-1}(\partial_t t - t\partial_t)\partial_t^{-1} = \partial_t^{-2} = s^2.$$

and hence $[\partial_t^2 t, s] = 1$. □

The $\mathbb{C}\{t\}[\partial_t]$-module structure of the Gauß-Manin connection can be described in terms of the Jordan data of the monodromy.

Proposition 1.3.17. *Let $n_{\lambda,j}$, $j = 1, \dots, m_\lambda$, be the Jordan blocks sizes of M^λ and $-1 \leq \alpha_\lambda < 0$ with $\lambda = \exp(2\pi \mathrm{i}\alpha_\lambda)$. Then there is a $\mathbb{C}\{t\}[\partial_t]$-isomorphism*

$$G \cong_{\mathbb{C}\{t\}[\partial_t]} \bigoplus_{\lambda} \bigoplus_{j=1}^{m_\lambda} \mathbb{C}\{t\}[\partial_t]/\mathbb{C}\{t\}[\partial_t](t\partial_t - \alpha_\lambda)^{n_{\lambda,j}}.$$

Proof: Let

$$H_{\mathbb{C}}^{\lambda} \cong \bigoplus_{j=1}^{m_\lambda} H_{\mathbb{C}}^{\lambda,j}$$

be a decomposition of $H_{\mathbb{C}}^{\lambda}$ into Jordan blocks $H_{\mathbb{C}}^{\lambda,j}$ of M^λ of size $n_{\lambda,j} = \dim_{\mathbb{C}} H_{\mathbb{C}}^{\lambda,j}$ and

$$C^{\alpha,j} := \psi_\alpha\big(H_{\mathbb{C}}^{\lambda,j}\big)$$

such that

$$V^{-1} = \bigoplus_{\lambda} \bigoplus_{j=1}^{m_\lambda} \mathbb{C}\{t\} C^{\alpha_\lambda,j}.$$

Let $A_{\lambda,j} \in H_{\mathbb{C}}^{\lambda,j}$ be an N-cyclic vector. By corollary 1.3.5.2, $\partial_t : C^\alpha \xrightarrow[\sim]{} C^{\alpha-1}$ is bijective for $\alpha \neq 0$ and hence

$$\begin{aligned}
G &= \mathbb{C}\{t\}\big[t^{-1}\big]V^{-1} \\
&\cong \bigoplus_{\lambda} \bigoplus_{j=1}^{m_\lambda} \sum_{k=0}^{n_{\lambda,j}-1} \mathbb{C}\{t\}\big[t^{-1}\big] s_{\alpha_\lambda}(\mathrm{N}^k A_{\lambda,j}) \\
&= \bigoplus_{\lambda} \bigoplus_{j=1}^{m_\lambda} \sum_{k=0}^{n_{\lambda,j}-1} \mathbb{C}\{t\}[\partial_t](t\partial_t - \alpha_\lambda)^k s_{\alpha_\lambda}(A_{\lambda,j}) \\
&\cong \bigoplus_{\lambda} \bigoplus_{j=1}^{m_\lambda} \mathbb{C}\{t\}[\partial_t]/\mathbb{C}\{t\}[\partial_t](t\partial_t - \alpha_\lambda)^{n_{\lambda,j}}
\end{aligned}$$

as a $\mathbb{C}\{t\}[\partial_t]$-module. □

1.4 Brieskorn lattices

In this section, we introduce the Brieskorn lattices. By the De Rham isomorphism, the cohomology bundle can be described in terms of (relative) holomorphic differential forms. This defines locally free extensions of the cohomology bundle, the Brieskorn lattices. The (local) Brieskorn lattices are embedded in the (local) Gauß-Manin connection as $\mathbb{C}\{t\}$- and $\mathbb{C}\{\{s\}\}$-lattices. The $\mathbb{C}\{t\}[\partial_t]$-module structure of the (local) Gauß-Manin connection can be expressed in terms of holomorphic differential forms on the (local) Brieskorn lattices. The results in this section can be found in [Bri70, Seb70, Mal74].

We denote by $(\Omega^\bullet, \mathrm{d})$ the complex of sheaves of holomorphic differential forms. Since the Milnor fibre X_t, $t \in T'$, is a Stein complex manifold, the De Rham homomorphism

$$\mathrm{H}^n_{DR}(\Omega^\bullet_{X_t}) \xrightarrow[\sim]{\rho_t} \mathrm{H}^n(X_t, \mathbb{C}) = \mathrm{Hom}_{\mathbb{C}}(H_n(X_t, \mathbb{C}), \mathbb{C})$$

defined by

$$\rho_t\big([\omega]\big)(\delta) = \int_\delta \omega$$

is an isomorphism. We denote by

$$(\Omega^\bullet_{X/T}, \mathrm{d}) = (\Omega^\bullet_X/\mathrm{d}f \wedge \Omega^{\bullet-1}_X, \mathrm{d})$$

the complex of sheaves of relative holomorphic differential forms with respect to $f : X \longrightarrow T$. The isomorphisms ρ_t glue to a natural isomorphism

$$\mathrm{H}^n(f_*\Omega^\bullet_{X'/T'}) \xrightarrow{\ \rho\ } \mathscr{H}^n .$$

E. Brieskorn [Bri70] defined the following extensions of $\mathscr{H}^n$.

Definition 1.4.1. The $\mathscr{O}_T$-modules

$$\begin{aligned}
\mathscr{H} &:= \mathrm{H}^n(f_*\Omega^\bullet_{X/T}) = \mathrm{H}^n(f_*\Omega^\bullet_X/\mathrm{d}f \wedge f_*\Omega^\bullet_X)\\
\mathscr{H}' &:= f_*\Omega^n_{X/T}/\mathrm{d}(f_*\Omega^{n-1}_{X/T}) = f_*\Omega^n_X/(\mathrm{d}(f_*\Omega^{n-1}_X) + \mathrm{d}f \wedge f_*\Omega^{n-1}_X)\\
\mathscr{H}'' &:= f_*\Omega^{n+1}_X/\mathrm{d}f \wedge \mathrm{d}(f_*\Omega^{n-1}_X)
\end{aligned}$$

are called the **Brieskorn lattices**. We call their stalks

$$\begin{aligned}
H &:= \mathscr{H}_0,\\
H' &:= \mathscr{H}'_0,\\
H'' &:= \mathscr{H}''_0
\end{aligned}$$

at $0 \in T$ the **(local) Brieskorn lattices**. We refer to $\mathscr{H}''$ resp. H'' as the **(local) Brieskorn lattice**.

The following $\mathscr{O}_T$-module occurs naturally in the context of Brieskorn lattices.

Definition 1.4.2.

$$\Omega := f_*\Omega^{n+1}_X/\mathrm{d}f \wedge f_*\Omega^n_X.$$

The operators d and df define the following exact sequences.

Lemma 1.4.3 (Poincaré lemma).

$$0 \longrightarrow \mathbb{C} \longrightarrow \mathscr{O}_X \xrightarrow{\mathrm{d}} \Omega^1_X \xrightarrow{\mathrm{d}} \cdots \xrightarrow{\mathrm{d}} \Omega^{n+1}_X \longrightarrow 0$$

is an exact sequence of $\mathbb{C}$*-vectorspaces.*

Lemma 1.4.4 (De Rham lemma).

$$0 \longrightarrow \mathscr{O}_X \xrightarrow{\mathrm{d}f} \Omega_X^1 \xrightarrow{\mathrm{d}f} \cdots \xrightarrow{\mathrm{d}f} \Omega_X^{n+1} \longrightarrow \Omega \longrightarrow 0$$

is an exact sequence of $\mathscr{O}_X$-modules.

Theorem 1.4.5.

1. *$\mathscr{H}$, $\mathscr{H}'$, and $\mathscr{H}''$ are locally free $\mathscr{O}_T$-modules of rank μ and*

$$\begin{aligned} H &= \mathrm{H}^n(\Omega^\bullet_{X/T,0}) = \mathrm{H}^n(\Omega^\bullet_{X,0}/\mathrm{d}f \wedge \Omega^\bullet_{X,0}),\\ H' &= \Omega^n_{X/T,0}/\mathrm{d}\Omega^{n-1}_{X/T,0} = \Omega^n_{X,0}/(\mathrm{d}\Omega^{n-1}_{X,0} + \mathrm{d}f \wedge \Omega^{n-1}_{X,0}),\\ H'' &= \Omega^n_{X,0}/\mathrm{d}f \wedge \mathrm{d}\Omega^{n-1}_{X,0} \end{aligned}$$

   are their stalks at 0.

2. *The identity and* df *induce $\mathscr{O}_T$-inclusions*

$$\mathscr{H} \hookrightarrow \mathscr{H}' \xhookrightarrow{\mathrm{d}f} \mathscr{H}'' \tag{1.4.1}$$

   restricting to $\mathscr{O}_{T'}$-isomorphisms

$$\mathscr{H}^n \xleftarrow[\sim]{\rho} \mathscr{H}|_{T'} \xrightarrow[\sim]{} \mathscr{H}'|_{T'} \xrightarrow[\sim]{\mathrm{d}f} \mathscr{H}''|_{T'} \tag{1.4.2}$$

Proof:

1. [Bri70] and [Seb70, Cor. 1]
2. This follows from lemma 1.4.4.

□

By theorem 1.4.5.1, there is an embedding of $\mathscr{H}$, $\mathscr{H}'$, and $\mathscr{H}''$ in $i_*\mathscr{H}^n$ compatible with the inclusions (1.4.1). We consider $\mathscr{H}$, $\mathscr{H}'$, and $\mathscr{H}''$ as $\mathscr{O}_T$-submodules of $i_*\mathscr{H}^n$.

Proposition 1.4.6.

1. *The identity induces an $\mathscr{O}_T$-isomorphism*

$$\mathscr{H}''/\mathscr{H}' \xrightarrow[\sim]{} \Omega\,.$$

2. *The differential* d *induces an $\mathscr{O}_T$-isomorphism*

$$\mathscr{H}'/\mathscr{H} \xrightarrow[\sim]{\mathrm{d}} \Omega\,.$$

Proof:

1. This follows from the definition.

2. This follows from lemma 1.4.3.

□

By the isomorphisms (1.4.2), (local) sections $\eta \in \Gamma\big(U, f_*\Omega^n_{X'}\big)$ and $\omega \in \Gamma\big(U, f_*\Omega^{n+1}_{X'}\big)$ define (local) sections $s\big([\eta]\big) \in \Gamma(U, \mathscr{H}^n)$ and $s\big([\omega]\big) \in \Gamma(U, \mathscr{H}^n)$ such that

$$\begin{aligned} s\big([\eta]\big)(t) &= \rho_t\big([\eta|_{X_t}]\big), \\ s\big([\omega]\big)(t) &= \rho_t\Big(\Big[\frac{\omega}{\mathrm{d}f}\Big|_{X_t}\Big]\Big). \end{aligned}$$

Definition 1.4.7. For (local) sections $\eta \in \Gamma\big(U, f_*\Omega^n_{X'}\big)$ and $\omega \in \Gamma\big(U, f_*\Omega^{n+1}_{X'}\big)$, the (local) sections $s\big([\eta]\big)$ and $s\big([\omega]\big)$ are called **geometrical sections**.

The action of the derivative ∂_t on the Brieskorn lattices can be expressed in terms of holomorphic differential forms and the operators d and $\mathrm{d}f$.

Proposition 1.4.8.

$$\begin{array}{ccccc} \mathscr{H} & \xrightarrow[\sim]{\partial_t} & \mathscr{H}' & \xrightarrow[\sim]{\partial_t} & \mathscr{H}'' \\ [\eta] & \longmapsto & \big[\frac{\mathrm{d}\eta}{\mathrm{d}f}\big] & & \\ & & [\mathrm{d}f \wedge \eta] & \longmapsto & [\mathrm{d}\eta] \end{array}$$

Proof: We follow the proof by E. Brieskorn [Bri70, Satz 1]. Let $\eta \in \Gamma\big(U, f_*\Omega^{n+1}_{X'}\big)$ a (local) section and $\delta \in \Gamma(U, H_n)$ a flat (local) section. Let

$$\mathrm{H}_n(X_t, \mathbb{C}) \xrightarrow{\partial} \mathrm{H}_{n+1}(X \backslash X_t, \mathbb{C})$$

be the Leray coboundary. By shrinking U, we may assume that there is a $\partial(\delta) \in \mathrm{H}_{n+1}(X \backslash U, \mathbb{C})$ inducing $\partial(\delta(t))$ for all $t \in U$. Then the Leray residue formula implies that

$$\begin{aligned}\big(\nabla_{\partial_t} s([\eta])\big)(\delta) &= \partial_t\big(s([\eta])(\delta)\big) - s([\eta])\big(\nabla_{\partial_t}\delta\big) = \partial_t \int_\delta \eta \\ &= \frac{1}{2\pi \mathrm{i}}\partial_t \int_{\partial(\delta)} \frac{\mathrm{d}f \wedge \eta}{f-t} \\ &= \frac{1}{2\pi \mathrm{i}} \int_{\partial(\delta)} \partial_t \frac{\mathrm{d}f \wedge \eta}{f-t} \\ &= \frac{1}{2\pi \mathrm{i}} \int_{\partial(\delta)} \frac{\mathrm{d}f \wedge \eta}{(f-t)^2} \\ &= \frac{1}{2\pi \mathrm{i}} \int_{\partial(\delta)} \Big(\frac{\mathrm{d}\eta}{f-t} - \mathrm{d}\frac{\eta}{f-t}\Big) \\ &= \frac{1}{2\pi \mathrm{i}} \int_{\partial(\delta)} \frac{\mathrm{d}\eta}{f-t} \\ &= \int_\delta \frac{\mathrm{d}\eta}{\mathrm{d}f} = s\Big(\Big[\frac{\mathrm{d}\eta}{\mathrm{d}f}\Big]\Big)(\delta)\end{aligned}$$

and hence $\partial_t[\eta] = \big[\frac{\mathrm{d}\eta}{\mathrm{d}f}\big]$ and $\partial_t[\mathrm{d}f \wedge \eta] = [\mathrm{d}\eta]$. The bijectivity follows from lemma 1.4.3. □

By proposition 1.4.8, the derivative ∂_t has a pole on each of the Brieskorn lattices. The pole orders are the same and at most equal to the dimension of the singularity added by one.

Corollary 1.4.9.

1. *The minimal κ with $f^\kappa \in \langle \underline{\partial}(f) \rangle$ equals the minimal κ with*

$$t^\kappa \partial_t \mathscr{H} \subset \mathscr{H},$$
$$\textit{resp. } t^\kappa \partial_t \mathscr{H}' \subset \mathscr{H}',$$
$$\textit{resp. } t^\kappa \partial_t \mathscr{H}'' \subset \mathscr{H}''.$$

2.

$$1 \leq \kappa \leq n+1$$

and $\kappa = 1$ if and only if the singularity $\mathrm{V}(f)$ is quasihomogeneous.

Proof:

1. Since $\underline{0} \in \mathbb{C}^{n+1}$ is an isolated critical point of f, $f \in \langle \underline{x} \rangle$ and there is a $k \geq 0$ such that $\langle \underline{x} \rangle^k \subset \langle \underline{\partial}(f) \rangle$. Hence, there is a minimal $1 \leq \kappa < \infty$ with

$$f^\kappa \in \langle \underline{\partial}(f) \rangle.$$

We denote $\mathrm{d}\underline{x} = \mathrm{d}x_0 \wedge \cdots \wedge \mathrm{d}x_n$ and, for $0 \leq j \leq n$, $\mathrm{d}\underline{x}_{\widehat{j}} = \mathrm{d}x_0 \wedge \cdots \wedge \mathrm{d}x_{j-1} \wedge \mathrm{d}x_{j+1} \wedge \cdots \wedge \mathrm{d}x_n$. Since

$$f^k \Omega^{n+1}_{X,0} = f^k \mathbb{C}\{t\} \mathrm{d}\underline{x} = \langle f^k \rangle \mathrm{d}\underline{x},$$
$$\mathrm{d}f \wedge \Omega^n_{X,0} = \mathrm{d}f \wedge \sum_{j=0}^{n} \mathbb{C}\{t\} \mathrm{d}\underline{x}_{\widehat{j}} = \sum_{j=0}^{n} \mathbb{C}\{t\} \partial_j(f) \mathrm{d}\underline{x} = \langle \underline{\partial}(f) \rangle \mathrm{d}\underline{x},$$

κ is minimal with

$$f^\kappa \mathrm{d}\underline{x} \in \mathrm{d}f \wedge \Omega^n_{X,0}.$$

By proposition 1.4.8, κ is minimal with $t^\kappa H'' \subset \partial_t^{-1} H''$ or equivalently

$$t^\kappa \partial_t H'' \subset H''.$$

Then the claim follows from proposition 1.4.8.

2. By J. Briançon and H. Skoda [BS74], $\kappa \leq n+1$ and this bound is strict. By K. Saito [Sai71], $f \in \langle \underline{\partial}(f) \rangle$ if and only if the singularity $\mathrm{V}(f)$ is quasihomogeneous.

□

The geometrical sections have moderate growth with respect to the flat multivalued sections and the geometrical sections in $\mathscr{H}'$ tend to 0 as $t \in T'$ tends to $0 \in T$.

Theorem 1.4.10.

1. *$\mathscr{H}'' \subset \mathscr{G}$*

2. *$H'' \subset V^{>-1}$.*

Proof:

1. [Bri70, Satz 2]

2. [Mal74, lem. 4.5]

□

P. Pham [Pha77, Pha79] proved the following result in the context of the Gauß-Manin system.

Proposition 1.4.11.

1. *H, H', and H'' are free $\mathbb{C}\{\{s\}\}$-modules of rank μ.*

2.

$$\begin{array}{ccccc} H'' & \xrightarrow[\sim]{s} & H' & \xrightarrow[\sim]{s} & H \\ [\mathrm{d}\eta] & \longmapsto & [\mathrm{d}f\wedge\eta] & & \\ & & \left[\frac{\mathrm{d}\eta}{\mathrm{d}f}\right] & \longmapsto & [\eta] \end{array}$$

Proof: By theorem 1.4.5.2 and proposition 1.4.8,

$$sH'' = \partial_t^{-1}H'' = H' \subset H''$$

and hence $\mathbb{C}[s]H'' \subset H''$. By theorem 1.4.10.2, $H'' \subset V^{>-1}$ and, by proposition 1.3.11, $V^{>-1}$ is a free $\mathbb{C}\{\{s\}\}$-module. Since H'' and $V^{>-1}$ are $\mathbb{C}\{t\}$-lattices, there is a $k \geq 0$ such that

$$s^kH'' \subset s^kV^{>-1} = V^{>k-1} = t^kV^{>-1} \subset H''$$

and, by proposition 1.3.11, $V^{>k-1}$ is a $\mathbb{C}\{\{s\}\}$-module. Hence, H'' is a free $\mathbb{C}\{\{s\}\}$-module. By proposition 1.4.6.1,

$$H''/sH'' \cong H''/H' \cong \Omega_0$$

and $\dim_{\mathbb{C}} \Omega_0 = \mu$. By Nakayama's lemma, this implies that H'' is a free $\mathbb{C}\{\{s\}\}$-module of rank μ. Then the claim follows from proposition 1.4.8.

□

From now on, we abbreviate

$$\Omega := \Omega_0,$$
$$\Omega^\bullet := \Omega^\bullet_{X,0}.$$

1.5 Completion and (t,s)-module structure

In this section, we consider the completions of the Brieskorn lattices. By E. Brieskorn [Bri70], the $\langle \underline{x} \rangle$-adic and $\langle t \rangle$-adic topologies on the Brieskorn lattices coincide. We show that the $\langle \underline{x} \rangle$-adic and $\langle s \rangle$-adic topologies coincide on the Brieskorn lattices. The formal Brieskorn lattices are the completions of the Brieskorn lattices with respect to this topology. We give an explicit description of the formal Brieskorn lattice leading to the normal form algorithm in section 2.2.

Definition 1.5.1. We call the $\langle \underline{x} \rangle$-adic completions

$$\widehat{H} = \mathrm{H}^n(\widehat{\Omega}^\bullet/\mathrm{d}f \wedge \Omega^\bullet),$$
$$\widehat{H}' = \widehat{\Omega}^n/\mathrm{d}\widehat{\Omega}^{n-1} + \mathrm{d}f \wedge \widehat{\Omega}^{n-1},$$
$$\widehat{H}'' = \widehat{\Omega}^{n+1}/\mathrm{d}f \wedge \mathrm{d}\widehat{\Omega}^{n-1}$$

of the Brieskorn lattices the **formal Brieskorn lattices**. We refer to $\widehat{H}''$ as the **formal Brieskorn lattice**.

The following proposition is essential for Brieskorn's algorithm to compute the complex monodromy.

Theorem 1.5.2. *The $\langle t\rangle$-adic and $\langle \underline{x}\rangle$-adic topology on H, H', resp. H'' coincide. In particular, the $\langle t\rangle$-adic completion of H, H', resp. H'' is naturally isomorphic to $\widehat{H}$, $\widehat{H}'$, resp. $\widehat{H}''$.*

Proof: [Bri70, Prop. 3.3] □

Corollary 1.5.3. *$\widehat{H}$, $\widehat{H}'$, and $\widehat{H}''$ are free $\mathbb{C}[\![t]\!]$-modules of rank μ.*

Proof: Since completion is faithfully flat, this follows from proposition 1.4.5.1 and theorem 1.5.2. □

Proposition 1.5.4. *The $\langle s\rangle$-adic and $\langle \underline{x}\rangle$-adic topology on H, H', resp. H'' coincide. In particular, the $\langle s\rangle$-adic completion of H, H', resp. H'' is naturally isomorphic to $\widehat{H}$, $\widehat{H}'$, resp. $\widehat{H}''$.*

Proof: Let

$$\big[g\partial_i(f)\partial_j(f)\mathrm{d}\underline{x}\big] \in \big(\langle\underline{\partial}(f)\rangle^{2k}\mathrm{d}\underline{x} + \mathrm{d}f\wedge\mathrm{d}\Omega^{n-1}\big)/\mathrm{d}f\wedge\mathrm{d}\Omega^{n-1} \subset H''.$$

Then, by proposition 1.4.11.2,

$$\begin{aligned}\big[g\partial_i(f)\partial_j(f)\mathrm{d}\underline{x}\big] &= \big[(-1)^i\mathrm{d}f\wedge\big(g\partial_j(f)\mathrm{d}\underline{x}_{\widehat{i}}\big)\big]\\ &= s\big[(-1)^i\mathrm{d}\big(g\partial_j(f)\mathrm{d}\underline{x}_{\widehat{i}}\big)\big]\\ &= s\big[\partial_i\big(g\partial_j(f)\big)\mathrm{d}\underline{x}\big]\\ &= s\big[\big(\partial_i(g)\partial_j(f)+g\partial_i\partial_j(f)\big)\mathrm{d}\underline{x}\big]\\ &\in s\big(\big(\langle\underline{\partial}(f)\rangle^{2(k-1)}\mathrm{d}\underline{x}+\mathrm{d}f\wedge\mathrm{d}\Omega^{n-1}\big)/\mathrm{d}f\wedge\mathrm{d}\Omega^{n-1}\big)\end{aligned}$$

and hence by induction

$$\big(\langle\underline{\partial}(f)\rangle^{2k}\mathrm{d}\underline{x}+\mathrm{d}f\wedge\mathrm{d}\Omega^{n-1}\big)/\mathrm{d}f\wedge\mathrm{d}\Omega^{n-1} \subset s^kH''.$$

Since $\underline{0}$ is an isolated critical point of f, there is an $m \geq 1$ such that

$$\langle\underline{x}\rangle \supset \langle\underline{\partial}(f)\rangle \supset \langle\underline{x}\rangle^m$$

and hence

$$\begin{aligned}&\big(\langle\underline{x}\rangle^{2km}\mathrm{d}\underline{x}+\mathrm{d}f\wedge\mathrm{d}\Omega^{n-1}\big)/\mathrm{d}f\wedge\mathrm{d}\Omega^{n-1}\\ &\qquad\subset \big(\langle\underline{\partial}(f)\rangle^{2k}\mathrm{d}\underline{x}+\mathrm{d}f\wedge\mathrm{d}\Omega^{n-1}\big)/\mathrm{d}f\wedge\mathrm{d}\Omega^{n-1}.\end{aligned}$$

This implies that

$$\big(\langle\underline{x}\rangle^{2km}\mathrm{d}\underline{x}+\mathrm{d}f\wedge\mathrm{d}\Omega^{n-1}\big)/\mathrm{d}f\wedge\mathrm{d}\Omega^{n-1}\subset s^kH''$$
$$\subset\big(\langle\underline{x}\rangle^{k}\mathrm{d}\underline{x}+\mathrm{d}f\wedge\mathrm{d}\Omega^{n-1}\big)/\mathrm{d}f\wedge\mathrm{d}\Omega^{n-1}.$$

Hence, $\langle s\rangle$-adic and $\langle\underline{x}\rangle$-adic topology on H'' coincide. Then the claim follows from proposition 1.4.11.2. □

Corollary 1.5.5.

1. *$\widehat{H}$, $\widehat{H}'$, and $\widehat{H}''$ are free $\mathbb{C}[\![s]\!]$-modules of rank μ.*

2.
$$\begin{array}{ccccc}\widehat{H}'' & \xrightarrow[\sim]{s} & \widehat{H}' & \xrightarrow[\sim]{s} & \widehat{H}\\ [\mathrm{d}\eta] & \longmapsto & [\mathrm{d}f\wedge\eta] & & \\ & & \left[\frac{\mathrm{d}\eta}{\mathrm{d}f}\right] & \longmapsto & [\eta]\end{array}$$

Proof: Since completion is faithfully flat, this follows from proposition 1.4.11 and 1.5.4. □

The following description of the formal Brieskorn lattice leads to the normal form algorithm in section 2.2.

Proposition 1.5.6. *There is a $\mathbb{C}[\![s]\!]$-isomorphism*

$$\widehat{H}''\cong_{\mathbb{C}[\![s]\!]}\mathbb{C}[\![s,\underline{x}]\!]/\langle\underline{\partial}(f)-s\underline{\partial}\rangle\mathbb{C}[\![s,\underline{x}]\!].$$

Proof: Since

$$\mathrm{d}f\wedge\mathrm{d}\widehat{\Omega}^n=(\mathrm{d}f-s\mathrm{d})\mathrm{d}\widehat{\Omega}^{n-1}[\![s]\!]\subset(\mathrm{d}f-s\mathrm{d})\widehat{\Omega}^n[\![s]\!]$$

there is a natural map

$$\widehat{H}''\xrightarrow{\iota}\widehat{\Omega}^{n+1}[\![s]\!]/(\mathrm{d}f-s\mathrm{d})\widehat{\Omega}^n[\![s]\!]\,.$$

By corollary 1.5.5.2, ι is a $\mathbb{C}[\![s]\!]$-homomorphism. Let $\omega=\sum_{k=0}^\infty\omega_ks^k\in\widehat{\Omega}^n[\![s]\!]$ with $(\mathrm{d}f-s\mathrm{d})\omega\in\widehat{\Omega}^{n+1}$. Then $\mathrm{d}f\wedge\omega_{k+1}=\mathrm{d}\omega_k$ and hence, by proposition 1.5.5.2,

$$s[\mathrm{d}\omega_{k+1}]=[\mathrm{d}f\wedge\omega_{k+1}]=[\mathrm{d}\omega_k]\in\widehat{H}''$$

for all $k\geq 0$. Then $[\mathrm{d}\omega_0]\in\bigcap_{k\geq 0}s^k\widehat{H}''=\{0\}$ and hence, by definition of H'',

$$\mathrm{d}\omega_0\in\mathrm{d}f\wedge\mathrm{d}\widehat{\Omega}^{n-1}=\mathrm{d}(\mathrm{d}f\wedge\widehat{\Omega}^{n-1}).$$

By lemma 1.4.3, $\omega_0 \in \mathrm{d}\widehat{\Omega}^{n-1} + \mathrm{d}f \wedge \widehat{\Omega}^{n-1}$ and hence

$$(\mathrm{d}f - s\mathrm{d})\omega = \mathrm{d}f \wedge \omega_0 \in \mathrm{d}f \wedge \mathrm{d}\widehat{\Omega}^{n-1}.$$

This implies that

$$(\mathrm{d}f - s\mathrm{d})\widehat{\Omega}^n[\![s]\!] \cap \widehat{\Omega}^{n+1} = \mathrm{d}f \wedge \mathrm{d}\widehat{\Omega}^{n-1}$$

and hence ι is injective. By lemma 1.4.3, $\mathrm{d}\widehat{\Omega}^n = \widehat{\Omega}^{n+1}$ and hence, by corollary 1.5.5.2, ι is surjective. Note that ι is inverse to the canonical $\mathbb{C}[\![s]\!]$-projection

$$\widehat{\Omega}^{n+1}[\![s]\!]/(\mathrm{d}f - s\mathrm{d})\widehat{\Omega}^n[\![s]\!] \xrightarrow{\pi} \widehat{\Omega}^{n+1}/\mathrm{d}f \wedge \mathrm{d}\widehat{\Omega}^{n-1} .$$

Since $\widehat{\Omega}^n[\![s]\!]$ is a free $\mathbb{C}[\![s,\underline{x}]\!]$-module of rank 1 with generator $\mathrm{d}\underline{x}$, there is a $\mathbb{C}[\![s,\underline{x}]\!]$-isomorphism

$$\mathbb{C}[\![s,\underline{x}]\!] \xrightarrow[\sim]{\mathrm{d}\underline{x}} \widehat{\Omega}^{n+1}[\![s]\!] .$$

For $\eta = \sum_{j=0}^n (-1)^j g_j \mathrm{d}\underline{x}_{\hat{j}} \in \widehat{\Omega}^n[\![s]\!]$,

$$(\mathrm{d}f - s\mathrm{d})\eta = \sum_{j=0}^n (\partial_j(f)g_j - s\partial_j(g_j))\mathrm{d}\underline{x} = (\underline{\partial}(f) - s\underline{\partial})\underline{g}\mathrm{d}\underline{x}$$

and hence $\mathrm{d}\underline{x}$ induces a $\mathbb{C}[\![s]\!]$-isomorphism

$$\mathbb{C}[\![s,\underline{x}]\!]/\langle\underline{\partial}(f) - s\underline{\partial}\rangle\mathbb{C}[\![s,\underline{x}]\!] \xrightarrow[\sim]{\mathrm{d}\underline{x}} \widehat{\Omega}^{n+1}[\![s]\!]/(\mathrm{d}f - s\mathrm{d})\widehat{\Omega}^n[\![s]\!] .$$

□

D. Barlet [Bar93, Bar00] considered the following algebraic structure.

Definition 1.5.7. A **(t,s)-module** is a free $\mathbb{C}[\![s]\!]$-module of finite rank endowed with a $\mathbb{C}$-endomorphism t fulfilling $[t,s] = s^2$.

Corollary 1.5.8. *$\widehat{H}$, $\widehat{H}'$, and $\widehat{H}''$ are (t,s)-modules.*

Proof: This follows from lemma 1.3.16 and corollary 1.5.5.1. □

1.6 Lattices

In this section, we consider $\mathbb{C}\{t\}$- and $\mathbb{C}\{\{s\}\}$-lattices in the Gauß-Manin connection. A $\mathbb{C}\{t\}$-lattice in the Gauß-Manin connection is a free $\mathbb{C}\{t\}$-submodule of rank μ. We call a free $\mathbb{C}\{\{s\}\}$-submodule of rank μ a $\mathbb{C}\{\{s\}\}$-lattice. The V-filtration consists of $\mathbb{C}\{t\}$- and $\mathbb{C}\{\{s\}\}$-lattices and the Brieskorn lattices are $\mathbb{C}\{t\}$- and $\mathbb{C}\{\{s\}\}$-lattices.

Definition 1.6.1. We call a free $\mathbb{C}\{t\}$- resp. $\mathbb{C}\{\{s\}\}$-submodule of G resp. $\widetilde{G}$ of rank μ a $\mathbb{C}\{t\}$**-** resp. $\mathbb{C}\{\{s\}\}$**-lattice**. We call a $\mathbb{C}\{t\}$- resp. $\mathbb{C}\{\{s\}\}$-lattice a **lattice**.

From now on, we denote by V the V-filtration on G resp. $\widetilde{G}$.

Remark 1.6.2.

1. By proposition 1.3.8 and 1.3.10, the V^α are $\mathbb{C}\{t\}$- and $\mathbb{C}\{\{s\}\}$-lattices.

2. By proposition 1.4.5.1 and 1.4.11.1, H, H', and H'' are $\mathbb{C}\{t\}$- and $\mathbb{C}\{\{s\}\}$-lattices.

Lemma 1.6.3. *Let L be a $\mathbb{C}\{t\}$- resp. $\mathbb{C}\{\{s\}\}$-lattice. Then there are $\alpha, \beta \in \mathbb{Q}$ such that*

$$V^\alpha \supset L \supset V^\beta.$$

Proof: Let L and L' be two $\mathbb{C}\{t\}$- resp. $\mathbb{C}\{\{s\}\}$-lattices. Since $\mathbb{C}\{t\}$ resp. $\mathbb{C}\{\{s\}\}$ is a discrete valuation ring, there is a $k \geq 0$ such that $t^k L \subset L'$ resp. $s^k L \subset L'$. By lemma 1.3.14 and proposition 1.3.8 and 1.3.11, this implies the claim. □

Lemma 1.6.4.

