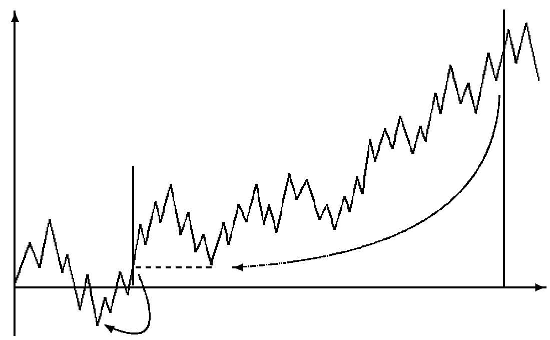
The genealogy of branching processes with continuous state space

Pascal Vogt



Diplomarbeit

Universität Kaiserslautern
Fachbereich Mathematik

Der Mensch wirft das Los;
Aber es fällt, wie der Herr will.

Salomon, 16.33

Introduction

Branching processes arise naturally to describe the random development in time of a population size. Heuristically, a branching process is a stochastic process $\{Z_t : t \in T\}$ such that at each time instance t the value Z_t is interpreted as the size of the population at this time. Although we have not given a rigorous mathematical definition of a branching process, we can already mention that this model does *not* contain the complete genealogical information about the population.

In this thesis, we present a method of coding the genealogical structure of branching processes and we mainly consider the case of branching processes with continuous state space. Nevertheless, we deal at first with the discrete setting, which is easier to handle, to motivate the construction in the continuous case.

Consider a discrete time, discrete state space branching processes. Then, it is natural to describe its genealogy by *family trees*. Nevertheless, in the continuous setting is *no* obvious and easy way to define any genealogical relations. Therefore, describing such a genealogy is a mathematically challenging task. Moreover, the construction automatically leads to a wide class of interesting questions on continuous state branching processes, which depend on their genealogy. Some of these questions are discussed in this thesis. Furthermore, one can use the genealogy of continuous state branching processes to give a *snakelike construction* of superprocesses as it was proposed by Le Gall for super Brownian motion with quadratic branching mechanisms in 1993 (see [LG93], [LGLY98b]).

Most of the work presented in this thesis relies on the work of Jean-François Le Gall, Yves Le Yan and Thomas Duquèsne ([LGLY98a], [LGLY98b], [LGD]). We now give a survey of the content of the individual chapters of this thesis.

The first chapter deals as a preliminary *tool collection* for the rest of the thesis. We introduce ***Lévy processes*** and make a particularly intensive look on the case of Lévy processes *without* negative jumps, which will be important for the later construction. Then, we consider ***local times*** of Markov processes as they were introduced by Blumenthal and Getoor. Finally, we introduce ***branching processes*** and state some of their basic properties.

The second chapter deals with the contruction of the genealogy. First, we consider the discrete case of Galton-Watson processes. Let μ be a probability measure on $\mathbb{N}_0 = \{0, 1, 2, \dots\}$, called the ***offspring distribution***, which we assume to be ***(sub)critical***, meaning that

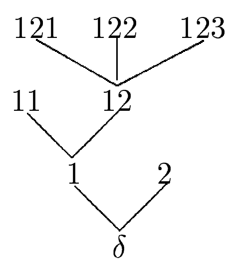
$$\sum_{k=1}^{\infty} k\mu(k) \leq 1.$$

If $\{X_j^n : j, n \in \mathbb{N}\}$ is an array of independent μ-distributed random variables, we call a stochastic process $\{G_n : n \geq 0\}$ defined by $G_0 = 0$ and

$$G_{n+1} = \sum_{j=1}^{G_n} X_j^n \quad \text{for } n \geq 1,$$

a μ-***Galton-Watson (branching) process***. Heuristically, G_n describes the size of a population at time n. As we already mentioned, it is an intuitive and natural idea to use *family trees* to describe the genealogy of a Galton-Watson process. Pictorially, we can construct a μ-***Galton-Watson tree*** τ via the following procedure: starting with one particle at time (generation) 0, this particle has offspring according to the distribution μ. These particles then live in generation 1 and each of them has again an offspring according to μ, independent from the others, and so on. Having constructed a Galton-Watson tree, it is easy to see that we can get a Galton-Watson branching process from this tree just by *counting* all particles alive in each generation.

Although Galton-Watson trees provide a nice and intuitive description of the genealogy in the discrete setting, it is *not at all* clear how to extend this contruction to the continuous case. The *main idea* behind this extension is to code Galton-Watson trees via a stochastic process $\{H_n : n \geq 0\}$, the so called ***discrete height process*** which can be defined by the following heuristic: identify each particle in the tree with an address $u \in \bigcup_{n=0}^{\infty} \mathbb{N}^n$ where $\mathbb{N}^0 := \{\delta\}$ and δ is the address of the root:



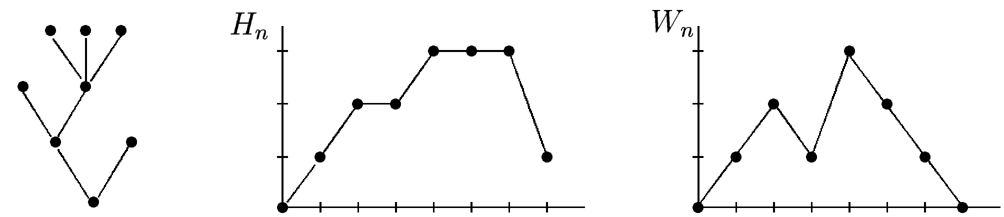
Now, let H_n describe the generation of the n-th vertex if we start at time 0 in the root and then, run through the tree according to the lexicographical order of the addresses. It is easy to see (e.g. by looking at the pictures below) that we can reconstruct the tree from the discrete height process. Moreover, we can construct the Galton-Watson process $\{G_n : n \geq 0\}$ from the height process just by summing up all particles in each level of generation, i.e.

$$G_n := \#\{s : H_s = n\}. \tag{0.1}$$

In general, the discrete height process can *not* be a Markov process. Nevertheless, we prove that we can construct the height process as a functional of a random walk $\{W_n : n \in \mathbb{N}\}$ with increment distribution $\nu(k) = \mu(k+1)$ for $k = -1, 0, 1, \ldots$, by

$$H_n = \#\left\{ j \in \{0, 1, \ldots, n-1\} : W_j = \inf_{j \leq k \leq n} W_k \right\}. \tag{0.2}$$

The following pictures may enlighten the key ideas:



μ-GW tree μ-height process ν-random walk

Why do we need such a complicated construction of the height process in terms of equation (0.2)? As we see, this equation motivates the definition of the continuous analogue of the discrete height process. At first, let us say a few words what is meant by continuous state branching processes.

A ***continuous state branching process*** $\{Z_t : t \geq 0\}$ is a strong Markov process with values in $[0, \infty)$, such that it fulfils the *branching property*

$$\mathcal{L}(Z_t \,|\, Z_0 = x + y) = \mathcal{L}(Z_t \,|\, Z_0 = x) * \mathcal{L}(Z_t \,|\, Z_0 = y),$$

where $*$ denotes convolution. This means that the branching process with initial value (population) $x + y$ behaves like the sum of two independent copies of Z started at x and y respectively.

It is well known that a continuous state branching process (shortly: CSBP) $\{Z_t : t \geq 0\}$ with start in $Z_0 = x > 0$ which is *(sub)critical*, meaning that

$$\mathbb{E}\{Z_t \,|\, Z_0 = x\} \leq x,$$

has the Laplace transform

$$\mathbb{E}\left\{ e^{-\lambda Z_t} \,\middle|\, Z_0 = x \right\} = \exp(-x u_t(\lambda)).$$

Moreover, u_t is the unique nonnegative solution of

$$\frac{\partial u_t(\lambda)}{\partial t} = -\psi(u_t(\lambda))$$

with initial condition $u_0(\lambda) = \lambda$ and a function $\psi : [0, \infty) \to [0, \infty)$ of the form

$$\psi(u) = \alpha u + \beta u^2 + \int_0^\infty (e^{-ru} - 1 + ru)\, \pi(dr), \tag{0.3}$$

where $\alpha, \beta \geq 0$ and π is a σ-finite measure on $(0, \infty)$ such that

$$\int (r \wedge r^2)\, \pi(dr) < \infty.$$

Conversely, for any choice of the parameters α, β and π in the function ψ, there is a CSBP with the corresponding Laplace transform. Hence, a CSBP is completely determined by defining such a function ψ. We call ψ the ***branching mechanism*** of the

ψ-CSBP $\{Z_t : t \geq 0\}$. Moreover, let us denote the class of ψ-CSBP by $\mathcal{C}$.

It turns out (and we show this in the first chapter) that these functions ψ are exactly the Laplace exponents of the *subclass* of Lévy processes $\mathcal{L}$ which are *spectrally positive* (meaning that they only make positive jumps), do *not tend to* ∞ and have *infinite variation*, i.e. if $\{X_t : t \geq 0\}$ is a Lévy process from the class $\mathcal{L}$, then for all $t \geq 0, \lambda \geq 0$,

$$\mathbb{E}\left\{e^{-\lambda X_t}\right\} = \exp(-t\psi(\lambda)),$$

where ψ is a function of the type (0.3). As the Laplace exponents determines a Lévy process uniquely, we can establish an interesting *bijection* between $\mathcal{C}$ and $\mathcal{L}$ in a *non-probabilistic* way, just in terms of the functions ψ.

The main idea behind the construction of the genealogy for ψ-CSBP is to replace the random walk W in formula (0.2) by a ψ-Lévy process $\{X_t : t \geq 0\}$ of the class $\mathcal{L}$. Then we can use this equation to define the continuous analogue of the discrete height process, the so called ***ψ-height process***. In analogy to the discrete case, the ψ-CSBP can be obtained from the *local times* of H as a function of the level parameter. We clearly have to go through some technical difficulties to get a *rigorous* definition of the continuous height process.

Let us consider a *special case* of the height processes, perhaps known by the reader, to describe the main ideas and results of the second chapter:

Let $\{X_t : t \geq 0\}$ be a *Brownian motion* which is contained in the class $\mathcal{L}$ and has Laplace exponent $\psi(\lambda) = \frac{1}{2}\lambda^2$. We show that the corresponding continuous height process $\{H_t : t \geq 0\}$ is distributed as a scaled *reflected Brownian motion.*

By the well known classical (first) ***Ray-Knight theorem***, one can construct a continuous state branching process from reflected Brownian motion. To be specific, denote by $\{L_t^a : t \geq 0, a \geq 0\}$ the *local time* of the reflected Brownian motion $\{H_t : t \geq 0\}$ and denote by

$$T_x := \inf\{s > 0 : L_s^0 = x\}$$

the first hitting time of x by the local time at 0, then, the process

$$\{L_{T_x}^a : a \geq 0\} \tag{0.4}$$

is a ψ-CSBP with start in x and branching mechanism $\psi(\lambda) = \frac{1}{2}\lambda^2$, known as ***Feller's diffusion***.

As we can think heuristically of the local time at level a as *counting* the time instants when $H_t = a$, the complicated looking equation (0.4) turns out to be the natural analogue of the discrete formula (0.1).

Moreover, it is known for several years (see [Al93],[LG93]) that one can use the *excursions* of reflected Brownian motion to code the genealogy of the Feller diffusion. Denote by H_t an excursion of reflected Brownian motion and by σ the length of this excursion. Then this excursion codes a ***continuum random tree*** by the following rules:

(i) each $s \in [0, \sigma]$ corresponds to a ***particle*** of generation H_s

(ii) if $s \leq s' \in [0, \sigma]$, then the particle s is called an ***ancestor*** of the particle s' if

$$H_s = \inf_{s \leq r \leq s'} H_r.$$

The major aim for the second chapter is to generalize this construction to *all* branching mechanisms corresponding to the class of (sub)critical CSBP $\mathcal{C}$ (as it is done in [LGLY98a], [LGD]). Hence, we see in Chapter 2 that for every ψ we can construct a ψ-CSBP from the ψ-height process in the same way as in the classical Ray-Knight case, and this ψ-height process codes the genealogy of the ψ-CSBP.

Moreover, (this is already done in [LGD]) we derive a criterion for the ψ-height process to have *continuous sample paths* which turns out to be important for many interesting purposes (see below).

Another interesting task, which is *not* treated by Le Gall and Le Yan, is to compute the Haussdorff dimension of the levelsets of the height process, i.e.

$$\dim\{t : H_t = a\}.$$

It turns out not to be too hard to derive an explicit formula for the dimension of the zeroset. Nevertheless, as the height process is *not* a Markov process, it is very difficult (and still an open question) to prove the conjecture that all other levelsets have the same dimension as the zeroset.

Having constructed the genealogy in terms of the ψ-height process, we see that there is a natural *duality* between the path properties of the continuous height process at the one hand, and the properties of the ψ-CSBP which depend on their genealogy at the other hand.

For example, and this is *not* treated in the literature so far, let us denote by $\mathcal{A}^a_{a-\varepsilon}$ the number ancestors of generation a alive in generation $a - \varepsilon$. Then, a natural and interesting question is to characterize those CSBP with the property that almost surely for all $a > 0$ and $\varepsilon \in (0, a]$,

$$|\mathcal{A}^a_{a-\varepsilon}| < \infty. \tag{0.5}$$

If a ψ-CSBP fulfils condition (0.5), i.e. for all generations, the number of ancestors of the complete population in any previous generation is finite, then we say that the

CSBP has ***finite biodiversity***.

We treat the characterization of CSBP with finite biodiversity in the third chapter and it turns out that the following statements are equivalent:

(i) the ψ-CSBP has finite biodiversity

(ii) the ψ-CSBP dies almost surely,

(iii) the sample paths of the ψ-height process are continuous almost surely and

(iv) the branching mechanism ψ fulfils the analytical condition

$$\int_1^\infty \frac{1}{\psi(\lambda)} d\lambda < \infty.$$

Moreover, in the case of finite biodiversity, we compute that the *number of ancestors* of any generation in some earlier generation is Poisson distributed.

In the fourth chapter, we present ***limit theorems*** that link the discrete and the continuous setting. It is well known [Lam67] that a sequence of suitable rescaled Galton-Watson processes converges towards a CSBP. We see that in this case, we also have convergence of the associated height processes at least in the sense of weak convergence of the finite dimensional marginal distributions.

In the case when the height process has continuous sample paths, we also present a functional convergence theorem. In particular, these limit theorems give an *a posteriori legitimation* for the choice of the height process as *the* natural candidate to code the genealogy of CSBPs.

The fifth and last chapter then treats ***Zubkov's theorem*** for CSBP, i.e. we compute the distribution of the *most recent common ancestor* conditioned on the survival of the CSBP (this is already done in [LGD]). In particular, it turns out that exactly in the case of stable branching mechanisms $\psi(\lambda) = \lambda^\beta, \beta \in (1, 2]$, the distribution of the most recent common ancestor of the particles alive in generation a is uniformly distributed over $[0, a]$ under the excursion measure of the height process.

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Landau-Arzheim, April 2001
Pascal Vogt

Contents

Chapter 1

Preparation and machinery

Before we can start with the main work, we have to introduce some background machinery which we then use throughout this thesis. We assume that the reader has basic knowledge of probability and stochastic processes and give a short overview on Poisson point, Lévy, branching processes and local times of Markov processes.

1.1 Poisson point and Lévy processes

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $(S, \mathcal{S})$ a measurable space. If we denote by $C(S)$ the set of all finite subsets of S, then $(C(S), \mathcal{C}(S))$, with a suitable σ-algebra $\mathcal{C}(S)$, also becomes a measurable space. Let μ be a measure on $(S, \mathcal{S})$. Then, we call a random variable

$$\Delta : (\Omega, \mathcal{F}, P) \to (C(S), \mathcal{C}(S))$$

a ***Poisson point process*** on $(S, \mathcal{S})$ with intensity measure μ if for all $A \in \mathcal{S}$ the counting functions

$$N_A : (\Omega, \mathcal{F}, P) \to \mathbb{N}$$

defined by $N_A(\omega) := |\Delta(\omega) \cap A|$ are measurable and for disjoint $A_1, \ldots, A_n \in \mathcal{S}$, the random variables $N_{A_1}, \ldots, N_{A_n}$ are independent and Poisson distributed with parameters $\mu(A_1), \ldots, \mu(A_n)$.

Sometimes, it is reasonable to look at Poisson point processes not as a random countable subset of some state space S, but in the slighly different way that the counting functions serve as *random measures* on S. In this sense, we call a measurable mapping Γ from Ω in the space of counting measures on S a ***Poisson point measure*** if the following conditions are satisfied:

(i) for all $A \in \mathcal{S}$, we have that

$$\mu(A) := \mathbb{E}\{\Gamma(A)\} < \infty$$

and $\Gamma(A)$ is Poisson distributed with intensity $\mu(A)$.

(ii) for all pairwise disjoint $A_1, \dots, A_n \in \mathcal{S}$ the random variables $\Gamma(A_1), \dots, \Gamma(A_n)$ are independent.

As Poisson point processes and Poisson point measures can be seen as describing the *same thing* from a different point of view, it is clear that we only need to formulate any theorems for just one perspective.

Theorem* 1.1 *(Existence of Poisson point processes)
If μ is a non atomic, σ-finite measure on $(S, \mathcal{S})$, then, there is a Poisson point process Δ on $(S, \mathcal{S})$ with intensity measure μ.

The condition of σ-finiteness can be slightly weakened, which is not really important for our purpose (for further details and a proof of the theorem see e.g. [Kgm]). For the work with Lévy- and branching processes, we need the following basic results from the theory of point processes which can all be found e.g. in [Kgm].

Theorem* 1.2 *(Mapping Theorem)
Let Δ be a Poisson point process on $(S, \mathcal{S})$ with σ-finite intensity measure μ. Furthermore, let

$$f : (S, \mathcal{S}, \mu) \to (T, \mathcal{T})$$

be measurable. Assume that the measure $\mu^* := \mu \circ f^{-1}$ induced by f on the space $(T, \mathcal{T})$ is non atomic. Then, $f(\Delta)$ is a Poisson point process with state space T and intensity measure μ^*.

Theorem* 1.3 *(Campbell's Theorem)
Let Δ be a Poisson point process on $(S, \mathcal{S})$ with intensity measure μ and let $f : S \to \mathbb{R}$ be a measurable function. Then, the sum

$$\sum_{x \in \Delta} f(x)$$

converges absolutely almost surely if and only if

$$\int_S (1 \wedge |f(x)|) \mu(dx) < \infty.$$

In this case, we have for all $\theta \in \mathbb{C}$

$$\mathbb{E}\left\{e^{\theta \sum_{x \in \Delta} f(x)}\right\} = exp\left(\int_S (e^{\theta f(x)} - 1)\, \mu(dx)\right) \tag{1.1}$$

if one of these expressions exists, and

$$\mathbb{E}\left\{\sum_{x \in \Delta} f(x)\right\} = \int_S f(x)\, \mu(dx). \tag{1.2}$$

Let $\mathcal{M}_f(\mathbb{R})$ be the set of all finite Borel measures on $\mathbb{R}$. Then, the ***Fourier transform*** of a measure $\mu \in \mathcal{M}_f(\mathbb{R})$ is given by

$$\hat{\mu}(\lambda) = \mathcal{F}\mu(\lambda) := \int_{\mathbb{R}} \exp(ix\lambda)\, \mu(dx),$$

for all $\lambda \in \mathbb{R}$. In particular, Fourier transforms are continuous and every $\mu \in \mathcal{M}_f(\mathbb{R})$ is uniquely determined by $\mathcal{F}\mu$ (see e.g. [Bau91]).

Moreover, $\mu \in \mathcal{M}_f(\mathbb{R})$ is said to be ***infinitely divisible*** if for all $n \in \mathbb{N}$, there exists a measure $\nu \in \mathcal{M}_f(\mathbb{R})$, such that $\hat{\mu} = \hat{\nu}^n$. For every infinitely divisible measure $\mu \in \mathcal{M}_f(\mathbb{R})$, it holds that $\hat{\mu}(\lambda) \neq 0$ for all $\lambda \in \mathbb{R}$ (see [Sato99]) and therefore, there is a uniquely determined and continuous function $\psi : \mathbb{R} \to \mathbb{C}$ with $\psi(0) = 0$ and $\hat{\mu}(\lambda) = \exp(-\psi(\lambda))$ for all $\lambda \in \mathbb{R}$. We call this mapping ψ ***characteristic exponent*** of the measure μ. We now turn our attention to the famous Lévy-Khintchine formula which characterizes infinitely divisible distributions. A proof can be found for example in [Fe71].

Theorem* 1.4 *(Lévy-Khintchine formula)
A function $\psi : \mathbb{R} \to \mathbb{C}$ is the characteristic exponent of an infinitely divisible measure μ on $\mathbb{R}$ if and only if there exists $\alpha \in \mathbb{R}$, $\beta > 0$ and a measure π on $\mathbb{R}\backslash\{0\}$ with $\int(1 \wedge x^2)\,\pi(dx) < \infty$, such that for all $\lambda \in \mathbb{R}$

$$\psi(\lambda) = i\alpha\lambda + \beta\lambda^2 + \int_{\mathbb{R}} (1 - e^{i\lambda x} + i\lambda x 1_{\{|x|<1\}})\, \pi(dx). \tag{1.3}$$

One has to remark that α, β and π are uniquely determined. The measure π is called ***Lévy measure*** with respect to μ.

We now introduce a very important class of continuous time stochastic processes, the so called Lévy processes, which can be interpreted heuristically as the continuous time analogue of random walks.

***Definition* 1.5** *A real valued stochastic process $\{X_t : t \geq 0\}$, which can be defined on some probability space $(\Omega, \mathcal{F}, P)$ starting at $X_0 = 0$ with stationary, independent increments and almost surely càdlàg paths, is called a **Lévy process**.*

More formally, $\{X_t : t \geq 0\}$ is a Lévy process if

(a) $X_0 = 0$ almost surely,

(b) for any $n \geq 1$ and for any choice of $0 \leq t_0 \leq t_1 \leq \cdots \leq t_n$, the random variables $X_{t_0}, X_{t_1} - X_{t_0}, \ldots, X_{t_n} - X_{t_{n-1}}$ are independent ***(independence of the increments)***,

(c) $\mathcal{L}(X_{s+t} - X_s)$, i.e. the distribution of $X_{s+t} - X_s$, does not depend on s ***(stationarity)***,

(d) there is a $\Omega_0 \in \mathcal{F}$ with $P(\Omega_0) = 1$, such that for every $\omega \in \Omega_0$, the paths $t \mapsto X_t(\omega)$ are right continuous with left limits as functions of t ***(càdlàg paths)***.

If X is a Lévy process on $(\Omega, \mathcal{F}, P)$, we call $X + x$ for $x \in \mathbb{R}$ a Lévy process *started at* x and we denote its distribution (that is the law of $X + x$ under P) by P_x.

Because of (d), it is *natural* to think of a Lévy process as a random function which is right continuous and has left limits. Hence, we can assume that the underlying probability space Ω is the ***Skorokhod space*** $\mathcal{D}$ of càdlàg functions $\omega : [0, \infty) \to \mathbb{R}$ equipped with the Borel-σ-field $\mathcal{F}$ generated with respect to the ***Skorokhod topology*** on Ω. This is a metrizable topology which can be described as follows: Define

$$\Lambda := \{g : \mathbb{R}_+ \to \mathbb{R}_+ : \text{ continuous with } g(0) = 0 \text{ and } g(s) \uparrow \infty \text{ as } s \uparrow \infty\}$$

to be the set of time changes, then a sequence $(f_n) \subseteq \mathcal{D}$ converges to an element $f \in \mathcal{D}$ in the Skorokhod topology if and only if there exists a sequence of time changes $(g_n) \subseteq \Lambda$, such that

(i) $\sup_{s \geq 0} |g_n(s) - s| \overset{n \uparrow \infty}{\longrightarrow} 0$ and

(ii) $\sup_{s \leq N} |f_n(g_n(s)) - f(s)| \overset{n \uparrow \infty}{\longrightarrow} 0$ for all $N \in \mathbb{N}$.

Moreover, one can show that $\mathcal{D}$ endowed with this topology becomes a complete, separable metric space, i.e. a ***Polish space***. For more details see [JaS87].

***Lemma* 1.6** *Let $\{X_t : t \in [0, \infty)\}$ be a Lévy process on $(\Omega, \mathcal{F}, P)$. Then, the one dimensional distributions P^t are infinitely divisible for all $t \geq 0$. Moreover, there are $\alpha \in \mathbb{R}, \beta > 0$ and a measure π on $\mathbb{R} \backslash \{0\}$, such that with*

$$\psi(\lambda) := i\alpha\lambda + \beta\lambda^2 + \int_{\mathbb{R}} (1 - e^{i\lambda x} + i\lambda x 1_{\{|x|<1\}})\, \pi(dx),$$

we have that $\mathbb{E}(e^{i\lambda X_t}) = \exp(-t\psi(\lambda))$ for all $\lambda \in \mathbb{R}$.

Proof: Consider for $t > 0$ and $n \in \mathbb{N}$ the decomposition

$$X_t = X_{\frac{t}{n}} + \left(X_{\frac{2t}{n}} - X_{\frac{t}{n}}\right) + \cdots + \left(X_{\frac{nt}{n}} - X_{\frac{(n-1)t}{n}}\right).$$

The stationarity of X implies that

$$\mathcal{F}P^t = \left(\mathcal{F}P^{\frac{t}{n}}\right)^n,$$

hence, P^t is infinitely divisible. In particular, $\mathbb{E}\{\exp(i\lambda X_t)\} \neq 0$ for all $t \geq 0$ and P^1 is infinitely divisible. By the Lévy-Khintchine formula, there is a continuous ψ :

$\mathbb{R} \to \mathbb{C}$, with the stated properties, such that $\mathbb{E}\{e^{i\lambda X_1}\} = \exp(-\psi(\lambda))$ for all $\lambda \in \mathbb{R}$. Furthermore, the mapping $t \mapsto \log \mathbb{E}\{e^{i\lambda X_t}\}$ is linear, since

$$\begin{aligned}
\log \mathbb{E}\left\{e^{i\lambda X_{t+s}}\right\} &= \log \mathbb{E}\left\{e^{i\lambda(X_{t+s}+X_t-X_t)}\right\} \\
&= \log \mathbb{E}\left\{e^{i\lambda(X_{t+s}-X_t)}\right\} + \log \mathbb{E}\left\{e^{i\lambda X_t}\right\} \\
&= \log \mathbb{E}\left\{e^{i\lambda X_s}\right\} + \log \mathbb{E}\left\{e^{i\lambda X_t}\right\}.
\end{aligned}$$

As $\log \mathbb{E}\{e^{i\lambda X_1}\} = -\psi(\lambda)$, we can complete the proof by

$$\begin{aligned}
\log \mathbb{E}\left\{e^{i\lambda X_t}\right\} &= t \cdot \log \mathbb{E}\left\{e^{i\lambda X_1}\right\} \\
&= -t\psi(\lambda).
\end{aligned}$$

□

The function ψ is called ***characteristic exponent*** of the Lévy process X. The following theorems state together that a Lévy process could be described *completely* in terms of the characteristic exponent. At first, it seems to be reasonable to give some examples of Lévy processes.

Examples (i) The easiest examples are the deterministic pure drift ***Lévy processes*** $X_t = at$ for some $a \in \mathbb{R}$. It is clear that all requirements of Definition 1.5 are fulfilled and that $P^t = \delta_{at}$, where δ_{at} denotes the Dirac point mass in at. Therefore, by

$$\begin{aligned}
\mathbb{E}\left\{e^{i\lambda X_t}\right\} &= \int e^{i\lambda x}\delta_{at}(dx) \\
&= e^{i\lambda at},
\end{aligned}$$

we can easily compute the characteristic exponent of a trivial Lévy process to $\psi(\lambda) = -i\lambda a$.

(ii) Another well known example is ***standard Brownian motion*** which has the characteristic exponent $\psi(\lambda) = \frac{1}{2}\lambda^2$. This can be seen easily by substitution:

$$\begin{aligned}
\mathbb{E}\left\{e^{i\lambda X_1}\right\} &= \frac{1}{\sqrt{2\pi}}\int e^{i\lambda x} \cdot e^{-\frac{x^2}{2}}\,dx \\
&= \frac{1}{\sqrt{2\pi}}\int e^{i\lambda(z+i\lambda)}e^{-\frac{1}{2}(z+i\lambda)^2}\,dz \\
&= \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\lambda^2}\int e^{-\frac{1}{2}z^2}\,dz \\
&= e^{-\frac{1}{2}\lambda^2}.
\end{aligned}$$

More general, if $\{B_t : t \geq 0\}$ is a standard Brownian motion, then $Y_t := \sqrt{\beta}B_t - \alpha t$ for $\beta > 0$ and $\alpha \in \mathbb{R}$, is a Brownian motion with diffusion parameter β and drift α. An easy modification of the last computations shows that the characteristic exponent of Y is just

$$\psi(\lambda) = i\alpha\lambda + \beta\lambda^2.$$

We need both examples for the first step in the proof of Theorem 1.8 which will give a nice construction of Lévy processes. First, we notice that the distribution of a Lévy process is uniquely determined by his characteristic exponent.

***Theorem* 1.7** *Let X and $\tilde{X}$ be two Lévy processes with the same characteristic exponent ψ, then X and $\tilde{X}$ have the same distribution.*

Proof: It follows readily from the uniqueness of the Fourier transform that the one dimensional marginal distributions agree for every fixed $t \geq 0$. Moreover, the increments of the processes X and $\tilde{X}$ are stationary and independent, hence, the finite dimensional distributions also agree. This gives the statement because of the right continuity of the paths (for more details, that the finite dimensional distributions determine laws on $\mathcal{D}$ see [JaS87] p.314). $\square$

The next theorem guarantees the existence of Lévy processes for a given characteristic exponent. Moreover, the proof gives a probabilistic interpretation of Lévy processes, which is sometimes refered as the ***Lévy-Itô decomposition*** (see [Itô42]): informally, we can think of a Lévy process as the sum of a Brownian motion with drift and a pure jump process. In particular, the proof also shows that any Lévy process is also a semimartingale.

