

**A: Multiresolution Data Analysis – Numerical Realization
by Use of Domain Decomposition Methods
and Fast Multipole Techniques**

**B: Multiscale Solution of Oblique Boundary-Value
Problems by Layer Potentials**

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Multiresolution Data Analysis – Numerical Realization by Use of Domain Decomposition Methods and Fast Multipole Techniques

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Abstract.

This survey paper deals with multiresolution analysis from geodetically relevant data and its numerical realization for functions harmonic outside a (Bjerhammar) sphere inside the Earth. Harmonic wavelets are introduced within a suitable framework of a Sobolev-like Hilbert space. Scaling functions and wavelets are defined by means of convolutions. A pyramid scheme provides efficient implementation und economical computation. Essential tools are the multiplicative Schwarz alternating algorithm (providing domain decomposition procedures) and fast multipole techniques (accelerating iterative solvers of linear systems).

Keywords. Multiresolution analysis, harmonic wavelets, reconstruction formula, pyramid schemes, domain decomposition methods, fast multipole techniques.

Gravity field modelling, although it is always governed by the same classical Newton's law, changes its nature when it is seen from different spatial scales. To be more specific, if one looks at gravity field determination on the basis of an increasing spatial magnification and accuracy, we have to go from something that is suitably characterized by a simple mass point, on astronomical scale, to what is described by a global truncated multipole (i.e., outer harmonic) model, at scales corresponding to satellite altimetry, down to wavelengths of about 100 km. By further *zooming in* we can reach a spatial resolution of about 1 km showing a very complicated pattern, strongly related to the shape of the Earth and to irregular masses inside the Earth's crust. Simultaneously the error in the knowledge of the gravity field model goes from 5 Gal, the flattening effect, down to 10 mGal in a today's global model, down to about 10^{-1} mGal at the future 1 km spatial resolution. There is also a change of the gravity field in the time scale depending on the time interval under consideration, but this

aspect will not be discussed here. Thus, gravity field modelling as scientific object to be investigated in this approach is by definition the stationary gravity field with a spatial resolution from a worldwide to about 1 km scale and from about 1000 Gal of the full field down to, at least 10^{-1} mGal in future.

What we would like to present in this note are multiscale structures by which the gravitational part of the gravity field can be approximated progressively better and better, reflecting an increasing flow of observations of terrestrial, airborne and/or satellite type (e.g., terrestrial gravimetry, airborne gravimetry, satellite-to-satellite tracking (SST), satellite gravity gradiometry (SGG), etc). More precisely, we shall try to outline the canonical bridge of gravitational field determination from the well-established global outer harmonic approximation corresponding to a spherical Earth to modern wavelet methods based on the actual geometry of the Earth's surface and geodetically relevant observables (thereby always neglecting the small effect of the atmosphere in the outer space).

At this stage some remarks should be made: First, every geodetic measurement is a functional which may be assumed to be suitably linearizable in case of non-linearity when the anomalous potential is considered. In other words, the relation between the object function, i.e., the anomalous potential and the data, may be supposed to be linear (for more details see, for example, W.A. Heiskanen, H. Moritz (1966)). Second, more and more measurements refer to satellites and cannot be modelled as functionals of the gravitational potential on the boundary, i.e. the Earth's surface. The satellite observations, which certainly increase in importance in the future, are much more difficult to handle, since they are exponentially smoothed while moving to the outer space. As a consequence, essential knowledge of the gravitational potential should be based on the combination of ground observations with satellite information.

Table 1: *Gravitational field observables and their location*

observables	location	measurement method
<i>terrestrial observables</i>		
$ \nabla V(x) $	$x \in \text{continent}$	gravimetry
$\frac{\nabla V(x)}{ \nabla V(x) }$	$x \in \text{continent}$	geometric–astronomic levelling
$V(x)$	$x \in \text{ocean}$	satellite altimetry
x, y	$x, y \in \text{surface of the Earth}$	Doppler, GPS, LASER, RADAR, VLBI techniques
<i>spaceborne techniques</i>		
$\nabla V(x)$	$x \in \text{outer space of the Earth}$	satellite-to-satellite tracking
$(\nabla \otimes \nabla)V(x)$	$x \in \text{outer space of the Earth}$	satellite gravity gradiometry
<i>seismic information</i>		
$\rho(x)$	$x \in \text{Earth's crust}$	seismic tomography, etc

A very rough overview of what we consider nowadays as relevant geodetic observables can be taken from a classification due to H. Nutz (2002) (see Table 1).

The main interest in modelling data associated to the anomalous potential is a mathematical model that is characterized by two aspects, on the one hand side by realistic geodetic assumptions and on the other hand side by an acceptable numerical complexity. It is very helpful from the point of approximation theory to require the class \mathcal{H} of object functions to constitute a Hilbert space of potentials F being harmonic down to an internal sphere A (with radius α around the origin). In doing so, we are led to a sphere-oriented Runge-Walsh approach (see W. Freeden, F. Schneider (1998)) which tells us that mean squares approximation on the internal ("Bjerhammar") sphere implies uniform approximation on and outside the Earth's surface Σ , a feature that is of exceptional significance. The price that must be paid when approximately replacing the actual gravitational potential V of the Earth by an object function $F \in \mathcal{H}$ is the smoothness of the gravitational potential V on the Earth's surface which is clearly acceptable seen from the viewpoint of numerical analysis. In addition, we are confronted with the fact that the Hilbert space \mathcal{H} of potentials in $\overline{A_{\text{ext}}}$ is infinite-dimensional, briefly formulated (for more details see W. Freeden (1999)) \mathcal{H} is the space of potentials in $\overline{A_{\text{ext}}}$ of the form $\sum_{n=0}^{\infty} \sum_{k=1}^{2n+1} \frac{1}{A_n} F^{\wedge}(n, k) H_{-n-1, k}(\alpha; \cdot)$

satisfying $\sum_{n=0}^{\infty} \sum_{k=1}^{2n+1} (F^{\wedge}(n, k))^2 < \infty$, where $\{A_n\}_{n \in \mathbb{N}_0}$ is a sequence of non-vanishing real numbers and $\{F^{\wedge}(n, k)\}_{\substack{n=0,1,\dots \\ k=1,\dots,2n+1}}$ given by

$$F^{\wedge}(n, k) = \int_A F(x) H_{-n-1, k}(\alpha; x) d\omega(x)$$

($d\omega$: the surface element on A) is the table of Fourier coefficients $F^{\wedge}(n, k)$ of F on A with respect to the outer harmonics $H_{-n-1, k}(\alpha; \cdot)$ of degree n and order k . (Note that A_{ext} is the outer space of the sphere A with radius α around the origin 0; $\overline{A_{\text{ext}}} = A_{\text{ext}} \cup A$). In other words, the information about the object function $V \in \mathcal{H}$ is incomplete, since only a finite number of data is available. Moreover, the data are of heterogeneous type relating to different positions in $\overline{A_{\text{ext}}}$.

1 Outer Harmonic Multiresolution

It is fortunate that the Hilbert space \mathcal{H} contains as subspaces all $(2j+1)$ -dimensional spaces $Harm_j$ of outer harmonics $H_{-j-1, l}(\alpha; \cdot)$ of order j and degree l . More explicitly, we are able to make profit of the following (in physical geodesy well-known) *multiresolution analysis in terms of*

outer harmonics:

- $\mathcal{V}_0 \subset \mathcal{V}_1 \subset \dots \subset \mathcal{H},$
 $\overline{\|\cdot\|_{\mathcal{H}}}$
- $\bigcup_{j=0} \mathcal{V}_j = \mathcal{H},$
- $\mathcal{V}_{j+1} = \mathcal{V}_0 \oplus \bigoplus_{l=0}^j \mathcal{W}_l = \mathcal{V}_{J_0} \oplus \bigoplus_{l=J_0}^j \mathcal{W}_l,$
 $0 \leq J_0 \leq j,$

where the *scale spaces* \mathcal{V}_j and the *detail spaces* \mathcal{W}_j are given by

$$\mathcal{V}_j = \bigoplus_{l=0}^j \text{Harm}_l = \text{Harm}_{0,\dots,j}$$

and

$$\mathcal{W}_j = \mathcal{V}_{j+1} \ominus \mathcal{V}_j = \text{Harm}_{j+1},$$

respectively. Advantages may be characterized as follows: Any member F of \mathcal{V}_j (that means any band-limited element of \mathcal{H}) is representable as a finite Fourier (orthogonal) *expansion*

$$F \in \mathcal{V}_j \iff \sum_{n=0}^j \sum_{l=1}^{2n+1} F^\wedge(n, l) H_{-n-1, l}(\alpha; \cdot)$$

in terms of the $\mathcal{L}^2(A)$ -orthonormal outer harmonics of order $\leq j$. It is well-known that each outer harmonic localizes ideally in the frequency space; the "polynomial structure" of the outer harmonics produces a good global "low frequency" approximation; each Fourier coefficient (i.e., potential coefficient) of the orthogonal expansion may be expressed exactly as finite linear combination of data functionals applied to outer harmonics. In conclusion, multiresolution analysis in terms of outer harmonics has proven to be very useful in physical geodesy for the purpose of global modelling. Nowadays, all updated global models include orthogonal coefficients up to the order 360 (for example, EGM96). The Fourier coefficients are derivable in different ways on certain spectral domains (by global combination of terrestrial and spaceborne data).

2 Wavelet Multiresolution Analysis

The fundamental difficulty of multiresolution in terms of outer harmonics, however, is that we have to deal with trial potentials which do not show any phenomenon of space localization.

Consequently, any local change of the potential affects the whole table of Fourier coefficients. In consequence, local modelling is hardly treatable. The essential reason is that the consecutive scale spaces \mathcal{V}_j and \mathcal{V}_{j+1} in outer harmonic multiresolution are not related by a dilation and shifting (translation) procedure. In other words, the spaces \mathcal{V}_j do not consist of dilated and shifted copies of one fixed function, i.e., there is *no "mother outer harmonic"* from which all other outer harmonics may be generated by dilation and translation. For these reasons we are convinced that local modelling cannot be performed adequately within the framework of outer harmonics. A new component of approximation has to come into play, viz. harmonic wavelets (see W. Freeden (1999)).

The power of wavelets is based on a multiresolution analysis which enables both frequency as well as space localization. In fact, wavelets may be used as mathematical means for breaking up the complicated structure of the Earth's anomalous field into many simple pieces at different scales and positions. Basically this is done by filtering, i.e. forming convolutions $\Psi_j * F$, $j \in \mathbb{N}_0$, of a harmonic function F such as the anomalous potential against "dilated" and "shifted" versions Ψ_j of one fixed function, viz. the *mother wavelet* Ψ . In consequence, the anomalous potential is represented by a two-parameter family reflecting the different localization and different levels of resolution. The definition of wavelets has to be given in close connection with a so-called scaling function $\{\Phi_j\}_{j \in \mathbb{N}_0}$, approaching formally in the limit case the "Dirac function(al)" Φ_∞ given by

$$\begin{aligned} \Phi_\infty(x, y) &= \sum_{n=0}^{\infty} \frac{1}{A_n^2} \sum_{k=1}^{2n+1} H_{-n-1}(\alpha; x) H_{-n-1}(\alpha; y), \end{aligned}$$

$x, y \in \overline{A_{\text{ext}}}$, of the Sobolev space \mathcal{H} under consideration. Observing the addition theorem of spherical harmonics (see, for example, W. Freeden et al. (1998)) we get

$$\begin{aligned} \Phi_\infty(x, y) &= \sum_{n=0}^{\infty} \frac{2n+1}{4\pi\alpha^2} \frac{1}{A_n^2} \left(\frac{\alpha^2}{|x||y|} \right)^{n+1} P_n \left(\frac{x}{|x|} \cdot \frac{y}{|y|} \right), \end{aligned}$$

where P_n is the Legendre polynomial of degree n .