1. *A $\mathbb{C}\{t\}$-lattice L is a $\mathbb{C}\{\{s\}\}$-lattice if and only if $sL \subset L$.*

2. *A $\mathbb{C}\{\{s\}\}$-lattice L is a $\mathbb{C}\{t\}$-lattice if and only if $tL \subset L$.*

Proof: This follows from lemma 1.6.3 as in the proof of proposition 1.4.11. □

1.6.1 Saturation and resonance

In this subsection, we consider saturated and non-resonant lattices. The V-filtration consists of saturated non-resonant lattices. By the regularity of the Gauß-Manin connection, the saturation of a lattice by the operator $t\partial_t = s\partial_s - 1$ is a lattice. The complex monodromy on the cohomology of the Milnor fibre corresponds to this operator on $L/tL = L/sL$ for a saturated non-resonant lattice L.

Definition 1.6.5. Let L be a $\mathbb{C}\{t\}$- resp. $\mathbb{C}\{\{s\}\}$-lattice.

1. If $t\partial_t L = (s\partial_s - 1)L \subset L$ then L is called **saturated**.

2. If L is saturated then the endomorphism $\operatorname{res}_L \in \operatorname{End}_{\mathbb{C}}(L/tL)$ resp. $\operatorname{res}_L \in \operatorname{End}_{\mathbb{C}}(L/sL)$ induced by $t\partial_t = s\partial_s - 1$ is called the **residue** of L.

3. If res_L has non-zero integer differences of eigenvalues then L is called **resonant**.

Note that the residue is independent of the coordinate t.

Remark 1.6.6.

1. By lemma 1.3.14.2 and 1.3.14.3, the V^α are saturated lattices.
2. By corollary 1.4.9.2, H, H', and H'' are saturated if and only if the singularity $\mathrm{V}(f)$ is quasihomogeneous.

Lemma 1.6.7. *Let L be a saturated lattice.*

1. *If L is a $\mathbb{C}\{t\}$- and $\mathbb{C}\{\{s\}\}$-lattice then*
$$tL = sL.$$
2. *If $V^\alpha \supset L \supset V^\beta$ then*
$$L = \Big(\bigoplus_{\alpha \leq \gamma < \beta} L \cap C^\gamma \Big) \oplus V^\beta$$
and the $L \cap C^\gamma$ are N*-invariant.*
3. *If the eigenvalues of* res_L *are in $[\alpha, \beta]$ resp. $[\alpha, \beta)$ then*
$$\begin{aligned} & V^\alpha \supset L \supset V^{>\beta-1} \\ \textit{resp.} \quad & V^\alpha \supset L \supset V^{\beta-1}. \end{aligned}$$

Proof:

1. Since L is saturated and $[\partial_t, t] = 1$, $s^{-1}tL = \partial_t tL \subset L$ and hence $tL \subset sL$. By Nakayama's lemma,
$$\dim_{\mathbb{C}}(L/tL) = \mu = \dim_{\mathbb{C}}(L/sL)$$
and hence $tL = sL$.
2. Since $V^\alpha \supset L \supset V^\beta$,
$$L = L \cap V^\alpha = \Big(L \cap \bigoplus_{\alpha \leq \gamma < \beta} C^\gamma\Big) \oplus V^\beta.$$
Since L is saturated, $L \cap \bigoplus_{\alpha \leq \gamma < \beta} C^\gamma$ is $t\partial_t$-invariant and $L \cap C^\gamma$ is the γ-eigenspace of $t\partial_t$. Hence, by lemma 1.3.4.3, $L \cap C^\gamma$ is N-invariant.

3. Since $L \cap C^\gamma$ is the γ-eigenspace of $t\partial_t$ and res_L is induced by $t\partial_t$, this follows from 2.

□

Definition 1.6.8. Let L be a lattice, $L_0 := L$, and

$$L_k := \sum_{j=0}^{k} (t\partial_t)^j L = L_{k-1} + t\partial_t L_{k-1}$$

for $k \geq 1$. Then

$$L_\infty := \bigcup_{k \geq 0} L_k$$

is called the **saturation** of L.

Lemma 1.6.9. *Let L be a lattice. Then L_k and L_∞ are lattices.*

Proof: We may assume that L is a $\mathbb{C}\{t\}$-lattice. By induction on k, we may assume that L_{k-1} is a $\mathbb{C}\{t\}$-lattice. Let $g \in \mathbb{C}\{t\}$ and $v \in L_{k-1}$. Then

$$gt\partial_t v = t\partial_t gv - t\partial_t(g)v \in t\partial_t L_{k-1} + L_{k-1} = L_k.$$

Hence, $\mathbb{C}\{t\}L_k \subset L_k$ and L_k is a $\mathbb{C}\{t\}$-lattice. By lemma 1.6.3, there is an $\alpha \in \mathbb{Q}$ such that $L \subset V^\alpha$. Since V^α is saturated, $L_k \subset V^\alpha$ for all $k \geq 0$. Since $\mathbb{C}\{t\}$ is Noetherian and V^α a finite $\mathbb{C}\{t\}$-module, V^α is Noetherian and hence the sequence of $\mathbb{C}\{t\}$-submodules

$$L_0 \subset L_1 \subset L_2 \subset \cdots \subset V^\alpha$$

is stationary. Hence, there is a $k \geq 0$ such that $L_\infty = L_k$ and L_∞ is a $\mathbb{C}\{t\}$-lattice. □

The following bound for the minimal $k \geq 0$ with $L_k = L_\infty$ is due to R. Gérard and A.H.M. Levelt [GL73, Thm. 4.2]. We give an elementary proof.

Proposition 1.6.10. *Let L be a lattice. Then*

$$L_\infty = L_{\mu-1}$$

Proof: We may assume that L is a $\mathbb{C}\{t\}$-lattice. By lemma 1.6.3, there is an $\alpha \in \mathbb{Q}$ such that $L \subset V^\alpha$. Let $\underline{e} = (e_1, \ldots, e_\mu)$ be a Jordan $\mathbb{C}$-basis of

$$\bigoplus_{\alpha \leq \beta < \alpha+1} C^\beta \cong V^\alpha / tV^\alpha$$

for res_L and $e_j \in C^{\alpha_j}$. By Nakayama's lemma, $\underline{e}$ is a $\mathbb{C}\{t\}$-basis of V^α. Let

$$E_j := \bigoplus_{j<i} \mathbb{C}\{t\} e_i.$$

After reordering $\underline{e}$, we may assume that

$$\mathrm{N} E_j \subset E_{j+1}.$$

Let $v = \sum_{j=1}^{\mu} \sum_{i=0}^{\infty} v_i^j t^i e_j \in E_j \cap L_k \backslash E_{j+1}$. After multiplication with a unit in $\mathbb{C}\{t\}$, we may assume that $v \in v_i^j t^i e_j + E_{j+1}$ with $v_i^j \neq 0$. Then

$$\begin{aligned} t\partial_t v &\in ((i+\alpha_j) v_i^j t^i e_j + E_{j+1}) \cap L_{k+1} \\ &= ((i+\alpha_j) v + E_{j+1}) \cap L_{k+1} \\ &\subset (L_k + E_{j+1}) \cap L_{k+1}. \\ &= L_k + E_{j+1} \cap L_{k+1}. \end{aligned}$$

By induction over $j = k$, $L_{k+1} \subset L_k + E_{k+1} \cap L_{k+1}$ and hence, since $E_\mu = 0$,

$$L_\mu \subset L_{\mu-1} + E_\mu \cap L_\mu = L_{\mu-1}.$$

□

By lemma 1.3.4.4, $\psi := \bigoplus_{-1<\alpha\leq 0} \psi_\alpha$ defines a $\mathbb{C}$-isomorphism

$$H_\mathbb{C} = \bigoplus_{-1<\alpha\leq 0} H_\mathbb{C}^{\lambda_\alpha} \xrightarrow[\sim]{\psi} \bigoplus_{-1<\alpha\leq 0} C^\alpha = V^{>-1}/tV^{>-1} \tag{1.6.1}$$

such that

$$\psi \circ \mathrm{M} = \exp(-2\pi \mathrm{i}\, \mathrm{res}_{V^{>-1}}) \circ \mathrm{M}.$$

A $\mathbb{C}\{t\}$- resp. $\mathbb{C}\{\{s\}\}$-lattice L defines an increasing filtration $F^L = (F_k^L)_{k\in\mathbb{Z}}$ on G resp. $\widetilde{G}$ by $\mathbb{C}\{t\}$- resp. $\mathbb{C}\{\{s\}\}$-lattices

$$\begin{aligned} F_k^L &:= t^{-k} L \\ \text{resp.} \quad \widetilde{F}_k^L &:= \widetilde{G} \cap s^{-k} L. \end{aligned}$$

Lemma 1.6.11. *Let L be a $\mathbb{C}\{t\}$- resp. $\mathbb{C}\{\{s\}\}$-lattice.*

1. *If L is a $\mathbb{C}\{t\}$- and $\mathbb{C}\{\{s\}\}$-lattice then, for $\alpha > -1$ or $\alpha \notin \mathbb{Z}$,*

$$\prod_{j=1}^{k} (\alpha + j + \mathrm{N}) F_k^L C^\alpha = \widetilde{F}_k^L C^\alpha.$$

2. *If L is a saturated $\mathbb{C}\{t\}$- and $\mathbb{C}\{\{s\}\}$-lattice L then the filtrations F^L and $\widetilde{F}^L$ coincide.*

3. *If L is saturated then the filtration $F^L \operatorname{gr}_V^\alpha G$ resp. $\widetilde{F}^L \operatorname{gr}_V^\alpha \widetilde{G}$ on C^α is* N*-invariant.*

Proof:

1. This follows from 1.3.6.
2. This follows from 1.6.7.1
3. This follows from 1.6.7.2.

□

The filtration F resp. $\widetilde{F}$ induces an increasing filtration

$$F^L := \psi^{-1}\big(F^L(V^{>-1}/tV^{>-1})\big)$$
$$\text{resp.}\quad \widetilde{F}^L := \psi^{-1}\big(\widetilde{F}^L(V^{>-1}/sV^{>-1})\big)$$

by $\mathbb{C}$-vectorspaces on $H_\mathbb{C}$. Note that, by lemma 1.3.4.3 and 1.6.11.3, if L is saturated then F^L resp. $\widetilde{F}^L$ on $H_\mathbb{C}$ is M-invariant.

Proposition 1.6.12. *Let L be a saturated $\mathbb{C}\{t\}$- resp. $\mathbb{C}\{\{s\}\}$-lattice. Then*

$$\psi_L := \bigoplus_{-1<\alpha\leq 0} \bigoplus_k \operatorname{gr}_k^{F^L}(\psi_{\alpha+k})$$
$$\textit{resp.}\quad \widetilde{\psi}_L := \bigoplus_{-1<\alpha\leq 0} \bigoplus_k \operatorname{gr}_k^{\widetilde{F}^L}(\psi_{\alpha+k})$$

defines a $\mathbb{C}$-isomorphism

$$\operatorname{gr}^{F^L}(H_\mathbb{C}) = \bigoplus\nolimits_{-1<\alpha\leq 0} \bigoplus\nolimits_k \operatorname{gr}_k^{F^L}(H_\mathbb{C}^{\lambda_\alpha}) \xrightarrow[\sim]{\psi_L} L/tL$$
$$\textit{resp.}\quad \operatorname{gr}_k^{\widetilde{F}^L}(H_\mathbb{C}) = \bigoplus\nolimits_{-1<\alpha\leq 0} \bigoplus\nolimits_k \operatorname{gr}_k^{\widetilde{F}^L}(H_\mathbb{C}^{\lambda_\alpha}) \xrightarrow[\sim]{\widetilde{\psi}_L} L/sL$$

such that

$$\psi_L \circ \operatorname{gr}^{F^L}(\mathrm{M}) = \exp(-2\pi\mathrm{i}\operatorname{res}_L) \circ \psi_L$$
$$\textit{resp.}\quad \widetilde{\psi}_L \circ \operatorname{gr}^{\widetilde{F}^L}(\mathrm{M}) = \exp(-2\pi\mathrm{i}\operatorname{res}_L) \circ \widetilde{\psi}_L.$$

Proof: We may assume that L is a $\mathbb{C}\{t\}$-lattice. By lemma 1.6.3, there are $\alpha, \beta \in \mathbb{Q}$ such that

$$V^\alpha \supset L \supset V^\beta.$$

By lemma 1.6.7.2,

$$L = \Big(\bigoplus_{\alpha\le\gamma<\beta} L\cap C^\gamma\Big)\oplus V^\beta$$

and hence

$$\begin{aligned}
L/tL &\cong \bigoplus_{\alpha\le\gamma<\beta+1} (L\cap C^\gamma)/t(L\cap C^{\gamma-1})\\
&= \bigoplus_{-1<\gamma\le 0}\ \bigoplus_{\alpha\le\gamma+k<\beta+1} (L\cap C^{\gamma+k})/t(L\cap C^{\gamma+k-1})\\
&\cong \bigoplus_{-1<\gamma\le 0}\ \bigoplus_{\alpha\le\gamma+k<\beta+1} t^k\big((t^{-k}L\cap C^\gamma)/(t^{-k+1}L\cap C^\gamma)\big)\\
&= \bigoplus_{-1<\gamma\le 0}\ \bigoplus_{\alpha\le\gamma+k<\beta+1} t^k\,\mathrm{gr}_k^{F^L}(C^\gamma)\\
&= \bigoplus_{-1<\gamma\le 0}\ \bigoplus_{\alpha\le\gamma+k<\beta+1} t^k\circ\mathrm{gr}_k^{F^L}(\psi_\gamma)\big(\mathrm{gr}_k^{F^L}(H_{\mathbb{C}}^{\lambda_\gamma})\big)\\
&\cong \bigoplus_{-1<\gamma\le 0}\ \bigoplus_{\alpha\le\gamma+k<\beta+1} \mathrm{gr}_k^{F^L}(\psi_{\gamma+k})\big(\mathrm{gr}_k^{F^L}(H_{\mathbb{C}}^{\lambda_\gamma})\big)\\
&= \psi_L\big(\mathrm{gr}^{F^L}(H_{\mathbb{C}})\big).
\end{aligned}$$

Then the claim follows from lemma 1.3.4.4. □

The complex monodromy is determined by the residue of a saturated non-resonant lattice.

Corollary 1.6.13. *Let L be a saturated non-resonant lattice. Then*

$$\mathrm{M}\cong\exp(-2\pi\mathrm{i}\,\mathrm{res}_L).$$

Proof: Since L is non-resonant, the filtration F^L resp. $\widetilde{F}^L$ is trivial. Then the claim follows from proposition 1.6.12. □

1.6.2 Basis representations

In this subsection, we consider basis representations of the operator ∂_t resp. t with respect to $\mathbb{C}\{t\}$ resp. $\mathbb{C}\{\{s\}\}$-bases of lattices. Since ∂_t resp. t is a differential operator, its basis representation with respect to a basis of the lattice is determined by a matrix.

Let L be a $\mathbb{C}\{t\}$-lattice. By Nakayama's lemma, a $\mathbb{C}$-section

$$L \underset{\tau}{\overset{\curvearrowleft}{\twoheadrightarrow}} L/tL$$

of the canonical projection defines a $\mathbb{C}\{t\}$-isomorphism

$$\mathbb{C}\{t\} \otimes_{\mathbb{C}} L/tL \xrightarrow[\sim]{\widetilde{\tau}} L$$

extending to a $\mathbb{C}\{t\}[t^{-1}]$-isomorphism

$$\mathbb{C}\{t\}[t^{-1}] \otimes_{\mathbb{C}} L/tL \xrightarrow[\sim]{\widetilde{\tau}} G\,.$$

We define the $\mathbb{C}$-homomorphism B^τ by the diagram

$$\begin{array}{ccc} L & \xrightarrow{\partial_t} & G \\ {\scriptstyle\tau}\uparrow & & \uparrow{\scriptstyle\widetilde{\tau}}\,{\scriptstyle\sim} \\ L/tL & \xrightarrow{B^\tau} & \mathbb{C}\{t\}[t^{-1}] \otimes_{\mathbb{C}} L/tL \end{array}$$

and extend it to a $\mathbb{C}\{t\}$-homomorphism

$$\mathbb{C}\{t\} \otimes_{\mathbb{C}} L/tL \xrightarrow{\widetilde{B}^\tau} \mathbb{C}\{t\}[t^{-1}] \otimes_{\mathbb{C}} L/tL\,.$$

By lemma 1.6.3, there is a $k \geq 0$ such that $t^k\partial_t L \subset L$ and hence

$$\operatorname{im} B^\tau \subset t^{-k}\mathbb{C}\{t\} \otimes L/tL.$$

Definition 1.6.14. We call

$$B^\tau = \sum_{j=-k}^{\infty} t^j B_j^\tau$$

the τ-**representation** of ∂_t. If τ is composed with a $\mathbb{C}$-basis $\mathbb{C}^\mu \cong_{\mathbb{C}} L/tL$ then we call B^τ the τ-**matrix** of ∂_t.

Proposition 1.6.15.

$$\partial_t \circ \widetilde{\tau} = \widetilde{\tau} \circ \big(\widetilde{B}^\tau + \partial_t \otimes 1\big)$$

Proof: For $g \otimes v \in \mathbb{C}\{t\}[t^{-1}] \otimes_{\mathbb{C}} L/tL$,

$$\begin{aligned} \partial_t\widetilde{\tau}(g \otimes v) &= \partial_t(g\tau(v)) \\ &= g\partial_t\tau(v) + \partial_t(g)\tau(v) \\ &= g\widetilde{\tau}B^\tau(v) + \partial_t(g)\tau(v) \\ &= g\widetilde{\tau}\widetilde{B}^\tau(1 \otimes v) + \widetilde{\tau}\big(\partial_t(g) \otimes v\big) \\ &= \widetilde{\tau}\big(\widetilde{B}^\tau(g \otimes v) + \partial_t(g) \otimes v\big) \\ &= \widetilde{\tau} \circ \big(\widetilde{B}^\tau + \partial_t \otimes 1\big)(g \otimes v). \end{aligned}$$

□

Remark 1.6.16.

1. If τ is the canonical $\mathbb{C}$-section
$$\tau : V^\alpha/tV^\alpha \cong \bigoplus_{\alpha\le\beta<\alpha+1} C^\beta \hookrightarrow V^\alpha$$
then $B^\tau = t^{-1}B^\tau_{-1}$ and, by corollary 1.6.13, $\mathrm{M} \cong \exp(-2\pi\mathrm{i}B^\tau_{-1})$.

2. Note that
$$H''/t^K H'' \cong \Omega^{n+1}/f^K\Omega^{n+1} + \mathrm{d}f \wedge \Omega^{n+1}.$$
By theorem 1.5.2, for any K, there is an $N(K)$ such that
$$\langle \underline{x} \rangle^{N(K)}\Omega^{n+1} \subset f^K\Omega^{n+1} + \mathrm{d}f \wedge \Omega^{n+1}.$$
Hence, one can compute a $\mathbb{C}$-section τ of the canonical projection
$$H'' \overset{\tau}{\underset{\longleftarrow}{\twoheadrightarrow}} H''/tH''$$
by linear algebra. By proposition 1.4.8 and corollary 1.4.9, one can compute the τ-representation
$$B^\tau = \sum_{j=-\kappa}^{\infty} t^j B^\tau_j$$
of ∂_t up to any degree in t by linear algebra. This method is part of Brieskorn's algorithm [Bri70, 3.5] to compute the complex monodromy.

Let L be a $\mathbb{C}\{t\}$- and $\mathbb{C}\{\{s\}\}$-lattice. By Nakayama's lemma, a $\mathbb{C}$-section
$$L \overset{\sigma}{\underset{\longleftarrow}{\twoheadrightarrow}} L/sL$$
of the canonical projection defines a $\mathbb{C}\{\{s\}\}$-isomorphism
$$\mathbb{C}\{\{s\}\} \otimes_{\mathbb{C}} L/sL \xrightarrow[\sim]{\widetilde{\sigma}} L\,.$$
We define the $\mathbb{C}$-homomorphism A^σ by the diagram
$$\begin{array}{ccc} L & \xrightarrow{\quad t \quad} & L \\ \sigma\uparrow & & \uparrow\widetilde{\sigma} \\ L/sL & \xrightarrow{A^\sigma} & \mathbb{C}\{\{s\}\} \otimes_{\mathbb{C}} L/sL \end{array}$$
and extend it to a $\mathbb{C}\{\{s\}\}$-endomorphism
$$\mathbb{C}\{\{s\}\} \otimes_{\mathbb{C}} L/sL \xrightarrow{\widetilde{A}^\sigma} \mathbb{C}\{\{s\}\} \otimes_{\mathbb{C}} L/sL\,.$$

Definition 1.6.17. We call

$$A^\sigma = \sum_{j=0}^{\infty} s^j A_j^\sigma$$

the **σ-representation** of t. If σ is composed with a $\mathbb{C}$-basis $\mathbb{C}^\mu \cong_{\mathbb{C}} L/sL$ then we call A^σ the **σ-matrix** of t.

Proposition 1.6.18.

$$t \circ \widetilde{\sigma} = \widetilde{\sigma} \circ \left(\widetilde{A}^\sigma + (s^2\partial_s) \otimes 1\right)$$

Proof: This follows from lemma 1.3.16 as in the proof of proposition 1.6.15.

□

Remark 1.6.19.

1. If $\alpha > -1$ and σ is the canonical $\mathbb{C}$-section

   $$\sigma : V^\alpha/tV^\alpha \cong \bigoplus_{\alpha\leq\beta<\alpha+1} C^\beta \hookrightarrow V^\alpha$$

   then $A^\sigma = sA_1^\sigma$ and, by corollary 1.6.13, $\mathrm{M} \cong \exp(-2\pi \mathrm{i} A_1^\sigma)$.

2. Note that, by proposition 1.4.6.1,

   $$H''/sH'' \cong \Omega.$$

   By algorithm 2.2.11 based on proposition 1.5.6, one can compute a $\mathbb{C}$-section σ of the canonical projection

   $$H'' \twoheadrightarrow H''/sH'' \quad (\sigma : H''/sH'' \to H'')$$

   and the σ-representation

   $$A^\sigma = \sum_{j=0}^{\infty} s^j A_j^\sigma$$

   of t up to any degree in s.

1.7 Mixed Hodge structure

In this section, we introduce the mixed Hodge structure on the cohomology of the Milnor fibre. The nilpotent operator N defines the weight filtration and the Brieskorn lattice the Hodge filtration on the Gauß-Manin connection. The weight and Hodge filtration define a mixed Hodge structure on the cohomology of the Milnor fibre. The results in this section can be found in [Del72, Ste76, Var82a, SS85].

1.7.1 Weight and Hodge filtrations

In this subsection, we define the weight and Hodge filtrations. The weight filtration of the nilpotent operator N defines a t- and s-invariant filtration on the Gauß-Manin connection, the weight filtration. It induces the weight filtration on the cohomology of the Milnor fibre. The Brieskorn lattice is a $\mathbb{C}\{t\}$- and $\mathbb{C}\{\{s\}\}$-lattice and defines a filtration by $\mathbb{C}\{t\}$-lattices and a filtration by $\mathbb{C}\{\{s\}\}$-lattices on G, the Hodge filtrations.

By the theorem 1.2.5, N is a nilpotent operator with

$$\mathrm{N}^{n+1} = 0,$$

$\left(\mathrm{N}|_{H^1_{\mathbb{C}}}\right)^n = 0$, and $\left(\mathrm{N}|_{C^\alpha}\right)^n = 0$ for $\alpha \in \mathbb{Z}$. Note that

$$-2\pi \mathrm{i} \mathrm{N} = \log \mathrm{M}_u \in \mathrm{End}_{\mathbb{Q}}(H_{\mathbb{Q}})$$

is defined over $\mathbb{Q}$.

Lemma 1.7.1. *Let* $\mathrm{N} \in \mathrm{End}(V)$ *be a nilpotent operator of a finite dimensional vectorspace* V *with* $\mathrm{N}^{n+1} = 0$. *Then there is a unique increasing filtration* $W = (W_l)_{l\in\mathbb{Z}}$ *such that* $\mathrm{N}(W_k) \subset W_{k-2}$,

$$0 = W_{-n-1} \subset W_{-n} \subset W_{-n+1} \subset \cdots \subset W_{n-2} \subset W_{n-1} \subset W_n = V,$$

and N^k *induces an isomorphism*

$$W_k/W_{k-1} \xrightarrow[\sim]{\mathrm{N}^k} W_{-k}/W_{-k-1}$$

for all $1 \leq k \leq n$.

Definition 1.7.2.

1. $W[-k]$ is called the **weight filtration** of N centered at k.

2. For $v \in V$,

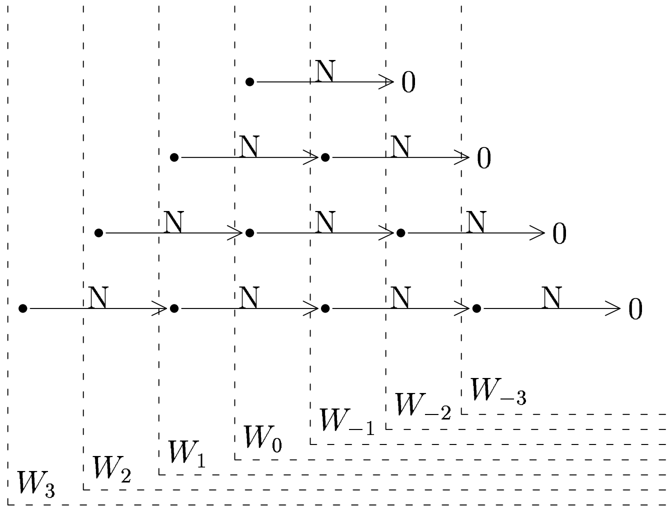
$$\min\{l \in \mathbb{Z} | v \in W_l\}$$

is called the **weight** of v.

Figure 1.6 shows the weight filtration centered at 0. We denote a 1-dimensional subvectorspace by a bullet, the operator N by an arrow, and the subvectorspace W_k by dashed lines. We consider each row of bullets as a Jordan block of N.

Remark 1.7.3.

Figure 1.6: The weight filtration centered at 0



1. A Jordan basis of V with respect to N defines a splitting of the weight filtration. If $v \in V$ is a $\mathbb{C}[\mathrm{N}]$-generator of a Jordan blocks of size $k \times k$ then
$$\mathrm{N}^j v \in W_{k-1-2j} \backslash W_{k-2j}.$$
2. The weight filtration of N determines the Jordan normal form of N. The number of Jordan blocks of N of size $k \times k$ is
$$\dim_{\mathbb{C}} \mathrm{gr}^W_{k-1} V - \sum_{j>0} \dim_{\mathbb{C}} \mathrm{gr}^W_{k-1+2j} V.$$

Lemma 1.7.4.

1. *t induces a filtered isomorphism*
$$(C^\alpha, WC^\alpha) \xrightarrow[\sim]{t} (C^{\alpha+1}, WC^{\alpha+1}) .$$
2. *For $\alpha > -1$ or $\alpha \notin \mathbb{Z}$, s induces a filtered isomorphism*
$$(C^\alpha, WC^\alpha) \xrightarrow[\sim]{s} (C^{\alpha+1}, WC^{\alpha+1}) .$$

Proof:

1. This follows from corollary 1.3.5.1.

2. This follows from corollary 1.3.5.2.

□

Definition 1.7.5.

1. The **weight filtration** $W = (W_k)_{k\in\mathbb{Z}}$ on G is the increasing filtration
$$W := \bigoplus_{-1<\alpha\leq 0} \mathbb{C}\{t\}[t^{-1}]W[-n]C^\alpha$$
by $\mathbb{C}\{t\}[t^{-1}]$-vectorspaces.

2. The **weight filtration** $W = (W_k)_{k\in\mathbb{Z}}$ on $H_\mathbb{Q}$ is the increasing filtration
$$W := W[-n-1]H^1_\mathbb{Q} \oplus W[-n]H^{\neq 1}_\mathbb{Q}$$
by $\mathbb{Q}$-vectorspaces.

Remark 1.7.6.

1. By lemma 1.7.4.2, the weight filtration
$$W\widetilde{G} = \mathbb{C}\{\{s\}\}W[-n]C^0 \oplus \bigoplus_{-1<\alpha<0} \mathbb{C}\{\{s\}\}[s^{-1}]W[-n]C^\alpha$$
is an increasing filtration by $\mathbb{C}\{\{s\}\}$-modules.

2. The weight filtrations on $H_\mathbb{C}$ and on G correspond by
$$\psi_\alpha(WH_\mathbb{C}^{\lambda_\alpha}) = \begin{cases} WC^\alpha, & \alpha \notin \mathbb{Z}, \\ W[-1]C^\alpha, & \alpha \in \mathbb{Z}. \end{cases}$$

3. By theorem 1.2.5, for $\alpha \notin \mathbb{Z}$ and $k \in \mathbb{Z}$,
$$\begin{aligned} W_{-1}C^\alpha &= 0, & W_0C^k &= 0, \\ W_{-1}H_\mathbb{Q}^{\lambda_\alpha} &= 0, & W_1H_\mathbb{Q}^{\lambda_k} &= 0, \\ W_{2n}C^\alpha &= C^\alpha, & W_{2n-1}C^k &= C^k, \\ W_{2n}H_\mathbb{Q}^{\lambda_\alpha} &= H_\mathbb{Q}^{\lambda_\alpha}, & W_{2n}H_\mathbb{Q}^{\lambda_k} &= H_\mathbb{Q}^{\lambda_k}. \end{aligned}$$

4. By remark 1.7.3.1 and lemma 1.7.4, the canonical splitting of the V-filtration can be refined to a simultaneous splitting of the V- and weight filtration on G. In particular,
$$\mathrm{gr}_V \mathrm{gr}^W = \mathrm{gr}^W \mathrm{gr}_V .$$

By proposition 1.4.5.1 and 1.4.11.1, H'' is a $\mathbb{C}\{t\}$- and $\mathbb{C}\{\{s\}\}$-lattice and defines a filtration $F^{H''}$ by $\mathbb{C}\{t\}$-lattices and a filtration $\widetilde{F}^{H''}$ by $\mathbb{C}\{\{s\}\}$-lattices on G.

Definition 1.7.7.

1. The increasing filtration $F := F^{H''}$ on G by $\mathbb{C}\{t\}$-modules

$$F_p = F^{n-p} := t^{-p}H''$$

   is called **Varchenko's Hodge filtration**. The filtration F was defined by A.N. Varchenko [Var82a, 1.2].

2. The increasing filtration $\widetilde{F} := \widetilde{F}^{H''}$ on $\widetilde{G}$ by $\mathbb{C}\{\{s\}\}$-modules

$$\widetilde{F}_p = \widetilde{F}^{n-p} := \widetilde{G} \cap s^{-p}H''$$

   is called **Steenbrink's Hodge filtration**. The filtration $\widetilde{F}$ was defined by J. Scherk and J.H.M. Steenbrink [SS85, Lem. 3.4].

Figure 1.7 shows the Hodge filtration on $\mathbb{C}\{t\}[t^{-1}]C^\alpha$.

The Hodge filtrations on the Gauß-Manin connection induce filtrations on the cohomology of the Milnor fibre via the isomorphism (1.6.1).

Definition 1.7.8. Varchenko's resp. **Steenbrink's Hodge filtration** F resp. $\widetilde{F}$ on $H_\mathbb{C}$ is defined by

$$F_pH_\mathbb{C} := \bigoplus_{-1<\alpha\leq 0} \psi_\alpha^{-1}(F_pC^\alpha),$$

$$\text{resp. } \widetilde{F}_pH_\mathbb{C} := \bigoplus_{-1<\alpha\leq 0} \psi_\alpha^{-1}(\widetilde{F}_pC^\alpha).$$

The Hodge filtrations on the cohomology of the Milnor fibre are not equal but coincide on the graded parts of the weight filtration.