***Theorem* 1.8** *Let $\alpha \in \mathbb{R}$, $\beta \geq 0$ and π be a σ-finite measure on $\mathbb{R}\backslash\{0\}$, such that*

$$\int (1 \wedge |x|^2)\, \pi(dx) < \infty.$$

For $\lambda \in \mathbb{R}$, define

$$\psi(\lambda) := i\alpha\lambda + \beta\lambda^2 + \int (1 - e^{i\lambda x} + i\lambda x 1_{\{|x|<1\}})\, \pi(dx).$$

Then, there exists a probability space $(\Omega, \mathcal{F}, P)$ and a Lévy process $\{X_t : t \geq 0\}$ on $(\Omega, \mathcal{F}, P)$ with characteristic exponent ψ.

We do not prove the theorem in complete detail. Nevertheless, it seems to be important to understand the key ideas of the construction of Lévy processes.

Idea of the proof: Let $(\Omega, \mathcal{F}, P)$ be a probability space on which we can define a Poisson point process Δ with state space $\mathbb{R} \times \mathbb{R}_0^+$ and intensity measure $\pi \otimes m$, where m denotes the Lebesgue measure, and a standard Brownian motion $\{B_t : t \geq 0\}$, which should be independent of Δ. The key idea of the proof is to construct a Lévy process with the given characteristic exponent from these probabilistic objects.

Define $X_t^{(1)} := \sqrt{\beta} B_t - \alpha t$, then $\{X_t^{(1)} : t \geq 0\}$ is a Lévy process with characteristic exponent $\psi^{(1)} = i\alpha\lambda + \beta\lambda^2$. By the mapping theorem, the restriction

$$\Delta^{(2)} := \{(x, s) \in \Delta : x \geq 1\}$$

of Δ is also a Poisson point process with intensity measure $1_{\{(x,s):x\geq 1\}}\,\pi(dx)\,m(ds)$. Now, we use this restriction $\Delta^{(2)}$ to define the process

$$X_t^{(2)} := \sum_{(x,s)\in\Delta^{(2)},s\leq t} x.$$

Then $X^{(2)}$ is a Lévy process independent of $X^{(1)}$. To compute the characteristic exponent of $X^{(2)}$, it is enough to consider the process at time $t=1$,

$$X_1^{(2)} = \sum_{(x,s)\in\Delta^{(2)}} x1_{\{(x,s)\in\Delta^{(2)}:0\leq s\leq 1\}}.$$

Since $\int(1\wedge x^2)\pi(dx)<\infty$, we get that

$$\iint (1\wedge x)1_{\{0\leq s\leq 1;1\leq x\}}(x,s)\,\pi(dx)\,m(ds) = \int_1^\infty \pi(dx) < \infty.$$

Hence, we can apply Campbells Theorem and the characteristic exponent of $X^{(2)}$ computes to

$$\begin{aligned}
\psi^{(2)}(\lambda) &= \iint (1-e^{i\lambda x 1_{\{(x,s):0\leq s\leq 1\}}})1_{\{(x,s):x\geq 1\}}\,\pi(dx)\,m(ds) \\
&= \int_0^1 m(ds)\int(1-e^{i\lambda x})1_{\{x\geq 1\}}\,\pi(dx) \\
&= \int(1-e^{i\lambda x})1_{\{x\geq 1\}}\,\pi(dx).
\end{aligned}$$

Finally, let $\Delta^{(3)} := \{(x,s)\in\Delta : x<1\}$. Then $\Delta^{(3)}$ is a Poisson point process with intensity measure $1_{\{(x,s):x<1\}}\,\pi(dx)\,m(ds)$, which is obviously independent of $\Delta^{(2)}$. For $\varepsilon>0$, the process

$$X_t^{(3),\varepsilon} := \sum_{(x,s)\in\Delta^{(3)},s\leq t} x1_{\{(x,s):\varepsilon<x<1\}} - t\int x1_{\{\varepsilon<x<1\}}\,\pi(dx)$$

is a Lévy process independent of $X^{(1)}$, $X^{(2)}$ and expectation $\mathbb{E}\{X_t^{(3),\varepsilon}\}=0$ for all $t\geq 0$. Hence, $X^{(3),\varepsilon}$ is a martingale with respect to the filtration $\mathcal{G}_t$ which is defined to be the smallest σ-field, such that all counting functions $N_{A\times C}$ for $A\in\mathcal{B}([0,t]), C\in\mathcal{B}(\mathbb{R})$ of the Poisson point process Δ are measurable.
With a similar argument as before, one can show that Campbells Theorem is applicable and the characteristic exponent of $X^{(3),\varepsilon}$ computes to

$$\begin{aligned}
\psi^{(3),\varepsilon}(\lambda) &= \iint (1-e^{i\lambda x 1_{\{(x,s):\varepsilon<x<1;0\leq s\leq 1\}}})\,\pi(dx)\,m(ds) \\
&\quad + \int i\lambda x 1_{\{\varepsilon<x<1\}}\,\pi(dx) \\
&= \int(1-e^{i\lambda x}+i\lambda x)1_{\{\varepsilon<x<1\}}\,\pi(dx).
\end{aligned}$$

For $\eta \in (0, \varepsilon)$, $X^{(3),\varepsilon} - X^{(3),\eta}$ is also a $\mathcal{G}_t$-martingale with

$$\mathbb{E}|X_t^{(3),\varepsilon} - X_t^{(3),\eta}|^2 < \infty, \qquad \text{for all } t \geq 0.$$

Hence, we get by Doob's inequality

$$\begin{aligned}
&\mathbb{E}\left\{\sup_{s\leq t}|X_s^{(3),\varepsilon} - X_s^{(3),\eta}|^2\right\} \\
= \; &\mathbb{E}\left\{\sup_{s\leq t}\left|\sum_{(x,s)\in\Delta^{(3)},s\leq t} x1_{\{(x,s):\eta<x<\varepsilon\}} - t\int x1_{\{\eta<x<\varepsilon\}}\,\pi(dx)\right|^2\right\} \\
\leq \; & 4t\int x^2 1_{\{\eta<x<\varepsilon\}}\,\pi(dx) \overset{\varepsilon\downarrow 0}{\to} 0.
\end{aligned}$$

Therefore, for any sequence $(\varepsilon_k) \downarrow 0$, the processes $X^{(3),\varepsilon_k}$ form a Cauchy sequence in the space of càdlàg processes with respect to the locally uniform convergence in probability (lup, or even in L^1). As this space is complete and every lup-convergent sequence has a subsequence which converges locally uniformly almost surely, we can choose a sequence (ε_k) such that

$$\lim_{\varepsilon\downarrow 0} X^{(3),\varepsilon} =: X^{(3)}$$

locally uniformly almost surely (see e.g. [vWW]). Moreover, one can show that $X^{(3)}$ inherits the martingale property of $X^{(3),\varepsilon_k}$ and it is a process with stationary, independent increments and characteristic exponent

$$\psi^{(3)}(\lambda) = \int (1 - e^{i\lambda x} + i\lambda x)1_{\{x<1\}}\,\pi(dx).$$

If we define $X := X^{(1)} + X^{(2)} + X^{(3)}$, then X is a Lévy process with characteristic exponent $\psi = \psi^{(1)} + \psi^{(2)} + \psi^{(3)}$. $\square$

To construct a Lévy process, it suffices by Theorem 1.8 to define a suitable characteristic exponent. Consider the following important examples.

Examples: (i) Suppose that $\alpha = \beta = 0$ and $\pi = c\delta_1$ for some $c \in \mathbb{R}$. Then,

$$\begin{aligned}
\mathbb{E}\{e^{i\lambda X_t}\} &= \exp(-t\psi(\lambda)) \\
&= \exp\left(-tc\int (1 - e^{i\lambda x} + i\lambda x1_{|x|<1})\,\delta_1(dx)\right) \\
&= \exp\left(-tc(1 - e^{i\lambda})\right),
\end{aligned}$$

which is the Fourier transform of a classical ***Poisson process***.

(ii) Let now $c > 0$ and σ be a probability measure on $\mathbb{R}$ with $\sigma(\{0\}) = 0$. Then, with $\alpha = \beta = 0$ and $\pi = c\sigma$, one can easily compute that

$$\mathbb{E}\left\{e^{i\lambda X_t}\right\} = \exp\left(-tc\int(1-e^{i\lambda x})\,\sigma(dx)\right),$$

which uniquely determines the so called ***compound Poisson process***.

As it is described e.g. in [Bau91] or [ReY99], there is another way to construct Lévy processes using ***convolution semigroups*** and the Daniell-Kolmogoroff Extension Theorem. In this context, there is a one-to-one correspondence between Lévy processes and convolution semigroups. In other words, for any convolution semigroup of probability measures $\{\mu_t : t \geq 0\}$, there exists a Lévy process such that

$$\mathcal{L}(X_t - X_s) = \mu_{t-s}$$

and any Lévy process has transition kernels given by a convolution semigroup. This construction becomes clear if we think of the Lévy Khintchine formula and remark that every infinitely divisible probability distribution can be embedded in a unique convolution semigroup (see for instance [ReY99] or [RW00a]).

A Lévy process is said to be of ***finite variation*** (on compact intervals) if almost surely all paths are of finite variation on compact intervals, i.e. if there is a $\Omega_0 \subseteq \Omega$ such that $P(\Omega_0) = 1$ and for all $\omega \in \Omega_0$ and all compact intervals $[a, b]$, we have

$$\sup\left\{\sum_{i=1}^{n}|X_{t_i}(\omega) - X_{t_{i-1}}(\omega)| : a = t_0 \leq \cdots \leq t_n = b, n \in \mathbb{N}\right\} < \infty. \tag{1.4}$$

The following Lemma gives a useful criterion.

***Corollary* 1.9** *A Lévy process $\{X_t : t \geq 0\}$ on $(\Omega, \mathcal{F}, P)$ with Lévy measure π and characteristic exponent ψ is of finite variation if and only if*

$$b = 0 \textit{ and } \int(1 \wedge |x|)\,\pi(dx) < \infty.$$

Proof: Since Brownian motion is of infinite variation, $b = 0$ is clearly a necessary condition. Let $\Delta \subseteq \mathbb{R}\times\mathbb{R}_0^+$ be the Poisson point process associated to X with intensity measure $\pi \otimes l$. Then, X is of finite variation if for all $t > 0$

$$\sum_{(x,s)\in\Delta, s\leq t} |x| < \infty \text{ almost surely.}$$

By Campbell's theorem, this is fulfiled if and only if $\int(1 \wedge |x|)\,\pi(dx) < \infty$. □

We have to remark that with this knowledge, we see that the proof of Theorem 1.8 shows in particular that any Lévy process $\{X_t : t \geq 0\}$ is a semimartingale, as we can decompose X even in an martingale part (the Brownian motion and the part which comes from the jumps less than 1) and a part of finite variation (the part from the jumps which are bigger than 1 and the drift).

***Corollary* 1.10** *Let $\{X_t : t \geq 0\}$ be a Lévy process with Lévy measure π and finite variation. Then there is a $\beta \in \mathbb{R}$ and a Poisson point process $\Delta \subseteq \mathbb{R} \times \mathbb{R}_0^+$ with intensity measure $\pi \otimes m$, such that*

$$X_t = \beta t + \sum_{(x,s)\in\Delta, s\leq t} x.$$

The opposite direction of the corollary is obviously *not* true, i.e. there are Lévy processes X without Gaussian part ($\beta = 0$) and infinite variation. For example, let $\pi(dx) := \frac{1}{x^2}dx$. Then, one can compute easily that

$$\int (1 \wedge |x|)\, \pi(dx) = \infty \quad \text{and} \quad \int (1 \wedge x^2)\, \pi(dx) < \infty.$$

Suppose for now (and forever if it is not explicitly otherwise stated), that the underlying probability space Ω is the Skorokhod space $\mathcal{D}$ of real valued functions $\omega : \mathbb{R}_0^+ \to \mathbb{R}$ which are right continuous and have left limits, equipped with the Borel-σ-field $\mathcal{F}$, generated with respect to the open sets in the Skorokhod topology and the distributions P_x of a Lévy process started in x. On Ω, we introduce for all $s > 0$ the shift operators

$$\theta_s, \Theta_s : \Omega \to \Omega$$

by $\theta_s\omega(t) = \omega(t+s)$ and $\Theta_s\omega(t) = \omega(t+s) - \omega(s)$. Clearly, we can consider θ_s and Θ_s as Ω-valued random variables and thus as stochastic processes. Now, we introduce a filtration $\{\mathcal{F}_t : t \geq 0\}$ where $\mathcal{F}_t$ is the P-completed σ-field generated by $\{X_s : s \leq t\}$. Then, X is adapted to this filtration and $(\Omega, \mathcal{F}, P_x, \{\mathcal{F}_t\})$ forms the canocical ***stochastic basis*** to treat Lévy processes.

In fact, one can show that the right continuity of the paths of X imply that the filtration $\mathcal{F}_t$ is right continuous in the sense that

$$\mathcal{F}_t = \bigcap_{s>t} \mathcal{F}_s, \qquad \text{for all } t \geq 0. \tag{1.5}$$

Moreover, the well known Blumenthals 01-law (which we know from Brownian motion, see e.g. [Mö00]), i.e. that $\mathcal{F}_0$ is P-trivial, *also* holds for Lévy processes (see e.g. [Ber96]).

A random variable T with values in $[0, \infty]$ is called a ***stopping time*** if $\{T < t\} \in \mathcal{F}_t$ for all $t \geq 0$. By the right continuity of the filtration $\{\mathcal{F}_t : t \geq 0\}$, one sees that T is a stopping time if and only if $\{T \leq t\} \in \mathcal{F}_t$ for all $t \geq 0$, i.e. if T is a ***strict stopping time***. For a stopping time T, define the σ-field

$$\mathcal{F}_T := \{A \in \mathcal{F} : \{T < t\} \cap A \in \mathcal{F}_t \text{ for all } t \geq 0\}. \tag{1.6}$$

We now turn our attention to a very useful property of Lévy processes, the so called ***Markov property***, which pictorially states that the future behaviour of the Lévy process *only* depends on the current state and *not* on the past of the process.

Theorem* 1.11 *(Strong Markov property)
Let $\{X_t : t \geq 0\}$ be a Lévy process on $(\Omega, \mathcal{F}, P)$ with start in x. Then, for every almost surely finite stopping time T and every bounded random variable $Y : (\Omega, \mathcal{F}, P) \to \mathbb{R}$, we have almost surely

$$\mathbb{E}_x \{Y \circ \Theta_T | \mathcal{F}_T\} = \mathbb{E}_{X_T} Y.$$

In particular, $\{X_{T+t} - X_T : t \geq 0\}$ is a Lévy process independent of $\mathcal{F}_T$ and with the same distribution as X.

The proof of the strong Markov property of Lévy processes is very similar to the one for ordinary Brownian motion as it is done e.g. in [Mö00]. See for instance [Ber96] or [Sa99].

1.2 Lévy processes without negative jumps

Let us introduce an important subclass of Lévy processes, namely Lévy processes *without* negative jumps. We will later see why it is important to restrict our attention to this case. For the moment, just keep the following in mind:

Definition* 1.12** *A Lévy process $\{X_t : t \geq 0\}$ with characteristic exponent ψ is called* ***spectrally positive *if its Lévy measure π is supported in $(0, \infty)$, i.e. X makes no negative jumps.*

To prepare the linking between Lévy processes and continuous state branching processes in Chapter 2, we need to characterize spectrally positive Lévy processes in terms of their Laplace transforms. Hence, we need the following fact which is proved in [Ber96].

***Lemma* 1.13** *Let $\{X_t : t \geq 0\}$ be a spectrally positive Lévy process. Then,*

$$\mathbb{E}\{\exp(-\lambda X_t)\} < \infty \textit{ for all } \lambda, t \geq 0.$$

Now, let $\{X_t : t \geq 0\}$ be a spectrally positive Lévy process with characteristic exponent ψ, i.e.

$$\mathbb{E}\{\exp(i\lambda X_t)\} = \exp(-t\psi(\lambda)) \text{ for all } \lambda \in \mathbb{R},$$

where ψ is a function of the type

$$\psi(\lambda) := i\alpha\lambda + \beta\lambda^2 + \int (1 - e^{i\lambda x} + i\lambda x 1_{\{x<1\}})\, \pi(dx), \tag{1.7}$$

with $\alpha \in \mathbb{R}$, $\beta \geq 0$ and π a σ-finite measure on $\mathbb{R}\backslash\{0\}$ which is supported on $(0, \infty)$ and fulfils the integrability condition $\int (1 \wedge x^2)\, \pi(dx) < \infty$.

Our aim is to extend $\lambda \mapsto \mathbb{E}\{\exp(i\lambda X_t)\}$, $\lambda \in \mathbb{R}$, to define an analytic function on the upper complex halfplane

$$\mathcal{U} := \{\lambda \in \mathbb{C} : \mathrm{Im}(\lambda) \geq 0\}.$$

Since $\mathrm{supp}(\pi) \subseteq (0, \infty)$, and by the Lévy Khintchine formula, ψ is well-defined and analytic on $\mathcal{U}$. Hence, using analytic continuation, we can define $\psi^L : [0, \infty) \to \mathbb{R}$ by

$$\begin{aligned}\psi^L(\lambda) &:= -\psi(i\lambda) \\ &= -\left[i\alpha(i\lambda) + \beta(i\lambda)^2 + \int_0^\infty (1 - e^{i(i\lambda)x} + i(i\lambda)x1_{\{x<1\}})\, \pi(dx)\right] \\ &= \alpha\lambda + \beta\lambda^2 + \int_0^\infty (e^{-\lambda x} - 1 + \lambda x1_{\{x<1\}})\, \pi(dx), \end{aligned} \tag{1.8}$$

and therefore,

$$\mathbb{E}\{\exp(-\lambda X_t)\} = \exp(-t\psi^L(\lambda)) \text{ for all } \lambda \geq 0$$

and we call ψ^L the ***Laplace exponent*** of X. To simplify notation, let us still write ψ instead of ψ^L.

Now, we go one step ahead and consider again a subclass of spectrally positive Lévy processes, the so called *subordinators*, which are Lévy processes with *non decreasing* paths. For the moment, it may look quite artificial to be interested in such processes. But, as we will see, they appear very naturally in many situations. For example, one can prove that the *first passage time* process of Brownian motion is such a subordinator (see Lemma 1.16). Moreover, we see that subordinators are closely related to *local times* of Markov processes (see Section 1.3). It is also worth to remark, that any Markov process time changed by a subordinator is still a Markov process.

Definition* 1.14** *A stochastic process $\{\sigma_t : t \geq 0\}$ with values in $[0, \infty]$ and lifetime $\xi := \inf\{t > 0 : \sigma_t = \infty\}$ is called a* ***subordinator *if $\{\sigma_t : 0 \leq t \leq \xi\}$ is a Lévy process with non decreasing paths.*

In particular, subordinators are of finite variation. So, using Corollary 1.10, there is a $d > 0$ and a Poisson point process $\Delta \subseteq \mathbb{R}_0^+ \times \mathbb{R}_0^+$ with intensity measure $\pi \otimes m$, such that

$$\sigma_t = d \cdot t + \sum_{(x,s)\in\Delta, s\leq t} x,$$

for $t < \xi$ and $\sigma_t = \infty$ for $t \geq \xi$. This representation of σ is often referred as the Lévy-Itô decomposition of subordinators. Moreover, the parameter d is called the ***drift*** of the subordinator.

Now, let $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ be the Laplace exponent of σ, i.e.

$$\mathbb{E}\{\exp(-\lambda\sigma_t)\} = \exp(-t\psi(\lambda))$$

for $\lambda \in \mathbb{R}$. Then, by dominated convergence, we get

$$\begin{aligned} P\{\xi > t\} &= \mathbb{E}\{1_{\{\xi>t\}}\} \\ &= \lim_{\lambda\to 0} \mathbb{E}\{\exp(-\lambda\sigma_t)\} \\ &= \lim_{\lambda\to 0} \exp(-t\psi(\lambda)) \\ &= \exp(-t\psi(0)), \end{aligned}$$

and therefore, ξ is *exponentially* distributed with parameter $\phi(0) =: k$, the so called ***killing rate*** of the subordinator σ. Using Campbell's theorem, we can easily compute the Laplace exponent of σ to

$$\psi(\lambda) = k + d\lambda + \int (1 - e^{-\lambda x})\,\pi(dx),$$

for $\lambda \leq 0$. Conversely, a mimic of the proof of Theorem 1.8 shows that for any choice of $k, d \geq 0$ and σ-finite π with $\int (1 \wedge x)\,\pi(dx) < \infty$, there is a subordinator with the corresponding Laplace exponent.

Important examples of subordinators are the so called ***α-stable subordinators*** (for $\alpha \in (0,1)$) which are characterized by their Laplace transform

$$\psi(\lambda) = \lambda^\alpha = \frac{\alpha}{\Gamma(1-\alpha)} \int_0^\infty (1 - e^{-\lambda x}) x^{-\alpha-1} dx.$$

If $\{\sigma_t : t \geq 0\}$ is a subordinator with drift d, then

$$\mathcal{R} := \{\sigma_t : t \geq 0\}^{cl}$$

is called the ***closed range*** of the subordinator σ. If we introduce the right continuous inverse $L_t := \inf\{s > 0 : \sigma_s > t\}$ of σ, then we have the following nice characterization of the size of the range (which can be found for example in [Ber00]).

***Theorem* 1.15** *Almost surely for all $t \geq 0$, we have*

$$m\{\mathcal{R} \cap [0,t]\} = d \cdot L_t,$$

where m denotes the Lebesgue measure on the real line. In particular, $\mathcal{R}$ has zero Lebesgue measure if and only if $d = 0$.

Proof: Without loss of generality, we can assume that $k = 0$. We use the Lévy-Itô decomposition of σ,

$$\sigma_t = d \cdot t + \sum_{(x,s)\in\Delta, s\leq t} x, \tag{1.9}$$

for $t < \xi$ and a suitable Poisson point process Δ. Denote by

$$\mathcal{J} := \{s < \xi : (x,s) \in \Delta \text{ and } x > 0 \text{ for some } x\}$$

the set of jump times of Δ. Then, we can decompose $\mathcal{R}^c$ by

$$\mathcal{R}^c = \bigcup_{s \in \mathcal{J}} (\sigma_{s^-}, \sigma_s)$$

and one gets almost surely for all $t \geq 0$

$$\begin{aligned} m\{\mathcal{R} \cap [0, \sigma_t]\} &= \sigma_t - m\left\{ \bigcup_{s \leq t, s \in \mathcal{J}} (\sigma_{s^-}, \sigma_s) \right\} \\ &= \sigma_t - \sum_{(x,s) \in \Delta, s \leq t} x = d \cdot t. \end{aligned}$$

Clearly, $\mathcal{R} \cap (\sigma_{t^-}, \sigma_t) = \{\sigma_t\}$ and therefore, $m\{\mathcal{R} \cap [0, \sigma_{t^-}]\} = d \cdot t$. Note that $s \in (\sigma_{t^-}, \sigma_t)$ if and only if $L_s = t$, which leads to

$$\begin{aligned} m\{\mathcal{R} \cap [0, s]\} &= d \cdot t \\ &= d \cdot L_s, \end{aligned}$$

almost surely for all $s \geq 0$ which gives the statement. □

The next lemma gives the key to a wide and interesting class of examples for subordinators. We prove that the first passage time process of a spectrally positive Lévy process is a subordinator.

Let $\psi : [0, \infty) \to \mathbb{R}$ be the Laplace exponent of a spectrally positive Lévy process, then the function ψ is convex and

$$\lim_{\lambda \to \infty} \psi(\lambda) = \infty$$

(see [Ber00]). Denote by $\phi(0)$ the largest root of ψ. As $\psi(0) = 0$, the convexity of ψ implies that 0 and $\phi(0)$ are the only roots in the case of $\phi(0) > 0$. Hence,

$$\psi : [\phi(0), \infty) \to [0, \infty)$$

is a continuous bijection which is invertible, and we denote by

$$\phi : [0, \infty) \to [\phi(0), \infty)$$

the *inverse* function of ψ, i.e. the unique function ϕ, such that $\psi \circ \phi(\lambda) = \lambda$, for all $\lambda \geq 0$. Now, we are ready to formulate the promised first passage time lemma:

Lemma* 1.16 *(First passage times)
Let X be a spectrally positive Lévy process with Laplace exponent ψ and denote by ϕ the inverse function of ψ. Then, the process $\{T_x : x \geq 0\}$ defined by

$$T_x := \inf\{s \geq 0 : X_s = -x\} \text{ for } x \geq 0,$$

is a subordinator with Laplace exponent ϕ.

Proof: It is clear that T is non-decreasing and by the strong Markov property of the Lévy process X the first passage process T inherits the independence and stationarity of the increments from X. So, $\{T_x : x \geq 0\}$ is a subordinator. But we still have to compute its Laplace transform.
For all $\lambda > 0$, let $M_t := \exp(-\phi(\lambda)X_t - \lambda t)$. Then for all $s < t$, the stationarity and independence of the increments of X and the fact that

$$\mathbb{E}\{e^{-\phi(\lambda)X_t}\} = \exp(-\lambda t)$$

implies that

$$\begin{aligned}
\mathbb{E}\{M_t|\mathcal{F}_s\} &= \mathbb{E}\{\exp(-\phi(\lambda)X_t - \lambda t)|\mathcal{F}_s\} \\
&= \exp(-\phi(\lambda)X_s - \lambda s)\mathbb{E}\{\exp(-\phi(\lambda)X_t - \lambda t - (-\phi(\lambda)X_s - \lambda s)\} \\
&= \exp(-\phi(\lambda)X_s - \lambda s) \cdot 1 = M_s
\end{aligned}$$

and hence, $\{M_t : t \geq 0\}$ is a martingale. Note that $M_{T_x \wedge t}$ is a uniformly integrable martingale and as X is assumed to be spectrally positive, $X_{T_x} = -x$ for all $x \geq 0$. By the martingale convergence theorem, $M_{t\wedge T_x}$ converges to $\exp(\phi(\lambda)x - \lambda T_x)$ on $\{T_x < \infty\}$ almost surely and in L^1. Hence,

$$\begin{aligned}
1 = \mathbb{E}\{M_0\}\mathbb{E}\{M_{T_x}\} &= \mathbb{E}\{\exp(\phi(\lambda)x - \lambda T_x)\} \\
&= \mathbb{E}\{\exp(-\lambda T_x)\} \cdot \exp(\phi(\lambda)x).
\end{aligned}$$

□

In particular, this Lemma generalizes the well known fact that the first passage time process of Brownian motion (BM is of course spectrally positve) is a stable subordinator of index $\frac{1}{2}$.

1.3 Local times

Before we can introduce local times, we first have to fix a suitable class of processes for which the construction works. In general, one could use methods of stochastic calculus to introduce local times for semimartingales (this could be achieved via the Meyer-Tanaka-formula for the case of continuous semimartingales, see e.g. [Mö00b]). Nevertheless, we will follow the different way of Blumenthal and Getoor who introduce local times for *Markov processes* which have some regularity condition. Throughout this section, let $(E, \mathcal{B}(E))$ a Polish space with Borel-σ-field.

Definition* 1.17** *A family of functions $P_t : E \times \mathcal{B}(E) \to [0,1]$ for $t \geq 0$, is called a family of* ***transition functions *(sometimes also called kernels), if the following conditions are satisfied*

(i) for $t \geq 0$ and any $x \in E$, $P_t(x, \cdot)$ is a probability measure on E,

(ii) for $t \geq 0$ and $A \in \mathcal{B}(E)$, $P_t(\cdot, A)$ is $\mathcal{B}(E)$-measurable and

(iii) for $s, t \geq 0, x \in E$ and $A \in \mathcal{B}(E)$, we have the so called ***Chapman-Kolmogoroff-equation***

$$P_{s+t}(x, A) = \int_E P_t(y, A) P_s(x, dy). \tag{1.10}$$

In particular, the Chapman-Kolmogoroff-equation can be seen as the basis for the weak Markov property. Let us now define what we mean by a Markov process.

Definition* 1.18** *An E-valued stochastic process $\{M_s : s \geq 0\}$, defined on some filtered probability space $(\Omega, \mathcal{G}, (\mathcal{G}_t), P_x)$ which is adapted to the filtration $(\mathcal{G}_t)$, is called a* ***Markov process *if for every $t > 0, s > 0$ and any Borel set $A \in \mathcal{B}(E)$*

$$P_x\{M_{t+s} \in A \,|\, \mathcal{G}_t\} = P_{M_t}\{M_s \in A\}$$

P_x-almost surely for all $x \in E$.

As it is shown for example in [EK86] (p.157), for any given family of transition kernels, there exists a probability space $(\Omega, \mathcal{G}, (\mathcal{G}_t), P_x)$ and an E-valued Markov process $\{M_s : s \geq 0\}$, which can be defined on this space, such that

$$P_x(M_t \in A) = P_t(x, A),$$

for all $t \geq 0, x \in E$ and $A \in \mathcal{B}(E)$.

If we could replace the deterministic t's in the definition of Markov processes by any almost surely finite $\mathcal{G}_t$-stopping time T, then we call M a ***strong Markov process***. So in particular, we already saw by Theorem 1.11 that every Lévy process is a strong Markov process.

Now assume that $\{M_t : t \geq 0\}$ is a Markov process with values in E which can be defined on the stochastic basis $(\Omega, \mathcal{G}, (\mathcal{G}_t), P_a)$. (The following description of local times can be similarly found in [Ber96] and [Ber00].)