The scaling (kernel) function is given as series expansion in terms of outer harmonics as follows:

$$\begin{aligned} \Phi_j(x, y) &= \sum_{n=0}^{\infty} \frac{\varphi_j(n)}{A_n^2} \sum_{k=1}^{2n+1} H_{-n-1}(\alpha; x) H_{-n-1}(\alpha, y) \end{aligned}$$

$x, y \in \overline{A_{\text{ext}}}$, where the family $\{\{\varphi_j(n)\}_{n \in \mathbb{N}_0}\}_{j \in \mathbb{N}_0}$ is (usually) assumed to satisfy the following properties:

(i) for all $j \in \mathbb{N}_0$,

$$\varphi_j(0) = 1$$

(ii) for all $j, j' \in \mathbb{N}_0$ with $j \leq j'$ and all $n \in \mathbb{N}$

$$\varphi_j(n) \leq \varphi_{j'}(n)$$

(iii) for all $n \in \mathbb{N}$

$$\lim_{j \rightarrow \infty} (\varphi_j(n)) = 1$$

(iv) for all $j \in \mathbb{N}_0$,

$$\sum_{n=1}^{\infty} \frac{\varphi_j(n)}{A_n^2} \frac{2n+1}{4\pi\alpha^2} < \infty.$$

The aforementioned convolutions in \mathcal{H} are understood in the following way:

$$F * G = \sum_{n=0}^{\infty} \sum_{k=1}^{2n+1} A_n^2 F^\wedge(n, k) G^\wedge(n, k)$$

for $F, G \in \mathcal{H}$. In particular, for all $x \in \overline{A_{\text{ext}}}$ and all $F \in \mathcal{H}$,

$$\begin{aligned} (\Phi_j * F)(x) &= \sum_{n=0}^{\infty} \sum_{k=1}^{2n+1} A_n^2 \varphi_j(n) F^\wedge(n, k) H_{-n-1, k}(\alpha; x). \end{aligned}$$

Wavelet function $\{\Psi_j\}_{j \in \mathbb{N}_0}$ and scaling function $\{\Phi_j\}_{j \in \mathbb{N}_0}$ are related via their symbols $\{\varphi_j(n)\}_{n \in \mathbb{N}_0}$, $\{\psi_j(n)\}_{n \in \mathbb{N}_0}$ via the *scaling equation*

$$\psi_j(n) = \varphi_{j+n}(n) - \varphi_j(n), \quad n \in \mathbb{N}_0.$$

Starting from the mother kernels Φ_0 and Ψ_0 , respectively, the j -th level kernels Φ_j and Ψ_j are obtained by dilation. The scale index j serves

as a "measure for decreasing frequency localization" or equivalently as a "measure for increasing space localization".

To be more specific, observing the addition theorem of spherical harmonics the kernels Φ_j and Ψ_j , respectively, can be written as follows:

$$\begin{aligned} \Phi_j(x, y) &= \sum_{n=0}^{\infty} \frac{\varphi_j(n)}{A_n^2} \frac{2n+1}{4\pi\alpha^2} \left(\frac{\alpha^2}{|x||y|} \right)^{n+1} P_n \left(\frac{x}{|x|} \cdot \frac{y}{|y|} \right), \end{aligned}$$

$$\begin{aligned} \Psi_j(x, y) &= \Phi_{j+1}(x, y) - \Phi_j(x, y) \\ &= \sum_{n=0}^{\infty} \frac{\Psi_j(n)}{A_n^2} \frac{2n+1}{4\pi\alpha^2} \left(\frac{\alpha^2}{|x||y|} \right)^{n+1} P_n \left(\frac{x}{|x|} \cdot \frac{y}{|y|} \right), \end{aligned}$$

$$(x, y) \in \overline{A_{\text{ext}}} \times \overline{A_{\text{ext}}}.$$

The so-called *dilation operator* D_j is defined in the following way. $D_j : \Phi_0(\cdot, \cdot) \mapsto D_j \Phi_0(\cdot, \cdot) = \Phi_j(\cdot, \cdot)$, $j \in \mathbb{N}_0$. If y is a point of $\overline{A_{\text{ext}}}$, then the *shift operator* S_y is defined by $S_y \Phi_0(\cdot, \cdot) = \Phi_0(\cdot, y)$, $y \in \overline{A_{\text{ext}}}$. Combining both operators we find $\Phi_j(\cdot, y) = S_y \Phi_j(\cdot, \cdot) = S_y D_j \Phi_0(\cdot, \cdot)$ and $\Psi_j(\cdot, y) = S_y \Psi_j(\cdot, \cdot) = S_y D_j \Psi_0(\cdot, \cdot)$.

The *multiresolution analysis in terms of harmonic wavelets* for a potential $F \in \mathcal{H}$ (such as the anomalous potential) is illustrated by the scheme below:

- $\Phi_j * F \in \mathcal{V}_j$, $j = 0, 1, \dots$,
- $\Phi_j * F \xrightarrow{j \rightarrow \infty} F$,
- $\mathcal{V}_0 \subset \dots \subset \mathcal{V}_j \subset \mathcal{V}_{j+1} \subset \mathcal{V}_{j+2} \subset \dots \subset \mathcal{H}$,
- $\mathcal{V}_{j+1} = \mathcal{V}_j + \mathcal{W}_j$,
- $\mathcal{V}_{j+1} = \mathcal{V}_{J_0} + \sum_{l=J_0}^j \mathcal{W}_l$, $J_0 \geq 0$,
- $\overline{\bigcup_{j=0}^{\infty} \mathcal{V}_j}^{\|\cdot\|_{\mathcal{H}}} = \mathcal{H}$,
- $\Psi_j * F \in \mathcal{W}_j$,
- $\Phi_{J_0} * F + \sum_{l=J_0}^J \Psi_l * F \xrightarrow{J \rightarrow \infty} F$.

The last result is known in the wavelet terminology as *reconstruction formula*. Explicitly written out it reads:

$$\lim_{J \rightarrow \infty} \left\| \Phi_{J_0} * F + \sum_{l=J_0}^J \Psi_l * F - F \right\|_{\mathcal{H}} = 0.$$

In terms of filtering, the sequences $\{\Phi_j\}_{j \in \mathbb{N}_0}$ and $\{\Psi_j\}_{j \in \mathbb{N}_0}$, respectively, may be interpreted as *low-pass filter* and *band-pass filter*. The convolutions $\Psi_j * F$ may be understood as a version of F blurred to the scale j . It describes the "detail structure" of F at scale j . The *detail space* \mathcal{W}_j contains the detail information needed to go from an approximation at resolution j to an approximation at resolution $j + 1$. For more theoretical details confer W. Freeden (1999); for graphical illustrations see e.g. Figure 2 of this paper.

3 Some Wavelet Examples

Next some examples of scaling functions and associated wavelets will be presented.

Example 1. Let $\{\gamma_j\}_{j \in \mathbb{N}_0}$ be a strict monotonically decreasing sequence of positive real numbers with $\lim_{j \rightarrow \infty} \gamma_j = 0$. The generating symbol $\{\{\varphi_j(n)\}_{n \in \mathbb{N}_0}\}_{j \in \mathbb{N}_0}$ of the *Shannon scaling function* $\{\varphi_j\}_{j \in \mathbb{N}_0}$ is given by

$$\varphi_j(n) = \begin{cases} 1 & , \quad n \in [0, \gamma_j^{-1}) \\ 0 & , \quad n \in [\gamma_j^{-1}, \infty). \end{cases}$$

The Shannon scaling functions constitutes a bandlimited wavelet function.

The following example is of non-bandlimited nature.

Example 2. Let $\{\gamma_j\}_{j \in \mathbb{N}_0}$ be a strict monotonically decreasing sequence of positive real numbers with $\lim_{j \rightarrow \infty} \gamma_j = 0$. The generating symbol $\{\{\varphi_j(n)\}_{n \in \mathbb{N}_0}\}_{j \in \mathbb{N}_0}$ of the *Tikhonov scaling function* $\{\Phi_j\}_{j \in \mathbb{N}_0}$ is given by

$$\varphi_j(n) = \frac{\sigma_n^2}{\sigma_n^2 + \gamma_j},$$

where the sequence $\{\sigma_n\}_{n \in \mathbb{N}_0}$ has to satisfy $\sigma_n \neq 0$, $n \in \mathbb{N}_0$, and $\sum_{n=0}^{\infty} (2n+1)\sigma_n^2 < \infty$.

In particular, the sequence $\{\gamma_j\}_{j \in \mathbb{N}_0}$ may be chosen in dyadic way: $\gamma_j = 2^{-j}\alpha$, $j \in \mathbb{N}_0$, for some constant $\alpha > 0$.

Finally, another type of a non-bandlimited scaling function should be listed that turns out to be of particular importance in numerical computations.

Example 3. Let $\{\gamma_j\}_{j \in \mathbb{N}_0}$ be a strict monotonically decreasing sequence of positive real numbers with $\lim_{j \rightarrow \infty} \gamma_j = 0$. Let $Q : [0, \infty) \rightarrow [0, \infty)$, $t \mapsto Q(t)$, be a function with the following properties:

- (α) $Q \in C^{(\infty)}[0, \infty)$,
- (β) $Q(0) = 0$,
- (γ) $Q(t) > 0$, $t > 0$
- (δ) $Q(t) < Q(t')$, $0 < t < t'$.

Then the generating symbol $\{\{\varphi_j(n)\}_{n \in \mathbb{N}_0}\}_{j \in \mathbb{N}_0}$ of the *exponential scaling function* $\{\Phi_j\}_{j \in \mathbb{N}_0}$ is given by

$$\varphi_j(n) = h^{n/2} e^{-\gamma_j Q(n)}, \quad h \in (0, 1].$$

Of particular significance are kernels Φ_j that are available as elementary functions.

Abel–Poisson kernel: $A_n = 1$, $Q(t) = Rt$, $R > 0$.

$$\Phi_j(x, y) = \frac{1}{4\pi} \frac{|x|^2 |y|^2 - h^2 \alpha^4 e^{-2\gamma_j R}}{(|x|^2 |y|^2 + h^2 \alpha^4 e^{-2\gamma_j R} - 2h\alpha^2 e^{-\gamma_j R}(x \cdot y))^{3/2}}.$$

Singularity kernel: $A_n = (n+1/2)^{1/2}$, $Q(t) = Rt$, $R > 0$.

$$\Phi_j(x, y) = \frac{1}{2\pi} \frac{1}{(|x|^2 |y|^2 + h^2 \alpha^4 e^{-2\gamma_j R} - 2h\alpha^2 e^{-\gamma_j R}(x \cdot y))^{1/2}}.$$

A fundamental aspect that should be mentioned is that a combined concept of outer harmonic and wavelet expansion may be formulated, the outer harmonic expansion being responsible for the global modelling and the wavelet expansion being appropriate for local modelling involving a zooming-in procedure. (For more details the reader is referred to W. Freeden (1999)).