Proposition 1.7.9. *For* $-1 < \alpha \leq 0$*,*

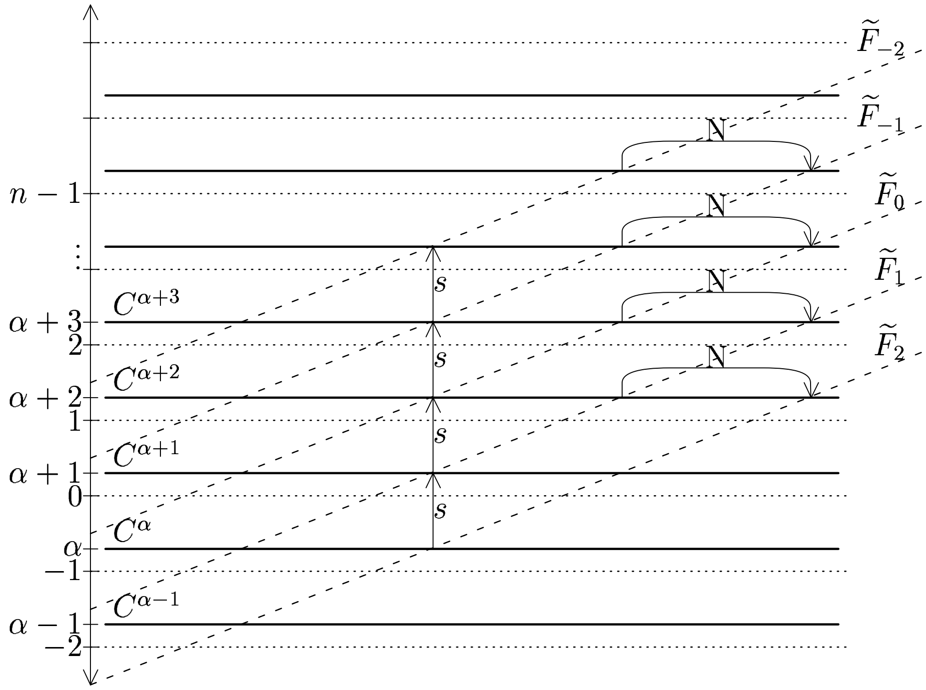
$$\prod_{q=1}^{p}(\alpha + q + \mathrm{N})F_pH_\mathbb{C}^{\lambda_\alpha} = \widetilde{F}_pH_\mathbb{C}^{\lambda_\alpha}.$$

In particular,

$$\widetilde{F}\operatorname{gr}^W H_\mathbb{C} = F\operatorname{gr}^W H_\mathbb{C}.$$

Proof: This follows from lemma 1.6.11.1. □

Figure 1.7: Steenbrink's Hodge filtration on $\mathbb{C}\{t\}[t^{-1}]C^{\alpha}$



1.7.2 Mixed Hodge structures

In this subsection, we introduce (mixed) Hodge structures and strict morphisms. A mixed Hodge structure consists of a $\mathbb{Q}$-vectorspace with a weight filtration and a Hodge filtration defined over $\mathbb{C}$ which is opposite to its conjugate shifted by the weight on the graded parts of the weight filtration. Since there is a simultaneous splitting of the weight and Hodge filtration, a morphism of mixed Hodge structures is strict. The weight filtration defined by N and Varchenko's resp. Steenbrink's Hodge filtration define a mixed Hodge structure on the cohomology of the Milnor fibre and N is a morphism of mixed Hodge structures.

Definition 1.7.10. Let (H, F) be a filtered vectorspace and let $\mathrm{N} \in \mathrm{End}(H)$ be an endomorphism.

1. We call N of **F-type** k if $\mathrm{N}(F) \subset F[k]$, that is, $\mathrm{N}(F^p) \subset F^{p+k}$ for all p.

2. We call N F-**strict** if there is a splitting $F^p = I^p \oplus F^{p+1}$ of F such that $\mathrm{N}(I) \subset I[k]$, that is, $\mathrm{N}(I^p) \subset I^{p+k}$ for all p. In this case, we call I an N-splitting of F.

Remark 1.7.11. Let H be a finite dimensional vectorspace, F an increasing filtration on H, and $\mathrm{N} \in \mathrm{End}(H)$ an F-strict endomorphism type $k \geq 0$. For $k = 0$, one can compute an N-splitting of F by computing the quotients $F_p/F_{p-1} \cong I_p \subset F_p$ of vectorspaces. For $k = 1$, there is an N-splitting I of F, that is,

$$F_p = \bigoplus_{q=0}^{p} I_q,$$
$$\mathrm{N}(I_p) \subset I_{p+1}$$

and hence

$$F_p \Big/ \bigoplus_{q=0}^{p} \mathrm{N}^q(F_0) = \bigoplus_{q=1}^{p} I_q/\mathrm{N}^q(F_0),$$
$$\mathrm{N}\big(I_q/\mathrm{N}^q(F_0)\big) \subset I_{q+1}/\mathrm{N}^{q+1}(F_0).$$

By induction on the length of F, this implies that one can compute an N-splitting of F by computing images of linear maps and quotients of vectorspaces. If $k \geq 2$ then (H, F, N) is a direct sum with k summands and the above argument applies simultaneously to each of them.

Definition 1.7.12. For $i = 1, 2$, let (H_i, F_i) be a filtered vectorspace and let $\mathrm{N} \in \mathrm{Hom}(H_1, H_2)$ be a homomorphism.

1. We call N of (F_1, F_2)-**type** k if $\mathrm{N}(F_1) \subset F_2[k]$, that is, $\mathrm{N}(F_1^p) \subset F_2^{p+k}$ for all p.

2. We call N (F_1, F_2)-**strict** if, for $i = 1, 2$, there is a splitting $F_i^p = I_i^p \oplus F_i^{p+1}$ of F_i such that $\mathrm{N}(I_1) \subset I_2[k]$, that is, $\mathrm{N}(I_1^p) \subset I_2^{p+k}$ for all p. In this case, we call (I_1, I_2) an N-splitting of (F_1, F_2).

For a $\mathbb{Q}$-vectorspace H and a field extension $\mathbb{Q} \subset K$, we denote by

$$H_K := H \otimes_{\mathbb{Q}} K$$

the corresponding extension of scalars.

Definition 1.7.13. Let H be a finite dimensional $\mathbb{Q}$-vectorspace.

1. A **Hodge structure** of weight $l \in \mathbb{Z}$ on H is given by a decreasing **Hodge filtration** F on $H_\mathbb{C}$ such that
$$H_\mathbb{C} = F^p \oplus \overline{F^{l-p+1}} \tag{1.7.1}$$
for all p.

2. For $i = 1, 2$, let (H_i, F_i) be a Hodge structure of weight l_i. A **morphism**
$$(H_1, F_1) \xrightarrow{\mathrm{N}} (H_2, F_2)$$
of Hodge structures of type r is a $\mathbb{Q}$-homomorphism of (F_1, F_2)-type r, that is
$$\mathrm{N}(F_1) \subset F_2[r] \tag{1.7.2}$$

Remark 1.7.14. A Hodge filtration F on $H_\mathbb{C}$ of weight l has a canonical splitting
$$H^{p,l-p} := F^p \cap \overline{F^{l-p}} \cong F^p / F^{p+1}.$$
Then (1.7.1) is equivalent to $H^{p,q} = \overline{H^{q,p}}$ and (1.7.2) to $\mathrm{N}(H_1^{p,q}) \subset H_2^{p+r,q-r}$. In particular, a morphism of Hodge structures is strict with respect to the Hodge filtration. Note that
$$H_\mathbb{C} = \bigoplus_p H^{p,l-p},$$
$$F^p = \bigoplus_{p \leq q} H^{q,l-q}.$$

Definition 1.7.15. Let H be a finite dimensional $\mathbb{Q}$-vectorspace.

1. A **mixed Hodge structure** on H consists of an increasing **weight filtration** W on H and a decreasing **Hodge filtration** F on $H_\mathbb{C}$ such that $F \operatorname{gr}_l^{W_\mathbb{C}} H_\mathbb{C}$ defines a Hodge structure of weight l on $\operatorname{gr}_l^W H$.

2. For $i = 1, 2$, let (H_i, W_i, F_i) be a mixed Hodge structure. A **morphism**
$$(H_1, W_1, F_1) \xrightarrow{\mathrm{N}} (H_2, W_2, F_2)$$
of mixed Hodge structures of type $(r, k - r)$ is a $\mathbb{Q}$-homomorphism of (F_1, F_2)-type r and (W_1, W_2)-type m, that is
$$\mathrm{N}(F_1) \subset F_2[r],$$
$$\mathrm{N}(W_1) \subset W_2[k].$$

There is a generalization of remark 1.7.14 for mixed Hodge structures.

Proposition 1.7.16. *Let (H, W, F) a mixed Hodge structure. Then the $\mathbb{C}$-vectorspaces*

$$I^{p,l-p} := (F^p \cap W_l) \cap \left(\overline{F^{l-p}} \cap W_l + \sum_{k>0} \overline{F^{l-p-k}} \cap W_{l-k-1}\right)$$

define a simultaneous splitting

$$F^p = \bigoplus_l \bigoplus_{p \leq q} I^{q,l-q},$$
$$W_l = \bigoplus_{k \leq l} \bigoplus_p I^{p,k-p}$$

of the weight and Hodge filtration. In particular, a morphism of mixed Hodge structures is strict with respect to the weight and Hodge filtration.

Proof: [Del72, Lem. 1.2.8] □

Theorem 1.7.17. *The weight filtration W and the Hodge filtration F resp. $\widetilde{F}$ define a mixed Hodge structure on the cohomology of the Milnor fibre $H_\mathbb{Q}$ and $\log \mathrm{M}_u$ is a morphism of mixed Hodge structures of type $(-1,-1)$.*

Proof: [Var82a, Ste76, SS85] □

Remark 1.7.18.

1. By proposition 1.7.9, the Hodge structures defined by F and $\widetilde{F}$ on $\mathrm{gr}^W H_\mathbb{Q}$ coincide.

2. By theorem 1.7.17 and proposition 1.7.16, N is strict with respect to W, F, and $\widetilde{F}$.

1.8 Hodge numbers and spectral pairs

In this section, we define and relate the Hodge numbers and spectral pairs. The Hodge numbers are defined by the dimensions of the graded parts of the Hodge filtrations on the graded parts of the weight filtration on the cohomology of the Milnor fibre. The dimensions of the graded parts of the V- resp. V- and weight filtration on the Brieskorn lattice define the spectral numbers resp. pairs. The spectral pairs correspond to the Hodge numbers and inherit their symmetries coming from the mixed Hodge structure. The spectral pairs in a family have a semicontinuity property and are constant in a μ-constant family. C. Hertling conjectured a bound for the variance of the spectral numbers which is strict for quasihomogeneous singularities. The results in this section can be found in [AGZV88, Var82b, Ste85, Her01, Sai].

Definition 1.8.1.

1. The **Hodge numbers** are defined by
$$h_\lambda^{p,l-p} := \dim_{\mathbb{C}} \operatorname{gr}_F^p \operatorname{gr}_l^W H_{\mathbb{C}}^\lambda.$$

2. The **spectral numbers** are those $\alpha \in \mathbb{Q}$ with positive multiplicity
$$d^\alpha := \dim_{\mathbb{C}} \operatorname{gr}_V^\alpha \operatorname{gr}_0^F G = \dim_{\mathbb{C}} \operatorname{gr}_V^\alpha \operatorname{gr}_0^{\widetilde{F}} \widetilde{G}$$
and form the **(singularity) spectrum**
$$\operatorname{Sp}(f) = (d^\alpha)_{\alpha\in\mathbb{Q}} \in \mathbb{N}^{\mathbb{Q}}.$$

3. The **spectral pairs** are those $(\alpha, l) \in \mathbb{Q} \times \mathbb{Z}$ with positive multiplicity
$$d_l^\alpha := \dim_{\mathbb{C}} \operatorname{gr}_l^W \operatorname{gr}_V^\alpha \operatorname{gr}_0^F G = \dim_{\mathbb{C}} \operatorname{gr}_l^W \operatorname{gr}_V^\alpha \operatorname{gr}_0^{\widetilde{F}} \widetilde{G}$$
and form the **weighted (singularity) spectrum**
$$\operatorname{Spp}(f) = (d_l^\alpha)_{(\alpha,l)\in\mathbb{Q}\times\mathbb{Z}} \in \mathbb{N}^{\mathbb{Q}\times\mathbb{Z}}.$$

Note that
$$d^\alpha = \sum_l d_l^\alpha.$$

Lemma 1.8.2.

1. *For $p \in \mathbb{Z}$, t^p induces a $\mathbb{C}$-isomorphism*
$$\operatorname{gr}_p^F \operatorname{gr}_V^\alpha \operatorname{gr}_l^W G \xrightarrow[\sim]{t^p} \operatorname{gr}_V^{\alpha+p} \operatorname{gr}_l^W \operatorname{gr}_0^F G\ .$$

2. *For $\alpha > -1$ or $\alpha \notin \mathbb{Z}$ and $p \in \mathbb{Z}$, s^p induces a $\mathbb{C}$-isomorphism*
$$\operatorname{gr}_p^{\widetilde{F}} \operatorname{gr}_V^\alpha \operatorname{gr}_l^W \widetilde{G} \xrightarrow[\sim]{s^p} \operatorname{gr}_V^{\alpha+p} \operatorname{gr}_l^W \operatorname{gr}_0^{\widetilde{F}} \widetilde{G}\ .$$

Proof: This follows from remark 1.7.6.4. □

The spectral pairs determine the complex monodromy and the spectral numbers determine the eigenvalues of the complex monodromy.

Proposition 1.8.3. *Let $(\alpha_i, l_i)_{1\le i\le\mu}$ be the spectral pairs. Then $(\lambda_{\alpha_i}, l_i)_{1\le i\le\mu}$ are the eigenvalues and weights of the monodromy.*

Proof: By lemma 1.8.2,

$$\begin{aligned}
\dim_{\mathbb{C}} \mathrm{gr}_l^W C^\alpha &= \dim_{\mathbb{C}} \mathrm{gr}_V^\alpha \mathrm{gr}_l^W G \\
&= \sum_{p\in\mathbb{Z}} \dim_{\mathbb{C}} \mathrm{gr}_p^F \mathrm{gr}_V^\alpha \mathrm{gr}_l^W G \\
&= \sum_{p\in\mathbb{Z}} \dim_{\mathbb{C}} \mathrm{gr}_V^{\alpha+p} \mathrm{gr}_l^W \mathrm{gr}_0^F G \\
&= \sum_{p\in\mathbb{Z}} d_l^{\alpha+p}.
\end{aligned}$$

Then the claim follows from lemma 1.3.4.4. □

The spectral pairs correspond to the Hodge numbers.

Lemma 1.8.4. *For $-1 < \alpha < 0$ and $p \in \mathbb{Z}$,*

$$\begin{aligned}
d_l^{\alpha+p} &= h_{\lambda_\alpha}^{n-p,l-n+p}, \\
d_l^p &= h_1^{n-p,l+1-n+p}.
\end{aligned}$$

Proof: This follows from lemma 1.8.2. □

The spectral pairs inherit the symmetries of the Hodge numbers coming from the mixed Hodge structure.

Proposition 1.8.5. *The spectral pairs have the three symmetry properties*

$$\begin{aligned}
d_l^\alpha &= d_l^{2n-l-1-\alpha}, \\
d_l^\alpha &= d_{2n-l}^{\alpha-n+l}, \\
d_l^\alpha &= d_{2n-l}^{n-1-\alpha}
\end{aligned}$$

and each of them follows from the other two. In particular, the spectral numbers have the symmetry property

$$d^\alpha = d^{n-1-\alpha}.$$

Proof: By theorem 1.7.17 and lemma 1.8.4, for $-1 < \alpha < 0$,

$$\begin{aligned}
d_l^{\alpha+n-p} = h_{\lambda_\alpha}^{p,l-p} = h_{\overline{\lambda_\alpha}}^{l-p,p} = h_{\lambda_{-1-\alpha}}^{n-(n-l+p),l-n+(n-l+p)} = d_l^{2n-l-1-(\alpha+n-p)}, \\
d_{n+l}^{\alpha+n-p} = h_{\lambda_\alpha}^{p,n+l-p} = h_{\lambda_\alpha}^{p-l,n-p} = h_{\lambda_\alpha}^{n-(n-p+l),-l+(n-p+l)} = d_{n-l}^{\alpha+n-p+l}, \\
d_l^{n-p} = h_1^{p,l+1-p} = h_1^{l+1-p,p} = h_1^{n-(-l-1+n+p),(l+1-n)+(-l-1+n+p)} = d_l^{2n-l-1-(n-p)}, \\
d_{n+l}^{n-p} = h_1^{p,n+l+1-p} = h_1^{p-l,n+1-p} = h_1^{n-(n-p+l),(-l+1)+(n-p+l)} = d_{n-l}^{n-p+l}.
\end{aligned}$$

□

Corollary 1.8.6. *If $\alpha \notin (-1, n)$ or $l \notin [0, 2n]$ or $\alpha \in \mathbb{Z}$ and $l \notin [1, 2n-1]$ then $d_l^\alpha = 0$. In particular,*

$$V^{>-1} \supset H'' \supset V^{n-1}.$$

Proof: By theorem 1.4.10.2, $V^{>-1} \supset H''$ and hence $d_l^\alpha = 0$ for $\alpha \leq -1$. By proposition 1.8.5, this implies that $d_l^\alpha = 0$ for $\alpha \geq n$ and hence $H'' \supset V^{n-1}$. By remark 1.7.6.3, if $l \notin [0, 2n]$ or $\alpha \in \mathbb{Z}$ and $l \notin [1, 2n-1]$ then $d_l^\alpha = 0$. □

The spectral numbers in a family have a semicontinuity property and are constant in a μ-constant family.

Definition 1.8.7.

1. A germ of a holomorphic function

$$(\mathbb{C}^{n+1} \times \mathbb{C}^r, \underline{0}) \xrightarrow{F} (\mathbb{C}, 0)$$

is called an r-parameter **unfolding** of f if $F_{\underline{0}} = f$ where

$$F_{\underline{t}} := F|_{\mathbb{C}^{n+1} \times \{\underline{t}\}}.$$

2. Let

$$(\mathbb{C}^{n+1} \times \mathbb{C}^r, \underline{0}) \xrightarrow{\pi} (\mathbb{C}^r, \underline{0})$$

be the canonical projection. Then the critical space $\mathrm{V}(\underline{\partial}(F))$ of the map (F, π) is called the **critical space** of F.

Let F be a r-parameter unfolding of f. Then the critical space C is an r-dimensional complete intersection and

$$\pi_C := \pi|_C$$

is finite and flat of degree μ. Hence,

$$\mu = \sum_{\underline{x} \in \pi_C^{-1}(\underline{t})} \mu_{\underline{x}}$$

where $\mu_{\underline{x}}$ denotes the Milnor number of $F_{\underline{t}}$ at $\underline{x}$.

Theorem 1.8.8. *If $F_{\underline{t}}(\pi_C^{-1}(\underline{t}))$ is a point then $\pi_C^{-1}(\underline{t})$ is a point.*

Proof: [Gab74, Laz73, Tra73] □

Definition 1.8.9. An unfolding is called **μ-constant** if $\pi_C^{-1}(\underline{t})$ is a point for all $\underline{t}$.

Theorem 1.8.10. *The spectral numbers and the spectral pairs are constant in a μ-constant unfolding.*

Proof: [Var82b] □

Theorem 1.8.11. *The number of spectral numbers contained in an half open interval of length 1 depends upper semicontinuously on the parameters of an unfolding.*

Proof: [Ste85] □

By proposition 1.8.5, the spectral numbers are symmetric around $\frac{n-1}{2}$. C. Hertling conjectured a bound for their variance which is strict for quasihomogeneous singularities.

Definition 1.8.12. Let $\alpha_1 \le \cdots \le \alpha_\mu$ be the spectral numbers. Then

$$\gamma := -\frac{1}{4}\sum_{i=1}^{\mu}\Big(\alpha_i - \frac{n-1}{2}\Big)^2 + \frac{\alpha_\mu - \alpha_1}{48}\mu$$

is called **Hertling's γ-invariant**.

Conjecture 1.8.13 (Hertling's conjecture). $\gamma \ge 0$

Theorem 1.8.14. *Hertling's conjecture holds for quasihomogeneous singularities.*

Proof: [Her01] □

Remark 1.8.15. For a singularity of type $T_{p,q,r}$, the γ-invariant is

$$\gamma = \frac{1}{24}\Big(1 - \frac{1}{p} - \frac{1}{q} - \frac{1}{r}\Big) \ge 0$$

M. Saito expressed the spectral numbers of an irreducible plane curve singularity in terms of its Puiseux pairs and proved Hertling's conjecture in this case.

Theorem 1.8.16. *Hertling's conjecture holds for irreducible plane curve singularities.*

Proof: [Sai] □

1.9 Newton filtration

In this section, we introduce the Newton filtration and relate it to the V-filtration. The Newton polyhedron of the defining power series of the singularity defines a decreasing filtration by $\mathbb{C}\{\underline{x}\}$-modules on $\mathbb{C}\{\underline{x}\}$ and Ω^{n+1}, the Newton filtration. For Newton non-degenerate singularities, the Newton filtration induces the V-filtration on the Brieskorn lattice. This leads to Endrass' algorithm to compute the spectral numbers. The results in this section can be found in [Sai88, VK85].

Definition 1.9.1. Let $f = \sum f_{\underline{\alpha}} \underline{x}^{\underline{\alpha}} \in \mathbb{C}\{\underline{x}\}$.

1. $$\mathrm{supp}(f) := \{\underline{\alpha} \in \mathbb{N}^{n+1} | f_{\underline{\alpha}} \neq 0\}$$
   is called the **support** of f.

2. The convex hull Γ of
   $$\bigcup_{\underline{\alpha} \in \mathrm{supp}(f)} \underline{\alpha} + \mathbb{N}^{n+1}$$
   is called the **Newton polyhedron**.

3. The set of compact faces of Γ is called the **Newton boundary** and we denote, for $\sigma \in \partial\Gamma$, the restriction of f to σ by
   $$f_\sigma := \sum_{\underline{\alpha} \in \sigma} f_{\underline{\alpha}} \underline{x}^{\underline{\alpha}}.$$

4. The singularity $\mathrm{V}(f)$ is called **Newton non-degenerate** if
   $$\mathrm{V}(\underline{\partial}(f_\sigma)) \subset \bigcup_{i=0}^{n} \mathrm{V}(x_i)$$
   for all $\sigma \in \partial\Gamma$.

We identify $\sigma \in \partial\Gamma$ with the linear function

$$\mathbb{Q}^{n+1} \xrightarrow{\sigma} \mathbb{Q}$$

with $\sigma \subset \sigma^{-1}(1)$. By theorem 1.1.2, we may assume that f is **convenient**, that is, for all $0 \leq i \leq n$, there is a k_i such that $x_i^{k_i} \in \mathrm{supp}(f)$.

Definition 1.9.2. The **Newton order**

$$\mathbb{C}\{\underline{x}\} \xrightarrow{\nu} \mathbb{Q}_{\geq 0} \cup \{\infty\}\,,$$
$$\Omega^{n+1} \xrightarrow{\nu} \mathbb{Q}_{\geq 0} \cup \{\infty\}$$

is defined by

$$\begin{aligned}\nu(g) &:= \min\{\sigma(\underline{\alpha})|\sigma \in \partial\Gamma, \underline{x}^{\underline{\alpha}} \in \mathrm{supp}(g)\},\\ \nu(\mathrm{d}\underline{x}) &:= \nu(x_0 \cdots x_n) - 1\end{aligned}$$

for $\underline{g} \in \mathbb{C}\{\underline{x}\}$.

Lemma 1.9.3. *For* $g_1, g_2 \in \mathbb{C}\{\underline{x}\}$,

$$\begin{aligned}\nu(g_1 + g_2) &\geq \min\{\nu(g_1), \nu(g_2)\},\\ \nu(g_1 g_2) &\geq \nu(g_1) + \nu(g_2)\end{aligned}$$

and

$$\nu(g_1 + g_2) = \min\{\nu(g_1), \nu(g_2)\}$$

if $\nu(g_1) = \nu(g_2)$.

Definition 1.9.4. The **Newton filtration** $N = (N^\alpha)_{\alpha\in\mathbb{Q}}$ on $\mathbb{C}\{\underline{x}\}$ and Ω^{n+1} is defined by

$$N^\alpha := \nu^{-1}(\alpha + \mathbb{Q}_{\geq 0}).$$

and induces the Newton filtration on the Jacobian algebra $\mathbb{C}\{\underline{x}\}/\langle\underline{\partial}(f)\rangle$ and on the Brieskorn lattice $H'' = \Omega^{n+1}/\mathrm{d}f \wedge \Omega^{n-1}$.

Theorem 1.9.5. *For a Newton non-degenerate singularity, the Newton filtration coincides with the V-filtration on the Brieskorn lattice.*

Proof: [Sai88, VK85] □

Remark 1.9.6. Based on theorem 1.9.5, S. Endrass [End02] implemented an algorithm to compute the singularity spectrum of a Newton non-degenerate singularity in the computer algebra system SINGULAR [GPS02].

1.10 V-filtration standard bases

In this section, we introduce standard bases with respect to the V-filtration. A t- resp. s-invariant splitting of a refinement of the V-filtration defines the leading term and the higher terms of an element of the Gauß-Manin connection. We consider $\mathbb{C}\{t\}$- resp. $\mathbb{C}\{\{s\}\}$-submodules of the Gauß-Manin connection. The leading module is the module generated by the leading terms of the elements. A standard basis is a set of generators such that their leading terms generate the leading module. It is reduced if its higher terms are contained in the complement of the leading module. We show that the spectral pairs are the orders of a standard basis of the Brieskorn lattice with respect to a splitting of the weight refined V-filtration. We show that M. Saito's basis of the Brieskorn lattice is a reduced standard basis with respect to a Hodge splitting of the V-filtration.

In order to refine the V-filtration, we order the index set $\mathbb{Q} \times \mathbb{Z}$ by the block ordering

$$<_{\mathbb{Q}\times\mathbb{Z}} := (>_{\mathbb{Q}}, >_{\mathbb{Z}})$$

of $>_{\mathbb{Q}}$ and $>_{\mathbb{Z}}$. Let $V^{\alpha,p}$ define a refinement of the V-filtration on G resp. $\widetilde{G}$ compatible with t resp. s, that is,

$$tV^{\alpha,p} = V^{\alpha+1,p}$$
$$\text{resp. } sV^{\alpha,p} = V^{\alpha+1,p}.$$

Let $C^{\alpha,p}$ define a splitting of this refined V-filtration, that is,

$$V^{\alpha,p}C^{\alpha} = \bigoplus_{p\leq q} C^{\alpha,q}$$

compatible with t resp. s, that is,

$$tC^{\alpha,p} = C^{\alpha+1,p}$$
$$\text{resp. } sC^{\alpha,p} = C^{\alpha+1,p}.$$

Remark 1.10.1.

1. By lemma 1.7.4,

$$V^{\alpha,l} := W_{-l}V^{\alpha} + V^{>\alpha} = W_{-l}C^{\alpha} \oplus V^{>\alpha}$$

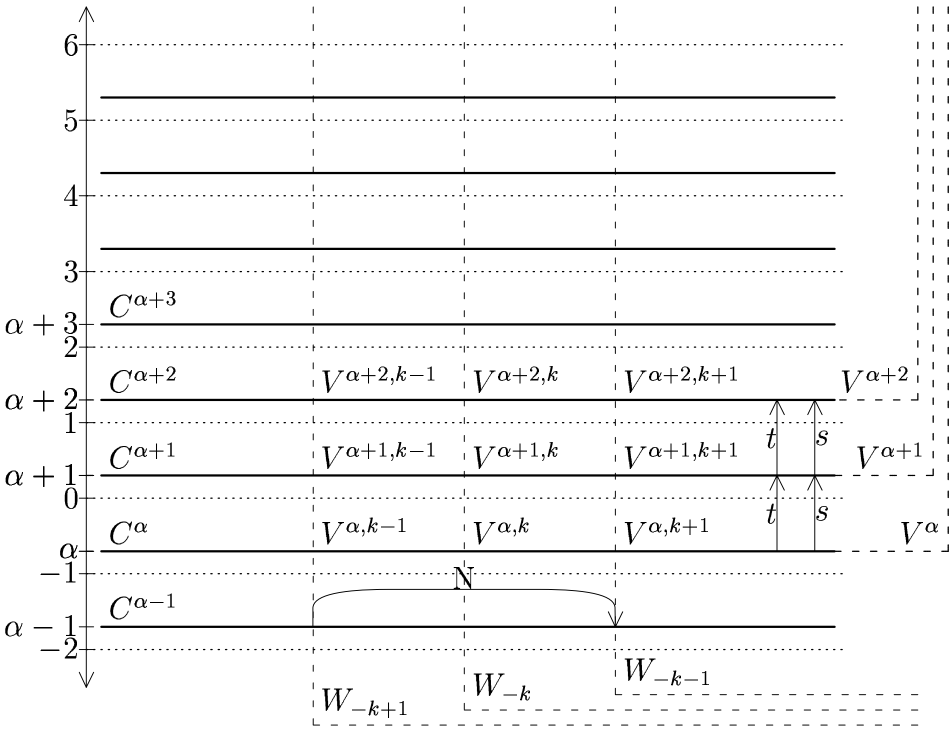
defines a refinement of the V-filtration on G resp. $\widetilde{G}$ compatible with t resp. s. We call it the **weight refinement** of the V-filtration.

Figure 1.8 shows the weight refinement of the V-filtration. We denote the submodules V^{α} and W_k and their intersections $V^{\alpha,k}$ by dashed lines.

2. By definition of the Hodge filtrations,

$$V^{\alpha,p} := F_pV^{\alpha} + V^{>\alpha} = F_pC^{\alpha} \oplus V^{>\alpha}$$
$$\text{resp. } V^{\alpha,p} := \widetilde{F}_pV^{\alpha} + V^{>\alpha} = \widetilde{F}_pC^{\alpha} \oplus V^{>\alpha}$$

Figure 1.8: The weight refinement of the V-filtration on $\mathbb{C}\{t\}[t^{-1}]C^\alpha$



defines a refinement of the V-filtration on G resp. $\widetilde{G}$ compatible with t resp. s, that is,

$$tV^{\alpha,p} = V^{\alpha+p,p-1}$$
$$\text{resp. } sV^{\alpha,p} = V^{\alpha+p,p-1}$$

We call it the **Hodge refinement** of the V-filtration. By proposition 1.7.16 and theorem 1.7.17, there is an N-splitting of the Hodge filtration on C^α compatible with t resp. s, that is,

$$F_pC^\alpha = \bigoplus_{q\leq p} C^{\alpha,q}$$
$$\text{resp. } \widetilde{F}_pC^\alpha = \bigoplus_{q\leq p} C^{\alpha,q}$$

with

$$\mathrm{N}(C^{\alpha,p}) \subset C^{\alpha,p+1}$$

and

$$tC^{\alpha,p} \subset C^{\alpha+1,p-1}$$
$$\text{resp. } sC^{\alpha,p} \subset C^{\alpha+1,p-1}.$$

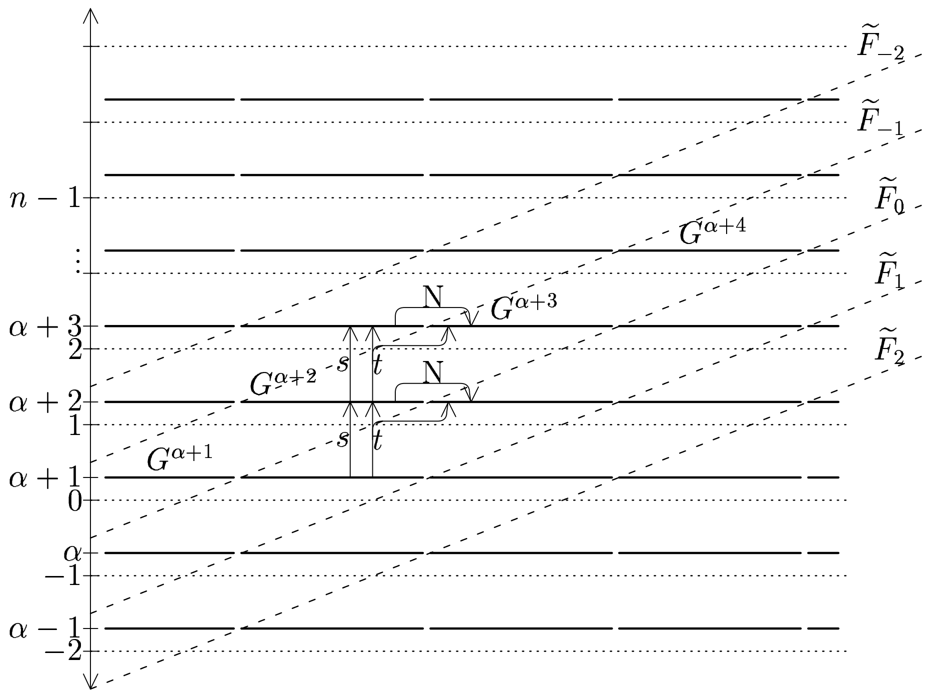
We call it a **Hodge splitting** of the V-filtration. Let

$$G^{\alpha} := C^{\alpha,0}$$

such that $t^p G^{\alpha} = C^{\alpha+p,-p}$ resp. $s^p G^{\alpha} = C^{\alpha+p,-p}$.

Figure 1.9 shows a Hodge splitting of the V-filtration. We denote the finite dimensional $\mathbb{C}$-subvectorspaces $s^p G^{\alpha}$ by horizontal line segments and consider their lengths as the corresponding dimensions.

Figure 1.9: A Hodge splitting of the V-filtration on $\mathbb{C}\{\{s\}\}[s^{-1}]C^{\alpha}$



For $0 \neq v \in G$, there is a maximal (α, p) with $v \in V^{\alpha,p}$ and there are unique $v^{\beta,q} \in C^{\beta,q}$ such that

$$v \equiv \sum_{\alpha \leq \beta < \gamma} \sum_{q} v^{\beta,q} \mod V^{\gamma}$$

for all $\gamma \geq \alpha$.

Definition 1.10.2. Let $v = \sum_\beta \sum_q v^{\beta,q} \in G$. We call $v^{\beta,q}$ the (β, q)**-term**,

$$\mathrm{supp}(v) := \{(\beta, q) | v^{\beta,q} \neq 0\}$$

the **support**, and

$$\deg(v) := \min\{(\beta, q) \mid v \in V^{\beta,q}\}$$

the **order** of v. If $v \neq 0$ then we call

$$\mathrm{lead}(v) := v^{\deg(v)}$$

the **leading term** of v. If the refinement is trivial then we call $\mathrm{supp}_V :=$ supp, $\deg_V :=$ deg resp. $\mathrm{lead}_V :=$ lead the **V-support**, **V-order** resp. **V-leading term**.