A point $a \in E$ is called ***regular*** if

$$P_a\{\inf\{t > 0 : M_t = a\} = 0\} = 1.$$

Informally, a is a regular point for the Markov process M if the process returns to a at arbitrarily small times almost surely. Our motivation for this section is to find a method how to measure the size of sets of those times, when M returns to a. As it was shown by Blumenthal and Getoor, there is an interesting link of this question and subordinators. They proved that the closure of the set of times when M started at a returns to a, can be identified as the closed range of some subordinator σ^a defined on the same probability space, i.e.

$$\mathcal{R} := \{s \geq 0 : M_s = a\}^{cl} = \{\sigma_t^a : t \geq 0\}^{cl}.$$

Moreover, the subordinator σ^a is uniquely determined by M up to a multiplicative constant, i.e. if σ' is another subordinator with close range $\mathcal{R}$, then there exists a $c > 0$, such that $\sigma'_t = \sigma^a_{ct}$, for all $t \geq 0$ almost surely.

Now, denote by L^a_t the inverse of σ given by

$$L^a_t := \inf\{s > 0 : \sigma^a_s > t\}.$$

Then, one can show that $\{L^a_t : t \geq 0\}$ has the following properties

(i) $\{L^a_t : t \geq 0\}$ is $\mathcal{G}_t$-adapted,

(ii) $L^a_0 = 0$ almost surely and the paths $t \mapsto L^a_t$ of L^a are almost surely *continuous* and *non decreasing*,

(iii) for all $t, s \geq 0$, we have the *additivity property*

$$L^a_{t+s} = L^a_s + \tilde{L}^a_t$$

almost surely, where $\tilde{L}$ refers to the shifted Markov process $M^{(s)}$,

(iv) the support of the Stieltjes measure dL^a is $\{t : M_t = a\}^{cl}$ and

(v) $\{L^a_t : t \geq 0\}$ is *uniquely* defined up to a multiplicative constant.

Definition* 1.19** *We call the* ***continuous additive functional $\{L^a_t : t \geq 0\}$ *that increases only on the set of times* $t > 0$*, when* $M_t = a$ *the* ***local time*** *at level* a *of the Markov process* M*.*

Moreover, this construction can be achieved simultaneously for all $a \in E$ and sometimes the jointly measurable two parameter random field

$$\{L^a_t : t \geq 0, a \in E\}$$

is called local time of the Markov process M.

In this thesis, we are particularly interested to study the local time at 0 of a subclass of real-valued Markov processes for which 0 necessarily is a regular point. Consider the closure of the zeros of a given Markov process $\{t : M_t = 0\}^{cl}$. Clearly, if M is continuous the set of zeros is automatically closed. Then its complement can be written as the union of countably many open intervals

$$\{t : M_t = 0\}^{cl} = \bigcup_{j \in J} (\alpha_j, \beta_j), \tag{1.11}$$

for some $J \subseteq \mathbb{N}$. Recall that $\mathcal{D}$ is the Skorokhod space of càdlàg functions and define $\omega_j \in \mathcal{D}$ by

$$\omega_j(s) := M_{(\alpha_j + s) \wedge \beta_j}. \tag{1.12}$$

We call the $\omega_j, j \in J$ the ***excursions*** of the Markov process M associated to the ***excursion intervals*** (α_j, β_j).

The following theorem due to Itô forms the basis of excursion theory:

Theorem* 1.20** ***(Point process of excursions)
There exists a σ-finite measure N on $\mathcal{D}$, called the ***excursion measure****, such that the countable random set*

$$\{(L^0_{\alpha_j}, \omega_j) : j \in J\}$$

forms a Poisson point process on $\mathbb{R}_+ \times \mathcal{D}$ with intensity measure $dl \otimes dN$.

As a preparation for the later chapters, we now introduce local times for a special class of Markov processes.

Therefore, let $\{X_t : t \geq 0\}$ be a Lévy process. Then, the processes $\{S_t : t \geq 0\}$ and $\{I_t : t \geq 0\}$ given by

$$S_t := \sup_{0 \leq s \leq t} X_s \quad \text{and} \quad I_t := \inf_{0 \leq s \leq t} X_s,$$

are called ***supremum*** and ***infimum process*** of X. Note that S and I are both no Markov processes, so they do *not* belong to any class of processes we introduced so far. Nevertheless, our aim is to define a local time at 0 of the reflected processes $S - X$ and $X - I$. Due to the last section, we need the following:

***Lemma* 1.21** *Let $\{X_t : t \geq 0\}$ be a Lévy process. Then, the processes $\{S_t - X_t : t \geq 0\}$ and $\{X_t - I_t : t \geq 0\}$ are strong Markov processes in $\mathbb{R}_+$. Moreover, if X has infinite variation, then 0 is a regular point for the Markov processes $S - X$ and $X - I$.*

The fact, that both reflected processes are strong Markov processes is proved in [Ber96], p.156, and could be done by standard arguments. To proof the regularity of the point 0, one needs some more machinery about Lévy processes which we want to omit here (see [Ber00], p.26).

As the regularity of 0 is *necessary* to define local times, we assume that $\{X_t : t \geq 0\}$ is a Lévy process with *infinite variation* for the rest of the thesis. Then, by the previous statements, there exists a local time at 0 denoted by $\{L_t : t \geq 0\}$ of the reflected Lévy process $S - X$, which is uniquely determined up to a multiplicative constant.

With the help of the next theorem, we want to fix this constant for the rest of this thesis. A proof of this theorem can be found in [LGD].

Theorem* 1.22 *(Normalization of local time)
We can fix the normalization of $\{L_t : t \geq 0\}$ *such that for all* $t \geq 0$*, in probability*

$$L_t = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_0^t 1_{\{S_s - X_s < \varepsilon\}} ds. \tag{1.13}$$

In particular, for every $t \geq 0$, we can choose a sequence ε_k converging to zero, such that

$$L_t = \lim_{k \to \infty} \frac{1}{\varepsilon_k} \int_0^t 1_{\{S_s - X_s < \varepsilon_k\}} \, ds \tag{1.14}$$

holds almost surely. Moreover, using the monotonicity of $t \mapsto L_t$, we could choose a suitable subsequence of (ε_k) such that (1.14) holds almost surely for all $t \geq 0$.

Let $L^{-1}(t) := \inf\{s : L_s > t\}$ be the inverse of the local time with the convention that $X_{L^{-1}(t)} = \infty$ if $t \geq L_\infty$. Then, we have the following useful fluctuation formula:

Theorem* 1.23 *(Fluctuation formula)
Let $\{X_t : t \geq 0\}$ *be a spectrally positive Lévy process with Laplace exponent* ψ*. Then,* $\{X_{L^{-1}(t)} : t \geq 0\}$ *is a subordinator with Laplace exponent*

$$\tilde{\psi}(\lambda) = \frac{\psi(\lambda)}{\lambda} = \alpha + \beta\lambda + \int_0^\infty (1 - e^{-\lambda r})\pi([r, \infty)) \, dr. \tag{1.15}$$

In particular in the case $\alpha \neq 0$, we have by dominated convergence that

$$\begin{aligned} P\{L^{-1}(t) < \infty\} &= P\{X_{L^{-1}(t)} < \infty\} \\ &= \mathbb{E}\left\{1_{\{X_{L^{-1}(t)} < \infty\}}\right\} \\ &= \mathbb{E}\left\{\lim_{\lambda \downarrow 0} (\exp -\lambda X_{L^{-1}(t)})\right\} \\ &= \lim_{\lambda \downarrow 0} \exp(-t\tilde{\psi}(\lambda)) \\ &= \exp(-t\alpha), \end{aligned} \tag{1.16}$$

which shows that L_∞ is *exponentially* distributed with parameter $\alpha > 0$.

Theorem 1.23 is due to Bingham (see [Bi76]). It originally states that

$$\mathbb{E}\left\{e^{-\lambda X_{L^{-1}(t)}}\right\} = \exp(-ct\tilde{\psi}(\lambda)),$$

for some $c > 0$ which depends on the normalization of the local time L. One can prove (this is done in [LGD]) that, due to our choice of the normalization in Theorem 1.22, we get that $c = 1$. This is *one* motivation to choose the normalization in the way of Theorem 1.22.

In particular, if we assume that $\beta > 0$, the subordinator $X_{L^{-1}(t)}$ has drift β and we can use Theorem 1.15 to measure the range of $X_{L^{-1}(t)}$ by Lebesgue measure, which leads to a nice modification of the local time L.

***Corollary* 1.24** *Let $\{X_t : t \geq 0\}$ be a spectrally positive Lévy process with Laplace exponent ψ as defined by (1.8) with $\beta > 0$ and let $\{L_t : t \geq 0\}$ be the local time at 0 of $S - X$. Then for every $t \geq 0$, almost surely,*

$$L_t = \frac{1}{\beta} m\{S_r : r \leq t\}.$$

Proof: We have just shown that $\{X_{L^{-1}(t)} : t \geq 0\}$ is a subordinator. Hence we can apply Theorem 1.15 to obtain, almost surely for all $u \geq 0$

$$m\{X_{L^{-1}(t)} : t \leq u, L^{-1}(t) < \infty\} = \beta(u \wedge L_\infty).$$

Moreover, the set $\{X_{L^{-1}(t)} : t \leq u, L^{-1}(t) < \infty\}$ coincides with the set $\{S_r : r \leq L^{-1}(u)\}$ except for at most countably many points, which do not affect the Lebesgue measure. Hence, we get

$$m\{S_r : r \leq t\} = \beta L_t.$$

□

1.4 Branching processes

Branching processes can be seen as describing the random evolution of a population size. Therefore, it is natural to assume, that if one divides the population at any time in two parts and let them develop independently, then the sum of these two parts should behave like the original population process. This heuristic directly lead us to the formal definition of a branching process. Nevertheless, this description does *not* contain any genealogical information, meaning that looking at branching processes does not help to get information about the ancestrial lines of certain particles in the population. The main subject of this thesis is to develop structures which can be interpreted as coding the genealogy of branching processes. At first we like to introduce these branching processes formally.

***Definition* 1.25** *A strong Markov process $\{Z_t : t \in T\}$ with some ordered time set T and values in $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ is called **branching process**, if for all families*

$$\{P_t(x, \cdot) : t \geq 0, x \in E\}$$

*of transition kernels the **branching property** is fulfiled, i.e.*

$$P_t(x + y, \cdot) = P_t(x, \cdot) * P_t(y, \cdot).$$

The most important examples are the following ones:

Examples: (a) Suppose that μ is a probability distribution on $\mathbb{N}$ and let $\{X_j^n : n, j \in \mathbb{N}\}$ be an array of independent and identical μ-distributed random variables. Then the process $\{G_n : n \in \mathbb{N}\}$ given by $G_0 = 1$ and

$$G_{n+1} = \sum_{j=1}^{G_n} X_j^n.$$

is called a ***Galton-Watson process*** with offspring distribution μ. The class of Galton-Watson processes form the simplest class of branching processes. They describe the evolution in discrete time of a population with offspring (or birth) distribution μ and are intensively studied for more than 30 years. We refer to the books of Harris and Athreya/Ney for more information about Galton-Watson processes.

(b) A strong Markov process $\{Z_t : t \in \mathbb{R}_+\}$ with values in $\mathbb{R}_0^+$, whose transition kernels fulfil the branching property, is said to be a continuous time continuous state branching process or just ***continuous state branching process*** (shortly a ***CSBP***).

The CSBPs are the main subject of interest for the rest of this thesis. To motivate the methods used in the continuous case, we will nevertheless first develop the theory for Galton-Watson processes. The following theorem is originally due to Silverstein. A proof can be found in [LG99]. It will give a first idea how to relate Lévy- and continuous state branching processes.

Theorem 1.26 *Let $\{Z_t : t \geq 0\}$ be a CSBP with start in $x > 0$ which is critical or subcritical, i.e. for all $t > 0$*

$$\int y P_t(x, dy) \leq x.$$

Then, the Laplace transform of the transition kernels is given by

$$\int_0^\infty e^{-\lambda y} P_t(x, dy) = \exp(-x u_t(\lambda)),$$

where u_t is the unique non-negative solution of

$$\frac{\partial u_t(\lambda)}{\partial t} = -\psi(u_t(\lambda))$$

with initial condition $u_0(\lambda) = \lambda$ and a function $\psi : [0, \infty) \to [0, \infty)$ of the type

$$\psi(u) = \alpha u + \beta u^2 + \int_0^\infty (e^{-ru} - 1 + ru)\, \pi(dr),$$

where $\alpha, \beta \geq 0$ and π is a σ-finite measure on $(0, \infty)$, such that

$$\int (r \wedge r^2)\, \pi(dr) < \infty.$$

Conversely, for any choice of the parameters in ψ, there is a CSBP with the corresponding Laplace transform.

Therefore, one can describe any (sub)critical CSBP just by defining a suitable ψ. In this case, we just write ***ψ-CSBP*** for a continuous-state branching process $\{Z_t : t \geq 0\}$ and we call the function ψ the ***branching mechanism*** of Z. Moreover, we denote by $\mathcal{C}$ the class of (sub)critical CSBP.

An important example of a CSBP is the so called ***Feller diffusion***, the ψ-CSBP corresponding to $\psi(u) = \beta u^2$. Another way to obtain the Feller diffusion is via a stochastic differential equation (see e.g. [LG99]).

***Theorem* 1.27** *Let $\{B_t : t \geq 0\}$ be a one dimensional Brownian motion then the Feller diffusion is the unique strong solution of the stochastic differential equation*

$$dX_t = \sqrt{2\beta X_t}\, dB_t.$$

Intuitively, it is clear that if a CSBP reaches 0, then the process stays there forever and the population dies. To be riguros, let $\{Z_t : t \geq 0\}$ be a ψ-CSBP and let $T := \inf\{t \geq 0 : Z_t = 0\}$. Then, the strong Markov property of Z implies that $Z_t = 0$, for every $t > T$ almost surely. Hence, the following definition makes sense:

***Definition* 1.28** *Let $\{Z_t : t \geq 0\}$ be a CSBP, then, the time $T := \inf\{t \geq 0 : Z_t = 0\}$ is called **extinction time** of Z. Moreover, if $T < \infty$ almost surely, we say that the CSBP $\{Z_t : t \geq 0\}$ **dies in finite time**.*

The following theorem gives a necessary and sufficient condition for a CSBP to die in finite time in terms of the branching mechanism.

Theorem* 1.29** ***(Extinction of CSBP)
A ψ-CSBP $\{Z_t : t \geq 0\}$ dies in finite time if and only if

$$\int_1^\infty \frac{1}{\psi(\lambda)}\, d\lambda < \infty.$$

If this condition fails, then $T = \infty$ almost surely.

Proof: Using dominated convergence, we can compute that for any fixed $t > 0$,

$$\begin{aligned} P\{Z_t = 0\} &= \mathbb{E}\left\{1_{\{Z_t=0\}}\right\} \\ &= \lim_{\lambda\to\infty} \mathbb{E}\{e^{-\lambda Z_t}\} \\ &= \lim_{\lambda\to\infty} \exp(-x u_t(\lambda)). \end{aligned} \tag{1.17}$$

Moreover, the continuity of P and the last formula yields

$$\begin{aligned} P\{Z_t = 0 \text{ some } t > 0\} &= P\left\{\bigcup_{t\geq 0}\{Z_t = 0\}\right\} \\ &= \lim_{t\to\infty} P\{Z_t = 0\} \\ &= \lim_{t\to\infty}\lim_{\lambda\to\infty} \exp(-xu_t(\lambda)). \end{aligned} \tag{1.18}$$

Recall that $u_t(\lambda)$ is the unique non negative solution of

$$\frac{\partial}{\partial t}u_t(\lambda) = -\psi(u_t(\lambda))$$

for $t > 0$ and $u_0(\lambda) = \lambda$, hence,

$$-\frac{1}{\psi(u_t(\lambda))}\frac{\partial}{\partial t}u_t(\lambda) = 1$$

and we get by substitution that

$$\int_{u_t(\lambda)}^{\lambda} \frac{1}{\psi(s)}\, ds = -\int_0^t \frac{1}{\psi(u_s(\lambda))}\frac{\partial}{\partial s}u_s(\lambda)\, ds = t. \tag{1.19}$$

We now show that the right hand side in formula (1.18) is equal to 1 if and only if

$$\int_1^\infty \frac{1}{\psi(s)}\, ds < \infty. \tag{1.20}$$

As the function ψ has its only zero at 0, (1.20) holds if and only if, for all $\varepsilon > 0$,

$$\int_\varepsilon^\infty \frac{1}{\psi(s)}\, ds < \infty. \tag{1.21}$$

If $\lim_{t\to\infty} u_t(\infty) = 0$, then (1.19) implies that (1.21) holds for all $\varepsilon > 0$. Moreover, under the assumption that (1.21) holds, (1.19) implies that $\lim_{t\to\infty} u_t(\infty) < \varepsilon$ for all $\varepsilon > 0$ and we infer that $\lim_{t\to\infty} u_t(\infty) = 0$.
Furthermore, if (1.21) fails, then the right hand side in formula (1.18) is 0 as $u_t(\infty) = \infty$ for any $t \geq 0$. □

In particular, this condition is true in the case of the Feller diffusion and in the stable case, i.e. where the branching mechanisms are of the form $\psi(\lambda) = \lambda^{1+\beta}$ for $\beta \in (0, 1]$.

Recall that any critical or subcritical Galton-Watson process dies in finite time. Hence, the condition for the extinction of a CSBP, given by Theorem 1.29, is *different* from the discrete case, i.e. there are critical (or subcritical) CSBP which do *not* die. Moreover, another difference to the case of Galton-Watson processes is the following 01-law, which is directly inspired by Theorem 1.29.

Corollary* 1.30 *(01-law for the extinction time)
Let T be the extinction time of a CSBP, then $P\{T < \infty\} \in \{0,1\}$.

To finish this chapter, notice that the functions ψ in Theorem 1.26 which determine a CSBP in a unique way, look very similar to Laplace exponents of Lévy processes. In fact, we can find a subclass $\mathcal{L}$ of Lévy processes, such that there is a one-to-one correspondence between $\mathcal{L}$ and $\mathcal{C}$ just via the Laplace exponents respectively the branching mechanisms ψ. Our aim for the rest of this chapter is to define this subclass $\mathcal{L}$.

***Definition* 1.31** *Denote by $\mathcal{L}$ the class of all Lévy processes $\{X_t : t \geq 0\}$ defined on $(\Omega, \mathcal{F}, P)$ with start in $X_0 = 0$ with the following properties:*

(L1) X is spectrally positive,

(L2) X does not tend to $+\infty$,

(L3) the paths of X are of infinite variation almost surely.

As we already pointed out in Section 1.2, due to (L1), the Laplace transform of X is well defined and we have for $\lambda \geq 0, t \geq 0$,

$$\mathbb{E}\{\exp -\lambda X_t\} = \exp(-t\psi(\lambda)), \tag{1.22}$$

with a function ψ of the form

$$\psi(\lambda) = \alpha_0\lambda + \beta\lambda^2 + \int_{(0,\infty)} (e^{-\lambda r} - 1 + 1_{\{r<1\}}\lambda r)\, \pi(dr), \tag{1.23}$$

where $\alpha_0 \in \mathbb{R}, \beta \geq 0$ and π is a σ-finite measure on $(0,\infty)$, such that

$$\int_{(0,\infty)} (1 \wedge r^2)\, \pi(dr) < \infty.$$

To go ahead, we need the following equivalence for the fact that a Lévy process does *not* tend to ∞:

***Lemma* 1.32** *A Lévy process $\{X_t : t \geq 0\}$ does not tend to ∞ almost surely if and only if X_t has first moments and $\mathbb{E}\{X_t\} \leq 0$ for all $t \geq 0$.*

For a proof of this lemma see e.g. [Ber96]. Of course, we could replace (L2) by the condition in the lemma, but we think that this would be less intuitive to start with. Assume now, that (L2) holds. Then, due to the Lemma,

$$\begin{aligned}
\mathbb{E}\{X_t\} &= \frac{\partial}{\partial\lambda}\mathbb{E}\{e^{-\lambda X_t}\}\Big|_{\lambda=0} \\
&= \frac{\partial}{\partial\lambda}\exp(-t\psi(\lambda))\Big|_{\lambda=0} \\
&= -t\psi'(0) \leq 0. \qquad (1.24)
\end{aligned}$$

Therefore, we get

$$\psi'(0) = \alpha_0 - \int_{[1,\infty)} r\,\pi(dr) \geq 0,$$

and so $\int_{[1,\infty)} r\,\pi(dr)$ must be finite. Hence, with $\alpha := \alpha_0 - \int_{[1,\infty)} r\,\pi(dr) \geq 0$, we can rewrite $\psi : [0,\infty) \to [0,\infty)$ as

$$\psi(\lambda) = \alpha\lambda + \beta\lambda^2 + \int_{(0,\infty)} (e^{-\lambda r} - 1 + \lambda r)\,\pi(dr), \tag{1.25}$$

where π satisfies the stronger integrability condition

$$\int (r \wedge r^2)\,\pi(dr) < \infty.$$

Thus, we have established the promised *bijection*

$$\Psi : \mathcal{C} \to \mathcal{L}$$

between (sub)critical CSBP and certain subclass of Lévy processes via the branching mechanisms respectively the Laplace exponents ψ. The aim of Chapter 2 is to give an explicit pathwise construction of an element $Z \in \mathcal{C}$ from a certain functional of the corresponding Lévy process $\Psi(Z)$.

Chapter 2

Construction of the genealogy

In order to understand the coding of the genealogical structure of a CSBP, we first consider the discrete time, discrete state space case of Galton-Watson trees and branching processes. Later we will try to link both concepts considering the limit behaviour of the discrete setting.

2.1 Galton-Watson trees and random walks

Let $\mathbb{N} = \{1, 2, \dots\}$, μ be a probability measure on $\mathbb{N}_0 = \{0, 1, 2, \dots\}$ and

$$\mathcal{U} = \bigcup_{n=0}^{\infty} \mathbb{N}^n$$

where $\mathbb{N}^0 := \{\delta\}$. Each $u \in \mathbb{N}^n$ can be written as $u = u_1 \dots u_n$ with $u_i \in \mathbb{N}$ for all $i = 1, \dots, n$ and $|u| = n$. Formally let $|\delta| = 0$. Denote by uv the concatenation of u and v and introduce the canonical lexicographical order $<_l$ on $\mathcal{U}$.

***Definition* 2.1** *A **tree** τ is a subset of $\mathcal{U}$ such that*

(i) $\delta \in \tau$

(ii) if $v \in \tau$ and $v = uj$ for some $u \in \mathcal{U}$ and $j \in \mathbb{N}$ then $u \in \tau$

(iii) for all $u = u_1 \dots u_k \in \tau$ also $u_1 \dots u_{k-1}s \in \tau$ for all $1 \leq s \leq u_n$.

Denote by $\mathcal{T}$ the set of trees.

These requirements are intuitively clear if we think of $\mathcal{U}$ as an ***address space***. It is natural that we can *not* take every subset of $\mathcal{U}$ to get a tree-structure. Assumption (i) just means that every tree must contain a root δ. (ii) means that every element of the tree (except the root) contains a mother in the tree and the third requirement is that existent older brothers of an element should also be contained in the tree.

***Definition* 2.2** *Let $\{C(u) : u \in \mathcal{U}\}$ be a family of independent μ-distributed random variables and define*

$$\tau := \{u_1 \dots u_n \in \mathcal{U} : 1 \le u_k \le C(u_1 \dots u_{k-1}) \text{ for all } 1 \le k \le n\}.$$

*Then by construction τ is a random element of $\mathcal{T}$ and we call τ a μ-**Galton-Watson tree**.*

This construction induces a probability measure P^μ on the set of trees $\mathcal{T}$ (see e.g. [Ne86]). Intuitively, Galton-Watson trees can be seen as *family trees* of Galton-Watson process, which can be easily constructed from a given tree by *summing up* all particles in each generation

$$G_n := \#\{u \in \tau : |u| = n\},$$

Nevertheless, the genealogical information which was contained in the tree is lost if we consider just Galton-Watson processes. In other words, the Galton-Watson trees contain *more* information than the Galton-Watson processes alone, namely the *genealogy.*

In this thesis we will only consider ***critical*** or ***subcritical*** probability measures μ, i.e.

$$\sum_{k=0}^{\infty} k\mu(k) \le 1,$$

and we always exclude the boring case $\mu = \delta_1$. Moreover, if μ is (sub)critical we also call a μ-Galton-Watson tree or process (sub)critical. We start with the well known result, that in this case the corresponding trees are *finite* and the μ-Galton-Watson processes *die out* almost surely.

***Lemma* 2.3** *Suppose that τ is a (sub)critical μ-Galton-Watson tree, then τ is finite almost surely and the corresponding μ-Galton-Watson process $\{G_n : n \in \mathbb{N}_0\}$ dies out in finite time, i.e.*

$$P\{G_n = 0 \text{ for finally all } n\} = 1.$$

A proof of this lemma, which uses generating functions and standard analysis methods, can be found in almost any book on branching processes or probability.

Assume now that τ is a (sub)critical μ-Galton-Watson tree and let σ be the number of particles in the tree τ.

***Definition* 2.4** *Let $u(0) = \delta <_l u(1) <_l \dots <_l u(\sigma - 1)$ the addresses of all nodes in τ listed in lexicographical order. Define the process $\{H_n(\tau) : n \in \mathbb{N}_0\}$ by*

$$H_n(\tau) := \begin{cases} |u(n)| & \text{if } 0 \le n \le \sigma - 1 \\ \Delta & \text{otherwise} \end{cases},$$

*where Δ is an absorbing cementary point. We will the process $\{H_n(\tau) : n \in \mathbb{N}_0\}$ the **discrete height process** associated to a Galton-Watson tree τ.*

Pictorially, $H_n(\tau)$ describes the generation of the n-th vertex if we run through the tree in lexicographical order. It is clear that the discrete height process codes the genealogy of a Galton-Watson tree. Unfortunately, the height process is *not* a Markov process. However, we can describe H as a functional of a random walk:

***Theorem* 2.5** *Let $\{H_n(\tau) : n \in \mathbb{N}_0\}$ be the discrete height process associated to a (sub)critical μ-Galton-Watson tree. Then there exists a random walk $\{W_n : n \in \mathbb{N}_0\}$ with values in $\mathbb{N}_0 \cup \{-1\}$ and increment distribution $\nu(k) = \mu(k+1)$ for $k = -1, 0, 1, 2, \dots$ such that for every $n \in \mathbb{N}$,*

$$H_n = \# \left\{ j \in \{0, 1, \dots, n-1\} : W_j = \inf_{j \le k \le n} W_k \right\} \tag{2.1}$$

Proof: Let τ be a μ-Galton-Watson tree and let $u = u_1 \dots u_p \in \mathcal{U}$ be a vertex in τ. Then for every $k \le p$ the ancestor of u in the k-th generation is denoted by

$$[u]_k := u_1 \dots u_k.$$

For all $n \in \{0, 1, \dots, \sigma - 1\}$ and $j \in \{1, \dots, H_n(\tau)\}$ define

$$B_{n,j}(\tau) := \#\{v \in \tau : |v| = j, [u(n)]_{j-1} = [v]_{j-1}, u(n) <_l v\},$$

which pictorially describes the number of *younger brothers* of the ancestor of $u(n)$ in the j-th generation. Now, we introduce the process $\{\rho_n : n \in \mathbb{N}_0\}$ by setting

$$\rho_n := \begin{cases} (B_{n,1}, \dots, B_{n,H_n}) & \text{if } n < \sigma \\ \Delta & \text{otherwise.} \end{cases}$$

Then one can easily check, that $\{\rho_n : n \in \mathbb{N}\}$ is a Markov chain with values in $\mathcal{U} \cup \{\Delta\}$ whose transition probabilities Q for $b = (b_1 \dots b_n) \in \mathcal{U}$ and $k \in \mathbb{N}$ are given by

$$\begin{aligned} Q(b, (b,k)) &= \mu(k+1) \text{ and} \\ Q(b, \tilde{b}) &= \mu(0), \end{aligned}$$

where $\tilde{b} = (b_1 \dots b_{m-1} b_m - 1)$ if $m = \sup\{j : b_j > 0\}$ and $\tilde{b} = \Delta$ if $b_1 = \dots = b_n = 0$. Clearly, $Q(\Delta, \Delta) = 1$. To make the statement clear, think of $u(n)$ as the particle in the tree *visited* at time n. Then for every $l \in \mathbb{N}$, the probability that$u(n)$ has $l+1$ succestors is $\mu(l+1)$. In this case, $u(n+1) = u(n)1$ and $\rho_{n+1} = (\rho_n, l)$ by construction. Moreover, with probability $\mu(0)$ the particle $u(n)$ has nos child. Hence, $u(n+1)$ is the *first* younger brother of $u(n)$, respectively the first younger brother of the ancestor of$u(n)$ if $u(n)$ has no younger brother and so on.
With the help of this Markov chain, it is easy to define the process $\{W_n : n \in \mathbb{N}_0\}$ by

$$W_n := \sum_{j=1}^{H_n} B_{n,j} \text{ for all } n < \sigma$$

and *killed* at time σ. Then it is clear by construction that W_n is a random walk with jump distribution ν, but we still have to show equation (2.1). To this end, remark that $W_j = \inf\{W_k : j \leq k \leq n\}$ if and only if $n < \inf\{l > j : W_l < W_j\}$ which is equal to the first time of visit of an individual that is not a descendant of $u(j)$. Hence,

$$W_j = \inf_{j \leq k \leq n} W_k$$

if and only if $u(j)$ is an ascendent of $u(n)$, and therefore

$$\begin{aligned} \#\left\{j \in \{0, 1, \ldots, n-1\} : W_j = \inf_{j \leq k \leq n} W_k\right\} &= \#\{\text{ancestor of } u(n)\} \\ &= \text{generation of } u(n) = H_n, \end{aligned}$$

which finishes the proof. $\square$

A few remarks are in order here. At first, the whole procedure of the discrete coding may be enlighten by some pictures. As we know how the construction works for *one*

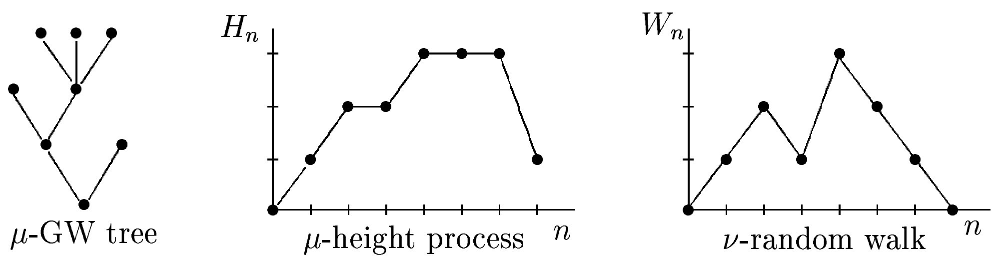


Figure 2.1: The coding of a GW tree

Galton-Watson tree (see Figure 2.1), one may wonder how to code a *sequence* (τ_n) of Galton-Watson trees. For example, consider the trees τ_1, τ_2, τ_3. We can code such a sequence of trees, just by *glueing together* the corresponding height processes (see Figure 2.2). Then each *excursion* of the height process corresponds to a tree. To get the corresponding random walk, it is *not* possible to get a unique representation by gluing together the random walks for each tree, as these random walks may take the value 0 several times while coding *one* tree. Therefore, we have to code every tree alone, then we shift the path of the k-th tree by $-(k-1)$ and then we obtain W by glueing together these shifted parts (see Figure 2.3).