4 A Tree Algorithm

For computational purposes it is useful to discretize the convolutions by approximate formulas based on bounded linear functionals $L_i^{N_j}$, $i = 1, \dots, N_j$, on \mathcal{H} characterizing the observables of the anomalous potential. What we are going to realize is a tree algorithm (*pyramid scheme*) with the following ingredients: Starting from a sufficiently large J such that

$$F(x) \simeq (\Phi_J * F)(x) \simeq \sum_{i=1}^{N_J} L_i^{N_J} \Phi_J(x, \cdot) a_i^{N_J},$$

$x \in \overline{A_{\text{ext}}}$, we are interested in showing that the coefficient vectors $a^{N_j} = (a_1^{N_j}, \dots, a_{N_j}^{N_j})^T \in \mathbb{R}^{N_j}$,

$j = J_0, \dots, J-1$ (being, of course, dependent on the potential $F \in \mathcal{H}$ under consideration) can be calculated such that the following statements hold true:

- (i) The vectors $a^{N_j}, j = J_0, \dots, J-1$, are obtainable by recursion from the values a^{N_J} .
- (ii) For $j = J_0, \dots, J$,

$$(\Phi_j * F)(x) \simeq \sum_{i=1}^{N_j} L_i^{N_j} \Phi_j(x, \cdot) a_i^{N_j},$$

$x \in \overline{A_{\text{ext}}}$. Analogously, we find for $j = J_0, \dots, J-1$,

$$(\Psi_j * F)(x) \simeq \sum_{i=1}^{N_j} L_i^{N_j} \Psi_j(x, \cdot) a_i^{N_j},$$

$x \in \overline{A_{\text{ext}}}$.

Our considerations leading to a tree algorithm are divided into two parts, viz. the initial step concerning the scale level J and the pyramid step establishing the recursion relation for $J-1, \dots, J_0$:

The Initial Step: For a suitably large integer J , $\Phi_J * F$ is sufficiently close to F on the whole space $\overline{A_{\text{ext}}}$. Formally understood, the kernel Φ_J replaces the "Dirac kernel" Φ_∞ . In other words,

$$L_i^{N_J} (\Phi_J * F) \simeq L_i^{N_J} F, \quad i = 1, \dots, N_J.$$

It remains to determine the coefficients $a_i^{N_J}$, $i = 1, \dots, N_J$. Different strategies can be applied. Dependent on the topology of the Hilbert space \mathcal{H} and the type of the linear functionals the coefficients $a_i^{N_J}$ can be calculated by numerical integration (as, for example, in the case of a spherical Earth) or by solving certain linear equations (for example, relating to points on the actual surface of the Earth or a satellite orbit).

Applying a general *interpolation procedure* we may base our initial step on the ansatz

$$a_i^{N_J} = \sum_{k=1}^{N_J} w_{i,k}^{N_J} L_k^{N_J} F, \quad i = 1, \dots, N_J,$$

where the coefficients $w_{i,k}^{N_J}$ satisfy the linear equations

$$\sum_{l=1}^{N_J} w_{l,k}^{N_J} L_l^{N_J} L_l^{N_J} \Phi_J(\cdot, \cdot) = \delta_{i,k}; \quad i, k = 1, \dots, N_J. \quad (1)$$

(It is worth mentioning that using numerical integration formulas, for example, in case of evaluation functionals $L_i^{N_J} : F \rightarrow F(x_i^{N_J}), x_i^{N_J} \in A$, we are canonically led to coefficients of the form

$$a_i^{N_J} = w_i^{N_J} F(x_i^{N_J}), \quad i = 1, \dots, N_J,$$

where $w_i^{N_J}$ are the weights and $x_i^{N_J}$ the knots of the approximate integration formula. This approach also leads to a tree algorithm; however, it will not be discussed here (see W. Freeden (1999)).

The Pyramid Step. The essential idea for the development of a pyramid scheme is the existence of a reproducing kernel function $K_{\mathcal{V}_j}$, $j = J_0, \dots, J$, of the detail spaces \mathcal{V}_j such that

$$\Phi_j \simeq K_{\mathcal{V}_j} * \Phi_j$$

and

$$K_{\mathcal{V}_j} \simeq K_{\mathcal{V}_{j+1}} * K_{\mathcal{V}_j}$$

for $j = J_0, \dots, J$. Observing our concept of discretizing convolutions by approximate formula based on interpolation we are led to

$$\begin{aligned} (\Phi_j * F)(x) &= (\Phi_j * K_{\mathcal{V}_j} * F)(x) \\ &\simeq \sum_{i=1}^{N_j} L_i^{N_j} \Phi_j(x, \cdot) a_i^{N_j}, \quad x \in \overline{A_{\text{ext}}}, \end{aligned} \quad (2)$$

where

$$a_i^{N_j} = \sum_{k=1}^{N_j} w_{i,k}^{N_j} L_k^{N_j} (K_{\mathcal{V}_j} * F), \quad i = 1, \dots, N_j,$$

and the coefficients $w_{i,k}^{N_j}$ satisfy the linear equations

$$\sum_{l=1}^{N_j} w_{l,k}^{N_j} L_l^{N_j} L_l^{N_j} \Phi_j(\cdot, \cdot) = \delta_{i,k}; \quad i, k = 1, \dots, N_j. \quad (3)$$

Now it follows by use of our approximate (interpolation based) formulae that

$$\begin{aligned}
a_i^{N_j} &= \sum_{k=1}^{N_j} w_{i,k}^{N_j} L_k^{N_j} (K_{V_j} * F) \\
&\simeq \sum_{k=1}^{N_j} w_{i,k}^{N_j} L_k^{N_j} (K_{V_j} * K_{V_{j+1}} * F) \\
&\simeq \sum_{k=1}^{N_j} w_{i,k}^{N_j} L_k^{N_j} \sum_{l=1}^{N_{j+1}} L_l^{N_{j+1}} K_{V_j}(\cdot, \cdot) a_l^{N_{j+1}} \\
&= \sum_{k=1}^{N_j} \sum_{l=1}^{N_{j+1}} w_{i,k}^{N_j} a_l^{N_{j+1}} L_k^{N_j} L_l^{N_{j+1}} K_{V_j}(\cdot, \cdot) \\
&= a_i^{N_{j+1}} \\
&+ \sum_{k=1}^{N_j} \sum_{l=N_{j+1}}^{N_{j+1}} w_{i,k}^{N_j} a_k^{N_{j+1}} L_k^{N_j} L_l^{N_{j+1}} K_{V_j}(\cdot, \cdot).
\end{aligned}$$

In other words, the coefficients $a_i^{N_{j-1}}$ can be calculated recursively starting from the values $a_i^{N_j}$ for the initial level J , $a_i^{N_{j-2}}$ can be deduced recursively from $a_i^{N_{j-1}}$, etc. Moreover, it is easy to see that the coefficients are independent of the chosen kernel. Consequently we have

$$(\Psi_j * F)(x) \simeq \sum_{i=1}^{N_j} L_i^{N_j} \Psi_j(x, \cdot) a_i^{N_j}, \quad x \in \overline{A_{\text{ext}}},$$

with the coefficients $a_i^{N_j}$, $i = 1, \dots, N_j$, given by

$$\begin{aligned}
a_i^{N_j} &\simeq a_i^{N_{j+1}} \\
&+ \sum_{k=1}^{N_j} \sum_{l=N_{j+1}}^{N_{j+1}} w_{i,k}^{N_j} a_k^{N_{j+1}} L_k^{N_j} L_l^{N_{j+1}} K_{V_j}(\cdot, \cdot).
\end{aligned}$$

Table 2: Decomposition Scheme

$$\begin{array}{ccccccc}
F & \rightarrow & a^{N_J} & \rightarrow & a^{N_{J-1}} & \rightarrow & \dots \rightarrow a^{N_{J_0}} \\
& & \downarrow & & \downarrow & & \downarrow \\
& & \Phi_J * F & & \Phi_{J-1} * F & & \Phi_{J_0} * F \\
& & \Psi_J * F & & \Psi_{J-1} * F & & \Psi_{J_0} * F
\end{array}$$

Table 3: Reconstruction Scheme

$$\begin{array}{ccccccc}
a^{N_{J_0}} & & a^{N_{J_0+1}} & & a^{N_{J_0+2}} & & \\
\downarrow & & & & & & \\
\Psi_{J_0} * F & & \Psi_{J_0+1} * F & & \Psi_{J_0+2} * F & & \\
\searrow & & \searrow & & \searrow & & \\
\Phi_{J_0} * F & \rightarrow + \rightarrow & \Phi_{J_0+1} * F & \rightarrow + \rightarrow & \Phi_{J_0+2} * F & \rightarrow + \rightarrow & \dots
\end{array}$$

From numerical point of view we have to deal with *two essential problems*: (i) Large linear systems have to be solved to determine the coefficients $w_{i,k}^{N_j}$, $j = J_0, \dots, J$. (ii) Large summations $\sum_{i=1}^{N_j} \dots$ must be performed during the solution process within the system of linear equations.

5 The Schwarz Alternating Algorithm: A Domain Decomposition Method

The pyramid scheme leads to a system of linear equations of the form (1) with a positive definite symmetric matrix $A = A_{i,j} = (L_i^{N_j} L_l^{N_j} \Phi_j(\cdot, \cdot))_{i,l=1,\dots,N_j}$, as far as the linear functionals under consideration are linearly independent. In other words, we have to solve a

linear system

$$\mathbf{A}x = b \quad (4)$$

where $\mathbf{A} \in \mathbb{R}^{N \times N}$, $\mathbf{A}^T = \mathbf{A}$, $(\mathbf{A}y, y) > 0$ for all $y \in \mathbb{R}^N \setminus \{0\}$, and $x, b \in \mathbb{R}^N$. Such a linear equation system can be solved with iterative solvers or direct solvers, but if N is large (for example, $N \geq 10000$), the runtime of an iterative solver without a suitable pre-conditioner or of a direct solver increases tremendously. Therefore, we need a more sophisticated method to

solve (4) for problems with a large number of given data. One such method is a multiplicative variant of the Schwarz alternating algorithm, a domain decomposition method, which allows to split the matrix \mathbf{A} in (4) into several smaller submatrices relating the linear functionals (see Table 1) to subdomains of the entire domain and thus providing a division in certain domains, which may (and will in numerical implementations) in general overlap. This multiplicative variant of the Schwarz alternating algorithm is an iterative method which solves in each iteration step successively a linear system with the matrices obtained from the splitting. This fact reduces both runtime and memory requirement drastically. (A further speed-up can be achieved if an additive variant of the Schwarz alternating algorithm is used which runs on parallel computers (see, for example, M. Griebel, P. Oswald (1995) and the references therein)).

The Schwarz alternating algorithm dates back to H.A. Schwarz' work, published in 1890, and has been investigated by many authors since then. A revived interest in variants of the Schwarz alternating method arose since 1985 due to the availability of fast modern and parallel computers. Roughly formulated, there are mainly two types of the Schwarz alternating algorithm: multiplicative variants (like the one used in this paper) and additive variants, which can be implemented on parallel computers and which are usually faster. For more information about the Schwarz alternating algorithm, the reader is referred to W. Frommer, H. Schwandt (1997), M. Griebel, P. Oswald (1995), and W. Freeden, K. Hesse (2002) and the references therein. In the last few years, a great interest has also been taken in the relation between the Schwarz alternating algorithm, multisplittings, multigrid methods, preconditioned conjugate gradient methods, as well as other iterative schemes.

The solution of (4) with a multiplicative variant of the Schwarz alternating algorithm, which will from now on be called briefly the multiplicative Schwarz alternating algorithm, is based on two facts: (i) every positive definite symmetric matrix is a Gram matrix, (ii) the convergence proof of the multiplicative Schwarz alternating algorithm is based on its formulation in terms of orthogonal projectors.