Definition 1.10.3. Let $H \subset G$ resp. $H \subset \widetilde{G}$ be a $\mathbb{C}\{t\}$- resp. $\mathbb{C}\{\{s\}\}$-submodule.

1. We call the $\mathbb{C}\{t\}$- resp. $\mathbb{C}\{\{s\}\}$-module

$$\mathrm{lead}(H) := \langle \mathrm{lead}(h) | h \in H \rangle$$

generated by all leading terms of elements of H the **leading module** of H.

2. We call a (minimal) set of elements $\underline{h} \subset H$ such that

$$\langle \mathrm{lead}(\underline{h}) \rangle = \mathrm{lead}(H)$$

a **(minimal) standard basis** of H.

Remark 1.10.4.

1. Since $\mathbb{C}\{t\}$ resp. $\mathbb{C}\{\{s\}\}$ is Noetherian, any $\mathbb{C}\{t\}$-submodule of G resp. $\mathbb{C}\{\{s\}\}$-submodule of $\widetilde{G}$ has a minimal standard basis.

2. A (minimal) standard basis with respect to a finer refinement is a (minimal) standard basis.

3. In remark 1.10.1.2,

$$\mathrm{lead}(H'') = \bigoplus_{\alpha_k} \mathbb{C}\{t\} G^{\alpha_k}$$

$$\text{resp. } \mathrm{lead}(H'') = \bigoplus_{\alpha_k} \mathbb{C}\{\{s\}\} G^{\alpha_k}.$$

Since $\mathbb{C}\{t\}$ resp. $\mathbb{C}\{\{s\}\}$, is a discrete valuation ring, a minimal standard basis is a $\mathbb{C}\{t\}$- resp. $\mathbb{C}\{\{s\}\}$-basis.

Lemma 1.10.5. *Let $H \subset G$ resp. $H \subset \widetilde{G}$ be a $\mathbb{C}\{t\}$- resp. $\mathbb{C}\{\{s\}\}$-submodule and let $\underline{h}$ be a standard basis of H. Then $\langle \underline{h} \rangle = H$. If $\underline{h}$ is a minimal then $\underline{h}$ is a $\mathbb{C}\{t\}$- resp. $\mathbb{C}\{\{s\}\}$-basis of H.*

Proof: We may assume that $H \subset G$ is a $\mathbb{C}\{t\}$-submodule. Since $\langle \operatorname{lead}(\underline{h}) \rangle = H$, there are $\alpha, \beta \in \mathbb{Q}$ such that

$$V^\alpha H \supset H \supset \langle \underline{h} \rangle \supset V^\beta H$$

and hence

$$V^\alpha H / V^\beta H \subset \bigoplus_{\alpha \leq \gamma < \beta} \bigoplus_q C^{\gamma,q}$$

is a finite dimensional $\mathbb{C}$-vectorspace. Since $\langle \operatorname{lead}(\underline{h}) \rangle = H$, this implies that $\langle \underline{h} \rangle = H$. Since $\operatorname{lead}(\underline{h})$ is a minimal set of generators and of $\operatorname{lead}(H)$, $\underline{h}$ is a minimal set of generators of H and hence, by Nakayama's lemma, a $\mathbb{C}\{t\}$-basis. □

Definition 1.10.6. We call a (minimal) standard basis $\underline{h}$ of $\langle \underline{h} \rangle \mathbb{C}\{t\}$ resp. $\langle \underline{h} \rangle \mathbb{C}\{\{s\}\}$ a **(minimal) standard basis**.

Remark 1.10.7. By the choice of a $\mathbb{C}$-basis $\mathbb{C}^\mu \cong H/tH$ resp. $\mathbb{C}^\mu \cong H/sH$, a minimal standard basis of H corresponds to a $\mathbb{C}$-section

$$H \underset{v}{\overset{\pi}{\rightleftarrows}} H/tH$$

$$\text{resp.} \quad H \underset{v}{\overset{\pi}{\rightleftarrows}} H/sH$$

of the canonical projection such that

$$v^{-1}(VH) = \pi(VH)$$

M. Saito [Sai89] calls such a section a **good section**.

Definition 1.10.8. We call a (minimal) standard basis $\underline{h}$ **reduced** if

$$(\underline{h} - \operatorname{lead}(\underline{h}))^{\alpha,p} \notin \operatorname{lead}(H)$$

for all $\alpha \in \mathbb{Q}$ and $p \in \mathbb{Z}$.

Remark 1.10.9. A reduced (minimal) standard basis with respect to a finer refinement is a reduced (minimal) standard basis.

Proposition 1.10.10. *Let $H \subset G$ resp. $H \subset \widetilde{G}$ be a $\mathbb{C}\{t\}$- resp. $\mathbb{C}\{\{s\}\}$-submodule. Then there is a reduced minimal standard basis of H.*

Proof: We may assume that $H \subset G$ is a $\mathbb{C}\{t\}$-submodule. By remark 1.10.9, we may assume that $\dim_{\mathbb{C}} C^{\alpha,p} = 1$. By lemma 1.10.5.2, there is a minimal standard basis $\underline{h}$ of H with decreasingly ordered $(\underline{\alpha}, \underline{p}) := \deg(\underline{h})$. Assume that, for some $k \in \mathbb{Z}$,

$$(\underline{h} - \operatorname{lead}(\underline{h}))^{\alpha_l + j, p_l} = 0, \text{ for } l < k \text{ and } j \geq 0. \tag{1.10.1}$$

Since $\deg(\underline{h})$ is decreasingly ordered, $(h_k)^{\alpha_l + j, p_l} = 0$ for $l < k$ and $j \in \mathbb{Z}$. We multiply h_k with a unit in $\mathbb{C}\{t\}$ such that $(h_k)^{\alpha_k + j, p_k} = 0$ for $j > 0$. For $k < l$, we add h_k times a unit in $\mathbb{C}\{t\}$ to h_l such that $(h_l)^{\alpha_k + j, p_k} = 0$ for $j \geq 0$. Then (1.10.1) holds for $k+1$ and the claim follows by induction. □

The spectral numbers resp. spectral pairs can be read off from a minimal standard basis of the Brieskorn lattice.

Proposition 1.10.11. *Let $\underline{h}$ be a minimal standard basis of H'' with respect to a splitting of a refinement of the V-filtration.*

1. *The V-leading terms* $\operatorname{lead}_V(\underline{h})$ *of* $\underline{h}$ *induce a* $\mathbb{C}$*-basis of* $\operatorname{gr}_V \operatorname{gr}_0^F G$ *resp.* $\operatorname{gr}_V \operatorname{gr}_0^{\widetilde{F}} \widetilde{G}$. *In particular,* $\deg_V(\underline{h})$ *are the spectral numbers.*

2. *Let the refinement be the weight refinement in remark 1.10.1.1. Then the leading terms* $\operatorname{lead}(\underline{h})$ *of* $\underline{h}$ *induce a* $\mathbb{C}$*-basis of* $\operatorname{gr}^W \operatorname{gr}_V \operatorname{gr}_0^F G$ *resp.* $\operatorname{gr}^W \operatorname{gr}_V \operatorname{gr}_0^{\widetilde{F}} \widetilde{G}$. *In particular,* $\deg(\underline{h})$ *are the spectral pairs.*

Proof:

1. For any splitting of the V-filtration,

$$\begin{aligned}
\operatorname{gr}_V^\alpha \operatorname{gr}_0^F G &= \operatorname{gr}_V^\alpha (H''/tH'') \\
&\cong \operatorname{gr}_V^\alpha (H'')/t \operatorname{gr}_V^{\alpha-1}(H'') \\
&\cong \operatorname{lead}_V(H'')^\alpha / t \operatorname{lead}_V(H'')^{\alpha-1}.
\end{aligned}$$

   resp. $\operatorname{gr}_V^\alpha \operatorname{gr}_0^{\widetilde{F}} \widetilde{G} \cong \operatorname{lead}_V(H'')^\alpha / s \operatorname{lead}_V(H'')^{\alpha-1}$.

2. Let the refinement be the weight refinement in remark 1.10.1.1. Then

$$\begin{aligned}
\operatorname{gr}_l^W \operatorname{gr}_V^\alpha \operatorname{gr}_0^F G &= \operatorname{gr}_V^{\alpha,l} (H''/tH'') \\
&\cong \operatorname{gr}_V^{\alpha,l}(H'')/t \operatorname{gr}_V^{\alpha-1,l}(H'') \\
&\cong \operatorname{lead}(H'')^{\alpha,l} / t \operatorname{lead}(H'')^{\alpha-1,l}.
\end{aligned}$$

   resp. $\operatorname{gr}_l^W \operatorname{gr}_V^\alpha \operatorname{gr}_0^{\widetilde{F}} \widetilde{G} \cong \operatorname{lead}(H'')^{\alpha,l} / s \operatorname{lead}(H'')^{\alpha-1,l}$.

□

M. Saito [Sai89] constructed a good $\mathbb{C}$-section

$$H'' \underset{}{\overset{v}{\twoheadrightarrow}} H''/sH''$$

of the canonical projection such that

$$tv = vA_0 + svA_1$$

for some $A_0, A_1 \in \mathrm{End}_{\mathbb{C}}(H''/sH'')$. By the choice of a $\mathbb{C}$-basis of $H''/sH'' \cong \mathbb{C}^\mu$, such a section corresponds to reduced minimal standard basis of the Brieskorn lattice with respect to a Hodge splitting of the V-filtration.

Figure 1.10 shows M. Saito's $\mathbb{C}\{\{s\}\}$-basis $\underline{h} = (h_i)_{1\leq i\leq\mu}$ of H''. We denote the finite dimensional $\mathbb{C}$-subvectorspaces $C^{\alpha,p}$ defining a Hodge splitting by horizontal line segments and consider their lengths as the corresponding dimensions. The support $\mathrm{supp}(h_i)$ of h_i is the set of indices (α, p) of $C^{\alpha,p}$ in the polygon.

Proposition 1.10.12. *Let $\underline{h}$ be a reduced minimal standard basis of H'' with respect to a Hodge splitting as in remark 1.10.1.2.*

1. *The $\underline{h}$-matrix of t has the form*

$$A^{\underline{h}} = A_0^{\underline{h}} + sA_1^{\underline{h}}$$

2. *$A_1^{\underline{h}}$ is semisimple with eigenvalues the spectral numbers added by one.*

3. *$\mathrm{gr}_V(A_0^{\underline{h}})$ can be identified with* N.

Proof: Let $(\underline{\alpha}, \underline{p}) = \deg(\underline{h})$. Then, by proposition 1.10.11.1, $\underline{\alpha}$ are the spectral numbers. By remark 1.10.4.2,

$$\mathrm{lead}(H'') = \bigoplus_{\alpha_k} \mathbb{C}\{\{s\}\}G^{\alpha_k}$$

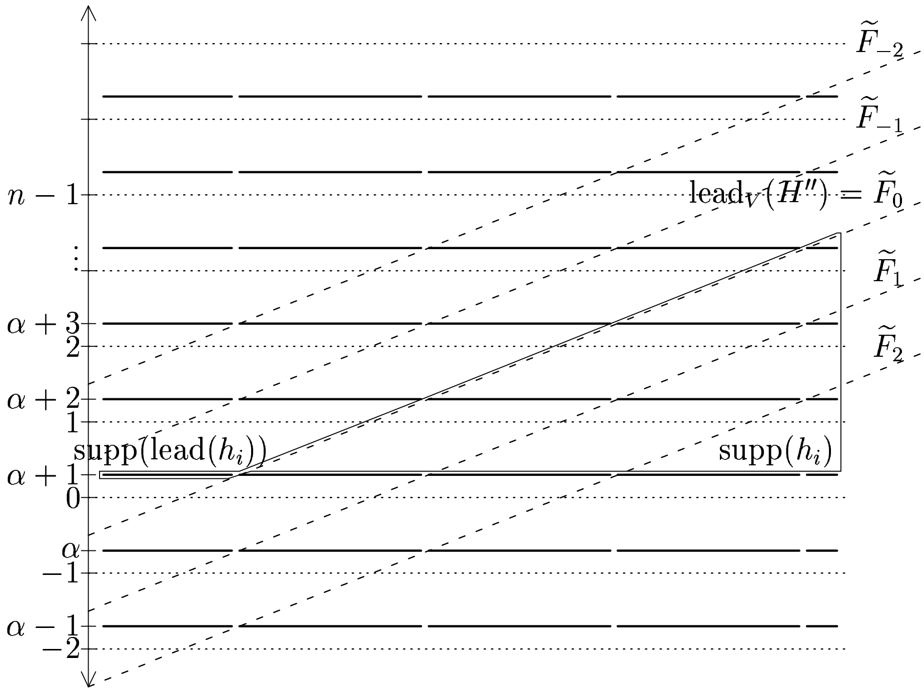
and hence $\underline{p} = \underline{0}$. Since $\underline{h}$ is reduced,

$$h_k \in G^{\alpha_k} \oplus \bigoplus_{\alpha_k < \alpha_l + j < \alpha_l} s^j G^{\alpha_l}.$$

Since $t = (\alpha + 1)s + s\mathrm{N}$ on C^α and $s\mathrm{N}(G^\alpha) \subset G^{\alpha+1}$, this implies that

$$th_k - (\alpha_k + 1)sh_k \in s\mathrm{N}(\mathrm{lead}(h_k)) + \bigoplus_{\alpha_k + 1 < \alpha_l + j \leq \alpha_l} s^j G^{\alpha_l}.$$

Figure 1.10: M. Saito's $\mathbb{C}\{\{s\}\}$-basis $\underline{h} = (h_i)_{1\leq i\leq\mu}$ of H'' on $\mathbb{C}\{t\}[t^{-1}]C^\alpha$



Then there are $a_k^l \in \mathbb{C}$ such that

$$g_k := th_k - (\alpha_k+1)sh_k - \sum_{\alpha_k+1\leq\alpha_l} a_k^l h_l \in \bigoplus_{\alpha_k+1<\alpha_l+j<\alpha_l} s^j G^{\alpha_l}$$

and

$$s\mathrm{N}(\mathrm{lead}(h_k)) = \sum_{\alpha_k+1=\alpha_l} a_k^l \,\mathrm{lead}(h_l). \tag{1.10.2}$$

Since $g_k \in H''$ and

$$\mathrm{lead}(H'') \cap \bigoplus_{\alpha_k+1<\alpha_l+j<\alpha_l} s^j G^{\alpha_l} = 0,$$

$g_k = 0$ and hence

$$th_k = (\alpha_k+1)sh_k + \sum_{\alpha_k+1\leq\alpha_l} a_k^l h_l. \tag{1.10.3}$$

By (1.10.3), $A_0^{\underline{h}} = (a_k^l)$ and

$$A_1^{\underline{h}} = \begin{pmatrix} \alpha_1 & & \\ & \ddots & \\ & & \alpha_\mu \end{pmatrix}.$$

Let

$$n_k^l := \begin{cases} a_k^l, & \alpha_k + 1 = \alpha_l, \\ 0, & \alpha_k + 1 < \alpha_l. \end{cases}$$

Then, by (1.10.2), $\mathrm{gr}_V(A_0^{\underline{h}}) = (n_k^l)$ is the lead($\underline{h}$)-matrix of $s\mathrm{N}$. □

We denote by min deg resp. max deg the minimal resp. the maximal degree of the components of a vector.

Remark 1.10.13. Let $\underline{h}$ be as in in proposition 1.10.12 and

$$\min \deg_V(\underline{h} - \underline{h}') > \max \deg_V(\underline{h}) + k.$$

Then $\underline{h}'$ is a minimal standard basis and

$$A_j^{\underline{h}} = A_j^{\underline{h}'}$$

for $j \leq k$. In particular, if $\underline{h}$ is correct up to V-order $\max \deg_V(\underline{h}) + 1$ then $A_0^{\underline{h}}$ and $A_1^{\underline{h}}$ are correct.

Chapter 2

Algorithms for the Gauß-Manin connection

In this chapter, we describe algorithmic methods for the Gauß-Manin connection to compute the invariants in chapter 1.

In section 2.1, we develop a theory of standard bases over formal power series rings. In section 2.2, we show that the formal Brieskorn lattice is a special case of a formal differential deformation and apply this theory to obtain a normal form algorithm for the formal Brieskorn lattice. In the following sections, we combine the description of the invariants in terms of standard bases with respect to the V-filtration in section 1.10 with the algorithmic methods from section 2.1 and 2.2. This leads to algorithms to compute the complex monodromy, the spectral pairs, and the (t,s)-module structure of the Brieskorn lattice in form of M. Saito's matrices A_0 and A_1.

An implementation [Sch02b] of these algorithms in the computer algebra system SINGULAR [GPS02] is documented in appendix A.

2.1 Formal standard bases

In this section, we develop a theory of standard bases over formal power series rings based on the idea of monomial orderings [Buc65, Buc85, GP96].

In section 1.10, a splitting of a refinement of the V-filtration defines the leading term of an element of the Gauß-Manin connection and hence the notion of a standard basis of a submodule. A lattice in the Gauß-Manin connection is a free module over a power series ring and its completion over a formal power series ring. In this section, we consider free modules over formal power series rings. A local monomial ordering defines a filtration by submodules with a splitting defined by the monomials. As in section 1.10, this defines the leading term of an element and hence the notion of a standard basis of a submodule. For global monomial orderings on a polynomial ring, Buchberger's algorithm [GP96] computes normal forms and standard bases. In general, it does not terminate for local monomial orderings on a formal power series ring. We describe a normal form and standard basis algorithm based on Buchberger's algorithm that converges with respect to the adic topology of the power series ring.

We denote by $<_{\mathbb{N}}$ resp. $<_{lex}$ the partial resp. the lexicographical ordering on $\mathbb{N}^n$ and by $\underline{e}^i = (\delta_{i,j})_j$ the i-th unit vector.

Definition 2.1.1.

1. A total ordering $<$ on $\mathbb{N}^n$ is called a **semigroup ordering** if

$$\underline{\alpha} < \underline{\beta} \ \Rightarrow\ \underline{\alpha} + \underline{\gamma} < \underline{\beta} + \underline{\gamma}.$$

   It is called **global** resp. **local** if $\underline{0} = \min \mathbb{N}^n$ resp. if $\underline{0} = \max \mathbb{N}^n$. If $\mathbb{N}^n$ is identified with a set of monomials $\{\underline{x}^{\underline{\alpha}}\}_{\underline{\alpha}\in\mathbb{N}}$ then a semigroup ordering is called a **monomial ordering**. If $<_m$ is a semigroup ordering on $\mathbb{N}^m$ and $<_n$ is a semigroup ordering on $\mathbb{N}^n$ then the **block ordering** $(<_m, <_n)$ of $<_m$ and $<_n$ defined by

$$(\underline{\alpha}, \underline{\beta})(<_m, <_n)(\underline{\alpha}', \underline{\beta}') \ :\Leftrightarrow\ \underline{\alpha} <_m \underline{\alpha}' \vee (\underline{\alpha} = \underline{\alpha}' \wedge \underline{\beta} <_n \underline{\beta}')$$

   is a semigroup ordering on $\mathbb{N}^{m+n}$.

2. A semigroup homomorphism

$$\mathbb{N}^n \xrightarrow{\deg} \mathbb{Q}$$

   is called a **degree** if $\deg(\underline{e}^i) \neq 0$ for all i and $\deg(\underline{e}^i)$ is called the **weight** of $\underline{e}^i$. It is called **global** resp. **local** if $\deg(\underline{e}^i) > 0$ resp. $\deg(\underline{e}^i) < 0$ for all i. If $\deg_m$ is a degree on $\mathbb{N}^m$ and $\deg_n$ is a degree on $\mathbb{N}^n$ then the **block degree** $(\deg_m, \deg_n)$ defined by

$$(\deg_m, \deg_n)(\underline{\alpha}, \underline{\beta}) := \deg_m(\underline{\alpha}) + \deg_n(\underline{\beta})$$

   is a degree on $\mathbb{N}^{m+n}$.

3. Let $<$ be a semigroup ordering and deg a degree on $\mathbb{N}^n$. Then $(<, \deg)$ is called a **degree ordering** if

$$\deg(\underline{\alpha}) < \deg(\underline{\beta}) \ \Rightarrow\ \underline{\alpha} < \underline{\beta}.$$

Remark 2.1.2.

1. A semigroup ordering is global if and only if it is a well-ordering.
2. A degree ordering $(<, \deg)$ is global resp. local if and only if deg is global resp. local.

Using the following lemma, one can define the leading term and the order of a formal power series.

Lemma 2.1.3 (Dickson's lemma). *Let $\emptyset \neq S \subset \mathbb{N}^n$. Then there are $\underline{\alpha}^1, \ldots, \underline{\alpha}^k \in S$ such that*

$$S \subset \bigcup_{j=1}^{k} (\underline{\alpha}^j + \mathbb{N}^n).$$

Proof: The claim holds for $n = 1$. Assume that the claim holds for some n. Let

$$\mathbb{N}^{n+1} \xrightarrow{\pi} \mathbb{N}^n \times \{0\} = \mathbb{N}^n$$

be the canonical projection. By assumption, there are $\underline{\alpha}^1, \ldots, \underline{\alpha}^k$ such that

$$\pi(S) \subset \bigcup_{j=1}^{k} (\pi(\underline{\alpha}^j) + \mathbb{N}^n).$$

Let $N := \max\{\alpha^j_{n+1} | 1 \leq j \leq k\}$. Then

$$S \cap (\mathbb{N}^n \times \mathbb{N}_{\geq N}) \subset \bigcup_{j=1}^{k} (\underline{\alpha}^j + \mathbb{N}^{n+1})$$

and, for $0 \leq j < N$,

$$S \cap (\mathbb{N}^n \times \{j\}) \subset \mathbb{N}^n \times \{j\} = \mathbb{N}^n.$$

Since the claim holds for n, this implies that the claim holds for $n + 1$. □

Proposition 2.1.4. *Let $<$ be a local semigroup ordering on $\mathbb{N}^n$ and $\emptyset \neq S \subset \mathbb{N}^n$. Then $\max_< S$ exists.*

Proof: Since $<$ is a local semigroup ordering,

$$\underline{\alpha} = \max_{<}(\underline{\alpha} + \mathbb{N}^n)$$

and the claim follows from lemma 2.1.3. □

Let $\underline{s} = (s_1, \ldots, s_m)$ and $\underline{x} = (x_1, \ldots, x_n)$.

Definition 2.1.5.

1. We call a global resp. local monomial ordering on $\{\underline{x}^{\underline{\alpha}}\}_{\underline{\alpha}\in\mathbb{N}^n}$ a **(monomial) ordering** on $\mathbb{C}[\underline{x}]$ resp. $\mathbb{C}[\![\underline{x}]\!]$.
2. We call a global resp. local degree on $\{\underline{x}^{\underline{\alpha}}\}_{\underline{\alpha}\in\mathbb{N}}$ a **degree** on $\mathbb{C}[\underline{x}]$ resp. $\mathbb{C}[\![\underline{x}]\!]$.

Definition 2.1.6.

1. For $p \in \mathbb{C}[\![\underline{x}]\!]$,
$$\mathrm{supp}(p) := \{\underline{\alpha} \in \mathbb{N}^n \mid p_{\underline{\alpha}} \neq 0\}$$
is called the **support** of p.
2. Let $<$ be a monomial ordering on $\mathbb{C}[\![\underline{s}]\!]$. For $0 \neq p \in \mathbb{C}[\![\underline{s}, \underline{x}]\!]$,
$$\mathrm{lead}_<(p) := \sum_{\underline{\alpha}=\max_< \pi_{\mathbb{N}^n}(\mathrm{supp}(p))} p_{\underline{\alpha},\underline{\beta}}\underline{s}^{\underline{\alpha}}\underline{x}^{\underline{\beta}}$$
is called the **leading term** of p. We omit the lower index $<$ if there is no ambiguity.
3. Let deg be a degree on $\mathbb{C}[\![\underline{s}]\!]$. For $p \in \mathbb{C}[\![\underline{s}, \underline{x}]\!]$,
$$\deg(p) := \max \deg(\pi_{\mathbb{N}^n}(\mathrm{supp}(p)))$$
is called the **order** of p.

Definition 2.1.7. Let $<$ a monomial ordering on $\mathbb{C}[\![\underline{s}, \underline{x}]\!]$ and $H \subset \mathbb{C}[\![\underline{s}, \underline{x}]\!]$ a $\mathbb{C}[\![\underline{s}]\!]$-module.

1. The $\mathbb{C}[\![\underline{s}]\!]$-module
$$\mathrm{lead}(H) := \langle \mathrm{lead}(h) | h \in H\rangle \mathbb{C}[\![\underline{s}]\!]$$
generated by all leading terms of elements of H is called the **leading module** of H.
2. A (minimal) set of elements $\underline{h} \subset H$ such that
$$\langle \mathrm{lead}(\underline{h})\rangle \mathbb{C}[\![\underline{s}]\!] = \mathrm{lead}(H)$$
is called a **(minimal) standard basis** of H. It is called **reduced** if
$$(\underline{h} - \mathrm{lead}(\underline{h}))_{\underline{\alpha},\underline{\beta}} = 0$$
for all $\underline{s}^{\underline{\alpha}}\underline{x}^{\underline{\beta}} \in \mathrm{lead}(H)$.

Using the following lemma, one can show that a standard basis is a set of generators.

Lemma 2.1.8.

1. *Let*
$$(\mathbb{N}^n, <_{lex}) \xrightarrow{\underline{\alpha}} (\mathbb{N}^n, <_{\mathbb{N}})$$
be a strictly non-decreasing sequence. Then, for all $\underline{\gamma} \in \mathbb{N}^n$*, there is a* $\underline{k} \in \mathbb{N}^n$ *such that* $\underline{\gamma} \leq_{\mathbb{N}} \underline{\alpha}^{\underline{k}}$.

2. *Let* $<$ *be a local semigroup ordering on* $\mathbb{N}^n$ *and*
$$(\mathbb{N}^n, <_{lex}) \xrightarrow{\underline{\alpha}} (\mathbb{N}^n, >)$$
a strictly decreasing sequence. Then, for all $\underline{\gamma} \in \mathbb{N}^n$*, there is a* $\underline{k} \in \mathbb{N}^n$ *such that* $\underline{\gamma} > \underline{\alpha}^{\underline{k}}$.

Proof:

1. For $n = 1$, $<_{lex}=<_{\mathbb{N}}$ and α is a strictly increasing sequence. Hence, the claim holds in this case. Assume that the claim holds for some n but not for $n+1$ and let
$$\mathbb{N}^{n+1} \xrightarrow{\pi} \mathbb{N}^n \times \{0\} = \mathbb{N}^n$$
be the canonical projection. Let
$$(\mathbb{N}^{n+1}, <_{lex}) \xrightarrow{\underline{\alpha}} (\mathbb{N}^{n+1}, <_{\mathbb{N}})$$
is a strictly non-decreasing sequence and $\underline{\gamma} \in \mathbb{N}^{n+1}$. Then $(\underline{\alpha}^{\underline{k},k})_{k\geq 0}$ is unbounded and, by replacing $\underline{\alpha}$ by a subsequence, we may assume that $(\alpha_{n+1}^{\underline{k},k})_{k\geq 0}$ is unbounded for all $\underline{k} \in \mathbb{N}^n$. Since $\underline{\gamma} \not\leq_{\mathbb{N}} \underline{\alpha}^{\underline{k},k}$ for all k, there is a $<_{\mathbb{N}}$-maximal $\underline{\beta}^{\underline{k}} \in \mathbb{N}^n$ such that
$$(\underline{\beta}^{\underline{k}} - \mathbb{N}^n) \times \mathbb{N} \subset \bigcup_{k\geq 0} (\underline{\alpha}^{\underline{k},k} - \mathbb{N}^{n+1})$$
and hence $\underline{\beta}^{\underline{k}} \leq_{\mathbb{N}} \pi(\underline{\alpha}^{\underline{k},k})$ for all $k \geq 0$. Since $\underline{\alpha}$ is strictly non-decreasing,
$$(\mathbb{N}^n, <_{lex}) \xrightarrow{\underline{\beta}} (\mathbb{N}^n, <_{\mathbb{N}})$$
is a strictly non-decreasing sequence. By the assumption, there is a $\underline{k}$ such that $\pi(\underline{\gamma}) \leq_{\mathbb{N}} \underline{\beta}^{\underline{k}}$. Since $(\alpha_{n+1}^{\underline{k},k})_{k\geq 0}$ is unbounded, there is a $k \geq 0$ such that $\gamma_{n+1} \leq \alpha_{n+1}^{\underline{k},k}$ and hence $\underline{\gamma} \leq_{\mathbb{N}} \underline{\alpha}^{\underline{k},k}$. This is a contradiction to the assumption.

2. Since $\underline{\alpha} = \max_<(\underline{\alpha} + \mathbb{N}^n)$ this follows from 1.

□

Proposition 2.1.9. *Let $<$ a monomial ordering on $\mathbb{C}[\![\underline{s}, \underline{x}]\!]$ and $\underline{h} \subset H$ a standard basis of a $\mathbb{C}[\![\underline{s}]\!]$-module $H \subset \mathbb{C}[\![\underline{s}, \underline{x}]\!]$. Then $\langle \underline{h} \rangle \mathbb{C}[\![\underline{s}]\!] = H$.*

Proof: We denote $\underline{0}^n := \underline{0} \in \mathbb{N}^n$. Let $p \in H$ and $p^{\underline{0}^{m+n}} := p$. Assume that for $1 \leq l \leq m + n - 2$, $\underline{k} \in \mathbb{N}^l$, and $k, j \in \mathbb{N}$, $p^{\underline{k},k,j,\underline{0}^{m-l-2}} \in \mathbb{C}[\![\underline{s}, \underline{x}]\!]$ with $p - p^{\underline{k},k,j,\underline{0}^{m-l-2}} \in \langle \underline{h} \rangle \mathbb{C}[\![\underline{s}]\!]$ is already defined. Then $p^{\underline{k},k,j,\underline{0}^{m-l-2}} \in H$ and hence $\mathrm{lead}(p^{\underline{k},k,j,\underline{0}^{m-l-2}}) \in \mathrm{lead}(H)$. Since $\underline{h}$ is a standard basis of H, $\mathrm{lead}(H) = \langle \mathrm{lead}(\underline{h}) \rangle \mathbb{C}[\![\underline{s}]\!]$ and hence there are $\underline{\alpha}^{\underline{k},k,j} \in \mathbb{N}^m$ and $i_{\underline{k},k,j}$ such that

$$\mathrm{lead}(p^{\underline{k},k,j,\underline{0}^{m-l-2}}) = \underline{s}^{\underline{\alpha}^{\underline{k},k,j}} \,\mathrm{lead}(h_{i_{\underline{k},k,j}}).$$

Let

$$p^{\underline{k},k,j+1,\underline{0}^{m-l-2}} := p^{\underline{k},k,j,\underline{0}^{m-l-2}} - \underline{s}^{\underline{\alpha}^{\underline{k},k,j}} h_{i_{\underline{k},k,j}}.$$

Then $\{\mathrm{lead}(p^{\underline{k},k,j,\underline{0}^{m-l-2}})\}_{j \geq 0}$ is a strictly decreasing sequence and

$$p^{\underline{k},k+1,\underline{0}^{m-l-1}} := (p^{\underline{k},k,j,\underline{0}^{m-l-2}})_{j \geq 0} \in \mathbb{C}[\![\underline{s}, \underline{x}]\!]$$

defines an element of $\mathbb{C}[\![\underline{s}, \underline{x}]\!]$ with $p - p^{\underline{k},k+1,\underline{0}^{m-l-1}} \in \langle \underline{h} \rangle \mathbb{C}[\![\underline{s}]\!]$. Then

$$(\mathbb{N}^{m+n}, <_{lex}) \xrightarrow{\mathrm{lead}(p)} \big(\{\underline{s}^{\underline{\alpha}}, \underline{x}^{\underline{\beta}} \mid (\underline{\alpha}, \underline{\beta}) \in \mathbb{N}^{m+n}\}, >\big)$$

is a strictly decreasing sequence and

$$p^{\infty} := (p^{\underline{k}})_{\underline{k} \in \mathbb{N}^{m+n}} \in \mathbb{C}[\![\underline{s}, \underline{x}]\!]$$

defines an element of $p - \mathbb{C}[\![\underline{s}, \underline{x}]\!]$ with $p^{\infty} \in \langle \underline{h} \rangle \mathbb{C}[\![\underline{s}]\!]$. By lemma 2.1.8.2, $p^{\infty} = 0$ and hence $p \in \langle \underline{h} \rangle \mathbb{C}[\![\underline{s}]\!]$. □

Definition 2.1.10. We call a (reduced) (minimal) standard basis $\underline{h}$ of $\langle \underline{h} \rangle \mathbb{C}[\![\underline{s}]\!]$ a **(reduced) (minimal) standard basis**.

For a finitely generated $\mathbb{C}[\![s]\!]$-module, a minimal standard basis is a $\mathbb{C}[\![s]\!]$-basis.