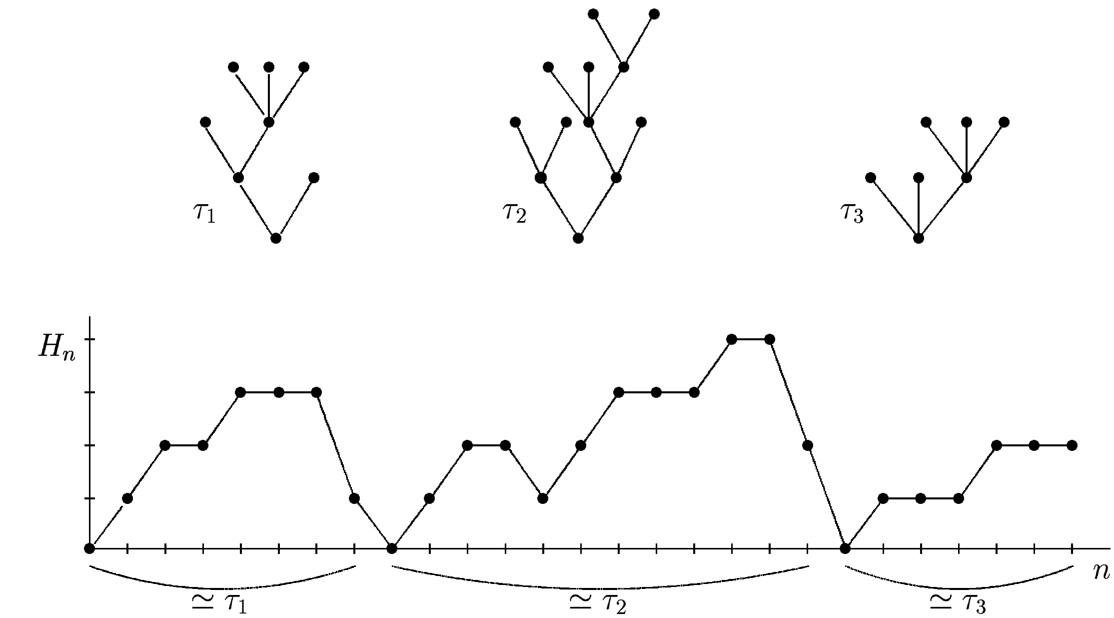


Figure 2.2: A sequence of μ-GW trees and the associated height process

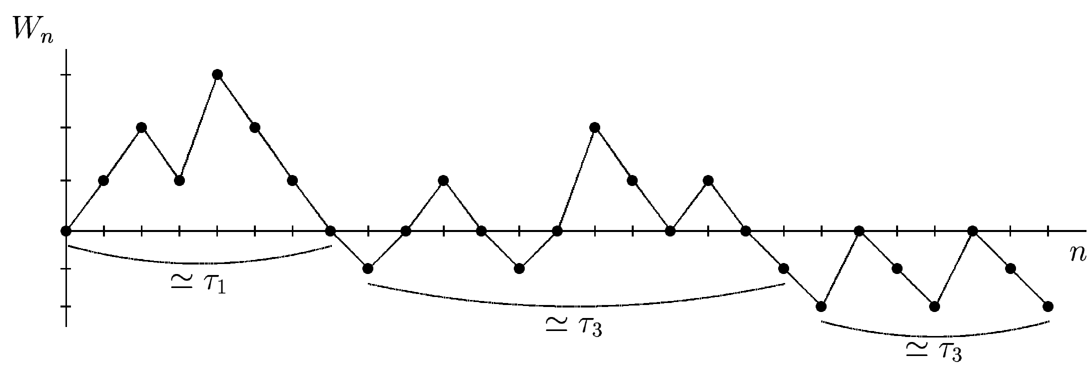


Figure 2.3: The random walk associated to τ_1, τ_2, τ_3

Moreover, we can easily obtain a μ-Galton-Watson process $\{G_n : n \in \mathbb{N}_0\}$ from a given height process just by counting the level sets of the discrete height process

$$G_n := \#\{s \in \mathbb{N} : H_s = n\}. \tag{2.2}$$

In the continuous time setting, we get a similar formula to count the particles in a given level-set. Nevertheless, as one could imagine, we have to deal with a lot of technical difficulties. In particular, Lebesgue measure does *not* work as a substitute for the counting in the above formula and we have to introduce *local times* of the continuous analogue of the height process to measure the size of each generation.

2.2 The coding of CSBP

Recall from Chapter 1 that there is a very natural bijection from the class of (sub)critial CSBP $\mathcal{C}$ and the class of spectrally positive Lévy processes $\mathcal{L}$ which have infinite variation and do *not* tend to ∞. This bijection can be realised via the branching mechanism respectively the Laplace exponents which are of the form

$$\psi(\lambda) = \alpha\lambda + \beta\lambda^2 + \int_{(0,\infty)} (e^{-\lambda r} - 1 + \lambda r)\pi(dr), \tag{2.3}$$

where $\alpha, \beta \geq 0$ and the Lévy measure π satisfies $\int (r \wedge r^2)\pi(dr) < \infty$.

The major aim in this chapter is to give an explicit *probabilistic* construction of a ψ-CSBP from a given ψ-Lévy process for any ψ of the form (2.3). Compared with the discrete setting, the Lévy process corresponds to the random walk W, which is *spectrally positive* in the sense that W makes only negative jumps of size 1. To get a formula analogous to (2.2) in the continuous setting, we have to introduce the continuous time analogue of the discrete height process. Clearly, it is *not* possible to define a continuous height process in the same way as in the discrete case in terms of a walk through a tree, because we do *not* know whatever a continuous tree should be (at least at the moment).

Nevertheless, Theorem 2.5 gives the major idea how to define a continuous height process. To get a formula, similar to (2.1), we first replace the random walk W by a Lévy process $X \in \mathcal{L}$. So, let $\{X_t : t \geq 0\}$ be a Lévy process in the class $\mathcal{L}$, which can be defined on the canonical stochastic basis $(\Omega, \mathcal{F}, P, \mathcal{F}_t)$ as described in Chapter 1.

***Definition* 2.6** *For every $t > 0$, define the* **time reversed process** $\{\hat{X}_s^{(t)} : s \in [0,t]\}$

$$\hat{X}_s^{(t)} := X_t - X_{(t-s)^-},$$

with the convention that $X_{0^-} := 0$, and its **supremum process** $\{\hat{S}_s^{(t)} : s \in [0,t]\}$ *by*

$$\hat{S}_s^{(t)} := \sup\{\hat{X}_r^{(t)} : r \leq s\}.$$

Although the definition of the time reversed process looks very strange at the first glance, we should keep the following picture in mind which helps us to understand the procedure:

Lemma* 2.7 *(Duality)
Let $\{X_t : t \geq 0\}$ be a Lévy process defined on $(\Omega, \mathcal{F}, P)$ and fix a $t > 0$. Then $\{X_s : 0 \leq s \leq t\}$ and $\{\hat{X}_s^{(t)} : 0 \leq s \leq t\}$ have the same distribution under P.

Proof: Let $t > 0$. It is clear that the one dimensional distributions agree, because for all $s \in [0,t]$ we have that

$$P_{X_t - X_{(t-s)^-}} = P_{X_s}.$$

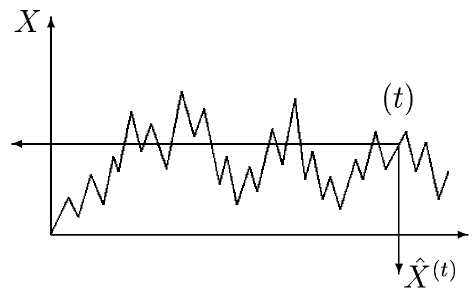


Figure 2.4: Construction of the time reversed process $\hat{X}$

Moreover, $\hat{X}^{(t)}$ is a process started at 0 with independent, homogeneous increments and almost surely càdlàg paths on $[0, t]$. Hence, $\{X_s : 0 \leq s \leq t\}$ and $\{\hat{X}_s^{(t)} : 0 \leq s \leq t\}$ must have the same distribution. □

As consequence of Lemma 1.21 the process $\{\hat{X}_s^{(t)} - \hat{S}_s^{(t)} : 0 \leq s \leq t\}$ is a strong Markov process with regular point 0, for every fixed $t \geq 0$. Hence, due to Section 1.3, the local time at 0 exists and we can pass to the following definition:

***Definition* 2.8** *Set $H_0 := 0$ and for every $t > 0$ let $H_t := \hat{L}_t^{(t)}$, where $\{\hat{L}_s^{(t)} : s \geq 0\}$ denotes the local time at 0 the the reflected, time reversed Lévy process*

$$\{\hat{X}_s^{(t)} - \hat{S}_s^{(t)} : s \leq t\}.$$

*We call the process $\{H_t : t \geq 0\}$ the ψ-**height process** associated with the underlying ψ-Lévy Process $\{X_t : t \geq 0\}$.*

Note that this definition is really motivated by the discrete formula (2.1). Let us elaborate a little more on this. The fact that the continuous formula looks different from the discrete one (at least at the first moment) is due to the following technical difficulties:

Suppose that the Lévy process X plays the role of the random walk W. It is clear, that we *cannot* replace the counting measure of the discrete setting by the Lebesgue measure m, i.e.

$$m\left\{s \leq t : X_s = \inf_{s \leq \eta \leq t} X_\eta\right\}$$

because this measure is to crude and would always lead to the value 0. Therefore, we use the *local time* to measure the size those sets. Nevertheless, the process

$$\left\{X_s - \inf_{s \leq \eta \leq t} X_\eta : 0 \leq s \leq t\right\}$$

is *not* a Markov process and we can *not* construct local times. To get out of this dilemma, we do *time reversal*. Note that

$$\left\{s \leq t : X_s = \inf_{s \leq \eta \leq t} X_\eta\right\} = \left\{s \leq t : \hat{X}_s^{(t)} = \hat{S}_s^{(t)}\right\},$$

and so our definition of the continuous height process is the *natural* continuous analogue to the discrete time formula (2.1).

To simplify notation, let us write

$$I_t^s := \inf_{s \le \eta \le t} X_\eta,$$

and note that $I_t = I_t^0$. For the local time, we use the normalization provided by Theorem 1.22, hence we may choose a sequence $(\varepsilon_k)_{k\ge 1}$ of positive real numbers such that for all fixed $t > 0$, almost surely

$$H_t = \lim_{k\to\infty} \frac{1}{\varepsilon_k} \int_0^t 1_{\{\hat{S}_s^{(t)} - \hat{X}_s^{(t)} < \varepsilon_k\}}\, ds. \tag{2.4}$$

Hence, we see in particular, that H_t is $\mathcal{F}_t$-measurable for all t and that we can choose a modification of the height process with values in $[0, \infty]$ by setting

$$H_t = \liminf_{k\to\infty} \frac{1}{\varepsilon_k} \int_0^t 1_{\{\hat{S}_s^{(t)} - \hat{X}_s^{(t)} < \varepsilon_k\}}\, ds. \tag{2.5}$$

Although this approximation of the height process is very valuable, we may wonder if one could do an approximation of H_t which is directly motivated by the discrete formula (2.1). Heuristically, by *counting* the time instants $r < t$, for which $X_r = I_t^r$. Let us fix $t > 0$. Using substitution, (2.4) implies that,

$$H_t = \lim_{k\to\infty} \frac{1}{\varepsilon_k} \int_0^t 1_{\{\hat{S}_{t-r}^{(t)} - \hat{X}_{t-r}^{(t)} < \varepsilon_k\}}\, dr.$$

As $\hat{S}_{t-r}^{(t)} = X_t - I_t^r$ and $\hat{X}_{t-r}^{(t)} = X_t - X_r$ for all $r \le t$ we get

$$H_t = \lim_{k\to\infty} \frac{1}{\varepsilon_k} \int_0^t 1_{\{X_r < I_t^r + \varepsilon_k\}}\, dr, \tag{2.6}$$

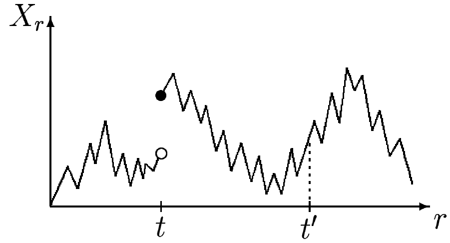
almost surely for all fixed $t > 0$. Recall from Chapter 1, that we denote by $\{L_t : t \ge 0\}$ the local time at 0 of the reflected Lévy process $S - X$. As $t \mapsto L_t$ is *monotone*, we can choose a suitable subsequence of (ε_k) (also denoted by (ε_k)) such that the convergence

$$L_t = \lim_{k\to\infty} \frac{1}{\varepsilon_k} \int_0^t 1_{\{S_r - X_r < \varepsilon_k\}}\, dr \tag{2.7}$$

holds almost surely for all $t > 0$. In the case of the height process, $t \mapsto H_t$ is *not* monotone. Hence, it is *not* possible to get neither the approximation (2.4) nor (2.6) for all t simultaneously for a single zero set in Ω. Nevertheless, we get the stronger statement if we restrict ourselves to special time points.

***Definition* 2.9** *A time instant $t \ge 0$ is called a* **low point** *of the path $r \mapsto X_r$ if either $\Delta X_t > 0$ or there is an $s > t$ such that $X_{t-} \le I_s^t$.*

The definition may be illustrated by the following picture which shows two examples t, t' of low points:



The next lemma shows that the disered approximation holds almost surely for all low points.

***Lemma* 2.10** *There exists a sequence (ε_k) decreasing to 0 such that almost surely for all low points $t \geq 0$,*

$$H_t = \lim_{k\to\infty} \frac{1}{\varepsilon_k} \int_0^t 1_{\{X_r < I_t^r + \varepsilon_k\}} dr < \infty.$$

Proof: Assume that $t \geq 0$ is a low point, then either $\Delta X_t > 0$ or there is an $s > t$ such that $X_{t-} \leq I_s^t$. In both cases, there is a rational $u > t$ such that $X_{t-} \leq I_u^t$. Recall that we denote by $\hat{L}_r^{(t)}$ the local time at 0 of $\hat{X}^{(t)} - \hat{S}^{(t)}$ at time r, so in particular $H_t = \hat{L}_t^{(t)}$. Using (2.4) there is a sequence (ε_k) decreasing to 0 such that almost surely, for all low points $t > 0$,

$$\begin{aligned} H_t &= \hat{L}_t^{(t)} \\ &= \hat{L}_u^{(u)} - \hat{L}_{u-t}^{(u)} \\ &= \lim_{k\to\infty} \frac{1}{\varepsilon_k} \int_{u-t}^u 1_{\{\hat{S}_r^{(u)} - \hat{X}_r^{(u)} < \varepsilon_k\}} dr. \end{aligned}$$

As by our construction $\hat{S}_{u-r}^{(u)} - \hat{X}_{u-r}^{(u)} = X_r - I_t^r$ for all $r \in [0, t]$, we get using substitution

$$\begin{aligned} H_t &= \lim_{k\to\infty} \frac{1}{\varepsilon_k} \int_0^t 1_{\{\hat{S}_{u-r}^{(u)} - \hat{X}_{u-r}^{(u)} < \varepsilon_k\}} dr \\ &= \lim_{k\to\infty} \frac{1}{\varepsilon_k} \int_0^t 1_{\{X_r < I_t^r + \varepsilon_k\}} dr. \end{aligned}$$

□

Although these approximation results are quite nice, it is still *very* unclear from the definition, if the height process has *any* nice path properties. In a special case, it is nevertheless easy to get a continuous modification:

***Lemma* 2.11** *Let $\{H_t : t \geq 0\}$ be a ψ-height process with a function ψ of the type (2.3) and suppose $\beta > 0$. Then H has a continuous modification given by*

$$H_t = \frac{1}{\beta} m\{I_t^s : 0 \leq s \leq t\}, \tag{2.8}$$

where m denotes Lebesgue measure on the real line.

Proof: Let $t \geq 0$. By Corollary 1.24, we know that

$$H_t = \hat{L}_t^{(t)} = \frac{1}{\beta} m\{\hat{S}_r^{(t)} : r \leq t\}.$$

As $X_t - I_t^r = \hat{S}_{t-r}^{(t)}$ for all $r \leq t$, we see that

$$m\{\hat{S}_r^{(t)} : r \leq t\} = m\{I_t^r : r \leq t\}$$

almost surely, which completes the proof. □

Example: Let $\{X_t : t \geq 0\}$ be standard Brownian motion. Then we already computed the characteristic exponent to $\psi(\lambda) = \frac{1}{2}\lambda^2$. Hence $\beta = \frac{1}{2}$ and we get

$$\begin{aligned} H_t &= 2m\{I_t^r : r \leq t\} \\ &= 2(X_t - I_t) \end{aligned}$$

because of the continuity of X. Hence, $\{H_t : t \geq 0\}$ is distributed as scaled reflected Brownian motion by a theorem of Paul Lévy (see [ReY99] or [RW00b]).

In the case when the Laplace exponent of the underlying Lévy process has *no* Brownian part, i.e. $\beta = 0$, it is very unclear at the moment, if H has any nice properties. In a later section we will derive a necessary and sufficient condition for H having continuous sample paths also in that case.

In general, the height process is *not* a Markov process. Nevertheless, we see that we can *enlarge* the state space of the height process and define a measure valued process ρ, the so called exploration process, such that

(i) the exploration process *contains* the height process, and

(ii) posseses the Markov property.

As we see, this process is a valuable tool to gain path properties of the height process. In this thesis we use both notations $\int f\, d\mu$ and $\langle \mu, f\rangle$ to denote integration of f with respect to the measure μ.

Definition* 2.12** *The* ***exploration process $\{\rho_t : t \geq 0\}$ *is the process with values in the space* $\mathcal{M}_f(\mathbb{R}_+)$ *of finite measures on* $\mathbb{R}_+$*, equipped with the topology of weak convergence, defined by*

$$\langle \rho_t, f\rangle = \int_{[0,t]} f(H_s)\, d_s I_t^s \tag{2.9}$$

where f *is any nonnegative measurable function and* $d_s I_t^s$ *is the measure which is associated with the càdlàg increasing function* $s \mapsto I_t^s$*.*

A few remarks seem to be in order here. Let $0 \neq \mu \in \mathcal{M}_f(\mathbb{R}_+)$ then there is a biggest open set $G \subseteq \mathbb{R}$ such that $\mu(G) = 0$. We call $G^c =: \operatorname{supp} \mu$ the ***support*** of the measure μ. As the height process is always positve, we have for every $t > 0$ that $\operatorname{supp} \rho_t \subseteq [0, \infty)$ almost surely. Moreover, it is easy to compute the *total mass* process of ρ by

$$\begin{aligned} \|\rho_t\| = \langle \rho_t, 1 \rangle &= \int_{[0,t]} 1 \, d_s I_t^s \\ &= I_t^t - I_t^0 = X_t - I_t. \end{aligned} \tag{2.10}$$

In particular, $\rho_t = 0$ if and only if $X_t = I_t$ for all $t \geq 0$. Also note, that the exploration process is by definition adapted to the filtration $\{\mathcal{F}_t : t \geq 0\}$.

Sometimes, it is useful to rewrite the definition of ρ_t to,

$$\langle \rho_t, f \rangle = \int_{[0,t]} f\left(\hat{L}_t^{(t)} - \hat{L}_s^{(t)}\right) d_s \hat{S}_s^{(t)}, \tag{2.11}$$

where $\hat{L}^{(t)}$ denotes the local time of the time reversed process $\hat{S}^{(t)} - \hat{X}^{(t)}$. To see (2.11), fix $t > 0$ and note that $\hat{S}_{t-s}^{(t)} = X_t - I_t^s$ for all $s \leq t$. As $H_s = \hat{L}_s^{(s)} = \hat{L}_t^{(t)} - \hat{L}_{t-s}^{(t)}$, we get

$$\langle \rho_t, f \rangle = \int_{[0,t]} f\left(\hat{L}_t^{(t)} - \hat{L}_{t-s}^{(t)}\right) d_s \left(X_t - \hat{S}_{t-s}^{(t)}\right),$$

and (2.11) follows by substitution. Moreover, we can give the following example:

Example: In the case $\beta > 0$, we can obtain a more explicit formula for ρ_t. Note that

$$\Delta := \{s \leq t : X_{s-} < I_t^s\}$$

are exactly the jumptimes of the function $s \mapsto I_t^s$. Therefore we get using Lemma 2.11 and a change of variables,

$$\begin{aligned} \langle \rho_t, f \rangle &= \int_{[0,t]} f(H_s) \, d_s I_t^s \\ &= \int_{[0,t]} f\left(\beta^{-1} m\{I_s^r : r \leq s\}\right) d_s I_t^s \\ &= \int_{[0,t]} f\left(\beta^{-1} m\{I_t^r : r \leq s\}\right) d_s I_t^s \\ &= \beta \int_{[0,H_t]} f(a) \, da + \sum_{s \in \Delta} (I_t^s - X_{s-}) f(H_s). \end{aligned} \tag{2.12}$$

Hence, it follows that

$$\rho_t(da) = \beta 1_{[0,H_t]}(a) \, da + \sum_{s \in \Delta} (I_t^s - X_{s-}) \, \delta_{H_s}(da).$$

In particular, we see that in the case of a quadratic branching mechanism $\psi(\lambda) = \beta \lambda^2$, the exploration process ρ_t is just a multiple of the Lebesgue measure restricted to the random interval $[0, H_t]$.

***Definition* 2.13** *The variation distance of two finite measures $\mu, \nu \in \mathcal{M}_f(\mathbb{R}_+)$ is given by*

$$d_v(\mu, \nu) := \sup_{A \in \mathcal{B}(\mathbb{R}_+)} |\mu(A) - \nu(A)|.$$

We also want to mention that $\mathcal{M}_f(\mathbb{R}_+)$ becomes a Polish space with respect to this metric and that convergence with respect to the variation norm implies weak convergence. We are now able to observe the first nice property of the exploration process, namely that it has càdlàg paths with respect to the variation distance.

***Lemma* 2.14** *The exploration process $\{\rho_t : t \geq 0\}$ has càdlàg paths with respect to the variation distance of finite measures. Moreover, ρ has the same discontinuity times as the underlying Lévy process $\{X_t : t \geq 0\}$ and*

$$\rho_t = \rho_{t-} + (\Delta X_t)\, \delta_{H_t}, \tag{2.13}$$

where $\Delta X_t > 0$ denotes the height of the jump of X at time t.

Proof: We first show that $\lim_{t' \downarrow t} d_v(\rho_{t'}, \rho_t) = 0$. To this end, note that for $t' > t$ we have that $I^s_{t'} \leq I^s_t$ and therefore,

$$\begin{aligned}
\sup_{A \in \mathcal{B}(\mathbb{R}_+)} |\rho_{t'}(A) - \rho_t(A)| &= \sup_{A \in \mathcal{B}(\mathbb{R}_+)} \left| \int_{[0,t']} 1_A(H_s) d_s I^s_{t'} - \int_{[0,t]} 1_A(H_s) d_s I^s_t \right| \\
&\leq \sup_{A \in \mathcal{B}(\mathbb{R}_+)} \left| \int_{(t,t']} 1_A(H_s) d_s I^s_t \right| \\
&\leq |I^{t'}_t - I^t_t|,
\end{aligned}$$

which converges to 0 as $t' \downarrow t$. Similarly for $t' < t$,

$$d_v(\rho_{t'}, \rho_t - \Delta X_t \delta_{H_t}) \leq \sup_{A \in \mathcal{B}(\mathbb{R}_+)} \left| \Delta X_t \delta_{H_t}(A) - \int_{(t',t]} 1_A(H_s) d_s I^s_{t'} \right|,$$

which converges to zero as $t' \uparrow t$ as

$$\lim_{t' \uparrow t} \int_{(t',t]} d_s I^s_{t'} = \int_{(t^-,t]} d_s I^s_{t'} = \Delta X_t.$$

This yields the existence of the left limits and the explicit form of the jumps. □

Our next aim is to show, how the height process is *contained* in the exploration process. To this end we have to introcude some notation. Define a function $H : \mathcal{M}_f(\mathbb{R}_+) \to \mathbb{R}$, by

$$H(\mu) := \sup\{\operatorname{supp} \mu\}.$$

By convention let $H(0) := 0$ for $0 \in \mathcal{M}_f(\mathbb{R}_+)$. The following lemma enlights the relationship between the height and the exploration process.

***Lemma* 2.15** *The process $\{H(\rho_t) : t \geq 0\}$ is a modification of the height process and we have almost surely for every $t > 0$,*

(i) $\rho_t\{0\} = 0$,

(ii) *supp* $\rho_t = [0, H(\rho_t)]$ *if* $\rho_t \neq 0$.

Moreover, we have almost surely for all low points $t > 0$ that $H(\rho_t) = H_t$.

Proof: Recall that $\hat{L}^{(t)} = \{\hat{L}_s^{(t)} : 0 \leq s \leq t\}$ denotes the local time at 0 of $\hat{S}^{(t)} - \hat{X}^{(t)}$. In particular we have by definition $H_t = \hat{L}_t^{(t)}$. By (2.11) we can rewrite the definition of the exploration process as

$$\langle \rho_t, f\rangle = \int_{[0,t]} f\left(\hat{L}_t^{(t)} - \hat{L}_s^{(t)}\right) d_s\hat{S}_s^{(t)}. \tag{2.14}$$

Because the monotone increasing functions $s \mapsto \hat{L}_s^{(t)}$ and $s \mapsto \hat{S}_s^{(t)}$ have the same points of increase, the associated random measures $d_s\hat{L}_s^{(t)}$ and $d_s\hat{S}_s^{(t)}$ have the same support almost surely. Hence, we get for every fixed $t \geq 0$ that

$$\begin{aligned} \operatorname{supp} \rho_t &= \operatorname{supp} d_s\hat{S}_s^{(t)} \\ &= \operatorname{supp} d_s\hat{L}_s^{(t)} \\ &= [0, H_t] \end{aligned} \tag{2.15}$$

because $s \mapsto \hat{L}_s^{(t)}$ is continuous and increasing with $\hat{L}_t^{(t)} = H_t$. In particular, we have for every $t \geq 0$, almost surely

$$H(\rho_t) = \sup\{\operatorname{supp} \rho_t\} = H_t, \tag{2.16}$$

hence $\{H(\rho_t) : t \geq 0\}$ is a modification of the height process $\{H_t : t \geq 0\}$. We still have to prove the properties (i)-(iii). Let us first show (i). By formula (2.14) we get for any fixed $t > 0$,

$$\rho_t(\{0\}) = \int_{[0,t]} 1_{\{0\}}\left(\hat{L}_t^{(t)} - \hat{L}_s^{(t)}\right) d\hat{S}_s^{(t)} = 0,$$

almost surely. Hence we get the statement almost surely for all rationals. Assume that

$$P\{\rho_t(\{0\}) > 0 \text{ for some } t > 0\} > 0.$$

then by the right continuity of the paths of ρ (Theorem 2.14), we would get the contradiction

$$P\{\rho_q(\{0\}) > 0 \text{ for some rational } q > t\} > 0,$$

hence (i) holds.

To prove (ii) requires more work. We already saw that (ii) holds for any fixed $t > 0$. Hence, (ii) must hold almost surely for all rationals. We now argue on this set of full measure. Let $t > 0$ with $\rho_t \neq 0$, i.e. $X_t > I_t$ and set

$$\gamma_t := \sup\{s < t : I_t^s < X_t\}. \tag{2.17}$$

We then have that $X_{\gamma_t-} \leq X_t$, and we treat both cases seperately. Assume first that $X_{\gamma_t-} < X_t$ (this is in particular true when X has a jump at time t). Hence, there exists a rational $q > t$ sufficiently close to t such that $X_{\gamma_t-} < X_q$ and either $X_t < X_s$ for all $s \in (t, q]$ or $X_q - I_q^s \leq 0$ for all $s \in (t, q]$. As in the first case we also have that $H_t < H_s$ for all $s \in (t, q]$, we get that in both cases ρ_q and ρ_t have the same restriction to $[0, H_t)$. Hence we know, that

$$\operatorname{supp} \rho_q = [0, H(\rho_q)] \quad \text{and} \quad \rho_{q\,|[0,H_t)} = \rho_{t\,|[0,H_t)}$$

for all rationals sufficiently close to t. Therefore, we also have $\operatorname{supp} \rho_t = [0, H(\rho_t)]$ and we see in particular that $H(\rho_t) = H_t$ in that case.