The matrix $\mathbf{A} = (A_{i,j})_{i,j=1,\dots,N}$ in (4) is positive definite and symmetric. Within the context

of Cholesky factorization, there exists a uniquely determined invertible lower triangular matrix \mathbf{L} with positive diagonal entries, such that

$$\mathbf{A} = \mathbf{L}\mathbf{L}^T. \quad (5)$$

Denote the row vectors of \mathbf{L} by v_1, \dots, v_N . Then (5) implies that

$$A_{ij} = v_i \cdot v_j = (v_i, v_j), \quad i, j = 1, \dots, N.$$

Thus, \mathbf{A} is the Gram matrix of the basis $\{v_1, \dots, v_N\}$ of \mathbb{R}^N , and the solution $x = (x_1, \dots, x_N)^T$ of the linear equation system (4) is the solution of the following orthogonal projection problem: Find $x = (x_1, \dots, x_N)^T \in \mathbb{R}^N$ such that $f \in \mathbb{R}^N$ with $(f, v_i) = b_i, i = 1, \dots, N$, has the representation

$$f = \sum_{i=1}^N x_i v_i. \quad (6)$$

Indeed, the solution of this problem demands the solution of the linear system

$$\sum_{i=1}^N x_i (v_i, v_j) = (f, v_j) = b_j, \quad j = 1, \dots, N, \quad (7)$$

which is just the linear system (4). The orthogonal projection operator corresponding to (6) is, of course, the identity operator: We seek a representation of $f = \text{Id}_{\mathbb{R}^N} f$ with respect to the basis $\{v_1, \dots, v_N\}$. Now we split the basis $\{v_1, \dots, v_N\}$ into several smaller possibly overlapping subsets $\Xi_r^{N_r} = \{v_1^r, \dots, v_{N_r}^r\} \subset \{v_1, \dots, v_N\}, r = 1, \dots, M$, such that

$$\bigcup_{r=1}^M \Xi_r^{N_r} = \{v_1, \dots, v_N\}.$$

This union will, in general, not be a disjoint, and we speak of overlapping subsets, if there are at least two subsets $\Xi_r^{N_r}, \Xi_k^{N_k}$ with $\Xi_r^{N_r} \cap \Xi_k^{N_k} \neq \emptyset$ and $k \neq r$.

Denote the orthogonal projector from \mathbb{R}^N onto $\text{span}(\Xi_r^{N_r})$ by

$$P_r : \mathbb{R}^N \rightarrow \text{span}\{v_1^r, \dots, v_{N_r}^r\}, g \mapsto P_r g, \quad (8)$$

i.e., $P_r = P_r \circ P_r$ and $(P_r v, w) = (v, P_r w)$ for all $v, w \in \mathbb{R}^N$. In order to compute $P_r g$, we assume again that $(g, v_i), i = 1, \dots, N$, is known. We want to calculate the coefficient vector $y = (y_1, \dots, y_{N_r})$ of the representation

$$P_r g = \sum_{i=1}^{N_r} y_i v_i^r.$$

Taking the inner product with $v_1^r, \dots, v_{N_r}^r$ successively leads to the linear equation system

$$\sum_{i=1}^{N_r} y_i (v_i^r, v_j^r) \stackrel{!}{=} (P_r g, v_j^r) = (g, P_r v_j^r) = (g, v_j^r), \quad (9)$$

$j = 1, \dots, N_r$. Clearly the matrix $\mathbf{A}_r = ((v_i^r, v_j^r))_{i,j=1,\dots,N_r}$ is a submatrix of the matrix \mathbf{A} of the linear equation system (4).

Algorithm 5.1 (*Multiplicative Schwarz Alternating Algorithm*)

set $f_0 = f \in \mathbb{R}^N$ and $s_0^f = 0$
for $n = 0, 1, 2, \dots$ do

 for $r = 1, \dots, M$ do

$$\text{calculate } s_{nM+r}^f = s_{nM+(r-1)}^f + P_r(f_{nM+(r-1)})$$

$$\text{update } f_{nM+r} = f_{nM+(r-1)} - P_r(f_{nM+(r-1)})$$

 until

$$\frac{|((f_{(n+1)M}, v_1), \dots, (f_{(n+1)M}, v_N))^T|}{|((f, v_1), \dots, (f, v_N))^T|} \leq \varepsilon$$

Recently, W. Freeden, K. Hesse (2002) have shown that the sequence of iterates $\{s_{nM}^f\}_{n \in \mathbb{N}_0}$ converges to f as $n \rightarrow \infty$.

Algorithm 5.1 will now be transformed into a matrix formulation via (7) and (9). For this purpose, we need the following restriction operators $R_r : \mathbb{R}^N \rightarrow \mathbb{R}^{N_r}$, $w \mapsto R_r(w) = ((R_r(w))_1, \dots, (R_r(w))_{N_r})^T$, and embedding operators $I_r : \mathbb{R}^{N_r} \rightarrow \mathbb{R}^N$, $z \mapsto I_r(z) = ((I_r(z))_1, \dots, (I_r(z))_N)^T$, corresponding to the subspaces \mathbb{R}^{N_r} of the subproblems (9). They are defined by

$$\begin{aligned} (R_r(w))_i &= w_j \text{ for the index } i \in \{1, \dots, N_r\} \text{ with } v_i^r = v_j, \\ (I_r(z))_i &= \begin{cases} z_j & \text{if there is } j \in \{1, \dots, N_r\} \text{ with } v_j^r = v_i \\ 0 & \text{else} \end{cases} \end{aligned}$$

Algorithm 5.2

(*matrix formulation of Algorithm 5.1*)

Define the matrices $\mathbf{A}_r = ((v_i^r, v_j^r))_{i,j=1,\dots,N_r}$, $r = 1, \dots, M$

set $\tilde{f}_0 = ((f, v_1), \dots, (f, v_N))^T$, $a_0 = (0, \dots, 0)^T \in \mathbb{R}^N$, where $f \in \mathbb{R}^N$

for $n = 0, 1, 2 \dots$ do

 for $r = 1, \dots, M$ do

$$\text{solve } \mathbf{A}_r d = R_r(\tilde{f}_{nM+(r-1)}), d =$$

$$\begin{aligned} &(d_1, \dots, d_{N_r})^T \in \mathbb{R}^{N_r} \\ \text{update } a_{nM+r} &= a_{nM+(r-1)} + I_r(d) \\ \text{update } \tilde{f}_{nM+r} &= \tilde{f}_{nM+(r-1)} - \left(\left(\sum_{i=1}^{N_r} d_i (v_i^r, v_k) \right)_{k=1,\dots,N} \right)^T \end{aligned}$$

until

$$\frac{|\tilde{f}_{n+1M}|}{|\tilde{f}_0|} \leq \varepsilon$$

compute

$$s_{(n+1)M}^f = \sum_{i=1}^M (a_{(n+1)M})_i v_i.$$

We stress that all the computations (except the computation of $s_{(n+1)M}^f$) in Algorithm 5.2 can be performed without actually computing $v_1, \dots, v_N \in \mathbb{R}^N$, i.e., we do not need the Cholesky factorization of \mathbf{A} : The matrices \mathbf{A}_r are available as submatrices of \mathbf{A} , and the update involves a matrix vector multiplication with the matrix $((v_k, v_i^r))_{\substack{k=1,\dots,N \\ i=1,\dots,N_r}}$, which is also a submatrix of \mathbf{A} .

Summarizing our result we obtain the following corollary.

Corollary 5.3 *Let the notation and the assumptions be the same as in Algorithm 5.2. Then the sequence $\{a_{nM}\}_{n \in \mathbb{N}_0} \subset \mathbb{R}^N$ converges to the solution $x \in \mathbb{R}^N$ of the linear equation system $\mathbf{A}x = b$, where $\mathbf{A} = ((v_i, v_j))_{i,j=1,\dots,N}$ and $b = ((f, v_1), \dots, (f, v_N))^T$.*

Finally, we want to give some comments concerning the implementation of Algorithm 5.2 for the solution of the aforementioned linear equation systems. In case of Example 2 or 3 the matrix entries are available as elementary function. This enables us to evaluate a matrix entry with small computational effort (in comparison to the generation of kernels by aid of series expansions in terms of outer harmonics). In an implementation of Algorithm 5.2 we will generate only the small matrices A_r in advance, compute, for example, their Cholesky factorization in a preprocessing step, and keep the matrices of the Cholesky factorizations of the A_r in the memory. The other matrix entries of A , which will be needed for the update (computation of the new residual), are generated while the update is performed. The update is the time-consuming task, whereas the smaller equation systems can now be solved extremely fast.

If fast multipole methods (fast summation techniques) are available for the type of kernel, which determines the matrix entries, the update can even be accelerated. In what follows this numerical method will be explained in more detail.

6 A Fast Multipole Technique

Although the fast multipole technique may also be applied to other types of kernels we restrict our considerations here to the singularity kernel (see Example 3) which is the reproducing kernel of the Sobolev space \mathcal{H} associated to the sequence $\{A_n\}_{n \in \mathbb{N}_0}$ given by

$$A_n = \left(n + \frac{1}{2}\right)^{1/2} h^{-n/2}, n = 0, 1, \dots; 0 < h < 1.$$

This particular choice yields the following kernel representation for the singularity kernel

$$K_{\mathcal{H}}(x, y) = \frac{1}{2\pi\alpha^2} \sum_{n=0}^{\infty} h^n \left(\frac{\alpha^2}{|x||y|}\right)^{n+1} P_n\left(\frac{x}{|x|} \cdot \frac{y}{|y|}\right),$$

$x, y \in \overline{A_{\text{ext}}}$. Applying the Kelvin transform of potential theory (see, for example, O.D. Kellogg (1929)) with respect to the variable y and the sphere A we obtain by setting $y^* = h \frac{\alpha^2}{|y|^2} y$

$$\begin{aligned} K_{\mathcal{H}}(x, y) &= \frac{1}{2\pi\alpha^2} \frac{|y|}{h} \sum_{n=0}^{\infty} \frac{|y^*|^n}{|x|^{n+1}} P_n\left(\frac{x}{|x|} \cdot \frac{y}{|y|}\right) \\ &= \frac{1}{2\pi\alpha^2} \frac{|y|}{h} \frac{1}{|x - y^*|}. \end{aligned}$$

This shows us, for example, in case of *evaluation functionals* $L_i F = F(x_i)$, $x_i \in \overline{A_{\text{ext}}}$,

$$L_i L_j K_{\mathcal{H}}(\cdot, \cdot) = \frac{1}{2\pi\alpha^2} \frac{|x_j|}{h} \frac{1}{|x_i - x_j^*|}, \quad (10)$$

and for functionals of *oblique derivatives* given by $L_i F = ((x \cdot \nabla F)(x))|_{x=x_i}$

$$\begin{aligned} L_i L_j K_{\mathcal{H}}(\cdot, \cdot) &= \frac{1}{2\pi|y^*|} \left((y^* \cdot \nabla_{y^*}) \left(\frac{1}{|x - y^*|} \right) \right. \\ &\quad \left. + 2(y^* \cdot \nabla_{y^*}) \left(\frac{1}{|x - y^*|} \right) + \frac{1}{|x - y^*|} \right) \Big|_{(x_i, x_j^*)}. \end{aligned} \quad (11)$$

Since the entries of the matrix consist of the fundamental solution of the Laplace equation and directional derivatives of it, the solution of

the linear system by a fast multipole accelerated solver becomes possible.

In order to reduce the computational effort and the storage requirements for solving the linear system (4) the Geomathematics Group Kaiserslautern adapted the fast multipole technique to accelerate the matrix multiplication in an iterative (GMRES) solver. Theory and algorithmic aspects of the FMM are very sophisticated and their description would go beyond the scope of this article. In this approach we merely outline the ideas and the basic aspects of the fast multipole method (FMM).

The subject of interest is the fast evaluation of a matrix multiplication of the form $\sum_{i=1}^N a_i L_i L_j K_{\mathcal{H}}(\cdot, \cdot)$, $j = 1, \dots, N$, N large. The idea of the FMM is based on the localization property of the kernel defining the entries $L_i L_j K_{\mathcal{H}}(\cdot, \cdot)$, i.e. on the fact that most of the energy is contained within a small vicinity of x_j^* . The sum is split up into a near-field part (of course, defined with respect to the linear functional associated to x_j^*), which is evaluated explicitly, and a far-field part, for which we seek a "fast" approximation with "sufficient accuracy". The algorithm is started by embedding the computational domain into an (initial) cube, followed by an adaptive and hierarchical subdivision of each cube into eight child cubes. Adaptivity means that a cube is only subdivided if it contains a minimum number of nodal points and is essential for the efficiency.