Lemma 2.1.11. *Let $<_s$ be the monomial ordering on $\mathbb{C}[\![s]\!]$, $<_{\underline{x}}$ a monomial ordering on $\mathbb{C}[\![\underline{x}]\!]$, $< := (<_s, <_{\underline{x}})$ the block ordering of $<_s$ and $<_{\underline{x}}$ on $\mathbb{C}[\![s, \underline{x}]\!]$, and $V \subset \mathbb{C}[\![\underline{x}]\!]$ a finite dimensional monomial $\mathbb{C}$-vectorspace. Then a minimal standard basis of a $\mathbb{C}[\![s]\!]$-module $H \subset \mathbb{C}[\![\underline{s}]\!]V$ is a $\mathbb{C}[\![s]\!]$-basis of H.*

Proof: Let $\underline{h}$ be a minimal standard basis of H. Then, by proposition 2.1.9, $\langle\underline{h}\rangle\mathbb{C}[\![s]\!] = H$. Assume that $\underline{h}$ is not a $\mathbb{C}[\![s]\!]$-basis of H. Then there is a $\overline{g} \subset \mathbb{C}[\![s]\!]$ such that $\underline{h}\overline{g} = 0$. By reordering $\overline{g}$ and $\underline{h}$, we may assume that

$$\sum_{j=1}^{k} \operatorname{lead}(g^j) \operatorname{lead}(h_j) = \operatorname{lead}(\underline{h}) \operatorname{lead}(\overline{g}) = 0$$

for some $k \geq 2$. By dividing by a power of s, we may assume that $\operatorname{lead}(g^k) = 1$ and hence

$$\operatorname{lead}(h_k) = -\sum_{j=1}^{k-1} \operatorname{lead}(g^j) \operatorname{lead}(h_j).$$

This is a contradiction to the minimality of the standard basis $\underline{h}$. □

We denote by $\complement$ the complement with respect to the monomial $\mathbb{C}$-basis of $\mathbb{C}[\![\underline{s}, \underline{x}]\!]$.

Definition 2.1.12. A $\mathbb{C}[\![\underline{s}]\!]$**-normal form** resp. a **reduced** $\mathbb{C}[\![\underline{s}]\!]$**-normal form** with respect to $\underline{h} \subset \mathbb{C}[\![\underline{s}, \underline{x}]\!]$ is a map

$$\mathbb{C}[\![\underline{s}, \underline{x}]\!] \xrightarrow{\Phi} \mathbb{C}[\![\underline{s}, \underline{x}]\!]$$

such that

1. $p = \Phi(p) + \underline{h}\overline{a}$ for some $\overline{a} \subset \mathbb{C}[\![\underline{s}]\!]$ with $\operatorname{lead}(p) \geq \operatorname{lead}(h_j a^j)$ for all j,
2. if $\Phi(p) \neq 0$ then $\operatorname{lead}\Phi(p) \in \complement\langle\operatorname{lead}(\underline{h})\rangle\mathbb{C}[\![\underline{s}]\!]$ resp. $\Phi(p) \in \complement\langle\operatorname{lead}(\underline{h})\rangle\mathbb{C}[\![\underline{s}]\!]$.

Lemma 2.1.13. *Let Φ be a $\mathbb{C}[\![\underline{s}]\!]$-normal form with respect to a standard basis $\underline{h} \subset \mathbb{C}[\![\underline{s}, \underline{x}]\!]$.*

1. $\Phi^{-1}(0) = \langle\underline{h}\rangle\mathbb{C}[\![\underline{s}]\!]$.
2. *If* $\operatorname{lead}(p) \in \complement\langle\operatorname{lead}(\underline{h})\rangle\mathbb{C}[\![\underline{s}]\!]$ *then* $\operatorname{lead}\Phi(p) = \operatorname{lead}(p)$.
3. $\operatorname{lead}\Phi$ *is uniquely determined.*

Proof: This follows from definition 2.1.12.

Proposition 2.1.14. *Let Φ be a reduced $\mathbb{C}[\![\underline{s}]\!]$-normal form with respect to a standard basis $\underline{h}$ of a $\mathbb{C}[\![\underline{s}]\!]$-module $H \subset \mathbb{C}[\![\underline{s}, \underline{x}]\!]$. Then Φ induces the $\mathbb{C}$-section*

$$\mathbb{C}[\![\underline{s}, \underline{x}]\!] \overset{\Phi}{\underset{}{\leftarrow\!\!\!\twoheadrightarrow}} \mathbb{C}[\![\underline{s}, \underline{x}]\!]/H$$

of the canonical projection with image $\complement\operatorname{lead}(H)$. In particular, Φ is uniquely determined.

Proof: This follows from proposition 2.1.9, definition 2.1.12, and lemma 2.1.13.1.

A monomial ordering $<$ on $\mathbb{C}[\![\underline{x}]\!]$ defines an increasing filtration $M = (M^{\underline{\alpha}})_{\underline{\alpha}\in\mathbb{N}^n}$ on $\mathbb{C}[\![\underline{x}]\!]$ by $\mathbb{C}[\![\underline{x}]\!]$-submodules

$$M^{\underline{\alpha}} := \bigoplus_{\underline{\alpha}\geq\underline{\beta}} \mathbb{C}\underline{x}^{\underline{\beta}},$$
$$M^{<\underline{\alpha}} := \bigoplus_{\underline{\alpha}>\underline{\beta}} \mathbb{C}\underline{x}^{\underline{\beta}}.$$

The monomials define a canonical splitting of M by

$$M_{\underline{\alpha}}/M_{<\underline{\alpha}} \cong \mathbb{C}\underline{x}^{\underline{\alpha}} \subset M_{\underline{\alpha}}.$$

Buchberger's normal form algorithm [GP96] computes a $\mathbb{C}[\![\underline{s}]\!]$-normal form with respect to $\underline{h}$ if it converges with respect to the $\langle\underline{s},\underline{x}\rangle$-adic topology on $\mathbb{C}[\![\underline{s},\underline{x}]\!]$. For certain monomial orderings on $\mathbb{C}[\![\underline{s},\underline{x}]\!]$, it converges for any $\underline{h} \subset \mathbb{C}[\![\underline{s}]\!]$.

Definition 2.1.15. We call a monomial ordering on $\mathbb{C}[\![\underline{x}]\!]$ **algorithmic** if $(M_{\underline{\alpha}})_{\underline{\alpha}}$ is a basis of the $\langle\underline{x}\rangle$-adic topology and $\dim_{\mathbb{C}} M_{\underline{\alpha}}/M_{\underline{\beta}} < \infty$ for $\underline{\alpha} > \underline{\beta}$.

Remark 2.1.16.

1. Any local degree ordering is algorithmic.

2. For an algorithmic ordering on $\mathbb{C}[\![\underline{s},\underline{x}]\!]$, Buchberger's normal form algorithm [GP96] converges for any $\underline{h} \subset \mathbb{C}[\![\underline{s}]\!]$.

For arbitrary monomial orderings, there is a sufficient condition on the monomial ordering and on $\underline{h} \subset \mathbb{C}[\![\underline{s},\underline{x}]\!]$ for the convergence of Buchberger's normal form algorithm.

Definition 2.1.17. Let $U = (U_k)_{k\leq 0}$ be an increasing filtration on $\mathbb{C}[\![\underline{s},\underline{x}]\!]$ by $\mathbb{C}[\![\underline{s}]\!]$-modules U_k.

1. We call U **$\langle\underline{s},\underline{x}\rangle$-adic** if it is a basis of the $\langle\underline{s},\underline{x}\rangle$-adic topology.

2. Let $<$ be a monomial ordering on $\mathbb{C}[\![\underline{s},\underline{x}]\!]$ and $\underline{h} \subset \mathbb{C}[\![\underline{s},\underline{x}]\!]$. We call U **$(<,\underline{h})$-stable** if $\underline{s}^{\underline{\alpha}}\operatorname{lead}(h_j) \in U_k$ implies that $\underline{s}^{\underline{\alpha}}h_j \in U_k$ for all j and k.

Remark 2.1.18.

1. Let $<$ be an algorithmic ordering and $(\underline{\alpha}^k, \underline{\beta}^k)_{k \leq 0}$ an increasing enumeration of $\mathbb{N}^{m+n}$. Then $U_k := M^{(\underline{\alpha}^k, \underline{\beta}^k)}$ defines a $\langle \underline{s}, \underline{x} \rangle$-adic $\mathbb{C}[\![\underline{s}]\!]$-filtration $U = (U_k)_{k \leq 0}$ which is $(<, \underline{h})$-stable for all $\underline{h} \subset \mathbb{C}[\![\underline{s}, \underline{x}]\!]$.

2. Let $(<_{\underline{s}}, \deg_{\underline{s}})$ be a degree ordering on $\mathbb{C}[\![\underline{s}]\!]$, $<_{\underline{x}}$ a monomial ordering on $\mathbb{C}[\![\underline{x}]\!]$, $<:= (<_{\underline{s}}, <_{\underline{x}})$ the block ordering of $<_{\underline{s}}$ and $<_{\underline{x}}$ on $\mathbb{C}[\![\underline{s}, \underline{x}]\!]$, and $V \subset \mathbb{C}[\![\underline{x}]\!]$ a finite dimensional monomial $\mathbb{C}$-vectorspace. Then there is a $\langle \underline{s}, \underline{x} \rangle$-adic filtration $U = (U_k)_{k \leq 0}$ which is $(<, \underline{h})$-stable for all $\underline{h} \subset \mathbb{C}[\![\underline{s}]\!]V$ and such that $\deg_{\underline{s}}(p) < k$ for all $p \in U_k \cap \mathbb{C}[\![\underline{s}]\!]V$.

The following algorithm is a modification of Buchberger's normal form algorithm for power series rings. For a monomial $\mathbb{C}$-vectorspace $U \subset \mathbb{C}[\![\underline{s}, \underline{x}]\!]$, we denote by

$$\mathbb{C}[\![\underline{s}, \underline{x}]\!] \xrightarrow{\pi_U} U$$

the canonical projection.

Definition 2.1.19. Let $<$ be a monomial ordering on $\mathbb{C}[\![\underline{s}, \underline{x}]\!]$, $\underline{h} \subset \mathbb{C}[\![\underline{s}, \underline{x}]\!]$, and $U = (U_k)_{k \leq 0}$ a $\langle \underline{s}, \underline{x} \rangle$-adic $(<, \underline{h})$-stable filtration.

1. We define a **normal form algorithm** $\mathsf{nf} = (\mathsf{nf}_1, \mathsf{nf}_2, \mathsf{nf}_3)$ by

   proc $\mathsf{nf}(p, \underline{h}, k) \equiv$
   $q := \pi_{U_k}(p)$;
   $r := p - q$;
   if $r \neq 0 \wedge \mathrm{lead}(r) \in \langle \mathrm{lead}(\underline{h}) \rangle \mathbb{C}[\![\underline{s}]\!]$
   then
   $j := \min\{i \mid \mathrm{lead}(r) \in \mathbb{C}[\![\underline{s}]\!] \, \mathrm{lead}(h_i)\}$;
   $r, \overline{a}, q' := \mathsf{nf}\big(r - \frac{\mathrm{lead}(r)}{\mathrm{lead}(h_j)} h_j, \underline{h}, k\big)$;
   $q := q + q'$;
   $\overline{a} := \overline{a} + \frac{\mathrm{lead}(r)}{\mathrm{lead}(h_j)} \overline{e}_j$
   fi
   $\mathsf{nf}_1, \mathsf{nf}_2, \mathsf{nf}_3 := r, \overline{a}, q$.

   for $p \in \mathbb{C}[\![\underline{s}, \underline{x}]\!]$ and $k \leq 0$.

2. We define a **reduced normal form algorithm** $\mathsf{rnf} = (\mathsf{rnf}_1, \mathsf{rnf}_2, \mathsf{rnf}_3)$ by

   proc $\mathsf{rnf}(p, \underline{h}, k) \equiv$
   $q := \pi_{U_k}(p)$;
   $r := p - q$;
   if $r \neq 0$ then

```
            if lead(r) ∈ ⟨lead(h)⟩C[[s]]
              then
                  j := min{i| lead(r) ∈ C[[s]] lead(h_i)};
                  r, ā, q' := rnf(r − (lead(r)/lead(h_j)) h_j, h, k);
                  ā := ā + (lead(r)/lead(h_j)) ē_j
               else
                  r', ā, q' := rnf(r − lead(r), h, k);
                  r := lead(r) + r';
            fi
            q := q + q';
   fi
   rnf_1, rnf_2, rnf_3 := r, ā, q.
```

for $p \in \mathbb{C}[\![\underline{s}, \underline{x}]\!]$ and $k \leq 0$.

Note that nf and rnf depend on $<$ and U.

Lemma 2.1.20. nf *and* rnf *terminate.*

Proof: Since $U = (U_k)_{k \leq 0}$ is a basis of the $\langle \underline{s}, \underline{x} \rangle$-adic topology, there are only finitely many monomials in $\complement U_k$ for all k. By definition of nf and rnf, $\operatorname{supp}(r) \subset \complement U_k$ and $\operatorname{lead}(r)$ is strictly decreasing. Hence, nf and rnf terminate.

Lemma 2.1.21. *Let* $l < k \leq 0$, $p \in U_k$ *and* $(r, \overline{a}, q) = \mathsf{nf}(p, \underline{h}, l)$ *resp.* $(r, \overline{a}, q) = \mathsf{rnf}(p, \underline{h}, l)$. *Then*

1. $p = \underline{h}\overline{a} + r + q$ *with* $a^j \in \mathbb{C}[\underline{s}]$ *and* $\operatorname{lead}(p) \geq \operatorname{lead}(h_j a^j)$ *for all* j,
2. *if* $r \neq 0$ *then* $\operatorname{lead}(r) \in \complement\langle \operatorname{lead}(\underline{h}) \rangle \mathbb{C}[\![\underline{s}]\!]$ *resp.* $r \in \complement\langle \operatorname{lead}(\underline{h}) \rangle \mathbb{C}[\![\underline{s}]\!]$,
3. *if* $\operatorname{lead}(p) \in \complement\langle \operatorname{lead}(\underline{h}) \rangle \mathbb{C}[\![\underline{s}]\!] + U_l$ *resp.* $p \in \complement\langle \operatorname{lead}(\underline{h}) \rangle \mathbb{C}[\![\underline{s}]\!] + U_l$ *then* $p = r + q$,
4. $r \in U_k \cap \complement U_l$ *and* $q \in U_l$.

In particular, $\mathsf{nf}_1(\cdot, \underline{h}, k)$ *resp.* $\mathsf{rnf}_1(\cdot, \underline{h}, k)$ *is a* $\mathbb{C}[\![\underline{s}]\!]$*-normal form resp. a reduced* $\mathbb{C}[\![\underline{s}]\!]$*-normal form with respect to* $\underline{h}$ *modulo* U_k.

Proof: This follows from the definition of nf resp. rnf.

By proposition 2.1.21.4, nf resp. rnf converges with respect to the $\langle \underline{s}, \underline{x} \rangle$-adic topology.

Definition 2.1.22. Let $<$ be a monomial ordering on $\mathbb{C}[\![\underline{s},\underline{x}]\!]$, $\underline{h} \subset \mathbb{C}[\![\underline{s},\underline{x}]\!]$, $U = (U_k)_{k\leq 0}$ a $\langle \underline{s},\underline{x}\rangle$-adic $(<,\underline{h})$-stable filtration, $k = (k_j)_{j\geq 0}$ a strictly decreasing sequence, $p_0 := p \in U_{k_0}$, and

$$r_{j+1}, \overline{a}_{j+1}, p_{j+1} := \mathsf{nf}(p_j, \underline{h}, k_{j+1})$$
$$\text{resp. } r_{j+1}, \overline{a}_{j+1}, p_{j+1} := \mathsf{rnf}(p_j, \underline{h}, k_{j+1})$$

for $j \geq 0$. We define a **normal form algorithm** $\mathsf{nf} = (\mathsf{nf}_1, \mathsf{nf}_2)$ resp. a **reduced normal form algorithm** $\mathsf{rnf} = (\mathsf{rnf}_1, \mathsf{rnf}_2)$ by

$$\mathsf{nf}(p,\underline{h}) := (r, \overline{a})$$
$$\text{resp. } \mathsf{rnf}(p,\underline{h}) := (r, \overline{a})$$

where $\overline{a} := \sum_{j\geq 0} \overline{a}_j \in \mathbb{C}[\![\underline{s}]\!]$ and $r := \sum_{j\geq 0} r_j \in \mathbb{C}[\![\underline{s},\underline{x}]\!]$. Note that nf and rnf depend on $<$, U, and $(k_j)_{j\geq 0}$.

Proposition 2.1.23. *Let $<$ be a monomial ordering on $\mathbb{C}[\![\underline{s},\underline{x}]\!]$, $\underline{h} \subset \mathbb{C}[\![\underline{s},\underline{x}]\!]$, $U = (U_k)_{k\leq 0}$ a $\langle \underline{s},\underline{x}\rangle$-adic $(<,\underline{h})$-stable filtration, and $(r,\overline{a}) = \mathsf{nf}(p,\underline{h})$ resp. $(r,\overline{a}) = \mathsf{rnf}(p,\underline{h})$. Then*

1. $p = \underline{h}\overline{a} + r$ *with* $a^j \in \mathbb{C}[\![\underline{s}]\!]$ *and* $\operatorname{lead}(p) \geq \operatorname{lead}(h_j a^j)$ *for all* i,
2. *if* $r \neq 0$ *then* $\operatorname{lead}(r) \in \complement\langle \operatorname{lead}(\underline{h})\rangle \mathbb{C}[\![\underline{s}]\!]$ *resp.* $r \in \complement\langle \operatorname{lead}(\underline{h})\rangle \mathbb{C}[\![\underline{s}]\!]$,
3. *if* $\operatorname{lead}(p) \in \complement\langle \operatorname{lead}(\underline{h})\rangle \mathbb{C}[\![\underline{s}]\!]$ *resp.* $p \in \complement\langle \operatorname{lead}(\underline{h})\rangle \mathbb{C}[\![\underline{s}]\!]$ *then* $p = r$,
4. *if* $p \in U_k$ *then* $r \in U_k$.

In particular, $\mathsf{nf}_1(\cdot,\underline{h})$ *resp.* $\mathsf{rnf}_1(\cdot,\underline{h})$ *is a* $\mathbb{C}[\![\underline{s}]\!]$*-normal form resp. a reduced* $\mathbb{C}[\![\underline{s}]\!]$*-normal form with respect to* $\underline{h}$.

Proof: This follows from lemma 2.1.21.

Remark 2.1.24. By definition of nf resp. rnf, nf_1 resp. rnf_1 is $\mathbb{C}$-scalar multiplicative in the first argument. But, in general, it is not $\mathbb{C}[\![\underline{s}]\!]$-scalar multiplicative and not additive.

Buchberger's standard basis algorithm [GP96] computes standard bases using a $\mathbb{C}[\![\underline{s}]\!]$-normal form.

Proposition 2.1.25. *Let $<$ be a monomial ordering on $\mathbb{C}[\![\underline{s},\underline{x}]\!]$ and $U = (U_k)_{k\leq 0}$ a $\langle \underline{s},\underline{x}\rangle$-adic filtration which is $(<,\underline{h})$-stable for all $\underline{h} \subset \mathbb{C}[\![\underline{s},\underline{x}]\!]$. Then there is a procedure* sb *resp.* rsb *to compute from $\underline{h} \subset \mathbb{C}[\![\underline{s},\underline{x}]\!]$ a standard basis resp. a reduced standard basis $\underline{g} \subset \mathbb{C}[\![\underline{s},\underline{x}]\!]$ of $\langle \underline{h}\rangle \mathbb{C}[\![\underline{s}]\!]$ and a matrix B with coefficients in $\mathbb{C}[\![\underline{s}]\!]$ such that $\underline{g} = \underline{h}B$.*

Proof: By proposition 2.1.23, nf_1 computes a $\mathbb{C}[\![\underline{s}]\!]$-normal form with respect to any $\underline{h} \subset \mathbb{C}[\![\underline{s},\underline{x}]\!]$. Buchberger's standard basis algorithm [GP96] using the normal form algorithm nf_1 computes from $\underline{h}$ a standard basis $\underline{g} \subset \mathbb{C}[\![\underline{s},\underline{x}]\!]$ of $\langle\underline{h}\rangle\mathbb{C}[\![\underline{s}]\!]$ and a matrix B with coefficients in $\mathbb{C}[\![\underline{s}]\!]$ such that $\underline{g} = \underline{h}B$. Let $(\underline{r}, A) := \mathsf{rnf}(\underline{g}-\mathrm{lead}(\underline{g}), \underline{g})$. Then, by proposition 2.1.23, $\underline{g}-\mathrm{lead}(\underline{g}) = \underline{r}+\underline{h}A$ and $\mathrm{lead}(\underline{g}) + \underline{r} = \underline{h}(B - A)$ is a reduced standard basis of $\langle\underline{h}\rangle\mathbb{C}[\![\underline{s}]\!]$.

Proposition 2.1.26. *In the situation of remark 2.1.18.2, there is a procedure* sb *to compute from* $\underline{h} \in \mathbb{C}[\underline{s}]V$ *a standard basis* $\underline{g} \in \mathbb{C}[\underline{s}]V$ *of* $\langle\underline{h}\rangle\mathbb{C}[\![\underline{s}]\!]$ *and a matrix* B *with coefficients in* $\mathbb{C}[\underline{s}]$ *such that* $\underline{g} = \underline{h}B$.

Proof: Buchberger's standard basis algorithm using a weak normal form algorithm [GP96] computes from $\underline{h} \in \mathbb{C}[\underline{s}]V$ a standard basis $\underline{g} \subset \mathbb{C}[\underline{s}]V$ of $\langle\underline{h}\rangle\mathbb{C}[\underline{s}]_{\langle\underline{s}\rangle}$ and a matrix B with coefficients in $\mathbb{C}[\underline{s}]$ such that $\underline{g} = \underline{h}B$. Since lead commutes with completion, $\underline{h}$ is a standard basis of $\langle\underline{h}\rangle\mathbb{C}[\![\underline{s}]\!]$. □

2.2 Formal differential deformations

In this section, we introduce the notion of a formal differential deformation. A formal differential deformation is a formal deformation of a local Artinian $\mathbb{C}$-algebra by differential operators and hence a finitely generated module over a power series ring. This notion is motivated by proposition 1.5.6 and the formal Brieskorn lattice is a free formal differential deformation of rank μ. We specialize the normal form algorithm in section 2.1 to an algorithm to compute presentations of formal differential deformations. In particular, this algorithm computes $\mathbb{C}[\![s]\!]$-basis representations in the formal Brieskorn lattice.

Let $\underline{x} = (x_1, \ldots, x_n)$ and $\underline{\partial} := (\partial_1, \ldots, \partial_n)$ where $\partial_j := \partial_{x_j}$. Let $\langle\underline{f}\rangle \subset \mathbb{C}[\![\underline{x}]\!]$ be an ideal such that $\mathbb{C}[\![\underline{x}]\!]/\langle\underline{f}\rangle$ is a local Artinian $\mathbb{C}$-algebra, that is,

$$\dim_{\mathbb{C}}(\mathbb{C}[\![\underline{x}]\!]/\langle\underline{f}\rangle) < \infty.$$

Then $\mathbb{C}[\![\underline{x}]\!][\underline{\partial}]/\langle\underline{f}\rangle\mathbb{C}[\![\underline{x}]\!][\underline{\partial}]$ is a right $\mathbb{C}[\![\underline{x}]\!][\underline{\partial}]$-module. Let $\underline{s} = (s_1, \ldots, s_m)$ and $D = (\underline{d}^j)$ be a matrix with coefficients in $\mathbb{C}[\![\underline{s},\underline{x}]\!][\underline{\partial}]$. Then the $\mathbb{C}[\![\underline{s}]\!]$-module

$$\mathbb{C}[\![\underline{s},\underline{x}]\!][\underline{\partial}]/\langle\underline{f} - \underline{s}D\rangle\mathbb{C}[\![\underline{s},\underline{x}]\!][\underline{\partial}]$$

is a formal deformation of $\mathbb{C}[\![\underline{s},\underline{x}]\!][\underline{\partial}]/\langle\underline{f}\rangle\mathbb{C}[\![\underline{s},\underline{x}]\!][\underline{\partial}]$ and

$$\begin{aligned}\mathbb{C}[\![\underline{s},\underline{x}]\!][\underline{\partial}]/\langle\underline{f} - \underline{s}D\rangle\mathbb{C}[\![\underline{s},\underline{x}]\!][\underline{\partial}] \otimes_{\mathbb{C}[\![\underline{x}]\!][\underline{\partial}]} \mathbb{C}[\![\underline{x}]\!][\underline{\partial}]/\mathbb{C}[\![\underline{x}]\!][\underline{\partial}]\langle\underline{\partial}\rangle \\ \cong_{\mathbb{C}[\![\underline{s}]\!]} \mathbb{C}[\![\underline{s},\underline{x}]\!]/\langle\underline{f} - \underline{s}D\rangle\mathbb{C}[\![\underline{s},\underline{x}]\!]\end{aligned}$$

with the coefficients of D acting on $\mathbb{C}[\![\underline{s},\underline{x}]\!]$ as differential operators.

Definition 2.2.1. We call a $\mathbb{C}[\![\underline{s}]\!]$-module of the form

$$\mathbb{C}[\![\underline{s},\underline{x}]\!]/\langle \underline{f} - \underline{s}D\rangle\mathbb{C}[\![\underline{s},\underline{x}]\!]$$

a **(formal) differential deformation** of $\mathbb{C}[\![\underline{x}]\!]/\langle \underline{f}\rangle$.

Remark 2.2.2. By proposition 1.5.6,

$$\widehat{H}'' \cong_{\mathbb{C}[\![s]\!]} \mathbb{C}[\![s,\underline{x}]\!]/\langle \underline{\partial}(f) - s\underline{\partial}\rangle\mathbb{C}[\![s,\underline{x}]\!]$$

is a differential deformation with $\underline{f} = \underline{\partial}(f)$ and $D = \underline{\partial}$.

Let

$$H = \mathbb{C}[\![\underline{s},\underline{x}]\!]/\langle \underline{f} - \underline{s}D\rangle\mathbb{C}[\![\underline{s},\underline{x}]\!]$$

be a differential deformation and let

$$\mathbb{C}[\![\underline{s},\underline{x}]\!] \xrightarrow{\pi_H} H$$

be the canonical projection. Let $(<_{\underline{x}}, \deg_{\underline{x}})$ be a degree ordering on $\mathbb{C}[\![\underline{x}]\!]$ and

$$\deg_{\underline{x}}(\underline{\partial}) := -\deg_{\underline{x}}(\underline{x}) > \underline{0}.$$

Then the order $\deg_{\underline{x}}$ is well-defined on $\mathbb{C}[\![\underline{x}]\!][\underline{\partial}]$. Let $\underline{g}$ be a standard basis of $\langle \underline{f}\rangle\mathbb{C}[\![\underline{x}]\!]$ and

$$\underline{m} = (m_j)_j := (\underline{x}^{\underline{\beta}})_{\underline{x}^{\underline{\beta}}\notin\langle \mathrm{lead}(\underline{g})\rangle}$$

decreasingly ordered. Then $\mathrm{lead}(\langle \underline{f}\rangle\mathbb{C}[\![\underline{x}]\!]) = \langle \mathrm{lead}(\underline{g})\rangle\mathbb{C}[\![\underline{x}]\!]$, $\langle \underline{f}\rangle\mathbb{C}[\![\underline{x}]\!] = \langle \underline{g}\rangle\mathbb{C}[\![\underline{x}]\!]$, and

$$\mathbb{C}[\![\underline{x}]\!] = \langle \underline{m}\rangle\mathbb{C} \oplus \langle \underline{g}\rangle\mathbb{C}[\![\underline{x}]\!]. \tag{2.2.1}$$

Hence, $[\underline{m}]$ is a $\mathbb{C}$-basis of

$$H/\langle \underline{s}\rangle H = \mathbb{C}[\![\underline{x}]\!]/\langle \underline{f}\rangle\mathbb{C}[\![\underline{x}]\!] = \langle \underline{m}\rangle\mathbb{C} \tag{2.2.2}$$

and, by Nakayama's lemma, $[\underline{m}]$ is a minimal set of $\mathbb{C}[\![\underline{s}]\!]$-generators of H. Let $B = (\underline{b}_j)$ a matrix such that

$$\underline{g} = \underline{f}B. \tag{2.2.3}$$

By remark 2.1.18.2 and proposition 2.1.25, one can compute $\underline{g} \subset \mathbb{C}[\![\underline{x}]\!]$ and B with coefficients in $\mathbb{C}[\![\underline{x}]\!]$. By proposition 2.1.26, if $\underline{f} \subset \mathbb{C}[\underline{x}]$ then one can compute $\underline{g} \subset \mathbb{C}[\underline{x}]$ and B with coefficients in $\mathbb{C}[\underline{x}]$. Let $(<_{\underline{s}}, \deg_{\underline{s}})$ be a degree ordering on $\mathbb{C}[\![\underline{s}]\!]$,

$$< := (<_{\underline{s}}, <_{\underline{x}})$$

be the block ordering of $<_{\underline{s}}$ and $<_{\underline{x}}$ on $\mathbb{C}[\![\underline{s},\underline{x}]\!]$, and

$$\deg := (\deg_{\underline{s}}, \deg_{\underline{x}})$$

the order with respect to the block degree of $\deg_{\underline{s}}$ and $\deg_{\underline{x}}$. We abbreviate $\mathrm{lead}_{\underline{s}} := \mathrm{lead}_{<_{\underline{s}}}$ and $\mathrm{lead} := \mathrm{lead}_{<}$. Let $(\underline{\alpha}^k)_{k\geq 0}$ be a decreasing enumeration of $\mathbb{N}^n$. For $k \geq 1$, we identify

$$\mathbb{C}[\![\underline{s}]\!]^k = \bigoplus_{1\leq j\leq k} \mathbb{C}[\![\underline{s}]\!]\underline{x}^{\underline{\alpha}^j}.$$

Definition 2.2.3. We define

$$\underline{h} = (h_{k,\underline{\beta}})_{k,\underline{\beta}} := \left((g_k - \underline{s}D\overline{b}^k)\underline{x}^{\underline{\beta}}\right)_{k,\underline{\beta}} \in \langle \underline{f} - \underline{s}D\rangle \mathbb{C}[\![\underline{s},\underline{x}]\!].$$

Then $\{\underline{h}(\underline{0},\underline{x})\} = \langle \underline{g}\rangle\mathbb{C}[\![\underline{x}]\!]$,

$$\langle \mathrm{lead}(\underline{h})\rangle \mathbb{C}[\![\underline{s}]\!] = \langle \mathrm{lead}(\underline{g})\rangle \mathbb{C}[\![\underline{s},\underline{x}]\!], \tag{2.2.4}$$

and, by (2.2.1),

$$\complement\langle \mathrm{lead}(\underline{h})\rangle \mathbb{C}[\![\underline{s}]\!] = \langle \underline{m}\rangle \mathbb{C}[\![\underline{s}]\!]. \tag{2.2.5}$$

Proposition 2.2.4. *H is free if and only if $\underline{h}$ is a standard basis.*

Proof: By (2.2.2) and Nakayama's lemma, $[\underline{m}]$ is a minimal set of generators of H. Since $H = \mathbb{C}[\![\underline{s},\underline{x}]\!]/\langle \underline{f} - \underline{s}D\rangle\mathbb{C}[\![\underline{s},\underline{x}]\!]$,

$$\mathbb{C}[\![\underline{s},\underline{x}]\!] \cong_{\mathbb{C}[\![\underline{s}]\!]} \langle \underline{m}\rangle \mathbb{C}[\![s]\!] + \langle \underline{f} - \underline{s}D\rangle \mathbb{C}[\![\underline{s},\underline{x}]\!]$$

and H is free if and only if

$$\langle \underline{f} - \underline{s}D\rangle \mathbb{C}[\![\underline{s},\underline{x}]\!] \cap \langle \underline{m}\rangle \mathbb{C}[\![s]\!] = 0.$$

Since $\langle \mathrm{lead}(\underline{g})\rangle \mathbb{C}[\![\underline{s},\underline{x}]\!] = \langle \mathrm{lead}(\underline{h})\rangle\mathbb{C}[\![\underline{s}]\!]$, this implies the claim. □

The following proposition characterizes free differential deformations in terms of flatness instead of standard bases.