Now, assume that $X_{\gamma_t-} = X_t$. Then define for all $\varepsilon > 0$,

$$\langle \rho_t^\varepsilon, f\rangle = \int_{[0,t]} 1_{\{I_t^s < X_t - \varepsilon\}} f(H_s)\, d_s I_t^s. \tag{2.18}$$

Then ρ_t^ε converges to ρ_t in variation norm, as ε tends to 0. Moreover, for every $\varepsilon > 0$ there exists a rational $q > t$ such that $I_q^t > X_t - \varepsilon$. Hence,

$$\begin{aligned}
\langle \rho_q^{\varepsilon + X_q - X_t}, f\rangle &= \int_{[0,q]} 1_{\{I_q^s < X_t - \varepsilon\}} f(H_s)\, d_s I_q^s \\
&= \int_{[0,t]} 1_{\{I_t^s < X_t - \varepsilon\}} f(H_s)\, d_s I_t^s \;=\; \rho_t^\varepsilon.
\end{aligned}$$

As we know, that (ii) holds for all rationals, we have

$$\rho_t^\varepsilon = \rho_q^{\varepsilon + X_q - X_t} = [0, a],$$

for some $a > 0$. And as ρ_t^ε tends to ρ_t in variation norm, we get the desired result.

To complete the proof, we have to show that almost surely for all low points $t > 0$ we have $H(\rho_t) = H_t$. By the first part of the proof we know this holds for all rationals outside a single set of zero measure. For any $t > 0$, we have also seen that that $H(\rho_t) = H_t$ in the case $\Delta X_t > 0$. Now assume that there is an $s > t$ such that $X_{t-} \leq I_s^t$. As in the proof of (ii) we can again find rationals $q \in (t, s)$ such that ρ_q and ρ_t have the same restriction to $[0, H_t)$ and we get the assertion using (ii). $\square$

Lemma 2.16 *Let T be a finite $\mathcal{F}_t$-stopping time and fix $t > 0$. Then almost surely for all $r \in [0, t]$*

$$H_{T+r} = H_u + H_r^{(T)}, \tag{2.19}$$

where $u := \sup\{s \in (0, T] : X_{s-} < I_{T+t}^T\}$ with the usual convention that $\sup \emptyset = 0$ and $\{H_t^{(T)} : t \geq 0\}$ denotes the height process associated with the shifted Lévy process $X^{(T)} := \{X_{T+t} - X_T : t \geq 0\}$.

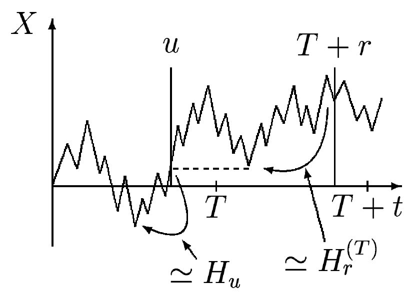


Figure 2.5: What's going on in Lemma 2.16

Proof: As u is a low point by definition, we get by Lemma 2.10 that there exists a sequence $(\varepsilon_k) \downarrow 0$ with

$$H_u = \lim_{k\to\infty} \frac{1}{\varepsilon_k} \int_0^u 1_{\{X_s < I_u^s + \varepsilon_k\}}\, ds. \tag{2.20}$$

Let $r \in [0, t]$ and note that $I_u^s = I_{T+r}^s$ for all $s \in (0, u)$. Hence

$$\begin{aligned} H_u &= \lim_{k\to\infty} \frac{1}{\varepsilon_k} \int_0^u 1_{\{X_s < I_{T+r}^s + \varepsilon_k\}}\, ds \\ &= \lim_{k\to\infty} \frac{1}{\varepsilon_k} \left(\int_0^{T+r} 1_{\{X_s < I_{T+r}^s + \varepsilon_k\}}\, ds - \int_u^T 1_{\{X_s < I_{T+r}^s + \varepsilon_k\}}\, ds - \int_T^{T+r} 1_{\{X_s < I_{T+r}^s + \varepsilon_k\}}\, ds \right). \end{aligned}$$

As $X_s \geq I_{T+r}^s$ for all $s \in [u, T]$, the second integral is equal to 0. To handle the third integral, observe that $I_{T+r}^{T+s} = I_r^{(T),s} + X_T$, for all $s \in [0, r]$ where $I_r^{(T),s} = \inf_{s\leq\eta\leq r} X_\eta^{(T)}$. Using substitution leads to,

$$\begin{aligned} \int_T^{T+r} 1_{\{X_s < I_{T+r}^s + \varepsilon_k\}}\, ds &= \int_0^r 1_{\{X_{T+s} < I_{T+r}^{T+s} + \varepsilon_k\}}\, ds \\ &= \int_0^r 1_{\{X_{T+s} - X_T < I_r^{(T),s} + \varepsilon_k\}}\, ds \\ &= \int_0^r 1_{\{X_s^{(T)} < I_r^{(T),s} + \varepsilon_k\}}\, ds. \end{aligned} \tag{2.21}$$

Hence, plugging (2.21) in the above approximation yields,

$$H_u = \lim_{k\to\infty} \frac{1}{\varepsilon_k} \left(\int_0^{T+r} 1_{\{X_s < I^s_{T+r} + \varepsilon_k\}}\, ds - \int_0^r 1_{\{X^{(T)}_s < I^{(T),s}_r + \varepsilon_k\}}\, ds \right) \tag{2.22}$$

As

$$H_u + \liminf_{k\to\infty} \frac{1}{\varepsilon_k} \left(\int_0^r 1_{\{X^{(T)}_s < I^{(T),s}_r + \varepsilon_k\}}\, ds \right) = \liminf_{k\to\infty} \left(\int_0^{T+r} 1_{\{X_s < I^s_{T+r} + \varepsilon_k\}}\, ds \right),$$

we get $H_u = H_{T+r} - H^{(T)}_r$ using (2.5). □

As already pointed out, H is in general not a Markov process. However, we will see later that H has some *Markov-style* properties. At this stage, we present the promised Markov property of the exploration process. But first, we need to say what is meant by ρ *started at an arbitrary initial value* $\mu \in \mathcal{M}_f(\mathbb{R}_+)$.

To this end, let $\mu \in \mathcal{M}_f(\mathbb{R}_+)$ and denote by $\|\mu\| := \mu(\mathbb{R}_+) < \infty$ the total mass of the measure μ. Let $a \geq 0$. If $a \leq ||\mu||$, we define $k_a\mu$ to be the unique finite measure on $\mathbb{R}_+$ such that for every $r \geq 0$,

$$k_a\mu([0,r]) = \mu([0,r]) \wedge (||\mu|| - a). \tag{2.23}$$

In the case $a \geq ||\mu||$ let $k_a\mu = 0 \in \mathcal{M}_f(\mathbb{R}_+)$. So, we have in particular $||k_a\mu|| = ||\mu|| - a$ and we can think of the $k_a\mu$ as μ *cut off* at the value a.

If $\mu \in \mathcal{M}_f(\mathbb{R}_+)$ has compact support and $\nu \in \mathcal{M}_f(\mathbb{R}_+)$ define the *concatenation* $[\mu, \nu] \in \mathcal{M}_f(\mathbb{R}_+)$ of μ and ν by

$$\int f(r)\, [\mu,\nu](dr) = \int f(r)\, \mu(dr) + \int f(H(\mu) + r)\, \nu(dr) \tag{2.24}$$

for any nonnegative measurable function f. The law of the process $\{\rho_t : t \geq 0\}$ started at $\rho_0 = \mu \in \mathcal{M}_f(\mathbb{R}_+)$ is then the distribution of the process ρ^μ defined by

$$\rho^\mu_t = [k_{-I_t}\mu, \rho_t] \tag{2.25}$$

for $t > 0$. Note that $k_{-I_t}\mu$ has compact support, hence (2.25) is well defined. It is easy to compute the total mass of ρ^μ_t by

$$\begin{aligned}
||\rho^\mu_t|| &= ||[k_{-I_t}\mu, \rho_t]|| \\
&= ||k_{-I_t}\mu|| + ||\rho_t|| \\
&= \begin{cases} (\|\mu\| + I_t) + (X_t - I_t) & \text{if } \|\mu\| + I_t \geq 0 \\ X_t - I_t & \text{otherwise} \end{cases} && (2.26) \\
&= \begin{cases} X_t + \|\mu\| & \text{if } \|\mu\| + I_t \geq 0 \\ X_t - I_t & \text{otherwise} \end{cases}. && (2.27)
\end{aligned}$$

Let us now prove the promised Markov property of the exploration process.

Theorem* 2.17 *(Markov property)
The exploration process $\{\rho_t : t \geq 0\}$ is a càdlàg strong $\mathcal{F}_t$-Markov process in the space of finite measures $\mathcal{M}_f(\mathbb{R}_+)$.

Proof: Let T be a finite $\mathcal{F}_t$-stopping time. The idea behind the proof is to express ρ_{T+t} in terms of ρ_T and the shifted process $X^{(T)} = \{X_{T+t} - X_T : t \geq 0\}$. Then the Markov property of X implies the Markov property of ρ. To do so, consider

$$\begin{aligned}\langle \rho_{T+t}, f\rangle &= \int_{[0,T+t]} f(H_s)\, d_s I^s_{T+t} \\ &= \int_{[0,T]} f(H_s)\, d_s I^s_{T+t} + \int_{(T,T+t]} f(H_s)\, d_s I^s_{T+t}. \qquad (2.28)\end{aligned}$$

We will deal the two summands in (2.28) separately. First, we show that

$$\int_{[0,T]} f(H_s)\, d_s I^s_{T+t} = \langle k_{-I_t^{(T)}} \rho_T, f\rangle \qquad (2.29)$$

for all nonnegative measurable f, where $I^{(T)}$ denotes the infimum process of the shifted Lévy process $X^{(T)}$.

By the standard trick, using approximation and the monotone convergence theorem, it is enough to prove (2.29) for indicator functions $f = 1_{[0,r]}$. Let

$$u := \sup\left\{r \in (0,T] : X_{r^-} < I^T_{T+t}\right\}$$

and as usual $\sup \emptyset = 0$. Then we have

$$I^s_{T+t} = I^s_T \text{ for all } s \in [0,u) \text{ and } I^s_{T+t} = I^T_{T+t} \text{ for all } s \in [u,T]. \qquad (2.30)$$

Consider the case when $u = 0$. Then $I^0_T \geq I^T_{T+t}$ which implies that $\|\rho_T\| + I_t^{(T)} \leq 0$ and the right hand side in formula (2.29) is equal to zero. But $I^0_T \geq I^T_{T+t}$ also implies that $I^T_{T+t} = I^0_{T+t}$ then also the left hand side in formula 2.29 is equal to zero because

$$\int_{[0,T]} 1_{[0,r]}(H_s)\, d_s I^s_{T+t} \leq I^T_{T+t} - I^0_{T+t} = 0.$$

If $u > 0$, then $I_T^{u^-} = I^T_{T+t}$ and $I^0_{T+t} = I^0_T$. Moreover, we have $I^T_{T+t} - I^0_T = I_T^{u^-} - I^0_T \geq 0$ and

$$\int_{[0,T]} 1_{[0,r]}(H_s) d_s I^s_{T+t} \leq I^T_{T+t} - I^0_{T+t} = I_T^{u^-} - I^0_T.$$

On the other hand, $I^T_{T+t} - I^0_{T+t} = \|\rho_T\| + I^{(T)}_t$ and (2.30) implies that

$$\begin{aligned}
\int_{[0,T]} 1_{[0,r]}(H_s) d_s I^s_{T+t} &= \int_{[0,u)} 1_{[0,r]}(H_s) d_s I^s_{T+t} + \int_{[u,T]} 1_{[0,r]}(H_s) d_s I^s_{T+t} \\
&= \int_{[0,u)} 1_{[0,r]}(H_s) d_s I^s_T, \\
&= \rho_T[0,r] \wedge (I^{u^-}_T - I^0_T) \\
&= \rho_T[0,r] \wedge \left(\|\rho_T\| + I^{(T)}_t\right) \\
&= \left(k_{-I^{(T)}_t}\rho_T\right)[0,r],
\end{aligned} \tag{2.31}$$

hence equation (2.29) follows.

The second summand of (2.28) is much easier to handle. Using Lemma 2.16 we see that

$$\begin{aligned}
\int_{(T,T+t]} f(H_s)\, d_s I^s_{T+t} &= \int_{(T,T+t]} f(H_u + H^{(T)}_{s-T})\, d_s I^s_{T+t} \\
&= \int f(H_u + x)\, \rho^{(T)}_t(dx).
\end{aligned} \tag{2.32}$$

Therefore we get the following expression for ρ_{T+t},

$$\langle \rho_{T+t}, f\rangle = \langle k_{-I^{(T)}_t}\rho_T, f\rangle + \int f(H_u + x)\, \rho^{(T)}_t(dx),$$

hence by the definition of the concatenation of measures, i.e. formula (2.24), it follows that

$$\rho_{T+t} = \left[k_{-I^{(T)}_t}\rho_T, \rho^{(T)}_t\right]. \tag{2.33}$$

The strong Markov property now follows immediately, using the the strong Markov property of X and the definition of the exploration process started at an arbitrary inital value. $\square$

Equation (2.33) is the key to the strong Markov property of the exploration process and is often used in later proofs. Therefore we formulate again:

***Corollary* 2.18** *Let T be a finite $\mathcal{F}_t$-stopping time. Then almost surely, for all $t > 0$*

$$\rho_{T+t} = \left[k_{-I^{(T)}_t}\rho_T, \rho^{(T)}_t\right].$$

The reader who is familiar with the *Brownian snake* should note that equation (2.33), is important to extend the Brownian Snake construction to general branching mechanisms given by the class $\mathcal{C}$. To be more specific: As

$$\inf_{T\le s\le T+t} H(\rho_s) = H\left(k_{-I^{(T)}_t}\rho_T\right) =: m(T, T+t),$$

the evolution of $\{\rho_{T+t} : t \geq 0\}$ can be interpreted as follows: We obtain the measure ρ_{T+t} by first restricting ρ_T to the interval $[0, m(T, T+t)]$ and then we concatenate this restricted measure with the random measure $\rho_t^{(T)}$ which is independent of the past up to time T and has the same distribution as ρ_t. We refer to [LGLY98b] for the extension of this idea to construct the generalized Brownian Snake the so called *Lévy Snake.*

Recall that a function $f : \mathbb{R} \to \mathbb{R}_0^+$ is called ***lower semicontinuous*** if

$$K_c(f) := \{x : f(x) > c\}$$

is an open set for all $c > 0$. We will now proof that the modification of the height process, which we obtained using the exploration process has lower semicontinous paths.

***Corollary* 2.19** *The modification of H given by $\{H(\rho_t) : t \geq 0\}$ in Lemma 2.15 has lower-semicontinuous paths.*

Proof: Let $c > 0$, $t \in K_c(H(\rho))$ and recall that $H(\rho_t) = \sup\{\operatorname{supp} \rho_t\}$ by definition. By Lemma 2.14 the exploration process ρ has càdlàg paths with respect to the variation distance. As convergence with respect to the variation distance implies weak convergence, it follows using the Portmanteaux theorem,

$$\liminf_{t' \downarrow t} \rho_{t'}(x, \infty) \geq \rho_t(x, \infty),$$

for all $x \geq 0$. Note that $\sup\{\operatorname{supp} \rho_s\} \leq x$ if and only if $\rho_s(x, \infty) = 0$ for all $s > 0$. Hence $\sup\{\operatorname{supp} \rho_{t'}\} \leq x$ for all t' sufficiently close to t implies that $\rho_{t'}(x, \infty) = 0$ for such t'. Hence we also have $\rho_t(x, \infty) = 0$ and therefore $\sup\{\operatorname{supp} \rho_t\} \leq x$. In particular,

$$l := \liminf_{t' \downarrow t} \sup\{\operatorname{supp} \rho_{t'}\} \leq x$$

implies also $\sup\{\operatorname{supp} \rho_t\} \leq x$. Hence, letting x converge from above towards l leads

$$\liminf_{t' \downarrow t} \sup\{\operatorname{supp} \rho_{t'}\} \geq \sup\{\operatorname{supp} \rho_t\} > c.$$

Therefore, we can pick $\varepsilon_1 > 0$ such that $s \in K_c(H(\rho))$ for all $s \in [t, t + \varepsilon_1)$.
Assume that ρ jumps at time t. Then we get by (2.13) that

$$\operatorname{supp} \rho_t = \operatorname{supp} \rho_{t-} \cup \{H_t\},$$

and because $\sup\{\operatorname{supp} \rho_{t-}\} = H_t$ we get that $H(\rho_{t-}) = H(\rho_t) > c$ hence there is a $\varepsilon_2 > 0$ such that $s \in K_c(H(\rho))$ for all $s \in (t - \varepsilon_2, t]$. In the case when ρ does not jump at time t, the existence of such an ε_2 is clear, as we could do the same construction as above. With $\varepsilon := \varepsilon_1 \wedge \varepsilon_2$ we found an open ball $B_\varepsilon(t)$ around t which is contained in $K_c(H(\rho))$, so $K_c(H(\rho))$ is an open set. □

From now on, we will *always* use this lower-semicontinuous modification and write indifferently H_t or $H(\rho_t)$. The next lemma provides a nice path property of H which is reminiscent of the *intermediate value property* of continuous functions. Nevertheless, it turns out that H is *not* always continuous. Therefore, the next theorem tells us that the height process must behave very wildy in the case of noncontinuity.

Theorem 2.20 ***(Intermediate value property)***
Almost surely, for all $t < t'$ the process $\{H_t : t \geq 0\}$ takes all values between H_t and $H_{t'}$ on the time interval $[t, t']$.

Proof: We proceed in several steps. Let us first assume that $t \in \mathbb{Q}$ and that $H_t > H_{t'}$. Corollary 2.18 applied to t yields,

$$\rho_{t+r} = \left[k_{-I_r^{(t)}}\rho_t, \rho_r^{(t)}\right] \tag{2.34}$$

for all $r > 0$ almost surely. Denote by

$$\gamma_r := \inf\left\{s > 0 : I_s^{(t)} = -r\right\}$$

the first time instant, when the infimum process of the shifted Lévy process $X^{(t)}$ reaches $-r$. As $X_{\gamma_r}^{(t)} = I_{\gamma_r}^{(t)}$ implies that $\rho_{\gamma_r}^{(t)} = 0$, we have that $\rho_{t+\gamma_r} = k_r\rho_t$ and therefore $H_{t+\gamma_r} = H(k_r\rho_t)$ for all $r \geq 0$ almost surely. Now, (2.23) implies that the mapping

$$r \mapsto H(k_r\rho_t) \tag{2.35}$$

is continuous. If $r = 0$, then $H_{t+\gamma_r} = H_t$. If $r = X_t - I_{t'}^t = -I_{t'-t}^{(t)}$ we have that $\gamma_r \leq t' - t$. In any case $H_{t+\gamma_r} \leq H_{t'}$ by construction, and the continuity of the mapping (2.35) implies that every point between H_t and $H_{t'}$ is hitten on the time intervall $[t, t']$. By the lower-semicontinuity of the height process we get the statement for all $t > 0$.

Now assume that $H_t < H_{t'}$. In this case we have to argue differently. Again, we can assume that $t' \in \mathbb{Q}$. Recall the notation $\hat{L}^{(t')}$ for the local time of $\hat{S}^{(t')} - \hat{X}^{(t')}$ so that by definition of the height process $H_{t'} = \hat{L}_{t'}^{(t')}$. For $r \in [0, X_{t'} - I_{t'}]$ define

$$\sigma_r = \inf\{s \geq 0 : \hat{S}_s^{(t)} \geq r\}$$

as the first time instant, when the supremum process of the time reversed Lévy process reaches r and note that $r \mapsto \sigma_r$ is continuous. The continuity of the local time then implies that the mapping

$$r \mapsto \hat{L}_{\sigma_r}^{(t')}$$

is continuous for all $r \in [0, X_{t'} - I_{t'}]$ almost surely. Note that $H_{t'-\sigma_r} = \hat{L}_{t'}^{(t')} - \hat{L}_{\sigma_r}^{(t')}$ for all $r \in [0, X_{t'} - I_{t'}]$ almost surely, hence for $r = X_{t'} - I_{t'} = \hat{S}_{t'-t}^{(t')}$ we have $t' - \sigma_r \geq t$ and $H_{t'-\sigma_r} \leq H_t$ and also in this case every point between H_t and $H_{t'}$ is hitten on the time intervall $[t, t']$ which finishes the proof of the theorem. $\square$

For $a \geq 0$, define the following random times

$$\tau_t^a := \inf\left\{s \geq 0 : \int_0^s 1_{\{H_r > a\}}\, dr > t\right\}$$

and for $a > 0$

$$\tilde{\tau}_t^a := \inf\left\{ s \geq 0 : \int_0^s 1_{\{H_r \leq a\}}\, dr > t \right\}.$$

Then $\tau_\cdot^a, \tilde{\tau}_\cdot^a$ can be interpreted as the inverse of *clocks* that run only if the height process is above or below level a respectively. In order to use these random times in a meaningful way, it is important to know that they are almost surely finite. To proof this property we have to make the following observations, which are of independent interest:

As for any $t \geq 0$, the value of the height and the exploration process depends *only* on the excursion of $X - I$ that contains t, we can define both processes under the excursion measure N of the Markov process $X - I$ away from 0. Moreover, as 0 is a regular point for $X - I$, the measure 0 is a regular point for the exploration process and the excursion measure of ρ away from 0 is the *law* of ρ under N. Similarly, even if the height process is not a Markov process, we deal with the law of H under N as the *excursion measure* of H away from 0. Let us denote by $N[\cdot]$ an integration over $\mathcal{D}$ with respect to the σ-finite measure N and by σ the length of an excursion.

Let $\{U_t : t \geq 0\}$ be a subordinator with Laplace exponent $\frac{\psi(\lambda)}{\lambda} - \alpha$. And let for all $a \geq 0$

$$J_a(dr) := 1_{[0,a]}(r)\, dU_r$$

be a random element of $\mathcal{M}_f(\mathbb{R}_+)$. Recall, that we denote by $\{L_t : t \geq 0\}$ the local time at 0 of $S - X$. The following lemma turns out to be very useful:

***Lemma* 2.21** *We have for every nonnegative, measurable funtional $F : \mathcal{D} \to \mathbb{R}$ that*

$$N\left[\int_0^\sigma F(\{\hat{X}_s^{(t)} : 0 \leq s \leq t\})\, dt\right] = \mathbb{E}\left\{\int_0^{L_\infty} F(\{X_s : 0 \leq s \leq L^{-1}(a)\})\, da\right\}. \quad (2.36)$$

In particular, for every nonnegative measurable functional $\Phi : \mathcal{M}_f(\mathbb{R}_+) \to \mathbb{R}$ we have that

$$N\left[\int_0^\sigma \Phi(\rho_t)\, dt\right] = \int_0^\infty e^{-a\alpha}\mathbb{E}\{\Phi(J_a)\}\, da. \quad (2.37)$$

Proof: We only proof the *in particular* statement. The proof of the first assertion can be found in [LGD]. Recall that we can represent ρ_t by

$$\langle \rho_t, f\rangle = \int_{[0,t]} f\left(\hat{L}_t^{(t)} - \hat{L}_s^{(t)}\right) d\hat{S}_s^{(t)}, \quad (2.38)$$

hence we can express ρ_t as a functional Γ of the time reversed Lévy process $\{\hat{X}_s^{(t)} : 0 \leq s \leq t\}$. Therefore

$$\Phi \circ \Gamma : \mathcal{D} \to \mathbb{R}$$

and by (2.36) we get

$$
\begin{aligned}
N\left[\int_0^\sigma \Phi(\rho_t)\,dt\right] &= N\left[\int_0^\sigma \Phi\circ\Gamma(\{\hat{X}_s^{(t)} : 0 \le s \le t\})\,dt\right] \\
&= \mathbb{E}\left\{\int_0^{L_\infty} \Phi\circ\Gamma(\{X_s : 0 \le s \le L^{-1}(a)\})\,da\right\}. \qquad (2.39)
\end{aligned}
$$

If $a < L_\infty$ we get using (2.38) and substitution,

$$
\begin{aligned}
\int f d\Gamma(\{X_s : 0 \le s \le L^{-1}(a)\}) &= \int_0^{L^{-1}(a)} f(L_{L^{-1}(a)} - L_s)\,dS_s \\
&= \int_0^{L^{-1}(a)} f(a - L_s)\,dS_s \\
&= \int_0^a f(a-s)\,d_s S_{L^{-1}(s)}. \qquad (2.40)
\end{aligned}
$$

By the fluctuation formula in Chapter 1, we get that $P\{a < L_\infty\} = e^{-\alpha a}$ and that conditionally on $\{a < L_\infty\}$, $\{S_{L^{-1}(r)} : 0 \le r \le a\}$ is a subordinator with the same distribution as U. Hence, conditionally on $\{a < L_\infty\}$ the measure

$$\Gamma(\{X_s : 0 \le s \le L^{-1}(a)\})$$

has the same distribution as J_a. Therefore, we get using (2.39) and Fubinis theorem,

$$
\begin{aligned}
N\left[\int_0^\sigma \Phi(\rho_t)dt\right] &= \mathbb{E}\left\{\int_0^\infty 1_{\{a<L_\infty\}}\Phi(J_a)\,da\right\} \\
&= \int_0^\infty \mathbb{E}\left\{1_{\{a<L_\infty\}}\Phi(J_a)\right\}\,da \\
&= \int_0^\infty e^{-\alpha a}\mathbb{E}\{\Phi(J_a)\}\,da. \qquad (2.41)
\end{aligned}
$$

□

Now we can use this Lemma, to prove

Lemma 2.22 *For every $a \ge 0$ and $t \ge 0$ the random times τ_t^a are almost surely finite.*

Proof: Let $a \ge 0$. It is enough to show that the height process spends an infinite amount of time above level a, i.e. that almost surely

$$\int_0^\infty 1_{\{H_s > a\}} ds = \infty. \qquad (2.42)$$

Recall that $H(\rho_t) = H_t$. Then by Lemma 2.21 applied to $\Phi(\rho_t) = 1_{\{\sup\{\operatorname{supp}\rho_t\}>a\}}$, there are $\varepsilon > 0, \delta > 0$ such that

$$N\left[\int_0^\sigma 1_{\{H_t>a\}}\,dt > \varepsilon\right] > \delta.$$

Hence (2.42) follows using the Borel-Cantelli lemma. □

Recall that $\tilde{\tau}^a$ is the inverse of the clock that runs only if the height process is below level a. Denote for $a > 0$ the σ-field $\mathcal{H}_a$ generated by the càdlàg process $\{(X_{\tilde{\tau}^a_t}, \rho_{\tilde{\tau}^a_t}) : t \geq 0\}$ with values in $\mathbb{R} \times \mathcal{M}_f(\mathbb{R}_+)$ and the class of P-negligible sets of $\mathcal{F}$. If $a = 0$, let $\mathcal{H}_0$ be the σ-field generated by the class of P-negligible sets of $\mathcal{F}$.

***Theorem* 2.23** *Let $a > 0$ and for every $t \geq 0$, let ρ^a_t be the random measure on $\mathbb{R}_+$ defined by*

$$\langle \rho^a_t, f \rangle = \int_{(a,\infty)} f(r-a)\, \rho_{\tau^a_t}(dr). \tag{2.43}$$

Then $\{\rho^a_t : t \geq 0\}$ has the same distribution as $\{\rho_t : t \geq 0\}$ and is independent of $\mathcal{H}_a$.

Before we start to prove the theorem, we have to make an important remark. Theorem 2.23 tells us, that the process $\{H^a_t : t \geq 0\}$ defined by $H^a_t := H(\rho^a_t)$, which can be seen as *glueing* together the upward excursions of the height process above level a, has the same distribution as the original height process $\{H_t : t \geq 0\}$.

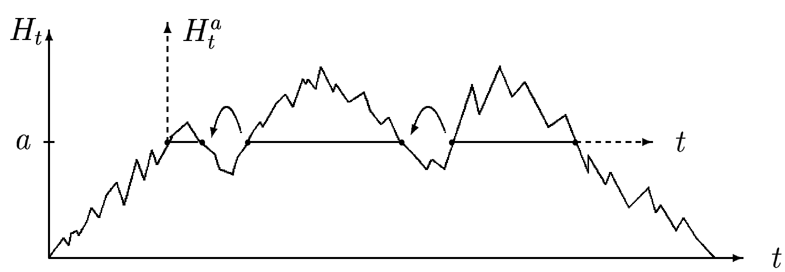


Figure 2.6: The idea behind Theorem 2.23

This property can be seen as a certain *pseudo Markov* property of the height process.

Proof: We proceed by several steps: First, we show that the total mass processes of ρ^a and ρ have the same distribution and then we will verify that ρ^a can be obtained from its total mass process in the same way as ρ. To complete the proof, we then show the independence of ρ^a of the σ-field $\mathcal{H}_a$.

So, let $a > 0$ and let us start in showing that $\{||\rho^a_t|| : t \geq 0\}$ has the same distribution as $\{||\rho_t|| : t \geq 0\}$.