The approximation of the far-field is performed on the basis of the well-known multipole expansion of $1/|x - y|$ in terms of inner and outer harmonics. The kernel is expanded for both variables with respect to the centers x_0 and y_0 of the cubes (instead of the origin) and the expansion is truncated at a certain (low) degree p . Conditions for the convergence of the expansion can easily be derived. For example, an expansion for $(y \cdot \nabla_y) K_{\mathcal{H}}(x, y)$ can be found indirectly by applying $(y \cdot \nabla_y)$ to the expansion of $K_{\mathcal{H}}(x, y)$. Since the error of the expansion is controllable only in a small vicinity of (x_0, y_0) , we have to perform several expansions with respect to different centers (x_0, y_0) to obtain a global approximation. A lot of work can be done in an a-priori step independent of the number and the location of the target points, resulting in a small set of so-called far-field coefficients containing the information from distant measurements. The hier-

archical subdivision in connection with the translation and conversion theorems of spherical harmonics allow to calculate the far-field coefficients in level l recursively from those of the parent cube in level $l + 1$ so that an explicit calculation is only needed for the finest level. Altogether we obtain an algorithm which performs a ma-

trix multiplication in $\mathcal{O}(N)$ operations and with order N storage requirements. Details and extensive numerical experiments concerning the efficiency of the method for satellite problems in physical geodesy can be found in the Diploma thesis D. Michel (2001) and the PhD thesis O. Glockner (2002).

7 Some Numerical Results

In the foregoing we have seen that harmonic wavelets are numerically efficient "building blocks" that enable fast modelling of geopotential data such as oblique derivative data of the anomalous potential. Now we discuss in more detail the concept of multiresolution data analysis. This method "looks at" the Earth's anomalous potential through a microscope, whose resolu-

tion gets finer and finer. Thus it associates to the anomalous potential a sequence of smoothed versions, labelled by the scale parameter j . This aspect is illustrated by the figures below (see Figure 2, (a)-(f)) from oblique derivative data on the actual Earth derived from the EGM96-model.

More explicitly, in our approach, we use the so-called TerrainBase data model of the National Geodetic Data Center in Boulder, Colorado, to have a representation of the actual surface of the Earth.

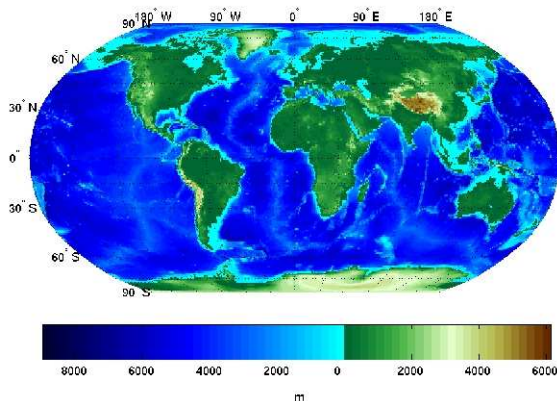
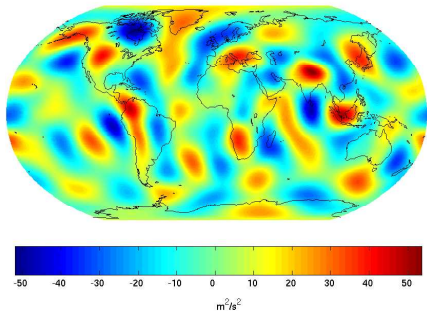


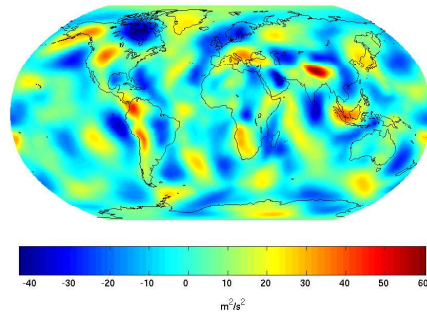
Figure 1: The Earth given by TerrainBase data.

The EGM96 model (F.G. Lemoine et al. (1996)) led us to create gradients of the anomalous potential on the actual Earth's surface, i.e. the TerrainBase (data) model. Hence, in an *a priori step*, oblique derivatives became available on a grid for the TerrainBase model of the actual Earth. Based on this material the interpolation oriented multiscale procedure described above could be used to model the anomalous potential on the Earth's surface and in the outer space of the Earth from prescribed oblique derivatives on

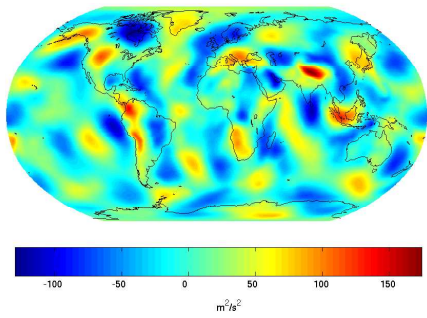
the Earth's surface. Actually, the interpolation procedure was executed on a (regular) grid of about 160 000 points, where the oblique derivatives were derived from the EGM96-model. It turned out that the multiplicative Schwarz alternating algorithm had to perform 45 iterations to get the results of the multiresolution data analysis (illustrated in Figure 4). For more computational details including an error analysis the reader is referred to the Diploma thesis due to M. Gutting (2002).



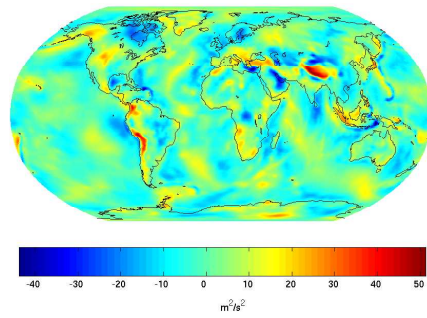
(a) $j=3$ (scaling function)



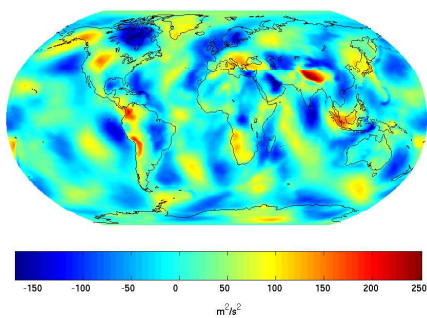
(b) $j=3$ (wavelet)



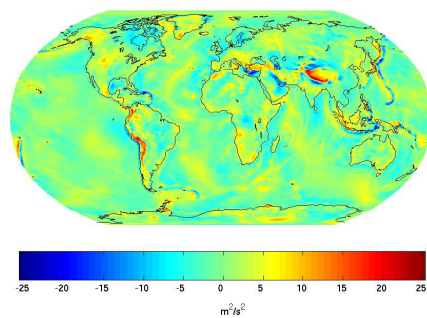
(c) $j=5$ (scaling function)



(d) $j=5$ (wavelet)



(e) $j=7$ (scaling function)



(f) $j=7$ (wavelet)

Figure 2: Dyadic multiresolution data analysis on the actual Earth's surface for the anomalous potential corresponding to oblique derivatives (derived from the EGM96-model).

Finally, we show the original potential and the approximate potential resulting from the multiresolution procedure.

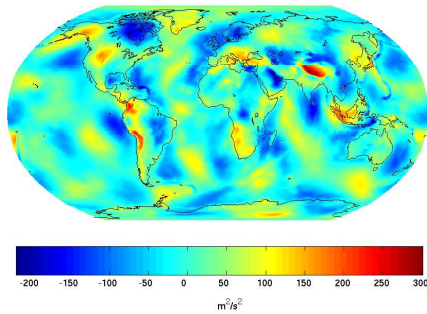


Figure 3: Original EGM96-potential on the actual Earth

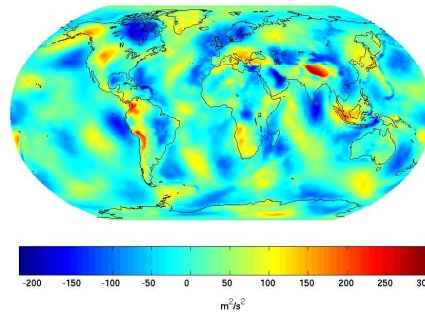


Figure 4: Approximate EGM96-potential from oblique derivatives on the actual Earth

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Multiscale Solution of Oblique Boundary-Value Problems by Layer Potentials

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Abstract. With the aid of classical results of potential theory, the limit- and jump-relations, a multiscale framework on geodetically relevant regular surfaces is established corresponding to oblique derivative data. By the oblique distance to the regular surface a scale factor in the kernel functions of the limit- and jump-operators is introduced, which connects these intergral kernels with the theory of scaling functions and wavelets.

As applications of the wavelet approach some numerical examples are presented on an ellipsoid of revolution. At the end we discuss a fast multiscale representation of the solution of the (exterior) oblique derivative (boundary-value) problem corresponding to geoscientifically relevant surfaces. A local as well as global reconstruction of the gravitational potential model EGM96 on the reference ellipsoid will illustrate the power of this approach.

Keywords. Scaling functions and wavelets on regular surfaces, potential operators, jump relations, multiscale analysis, (exterior) oblique derivative problem of potential theory

1 Introduction

Wavelets are known as mathematical means for breaking up a complicated function (signal) into many simple pieces at different scales and positions. Thus wavelets have become a powerful and flexible tool for scientific computation and data handling. Basically, wavelet analysis is done by convolving the function under consideration against 'dilated' and 'shifted' versions of one fixed function, viz. the 'mother wavelet'. Traditionally, applications of wavelets have been signal analysis, image processing, noise cancellation, etc, but there is also a growing interest in the numerical treatment of partial differential equations. However, wavelet methods are merely known for unfolding their computational economy and efficiency when applied to problems on

Euclidian spaces, the sphere or the torus. The aim of this article is to present a new wavelet approach to geodetically relevant surfaces. Our purpose is to develop a multiscale theory on regular surfaces by using results of classical potential theory.

The outline of this paper is as follows: First we introduce the notations and preliminaries that are needed for our wavelet approach. We specify regular surfaces on which our theory is established. Then we introduce potential operators with respect to an arbitrary non-tangential vector field which are the main ingredients of this work. We develop the oblique limit and jump relations of these potential operators formulated in the framework of the Hilbert space of square-integrable functions. The setup of a multiresolution analysis (i.e. scaling functions, scale spaces, wavelets, detail spaces) is defined by interpreting the kernel functions of the limit and jump integral operators as scaling functions on regular surfaces. The oblique distance to the parallel surfaces of the regular surface under consideration thereby represents the scale level in the scaling function. At the end we deal with the already mentioned discretization of Fredholm integral equations in order to give a multiscale representation of the solution of the (exterior) oblique derivative problem (EODP) in three dimensions corresponding to geoscientifically relevant regular surfaces. Furthermore, we discuss some numerical examples. In particular we are interested in the zoom-in property and the detection of a high frequency perturbation which are typical features within a wavelet framework.

2 Basic Concept

At first we introduce some settings which are standard in potential theory (see, for example, Kellogg (1929)).