Proposition 2.2.5. *If any relation of $\underline{f}$ lifts to a relation of $\underline{f} - \underline{s}D$ then H is free.*

Proof: Let $[\underline{v}] \subset \mathbb{C}[\![\underline{x}]\!]/\langle \underline{f}\rangle = H/\langle \underline{s}\rangle H$ be a $\mathbb{C}$-basis. Then, by Nakayama's lemma, $\langle [\underline{v}]\rangle \mathbb{C}[\![\underline{s}]\!] = H$. Assume that $\underline{v}\overline{g} \in \langle \underline{f} - \underline{s}D\rangle \mathbb{C}[\![\underline{s},\underline{x}]\!]$ for some $\overline{0} \neq \overline{g} \subset \mathbb{C}[\![\underline{s}]\!]$. Since $\underline{v} \subset \mathbb{C}[\![\underline{x}]\!]$ is $\mathbb{C}$-linearly independent, $0 \neq \underline{v}\,\mathrm{lead}_{\underline{s}}(\overline{g}) = \mathrm{lead}_{\underline{s}}(\underline{v}\overline{g})$. Hence, $\underline{v}\overline{g} = (\underline{f} - \underline{s}D)\overline{q}$ for some $\overline{q} \subset \mathbb{C}[\![\underline{s},\underline{x}]\!]$ with minimal $\deg_{\underline{s}}(\overline{q})$. If $\underline{f}\,\mathrm{lead}_{\underline{s}}(\overline{q}) = 0$ then $\mathrm{lead}_{\underline{s}}(\overline{q})$ is a relation of $\underline{f}$ and lifts to a relation of $\underline{f} - \underline{s}D$. Hence, by the minimality of $\deg_{\underline{s}}(\overline{q})$, we may assume that $\underline{f}\,\mathrm{lead}_{\underline{s}}(\overline{q}) \neq 0$. Then $\underline{v}\,\mathrm{lead}_{\underline{s}}(\overline{g}) = \underline{f}\,\mathrm{lead}_{\underline{s}}(\overline{q}) \in \langle \underline{f}\rangle\mathbb{C}[\![\underline{x}]\!]$ with $\overline{0} \neq \mathrm{lead}_{\underline{s}}(\overline{g}) \subset \mathbb{C}[\![\underline{s}]\!]$. This is a contradiction to the $\mathbb{C}$-linear independence of $[\underline{v}]$. Hence, $[\underline{v}]$ is a $\mathbb{C}[\![\underline{s}]\!]$-basis of H. □

By corollary 1.5.5 and remark 2.2.2, the differential deformation

$$\mathbb{C}[\![s,\underline{x}]\!]/\langle\underline{\partial}(f)-s\underline{\partial}\rangle\mathbb{C}[\![s,\underline{x}]\!]$$

is a free $\mathbb{C}[\![s]\!]$-module of rank μ. We give an elementary proof of this fact.

Lemma 2.2.6. *If* $0\neq p=(\underline{f}-\underline{s}D)\overline{q}$ *with* $\operatorname{lead}(p)\notin\langle\operatorname{lead}(\underline{h})\rangle\mathbb{C}[\![\underline{s}]\!]$ *then* $\operatorname{lead}_{\underline{s}}(\overline{q})$ *is a relation of* $\underline{f}$*, that is,* $\underline{f}\operatorname{lead}_{\underline{s}}(\overline{q})=0$.

Proof: If $0\neq p=(\underline{f}-\underline{s}D)\overline{q}$ with $\underline{f}\operatorname{lead}_{\underline{s}}(\overline{q})\neq 0$ then $\operatorname{lead}_{\underline{s}}(p)=\underline{f}\operatorname{lead}_{\underline{s}}(\overline{q})$ and hence $\operatorname{lead}(p)\in\langle\operatorname{lead}(\underline{h})\rangle\mathbb{C}[\![\underline{s}]\!]$. □

Proposition 2.2.7. *The differential deformation*

$$\mathbb{C}[\![s,\underline{x}]\!]/\langle\underline{\partial}(f)-s\underline{\partial}\rangle\mathbb{C}[\![s,\underline{x}]\!]$$

is a free $\mathbb{C}[\![s]\!]$*-module.*

Proof: Assume that $0\neq p=(\underline{\partial}(f)-s\underline{\partial})\overline{q}$ and $\operatorname{lead}(p)\notin\langle\operatorname{lead}(\underline{h})\rangle\mathbb{C}[\![s]\!]$ for some $\overline{q}\subset\mathbb{C}[\![\underline{s},\underline{x}]\!]$ with minimal $\deg_s(\overline{q})$. By lemma 2.2.6, $\underline{\partial}(f)\operatorname{lead}_s(\overline{q})=0$. By lemma 1.4.4, we may assume that there are $1\leq i<j\leq n$, $k\in\mathbb{N}$, and $r\in\mathbb{C}[\![\underline{x}]\!]$ such that

$$\operatorname{lead}_s(\overline{q})=s^k r\big(\partial_i(f)\overline{e}_j-\partial_j(f)\overline{e}_i\big).$$

This implies that

$$\underline{\partial}\operatorname{lead}_s(\overline{q})=s^k\underline{\partial}(f)\big(\partial_j(r)\overline{e}_i-\partial_i(r)\overline{e}_j\big)$$

and hence

$$p=(\underline{\partial}(f)-s\underline{\partial})\big(\overline{q}-\operatorname{lead}_s(\overline{q})-s^{k+1}\big(\partial_j(r)\overline{e}_i-\partial_i(r)\overline{e}_j\big)\big).$$

This is a contradiction to the minimality of $\deg_s(\overline{q})$. Hence, $\underline{h}$ is a standard basis and the claim follows from proposition 2.2.4. □

Using the following proposition, one can specialize the normal form algorithm from section 2.1 to formal differential deformations. We denote by min deg resp. max deg the minimal resp. the maximal degree of the components of a vector or a matrix.

Definition 2.2.8.

1. We define the weights $\deg(\underline{s})$ by

$$\deg(s_j):=\min\deg(\underline{m})+\min\deg(\underline{x})-\max\deg(\underline{d}^j)<0.$$

2. We define a sequence $N=(N_K)_{K\leq 0}$ by
$$N_K := -K\min\deg(\underline{s}) - \min\deg(\underline{x}) + \max\deg(D).$$

3. We define a filtration $V=(V_K)_{K\leq 0}$ on $\mathbb{C}[\![\underline{s},\underline{x}]\!]$ by $\mathbb{C}[\![\underline{s}]\!]$-modules
$$V_K := \{p\in\mathbb{C}[\![\underline{s},\underline{x}]\!]\,|\,\deg(p)<N_K\} + \langle\underline{s}\rangle^{-K}\mathbb{C}[\![\underline{s},\underline{x}]\!].$$

Remark 2.2.9. If $H=\mathbb{C}[\![\underline{s},\underline{x}]\!]/\langle\underline{\partial}(f)-s\underline{\partial}\rangle\mathbb{C}[\![s,\underline{x}]\!]\cong_{\mathbb{C}[\![s]\!]}\widehat{H}''$ then
$$\begin{aligned}\deg(s)&=\min\deg(\underline{m})+2\min\deg(\underline{x}),\\ N_K&=-K\deg(s)-2\min\deg(\underline{x}).\end{aligned}$$

The following proposition is the basis for a normal form algorithm for differential deformations.

Proposition 2.2.10. $V=(V_K)_{K\leq 0}$ *is a* $\langle\underline{s},\underline{x}\rangle$*-adic* $(<,\underline{h})$*-stable filtration on* $\mathbb{C}[\![\underline{s},\underline{x}]\!]$ *with*
$$\pi_H(V_K)=\langle\underline{s}\rangle^{-K}H.$$

Proof:

1. By definition of N and U, U is $\langle\underline{s},\underline{x}\rangle$-adic.

2. Since $\underline{g}$ is a standard basis,
$$\min\deg(\underline{m})+\min\deg(\underline{x})\leq\min\deg(\underline{g}) \tag{2.2.6}$$
and hence
$$\begin{aligned}\deg(\underline{s}D\overline{b}_k)&\leq\max\{\deg(s_j\underline{d}^j\overline{b}_k)|1\leq j\leq m\}\\ &\leq\max\{\deg(s_j)+\max\deg(\underline{d}^j)+\max\deg(\overline{b}_k)|1\leq j\leq m\}\\ &\leq\max\{\deg(s_j)+\max\deg(\underline{d}^j)|1\leq j\leq m\}\\ &\leq\min\deg(\underline{m})+\min\deg(\underline{x})\\ &\leq\min\deg(\underline{g})\\ &\leq\deg(g_k)\\ &=\deg\operatorname{lead}(g_k).\end{aligned}$$
Since $\underline{s}^{\underline{\alpha}}h_{k,\underline{\beta}}=\underline{s}^{\underline{\alpha}}(g_k+\underline{s}D\overline{b}_k)\underline{x}^{\underline{\beta}}$, this implies that
$$\deg(\underline{s}^{\underline{\alpha}}h_{k,\underline{\beta}})\leq\deg(\underline{s}^{\underline{\alpha}}\operatorname{lead}(g_k)\underline{x}^{\underline{\beta}})=\deg\operatorname{lead}(s^{\underline{\alpha}}h_{k,\underline{\beta}}).$$
Hence, U is $(<,\underline{h})$-stable.

3. Let $p \in V_K$ and $\underline{s}^{\underline{\alpha}} p_{\underline{\alpha}} := \operatorname{lead}_{\underline{s}}(p)$ with maximal $|\alpha| < -K$ for fixed p mod $\langle \underline{f} - \underline{s}D\rangle \mathbb{C}[\![\underline{s}, \underline{x}]\!]$. Then

$$\begin{aligned}\deg(p_{\underline{\alpha}}) &= \deg(\underline{s}^{\underline{\alpha}} p_{\underline{\alpha}}) - \deg(\underline{s}^{\underline{\alpha}}) \\ &< -(K + |\underline{\alpha}|) \min\deg(\underline{s}) - \min\deg(\underline{x}) + \max\deg(D) \\ &\leq \min\deg(\underline{s}) - \min\deg(\underline{x}) + \max\deg(D) \\ &= \min\deg(\underline{m})\end{aligned}$$

and, by (2.2.1), $p_{\underline{\alpha}} \in \langle \underline{g}\rangle \mathbb{C}[\![\underline{x}]\!]$. By remark 2.1.18.2 and proposition 2.1.23, there is a $\overline{q} \subset \mathbb{C}[\![\underline{x}]\!]$ with $p_{\underline{\alpha}} = \underline{g}\overline{q}$ and $\operatorname{lead}(p_{\underline{\alpha}}) \geq \operatorname{lead}(g_j q^j)$ for all j. . Then, by (2.2.3),

$$p_{\underline{\alpha}} = \underline{g}\overline{q} = \underline{f} B\overline{q} \equiv \underline{s}DB\overline{q} \mod \langle \underline{f} - \underline{s}D\rangle \mathbb{C}[\![\underline{s}, \underline{x}]\!] \tag{2.2.7}$$

and, since $(<_{\underline{x}}, \deg_{\underline{x}})$ is a degree ordering, $\deg(g_j) + \deg(q^j) \leq \deg(p_{\underline{\alpha}})$ for all j and hence

$$\max\deg(\overline{q}) \leq \deg(p_{\underline{\alpha}}) - \min\deg(\underline{g}). \tag{2.2.8}$$

Then, by (2.2.8) and (2.2.6),

$$\begin{aligned}\deg(\underline{s}DB\overline{q}) &\leq \max\deg(\underline{s}D) + \max\deg(B) + \max\deg(\overline{q}) \\ &\leq \max\deg(\underline{s}D) + \max\deg(\overline{q}) \\ &\leq \max\deg(\underline{s}D) - \min\deg(\underline{g}) + \deg(p_{\underline{\alpha}}) \\ &\leq \max\deg(\underline{s}D) - \min\deg(\underline{m}) - \min\deg(\underline{x}) + \deg(p_{\underline{\alpha}}) \\ &\leq \max\{\deg(s_j) + \max\deg(\underline{d}^j) | 1 \leq j \leq m\} \\ &- \min\deg(\underline{m}) - \min\deg(\underline{x}) + \deg(p_{\underline{\alpha}}) \\ &= \deg(p_{\underline{\alpha}}).\end{aligned}$$

Hence, by (2.2.7),

$$p' := p - \operatorname{lead}_{\underline{s}}(p) + \underline{s}^{\underline{\alpha}} \underline{s}DB\overline{b} \equiv p \mod \langle \underline{f} - \underline{s}D\rangle \mathbb{C}[\![\underline{s}, \underline{x}]\!]$$

with

$$\begin{aligned}\deg(p') \leq \deg(p) \leq N_K, \\ \operatorname{lead}_{\underline{s}}(p') <_{\underline{s}} \operatorname{lead}_{\underline{s}}(p).\end{aligned}$$

Since $(<_{\underline{s}}, \deg_{\underline{s}})$ is a degree ordering, this contradicts to the maximality of $|\alpha|$. Hence, $p \in \langle \underline{s}\rangle^{-K} + \langle \underline{f} - \underline{s}D\rangle \mathbb{C}[\![\underline{s}, \underline{x}]\!]$ and $\pi_H(V_K) = \langle \underline{s}\rangle^{-K} H$.

□

The following normal form algorithm is a specialization of the reduced normal form algorithm in section 2.1 based on proposition 2.2.10.

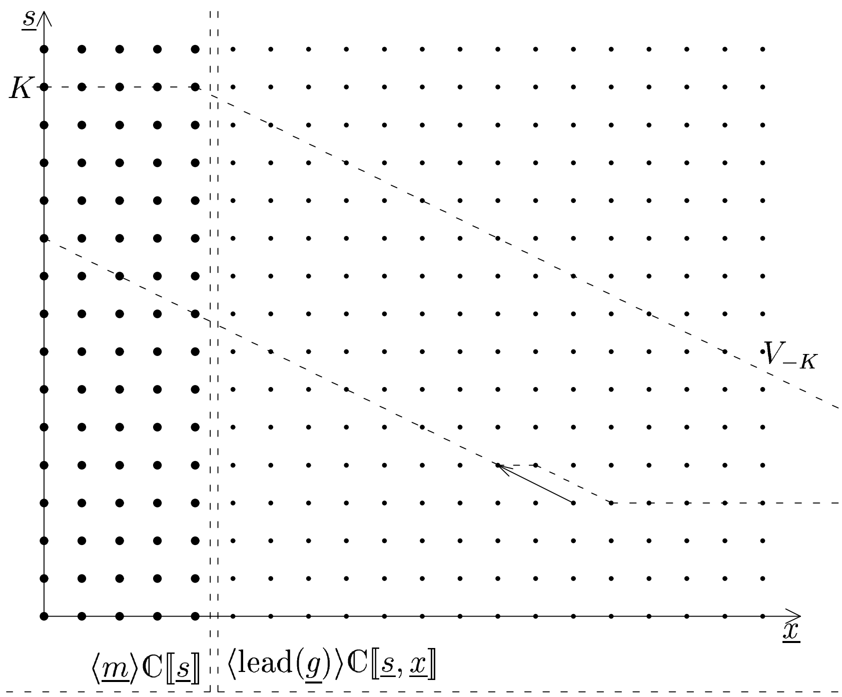
Definition 2.2.11. We define a **(reduced) normal form algorithm** for differential deformations $\mathsf{ddnf} = (\mathsf{ddnf}_1, \mathsf{ddnf}_2, \mathsf{ddnf}_3)$ by

$$\mathsf{ddnf}(p, \underline{g}, K) := \mathsf{rnf}(p, \underline{h}, -K),$$
$$\mathsf{ddnf}(p, \underline{g}) := \mathsf{rnf}(p, \underline{h})$$

with $<= (<_{\underline{s}}, <_{\underline{x}})$ and $U = V$ for $p \in \mathbb{C}[\![\underline{s}, \underline{x}]\!]$ and $K \geq 0$. Note that ddnf depends on $(k_j)_{j \geq 0}$ and D.

Figure 2.1 shows a reduction step in the normal form algorithm for differential deformations. We denote the 1-dimensional $\mathbb{C}$-vectorspace spanned by a monomial in $\langle \underline{m} \rangle \mathbb{C}[\![\underline{s}]\!]$ resp. in $\langle \mathrm{lead}(\underline{g}) \rangle \mathbb{C}[\![\underline{s}, \underline{x}]\!]$ by a bullet resp. a circle. The monomial at the tail of the arrow is replaced by a power series with support above the dashed line crossing the head of the arrow. The $\mathbb{C}[\![s]\!]$-submodule V_{-K} is invariant with respect to such a reduction step.

Figure 2.1: The differential deformation normal form algorithm



Remark 2.2.12. In practice, one should replace the computation of $\pi_{V_{-K}}$ by a sequence of $\mathrm{lead}_{\underline{s}}$ computations. This leads to the following equivalent definition of ddnf.

proc $\mathsf{ddnf}(p, \underline{g}, K) \ \equiv$
  if $\mathrm{lead}(p) \in \langle \underline{s} \rangle^K$ then
        $q := p$;
  elsif $\deg \mathrm{lead}(p) < N_{-K}$
    $q := \mathrm{lead}_{\underline{s}}(p)$;
  fi
  $r := p - q$;
  if $r \neq 0$ then
      if $\mathrm{lead}(r) \in \langle \mathrm{lead}(\underline{g}) \rangle \mathbb{C}[\![\underline{s}, \underline{x}]\!]$
        then
          $j := \min\{i | \, \mathrm{lead}(r) \in \mathbb{C}[\![\underline{s}, \underline{x}]\!] \, \mathrm{lead}(g_i)\}$;
          $r, q', \overline{a} := \mathsf{ddnf}\big(r - \frac{\mathrm{lead}(r)}{\mathrm{lead}(g_j)} g_j + \underline{s} D\big(\frac{\mathrm{lead}(r)}{\mathrm{lead}(g_j)} \overline{b}_j\big), K\big)$;
          $\overline{a} := \overline{a} + \frac{\mathrm{lead}(r)}{\mathrm{lead}(g_j)} \overline{e}_j$;
        else
          $r', q', \overline{a} := \mathsf{ddnf}(r - \mathrm{lead}(r), K)$;
          $r := \mathrm{lead}(r) + r'$;
      fi
      $q := q + q'$;
  fi
  $\mathsf{ddnf}_1, \mathsf{ddnf}_2, \mathsf{ddnf}_3 := r, q, \overline{a}$.

Definition 2.2.13. We define $\Phi : \mathbb{C}[\![\underline{s}, \underline{x}]\!] \longrightarrow \mathbb{C}[\![\underline{s}, \underline{x}]\!]$ by

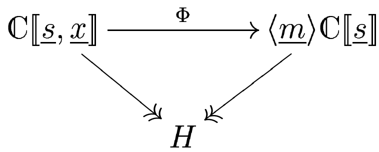
$$\Phi_K(p) := \mathsf{ddnf}_1(p, \underline{g}, K),$$
$$\Phi(p) := \mathsf{ddnf}_1(p, \underline{g})$$

for $p \in \mathbb{C}[\![\underline{s}, \underline{x}]\!]$ and $K \geq 0$.

Remark 2.2.14. By definition of ddnf, Φ is $\mathbb{C}[\![\underline{s}]\!]$-scalar multiplicative. But, in general, it is not additive.

Proposition 2.2.15.

1. *Φ is a $\mathbb{C}[\![\underline{s}]\!]$-scalar multiplicative map*



over H with $\mathrm{im}\, \Phi_K = \bigoplus_{|\alpha|<K} \underline{s}^{\alpha} \langle \underline{m} \rangle \mathbb{C}$.

2. *If H is a free $\mathbb{C}[\![\underline{s}]\!]$-module then Φ induces the $\mathbb{C}[\![\underline{s}]\!]$-section*

$$\mathbb{C}[\![\underline{s},\underline{x}]\!] \underset{\pi_H}{\overset{\Phi}{\rightleftarrows}} H$$

of the canonical projection π_H with $\operatorname{im}\Phi = \langle \underline{m}\rangle \mathbb{C}[\![\underline{s}]\!]$.

Proof:

1. This follows from remark 2.2.14, proposition 2.1.23 and 2.2.10, and (2.2.5).

2. This follows from remark 2.2.14, proposition 2.1.14, 2.1.23, 2.2.4, and 2.2.10, and (2.2.5).

Remark 2.2.16.

1. By proposition 2.2.15.1,

$$\Phi\big(\langle \underline{f} - \underline{s}D\rangle \mathbb{C}[\![\underline{s},\underline{x}]\!]\big) = \langle \underline{m}\rangle \mathbb{C}[\![\underline{s}]\!] \cap \langle \underline{f} - \underline{s}D\rangle \mathbb{C}[\![\underline{s},\underline{x}]\!]$$

and

$$\begin{aligned} H &\cong_{\mathbb{C}[\![\underline{s}]\!]} \langle \underline{m}\rangle \mathbb{C}[\![\underline{s}]\!] / \big(\langle \underline{m}\rangle \mathbb{C}[\![\underline{s}]\!] \cap \langle \underline{f} - \underline{s}D\rangle \mathbb{C}[\![\underline{s},\underline{x}]\!]\big) \\ &= \langle \underline{m}\rangle \mathbb{C}[\![\underline{s}]\!] / \Phi\big(\langle \underline{f} - \underline{s}D\rangle \mathbb{C}[\![\underline{s},\underline{x}]\!]\big) \end{aligned}$$

In particular, by proposition 2.1.23.4 and 2.2.10,

$$H/\langle \underline{s}\rangle^K H \cong_{\mathbb{C}[\![\underline{s}]\!]} \langle \underline{m}\rangle \mathbb{C}[\![\underline{s}]\!] / \big(\Phi_K\big(\langle \underline{f} - \underline{s}D\rangle \mathbb{C}[\![\underline{s},\underline{x}]\!] \backslash V_{-K}\big) + \langle \underline{s}\rangle^K \langle \underline{m}\rangle \mathbb{C}[\![\underline{s}]\!]\big)$$

and $\dim_{\mathbb{C}}\big(\langle \underline{f} - \underline{s}D\rangle \mathbb{C}[\![\underline{s},\underline{x}]\!] \backslash V_{-K}\big) < \infty$. Hence, one can compute a $\mathbb{C}[\![\underline{s}]\!]$-presentation of H.

2. By proposition 2.2.15.2, if H is a free $\mathbb{C}[\![\underline{s}]\!]$-module then Φ is the $\mathbb{C}[\![\underline{s}]\!]$-basis representation with respect to $[\underline{m}]$. By proposition 1.5.6, remark 2.2.2, and proposition 2.2.7, this is the case for the formal Brieskorn lattice $\widehat{H}''$.

2.3 Matrix A of t

In this section, we describe an algorithm to compute a basis representation of the operator t with respect to a $\mathbb{C}[\![s]\!]$-basis of the Brieskorn lattice. This algorithm is a specialization of the normal form algorithm for differential deformations in definition 2.2.11.

Let $\underline{x} = (x_0, \ldots, x_n)$ and $\underline{\partial} := (\partial_0, \ldots, \partial_n)$ where $\partial_j := \partial_{x_j}$. Let $f \in \mathbb{C}\{\underline{x}\}$ define an isolated hypersurface singularity with Milnor number

$$\mu = \dim_{\mathbb{C}} \mathbb{C}\{\underline{x}\}/\langle \underline{\partial}(f)\rangle < \infty.$$

By theorem 1.1.2, we may assume that $f \in \mathbb{C}[\underline{x}]$ and hence $\underline{\partial}(f) \subset \mathbb{C}[\underline{x}]$. Note that this assumption can be replaced by appropriate degree bounds. Since completion is faithfully flat, we may replace $\mathbb{C}\{\underline{x}\}$-, $\mathbb{C}\{t\}$-, resp. $\mathbb{C}\{\{s\}\}$-modules by their $\langle \underline{x}\rangle$-, $\langle t\rangle$-, resp. $\langle s\rangle$-adic completion. Let $<_{\underline{x}}$ be a degree ordering on $\mathbb{C}[\![\underline{x}]\!]$. By proposition 2.1.26, one can compute a polynomial standard basis $\underline{g}$ of $\langle \underline{\partial}(f)\rangle$ and a polynomial matrix B such that

$$\underline{g} = \underline{f}B.$$

Then $\operatorname{lead}\langle \underline{\partial}(f)\rangle = \langle \operatorname{lead}(\underline{g})\rangle$ and hence one can compute

$$\underline{m} = (m_j)_{1\le j\le \mu} := (\underline{x}^{\beta})_{\underline{x}^{\beta}\notin \operatorname{lead}\langle \underline{\partial}(f)\rangle}$$

with $m_1 > \cdots > m_\mu$. Then

$$\mathbb{C}[\![\underline{x}]\!] = \langle \underline{m}\rangle \mathbb{C} \oplus \langle \underline{\partial}(f)\rangle \mathbb{C}[\![\underline{x}]\!]$$

and $\underline{m}$ is a $\mathbb{C}$-basis of $\mathbb{C}[\![\underline{x}]\!]/\langle \underline{\partial}(f)\rangle$. By theorem 1.5.2 and proposition 1.5.4, $\widehat{H}''$ is the $\langle \underline{x}\rangle$-, $\langle t\rangle$-, and $\langle s\rangle$-adic completion of H'' and we may replace H'' by $\widehat{H}''$ for the computation. By corollary 1.5.3 and 1.5.5, H'' is a free $\mathbb{C}[\![t]\!]$- and $\mathbb{C}[\![s]\!]$-module of rank μ with

$$H''/sH'' \cong \mathbb{C}[\![\underline{x}]\!]/\langle \underline{\partial}(f)\rangle.$$

Hence, by Nakayama's lemma, $[\underline{m}]$ is a $\mathbb{C}[\![s]\!]$-basis of H''. For $k \in \mathbb{Z}$, the canonical projection

$$\mathbb{C}[\![\underline{s}]\!][s^{-1}] \xrightarrow{\operatorname{jet}_k} \bigoplus\nolimits_{j\le k} \mathbb{C}s^j$$

is called the k-**jet**. Let $A = \sum_{k\ge 0} s^j A_j$ be the $[\underline{m}]$-matrix of t defined by

$$[\underline{m}]A := t[\underline{m}] = [f\underline{m}].$$

Then, by proposition 1.6.18, $A + s^2\partial_s$ is the $[\underline{m}]$-basis representation of t. By proposition 2.2.15.2 and remark 2.2.16.2,

$$\underline{m}\operatorname{jet}_k(A) = \mathsf{ddnf}_1(f\underline{m}, \underline{g}, k+1)$$

with $D = \underline{\partial}$ and hence one can compute $\operatorname{jet}_k(A)$ for any $k \ge 0$. Let $<_s$ be the standard degree ordering on $\mathbb{C}[\![s]\!]$ and

$$<:= (<_s, <_{\underline{x}})$$

the block ordering of $<_s$ and $<_{\underline{x}}$. Then $<$ induces a monomial ordering on the $[\underline{m}]$-basis representation

$$\mathbb{C}[\![s]\!]^\mu = \langle \underline{m} \rangle \mathbb{C} \subset \mathbb{C}[\![s, \underline{x}]\!]$$

of H''.

2.4 Saturation of the Brieskorn lattice

In this section, we describe an algorithm to compute a basis representation of the Brieskorn lattice and of the operator t for a $\mathbb{C}[\![s]\!]$-basis of the saturation of the Brieskorn lattice.

By corollary 1.5.3 and 1.5.5, H'' is a $\mathbb{C}[\![t]\!]$- and $\mathbb{C}[\![s]\!]$-lattice. Hence, by lemma 1.6.9, H''_k and H''_∞ are $\mathbb{C}[\![t]\!]$- and $\mathbb{C}[\![s]\!]$-lattices. By proposition 1.6.10,

$$\kappa := \min\{k \geq 0 \mid H_k = H''_\infty\} \leq \mu - 1.$$

We consider the columns of a matrix as generators of a module. Let $E = (\delta_{i,j})^i_j$ where δ is the Kronecker symbol be the unit matrix of size μ. Then $\langle E \rangle = \mathbb{C}[\![\underline{s}]\!]^\mu$ is the $[\underline{m}]$-basis representation of H''. By proposition 1.6.18, $A + s^2\partial_s$ is the $[\underline{m}]$-basis representation of t and hence $s^{-1}A + s\partial_s$ is the $[\underline{m}]$-basis representation of $\partial_t t = t\partial_t + 1$. Let

$$\begin{aligned} Q_{-1} &:= E, \\ H_0 &:= E, \\ Q_k &:= (s^{-1}A + s\partial_s)Q_{k-1}, \\ H_{k+1} &:= (H_k | Q_k), \end{aligned}$$

for $k \geq 0$. Then $\langle H_k \rangle$ is the $[\underline{m}]$-basis representation of H''_k for $k \geq 0$. Since $\mathbb{C}[\![\underline{s}]\!]^\mu = \langle H_0 \rangle \subset \langle H_k \rangle$ for $k \geq 0$, we may redefine the generators H_k without changing the modules $\langle H_k \rangle$ by

$$\begin{aligned} Q_{-1} &:= E, \\ H_0 &:= E, \\ Q_k &:= \mathrm{jet}_{-1}\big((s^{-1}\mathrm{jet}_k(A) + s\partial_s)Q_{k-1}\big), \\ H_{k+1} &:= (H_k | Q_k), \end{aligned}$$

for $k \geq 0$. In order to compute H_{k+1}, it suffices to compute the k-jet of A. Hence, the generators H_k can be computed successively for increasing $k \geq 0$. In order to avoid poles in s, we redefine the generators H_k such that $\langle H_k \rangle$ is the $s^{-k}[\underline{m}]$-basis representation of H''_k. Then the generators H_k are defined by

$$\begin{aligned} Q_{-1} &:= E, \\ H_0 &:= E, \\ Q_k &:= \mathrm{jet}_k\big((\mathrm{jet}_k(A) + s^2\partial_s)Q_{k-1}\big), \\ H_{k+1} &:= (sH_k|Q_k), \end{aligned}$$

for $k \geq 0$ and the coefficients of H_k resp. Q_k are in $\bigoplus_{0\leq j\leq k} \mathbb{C}s^j$. By remark 2.1.18.2 and proposition 2.1.23 and 2.1.25, one can compute a standard basis of $\langle sH_k\rangle$ up to degree k and a normal form of Q_k with respect to this standard basis up to degree k. Since $s^k\mathbb{C}[\![s]\!] \subset \langle H_k\rangle$, the truncated standard basis is a polynomial standard basis and the normal form equals zero if and only if $\langle Q_k\rangle \subset \langle sH_k\rangle$ or equivalently $\langle H_{k+1}\rangle = \langle H_k\rangle$. Hence, one can compute κ and

$$\begin{aligned} H_\infty &:= H_\kappa, \\ H &:= s^\kappa E. \end{aligned}$$

Then $\langle H\rangle = s^\kappa\mathbb{C}[\![s]\!]^\mu$ is the $s^{-\kappa}[\underline{m}]$-basis representation of H''. By remark 2.1.18.2 and proposition 2.1.25, one can compute a minimal standard basis of $\langle H_\infty\rangle$ up to degree κ. Since $s^\kappa\mathbb{C}[\![s]\!] \subset \langle H_\infty\rangle$, the truncated standard basis is a polynomial minimal standard basis and, by lemma 2.1.11, it is a $\mathbb{C}[\![s]\!]$-basis. Since $s^\kappa\langle H_\infty\rangle \subset \langle H\rangle$ and hence $s^\kappa\mathbb{C}[\![s]\!]^\mu \subset \langle H_\infty^{-1}H\rangle$,

$$\langle H_\infty^{-1}H\rangle = \langle \mathrm{jet}_\kappa(H_\infty^{-1}H)\rangle.$$

By remark 2.1.18.2 and proposition 2.1.23, one can compute a normal form of H with respect to H_∞ up to degree κ in order to compute the H_∞-basis representation $\langle H_\infty^{-1}H\rangle$ of $\langle H\rangle$. The following procedure $\mathsf{saturate} = (\mathsf{saturate}_1, \mathsf{saturate}_2, \mathsf{saturate}_3)$ computes from the $[\underline{m}]$-matrix A of t a polynomial minimal standard basis H_∞ of the $s^{-\kappa}[\underline{m}]$-basis representation $\langle H_\infty\rangle$ of H''_∞, the $s^{-\kappa}[\underline{m}]H_\infty$-basis representation $\langle H\rangle$ of H'', and the $s^{-\kappa}[\underline{m}]$-matrix A of t.

```
proc saturate(A)
  H := E;
  Q := jet_0(A);
  κ := 0;
  while jet_κ(nf_1(Q, sb(sH), κ)) ≠ 0 do
        κ := κ + 1;
        H := (sH|Q);
        Q := jet_κ((jet_κ(A) + s^2 ∂_s)Q);
```

```
  od
  H_∞ := sb(H);
  H := jet_κ(nf_2(s^κ E, H_∞, κ));
  A := A − κsE;
  saturate_1, saturate_2, saturate_3 := H_∞, H, A, κ.
```

Since the $s^{-k}[\underline{m}]$- and $[\underline{m}]$-matrix of t differ by $-ksE$, the matrix A is transformed after the saturation process and the computation of A can be continued up to any degree after this transformation.

Let H_∞, H, A, κ be computed by saturate. Then $A = \sum_{j=0}^{\infty} s^j A_j$ is the $s^{-\kappa}[\underline{m}]$-matrix of t. By proposition 1.6.18, $A + s^2\partial_s$ is the $s^{-\kappa}[\underline{m}]$-basis representation of t and hence $H_\infty^{-1}(A + s^2\partial_s)H_\infty$ is the $s^{-\kappa}[\underline{m}]H_\infty$-matrix of t. Since $s^\kappa \mathbb{C}[\![s]\!]^\mu \subset \langle H_\infty \rangle$,

$$\mathrm{jet}_k\big(H_\infty^{-1}(A + s^2\partial_s)H_\infty\big) = \mathrm{jet}_k\big(H_\infty^{-1}(\mathrm{jet}_{k+\kappa}(A) + s^2\partial_s)H_\infty\big).$$

By remark 2.1.18.2 and proposition 2.1.23, one can compute a normal form of $(\mathrm{jet}_{k+\kappa}(A) + s^2\partial_s)H_\infty$ with respect to H_∞ up to degree $k + \kappa$ in order to compute $H_\infty^{-1}(A+s^2\partial_s)H_\infty$ up to degree k. The following procedure transform computes from H_∞, H, A, κ computed by saturate the $s^{-\kappa}[\underline{m}]H_\infty$-matrix A of t up to any degree k.

```
proc transform(H_∞, H, A, κ, k)
  A := jet_k(nf_2((jet_{k+κ}(A) + s^2 ∂_s)H_∞, H_∞, k + κ));
  transform := A.
```

2.5 Monodromy

In this section, we describe an algorithm to compute the complex monodromy. We show that one can compute a basis representation of the Brieskorn lattice and of the operator t with respect to a $\mathbb{C}[\![s]\!]$-basis of a saturated non-resonant lattice. In particular, by corollary 1.6.13, one can compute the complex monodromy.