First, we fix some notation. For $\varepsilon > 0$, define stopping times

$$\begin{aligned} S^1_\varepsilon &:= \inf\{s \geq 0 : \rho_s(a,\infty) \geq \varepsilon\} \\ T^k_\varepsilon &:= \inf\{s \geq S^k_\varepsilon : \rho_s(a,\infty) = 0\} \\ S^{k+1}_\varepsilon &:= \inf\{s \geq T^k_\varepsilon : \rho_s(a,\infty) \geq \varepsilon\}, \end{aligned}$$

for $k \geq 1$. Recall that 0 is a regular point for the reflected Lévy process $X - I$ and that $X_t - I_t = 0$ if and only if $H_t = 0$, moreover for any $a > 0$, the height process spends an infinite amount of time above level a, hence for all $k \geq 1$, the stopping times $S_\varepsilon^k, T_\varepsilon^k$ are almost surely finite and tend with k to infinity. By plugging S_ε^k in Corollary 2.18 we get

$$\rho_{S_\varepsilon^k+t} = \left[k_{-I_t^{(S_\varepsilon^k)}}\rho_{S_\varepsilon^{(k)}}, \rho_t^{(S_\varepsilon^k)}\right], \tag{2.44}$$

hence we obtain for every $k \geq 1$,

$$\begin{aligned} T_\varepsilon^k &= \inf\left\{s \geq S_\varepsilon^k : \rho_s(a,\infty) = 0\right\} \\ &= \inf\left\{s \geq 0 : \rho_{S_\varepsilon^k+s}(a,\infty) = 0\right\} \\ &= \inf\left\{s \geq 0 : k_{-I_s^{(S_\varepsilon^k)}}\rho_{S_\varepsilon^k}(a,\infty) = 0\right\} \\ &= \inf\left\{s \geq 0 : \|\rho_{S_\varepsilon^k}\| - \rho_{S_\varepsilon^k}[0,a] + I_s^{(S_\varepsilon^k)} = 0\right\} \\ &= \inf\left\{s \geq 0 : I_s^{(S_\varepsilon^k)} = -\rho_{S_\varepsilon^k}(a,\infty)\right\} \\ &= \inf\left\{s \geq 0 : X_{S_\varepsilon^k+s} - X_{S_\varepsilon^k} = -\rho_{S_\varepsilon^k}(a,\infty)\right\} \\ &= \inf\left\{s > S_\varepsilon^k : X_s = X_{S_\varepsilon^k} - \rho_{S_\varepsilon^k}(a,\infty)\right\}. \end{aligned} \tag{2.45}$$

In particular, we see that for every $s \in [0, T_\varepsilon^k - S_\varepsilon^k]$,

$$\begin{aligned} \rho_{S_\varepsilon^k+s}(a,\infty) &= \left(\rho_{S_\varepsilon^k}(a,\infty) + I_s^{(S_\varepsilon^k)}\right) + \left(X_s^{(S_\varepsilon^k)} - I_s^{(S_\varepsilon^k)}\right) \\ &= X_{S_\varepsilon^k+s} - (X_{S_\varepsilon^k} - \rho_{S_\varepsilon^k}(a,\infty)). \end{aligned} \tag{2.46}$$

Using (2.45) and (2.46) we proved in fact that the processes $\{Y^{k,\varepsilon} : s \geq 0\}$ defined by

$$Y_s^{k,\varepsilon} := \rho_{(S_\varepsilon^k+s)\wedge T_\varepsilon^k}(a,\infty)$$

are distributed, conditionally on $\mathcal{F}_{S_\varepsilon^k}$, like the underlying Lévy process X started at $\rho_{S_\varepsilon^k}(a,\infty)$ and stopped at its first hitting time of 0.

Moreover, conditionally on $\mathcal{F}_{S_\varepsilon^k}$, the processes $\{Z_s^{k,\varepsilon} : 0 \leq s \leq T_\varepsilon^k - S_\varepsilon^k\}$ given by

$$Z_s^{k,\varepsilon} := Y_s^{k,\varepsilon} - \inf_{0 \leq r \leq s} Y_r^{k,\varepsilon},$$

are distributed like the reflected process $X - I$, stopped when its local time at 0 (which is $-I$) hits $\rho_{S_\varepsilon^k}(a,\infty)$. By patching together the paths of the processes $Z^{k,\varepsilon}$ for $k \geq 1$ we obtain a process U^ε, which is distributed like the reflected Lévy process $X - I$.

Now, define

$$\tau_s^{a,\varepsilon} := \inf\left\{t \geq 0 : \sum_{k=1}^{\infty} \int_0^t 1_{[S_\varepsilon^k, T_\varepsilon^k]}(r)dr > s\right\}.$$

Recall that $Y^{k,\varepsilon}$ was exactly constructed in such a way, that the time changed process $\{\rho_{\tau_s^{a,\varepsilon}}(a,\infty) : s \geq 0\}$ can be obtained by patching together the paths of $Y^{k,\varepsilon}$ for all $k \geq 1$. Moreover, for every $k \geq 1$,

$$\begin{aligned} \sup_{0\leq s\leq T_\varepsilon^k - S_\varepsilon^k} (Y_s^{k,\varepsilon} - Z_s^{k,\varepsilon}) &= \sup_{0\leq s\leq T_\varepsilon^k - S_\varepsilon^k} \left(\inf_{0\leq r\leq s} (\rho_{(S_\varepsilon^k + s)\wedge T_\varepsilon^k}(a,\infty)) \right) \\ &= \rho_{S_\varepsilon^k}(a,\infty) \qquad (2.47) \\ &= Y_0^{k,\varepsilon}. \end{aligned}$$

Because $\tau_s^{a,\varepsilon}$ tends to τ_s^a as ε tends to 0 and U^ε is distributed like the reflected process $X - I$, we would complete the proof of the first step, if we could show that

$$\lim_{\varepsilon\downarrow 0} \left(\sup_{s\leq t} |U_s^\varepsilon - \rho_{\tau_s^{a,\varepsilon}}(a,\infty)| \right) = 0. \qquad (2.48)$$

Because then the total mass process of ρ^a

$$\{||\rho_s^a|| : s \geq 0\} = \{\rho_{\tau_s^a}(a,\infty) : s \geq 0\},$$

is distributed as $X - I$ which is known to be the total mass process of ρ.

But we still have to show formula (2.48). By (2.47), it is enough to show that for every $t \geq 0$,

$$\lim_{\varepsilon\downarrow 0} \sup_{k\geq 1,\, S_\varepsilon^k \leq t,\, 0\leq s\leq T_\varepsilon^k - S_\varepsilon^k} \left(Y_s^{k,\varepsilon} - Z_s^{k,\varepsilon}\right) = \lim_{\varepsilon\downarrow 0} \sup_{k\geq 1, S_\varepsilon^k\leq t} \rho_{S_\varepsilon^k}(a,\infty) = 0$$

almost surely. By Lemma 2.14, the mapping $t \mapsto \rho_t(a,\infty)$ is càdlàg and the discontinuity times are those times t such that $\Delta X_t > 0$ and $H_t > a$. Moreover, the height of the corresponding jump is exactly ΔX_t. Hence, we get

$$\sup_{k\geq 1, S_\varepsilon^k \leq t} \rho_{S_\varepsilon^k}(a,\infty) \leq \varepsilon + \sup\{\Delta X_s : s \leq t, H_s > a, \rho_s(a,\infty) \leq \varepsilon\}.$$

Because the sets

$$\{s \leq t : \Delta X_s > 0, H_s > a, \rho_s(a,\infty) \leq \varepsilon\}$$

decrease to $\emptyset$ as ε goes to 0 we get the statement.

Let us now show, that ρ^a can be obtained as a functional of the total mass process $||\rho^a||$ in the same way as ρ is obtained from $||\rho||$.

By the Markov property of the exploration process, it is enough to consider one excursion ω of $||\rho^a||$ away from 0. Let (u, v) be the corresponding excursion interval. Recall that by the lower-semicontinuity of the height process, the set $\{s : H_s > a\}$ is open and there is a unique open subinterval

$$(p, q) \subseteq \{s : H_s > a\}$$

such that $\tau^a_{u+r} = p + r$ for every $r \in [0, v-u)$ and $q = \tau^a_{v-}$. Heuristically, (u,v) and (p,q) describe the same times, measured by a different clock. The mean value property of the height process now implies that $H_p = H_q = a$. Moreover, we have $X_r > X_p$ for every $r \in (p,q)$ because otherwise we could find an $r \in (p,q)$ such that

$$X_r = \inf\{X_s : p \le s \le r\}$$

which would imply $H_r \le H_p = a$ and therefore contradicts the fact that $H_r > a$, as $r \in (p,q)$. Recall the definition of the exploration process

$$\langle \rho_t, f\rangle = \int_{[0,t]} f(H_s)\, d_s I^s_t.$$

Let $A \subseteq [0,a]$. As $X_r > X_p$ for all $r \in (p,q)$, we get

$$\begin{aligned} \rho_r(A) &= \int_{[0,r]} 1_A(H_s)\, d_s I^s_r \\ &= \int_{[0,p]} 1_A(H_s)\, d_s I^s_r + \int_{(p,r]} 1_A(H_s)\, d_s I^s_r \\ &= \int_{[0,p]} 1_A(H_s)\, d_s I^s_p. \end{aligned} \tag{2.49}$$

Hence, for every $r \in (p,q)$ the restriction of ρ_r to $[0,a]$ is just ρ_p. Now define,

$$\omega(r) := X_{(p+r)\wedge q} - X_p.$$

As $I_{(p+r)\wedge q} = I_p$ we can use the total mass formula to compute

$$\begin{aligned} \omega(r) &= X_{(p+r)\wedge q} - I_{(p+r)\wedge q} - X_p + I_p \\ &= \|\rho_{(p+r)\wedge q}\| - \|\rho_p\| \\ &= \|\rho^a_{(u+r)\wedge v}\|. \end{aligned} \tag{2.50}$$

Hence, $\omega(r)$ is the excursion of $\|\rho^a\|$ corresponding to the interval (u,v). Therefore we get for $0 < r < q - p = u - v$

$$\rho_{p+r} = [\rho_p, \rho_r(\omega)] \quad \text{and} \quad \rho^a_{u+r} = \rho_r(\omega),$$

where $\rho_r(\omega)$ denotes the exploration process constructed from the excursion $\omega(r)$. Hence we succeeded in showing that ρ^a has the same distribution as the original exploration process ρ.

We still have to show the independence of ρ^a from the σ-field $\mathcal{H}_a$. Let $\varepsilon > 0$ and denote by $\mathcal{H}^\varepsilon_a$ the σ-field generated by the processes

$$\left\{X_{(T^k_\varepsilon + s)\wedge S^{k+1}_s} : s \ge 0\right\},\ k \ge 1$$

and the negligible sets of $\mathcal{F}_\infty$. As for all $k \geq 1$, the processes

$$\left\{\rho_{\left(T_\varepsilon^k+s\right)\wedge S_\varepsilon^{k+1}} : s \geq 0\right\}$$

are $\mathcal{H}_a^\varepsilon$-measureable and the fact that $H_t > a$ for all $t \in (S_\varepsilon^k, T_\varepsilon^k)$, we get that $\mathcal{H}_a \subseteq \mathcal{H}_a^\varepsilon$ for all $\varepsilon > 0$. Conditionally on $\mathcal{H}_a^\varepsilon$ the processes $Z^{k,\varepsilon}$, $k \geq 1$, constructed in the first step of the proof, are independent and distributed as independent copies of the reflected process $X - I$ stopped when its local time at 0 hits $\rho_{S_\varepsilon^k}(a, \infty)$. As the process U^ε is obtained by glueing together the paths of $Z^{k,\varepsilon}$ we see that U^ε is independent of $\mathcal{H}_a^\varepsilon$ and as $\mathcal{H}_a \subseteq \mathcal{H}_a^\varepsilon$ also independent of $\mathcal{H}_a$. By passing to $\varepsilon \downarrow 0$, and using (2.48) it follows that the total mass process $\|\rho^a\|$ is independent of $\mathcal{H}_a$. Using our construction of the second part of the proof, where we constructed ρ^a from its total mass process, we get the independence of $\{\rho_s^a : s \geq 0\}$ from the σ-field $\mathcal{H}_a$. $\square$

Our next aim is to introduce *local times* for $\{H_t : t \geq 0\}$. As already pointed out, the height process is in general *neither* a Markov process *nor* a semimartingale. Therefore, we cannot use the standard machinery to ensure the existence of local times of the height process. Remember, that we introduced a process ρ which is measure valued and in fact Markovian. Hence the following definition makes sense.

***Definition* 2.24** *For every $a \geq 0$, let $\{l^a(s) : s \geq 0\}$ be the local time of ρ^a at level 0. We set*

$$L_t^a := l^a\left(\int_0^t 1_{\{H_r > a\}}\, dr\right), \tag{2.51}$$

*and call the process $\{L_s^a : s \geq 0\}$ the **local time** at level a of the height process $\{H_t : t \geq 0\}$.*

We justify the name *local time* of the height process by an *occupation time* formula (Theorem 2.26) below. To prepare this, we prove the following lemma, which also justifies our intuition, that $-I$ is a natural candidate to be the local time at 0 of the height process.

***Lemma* 2.25** *For all $t \geq 0$, we have*

$$\lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_0^t 1_{\{H_s \leq \varepsilon\}}\, ds = -I_t,$$

where the limit is taken in the L^1-sense.

Proof: Recall that we denote by N the excursion measure of $X - I$ away from 0 and that we can define the height process H under N because the value of H at time t does only depend on the excursion of $X - I$ that contains t. We denote by T_x the first hitting time of $-x$ by the Lévy process X. As $-I$ is the local time at 0 for $X - I$, the stopping time T_x is also the first time instant, when the local time of $X - I$ hits x. Also recall, that the local times at the beginning of an excursion and the excursions of $X - I$ away from 0 form a Poisson point process with intensity measure $m \otimes N$. The proof is done in several steps.

Step 1 *We show that for all $x > 0$*

$$\lim_{\varepsilon \downarrow 0} \mathbb{E}\left\{ \frac{1}{\varepsilon} \int_0^{T_x} 1_{\{H_s \leq \varepsilon\}} \, ds \right\} = x.$$

Define for any measurable real function g a mapping $\Phi_g : \mathcal{M}_f(\mathbb{R}_+) \to \mathbb{R}$ by

$$\Phi_g(\mu) := g\left(\sup\{\operatorname{supp} \mu\}\right).$$

From Lemma 2.21 follows that

$$\begin{aligned} N\left[\int_0^\sigma \Phi_g(\rho_t) \, ds\right] &= N\left[\int_0^\sigma g(H_s) \, ds\right] \\ &= \int_0^\infty e^{-\alpha x} \mathbb{E}\{g(x)\} \, dx \\ &= \int_0^\infty e^{-\alpha x} g(x) dx. \end{aligned} \tag{2.52}$$

Using Campbells theorem (especially formula 1.2) and (2.52) leads to

$$\mathbb{E}\left\{ \frac{1}{\varepsilon} \int_0^{T_x} 1_{\{H_s \leq \varepsilon\}} \, ds \right\} = \frac{x}{\varepsilon} N\left[\int_0^\sigma 1_{\{H_s \leq \varepsilon\}} \, ds\right] \tag{2.53}$$

$$= \frac{x}{\varepsilon} \left(\frac{1 - e^{-\alpha x}}{\alpha} \right) \leq x. \tag{2.54}$$

In particular, we get step 1 by letting ε decrease to zero in the previous formula.

Step 2 *We show that for any fixed $K > 0$*

$$A_{K,\varepsilon} := N\left[\left(\int_0^\sigma 1_{\{H_s \leq \varepsilon\}} 1_{\{X_s \leq K\}} ds\right)^2\right] = o(\varepsilon^2).$$

We clearly have that

$$\begin{aligned} A_{K,\varepsilon} &= N\left[\int_0^\sigma 1_{\{H_s \leq \varepsilon\}} 1_{\{X_s \leq K\}} \, ds \cdot \int_0^\sigma 1_{\{H_t \leq \varepsilon\}} 1_{\{X_t \leq K\}} \, dt\right] \\ &\leq 2N\left[\int_0^\sigma \int_s^\sigma 1_{\{H_s \leq \varepsilon\}} 1_{\{X_s \leq K\}} 1_{\{H_t \leq \varepsilon\}} \, dt \, ds\right]. \end{aligned} \tag{2.55}$$

Denote by $H^{(s)}$ the height process associated with the shifted Lévy process $X^{(s)}$. By the monotonicity of the local time, we have that $H^{(s)}_{t-s} \leq H_t$ for all $0 \leq s \leq t$. Using this bound and (2.53),

$$\begin{aligned} A_{K,\varepsilon} &\leq 2N\left[\int_0^\sigma \int_s^\sigma 1_{\{H_s\leq\varepsilon\}} 1_{\{X_s\leq K\}} 1_{\{H^{(s)}_{t-s}\leq\varepsilon\}}\, dt\, ds\right] \\ &= 2N\left[\int_0^\sigma 1_{\{H_s\leq\varepsilon\}} 1_{\{X_s\leq K\}}\, \mathbb{E}_{X_s}\left\{\int_0^{T_0} 1_{\{H_t\leq\varepsilon\}}\, dt\right\} ds\right]. \end{aligned}$$

Now, by (2.54) and Lemma 2.21, we see that the last equation is less or equal

$$\begin{aligned} &\leq 2\varepsilon N\left[\int_0^\sigma 1_{\{H_s\leq\varepsilon\}} 1_{\{X_s\leq K\}} X_s\, ds\right] \\ &= 2\varepsilon \int_0^\varepsilon \mathbb{E}\left\{X_{L^{-1}(y)} 1_{\{L^{-1}(y)<\infty, X_{L^{-1}(y)}\leq K\}}\right\} \\ &\leq 2\varepsilon^2\, \mathbb{E}\left\{X_{L^{-1}(\varepsilon)} \wedge K\right\}. \end{aligned}$$

Step 3 *We show that for all $K > 0$ one has*

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\int_0^{T_x} 1_{\{H_s\leq\varepsilon\}} 1_{\{X_s - I_s\leq K\}} ds = x$$

where the limit is in L^2 .

So fix $K > 0$. By the same arguments as they were explicitly done in step 1, it follows that

$$\lim_{\varepsilon\downarrow 0} \mathbb{E}\left\{\frac{1}{\varepsilon}\int_0^{T_x} 1_{\{H_s\leq\varepsilon\}} 1_{\{X_s - I_s\leq K\}}\, ds\right\} = x. \tag{2.56}$$

Moreover, as $X_s - I_s \geq X_s$ for all $s \geq 0$, step 2 implies that we also have,

$$N\left[\left(\int_0^\sigma 1_{\{H_s\leq\varepsilon\}} 1_{\{X_s - I_s\leq K\}}\, ds\right)^2\right] \leq 2\varepsilon^2 K. \tag{2.57}$$

Hence we get step 3, by combining (2.53), (2.56) and (2.57).

Step 4 *We show that*

$$\lim_{\varepsilon\downarrow 0} \frac{1}{\varepsilon}\int_0^t 1_{\{H_s\leq\varepsilon\}} ds = -I_t$$

in probability.

As step 3 holds for very $K > 0$, this particularly implies that

$$\lim_{\varepsilon\downarrow 0} \frac{1}{\varepsilon}\int_0^{T_x} 1_{\{H_s\leq\varepsilon\}}\, ds = x$$

in probability. As T_x is the first time, when the Lévy process X hits $-x$ respectively the first time when the infimum process $-I$ hits x we get step 4. As step 4 together with some boundedness condition would imply the assertion, we finish the proof by the following:

Step 5. *We show that $\frac{1}{\varepsilon}\int_0^t 1_{\{H_s\leq\varepsilon\}}ds$ is L^2-bounded for all $\varepsilon > 0$.*

Using Fubinis theorem and the fact that H_s has the same distribution as L_s we compute,

$$\begin{aligned}\mathbb{E}\left\{\int_0^t 1_{\{H_s\leq\varepsilon\}}\,ds\right\} &= \int_0^t P\{H_s\leq\varepsilon\}\,ds \\ &= \int_0^t P\{L_s\leq\varepsilon\}\,ds \\ &= \mathbb{E}\{L^{-1}(\varepsilon)\wedge t\}. \end{aligned} \tag{2.58}$$

As L^{-1} is a strictly increasing subordinator, there exists a constant C only depending on t, such that (2.58) is equal to $C\cdot\varepsilon$. We now use this estimate to get the L^2-boundedness via the following estimation

$$\begin{aligned}\mathbb{E}\left\{\left(\int_0^t 1_{\{H_s\leq\varepsilon\}}\,ds\right)^2\right\} &= 2\mathbb{E}\left\{\iint_{\{0<r<s<t\}} 1_{\{H_r\leq\varepsilon\}}1_{\{H_s\leq\varepsilon\}}\,dr\,ds\right\} \\ &\leq 2\mathbb{E}\left\{\iint_{\{0<r<s<t\}} 1_{\{H_r\leq\varepsilon\}}1_{\{H^{(r)}_{s-r}\leq\varepsilon\}}\,dr\,ds\right\} \\ &= 2\mathbb{E}\left\{\int_0^t 1_{\{H_r\leq\varepsilon\}}\mathbb{E}\left(\int_0^{t-r} 1_{\{H_s\leq\varepsilon\}}\,ds\right)\,dr\right\} \\ &\leq 2\left(\mathbb{E}\left\{\int_0^t 1_{\{H_r\leq\varepsilon\}}\right\}\right)^2 \\ &\leq 2C^2\varepsilon^2. \end{aligned} \tag{2.59}$$

Step 6 *(Conclusion)* Because $\frac{1}{\varepsilon}\int_0^t 1_{\{H_s\leq\varepsilon\}}ds$ is L^2-bounded (step 5), it is uniformly integrable and therefore we get L^1-convergence because we know the convergence in probability (Step 4). □

As promised earlier, the next theorem justifies the name *local time* for the process L^a in terms of an occupation time formula.

Theorem 2.26 (Occupation time formula)
There exists a jointly measurable modification of the collection $\{L^a_s : a\geq 0, s\geq 0\}$, which is continuous and nondecreasing in the variable s, such that almost surely for any nonnegative measurable function g on $\mathbb{R}_+$ and any $s\geq 0$ we have the ***occupation time*** *formula*

$$\int_0^s g(H_r)dr = \int_{\mathbb{R}_+} g(a)L^a_s da. \tag{2.60}$$

Proof: We split the proof into two parts. First we show that

$$\sup_{a\geq 0}\mathbb{E}\left\{\sup_{s\leq t}\left|\frac{1}{\varepsilon}\int_0^s 1_{\{a<H_r\leq a+\varepsilon\}}dr - L_s^a\right|\right\}\to 0 \tag{2.61}$$

as ε tends to 0. From this approximation result, we show then the existence of such a jointly measurable modification which fulfils the occupation time density formula as stated in the theorem.

First consider the case $a=0$. Then $\rho^0=\rho$ and $L_t^0=l_t^0=-I_t$. We now use Lemma 2.25 and a monotonicity argument to see that

$$\mathbb{E}\left\{\sup_{s\leq t}\left|\frac{1}{\varepsilon}\int_0^s 1_{\{0<H_r\leq \varepsilon\}}dr - L_s^0\right|\right\}\to 0 \text{ as } \varepsilon\downarrow 0. \tag{2.62}$$

Let $\delta>0$. The almost sure continuity of $s\mapsto L_s^0$ allows us to choose n large enough such that

$$\mathbb{E}\left\{\sup_{k\in\{1,\dots,n-1\}}\left|L_{kt/n}^0-L_{(k+1)t/n}^0\right|\right\}<\delta.$$

Moreover, we have that

$$\begin{aligned}\mathbb{E}&\left\{\sup_{kt/n:k\in\{1,\dots,n-1\}}\left|\frac{1}{\varepsilon}\int_0^{kt/n}1_{\{0<H_r\leq\varepsilon\}}\,dr - L_{kt/n}^0\right|\right\}\\ &\leq\sum_{k=1}^n\mathbb{E}\left\{\frac{1}{\varepsilon}\int_0^{kt/n}1_{\{0<H_r\leq\varepsilon\}}\,dr-L_{kt/n}^0\right\},\end{aligned}$$

which tends to 0 as $\varepsilon\downarrow 0$ using Lemma 2.25. Using the monotonicity of the function $s\mapsto L_s^0$, we get

$$\begin{aligned}&\mathbb{E}\left\{\sup_{s\leq t}\frac{1}{\varepsilon}\left|\int_0^s 1_{\{0<H_r\leq\varepsilon\}}\,dr-L_s^0\right|\right\}\\ \leq\;&\mathbb{E}\left\{\sup_{k\in\{1,\dots,n-1\}}\left|\frac{1}{\varepsilon}\int_0^{(k+1)t/n}1_{\{0<H_r\leq\varepsilon\}}\,dr-L_{kt/n}^0\right|\right\}\\ &+\mathbb{E}\left\{\sup_{k\in\{1,\dots,n-1\}}\left|\int_0^{kt/n}1_{\{0<H_r\leq\varepsilon\}}\,dr-L_{(k+1)t/n}^0\right|\right\}\\ \leq\;&\mathbb{E}\left\{\sup_{k\in\{1,\dots,n-1\}}\left|\frac{1}{\varepsilon}\int_0^{(k+1)t/n}1_{\{0<H_r\leq\varepsilon\}}\,dr-L_{(k+1)t/n}^0\right|\right\}\\ &+\mathbb{E}\left\{\sup_{k\in\{1,\dots,n-1\}}\left|L_{kt/n}^0-L_{(k+1)t/n}^0\right|\right\}+\mathbb{E}\left\{\sup_{k\in\{1,\dots,n-1\}}\left|\int_0^{kt/n}1_{\{0<H_r\leq\varepsilon\}}\,dr-L_{kt/n}^0\right|\right\}\\ \leq\;&3\delta,\end{aligned}$$

if n is large enough. Hence (2.62) follows.

Now let $a > 0$. Define $A_t^a := \int_0^t 1_{\{H_s > a\}}\, ds$ which can be seen as a *clock* that runs only if the height process is above a given level a. Recall that we can think of the height process as embedded in the exploration process as supremum of its support. So we get

$$\{a < H_s \le a + \varepsilon\} = \{\rho_s(a,\infty) > 0\} \cap \{\rho_s(a+\varepsilon,\infty) = 0\}. \tag{2.63}$$

Recall that we denote by $\tau_t^a = \inf\{s \ge 0 : \int_0^s 1_{\{H_r > a\}} dr > t\}$ the right continuous inverse of A_t^a.
By using (2.63), the substitution $s = \tau_r^a$ and the definition of ρ^a we get

$$\begin{aligned}
\int_0^t 1_{\{a < H_s \le a+\varepsilon\}}\, ds &= \int_0^t \{\rho_s(a,\infty) > 0\} \cap \{\rho_s(a+\varepsilon,\infty) = 0\}\, ds \\
&= \int_0^{A_t^a} 1_{\{\rho_{\tau_r^a}(a+\varepsilon,\infty) = 0\}}\, dr \\
&= \int_0^{A_t^a} 1_{\{\rho_r^a(\varepsilon,\infty) = 0\}}\, dr \\
&= \int_0^{A_t^a} 1_{\{0 < H_r^a \le \varepsilon\}}\, dr,
\end{aligned} \tag{2.64}$$

where $H_r^a = H(\rho_r^a)$. As we know that ρ^a has the same distribution as ρ, we can use formula (2.64) to get the desired aproximation

$$\begin{aligned}
\sup_{a \ge 0} \mathbb{E}\left\{ \sup_{s \le t} \left| \frac{1}{\varepsilon} \int_0^s 1_{\{a < H_r \le a+\varepsilon\}}\, dr - L_s^a \right| \right\} &= \sup_{a \ge 0} \mathbb{E}\left\{ \sup_{s \le t} \left| \frac{1}{\varepsilon} \int_0^{A_s^a} 1_{\{0 < H_r^a \le \varepsilon\}}\, dr - L_s^a \right| \right\} \\
&\le \sup_{a \ge 0} \mathbb{E}\left\{ \sup_{s \le t} \left| \frac{1}{\varepsilon} \int_0^s 1_{\{0 < H_r \le \varepsilon\}}\, dr - L_s^0 \right| \right\} \to 0,
\end{aligned}$$

where the last estimation also uses that $A_s^a \le s$.

By (2.62), we can choose a sequence $(\varepsilon_k) \downarrow 0$ independent from $a \ge 0$ to define

$$\tilde{L}_\cdot^a := \begin{cases} \lim_{n\to\infty} \left(\frac{1}{\varepsilon_k} \int_0^\cdot 1_{\{a < H_r \le a+\varepsilon_k\}}\, dr \right) & \text{if the limit exists} \\ 0 & \text{otherwise.} \end{cases}$$

Then $\{\tilde{L}_s^a : a \ge 0, s \ge 0\}$ provides a measureable modification of $\{L_s^a : a \ge 0, s \ge 0\}$ such that $s \mapsto \tilde{L}_s^a$ is continuous and non-decreasing. We use this modification from now on and write L instead of $\tilde{L}$.
To complete the proof, we still have to show formula (2.60). Let $A \in \mathcal{B}(\mathbb{R})$ by an open Borel set. Using

$$\begin{aligned}
\lim_{k\to\infty} \frac{1}{\varepsilon_k} \int 1_A(a) 1_{\{a < H_r \le a+\varepsilon_k\}}\, da &= \lim_{k\to\infty} \int 1_{A \cap [H_r - \varepsilon_k, H_r)}(a)\, da \\
&= \begin{cases} 1 & \text{if } H_r \in A \\ 0 & \text{otherwise,} \end{cases}
\end{aligned}$$

we get the desired result using the modification of L_t^a choosen above, Fubini's theorem and dominated convergence. $\square$

***Corollary* 2.27** *For every $t > 0$, we have*

$$\limsup_{\varepsilon \downarrow 0} \sup_{a \geq \varepsilon} \mathbb{E}\left\{ \sup_{s \leq t} \left| \frac{1}{\varepsilon} \int_0^s 1_{\{a-\varepsilon < H_r \leq a\}} \, dr - L_s^a \right| \right\} = 0.$$

For the proof of the next theorems, we need to extend the definition of local times to the exploration process started at an arbitrary initial value $\mu \in \mathcal{M}_f(\mathbb{R}_+)$ for which we only assume that μ is supported on $[0, a)$ for some $a > 0$. The main ingredient for the construction of local time was in fact Theorem 2.23, which remains valid, if one replaces ρ by ρ^μ, the exploration process started at μ. Therefore, we can also generalize Theorem 2.26 from which follows that, in probability,

$$L_s^a(\rho^\mu) = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_0^s 1_{\{a < H(\rho_r^\mu) < a+\varepsilon\}} \, dr.$$

For now, and the following proofs, let

$$\tau_0 := \inf\{s \geq 0 : \rho_s^\mu = 0\}$$

and denote by $\omega_j, (\alpha_j, \beta_j)$, $j \in J_{\tau_0}$ the excursions and excursion intervals of $X - I$ before time τ_0. Set $r_j := H(k_{-I_{\alpha_j}}\mu)$, and denote by $L_\cdot^{a-r_j}(\omega_j)$ the local time of the height process at level $a - r_j$ constructed from the excursion ω_j. Then one can show (we omit the proof and refer to [LGD]) that

$$L_{\tau_0}^a(\rho^\mu) = \sum_{j \in J_{\tau_0}} L_{\beta_j - \alpha_j}^{a-r_j}(\omega_j). \tag{2.65}$$

Now, we are ready to come to the main result of this chapter, namely the construction of a ψ-CSBP from a ψ-height process via a Ray-Knight Theorem. Therefore recall that T_x was defined to be the first hitting time of $-x$ by the underlying Lévy process X, or equivalently the first hitting time of x by the local time at 0 of the height process.