We begin our considerations by introducing the notation of a regular surface:

Definition 2.1 A surface $\Sigma \subset \mathbb{R}^3$ is called reg-

ular, if it satisfies the following properties:

- (i) Σ divides the three-dimensional Euclidean space \mathbb{R}^3 into the bounded region Σ_{int} (inner space) and the unbounded region Σ_{ext} (outer space) defined by $\Sigma_{\text{ext}} = \mathbb{R}^3 \setminus \overline{\Sigma_{\text{int}}}$, $\overline{\Sigma_{\text{int}}} = \Sigma_{\text{int}} \cup \Sigma$,
- (ii) Σ_{int} contains the origin,
- (iii) Σ is a closed and compact surface free of double points,
- (iv) Σ is locally of class $C^{(2,\mu)}$. (ν denotes the unit normal field on Σ pointing into the outer space Σ_{ext}).

Geoscientifically relevant regular surfaces are, for example, sphere, ellipsoid, spheroid, geoid, (regular) Earth's surface.

Given a regular surface, then there exist positive constants α, β such that

$$\alpha < \sigma^{\text{inf}} = \inf_{x \in \Sigma} |x| \leq \sup_{x \in \Sigma} |x| = \sigma^{\text{sup}} < \beta. \quad (1)$$

By λ we designate a $c^{(1,\mu)}$ -unit vector field on Σ satisfying $\inf_{x \in \Sigma} (\lambda(x) \cdot \nu(x)) > 0$. The set

$$\Sigma^{(\lambda)}(\tau) = \{x \in \mathbb{R}^3 \mid x = y + \tau\lambda(y), y \in \Sigma\}$$

generates an *oblique (parallel) surface* which is exterior to Σ for $\tau > 0$ and interior for $\tau < 0$. It is well known from differential geometry (see e.g. Freeden, Kersten (1980)) that if $|\tau|$ is sufficiently small, then the surface $\Sigma^{(\lambda)}(\tau)$ is regular.

As usual, by $C^{(0,\mu)}(\Sigma)$, $0 < \mu < 1$, we denote the space of all μ -Hölder continuous functions on the regular surface Σ and by $L^2(\Sigma)$ we denote the space of (Lebesgue) square-integrable functions on the regular surface Σ . $L^2(\Sigma)$ is a Hilbert space with respect to the inner product $(\cdot, \cdot)_{L^2(\Sigma)}$ and a Banach space with respect to the norm $\|\cdot\|_{L^2(\Sigma)}$. $L^2(\Sigma)$ is the completion of $C^{(0,\mu)}(\Sigma)$ with respect to the norm $\|\cdot\|_{L^2(\Sigma)}$:

$$\overline{C^{(0,\mu)}(\Sigma)}^{\|\cdot\|_{L^2(\Sigma)}} = L^2(\Sigma) .$$

3 Limit Formulae und Jump Relation

Next we want to formulate the classical limit and jump relations of potential theory in comprehensive manner. They will be the fundamental

framework for the development of scaling functions and wavelets on regular surfaces.

For $\tau \neq \sigma$ with $|\tau|, |\sigma|$ sufficiently small, the functions

$$(x, y) \mapsto \frac{1}{|x + \tau\lambda(x) - (y + \sigma\lambda(y))|},$$

for $(x, y) \in \Sigma \times \Sigma$ are continuous. Thus the *potential operators* $P^{(\lambda)}(\tau, \sigma)$ formally defined by

$$\begin{aligned} &P^{(\lambda)}(\tau, \sigma)F(x) \\ &= \int_{\Sigma} F(y) \frac{1}{|x + \tau\lambda(x) - (y + \sigma\lambda(y))|} d\omega(y) \end{aligned}$$

form mappings from $L^2(\Sigma)$ into $C^{(0,\mu)}(\Sigma)$ and are Hölder continuous with respect to $\|\cdot\|_{C^{(0,\mu)}(\Sigma)}$.

By formal operations we obtain for $F \in C^{(0,\mu)}(\Sigma)$

$$P^{(\lambda)}(\tau, 0)F(x) = \int_{\Sigma} F(y) \frac{1}{|x + \tau\lambda(x) - y|} d\omega(y)$$

($P^{(\lambda)}(\tau, 0)$: operator of the single-layer potential on Σ for values on $\Sigma^{(\lambda)}(\tau)$),

$$\begin{aligned} P_{|_2}^{(\lambda)}(\tau, 0)F(x) &= \frac{\partial}{\partial \sigma} P^{(\lambda)}(\tau, \sigma)F(x)|_{\sigma=0} \\ &= \int_{\Sigma} F(y) \frac{\lambda(y) \cdot (x + \tau\lambda(x) - y)}{|x + \tau\lambda(x) - y|^3} d\omega(y) \end{aligned}$$

($P_{|_2}^{(\lambda)}(\tau, 0)$: operator of the double-layer potential on Σ for values on $\Sigma^{(\lambda)}(\tau)$).

The notation $P_{|_i}^{(\lambda)}$ indicates differentiation with respect to the i -th variable. Analogously we get

$$\begin{aligned} P_{|_1}^{(\lambda)}(\tau, 0)F(x) &= \frac{\partial}{\partial \tau} P^{(\lambda)}(\tau, \sigma)F(x)|_{\sigma=0}, \\ &= - \int_{\Sigma} F(y) \frac{\lambda(x) \cdot (x + \tau\lambda(x) - y)}{|x + \tau\lambda(x) - y|^3} d\omega(y) \end{aligned}$$

and

$$P_{|_2|_1}^{(\lambda)}(\tau, 0)F(x) = \frac{\partial^2}{\partial \tau \partial \sigma} P^{(\lambda)}(\tau, \sigma)F(x)|_{\sigma=0}$$

for the *operators of the normal derivatives*.

The potential operators now enable us to give concise formulations of the classical *limit formulae* and *jump relations* in potential theory which are given in Table 1. Proofs of these relations are given in Freeden (1980), Kersten (1980).

$$\begin{aligned}
& \lim_{\substack{\tau \rightarrow 0 \\ \tau > 0}} \|P^{(\lambda)}(\pm\tau, 0)F - P^{(\lambda)}(0, 0)F\| = 0, \\
& \lim_{\substack{\tau \rightarrow 0 \\ \tau > 0}} \|P_{|1}^{(\lambda)}(\pm\tau, 0)F - P_{|1}^{(\lambda)}(0, 0)F \pm 2\pi(\lambda(x) \cdot \nu(x))F\| = 0, \\
& \lim_{\substack{\tau \rightarrow 0 \\ \tau > 0}} \|P_{|2}^{(\lambda)}(\pm\tau, 0)F - P_{|2}^{(\lambda)}(0, 0)F \mp 2\pi(\lambda(x) \cdot \nu(x))F\| = 0, \\
& \lim_{\substack{\tau \rightarrow 0 \\ \tau > 0}} \|P^{(\lambda)}(\tau, 0)F - P^{(\lambda)}(-\tau, 0)F\| = 0, \\
& \lim_{\substack{\tau \rightarrow 0 \\ \tau > 0}} \|P_{|1}^{(\lambda)}(\tau, 0)F - P_{|1}^{(\lambda)}(-\tau, 0)F + 4\pi(\lambda(x) \cdot \nu(x))F\| = 0, \\
& \lim_{\substack{\tau \rightarrow 0 \\ \tau > 0}} \|P_{|2}^{(\lambda)}(\tau, 0)F - P_{|2}^{(\lambda)}(-\tau, 0)F - 4\pi(\lambda(x) \cdot \nu(x))F\| = 0,
\end{aligned}$$

Table 1: Classical limit and jump relations of potential theory for $F \in C^{(0,\mu)}(\Sigma)$, respectively $F \in L^2(\Sigma)$. The norm $\|\cdot\|$ indicates the Hölder-norm, the $C^{(0)}(\Sigma)$ -norm or the $L^2(\Sigma)$ -norm. For proofs of these relations see Freeden, Kersten (1980).

4 Multiscale Modelling in $L^2(\Sigma)$

Since we are interested in a reconstruction of a function $F \in L^2(\Sigma)$ we will only take those cases in Table 1 into account, where the construction of an approximating identity for F is possible. These are the second and the third limit relation and the second and the third jump relation, which were simply deduced from the limit relations. The disadvantage of the two limit relations mentioned above is, that there appear strongly singular integral kernels in the potential operators ($P_{|1}^{(\lambda)}(0, 0)$, respectively $P_{|2}^{(\lambda)}(0, 0)$ understood in the sense of Cauchy), while the integral kernels of the jump relations are fully regular. This is the reason why we will only take into account the first two jump relations, the jump relation of the normal derivative of the single layer potential (labeled by 5) and the jump relation of the double layer potential (labeled by 6). It should be noted, that the following theory can be formulated in almost the same manner for the two limit relations.

Writing out the jump relations explicitly we obtain the following theorem.

Theorem 4.1 For $F \in L^2(\Sigma)$ and $i = 5, 6$

$$\lim_{\substack{\tau \rightarrow 0 \\ \tau > 0}} \int_{\Sigma} \Phi_{\tau}^i(\cdot, y) F(y) d\omega(y) = F$$

holds in the sense of the $\|\cdot\|_{L^2(\Sigma)}$ -norm, where the kernel functions Φ_{τ}^5 and Φ_{τ}^6 are known explicitly (see Table 2).

4.1 Scaling and Wavelet Functions

For $\tau > 0$ the family $\{\Phi_{\tau}^i\}_{\tau > 0}$ of kernels $\Phi_{\tau}^i : \Sigma \times \Sigma \rightarrow \mathbb{R}$ is called an *oblique Σ -scaling function of type i* . Moreover, $\Phi_1^i : \Sigma \times \Sigma \rightarrow \mathbb{R}$ (i.e.: $\tau = 1$) is called the *oblique mother kernel of the oblique Σ -scaling function of type i* .

Correspondingly, for $\tau > 0$ and $i = 5, 6$, the family $\{\Psi_{\tau}^i\}_{\tau > 0}$ of kernels $\Psi_{\tau}^i : \Sigma \times \Sigma \rightarrow \mathbb{R}$ given by the scaling equation

$$\Psi_{\tau}^i(x, y) = -\alpha(\tau)^{-1} \frac{d}{d\tau} \Phi_{\tau}^i(x, y), \quad (2)$$

for $x, y \in \Sigma$ is called an *oblique Σ -wavelet function of type i* .

In the remainder of this paper we particularly choose $\alpha(\tau) = \tau^{-1}$ (of course, other weight functions than $\alpha(\tau) = \tau^{-1}$ can be chosen in (2)). Moreover, $\Psi_1^i : \Sigma \times \Sigma \rightarrow \mathbb{R}$ (i.e.: $\tau = 1$) is called the *oblique mother kernel of the oblique Σ -wavelet function of type i* .

Definition 4.2 Let $\{\Phi_{\tau}^i\}_{\tau > 0}$ be a Σ -scaling function of type i . Then the associated oblique Σ -wavelet transform of type i is defined by

$$(WT)^{(i)} : L^2(\Sigma) \rightarrow L^2((0, \infty) \times \Sigma)$$

$$(WT)^{(i)}(F)(\tau, x) = \int_{\Sigma} \Psi_{\tau}^i(x, y) F(y) d\omega(y) .$$

Explicit formulations of the oblique Σ -wavelet function of type 5 and type 6 are given in Table 2.

It is not difficult to see that the wavelets Ψ_{τ}^i , $i = 5, 6$, behave like $O(\tau^{-1})$, hence, the conver-

gence of the integrals in the following *reconstruction theorem* is guaranteed.

Theorem 4.3 *Let $\{\Phi_\tau^i\}_{\tau>0}$ be an oblique Σ -scaling function of type i . Suppose that F is of class $L^2(\Sigma)$. Then the reconstruction formula*

$$\int_0^\infty (WT)^i(F)(\tau, \cdot) \frac{d\tau}{\tau} = F$$

holds for $i = 5, 6$ in the sense of $\|\cdot\|_{L^2(\Sigma)}$.