Let H_∞, H, and κ be computed by saturate and A be computed by transform. Then

$$\underline{h} := s^{-\kappa}[\underline{m}]H_\infty$$

is a $\mathbb{C}[\![s]\!]$-basis of the saturated $\mathbb{C}[\![t]\!]$- and $\mathbb{C}[\![s]\!]$-lattice H''_∞, $\langle H \rangle$ the $\underline{h}$-basis representation of H'', and $A = \sum_{j=0}^{\infty} s^j A_j$ the $\underline{h}$-matrix of t and hence $A_0 = 0$. By proposition 1.6.18, $A + s^2\partial_s$ is the $\underline{h}$-basis representation of t and hence $s^{-1}A + s\partial_s$ is the basis representation of $\partial_t t = t\partial_t + 1$. Hence, by lemma 1.6.7.1, $A_1 - E$ is the $[\underline{h}]$-basis representation of the residue $\mathrm{res}_{H''_\infty}$ of H''_∞.

By transforming A_1 to Hessenberg form, one can compute a product decomposition of the characteristic polynomial of A_1. In many cases A_1 is a sparse matrix and this product decomposition is non-trivial. By theorem 1.2.5, the eigenvalues of A_1 are rational and can be computed by univariate factorization of the factors of the characteristic polynomial. Let

$$\underline{\alpha} = (\alpha_i)_{1\le i\le\mu} := \mathsf{eigenvals}(A_1 - E)$$

be the eigenvalues of $\mathrm{res}_{H''_\infty}$. Then, by proposition 1.6.12, $\exp(-2\pi\mathrm{i}\underline{\alpha})$ are the eigenvalues of the monodromy M. Let

$$\delta(A) := \max\{\alpha_i - \alpha_j \mid \alpha_i - \alpha_j \in \mathbb{Z}\}$$

be the maximal integer difference of eigenvalues of A_1. By corollary 1.8.6, $V^{>-1} \supset H'' \supset V^{n-1}$. Since the V^α are saturated, $V^{>-1} \supset H''_\infty \supset sH''_\infty \supset V^n$ and hence,

$$\delta(A) \le n.$$

If $\delta(A) = 0$ then H''_∞ is a non-resonant lattice and, by corollary 1.6.13, $\exp(-2\pi\mathrm{i}A_1)$ is a monodromy matrix.

If $\delta(A) \neq 0$ then we proceed as follows. Since one can compute the eigenvalues of A_1, one can compute the generalized eigenspaces of A_1. By a $\mathbb{C}$-linear basis transformation, we may assume that

$$A = \begin{pmatrix} A^{1,1} & A^{1,2} \\ A^{2,1} & A^{2,2} \end{pmatrix}$$

such that $A_0 = 0$, $A_1^{1,2} = 0$, $A_1^{2,1} = 0$, and the eigenvalues of $A^{1,1}$ resp. $A^{2,2}$ are minimal resp. non-minimal in their class modulo $\mathbb{Z}$. Then the $\mathbb{C}[\![s]\!][s^{-1}]$-basis transformation

$$A := \begin{pmatrix} s & 0 \\ 0 & 1 \end{pmatrix}^{-1} \left(A + s^2\partial_s\right) \begin{pmatrix} s & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} A^{1,1} + s & s^{-1}A^{1,2} \\ sA^{2,1} & A^{2,2} \end{pmatrix}$$

decreases $\delta(A)$ and the degree up to which A is computed by 1. After at most n steps, $\delta(A) = 0$ and $\exp(-2\pi\mathrm{i}A_1)$ is a monodromy matrix. In order to compute A_1 and hence $\exp(-2\pi\mathrm{i}A_1)$, it suffices to compute the $[\underline{m}]$-matrix of t up to degree

$$\kappa + \delta(A) + 1 \le \mu + n.$$

2.6 Spectral pairs

In this section, we describe an algorithm to compute the spectral pairs. We show that one can compute a basis representation of the Brieskorn lattice and of the operator t with respect to a $\mathbb{C}[\![s]\!]$-basis of the weight refined V-filtration. In particular, one can compute a standard basis of the Brieskorn lattice with respect to the weight refined V-filtration and hence, by proposition 1.10.11.2, the spectral pairs.

Let H_∞, H, and κ be computed by saturate and A be computed by transform. By corollary 1.8.6, $V^{>-1} \supset H'' \supset V^{n-1}$. Since the V^α are saturated, $V^{>-1} \supset H''_\infty \supset sH''_\infty \supset V^n$ and hence,

$$\max \underline{\alpha} - \min \underline{\alpha} < n + 1.$$

If $\max \underline{\alpha} - \min \underline{\alpha} < 1$ then, by lemma 1.6.7.3, $\underline{h}$ is a $\mathbb{C}[\![s]\!]$-basis of $V^{\min \underline{\alpha}}$ and hence $A_1 - E$ is the $[\underline{h}]$-basis representation of the residue $\mathrm{res}_{V^{\min \underline{\alpha}}}$ of $V^{\min \underline{\alpha}}$. By lemma 1.3.4.3, the nilpotent part of A_1 is the $\underline{h}$-basis representation of the operator N defining the weight filtration. Since one can compute the eigenvalues of A_1, one can compute a $\mathbb{C}$-basis transformation $U = (\overline{u}_i)_{1 \le i \le \mu}$ such that

$$\begin{pmatrix} \alpha_1 & & \\ & \ddots & \\ & & \alpha_\mu \end{pmatrix} + N := U^{-1} A_1 U$$

has Jordan normal form. By remark 1.7.3.1, one can compute the weight filtration indices $\underline{l} = (l_i)_{1 \le i \le \mu}$ of the columns of U from N. We reorder the columns of U such that $(\underline{\alpha}, \underline{l})$ is ordered $<_{\mathbb{Q} \times \mathbb{Z}}$-decreasingly and redefine

$$H := U^{-1} H,$$
$$A := U^{-1} A U.$$

Then the $\mathbb{C}[\![s]\!]$-basis

$$\underline{v} := s^{-\kappa} [\underline{m}] H_\infty U$$

of V^{α_1} refines the weight refinement in remark 1.10.1.1, $\langle H \rangle$ is the $\underline{v}$-basis representation of H'', and A is the $\underline{v}$-matrix of t. By remark 2.1.18.2 and proposition 2.1.25, one can compute a minimal standard basis

$$H = (\overline{h}_i)_{1 \le i \le \mu} := \mathsf{sb}(H, \kappa)$$

of $\langle H \rangle$ up to degree κ and the leading terms

$$\mathrm{lead}(H) =: (s^{k_i} \overline{e}_{j_i})_{1 \le i \le \mu}.$$

Since $s^\kappa\mathbb{C}[\![s]\!]^\mu \subset \langle H\rangle$, the truncated standard basis H is a polynomial minimal standard basis of $\langle H\rangle$ and $\underline{v}H$ is a minimal standard basis of H'' with respect to the weight refinement in remark 1.10.1.1. Then, by proposition 1.10.11.2,

$$\deg(\underline{v}H) = (\alpha_{j_i} + k_i, l_{j_i})_{1\le i\le\mu}$$

are the spectral pairs and

$$\langle s^{-\kappa} H_\infty U\overline{h}_i | \alpha \le \alpha_{j_i} + k_i, l_{j_i} \le l\rangle\mathbb{C}$$

is the $[\underline{m}]$-basis representation of $V^\alpha W_l(H''/sH'')$. Hence, one can compute the V- and weight filtration on H''/sH''.

In order to compute the spectral numbers only, one can skip the computation of $\underline{l}$ and reorder the columns of U such that $\underline{\alpha}$ is ordered increasingly. Then, by proposition 1.10.11.1,

$$\deg_V(\underline{v}H) = (\alpha_{j_i} + k_i)_{1\le i\le\mu}$$

are the spectral numbers and

$$\langle s^{-\kappa} H_\infty U\overline{h}_i | \alpha \le \alpha_{j_i} + k_i\rangle\mathbb{C}$$

is the $[\underline{m}]$-basis representation of $V^\alpha(H''/sH'')$. Hence, one can compute the V-filtration on H''/sH''.

If $\max\underline{\alpha} - \min\underline{\alpha} \ge 1$ then we proceed as follows. By section 2.5, we may assume that

$$A = \begin{pmatrix} A^{1,1} & A^{1,2} \\ A^{2,1} & A^{2,2} \end{pmatrix}$$

such that $A_0 = 0$, $A_1^{1,2} = 0$, $A_1^{2,1} = 0$, and the eigenvalues of $A_1^{1,1}$ resp. $A_1^{2,2}$ are less than $\min\underline{\alpha} + 1$ resp. greater or equal to $\min\underline{\alpha} + 1$. Then the $\mathbb{C}[\![s]\!][s^{-1}]$-basis transformation

$$H \mapsto \begin{pmatrix} s & 0 \\ 0 & 1 \end{pmatrix}^{-1} H,$$

$$A \mapsto \begin{pmatrix} s & 0 \\ 0 & 1 \end{pmatrix}^{-1} (A + s^2\partial_s) \begin{pmatrix} s & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} A^{1,1} + s & s^{-1}A^{1,2} \\ sA^{2,1} & A^{2,2} \end{pmatrix}$$

decreases $\max\underline{\alpha} - \min\underline{\alpha}$ and the degree up to which A is computed by 1 and increases κ with $s^\kappa\mathbb{C}[\![s]\!]^\mu \subset \langle H\rangle$ by 1. After at most n steps, $\max\underline{\alpha} - \min\underline{\alpha} <$

1 and one can compute the spectral numbers resp. spectral pairs as before. In order to compute the spectral numbers resp. spectral pairs, it suffices to compute the $[\underline{m}]$-matrix of t up to degree

$$\kappa + \lfloor \max \underline{\alpha} - \min \underline{\alpha} \rfloor + 1 \leq \mu + n.$$

2.7 (t, s)-module structure

In this section, we describe an algorithm to compute the (t, s)-module structure of the Brieskorn lattice. We show that one can compute a basis representation of the Brieskorn lattice and of the operator t with respect to a $\mathbb{C}[\![s]\!]$-basis of a Hodge splitting. In particular, one can compute a reduced standard basis of the Brieskorn lattice with respect to a Hodge splitting and hence, by proposition 1.10.12 and remark 1.10.13, M. Saito's matrices A_0 and A_1.

Let H_∞, H, and κ be computed by saturate and A be computed by transform. Then $\langle H \rangle$ is the $\underline{v}$-basis representation of H'' and A is the $\underline{v}$-matrix of t. By section 2.6, we may assume that

$$A_1 = \begin{pmatrix} \alpha_1 & & \\ & \ddots & \\ & & \alpha_\mu \end{pmatrix} + N$$

with $\alpha_1 \leq \cdots \leq \alpha_\mu < \alpha_1 + 1$ and that $s^\kappa \mathbb{C}[\![s]\!] \subset \langle H \rangle$. Then the $\mathbb{C}[\![s]\!]$-basis $\underline{v}$ of V^{α_1} refines the V-filtration. Let $\underline{c}$ be the image of $\underline{v}$ under the map

$$V^{\alpha_1} \twoheadrightarrow V^{\alpha_1}/sV^{\alpha_1} \cong \bigoplus\nolimits_{\alpha_j} C^{\alpha_j} \hookrightarrow V^{\alpha_1}.$$

Then $\underline{c}$ is a $<_{\mathbb{Q}}$-increasingly ordered $\mathbb{C}$-basis of $\bigoplus_{\alpha_j} C^{\alpha_j}$ compatible with the direct sum. By Nakayama's lemma, $\underline{c}$ is a $\mathbb{C}[\![s]\!]$-basis of V^{α_1} and $s^k \mathbb{C}[\![s]\!]^\mu$ is the $\underline{c}$-basis representation of V^{α_1+k}. Let the $\mathbb{C}[\![s]\!]$-basis transformation $U = \sum_{j=0}^\infty s^j U_j$ with $U_0 = E$ be defined by

$$\underline{v}U := \underline{c}.$$

By lemma 1.3.4.3 and proposition 1.6.18,

$$UsA_1 = (A + s^2 \partial_s)U$$

or equivalently

$$\Big(E + \sum_{k=1}^\infty s^k U_k\Big) sA_1 = \Big(\sum_{i=0}^\infty s^i A_{i+1} + s^2 \partial_s\Big) \Big(E + \sum_{j=1}^\infty s^j U_j\Big)$$

and hence

$$([\cdot,A_1]-k)U_k=\sum_{j=0}^{k-1}A_{k-j+1}U_j$$

for all $k\geq 1$. The eigenvalues of the commutator $[\cdot,A_1]\in\mathrm{End}_{\mathbb{C}}(\mathbb{C}^{\mu^2})$ are the differences of the eigenvalues $\underline{\alpha}$ of A_1. Since $\underline{\alpha}\in[\alpha_1,\alpha_1+1)$, this implies that, for all $k\geq 1$, the eigenvalues of $[\cdot,A_1]-k$ are in $\mathbb{Q}^*$ and hence

$$U_k=([\cdot,A_1]-k)^{-1}\Big(\sum_{j=0}^{k-1}A_{k-j+1}U_j\Big).$$

Hence, one can compute $\mathrm{jet}_k(U)$ from $\mathrm{jet}_{k+1}(A)$ and one can compute U up to any degree. Since $s^\kappa\mathbb{C}[\![s]\!]^\mu\subset\langle H\rangle$ and hence $s^\kappa\mathbb{C}[\![s]\!]^\mu\subset\langle U^{-1}H\rangle$,

$$\langle U^{-1}H\rangle=\langle\mathrm{jet}_\kappa(\mathrm{jet}_\kappa(U)^{-1}H)\rangle.$$

is the $\underline{c}$-basis representation of H''. Since $U_0=E$ and $\kappa\geq 0$, $\mathrm{jet}_\kappa(U)$ is a minimal standard basis of $\langle E\rangle=\mathbb{C}[\![s]\!]^\mu$. By remark 2.1.18.2 and proposition 2.1.23, one can compute a normal form

$$H:=\mathrm{jet}_\kappa(\mathsf{nf}_2(H,\mathrm{jet}_\kappa(U),\kappa))$$

of H with respect to $\mathrm{jet}_\kappa(U)$ up to degree κ in order to compute the $\underline{c}$-basis representation $\langle H\rangle$ of H''. Then $\mathbb{C}[\![s]\!]^\mu\cap\langle s^{-p}H\rangle$ is the $\underline{c}$-basis representation of the p-part $\widetilde{F}_pV^{\alpha_1}$ of the Steenbrink's Hodge filtration $\widetilde{F}$ on V^{α_1}. By remark 2.1.18.2 and proposition 2.1.25, one can compute a standard basis

$$H=(\overline{h}_i)_{1\leq i\leq\mu}:=\mathsf{sb}(H,\kappa)$$

of $\langle H\rangle$ up to degree κ. Since $s^\kappa\mathbb{C}[\![s]\!]^\mu\subset\langle H\rangle$, the truncated standard basis H is a polynomial standard basis and $\underline{c}H$ is a standard basis with respect to the V-filtration. Since $\deg_V(s^kc_j)=\alpha_j+k$, one can compute $F=(\overline{f}_i)_{1\leq i\leq\mu}$ such that

$$\underline{c}F:=\mathrm{lead}_V(\underline{c}H)$$

are the V-leading terms of $\underline{c}H$ and the leading terms

$$(s^{k_i}\overline{e}_{j_i})_{1\leq i\leq\mu}:=\mathrm{lead}(F)$$

of F. Then $\langle F_k\rangle\mathbb{C}$ with

$$F_k:=\big(\mathrm{jet}_0(\overline{f}_i/s^{k_i})\big)_{k_i\leq k}$$

is the $\underline{c}$-basis representation of $\widetilde{F}_k(\bigoplus_{\alpha_j} C^{\alpha_j})$. Hence, one can compute Steenbrink's Hodge filtration $\widetilde{F}$ on $\bigoplus_{\alpha_j} C^{\alpha_j}$.

By lemma 1.3.4.3 and proposition 1.6.18,

$$A := sA_1 = \mathrm{jet}_1(A)$$

is the $\underline{c}$-matrix of t and

$$A_1 = \begin{pmatrix} \alpha_1 & & \\ & \ddots & \\ & & \alpha_\mu \end{pmatrix} + N.$$

By remark 1.7.3.1, one can compute the weight filtration indices of $\underline{c}$ from N. By remark 1.7.11, one can compute a $\mathbb{C}$-basis transformation U such that

$$\underline{c} := \underline{c}U$$

is a $<_{\mathbb{Q}\times\mathbb{Z}}$-decreasingly ordered $\mathbb{C}$-basis of $\bigoplus_{\alpha_j}\bigoplus_{p\in\mathbb{Z}} C^{\alpha_j,p}$ compatible with the direct sum for a Hodge splitting as in remark 1.10.1.2. Let

$$H := U^{-1}H,$$
$$A := U^{-1}AU.$$

Then $\langle H\rangle$ is the $\underline{c}$-basis representation of H'' with $s^\kappa\mathbb{C}[\![s]\!]^\mu \subset H$ and A is the $\underline{c}$-matrix of t. By remark 2.1.18.2 and proposition 2.1.25, one can compute a reduced standard basis

$$H := \mathsf{rsb}_1(H, \kappa+1)$$

of $\langle H\rangle$ up to degree $\kappa+1$. Since $s^\kappa\mathbb{C}[\![s]\!]^\mu \subset \langle H\rangle$, H is a standard basis and, by lemma 2.1.11, H is a $\mathbb{C}[\![s]\!]$-basis of H. Hence, $\underline{c}H$ is a $\mathbb{C}[\![s]\!]$-basis and a standard basis of H'' with respect to a Hodge splitting which is reduced up to V-order less then $\alpha_1+\kappa+2$. Since $s^\kappa\mathbb{C}[\![s]\!]^\mu \subset \langle H\rangle$,

$$V^{\alpha_1+\kappa} = \langle s^\kappa \underline{c}\rangle \subset \langle \underline{c}H\rangle = H''$$

and hence

$$\alpha_1+\kappa+1 > \max\deg_V(\underline{c}H).$$

By proposition 1.6.18, remark 2.1.18.2, and proposition 2.1.23, one can compute a normal form

$$A := \mathrm{jet}_1\big(\mathsf{nf}_2\big((\mathrm{jet}_{\kappa+1}(A) + s^2\partial_s)H, H, \kappa+1\big)\big)$$

of $(\mathrm{jet}_{\kappa+1}(A) + s^2\partial_s)H$ with respect to H up to degree $\kappa+1$ in order to compute the $\underline{c}H$-matrix A of t up to degree 1. By proposition 1.10.12 and remark 1.10.13,

$$A = A_0 + sA_1 = A^{\underline{h}}$$

for a reduced standard basis $\underline{h}$ of H'' with respect to a Hodge splitting and hence, by proposition 1.6.18,

$$(H'', t) \cong_{\mathbb{C}\{\{s\}\}} (\mathbb{C}\{\{s\}\}^\mu, A_0 + sA_1 + s^2\partial_s).$$

In order to compute A_0 and A_1, it suffices to compute the $[\underline{m}]$-matrix of t up to degree

$$2(\kappa + \lfloor \max \underline{\alpha} - \min \underline{\alpha} \rfloor) + 1 \leq 2\mu + 2n - 1.$$

Chapter 3

Applications and Examples

In this chapter, we give examples and applications for the algorithms in chapter 2.

For quasihomogeneous singularities, the invariants described in chapter 1 can be computed in terms of weights of a monomial vector space basis of the Jacobian algebra. The algorithms in chapter 2 are implemented in the computer algebra system SINGULAR [GPS02] by the author [Sch02a]. We demonstrate this implementation for a singularity with a 2×2 resp. 3×3 Jordan block of the monodromy. The the hypersurface singularities up to Milnor number 16 are classified by V.I. Arnold [AGZV85]. Using the SINGULAR implementation and Arnold's classification, we verify Hertling's conjecture 1.8.13 up to Milnor number 16.

3.1 Quasihomogeneous singularities

In this section, we compute the invariants described in chapter 1 in terms of weights of a monomial vector space basis of the Jacobian algebra.

Let f be a quasihomogeneous singularity. By a coordinate transformation, we may assume that $f \in \mathbb{C}[\underline{x}]$ is a quasihomogeneous polynomial with respect to a weight vector $\overline{w}$ with $\sum_{i=0}^{n} w^i = 1$. Then the matrix of t can be computed explicitly by the Euler relation

$$f = \sum_{i=0}^{n} w^i x_i \partial_i(f).$$

We denote by $\deg_{\overline{w}}$ the weighted degree with respect to $\overline{w}$. Let $[\underline{m}]$ be a monomial $\mathbb{C}$-basis of the Jacobian algebra $\mathbb{C}\{\underline{x}\}/\langle\underline{\partial}(f)\rangle$. By the isomorphism

$$H''/sH'' \cong \Omega^{n+1}/\mathrm{d}f \wedge \Omega^n \cong \mathbb{C}\{\underline{x}\}/\langle\underline{\partial}(f)\rangle,$$

$[\underline{m}]$ can be considered as a vectorspace basis $[\underline{m}\mathrm{d}\underline{x}]$ of H''/sH''. Then, by proposition 1.4.8,

$$\begin{aligned} t[m_j\mathrm{d}\underline{x}] &= [fm_j\mathrm{d}\underline{x}] \\ &= \Big[\sum_{i=0}^{n} w^i m_j x_i \partial_i(f)\mathrm{d}\underline{x}\Big] \\ &= \Big[\sum_{i=0}^{n} (-1)^i w^i m_j x_i \mathrm{d}f \wedge \mathrm{d}\underline{x}_{\widehat{i}}\Big] \\ &= s\Big[\sum_{i=0}^{n} \partial_i(w^i m_j x_i)\mathrm{d}\underline{x}\Big] \\ &= s(\deg_{\overline{w}}(m_j)+1)[m_j\mathrm{d}\underline{x}] \end{aligned}$$

and hence

$$t[m_j\mathrm{d}\underline{x}] = (\deg_{\overline{w}}(m_j)+1)s[m_j\mathrm{d}\underline{x}], \tag{3.1.1}$$

$$(t\partial_t - \deg_{\overline{w}}(m_j))[m_j\mathrm{d}\underline{x}] = 0. \tag{3.1.2}$$

Hence, for $\alpha > -1$ or $\alpha \notin \mathbb{Z}$, $(s^k[m_j\mathrm{d}\underline{x}])_{\deg_{\overline{w}}(m_j)+k=\alpha}$ is a $\mathbb{C}$-basis of C^α. By (3.1.2) and lemma 1.3.4.3, $\mathrm{N}=0$ and the weight filtration is trivial. Hence, $[\underline{m}\mathrm{d}\underline{x}]$ is a standard basis with respect to the weight refinement and with respect to a Hodge splitting as in remark 1.10.1. By proposition 1.10.11, $\deg_{\overline{w}}(\underline{m})$ are the spectral numbers and

$$(\deg_{\overline{w}}(m_i), n-1)_{1\le i\le\mu}$$

are the spectral pairs. In particular, since $\mathrm{N}=0$ and by lemma 1.8.3,

$$\begin{pmatrix} \mathrm{e}^{-2\pi\mathrm{i}\deg_{\overline{w}}(m_1)} & & \\ & \ddots & \\ & & \mathrm{e}^{-2\pi\mathrm{i}\deg_{\overline{w}}(m_\mu)} \end{pmatrix}$$

is a monodromy matrix. By (3.1.1), the $[\underline{m}\mathrm{d}\underline{x}]$-matrix of t is $A = sA_1$ with

$$A_1 = \begin{pmatrix} \deg_{\overline{w}}(m_1)+1 & & \\ & \ddots & \\ & & \deg_{\overline{w}}(m_\mu)+1 \end{pmatrix}.$$

By proposition 1.6.18, the diagram

$$\begin{array}{ccc} H'' & \xrightarrow{\quad t \quad} & H'' \\ {\scriptstyle[\underline{m}\mathrm{d}\underline{x}]}\uparrow{\scriptstyle\wr} & & {\scriptstyle[\underline{m}\mathrm{d}\underline{x}]}\uparrow{\scriptstyle\wr} \\ \mathbb{C}\{\{s\}\}^\mu & \xrightarrow{sA_1+s^2\partial_s} & \mathbb{C}\{\{s\}\}^\mu \end{array}$$

commutes.

3.2 Example with SINGULAR

The algorithms in chapter 2 are implemented in the computer algebra system SINGULAR [GPS02] in the library `gaussman.lib` [Sch02a]. We demonstrate the usage of the SINGULAR implementation by computing an example with a 2×2 Jordan block of the monodromy.

We consider the singularity of type $T_{2,5,5}$ defined by the polynomial

$$f = x^2y^2 + x^5 + y^5 \in \mathbb{C}\{x, y\}.$$

For the following computations, we use a PENTIUM III 850 with LINUX operating system.

First, we load the SINGULAR libraries `gaussman.lib` and `sing.lib`.

```
> LIB "gaussman.lib";
> LIB "sing.lib";
```

Then we define the ring $R = \mathbb{Q}[x, y]_{(x,y)}$ and the polynomial $f = x^5 + x^2y^2 + y^5 \in R$.

```
> ring R=0,(x,y),ds;
> poly f=x2y2+x5+y5;
```

The `ring` command defines and activates a polynomial ring over a field localized by a monomial ordering [GP96]. In the definition of `R`, `0` means that the base field has characteristic 0 and equals $\mathbb{Q}$, `(x,y)` means that the variables are x and y, and `ds` means that the monomial ordering is a local degree ordering. The `poly` command defines a polynomial in the active ring. Note that a monomial x^iy^j is denoted by `x`i`y`j.

Next, we compute the Milnor number μ using the `milnor` command.

```
> int mu=milnor(f);
> mu;
11
```

Hence, $\mu = 11$.

Next, we compute the Jordan data of the monodromy using the `monodromy` command.

```
> list l=monodromy(f);
> def e,s,d=l[1..3];
> e;
e[1]=1/2
e[2]=7/10
e[3]=9/10
e[4]=1
e[5]=11/10
e[6]=13/10
> s;
2,1,1,1,1,1
> d;
1,2,2,1,2,2
```

The computation takes less than one second. The result is a list consisting of

$$\begin{aligned}\underline{a} &= \frac{1}{2}, \frac{7}{10}, \frac{9}{10}, 1, \frac{11}{10}, \frac{13}{10},\\ \underline{s} &= 2,\ 1,\ 1, 1,\ 1,\ 1,\\ \underline{d} &= 1,\ 2,\ 2, 1,\ 2,\ 2.\end{aligned}$$

A triple (a_i, s_i, d_i) corresponds to d_i Jordan blocks of the monodromy of size $s_i \times s_i$ with eigenvalue $\mathrm{e}^{-2\pi \mathrm{i} a_i}$. Hence,

$$\begin{pmatrix}
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & \mathrm{e}^{-\pi \mathrm{i}\frac{7}{5}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & \mathrm{e}^{-\pi \mathrm{i}\frac{7}{5}} & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & \mathrm{e}^{-\pi \mathrm{i}\frac{9}{5}} & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & \mathrm{e}^{-\pi \mathrm{i}\frac{9}{5}} & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathrm{e}^{-\pi \mathrm{i}\frac{11}{5}} & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathrm{e}^{-\pi \mathrm{i}\frac{11}{5}} & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathrm{e}^{-\pi \mathrm{i}\frac{13}{5}} & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathrm{e}^{-\pi \mathrm{i}\frac{13}{5}}
\end{pmatrix}.$$

is a monodromy matrix in Jordan normal form. Note that the eigenvalues are roots of unity and that there is a Jordan block of size 2×2 with eigenvalue -1 which is the maximal size for 2 variables by theorem 1.2.5.

Next, we compute the weight refined V- filtration on H''/sH'' using the `vwfilt` command.

```
> list v=vwfilt(f);
> def a,l,d,V,m,j=v[1..6];
> a;
a[1]=-1/2
a[2]=-3/10
a[3]=-1/10
a[4]=0
a[5]=1/10
a[6]=3/10
a[7]=1/2
> l;
2,1,1,1,1,1,0
> d;
1,2,2,1,2,2,1
> V;
[1]:
   _[1]=gen(11)
[2]:
   _[1]=gen(10)
   _[2]=gen(6)
[3]:
   _[1]=gen(9)
   _[2]=gen(4)
[4]:
   _[1]=gen(5)
[5]:
   _[1]=gen(3)
   _[2]=gen(8)
[6]:
   _[1]=gen(2)
   _[2]=gen(7)
[7]:
   _[1]=gen(1)
> m;
m[1]=y5
m[2]=y4
m[3]=y3
m[4]=y2
```

```
m[5]=xy
m[6]=y
m[7]=x4
m[8]=x3
m[9]=x2
m[10]=x
m[11]=1
> j;
j[1]=2x2y+5y4
j[2]=2xy2+5x4
j[3]=5x5-5y5
j[4]=10y6+25x3y4
```

The computation takes less than one second. Note that the i-th unit vector $\overline{e}_i = (\delta_{i,j})^j$ where δ is the Kronecker symbol is denoted by gen(i). The result is a list consisting of

$$\begin{aligned}
\underline{a} =& -\frac{1}{2}, & -\frac{3}{10}, & & -\frac{1}{10}, & & 0, & & \frac{1}{10}, & & \frac{3}{10}, & & \frac{1}{2}, \\
\underline{l} =& \ 2, & 1, & & 1, & & 1, & & 1, & & 1, & & 0, \\
\underline{d} =& \ 1, & 2, & & 2, & & 1, & & 2, & & 2, & & 1, \\
\underline{V} =& (\overline{e}_{11}), & (\overline{e}_{10}, \overline{e}_6), & & (\overline{e}_9, \overline{e}_4), & & (\overline{e}_5), & & (\overline{e}_3, \overline{e}_8), & & (\overline{e}_2, \overline{e}_7), & & (\overline{e}_1),
\end{aligned}$$

and

$$\begin{aligned}
\underline{m} &= y^5, y^4, y^3, y^2, xy, y, x^4, x^3, x^2, x, 1, \\
\underline{j} &= 2x^2y + 5y^4, 2xy^2 + 5x^4, 5x^5 - 5y^5, 10y^6 + 25x^3y^4.
\end{aligned}$$

A triple (a_i, l_i, d_i) corresponds to d_i spectral pairs (a_i, l_i), $\underline{j}$ is a standard basis of the Jacobian ideal $\langle \partial_x(f), \partial_y(f) \rangle$, and $[\underline{m}]$ is a monomial vectorspace basis of the Jacobian algebra $\mathbb{C}/\langle \partial_x(f), \partial_y(f) \rangle$. Hence,

$$\begin{gathered}
\left(-\frac{1}{2}, 2\right), \left(-\frac{3}{10}, 1\right), \left(-\frac{3}{10}, 1\right), \left(-\frac{1}{10}, 1\right), \left(-\frac{1}{10}, 1\right), \\
(0, 1), \\
\left(\frac{1}{10}, 1\right), \left(\frac{1}{10}, 1\right), \left(\frac{3}{10}, 1\right), \left(\frac{3}{10}, 1\right), \left(\frac{1}{2}, 0\right)
\end{gathered}$$

are the spectral pairs. Note that they are symmetric around $(0, 1)$ by proposition 1.8.5. By the isomorphism

$$H''/sH'' \cong \Omega^{n+1}/\mathrm{d}f \wedge \Omega^n \cong \mathbb{C}\{x, y\}/\langle \partial_x(f), \partial_y(f) \rangle,$$

$[\underline{m}]$ can be considered as a vectorspace basis $[\underline{m}(\mathrm{d}x \wedge \mathrm{d}y)]$ of H''/sH''. Then V_i is the $[\underline{m}(\mathrm{d}x \wedge \mathrm{d}y)]$-basis representation of a vectorspace basis of the d_i-dimensional (a_i, l_i)-graded part of the weight refined V-filtration on H''/sH'', that is,

$$d_i = \dim_{\mathbb{C}}\big(\mathrm{gr}_V^{a_i}\,\mathrm{gr}_{l_i}^W(H''/sH'')\big),$$
$$\langle[\underline{m}(\mathrm{d}x \wedge \mathrm{d}y)]V_i\rangle\mathbb{C} = \mathrm{gr}_V^{a_i}\,\mathrm{gr}_{l_i}^W(H''/sH'').$$

Next, we compute the spectral numbers and Hertling's γ-invariant using the `spgamma` command.

```
> v=spnf(a,l,d);
> def a,d=v[1..2];
> a;
a[1]=-1/2
a[2]=-3/10
a[3]=-1/10
a[4]=0
a[5]=1/10
a[6]=3/10
a[7]=1/2
> d;
1,2,2,1,2,2,1
> spgamma(v);
1/240
```

Hence, the spectral numbers are

$$-\frac{1}{2}, -\frac{3}{10}, -\frac{3}{10}, -\frac{1}{10}, -\frac{1}{10}, 0, \frac{1}{10}, \frac{1}{10}, \frac{3}{10}, \frac{3}{10}, \frac{1}{2}$$

and the γ-invariant is

$$\gamma = \frac{1}{240}.$$

Finally, we compute the (t, s)-module structure of the Brieskorn lattice using the `tmatrix` command.

```
> list l=tmatrix(f);
> def A0,A1=l[1..2];
```

The computation takes about one second. The result is a list consisting of

$$A_0 = \begin{pmatrix} 0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0&0&0&0 \end{pmatrix},$$

$$A_1 = \begin{pmatrix} \frac{1}{2}&0&0&0&0&0&0&0&0&0&0\\ 0&\frac{7}{10}&0&0&0&0&0&0&0&0&0\\ 0&0&\frac{7}{10}&0&0&0&0&0&0&0&0\\ 0&0&0&\frac{9}{10}&0&0&0&0&0&0&0\\ 0&0&0&0&\frac{9}{10}&0&0&0&0&0&0\\ 0&0&0&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&\frac{11}{10}&0&0&0&0\\ 0&0&0&0&0&0&0&\frac{11}{10}&0&0&0\\ 0&0&0&0&0&0&0&0&\frac{13}{10}&0&0\\ 0&0&0&0&0&0&0&0&0&\frac{13}{10}&0\\ 0&0&0&0&0&0&0&0&0&0&\frac{3}{2} \end{pmatrix}.$$

Hence,

$$(H'', t) \cong (\mathbb{C}\{\{s\}\}^{\mu}, A_0 + sA_1 + s^2\partial_s)$$

as (t, s)-module, that is, there is a $\mathbb{C}\{\{s\}\}$-isomorphism ϕ such that the diagram

$$\begin{array}{ccc} H'' & \xrightarrow{\quad t \quad} & H'' \\ {\scriptstyle\phi}\uparrow{\scriptstyle\sim} & & {\scriptstyle\phi}\uparrow{\scriptstyle\sim} \\ \mathbb{C}\{\{s\}\}^{\mu} & \xrightarrow{A_0+sA_1+s^2\partial_s} & \mathbb{C}\{\{s\}\}^{\mu} \end{array}$$

commutes. Note that A_1 is semisimple with eigenvalues the spectral numbers added by one and that

$$A_0\overline{e}_1 = \overline{e}_{11},$$
$$A_0\overline{e}_{11} = 0$$

where

$$A_1\overline{e}_1 = \frac{1}{2}\overline{e}_1,$$
$$A_1\overline{e}_{11} = \Big(\frac{1}{2}+1\Big)\overline{e}_{11}.$$

By proposition 1.10.12, $\mathrm{gr}_V(A_0) = A_0$ can be identified with N. Hence, $\overline{e}_1$ can be identified with a $\mathbb{C}[\mathrm{N}]$-generator of the 2×2 Jordan block of the monodromy with eigenvalue $\mathrm{e}^{2\pi\mathrm{i}(\frac{1}{2}+\mathbb{Z})} = -1$.