Theorem* 2.28** ***(Le Gall, Le Yan, Duquèsne)
Let $x > 0$, then the process $\{L_{T_x}^a : a \geq 0\}$ is a ψ-CSBP adapted to the filtration $\{\mathcal{H}_a\}$ with start in x.

Example. Let the underlying Lévy process $\{X_t : t \geq 0\}$ be standard Brownian motion. We already computed that $\psi(\lambda) = \frac{1}{2}\lambda^2$ and that the height process H is distributed as reflected linear Brownian motion. Then $\{L_{T_x}^a : a \geq 0\}$ is a ψ-CSBP, in this case an example from Chapter 1, the so called Feller diffusion. This is known to be the classical first Ray-Knight Theorem.

To prove Theorem 2.28 we need the following Lemma which can be proved very similarly to Lemma 2.21. Therefore we omit the proof and refer to [LGD]. Recall that the random measure J_a is defined by

$$J_a(dr) = 1_{[0,a]}\, dU_r,$$

where $\{U_t : t \geq 0\}$ is a subordinator with Laplace exponent

$$\tilde{\psi}(\lambda) - \alpha = \frac{\psi(\lambda)}{\lambda} - \alpha.$$

***Lemma* 2.29** *Let $F : \mathcal{D} \to \mathbb{R}$ be any nonnegative, measurable functional, then we have for all $a \geq 0$*

$$N\left[\int_0^\sigma F(\hat{X}^{(t)}_{r\wedge s} : r \geq 0)\, dL^a_s\right] = \mathbb{E}\left\{1_{\{L^{-1}(a)<\infty\}} F(X_{r\wedge L^{-1}(a)} : r \geq 0)\right\}.$$

So, in particular we have for $\Phi : \mathcal{M}_f(\mathbb{R}_+) \to \mathbb{R}$ which are nonnegative and measurable

$$N\left[\int_0^\sigma \Phi(\rho_s)\, dL^a_s\right] = e^{-\alpha a}\mathbb{E}\{\Phi(J_a)\}.$$

Proof of Theorem 2.28: The idea of the proof is the following: We define for $a > 0$

$$u_a(\lambda) := N\left[1 - e^{-\lambda L^a_\sigma}\right], \tag{2.66}$$

where σ denotes, as usual, the length of an excursion. Then we proceed in two steps. First, we show that for all $a > 0, \lambda > 0$ the function $\{u_a(\lambda) : a \geq 0\}$ solves the equation

$$u_a(\lambda) + \int_0^a \psi(u_s(\lambda))ds = \lambda. \tag{2.67}$$

Then, we show that

$$\mathbb{E}\left\{ e^{-\lambda L^b_{T_x}} \,\middle|\, L^a_{T_x} \right\} = \exp\left(-L^a_{T_x} u_{b-a}(\lambda)\right), \tag{2.68}$$

and we are done, because by Theorem 1.26, $\{L^a_{T_x} : a \geq 0\}$ is a ψ-CSBP which is of course uniquely determined.

Before doing this, let us first see that for each $a \geq 0$, $L^a_{T_x}$ is $\mathcal{H}_a$-measurable. For $a = 0$ this is trivial because $L^0_{T_x} = x$ is constant by our construction. If $a > 0$, recall that we can use Corollary 2.27 to approximate $L^a_{T_x}$ in probability,

$$L^a_{T_x} = \lim_{\varepsilon\downarrow 0} \frac{1}{\varepsilon}\int_0^{T_x} 1_{\{a-\varepsilon<H_r\leq a\}}\, dr.$$

With $T^a_x := \inf\{s \geq 0 : X_{\tilde{\tau}^a_s} = -x\}$ we have using a change of variable,

$$\int_0^{T_x} 1_{\{a-\varepsilon<H_s\leq a\}}\, ds = \int_0^{T^a_x} 1_{\{a-\varepsilon<H_{\tilde{\tau}^a_r}\leq a\}}\, dr$$

and the right hand side is measurable with respect to $\mathcal{H}_a$. Plugging this in the approximation result yields also the $\mathcal{H}_a$- measurability of $L^a_{T_x}$.

Now, we can go ahead in proving that $u_a(\lambda)$ solves equation (2.67). For any $\mu \in \mathcal{M}_f(\mathbb{R}_+)$ supported on $[0,a)$ define

$$F(\mu) := \mathbb{E}\left\{\exp(-\lambda L^a_{\tau_0}(\rho^\mu))\right\}.$$

By an elementary integration and the strong Markov property of the exploration process ρ under the excursion measure, we get that for every $a > 0$,

$$\begin{aligned} u_a(\lambda) &= N\left[1 - e^{-\lambda L^a_\sigma}\right] \\ &= N\left[\int_0^\sigma \lambda \exp(-\lambda(L^a_\sigma - L^a_s))\, dL^a_s\right] \\ &= \lambda N\left[\int_0^\sigma F(\rho_s)\, dL^a_s.\right] \end{aligned} \tag{2.69}$$

Recall that by our extension of the local times for the exploration process with start at an arbitrary initial value we have that

$$L^a_{\tau_0}(\rho^\mu) = \sum_{j \in J_{\tau_0}} L^{a-r_j}_{\beta_j - \alpha_j}(\omega_j),$$

where $r_j = H(k_{-I_{\alpha_j}}\mu)$. Also recall, that we only consider the excursions of $X - I$ before time τ_0 indexed by J_{τ_0}. Moreover,

$$\begin{aligned} \tau_0 &= \inf\{t \geq 0 : \rho^\mu_t = 0\} \\ &= \inf\{t \geq 0 : [k_{-I_t}\mu, \rho_t] = 0\} \\ &= \inf\{t \geq 0 : -I_t = \|\mu\|\}. \end{aligned} \tag{2.70}$$

As $-I$ is the local time at 0 of $X - I$, the local times and excursions

$$\Delta := \left\{(L_{\alpha_j}, \omega_j) : j \in J_{\tau_0}\right\} \subseteq \mathbb{R}_+ \times \mathcal{D},$$

form a Poisson point process with intensity measure $1_{[0,\|\mu\|]}\, dl\, dN$. Hence, we can apply Campbells theorem to compute,

$$\begin{aligned} F(\mu) &= \mathbb{E}\left\{\exp(-\lambda L^a_{\tau_0}(\rho^\mu))\right\} \\ &= \mathbb{E}\left\{\exp\left(-\lambda \sum_{j \in J_{\tau_0}} L^{a-r_j}_{\beta_j - \alpha_j}(\omega_j)\right)\right\} \\ &= \exp\left(-\int N\left[1 - \exp(-\lambda L^{a-H(k_u\mu)}_\sigma)\right] 1_{[0,\|\mu\|]}(u)\, du\right). \end{aligned} \tag{2.71}$$

As $H(k_u\mu) = \sup\operatorname{supp}\{k_u\mu\}$, we get by change of variable,

$$F(\mu) = \exp\left(-\int N[1 - \exp(-\lambda L^{a-r}_\sigma)]\, \mu(dr)\right). \tag{2.72}$$

Plugging this into (2.69) and the use of Lemma 2.29 yields

$$\begin{aligned} u_a(\lambda) &= \lambda N\left[\int_0^\sigma \exp\left(-\int N[1-\exp(-\lambda L_\sigma^{a-r})]\,\rho_s(dr)\right)\,dL_s^a\right] \\ &= \lambda N\left[\int_0^\sigma \exp\left(-\int u_{a-r}(\lambda)\,\rho_s(dr)\right)\,dL_s^a\right] \\ &= \lambda e^{-\alpha a}\mathbb{E}\left\{\exp\left(-\int u_{a-r}(\lambda)\,J_a(dr)\right)\right\}, \end{aligned} \tag{2.73}$$

where J_a is the random element of $\mathcal{M}_f(\mathbb{R}_+)$ which is given by $J_a(dr) = 1_{[0,a]}(r)\,dU_r$ and U is a subordinator with Laplace exponent $\tilde{\psi}(\lambda) - \alpha$. Hence,

$$\begin{aligned} u_a(\lambda) &= \lambda e^{-\alpha a}\mathbb{E}\left\{\exp\left(-\int_0^a u_{a-r}(\lambda)\,dU_r\right)\right\} \\ &= \lambda e^{-\alpha a}\exp\left(-\int_0^a (\tilde{\psi}(u_{a-r}(\lambda)) - \alpha)\,dr\right) \\ &= \lambda \exp\left(-\int_0^a \tilde{\psi}(u_{a-r}(\lambda))\,dr\right), \end{aligned} \tag{2.74}$$

which solves equation (2.67). So, we succeeded in the first step of the proof.

Let us now proof the second step. Because $L_{T_x}^a$ is $\mathcal{H}_a$-measurable, it is enough to show that

$$\mathbb{E}\left\{\exp(-\lambda L_{T_x}^b) \mid \mathcal{H}_a\right\} = \exp(-L_{T_x}^a u_{b-a}(\lambda)). \tag{2.75}$$

Denote by $\tilde{L}_s^c$ the local time of $H_s^a = H(\rho_s^a)$ and recall that $A_s^a = \int_0^s 1_{\{H_r > a\}}\,dr$ is the clock that runs only if H is above level a. Recall that $l^a(\cdot)$ denotes the local time at 0 of ρ^a. Then using the definitions,

$$\begin{aligned} L_{T_x}^b &= l^b\left(\int_0^{T_x} 1_{\{H_r > b\}}\,dr\right) \\ &= l^{b-a}\left(\int_0^{A_{T_x}^a} 1_{\{H_r^a > b-a\}}\,dr\right) \\ &= \tilde{L}_{A_{T_x}^a}^{b-a}. \end{aligned} \tag{2.76}$$

Let $T_r^a := \inf\{t \geq 0 : l_t^a > r\}$. Then we have

$$l_{A_{T_x}^a}^a = L_{T_x}^a,$$

by the definition of $L_{T_x}^a$. Moreover, as

$$l_t^a > l_{A_{T_x}^a}^a$$

for all $t > A_{T_x}^a$ it follows that

$$\begin{aligned} A_{T_x}^a &= \inf\left\{t \geq 0 : l_t^a > L_{T_x}^a\right\} \\ &= T_{L_{T_x}^a}^a, \end{aligned} \tag{2.77}$$

which leads to

$$\begin{aligned}
\mathbb{E}\{\exp(-\lambda L^b_{T_x}) \mid \mathcal{H}_a\} &= \mathbb{E}\left\{\exp\left(-\lambda \tilde{L}^{b-a}_{A^a_{T_x}}\right) \middle| \mathcal{H}_a\right\} \\
&= \mathbb{E}\left\{\exp\left(-\lambda \tilde{L}^{b-a}_{T^a_{L^a_{T_x}}}\right) \middle| \mathcal{H}_a\right\} \\
&= \mathbb{E}\left\{\exp\left(-\lambda \tilde{L}^{b-a}_{T^a_{L^a_{T_x}}}\right)\right\}, \qquad (2.78)
\end{aligned}$$

since the process $\{\tilde{L}^{b-a}_{T^a_u} : u \geq 0\}$ is a functional of ρ^a and therefore independent of $\mathcal{H}_a$. Now, (2.76) implies that

$$\mathbb{E}\{\exp(-\lambda L^b_{T_x}) \mid \mathcal{H}_a\} = \mathbb{E}\left\{\exp(-\lambda L^{b-a}_{T_{L^a_{T_x}}})\right\}. \qquad (2.79)$$

As we can apply the mapping and Campbell theorem to the Poisson point process of local times and excursions

$$\{(L_{\alpha_i}, \omega_i) : i \in I\},$$

of $X - I$, we get

$$\begin{aligned}
\mathbb{E}\{\exp(-\lambda L^a_{T_x})\} &= \mathbb{E}\left\{\exp\left(-\lambda \sum_{i:L_{\alpha_i} \leq x} L^a_{T_x}(\omega_i)\right)\right\} \\
&= \exp\left(\iint \left(e^{-\lambda L^a_\sigma} - 1\right) 1_{\{l \leq x\}}\, dl\, dN\right) \\
&= \exp(-xN[1 - \exp(-\lambda L^a_\sigma)]) \\
&= \exp(-x u_a(\lambda)),
\end{aligned}$$

with which we can complete the proof by

$$\begin{aligned}
\mathbb{E}\{\exp(-\lambda L^b_{T_x}) \mid \mathcal{H}_a\} &= \mathbb{E}\left\{\exp\left(-\lambda L^{b-a}_{T_{L^a_{T_x}}}\right)\right\} \\
&= \exp\left(-L^a_{T_x} u_{b-a}(\lambda)\right).
\end{aligned}$$

□

Note, that we have particularly shown that

$$u_a(\lambda) := N\left[1 - e^{\lambda L^a_\sigma}\right],$$

solves the integral equation

$$u_a(\lambda) + \int_0^a \psi(u_s(\lambda))\, ds = \lambda. \qquad (2.80)$$

We will use this fact in Section 2.5 and in Chapter 5.

What have we done so far? In fact, we know that there is a natural bijection of a subclass of Lévy processes $\mathcal{L}$ and the (sub)critical CSBP $\mathcal{C}$ just via the Laplace exponents respectively the branching mechanism ψ. Then we have seen a probabilistic version of such a bijection by a direct construction of a ψ-CSBP from a certain local time functional of a ψ-Lévy process, the so called height-process:

$$\begin{tikzcd}
& \boxed{\psi\text{-Lévy process}} \arrow[dl, "\text{via } \psi"] \arrow[dr] & \\
\boxed{\psi\text{-CSBP}} & & \boxed{\psi\text{-height process}} \arrow[ll, "\text{via Ray-Knight}"]
\end{tikzcd}$$

In the next section, we give an interpretation of this height process as coding the genealogy of such a CSBP.

2.3 The genealogy given by the height process

We have seen in the last chapter , that one can use the height process to construct a CSBP in terms of a generalized Ray-Knight theorem. This construction is in particular similar to the discrete setting, where we constructed a Galton-Watson process from a given tree just by counting the number of particles on each tree level. Of course also in the continuous setting the height process contains more information than the CSBP, which we will now use to define the genealogical structure of a CSBP.

Let $\{H_t : t \geq 0\}$ be a continuous ψ-height process, respectively its lower semicontinuous version which is crucial in the sequel. Consider an excursion $H.$ of the height process away from 0 and again denote by σ the length of this excursion. Then $H.$ encodes a ***continuum random tree*** $\mathcal{T}_H$ by the following rules

- each $s \in [0, \sigma]$ corresponds to a ***vertex*** (or a particle) of generation H_s
- if $s \leq s' \in [0, \gamma]$, then the vertex s is called an ***ancestor*** of vertex s' if

$$H_s = \inf_{s \leq r \leq s'} H_r =: m(s, s').$$

In the left side picture, the particle s is an ancestor of s'. The right side picture shows two particles s and s' and the generation m of the most recent common ancestor of s and s'.

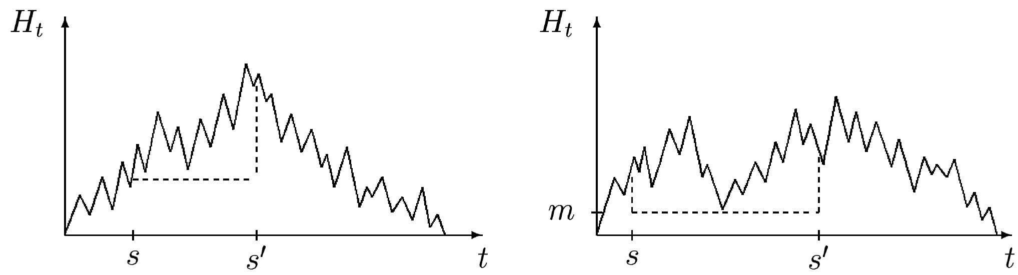


Figure 2.7: How does the height process code the genealogy?

In general, $m(s, s')$ is the generation of the last common ancestor of the particles s and s'. With

$$d(s, s') := H_s + H_{s'} - 2m(s, s') \text{ for } s, s' \in [0, \gamma],$$

$([0, \gamma], d)$ becomes a metric space on which we introduce the equivalence relation $s \sim s'$ if and only if $d(s, s') = 0$.

Definition* 2.30** *The quotient set $\mathcal{T}_H / \sim$ is called* ***continuum random tree *with respect to H.*

In this way, each excursion (α_i, β_i) of the height process codes a continuum random tree which represents the genealogical dependence of the descendents of a single particle α_i. In particular, this definition generalizes the continuum random tree of Aldous [Ald93] which coincides with the special case when H is reflected Brownian motion.

Let us make a ***very important remark:*** As one can imagine, it is difficult to deal directly with contiuum random trees. For example, the question of convergence of Galton-Watson trees towards continuum random trees. It seems to be very unclear how to make this convergence precise. But due to our coding of the genealogy, we do *not* really need to talk about trees, because the genealogical structure of Galton-Watson trees, resp. continuum random trees, is coded in the discrete and continuous height processes. These processes can be handled with the full machinery to establish for example limit theorems, which deal as a legetimation of the height process as *the* natural genealogy of CSBP (see Chapter 4). Moreover we can study paths properties of the height process which can then be interpreted as properties of CSBPs and their genealogy. A very nice example is given in the next two sections.

2.4 Continuity of the height process

We now turn back to the interesting search of a necessary and sufficient condition for the sample paths continuity of the height process. This will lead on the one hand to a nice interpretation if we think of the height process as the genealogical structure of CSBP and on the other hand it will provide a class of height processes which are possible limits in a functional convergence theorem.

Theorem* 2.31 *(Coninuity of the height process)
The height process $\{H_t : t \geq 0\}$ has continuous sample paths P-almost surely if and only if

$$\int_1^\infty \frac{1}{\psi(u)} du < \infty. \tag{2.81}$$

Remark. Recall that due to Theorem 1.29, the analytical condition for the ψ-height process to have continuous sample paths is the same as for the almost sure extinction of the associated ψ-CSBP.

***Corollary* 2.32** *The ψ-height process $\{H_t : t \geq 0\}$ has almost surely continuous sample paths if and only if the associated ψ-CSBP dies in finite time.*

For the proof of Theorem 2.31 we need the following key lemma.

***Lemma* 2.33** *For all $a \geq 0$ set*

$$v(a) := N\left[\sup_{0 \leq s \leq \sigma} H_s > a\right].$$

Then the following is true

(1) If $\int_1^\infty \frac{du}{\psi(u)} = \infty$, we have $v(a) = \infty$ for all $a > 0$.

(2) If $\int_1^\infty \frac{du}{\psi(u)} < \infty$, the function $\{v(a) : a \geq 0\}$ is determined by

$$\int_{v(a)}^\infty \frac{1}{\psi(u)} du = a. \tag{2.82}$$

Proof: Let $a \geq 0$ and recall the notation $A_t^a = \int_0^t 1_{\{H_s > a\}} ds$, for the clock that runs only if the height process is above the given level a. Using the approximation, in probability,

$$L_t = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_0^t 1_{\{a < H_s < a+\varepsilon\}}\, ds,$$

for the local time, we have that $A_t^a > 0$ if and only if $L_t^a > 0$, as

$$L_t^a = l^a\left(\int_0^t 1_{\{H_s > a\}}\, ds\right) = l^a(A_t^a).$$

Moreover, we see that $A_t^a > 0$ is equivalent to $\sup_{0\le s\le t} H_s > a$. If $\sup_{0\le s\le t} H_s > a$, then by the lower-semincontinuity of the height process, the set $\{s \le t : H_s > a\}$ is open and not empty. Hence, this set contains an open ball which implies that $A_t^a > 0$. The other implication is trivial.

As these statements are also true under the excursion measure, we get by dominated convergence

$$\begin{aligned} v(a) &= N\left[\sup_{0\le s\le\sigma} H_s > a\right] \\ &= N\left[L_\sigma^a > 0\right] \\ &= \lim_{\lambda\to\infty} N\left[1-e^{-\lambda L_\sigma^a}\right] \\ &= \lim_{\lambda\to\infty} u_a(\lambda), \end{aligned} \tag{2.83}$$

where $u_a(\lambda)$ solves (see remark after Theorem 2.28)

$$\int_{u_a(\lambda)}^{\lambda} \frac{1}{\psi(u)}\, du = a.$$

Hence we get the statement. □

Now, we are able to prove the continuity Theorem

Proof: (Theorem 2.31) Recall that T_x is the first hitting time of $-x$ of the underlying Lévy process X or equivalently, the first time when the local time at 0 of the height process reaches x. Campbells theorem applied to the Poisson point process of local times and excursions of the height process

$$\{(L_{\alpha_i}, H(\omega_i)) : i \in J\}$$

yields, (to simplify notation, $H(\omega_i)_s = H_s$)

$$\begin{aligned} P\left\{\sup_{0\le s\le T_x} H_s > a\right\} &= \lim_{\lambda\to\infty} \mathbb{E}\left\{1-\exp\left(-\lambda 1_{\{\sup_{0\le s\le T_x} H_s > a\}}\right)\right\} \\ &= \lim_{\lambda\to\infty} \mathbb{E}\left\{1-\exp -\lambda \sum_{i\in J, L_{\alpha_i}\le x} 1_{\{\sup_{0\le s\le\sigma} H_s > a\}}\right\} \\ &= 1-\exp\left(-xN\left[\sup_{0\le s\le\sigma}\omega_s > a\right]\right) \\ &= 1-\exp(-xv(a)). \end{aligned} \tag{2.84}$$

As we know by (2.33), that $v(a) = \infty$ for all $a > 0$ in the case that $\int_1^\infty \frac{1}{\psi(u)}\, du = \infty$, the height process H is almost surely unbounded on $[0, T_x]$ and thus cannot have continuous sample paths.

Assume now, that

$$\int_1^\infty \frac{1}{\psi(u)}\,du < \infty \tag{2.85}$$

holds. Our aim is to show that the height process has continuous paths almost surely. By Lemma 2.33, we have in the case that (2.85) holds, that $v(a) < \infty$ for all $a > 0$. Moreover, we have already shown in the first part of the proof that

$$P\left\{\sup_{0\le s\le T_x} H_s > a\right\} = 1 - \exp(-xv(a)).$$

Therefore, we get using that $T_x \downarrow 0$ as $x \downarrow 0$, almost surely

$$\lim_{t\downarrow 0} H_t = 0. \tag{2.86}$$

So the height process is right continuous at 0 almost surely. The continuity of H then follows, by the intermediate value theorem if we show that for every fixed interval $[a-\varepsilon, a], a > 0, \varepsilon \in (0, a]$, the number of upcrossings of this interval by the height process is almost surely finite.

Let $\gamma_0 := 0$ and define inductively

$$\delta_n := \inf\{t \ge \gamma_{n-1} : H_t \ge a\} \qquad \text{and} \qquad \gamma_n := \inf\{t \ge \delta_n : H_t \le a - \varepsilon\}.$$

Moreover, define the number of upcrossings

$$U_H[a-\varepsilon, a] := \sup\{n \in \mathbb{N} : \delta_n < T_x\}.$$

Because we need it later again, we proof that this number is finite almost surely as a Lemma which also completes the proof of Theorem 2.31. $\square$

***Lemma* 2.34** *If $\int_1^\infty \frac{du}{\psi(u)} < \infty$, then $U_H[a-\varepsilon, a] < \infty$ almost surely.*

Proof: A mimic of the proof of Lemma 2.16, shows that almost surely for every $t \ge 0$,

$$H_{\gamma_n + t} \le H_{\gamma_n} + H_t^{(\gamma_n)} \tag{2.87}$$

where $H^{(\gamma_n)}$ refers to the height process associated with the shifted Lévy process $X^{(\gamma_n)}$. Define

$$\kappa_n := \inf\{t \ge 0 : H_t^{(\gamma_n)} \ge \varepsilon\}.$$

By the strong Markov property of the underlying Lévy process X, the $\kappa_1, \kappa_2, \ldots$ form a sequence of independent identically distributed random variables. By the lower sem-incontinuity of H we have that $H_{\gamma_n} \le a - \varepsilon$. This implies with formula (2.87) that

$$\delta_{n-1} - \gamma_n \ge \kappa_n, \tag{2.88}$$

i.e. the time needed by the height process for the nth crossing of $[a-\varepsilon, a]$ is bigger than the time in which $H^{(\gamma_n)}$ reaches level ε. By (2.87) we get that $\kappa_n > 0$ almost surely as $\varepsilon > 0$. Hence δ_n tends to infinity almost surely. Because T_x is almost surely finite we get that

$$\sup\{n \in \mathbb{N} : \delta_n < T_x\} < \infty$$

almost surely, which completes the proof. $\square$

2.5 The dimension of the zerosets

Let $\{|B_t| : t \geq 0\}$ be a reflected Brownian motion, then it is well known how to compute the Hausdorff dimension of their levelsets. As the height process is distributed as a reflected Brownian motion in the case of a quadratic branching mechanism $\psi(\lambda) = \frac{1}{2}\lambda^2$, it is a very natural question to ask about the dimension of the levelsets

$$\{t : H_t = a\}$$

of an arbitrary continuous ψ-height process $\{H_t : t \geq 0\}$, a question which seems to be not treated in the literature so far.

Recall that, $-I$ is a local time for the height process in level 0 and that $H_t = 0$ if and only if $X_t - I_t = 0$. Moreover, because the underlying Lévy process X is assumed to be spectrally positive, we can write the inverse local time as

$$\begin{aligned} T_x &= \inf\{s \geq 0 : X_t \leq -x\} \\ &= \inf\{s \geq 0 : -I_t \geq x\} \end{aligned} \tag{2.89}$$

By Theorem 1.16, $\{T_x : x \geq 0\}$ is a subordinator with Laplace exponent ϕ, where ϕ is the inverse function of ψ. Hence, we have by our construction that

$$\{t : H_t = 0\} = \{T_x : x \geq 0\},$$

i.e. the zero-set of the height process agrees with the range of the inverse local time at zero. Therefore we can use the following lemma to compute the dimension of the zero-set (a proof can be found in [Ber96]):

***Lemma* 2.35** *Let $\{T_x : x \geq 0\}$ be a subordinator with Laplace exponent ϕ, then*

$$\dim\{T_x : x \geq 0\} = \sigma$$

almost surely, where $\sigma = \sup\{s > 0 : \lim_{\lambda\to\infty} \lambda^{-s}\phi(\lambda) = \infty\}$.

Using this lemma, directly yields the formula for the Hausdorff dimension of the zeroset of the height process:

Theorem* 2.36** ***(the zeroset dimension)
Let $\{H_t : t \geq 0\}$ be a ψ-height process with continuous sample paths, then almost surely,

$$\dim\{t : H_t = 0\} = \sup\left\{s > 0 : \lim_{\lambda\to\infty} \lambda^{-s}\phi(\lambda) = \infty\right\},$$

where ϕ denotes the inverse of the function ψ.

In particular, in the interesting case of stable branching mechanisms, i.e. $(\psi(\lambda) = \lambda^\alpha, \alpha \in (1,2])$, we get that $\phi(\lambda) = \lambda^{\frac{1}{\alpha}}$ and

$$\sigma = \sup\left\{s > 0 : \lim_{\lambda \to \infty} \lambda^{-s} \lambda^{\frac{1}{\alpha}}\right\} = \frac{1}{\alpha}.$$

This result also entails the well known fact, that the dimension of the zeroset of reflected Brownian motion is $1/2$ almost surely.

Because the height process is not Markovian it is *not* obvious how to extend this result for an arbitrary levelset. Nevertheless, this extension would be the basis to generalize the work of [Mö99] to get an alternative (and more general) proof for the dimension of the support of super Brownian motion as it is done in [Del96].

Chapter 3

Biodiversity of CSBP

Pictorially, we say that a CSBP has *finite biodiversity*, if all particles alive at any given time are descendants of only *finitely many* individuals alive in any fixed earlier generation - with respect to the genealogy provided by the height process. In this chapter, we want to show that the concept of finite biodiversity is equivalent to the almost sure extinction of the CSBP in finite time and thus to the continuity of the height process.

3.1 A characterization of finite biodiversity

Let $\{Z_a : a \geq 0\}$ be a ψ-CSBP from the class $\mathcal{C}$. Denote by $\{H_t : t \geq 0\}$ the associated height process that codes the genealogical structure of Z. Recall that

$$Z_a = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_0^{T_x} 1_{\{a < H_s < a+\varepsilon\}} ds,$$

in probability, where T_x is the first hitting time of $-x$ of the underlying ψ-Lévy process, respectively the first time, where the local time in level 0 of the height process reaches x.

Moreover, recall that we call s an *ancestor* of s' if

$$H_s = \inf_{s \leq r \leq s'} H_r,$$

and we denote by $\mathcal{A}(s)$ the set of all ancestors of s. Let

$$H^{-1}(a) := \{s \in [0, T_x] : H_s = a\} \text{ and}$$

$$\mathcal{A}^a_{a-\varepsilon} := \{s \in H^{-1}(a-\varepsilon) : \text{ there exists } s' \in H^{-1}(a) \text{ such that } s \in \mathcal{A}(s')\}.$$

Then $|\mathcal{A}^a_{a-\varepsilon}|$ represents the number of ancestors of generation a living in generation $a - \varepsilon$.

Definition* 3.1** *A ψ-CSBP $\{Z_a : a \geq 0\}$ is said to have* ***finite biodiversity*, with respect to the genealogy provided by the ψ-height process, if almost surely for all $a > 0$ and all $\varepsilon \in (0, a]$, we have $|\mathcal{A}^a_{a-\varepsilon}| < \infty$.*

So, finite biodiversity pictorially states that all particles alive at any given time have only finitely many ancestors in the generations before. The main result of this chapter is the following theorem which gives an analytical condition and relates the finite biodiversity of a CSBP to his almost sure extinction and to the continuity of the associated height process.