Proof. Let $R > 0$ be arbitrary. By observing Fubini's theorem and the identity

$$\Phi_R^i(x, y) = \int_R^\infty \Psi_\tau^i(x, y) \frac{d\tau}{\tau}, \quad (x, y) \in \Sigma \times \Sigma,$$

we obtain

$$\begin{aligned} & \int_R^\infty (WT)^i(F)(\tau, \cdot) \frac{d\tau}{\tau} \\ &= \int_R^\infty \int_\Sigma \Psi_\tau^i(\cdot, y) F(y) d\omega(y) \frac{d\tau}{\tau} \\ &= \int_\Sigma \int_R^\infty \Psi_\tau^i(\cdot, y) F(y) \frac{d\tau}{\tau} d\omega(y) \\ &= \int_\Sigma \Phi_R^i(\cdot, y) F(y) d\omega(y) . \end{aligned}$$

The limit $R \rightarrow 0$ in connection with Theorem 4.1 yields the desired result. \square

Note that the properties of the oblique Σ -wavelets of type i (analogously to variants of spherical wavelets developed in Freedman et al. (1998)) do not presume the zero-mean property of Ψ_τ^i . The wavelets constructed in this way, therefore, do not satisfy a substantial condition of the Euclidean concept.

4.2 Scale Discretized Reconstruction Formula

In what follows, scale discrete oblique Σ -scaling functions and wavelets of type i will be introduced. We start with the choice of a sequence which divides the continuous scale interval $(0, \infty)$ into discrete subintervals. More explicitly, $(\tau_j)_{j \in \mathbb{Z}}$ denotes a sequence of real numbers satisfying

$$\lim_{j \rightarrow \infty} \tau_j = 0 \quad \text{and} \quad \lim_{j \rightarrow -\infty} \tau_j = \infty . \quad (3)$$

For example, one may choose $\tau_j = 2^{-j}$, $j \in \mathbb{Z}$ (note that in this case, $2\tau_{j+1} = \tau_j$, $j \in \mathbb{Z}$).

Given an oblique Σ -scaling function $\{\Phi_\tau^i\}_{\tau>0}$ of type i , then we define the (scale) *discretized oblique Σ -scaling function* of type i by $\{\Phi_{\tau_j}^i\}_{j \in \mathbb{Z}}$. In doing so, we immediately get the following result.

Theorem 4.4 *For $F \in L^2(\Sigma)$*

$$\lim_{j \rightarrow \infty} \int_\Sigma \Phi_{\tau_j}^i(\cdot, y) F(y) d\omega(y) = F$$

holds for $i = 5, 6$ in the $\|\cdot\|_{L^2(\Sigma)}$ -sense.

Our procedure canonically leads us to the following type of scale discretized wavelets.

Definition 4.5 *Let $\{\Phi_{\tau_j}^i\}_{j \in \mathbb{Z}}$ be a discretized oblique Σ -scaling function of type i . Then the (scale) *discretized oblique Σ -wavelet function* of type i is defined by*

$$\Psi_{\tau_j}^i(\cdot, \cdot) = \int_{\tau_{j+1}}^{\tau_j} \Psi_\tau^i(\cdot, \cdot) \frac{d\tau}{\tau}, \quad j \in \mathbb{Z}.$$

In connection with (2) it follows that

$$\Psi_{\tau_j}^i = - \int_{\tau_{j+1}}^{\tau_j} \tau \frac{d}{d\tau} \Phi_\tau^i \frac{d\tau}{\tau} = \Phi_{\tau_{j+1}}^i - \Phi_{\tau_j}^i . \quad (4)$$

Formula (4) is called (scale) *discretized Σ -scaling equation of type i* .

Assume now that F is a function of class $L^2(\Sigma)$. Observing the discretized Σ -scaling equation of type i we get for $J \in \mathbb{Z}$ and $N \in \mathbb{N}$

$$\begin{aligned} & \int_\Sigma \Phi_{\tau_{J+N}}^i(\cdot, y) F(y) d\omega(y) \\ &= \int_\Sigma \Phi_{\tau_J}^i(\cdot, y) F(y) d\omega(y) \\ &+ \sum_{j=J}^{J+N-1} \int_\Sigma \Psi_{\tau_j}^i(\cdot, y) F(y) d\omega(y) . \end{aligned}$$

The (scale) *discretized oblique Σ -wavelet transform of type i* is defined by

$$(WT)^i : L^2(\Sigma) \mapsto \{H : \mathbb{Z} \times \Sigma \rightarrow \mathbb{R}\}$$

$$(WT)^i(F)(\tau_j; x) = \int_\Sigma \Psi_{\tau_j}^i(x, y) F(y) d\omega(y) .$$

As a final result we are able to formulate the following theorem.

Theorem 4.6 *Let $\{\Psi_{\tau_j}^i\}_{j \in \mathbb{Z}}$ be a (scale) *discretized oblique Σ -wavelet function* of type i . Then, for all $F \in L^2(\Sigma)$, the reconstruction formula*

$$\sum_{j=-\infty}^{+\infty} (WT)^i(F)(\tau_j; \cdot) = F$$

holds for $i = 5, 6$ in $\|\cdot\|_{L^2(\Sigma)}$ -sense.

4.3 Scale and Detail Spaces

Comparing the above result with the continuous analogue (Theorem 4.3) we notice that the subdivision of the continuous scale interval $(0, \infty)$ into discrete pieces means substitution of the integral over τ by an associated discrete sum.

As in the spherical theory of wavelets (see Freeden et al. (1996a), Freeden, Windheuser (1996b)), the operators $R_{\tau_j}^i, P_{\tau_j}^i$ defined by

$$\begin{aligned} R_{\tau_j}^i(F) &= \int_{\Sigma} \Psi_{\tau_j}^i(\cdot, y) F(y) \, d\omega(y), \\ P_{\tau_j}^i(F) &= \int_{\Sigma} \Phi_{\tau_j}^i(\cdot, y) F(y) \, d\omega(y), \end{aligned}$$

for $F \in L^2(\Sigma)$ may be understood as band pass and low pass filter, respectively. The *scale spaces* $V_{\tau_j}^i$ and the *detail spaces* $W_{\tau_j}^i$ of type i are defined as usual by

$$\begin{aligned} V_{\tau_j}^i &= P_{\tau_j}^i(L^2(\Sigma)) = \left\{ P_{\tau_j}^i(F) \mid F \in L^2(\Sigma) \right\}, \\ W_{\tau_j}^i &= R_{\tau_j}^i(L^2(\Sigma)) = \left\{ R_{\tau_j}^i(F) \mid F \in L^2(\Sigma) \right\}, \end{aligned}$$

respectively. From the identity

$$\Phi_{\tau_{j+1}}^i = \Phi_{\tau_j}^i + \Psi_{\tau_j}^i,$$

i.e.

$$P_{\tau_{j+1}}^i(F) = P_{\tau_j}^i(F) + R_{\tau_j}^i(F)$$

for all $J \in \mathbb{Z}$ it easily follows that

$$V_{\tau_{j+1}}^i = V_{\tau_j}^i + W_{\tau_j}^i. \quad (5)$$

However, it should be remarked that the sum (5) generally is neither direct nor orthogonal.

Equation (5) may be interpreted in the following way: The set $V_{\tau_j}^i$ contains a $P_{\tau_j}^i$ -filtered version of a function belonging to the class $L^2(\Sigma)$. The lower the scale, the stronger the intensity of filtering. By adding 'R $_{\tau_j}^i$ -details' contained in the space $W_{\tau_j}^i$ the space $V_{\tau_{j+1}}^i$ is created, which consists of a filtered versions at resolution $j + 1$. Obviously, for $i = 5, 6$,

$$\overline{\bigcup_{j=-\infty}^{\infty} V_{\tau_j}^i} \|_{L^2(\Sigma)} = \overline{\bigcup_{j=-\infty}^{\infty} W_{\tau_j}^i} \|_{L^2(\Sigma)} = L^2(\Sigma).$$

4.4 Solving BVPs by Wavelets

In what follows we want to show, how our multi-scale approach on geoscientifically relevant regular surfaces can be used to approximate the solution of the exterior oblique derivative problem (EODP). The problem can be formulated briefly as follows:

(EODP) Given a function F of class $C^{(0,\mu)}(\Sigma)$. Find a function $U \in C^{(1,\mu)}(\overline{\Sigma_{ext}})$ with

$$\begin{aligned} \Delta U &= 0, \quad x \in \Sigma_{ext}, \\ \frac{\partial U^+}{\partial \lambda}(x) &= \lim_{\substack{\tau \rightarrow 0 \\ \tau > 0}} \lambda(x) \cdot (\nabla U)(x + \tau \lambda(x)) \\ &= F(x), \quad x \in \Sigma, \end{aligned} \quad (6)$$

and U being regular at infinity (that is, $|U(x)| = O(|x|^{-1})$, $|\nabla U(x)| = O(|x|^{-2})$ for $|x| \rightarrow \infty$ uniformly with respect to all directions).

If the field λ coincides with the normal field ν on Σ , equation (6) becomes the boundary condition of the classical exterior Neumann problem.

For given $F \in C^{(0,\mu)}(\Sigma)$, the solution $U \in \text{Pot}^{(1,\mu)}(\overline{\Sigma_{ext}}) = \{U \in C^{(1,\mu)}(\overline{\Sigma_{ext}}) \mid \Delta U = 0 \text{ in } \Sigma_{ext}, U \text{ regular at infinity}\}$ of the (EODP) can be written as a layer potential,

$$U(x) = \int_{\Sigma} Q(y) \frac{1}{|x-y|} \, d\omega(y), \quad (7)$$

where the single layer $Q \in C^{(0,\mu)}(\Sigma)$ satisfies the Fredholm integral equation

$$\begin{aligned} -2\pi Q(x) (\lambda(x) \cdot \nu(x)) \\ + \int_{\Sigma} Q(y) \frac{\partial}{\partial \lambda(x)} \frac{1}{|x-y|} \, d\omega(y) &= F(x) \end{aligned} \quad (8)$$

for all $x \in \Sigma$. An approximation of scale J

$$P_{\tau_J}(Q)(x) = \sum_{l=1}^{N_J} a_l^{N_J} \Phi_{\tau_J}^i(x, y_l^{N_J}), \quad x \in \Sigma$$

(with $i \in \{5, 6\}$, $a_l^{N_J} \in \mathbb{R}$, $y_l^{N_J} \in \Sigma$ appropriate point system on Σ , $l = 1, \dots, N_J$ and $J, N_J \in \mathbb{N}$ sufficiently large) is deducible from (8) by solving a system of linear equations obtained by an appropriate approximation method such as collocation, Galerkin procedure, least squares approximation, etc.

For solving the linear systems fast multipole methods (FMM) are applicable (see e.g. Glockner (2001) and the reference therein).

Remark 4.7 *It should be noted that the singular kernel function $\frac{\partial}{\partial\lambda(x)}\frac{1}{|x-y|}$ in the boundary integral equation (8) is substituted (regularized) in numerical applications by a regular approximation of the form*

$$\frac{1}{2} \frac{\partial}{\partial\lambda(x)} \left(\frac{1}{|x + \tau_L\lambda(x) - y|} + \frac{1}{|x - \tau_L\lambda(x) - y|} \right)$$

with $L \in \mathbb{N}$ chosen large enough. For an explicit description of this type of regularization of the singular integral operator in (8) for Neumann boundary conditions the reader is referred to Freeden, Mayer (2001).