3.3 Example with 3×3 Jordan block

In this section, we use the SINGULAR implementation to compute a more complicated example with a 3×3 Jordan block of the monodromy.

We consider the singularity defined by the polynomial

$$f = x^2y^2z^2 + x^7 + y^7 + z^7 \in \mathbb{C}\{x, y, z\}.$$

The `milnor` command returns the Milnor number $\mu = 167$.

The `monodromy` command returns

$$\begin{aligned}
\underline{a} &= \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{4}{7}, \tfrac{9}{14}, \tfrac{9}{14}, \tfrac{5}{7}, \tfrac{11}{14}, \tfrac{11}{14}, \tfrac{6}{7}, \tfrac{13}{14}, \tfrac{13}{14}, 1, \tfrac{15}{14}, \tfrac{15}{14}, \tfrac{8}{7}, \tfrac{17}{14}, \tfrac{17}{14}, \tfrac{9}{7}, \tfrac{19}{14}, \tfrac{19}{14}, \tfrac{10}{7},\\
\underline{s} &= 1, 3, 1, 1, 2, 1, 1, 2, 1, 1, 2, 2, 1, 2, 1, 1, 2, 1, 1, 2, 1,\\
\underline{d} &= 18, 1, 3, 15, 3, 3, 15, 3, 3, 15, 3, 1, 15, 3, 3, 15, 3, 3, 15, 3, 3
\end{aligned}$$

after about $2\frac{1}{2}$ minutes. Hence, the Jordan data of the monodromy is

$$\begin{aligned}
&(-1,1)^{18}, (-1,3), (\mathrm{e}^{-\pi\mathrm{i}\frac{8}{7}},1)^3, (\mathrm{e}^{-\pi\mathrm{i}\frac{9}{7}},1)^{15}, (\mathrm{e}^{-\pi\mathrm{i}\frac{9}{7}},2)^3, (\mathrm{e}^{-\pi\mathrm{i}\frac{10}{7}},1)^3,\\
&\quad (\mathrm{e}^{-\pi\mathrm{i}\frac{11}{7}},1)^{15}, (\mathrm{e}^{-\pi\mathrm{i}\frac{11}{7}},2)^3, (\mathrm{e}^{-\pi\mathrm{i}\frac{12}{7}},1)^3, (\mathrm{e}^{-\pi\mathrm{i}\frac{13}{7}},1)^{15}, (\mathrm{e}^{-\pi\mathrm{i}\frac{13}{7}},2)^3,\\
&\quad\quad (1,2), (\mathrm{e}^{-\pi\mathrm{i}\frac{15}{7}},1)^{15}, (\mathrm{e}^{-\pi\mathrm{i}\frac{15}{7}},2)^3, (\mathrm{e}^{-\pi\mathrm{i}\frac{16}{7}},1)^3, (\mathrm{e}^{-\pi\mathrm{i}\frac{17}{7}},1)^{15},\\
&\quad\quad\quad (\mathrm{e}^{-\pi\mathrm{i}\frac{17}{7}},2)^3, (\mathrm{e}^{-\pi\mathrm{i}\frac{18}{7}},1)^3, (\mathrm{e}^{-\pi\mathrm{i}\frac{19}{7}},1)^{15}, (\mathrm{e}^{-\pi\mathrm{i}\frac{19}{7}},2)^3, (\mathrm{e}^{-\pi\mathrm{i}\frac{20}{7}},1)^3.
\end{aligned}$$

A triple $(\mathrm{e}^{-2\pi\mathrm{i}a}, s)^d$ denotes d Jordan blocks of the monodromy of size $s \times s$ with eigenvalue $\mathrm{e}^{-2\pi\mathrm{i}a}$. Note that the eigenvalues are roots of unity and that there is a Jordan block of size 2×2 resp. 3×3 with eigenvalue 1 resp. -1 which is the maximal size for 3 variables by theorem 1.2.5.

The `sppairs` command returns

$$\begin{array}{rl}
\underline{a} = & -\frac{1}{2}, -\frac{5}{14}, -\frac{3}{14}, -\frac{3}{14}, -\frac{1}{14}, -\frac{1}{14}, 0, \frac{1}{14}, \frac{1}{14}, \frac{1}{7}, \frac{3}{14}, \frac{3}{14}, \frac{2}{7}, \frac{5}{14}, \frac{5}{14}, \frac{3}{7}, \frac{1}{2}, \\
 & \frac{4}{7}, \frac{9}{14}, \frac{9}{14}, \frac{5}{7}, \frac{11}{14}, \frac{11}{14}, \frac{6}{7}, \frac{13}{14}, \frac{13}{14}, 1, \frac{15}{14}, \frac{15}{14}, \frac{17}{14}, \frac{17}{14}, \frac{19}{14}, \frac{3}{2}, \\
\underline{l} = & 4, 3, 3, 2, 3, 2, 3, 3, 2, 2, 3, 2, 2, 3, 2, 2, 2, \\
 & 2, 2, 1, 2, 2, 1, 2, 2, 1, 1, 2, 1, 2, 1, 1, 0, \\
\underline{d} = & 1, 3, 3, 3, 3, 6, 1, 3, 9, 3, 3, 12, 3, 3, 15, 3, 19, \\
 & 3, 15, 3, 3, 12, 3, 3, 9, 3, 1, 6, 3, 3, 3, 3, 1
\end{array}$$

after about $2\frac{1}{2}$ minutes. Hence, the spectral pairs are

$$\begin{gathered}
\left(-\tfrac{1}{2}, 4\right), \left(-\tfrac{5}{14}, 3\right)^3, \left(-\tfrac{3}{14}, 3\right)^3, \left(-\tfrac{3}{14}, 2\right)^3, \left(-\tfrac{1}{14}, 3\right)^3, \left(-\tfrac{1}{14}, 2\right)^6, \\
\left(0, 3\right), \left(\tfrac{1}{14}, 3\right)^3, \left(\tfrac{1}{14}, 2\right)^9, \left(\tfrac{1}{7}, 2\right)^3, \left(\tfrac{3}{14}, 3\right)^3, \\
\left(\tfrac{3}{14}, 2\right)^{12}, \left(\tfrac{2}{7}, 2\right)^3, \left(\tfrac{5}{14}, 3\right)^3, \left(\tfrac{5}{14}, 2\right)^{15}, \\
\left(\tfrac{3}{7}, 2\right)^3, \left(\tfrac{1}{2}, 2\right)^{19}, \left(\tfrac{4}{7}, 2\right)^3, \\
\left(\tfrac{9}{14}, 2\right)^{15}, \left(\tfrac{9}{14}, 1\right)^3, \left(\tfrac{5}{7}, 2\right)^3, \left(\tfrac{11}{14}, 2\right)^{12}, \\
\left(\tfrac{11}{14}, 1\right)^3, \left(\tfrac{6}{7}, 2\right)^3, \left(\tfrac{13}{14}, 2\right)^9, \left(\tfrac{13}{14}, 1\right)^3, \left(1, 1\right), \\
\left(\tfrac{15}{14}, 2\right)^6, \left(\tfrac{15}{14}, 1\right)^3, \left(\tfrac{17}{14}, 2\right)^3, \left(\tfrac{17}{14}, 1\right)^3, \left(\tfrac{19}{14}, 1\right)^3, \left(\tfrac{3}{2}, 0\right).
\end{gathered}$$

A triple $(a, l)^d$ denotes d spectral pairs (a, l). Note that

$$d_{l_i}^{a_i} := d_i$$

have the symmetry properties

$$\begin{aligned}
d_l^a &= d_l^{3-l-a}, \\
d_l^a &= d_{4-l}^{a-2+l}, \\
d_l^a &= d_{4-l}^{1-a}
\end{aligned}$$

by proposition 1.8.5.

3.4 Hertling's conjecture

The hypersurface singularities up to Milnor number 16 are classified by V.I. Arnold [AGZV88]. Using Arnold's classification and the SINGULAR implementation, we verify Hertling's conjecture 1.8.13 up to Milnor number 16.

By theorem 1.8.14, Hertling's conjecture holds for quasihomogeneous singularities and, by remark 1.8.15, for singularities of type $T_{p,q,r}$. We exclude these types of singularities and compute the spectral pairs and Hertling's γ-invariant for the remaining singularities. The result is contained in table 3.1, the singularity type in the first column, the defining polynomial in the second column, the spectral numbers in the third column, the γ-invariant in the fourth column, and the computation time in seconds in the last column. We omit the weight filtration indices since they are trivial.

By theorem 1.8.16, Hertling's conjecture holds for irreducible plane curve singularities. Apart from $V_{1,1}$ and $V_{1,1}^{\#}$, the spectra in table 3.1 occur already in the list of spectra of unimodal and bimodal singularities in [AGZV85].

Table 3.1: γ-invariant for Milnor number $\mu \leq 16$

type	polynomial	spectral numbers	γ	time
$Z_{1,1}$	$yx^3 + y^3x^2 + y^8$	$-\frac{4}{7}$, $-\frac{7}{16}$, $-\frac{5}{16}$, $-\frac{2}{7}$, $-\frac{3}{16}$, $-\frac{1}{7}$, $-\frac{1}{16}$, 0, 0, $\frac{1}{16}$, $\frac{1}{7}$, $\frac{3}{16}$, $\frac{2}{7}$, $\frac{5}{16}$, $\frac{7}{16}$, $\frac{4}{7}$	$\frac{1}{384}$	3
$W_{1,1}$	$x^4 + y^3x^2 + y^7$	$-\frac{7}{12}$, $-\frac{3}{7}$, $-\frac{1}{3}$, $-\frac{2}{7}$, $-\frac{1}{6}$, $-\frac{1}{7}$, $-\frac{1}{12}$, 0, 0, $\frac{1}{12}$, $\frac{1}{7}$, $\frac{1}{6}$, $\frac{2}{7}$, $\frac{1}{3}$, $\frac{3}{7}$, $\frac{7}{12}$	$\frac{1}{336}$	2
$W_{1,1}^{\#}$	$x^4 + 2y^3x^2 + y^6 + y^5x$	$-\frac{7}{12}$, $-\frac{11}{26}$, $-\frac{9}{26}$, $-\frac{7}{26}$, $-\frac{5}{26}$, $-\frac{3}{26}$, $-\frac{1}{12}$, $-\frac{1}{26}$, $\frac{1}{26}$, $\frac{1}{12}$, $\frac{3}{26}$, $\frac{5}{26}$, $\frac{7}{26}$, $\frac{9}{26}$, $\frac{11}{26}$, $\frac{7}{12}$	$\frac{7}{1872}$	4
$Q_{2,1}$	$z^2y + x^3 + y^2x^2 + y^7$	$-\frac{1}{12}$, $\frac{1}{14}$, $\frac{3}{14}$, $\frac{1}{4}$, $\frac{1}{3}$, $\frac{5}{14}$, $\frac{5}{12}$, $\frac{1}{2}$, $\frac{7}{12}$, $\frac{9}{14}$, $\frac{2}{3}$, $\frac{3}{4}$, $\frac{11}{14}$, $\frac{13}{14}$, $\frac{13}{12}$	$\frac{1}{336}$	3

Table 3.1: γ-invariant for Milnor number $\mu \leq 16$

$Q_{2,2}$	$z^2y + x^3 + y^2x^2 + y^8$	$-\frac{1}{12}, \frac{1}{16}, \frac{3}{16}, \frac{1}{4}, \frac{5}{16}, \frac{1}{3}, \frac{5}{12}, \frac{7}{16}, \frac{9}{16}, \frac{7}{12}, \frac{2}{3}, \frac{11}{16}, \frac{3}{4}, \frac{13}{16}, \frac{15}{16}, \frac{13}{12}$	$\frac{7}{1152}$	2
$S_{1,1}$	$z^2y + zx^2 + y^2x^2 + y^6$	$-\frac{1}{10}, \frac{1}{12}, \frac{1}{5}, \frac{1}{4}, \frac{3}{10}, \frac{2}{5}, \frac{5}{12}, \frac{1}{2}, \frac{7}{12}, \frac{3}{5}, \frac{7}{10}, \frac{3}{4}, \frac{4}{5}, \frac{11}{12}, \frac{11}{10}$	$\frac{1}{288}$	3
$S_{1,2}$	$z^2y + zx^2 + y^2x^2 + y^7$	$-\frac{1}{10}, \frac{1}{14}, \frac{1}{5}, \frac{3}{14}, \frac{3}{10}, \frac{5}{14}, \frac{2}{5}, \frac{1}{2}, \frac{1}{2}, \frac{3}{5}, \frac{9}{14}, \frac{7}{10}, \frac{11}{14}, \frac{4}{5}, \frac{13}{14}, \frac{11}{10}$	$\frac{1}{140}$	3
$S^{\#}_{1,1}$	$z^2y + zx^2 + zy^3 + y^4x$	$-\frac{1}{10}, \frac{1}{11}, \frac{2}{11}, \frac{3}{11}, \frac{3}{10}, \frac{4}{11}, \frac{5}{11}, \frac{1}{2}, \frac{6}{11}, \frac{7}{11}, \frac{7}{10}, \frac{8}{11}, \frac{9}{11}, \frac{10}{11}, \frac{11}{10}$	$\frac{1}{220}$	5
$S^{\#}_{1,2}$	$z^2y + zx^2 + zy^3 + y^3x^2$	$-\frac{1}{10}, \frac{1}{12}, \frac{1}{6}, \frac{1}{4}, \frac{3}{10}, \frac{1}{3}, \frac{5}{12}, \frac{1}{2}, \frac{1}{2}, \frac{7}{12}, \frac{2}{3}, \frac{7}{10}, \frac{3}{4}, \frac{5}{6}, \frac{11}{12}, \frac{11}{10}$	$\frac{13}{1440}$	5
$U_{1,1}$	$x^3 + xz^2 + xy^3 + y^2z^2$	$-\frac{1}{9}, \frac{1}{10}, \frac{1}{5}, \frac{2}{9}, \frac{3}{10}, \frac{2}{5}, \frac{4}{9}, \frac{1}{2}, \frac{5}{9}, \frac{3}{5}, \frac{7}{10}, \frac{7}{9}, \frac{4}{5}, \frac{9}{10}, \frac{10}{9}$	$\frac{11}{2160}$	5
$U_{1,2}$	$x^3 + xz^2 + xy^3 + y^4z$	$-\frac{1}{9}, \frac{1}{11}, \frac{2}{11}, \frac{2}{9}, \frac{3}{11}, \frac{4}{11}, \frac{4}{9}, \frac{5}{11}, \frac{6}{11}, \frac{5}{9}, \frac{7}{11}, \frac{8}{11}, \frac{7}{9}, \frac{9}{11}, \frac{10}{11}, \frac{10}{9}$	$\frac{1}{99}$	2
$V_{1,1}$	$yx^2 + z^4 + z^2y^2 + y^5$	$-\frac{1}{8}, \frac{1}{10}, \frac{1}{8}, \frac{1}{4}, \frac{3}{10}, \frac{3}{8}, \frac{3}{8}, \frac{1}{2}, \frac{1}{2}, \frac{5}{8}, \frac{5}{8}, \frac{7}{10}, \frac{3}{4}, \frac{7}{8}, \frac{9}{10}, \frac{9}{8}$	$\frac{1}{240}$	3
$V^{\#}_{1,1}$	$yx^2 + z^3y + y^4 + z^3x$	$-\frac{1}{8}, \frac{1}{9}, \frac{1}{8}, \frac{2}{9}, \frac{1}{3}, \frac{3}{8}, \frac{3}{8}, \frac{4}{9}, \frac{5}{9}, \frac{5}{8}, \frac{5}{8}, \frac{2}{3}, \frac{7}{9}, \frac{7}{8}, \frac{8}{9}, \frac{9}{8}$	$\frac{5}{864}$	5

Appendix A

SINGULAR implementation

This chapter contains the documentation of the SINGULAR [GPS02] implementation of the algorithms in chapter 2 in the SINGULAR documentation style. This implementation is contained in the SINGULAR library `gaussman.lib` and uses procedures from the SINGULAR library `linalg.lib` to compute eigenvalues and the Jordan normal form. SINGULAR including the above libraries is free software under the GNU Public License.

A.1 SINGULAR library `linalg.lib`

LIBRARY:	`linalg.lib` — Linear Algebra
AUTHOR:	Mathias Schulze
	(`mschulze@mathematik.uni-kl.de`)

PROCEDURE:	`hessenberg` – Hessenberg form
USAGE:	`hessenberg(M); matrix M`
ASSUME:	`M` constant square matrix
RETURN:	
`matrix H;`	Hessenberg form of `M`
EXAMPLE:	`example hessenberg;` shows examples

PROCEDURE:	`evnf` — eigenvalues normal form
USAGE:	`evnf(e[,m]); ideal e, intvec m`

ASSUME:	`ncols(e)==size(m)`
RETURN:	order eigenvalues `e` with multiplicities `m`
EXAMPLE:	`example evnf;` shows examples

PROCEDURE:	`eigenvals` — eigenvalues and multiplicities
USAGE:	`eigenvals(M); matrix M`
ASSUME:	eigenvalues of `M` in basefield
RETURN:	
`list l;`	
`ideal l[1];`	
`number l[1][i];`	i-th eigenvalue of `M`
`intvec l[2];`	
`int l[2][i];`	multiplicity of `i`-th eigenvalue of `M`
EXAMPLE:	`example eigenvals;` shows examples

PROCEDURE:	`jordan` — Jordan data
USAGE:	`jordan(M); matrix M`
ASSUME:	eigenvalues of `M` in basefield
RETURN:	
`list l;`	
`ideal l[1];`	
`number l[1][i];`	eigenvalue of `i`-th Jordan block of `M`
`intvec l[2];`	
`int l[2][i];`	size of `i`-th Jordan block of `M`
`intvec l[3];`	
`int l[3][i];`	multiplicity of `i`-th Jordan block of `M`
EXAMPLE:	`example jordan;` shows examples

PROCEDURE:	`jordanbasis` — Jordan basis
USAGE:	`jordanbasis(M); matrix M`

ASSUME:	eigenvalues of `M` in basefield
RETURN:	
`list l;`	
`module l[1];`	`inverse(l[1])*M*l[1]` in Jordan normal form
`intvec l[2];`	
`int l[2][i];`	weight filtration index of `l[1][i]`
EXAMPLE:	`example jordanbasis;` shows examples

PROCEDURE:	`jordanmatrix` — Jordan matrix
USAGE:	`jordanmatrix(e,s,m);`
	`ideal e, intvec s, intvec m`
ASSUME:	`ncols(e)==size(s)==size(m)`
RETURN:	
`matrix J;`	Jordan matrix with `list(e,s,m)==jordan(J)`
EXAMPLE:	`example jordanmatrix;` shows examples

PROCEDURE:	`jordannf` — Jordan normal form
USAGE:	`jordannf(M); matrix M`
ASSUME:	eigenvalues of `M` in basefield
RETURN:	
`matrix J;`	Jordan normal form of `M`
EXAMPLE:	`example jordannf;` shows examples

A.2 SINGULAR library gaussman.lib

LIBRARY:	`gaussman.lib` — Gauß-Manin Connection
AUTHOR:	Mathias Schulze
	(`mschulze@mathematik.uni-kl.de`)
OVERVIEW:	A library to compute Hodge-theoretic invariants of isolated hypersurface singularities
SEE ALSO:	`mondromy.lib`, `spectrum.lib`
KEYWORDS:	singularities; Gauß-Manin connection; Brieskorn lattice; monodromy; spectrum; spectral pairs; mixed Hodge structure; V-filtration; weight filtration

PROCEDURE:	`gmsring` — Gauß-Manin system
USAGE:	`gmsring(t,s); poly t, string s`
ASSUME:	characteristic `0`; local degree ordering;
	isolated critical point `0` of `t`
RETURN:	
`ring G;`	Gauß-Manin system of `t` with variable `s`
`poly gmspoly=t;`	
`ideal gmsjacob;`	Jacobian ideal
`ideal gmsstd;`	standard basis of Jacobian ideal
`matrix gmsmatrix;`	`matrix(gmsjacob)*gmsmatrix==matrix(gmsstd)`
`ideal gmsbasis;`	monomial vectorspace basis of Jacobian algebra
`int gmsmaxdeg;`	maximal weight of variables
NOTE:	`gmsbasis` is a $\mathbb{C}\{\{s\}\}$-basis of H'' and $[\mathtt{t},\mathtt{s}]=\mathtt{s}^2$
KEYWORDS:	singularities; Gauß-Manin connection; Brieskorn lattice
EXAMPLE:	`example gmsring;` shows examples

PROCEDURE:	`gmsnf` — Gauß-Manin normal form

USAGE:	`gmsnf(p,K); poly p, int K`
ASSUME:	`basering` returned by `gmsring`
RETURN:	
`list nf;`	
`ideal nf[1];`	projection of `p` to $\langle$`gmsbasis`$\rangle\mathbb{C}\{\{s\}\}$ mod $\mathtt{s}^{\mathtt{K}+1}$
`ideal nf[2];`	`p==nf[1]+nf[2]`
NOTE:	computation can be continued by setting `p=l[2]`
KEYWORDS:	singularities; Gauß-Manin connection; Brieskorn lattice
EXAMPLE:	`example gmsnf;` shows examples

PROCEDURE:	`gmscoeffs` — Gauß-Manin basis representation
USAGE:	`gmscoeffs(p,K); poly p, int K`
ASSUME:	`basering` returned by `gmsring`
RETURN:	
`list l;`	
`matrix l[1];`	$\mathbb{C}\{\{s\}\}$-basis representation of `p` mod $\mathtt{s}^{\mathtt{K}+1}$
`ideal l[2];`	`p==matrix(gmsbasis)*l[1]+l[2]`
NOTE:	computation can be continued by setting `p=l[2]`
KEYWORDS:	singularities; Gauß-Manin connection; Brieskorn lattice
EXAMPLE:	`example gmscoeffs;` shows examples

PROCEDURE:	`monodromy` — complex monodromy
USAGE:	`monodromy(t); poly t`
ASSUME:	characteristic `0`; local degree ordering;
	isolated critical point `0` of `t`
RETURN:	
`list l=jordan(M);`	Jordan data of monodromy matrix $\mathrm{e}^{-2\pi \mathrm{i}\mathtt{M}}$
SEE ALSO:	`mondromy.lib`, `linalg.lib`

KEYWORDS:	singularities; Gauß-Manin connection; Brieskorn lattice; monodromy
EXAMPLE:	`example monodromy;` shows examples

PROCEDURE:	`spectrum` — singularity spectrum
USAGE:	`spectrum(t); poly t`
ASSUME:	characteristic `0`; local degree ordering;
	isolated critical point `0` of `t`
RETURN:	
`list sp;`	singularity spectrum of `t`
`ideal sp[1];`	
`number sp[1][i];`	`i`-th spectral number
`intvec sp[2];`	
`int sp[2][i];`	multiplicity of `i`-th spectral number
SEE ALSO:	`spectrum.lib`
KEYWORDS:	singularities; Gauß-Manin connection; Brieskorn lattice; mixed Hodge structure; V-filtration; spectrum
EXAMPLE:	`example spectrum;` shows examples

PROCEDURE:	`sppairs` — spectral pairs
USAGE:	`sppairs(t); poly t`
ASSUME:	characteristic `0`; local degree ordering;
	isolated critical point `0` of `t`
RETURN:	
`list spp;`	spectral pairs of `t`
`ideal spp[1];`	
`number spp[1][i];`	V-filtration index of `i`-th spectral pair
`intvec spp[2];`	
`int spp[2][i];`	weight filtration index of `i`-th spectral pair
`intvec spp[3];`	

`int spp[3][i];`	multiplicity of `i`-th spectral pair
SEE ALSO:	`spectrum.lib`
KEYWORDS:	singularities; Gauß-Manin connection; Brieskorn lattice; mixed Hodge structure; V-filtration; weight filtration; spectrum; spectral pairs
EXAMPLE:	`example sppairs;` shows examples

PROCEDURE:	`spnf` — spectrum normal form
USAGE:	`spnf(a[,m][,V]);`
	`ideal a, intvec m, list V`
ASSUME:	`ncols(a)==size(m)==size(V);` `typeof(V[i])=="int"`
RETURN:	order (`a[i]`[,`V[i]`]) with multiplicity `m[i]` lexicographically
EXAMPLE:	`example spnf;` shows examples

PROCEDURE:	`sppnf` — spectral pairs normal form
USAGE:	`sppnf(a,w[,m][,V]);`
	`ideal a, intvec w, intvec m, list V`
ASSUME:	`ncols(a)==size(w)==size(m)==size(V);`
	`typeof(V[i])=="module"`
RETURN:	order (`a[i]`[,`w[i]`][,`V[i]`]) with multiplicity `m[i]` lexicographically
EXAMPLE:	`example sppnf;` shows examples

PROCEDURE:	`vfilt` — V-filtration
USAGE:	`vfilt(t); poly t`
ASSUME:	characteristic `0`; local degree ordering;
	isolated critical point `0` of `t`
RETURN:	
`list v;`	V-filtration on H''/sH''
`ideal v[1];`	

`number v[1][i];`	V-filtration index of `i`-th spectral number
`intvec v[2];`	
`int v[2][i];`	multiplicity of `i`-th spectral number
`list v[3];`	
`module v[3][i];`	vectorspace basis of `i`-th graded part in terms of `v[4]`
`ideal v[4];`	monomial vectorspace basis of H''/sH''
`ideal v[5];`	standard basis of Jacobian ideal
SEE ALSO:	`spectrum.lib`
KEYWORDS:	singularities; Gauß-Manin connection; Brieskorn lattice; mixed Hodge structure; V-filtration; spectrum
EXAMPLE:	`example vfilt;` shows examples

PROCEDURE:	`vwfilt` — weighted V-filtration
USAGE:	`vwfilt(t); poly t`
ASSUME:	characteristic `0`; local degree ordering; isolated critical point `0` of `t`
RETURN:	
`list vw;`	weighted V-filtration on H''/sH''
`ideal v[1];`	
`number v[1][i];`	V-filtration index of `i`-th spectral pair
`intvec v[2];`	
`int v[2][i];`	weight filtration index of `i`-th spectral pair
`intvec v[3];`	
`int v[3][i];`	multiplicity of `i`-th spectral pair
`list v[4];`	
`module v[4][i];`	vectorspace basis of `i`-th graded part in terms of `v[5]`
`ideal v[5];`	monomial vectorspace basis of H''/sH''

`ideal v[6];`	standard basis of Jacobian ideal
SEE ALSO:	`spectrum.lib`
KEYWORDS:	singularities; Gauß-Manin connection; Brieskorn lattice; mixed Hodge structure; V-filtration; spectrum
EXAMPLE:	`example vwfilt;` shows examples

PROCEDURE:	`tmatrix` — $\mathbb{C}\{\{s\}\}$-matrix of t on Brieskorn lattice
USAGE:	`tmatrix(t); poly t`
ASSUME:	characteristic `0`; local degree ordering;
	isolated critical point `0` of `t`
RETURN:	
`list A;`	$\mathbb{C}\{\{s\}\}$-matrix `A[1]+s*A[2]` of `t` on H''
`matrix A[1];`	
`matrix A[2];`	
KEYWORDS:	singularities; Gauß-Manin connection; Brieskorn lattice; mixed Hodge structure; opposite Hodge filtration; V-filtration;
EXAMPLE:	`example tmatrix;` shows examples

PROCEDURE:	`endfilt` — endomorphism V-filtration
USAGE:	`endvfilt(v); list v`
ASSUME:	`v` returned by `vfilt`
RETURN:	
`list ev;`	V-filtration of `t` on Jacobian algebra
`ideal ev[1];`	
`number ev[1][i];`	`i`-th V-filtration index
`intvec ev[2];`	
`int ev[2][i];`	`i`-th multiplicity
`list ev[3];`	
`module ev[3][i];`	vectorspace basis of `i`-th graded part

	in terms of `ev[4]`
`ideal ev[4];`	monomial vectorspace basis of Jacobian algebra
`ideal ev[5];`	standard basis of Jacobian ideal
KEYWORDS:	singularities; Gauß-Manin connection; Brieskorn lattice; mixed Hodge structure; V-filtration; endomorphism filtration
EXAMPLE:	`example endfilt;` shows examples

PROCEDURE:	`spprint` — print spectrum
USAGE:	`spprint(sp); list sp`
RETURN:	
`string s;`	spectrum `sp`
EXAMPLE:	`example spprint;` shows examples

PROCEDURE:	`sppprint` — print spectral pairs
USAGE:	`spprint(spp); list spp`
RETURN:	
`string s;`	spectral pairs `spp`
EXAMPLE:	`example sppprint;` shows examples

PROCEDURE:	`spadd` — sum of spectra
USAGE:	`spadd(sp1,sp2); list sp1, list sp2`
RETURN:	
`list sp;`	sum of spectra `sp1` and `sp2`
EXAMPLE:	`example spadd;` shows examples

PROCEDURE:	`spsub` — difference of spectra
USAGE:	`spsub(sp1,sp2); list sp1, list sp2`
RETURN:	
`list sp;`	difference of spectra `sp1` and `sp2`
EXAMPLE:	`example spsub;` shows examples

PROCEDURE: `spmul` — linear combination of spectra

USAGE: `spmul(sp0,k); list sp0, int[vec] k`

RETURN:

`list sp;` linear combination of spectra `sp0` with coefficients `k`

EXAMPLE: `example spmul;` shows examples

PROCEDURE: `spissemicont` — semicontinuity test of spectrum

USAGE: `spissemicont(sp[,1]); list sp`

RETURN:

`int k=1;` if sum of `sp` is positive on all intervals $[a, a+1)$ [and $(a, a+1)$]

`int k=0;` if sum of `sp` is negative on some interval $[a, a+1)$ [or $(a, a+1)$]

EXAMPLE: `example spissemicont;` shows examples

PROCEDURE: `spsemicont` — semicont. combinations of spectra

USAGE: `spsemicont(sp0,sp,k[,1]);`
`list sp0, list sp`

RETURN:

`list l;`

`intvec l[i];` if the spectra `sp0` occur with multiplicities `k` in a deformation of a [quasihomogeneous] singularity with spectrum `sp` then `k<=l[i]`

EXAMPLE: `example spsemicont;` shows examples

PROCEDURE: `spmilnor` — Milnor number of spectrum

USAGE: `spmilnor(sp); list sp`

RETURN:

`int mu;` Milnor number of spectrum `sp`

EXAMPLE: `example spmilnor;` shows examples

PROCEDURE:	`spgeomgenus` — geometrical genus of spectrum
USAGE:	`spgeomgenus(sp); list sp`
RETURN:	
`int g;`	geometrical genus of spectrum `sp`
EXAMPLE:	`example spgeomgenus;` shows examples

PROCEDURE:	`spgamma` — γ-invariant of spectrum
USAGE:	`spgamma(sp); list sp`
RETURN:	
`number gamma;`	γ-invariant of spectrum `sp`
EXAMPLE:	`example spgamma;` shows examples

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