As a final result we can formulate the following

Theorem 4.8 *For given $F \in C^{(0,\mu)}(\Sigma)$, let U be the potential of class $\text{Pot}^{(1,\mu)}(\overline{\Sigma_{ext}})$ with $\frac{\partial U^+}{\partial\lambda} = F$ on Σ . Then the function $U_J \in \text{Pot}^{(0,\mu)}(\overline{\Sigma_{ext}})$ given by*

$$U_J(x) = \sum_{l=1}^{N_J} a_l^{N_J} \int_{\Sigma} \Phi_{\tau_J}^i(y, y_l^{N_J}) \frac{1}{|x-y|} d\omega(y) \quad (9)$$

represents a J -scale approximation of U in the $\|\cdot\|_{C^0(\overline{K})}$ -sense for every $K \subset \Sigma_{ext}$ with $\text{dist}(\overline{K}, \Sigma) > 0$.

Remark 4.9 *It should be noted, that in numerical applications a similar regularization as in Remark 4.7 is applied to the kernel $\frac{1}{|x-y|}$ in (9). Then a fully discrete representation of (9) can be achieved by discretizing the integral with a suitable integration rule (for more details see Mayer (2001)).*

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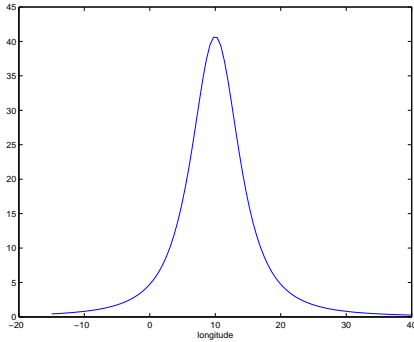
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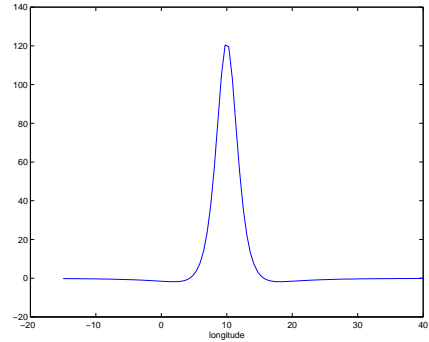
Appendix: Tables and Figures

$$\begin{aligned}
\Phi_\tau^5(x, y) &= \frac{1}{4\pi(\lambda(x) \cdot \nu(x))} \left(\frac{(x + \tau\lambda(x) - y) \cdot \lambda(x)}{|x + \tau\lambda(x) - y|^3} - \frac{(x - \tau\lambda(x) - y) \cdot \lambda(x)}{|x - \tau\lambda(x) - y|^3} \right), \\
\Phi_\tau^6(x, y) &= \frac{1}{4\pi(\lambda(x) \cdot \nu(x))} \left(\frac{(x + \tau\lambda(x) - y) \cdot \lambda(y)}{|x + \tau\lambda(x) - y|^3} - \frac{(x - \tau\lambda(x) - y) \cdot \lambda(y)}{|x - \tau\lambda(x) - y|^3} \right), \\
\Psi_\tau^5(x, y) &= \frac{-\tau}{4\pi(\lambda(x) \cdot \nu(x))} \left(\frac{1}{|x - \tau\lambda(x) - y|^3} + \frac{1}{|x - \tau\lambda(x) - y|^3} \right) \\
&\quad + \frac{3\tau}{4\pi(\lambda(x) \cdot \nu(x))} \left(\frac{((x + \tau\lambda(x) - y) \cdot \lambda(x))^2}{|x + \tau\lambda(x) - y|^5} + \frac{((x - \tau\lambda(x) - y) \cdot \lambda(x))^2}{|x - \tau\lambda(x) - y|^5} \right), \\
\Psi_\tau^6(x, y) &= \frac{-\tau}{4\pi(\lambda(x) \cdot \nu(x))} \left(\frac{\lambda(x) \cdot \lambda(y)}{|x + \tau\lambda(x) - y|^3} + \frac{\lambda(x) \cdot \lambda(y)}{|x - \tau\lambda(x) - y|^3} \right) \\
&\quad + \frac{3\tau}{4\pi(\lambda(x) \cdot \nu(x))} \left(\frac{((x + \tau\lambda(x) - y) \cdot \lambda(x))((x + \tau\lambda(x) - y) \cdot \lambda(y))}{|x + \tau\lambda(x) - y|^5} \right. \\
&\quad \left. + \frac{((x - \tau\lambda(x) - y) \cdot \lambda(x))((x - \tau\lambda(x) - y) \cdot \lambda(y))}{|x - \tau\lambda(x) - y|^5} \right),
\end{aligned}$$

Table 2: Explicit terms of the oblique Σ -scaling functions and wavelet functions defined in Theorem 4.1 and equation (2). In the approach presented here if restricted to the sphere, elementary representations in explicit form and spectral representations in terms of spherical harmonics are available for the Σ -scaling functions and the Σ -wavelets.



(a) scaling function



(b) wavelet function

Figure 1: Σ -scaling function Φ_τ^6 and Σ -wavelet-function Ψ_τ^6 (sectional illustration) for $\tau = 2^{-4}$.

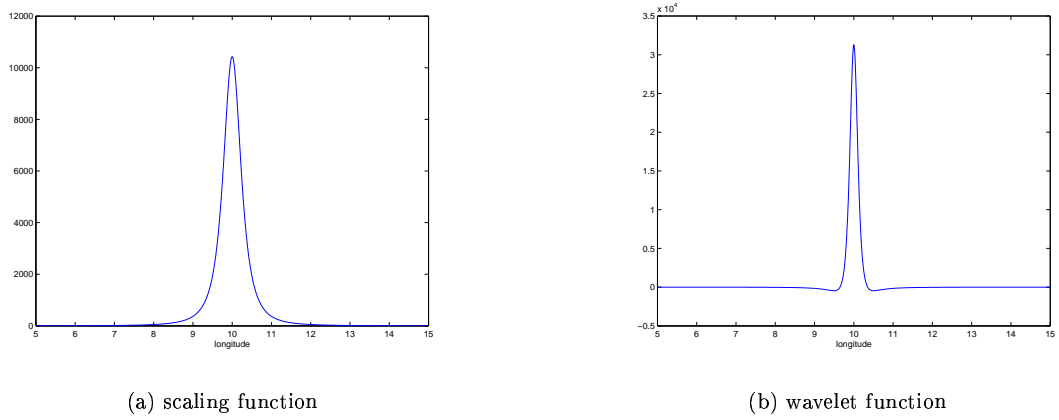


Figure 2: Oblique Σ -scaling function Φ_{τ}^6 and Σ -wavelet-function Ψ_{τ}^6 (sectional illustration) for $\tau = 2^{-8}$. Comparing to Figure 1 the property of space localization without the appearance of any oscillations which is a main disadvantage of some wavelet functions can clearly be seen.

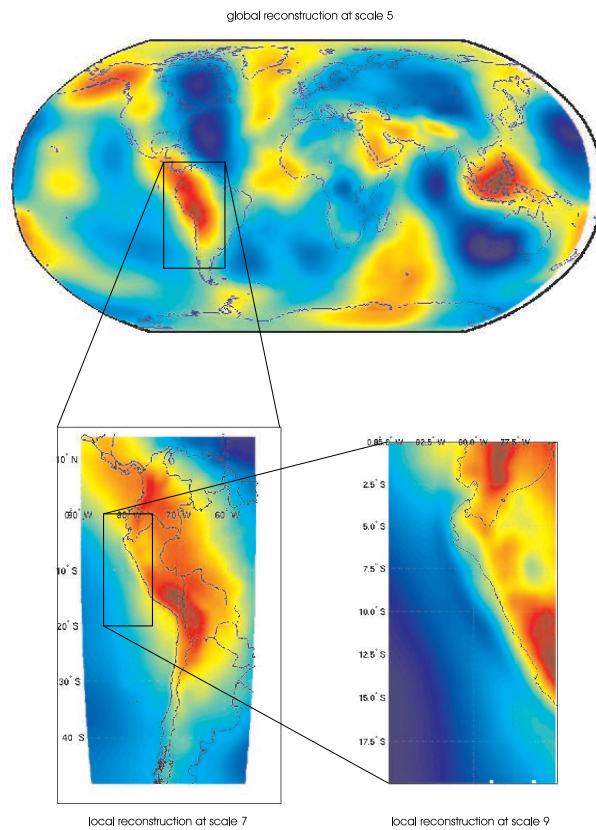
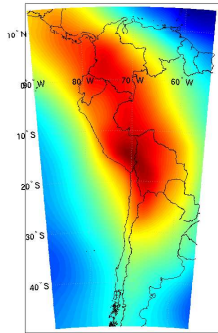
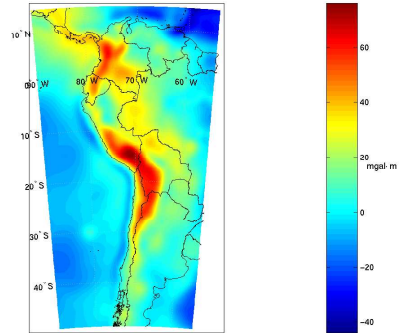


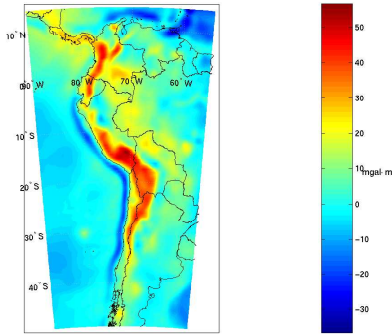
Figure 3: Illustration of the zoom-in property. In order to reconstruct a function on a local area, only data in a certain neighborhood of this area are used. Since global high-scale reconstruction of fine structure is very time-consuming, only the area of interest is reconstructed which can be done with a considerably fewer effort.



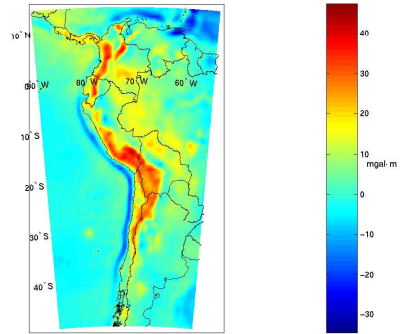
(a) local reconstruction with Σ -scaling function $\Phi_{\tau_j}^6$ at scale $j = 5$



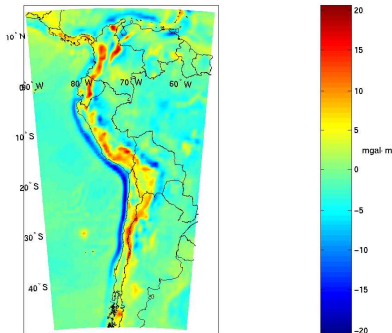
(b) local reconstruction with Σ -wavelet function $\Psi_{\tau_j}^6$ at scale $j = 5$



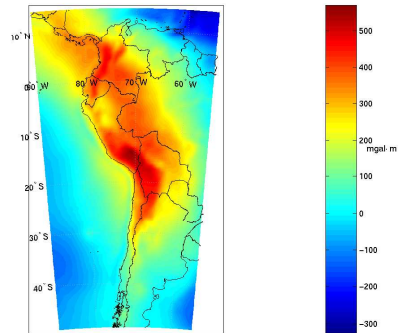
(c) local reconstruction with Σ -wavelet function $\Psi_{\tau_j}^6$ at scale $j = 6$



(d) local reconstruction with Σ -wavelet function $\Psi_{\tau_j}^6$ at scale $j = 7$



(e) local reconstruction with Σ -scaling function $\Psi_{\tau_j}^6$ at scale $j = 8$



(f) local reconstruction with Σ -scaling function $\Phi_{\tau_j}^6$ at scale $j = 5$

Figure 4: Detection of high frequency perturbation within a local area of the EGM96-geopotential model. The buried mass point at 80° West, 30° South is clearly detected, especially in the wavelet reconstruction at scale 8.