Theorem* 3.2 *(Finite biodiversity charaterization)
A ψ-CSBP $\{Z_a : a \geq 0\}$ has finite biodiversity if and only if

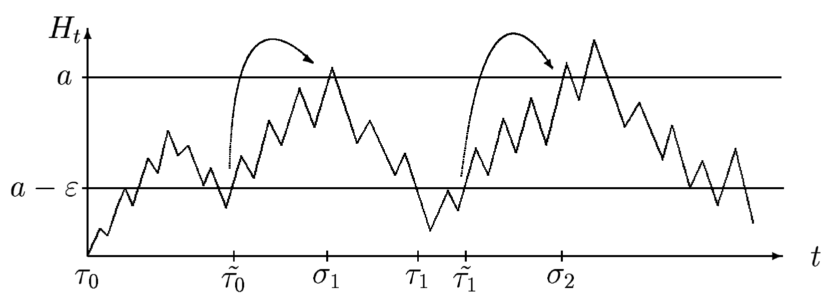
$$\int_1^\infty \frac{1}{\psi(\lambda)}\, d\lambda < \infty.$$

In particular, finite biodiversity is equivalent to the almost sure extinction of Z in finite time and to the property that the associated ψ-height process has continuous paths almost surely.

Before we start with the proof of Theorem 3.2, we fix some notation. Define the following random times $\tau_0 := 0$, $\sigma_n := \inf\{t > \tau_{n-1} : H_t \geq a\}$ and $\tau_n := \inf\{t > \sigma_n : H_t \leq a - \varepsilon\}$ for all $n \geq 1$. Then, $\{\tau_0, \sigma_1\}, \{\tau_1, \sigma_2\}, \ldots$ describe the *upcrossings* of the interval $[a - \varepsilon, a]$ by the height process and

$$U_H[a - \varepsilon, a] := \sup\{n \in \mathbb{N} : \sigma_n \leq T_x\}$$

denotes the *number* of such upcrossings. Note, that by the lower semicontinuity of the height process we have that $H_{\tau_n} \leq a - \varepsilon$ for all $n \geq 1$.



In this picture, $\{\tau_0, \sigma_1\}$ and $\{\tau_1, \sigma_2\}$ describe crossings of the interval $[a - \varepsilon, a]$ by the height process $\{H_t : t \geq 0\}$. Note, that σ_1 and σ_2 both live in generation a. Moreover, τ_0 is an ancestor of σ_1 living in generation 0 and τ_1 lives in generation $a - \varepsilon$, but is *not* an ancestor of σ_2. Nevertheless, both intervals $[\tau_0, \sigma_1]$ and $[\tau_1, \sigma_2]$ contain $\tilde{\tau}_0$ and $\tilde{\tau}_1$ living in generation $a - \varepsilon$ such that $\tilde{\tau}_0$ is an ancestor of σ_1 and $\tilde{\tau}_1$ is an ancestor of σ_2.

The following lemma gives the main idea how to prove Theorem 3.2.

Lemma* 3.3 *(Upcrossing lemma)
A ψ-CSBP $\{Z_a : a \geq 0\}$ has finite biodiversity if and only if almost surely for all $a > 0$ and $\varepsilon \in (0, a]$ the number of upcrossings of $[a-\varepsilon, a]$ by the associated height process $U_H[a-\varepsilon, a]$ is finite.

Proof: Let $a > 0$ and $\varepsilon \in (0, a]$. As $U_H[a-\varepsilon, a] \geq |\mathcal{A}^a_{a-\varepsilon}|$, $U_H[a-\varepsilon, a] < \infty$ implies that $|\mathcal{A}^a_{a-\varepsilon}| < \infty$.
Now assume that $U_H[a, a-\varepsilon] = \infty$. Let $n \in \mathbb{N}$ and consider the upcrossing $\{\tau_n, \sigma_{n+1}\}$. By the lower semicontinuity of the height process we know that $H_{\tau_n} \leq a - \varepsilon$. Furthermore, we have *either* $\tau_n < \sigma_{n+1}$ with $H_{\sigma_{n+1}} \geq a$ *or* we can find a $\sigma' > \sigma_{n+1}$ such that $H_{\sigma'} \geq a$. To simplify notation, let $\sigma' := \sigma_{n+1}$ even in the first case and in any case $\tau := \tau_n$. The idea of the proof is use the intermediate value property of the height process to construct $\tilde{\tau}, \tilde{\sigma} \in [\tau, \sigma']$ such that $H_{\tilde{\tau}} = a - \varepsilon$, $H_{\tilde{\sigma}} = a$ and $\tilde{\tau}$ is an *ancestor* of $\tilde{\sigma}$. Define

$$A := \{t \in [\tau, \sigma'] : H_t \leq a - \varepsilon\}.$$

As $\tau \in A$, we see that A is *not* empty. As A is bounded, we can define $\tau' := \sup A$. The lower semicontinuity of the height process implies that $H_{\tau'} \leq a - \varepsilon$. Moreover, by definition of τ' we see that $H_t > a - \varepsilon$ for all $t \in (\tau', \sigma']$. Hence we can use the intermediate value property of H to find first $\tilde{\tau} \in [\tau', \sigma']$ such that $H_{\tilde{\tau}} = a - \varepsilon$ and then $\tilde{\sigma} \in [\tilde{\tau}, \sigma']$ with $H_{\tilde{\sigma}} = a$. As $H_t > a - \varepsilon$ for all $t \in (\tilde{\tau}, \tilde{\sigma}]$, we see that

$$H_{\tilde{\tau}} = \inf_{\tilde{\tau} \leq s \leq \tilde{\sigma}} H_s.$$

Hence $\tilde{\tau}$ is an ancestor of $\tilde{\sigma}$ which completes the proof of the lemma. $\square$

Now we are ready to start with the proof of the characterization theorem.

Proof of Theorem 3.2: Recall from Lemma 2.34 that

$$\int_1^\infty \frac{1}{\psi(u)}\, du < \infty$$

implies that $U_H[a-\varepsilon, a] < \infty$, which implies finite biodiversity by the previous lemma. But we still have to prove the converse. Therefore, assume that

$$\int_1^\infty \frac{1}{\psi(u)}\, du = \infty.$$

Recall that in this case, the time of extinction of the associated ψ-CSBP is ∞ almost surely. Hence, the Ray-Knight Theorem implies that for all $x > 0$, almost surely

$$\sup_{0 \leq s \leq T_x} H_s = \infty.$$

Since $T_x \downarrow 0$ almost surely as $x \downarrow 0$, it follows that

$$P\left\{\sup_{0 \leq s \leq a} H_s = \infty\right\} = 1$$

for all $a > 0$. Now let $0 < a < b$ and let $H^{(a)}$ be the height process associated with the shifted Lévy process $X^{(a)}$. Recall that $H_{a+t} \geq H_t^{(a)}$ almost surely for all $t \geq 0$. Hence,

$$\sup_{a \leq s \leq b} H_s \geq \sup_{0 \leq s \leq b-a} H_s^{(a)} = \infty$$

almost surely. So, in particular

$$P\left\{ \sup_{a \leq s \leq b} H_s \neq \infty \text{ for all rationals } a < b \right\} = 1,$$

and hence, almost surely for all $a < b$,

$$\sup_{a \leq s \leq b} H_s = \infty. \tag{3.1}$$

This is in particular an alternative proof to show that H cannot be continuous in this case. But our aim here is now to show that we *cannot* have finite biodiversity.

Assume that there is a set of full P-measure such that for all $a > 0$ and $\varepsilon \in (0, a]$ we have $U_H[a - \varepsilon, a] < \infty$. Denote by $\{\tau, \sigma\}$ such an upcrossing, i.e. $H_\tau \leq a - \varepsilon$ and $\sigma = \inf\{t > \tau : H_t \geq a\}$. Note, that $\tau < \sigma$, because otherwise, we could find a stricly monotone sequence $\tilde{\sigma}_n \downarrow \sigma$ such that $\sigma_n > \sigma$ and $H_{\tilde{\sigma}_n} \geq a$. Moreover, by the intermediate value property, we can assume that $H_{\tilde{\sigma}_n} = a$. Using (3.1) and again the intermediate value property of the height process, we can find a $\delta > 0$, such that for every $n \geq 1$ we could find a $\tilde{\gamma}_n \in [\tilde{\sigma}_n, \tilde{\sigma}_{n+1}]$ with $H_{\tilde{\gamma}_n} \geq a + \delta$. Hence, $U_H[a, a + \delta] = \infty$ which contradicts our assumption.
Therefore we can assume that $(\tau, \sigma) \neq \emptyset$ and (3.1) leads to

$$\sup_{\tau \leq s \leq \sigma} H_s = \infty.$$

Using the intermediate value property, we can find $\tilde{\sigma} \in [\tau, \sigma]$ such that $H_{\tilde{\sigma}} = a$, which *contradicts* our choice of σ. Hence our assumption that $U_H[a - \varepsilon] < \infty$ for all $a > 0$ and all $\varepsilon \in (0, a]$ must be wrong and we cannot have finite biodiversity which completes the proof. $\square$

Corollary* 3.4 *(Finite biodiversity 01-law)
Let $\{Z_t : t \geq 0\}$ be a ψ-CSBP, then

$$P\{Z \textit{ has finite biodiversity}\} \in \{0, 1\}.$$

3.2 The distribution of the number of ancestors

Let $\{Z_a : a \geq 0\}$ be a ψ-CSBP of finite biodiversity, i.e. we have almost surely for all $a > 0$ and all $\varepsilon \in (0, a]$, that

$$|\mathcal{A}_{a-\varepsilon}^a| < \infty.$$

We will now prove that in this case, the random variables $|\mathcal{A}^a_{a-\varepsilon}|$ are *Poisson* distributed. Recall, that we denote by $\{L^0_t : t \geq 0\}$ the local time of the height process at level 0 and denote by

$$T_x = \inf\{s > 0 : L^0_t = x\},$$

it's right continuous inverse.

***Theorem* 3.5** *Let $\{Z_a : a \geq 0\}$ be a ψ-CSBP started in $Z_0 = 1$ with finite biodiversity. Then for every $a > 0$ and every $\varepsilon \in (0, a]$, we have*

$$P\left\{ |\mathcal{A}^a_{a-\varepsilon}| = k \,\middle|\, Z_{a-\varepsilon} > 0 \right\} = e^{-\lambda}\frac{\lambda^k}{k!}, \qquad \textit{for all } k \geq 0,$$

where $\lambda := \lim_{n\to\infty} np_n(\varepsilon)$ and $p_n(\varepsilon)$ is the probability that the height process reaches level ε in the time interval $[0, T_{1/n}]$.

Proof: Let $a > 0$ and consider at first the case $\varepsilon = a$. In particular, the conditioning in the assertion is void. For all $n \in \mathbb{N}$ and $i = 1, \ldots, n$ consider,

$$T_{i/n} = \inf\left\{ t > 0 : L^0_t = \frac{i}{n} \right\}.$$

In particular, we have $H_{T_{i/n}} = 0$ for all $n, i \leq n$. Now define

$$B^n_i := \begin{cases} 1 & \text{if there is an } s \in \left(T_{\frac{i-1}{n}}, T_{\frac{1}{n}}\right] \text{ such that } H_s = a \\ 0 & \text{otherwise.} \end{cases}$$

Since we know from Chapter 2, that the height process starts anew after a zero, $B^n_1, \ldots, B^n_n$ are independent identically distributed Bernoulli variables with

$$P\{B^n_i = 1\} =: p_n(a).$$

Hence, their sum

$$A_n := \sum_{i=1}^n B^n_i$$

is binomial $(n, p_n(a))$ distributed. Because we assume finite biodiversity, we have that $U_H[0, a] < \infty$ almost surely and therefore, there is an almost surely finite random variable $N(a)$ such that each interval $[T_{(i-1)/N(a)}, T_{i/N(a)}]$ contains at most one crossing of $[0, a]$ by the height process H. Therefore

$$P\{A_{N(a)+k} = A_{N(a)} \text{ for all } k \in \mathbb{N}\} = 1$$

and therefore $A_n = |\mathcal{A}^a_0|$ for eventually all n almost surely. In particular, A_n converges to $|\mathcal{A}^a_0|$ almost surely, and we also have for all $k \in \mathbb{N}$

$$\lim_{n\to\infty} P\{A_n = k\} = P\{|\mathcal{A}^a_0| = k\}.$$

To simplify the notation let $p_n = p_n(a)$ and we compute, using Euler's formula,

$$\begin{aligned}
\lim_{n\to\infty} P\{A_n = k\} &= \lim_{n\to\infty} \binom{n}{k} p_n^k (1-p_n)^{n-k} \\
&= \lim_{n\to\infty} \frac{1}{k!}\left(1-\frac{np_n}{n}\right)^{n-k} p_n^k \frac{n!}{(n-k)!} \\
&= \lim_{n\to\infty} \frac{1}{k!}\left(1-\frac{np_n}{n}\right)^{n-k} p_n^k \prod_{j=1}^{k}(n-k+j) \\
&= \lim_{n\to\infty} \frac{1}{k!}\left(1-\frac{np_n}{n}\right)^{n-k} \prod_{j=1}^{k}(n-k+j)p_n \\
&= \frac{1}{k!}\exp\left(-\lim_{n\to\infty} np_n\right)\left(\lim_{n\to\infty} np_n\right)^k .
\end{aligned}$$

In particular, $\lim np_n$ exists and must be finite because otherwise we could *not* have finite biodiversity. In the case when $\varepsilon < a$, recall that

$$H_t^{a-\varepsilon} := H(\rho_t^{a-\varepsilon}),$$

which is the process obtained by glueing together the upward excursions above level $a-\varepsilon$, has the same distribution as the original height process. Hence, the assertion, conditioned that the CSBP $\{Z_t : t \geq 0\}$ survives at least up to time $a-\varepsilon$, follows using the first part of the proof. □

Similar results could also be achieved using excursion theory (see e.g. Lemma 5.3). To finish this chapter, consider the following interesting example:

Example: Let us consider the Feller diffusion. Then

$$L_0^{-1}\left(\frac{1}{n}\right) = T_{1/n}$$

where $\{T_x : x \geq 0\}$ is the inverse local time of reflected Brownian motion at 0 i.e. the stable subordinator of index 1/2. Then we can compute (using a formula of [BS96])

$$p_n(a) = P\left\{\sup_{0\leq t\leq T_{1/n}} |B_t| \geq a\right\} = 1-\exp\left(-\frac{1}{2an}\right).$$

Using this, we can easily compute the parameter

$$\lim_{n\to\infty} np_n(a) = \frac{1}{2a}.$$

Hence, the number of ancestors $\mathcal{A}_0^a$ of generation a living in the starting generation is Poisson distributed with parameter $\frac{1}{2a}$. In particular, this result coincides with the well known fact, that the number of excursions of reflected Brownian motion that reach level a is Poisson distributed (see e.g. [RW00b]).

Chapter 4

Limit theorems

4.1 Motivation

It is well known for several years that a suitably rescaled sequence of Galton-Watson processes converges in the sense of the finite dimensional marginals towards a CSBP. This was first shown by Lamperti in 1967 (see [Lam67]).

At first, we recall and fix some notation. Let $\{Z_t : t \geq 0\}$ be a ψ-CSBP from the class $\mathcal{C}$ started at $Z_0 = 1$. Moreover, denote by $\{X_t : t \geq 0\}$ the associated ψ-Lévy process from the class $\mathcal{L}$. For a sequence (μ_p) of (sub)critical probability measures on $\mathbb{N}_0$, we denote by $\{G_n^p : n \in \mathbb{N}_0\}$ the associated μ_p-Galton-Watson processes started at $G_0^p = p$. Finally, we let $\{W_n^p : n \in \mathbb{N}_0\}$ be a sequence of random walks on $\mathbb{Z}$ starting at 0 with increment distribution $\nu_p(k) = \mu_p(k+1)$ for $k = -1, 0, 1, \ldots$.

The following theorem is a result due to Grimvall (see [Gr74] or [LGD]) which generalizes Lampertis work.

Theorem* 4.1** ***(GW Convergence) *Let $(c_p)_{p\in\mathbb{N}}$ be a nondecreasing sequence of positive integers converging to ∞. Then, the convergence in distribution on the Skorokhod space $\mathcal{D}$*

$$\left\{ \frac{1}{p} G^p_{[c_p t]} : t \geq 0 \right\} \xrightarrow[p\to\infty]{(d)} \{Z_t : t \geq 0\} \tag{4.1}$$

holds if and only if

$$\left\{ \frac{1}{p} W^p_{[p c_p t]} : t \geq 0 \right\} \xrightarrow[p\to\infty]{(d)} \{X_t : t \geq 0\}. \tag{4.2}$$

Knowing Theorem 4.1, it is a natural question to ask if there is a similar result for the convergence of (suitable rescaled) discrete height processes towards continuous height processes. This would give a certain *a-posteriori legitimation* of our genealogical model in terms of the continuous height process, i.e. if we think of a tree as the natural way to code a genealogical structure in the discrete case and these trees (coded in the discrete height process) converge towards a certain process, then, it is very natural to think of this process as the coding of a continuous tree.

The plan for this chapter is to prove first the convergence of the *finite dimensional marginals* of the suitably rescaled discrete height processes (see [LGLY98a]). As a second step we then prove the *tightness* under the weakest technical conditions (see [LGD]) to get a functional convergence theorem.

4.2 Convergence of the finite-dimensional marginals

Let us denote throughout the chapter by $\{H_n^p : n \in \mathbb{N}_0\}$ the discrete μ_p-height processes associated with the μ_p-Galton-Watson processes $\{G_n^p : n \in \mathbb{N}_0\}$. As usual, we denote by $\{H_t : t \geq 0\}$ the ψ-height process corresponding to the ψ-CSBP $\{Z_t : t \geq 0\}$.

Theorem* 4.2 *(fd-Convergence)
Under either one of the conditions (4.1) or (4.2), we have also

$$\left\{\frac{1}{c_p} H^p_{[pc_pt]} : t \geq 0\right\} \underset{p\to\infty}{\overset{(fd)}{\longrightarrow}} \{H_t : t \geq 0\}, \tag{4.3}$$

where the convergence holds in the sense of weak convergence of the finite dimensional distributions.

Proof: Let $\{H_k^p : k \geq 1\}$ be the sequence of discrete height processes associated with an independent sequence of GW-trees with offspring distribution μ_p. Recall that due to Theorem 2.5

$$H_k^p = \#\left\{j \in \{0, \dots, k-1\} : W_j^p = \inf_{j \leq l \leq k} W_l^p\right\}. \tag{4.4}$$

Since the time reversed random walk has the same distribution as W in analogy to Lemma 2.7, we have that for all $p \geq 1$ and $k \in \mathbb{N}$ that the random variable

$$\Lambda_k^p := \#\left\{j \in \{1, \dots, k\} : W_j^p = \sup_{0 \leq l \leq j} W_l^p\right\}$$

has the same distribution as the discrete height process H_k^p at time k. Denote by L_t the local time at level 0 of the reflected Lévy process $S - X$. Without loss of generality, we can assume, by Skorokhod embedding, that the convergence

$$\left\{\frac{1}{p} W^p_{[pc_pt]} : t \geq 0\right\} \underset{p\to\infty}{\longrightarrow} \{X_t : t \geq 0\}$$

holds almost surely in the sense of Skorokhods topology in $\mathcal{D}$ (see e.g. [RW00a, p.215]). Hence, to complete the proof of the theorem, it will be enough to show that

$$\frac{1}{c_p} \Lambda^p_{[pc_pt]} \overset{p\to\infty}{\longrightarrow} L_t \tag{4.5}$$

in probability for all fixed $t > 0$, because this would imply that

$$\frac{1}{c_p} H^p_{[pc_pt]} \stackrel{p\to\infty}{\longrightarrow} \hat{L}_t^{(t)} = H_t$$

in probability for all $t > 0$. Then the convergence of the finite dimensional distributions follows, as fd-convergence is weaker than the convergence in probability of the one dimensional marginals.

So, we still have to prove equation (4.5). In the case when X is a Brownian motion, then, (4.5) automatically holds true. To prove this relation in the non-Brownian case, which is in fact the key and the most difficult part of the proof of the convergence theorem, we need the following tool from the book of Jean Jacod [Ja85]. Recall that $\{X_t : t \geq 0\}$ is a Lévy process with Laplace exponent ψ. Moreover, recall that ψ and hence, its parameters α, β, π determine the law of X.

***Lemma* 4.3** *Let f_0 be a truncation function, i.e. a bounded, continuous function $f_0 : \mathbb{R} \to \mathbb{R}$, such that $f_0(x) = x$ for all x in a neighbourhood of 0. Then,*

$$\left\{ \frac{1}{p} W^p_{[pc_pt]} : t \geq 0 \right\} \xrightarrow[p\to\infty]{(d)} \{X_t : t \geq 0\} \tag{4.6}$$

if and only if the following three conditions are satisfied:

$$\text{(C1)} \quad \lim_{p\to\infty} pc_p \sum_{k=-1}^{\infty} f_0\left(\frac{k}{p}\right) \nu_p(k) = -\alpha + \int_0^\infty (f_0(r) - r)\, \pi(dr),$$

$$\text{(C2)} \quad \lim_{p\to\infty} pc_p \sum_{k=-1}^{\infty} f_0\left(\frac{k}{p}\right)^2 \nu_p(k) = 2\beta + \int_0^\infty f_0(r)^2\, \pi(dr),$$

$$\text{(C3)} \quad \lim_{p\to\infty} pc_p \sum_{k=-1}^{\infty} \varphi\left(\frac{k}{p}\right) \nu_p(k) = \int_0^\infty \varphi(r)\, \pi(dr),$$

for any bounded, continuous function φ on $\mathbb{R}$ that vanishes on a neighbourhood of 0.

We do not show the proof of Lemma 4.3 (see e.g. [Ja85]). Nevertheless, we use this result to prove equation (4.5).

At first, we fix some notation. Define inductively $\tau_0^p := 0$ and

$$\tau_{m+1}^p := \inf\left\{ n > \tau_m^p : W_n^p \geq W^p_{\tau_m^p} \right\}.$$

Then the Markov property of the random walk W^p implies that conditionally on the event $\{\tau_m^p < \infty\}$, the random variable

$$1_{\{\tau_{m+1}^p < \infty\}} \left(W^p_{\tau_{m+1}^p} - W^p_{\tau_m^p} \right)$$

is independent of the past of the process W^p up to time τ_m^p and has the same law as $1_{\{\tau_1^p<\infty\}}W^p_{\tau_1^p}$. Moreover, it is a classical result (see e.g. [Fe71b] or [Ber96] for the analogue formula in the continuous setting) that for $j \geq 0$,

$$P\left\{\tau_1^p < \infty, W^p_{\tau_1^p} = j\right\} = \sum_{i=j}^{\infty} \nu_p(i) = \nu_p([j,\infty)). \tag{4.7}$$

In particular, we have that

$$P\{\tau_1^p < \infty\} = \sum_{j=0}^{\infty} \nu_p[j,\infty) = \sum_{k=0}^{\infty} k\mu_p(k). \tag{4.8}$$

Define for $u > \delta > 0$,

$$\begin{aligned}\kappa(\delta, u) &:= \int_0^\infty \int_0^r 1_{(\delta,u]}(r)\,dx\,\pi(dr) \\ &= \int_0^\infty \big((r-\delta)^+ \wedge (u-\delta)\big)\,\pi(dr).\end{aligned} \tag{4.9}$$

Moreover, let

$$\kappa_p(\delta,u) = \frac{\sum_{p\delta<j\leq pu} \nu_p[j,\infty)}{\sum_{j\geq 0}\nu_p[j,\infty)}$$

Due to (4.7) and (4.8), we can rewrite the definition of $\kappa_p(\delta,u)$ in the more intuitive form

$$\kappa_p(\delta,u) = P\left\{p\delta < W^p_{\tau_1^p} \leq pu \,\middle|\, \tau_1^p < \infty\right\}. \tag{4.10}$$

Let us make two more definitions. Recall that we denote by $\{S_t : t \geq 0\}$ the supremum process of the Lévy process X. Then, define

$$\begin{aligned}L_t^{\delta,u} &:= \#\{s \leq t : \Delta S_{s^-} \in (\delta,u]\} \\ l_k^{p,\delta,u} &:= \#\{j < k : \overline{W}_j^p + p\delta < W_{j+1}^p \leq \overline{W}_j^p + pu\},\end{aligned}$$

where $\overline{W}_j^p := \sup\{W_i^p : 0 \leq i \leq j\}$.

Recall that we want to show that, for every $t > 0$,

$$\lim_{p\to\infty} \frac{1}{c_p}\Lambda^p_{[pc_n t]} = L_t$$

in probability. The *rought idea* to prove this relation is the following: at first, we approximate $L^{\delta,u}$ by $l^{p,\delta,u}$ letting p tend to infinity, then, we use $L^{\delta,u}$ (suitably normalized) for an approximation of L_t as $\delta \downarrow 0$. We then see that we can also relate $l^{p,\delta,u}$ and Λ^p in a suitable way, and finally, we put all these approximations together to get the result.

As we can assume that the convergence

$$\left\{\frac{1}{p}W^p_{[pc_pt]}, t \geq 0\right\} \to \{X_t : t \geq 0\}$$

holds almost surely in the sense of the Skorokhod topology, we also have that for every fixed $t > 0$

$$\lim_{p\to\infty} l^{p,\delta,u}_{[pc_pt]} = L^{\delta,u}_t \tag{4.11}$$

almost surely. Next, we want to show that almost surely, for all $u > 0$

$$\lim_{\delta\to 0} \frac{1}{\kappa(\delta, u)} L^{\delta,u}_t = L_t. \tag{4.12}$$

Recall that L_∞ is finite almost surely and exponentially distributed if the underlying Lévy process X drifts to $-\infty$ and $L_\infty = \infty$ in the recurrent case. Denote by $(g_i, d_i), i \in I$ the excursion intervals of $S - X$ away from 0. Then, classical results on excursion theory (see e.g. [Ro84]) imply that the point measure

$$\sum_{i\in I, d_i<\infty} \delta_{(L_{d_i}, \Delta S_{d_i}, \Delta X_{d_i})}$$

is distributed as $1_{\{l<\eta\}}\mathcal{N}(\, dl\, dx\, dy\,)$, where $\mathcal{N}$ is a Poisson point measure on $\mathbb{R}_+ \times \mathbb{R}^2_+$ with intensity measure $dl\, n(dx\, dy)$ and η is an independent exponential time with parameter α. Moreover, one can choose the normalization of L, such that

$$n(dx\, dy) = 1_{[0,y]}(x)\, dx\, \pi(dy).$$

Hence, due to our definition of $\kappa(\delta, u)$,

$$n(\, (\delta, u] \times \mathbb{R}_+\,) = \kappa(\delta, u). \tag{4.13}$$

Let $(\delta_n) \downarrow 0$ and choose disjoint $A^n_1, \ldots, A^n_n$, such that

$$(\delta_n, u] = \bigcup_{i=1}^n A^n_i \quad \text{and} \quad n(A^n_j \times \mathbb{R}_+) = \frac{1}{n}\kappa(\delta_n, u)$$

for all $j = 1, \ldots, n$. Then,

$$\#\{i \in I : L_{d_i} \leq u, \Delta S_{d_i} \in (\delta_n, u]\}$$

is the sum of n independent and identically distributed Poisson variables with parameter $u \cdot n(A^n_j \times \mathbb{R}_+) = u \cdot \frac{1}{n}\kappa(\delta_n, u)$. Therefore, the strong law of large number implies that almost surely

$$\lim_{n\to\infty} \frac{1}{\kappa(\delta_n, u)} \#\{i \in I : L_{d_i} \leq u, \Delta S_{d_i} \in (\delta_n, u]\} = u \wedge L_\infty.$$

Hence, we must also have almost surely for all $u > 0$,

$$L_\infty \wedge u = \lim_{\delta \downarrow 0} \#\{s \geq 0 : L_s \leq u, \Delta S_{s^-} \in (\delta, u]\}$$

as whenever $\Delta S_{s^-} > 0$ for some s, also a new excursion of $S - X$ starts. Hence, the disered result (4.12) follows.

The next step is to show that $\lim c_p \kappa_p(\delta, u) = \kappa(\delta, u)$, which is by the definition of $\kappa_p(\delta, u)$ equivalent to prove that

$$\lim_{p \to \infty} c_p \frac{\sum_{p\delta < j \leq pu} \nu_p[j, \infty)}{\sum_{j \geq 0} \nu_p[j, \infty)} = \kappa(\delta, u). \tag{4.14}$$

Now we need the help of Lemma 4.3. Consider at first the denominator of (4.14). Recall that

$$\kappa(\delta, u) = \int_0^\infty ((r - \delta)^+ \wedge (u - \delta))\, \pi(dr).$$

Then (C3) applied to the function $\varphi(x) = (x - \delta)^+ \wedge (u - \delta)$ shows that

$$\lim_{p \to \infty} p c_p \sum_{k=-1}^{\infty} \nu_p(k) \left(\left(\frac{k}{p} - \delta \right)^+ \wedge (u - \delta) \right) = \int_0^\infty \varphi(r)\, \pi(dr) = \kappa(\delta, u).$$

Therefore,

$$\left| p c_p \sum_{k=-1}^{\infty} \nu_p(k) \left(\left(\frac{k}{p} - \delta \right)^+ \wedge (u - \delta) \right) - c_p \sum_{p\delta < j \leq pu} \nu_p[j, \infty) \right| \leq c_p \sum_{k \geq \delta p} \nu_p(k),$$

which tends to 0 as $p \to \infty$ by (C3). Hence, we have

$$\lim_{p \to \infty} c_p \sum_{p\delta < j \leq pu} \nu_p[j, \infty) = \kappa(\delta, u).$$

Now, let us look at the enumerator of (4.14). Recall that the measures μ_p are (sub)critical, hence,

$$1 \geq \sum_{k=0}^{\infty} k \mu_p(k) = \sum_{k=-1}^{\infty} k \nu_p(k)$$

and by (C1) this tends to 0 as $p \uparrow \infty$. So,

$$\sum_{j=0}^{\infty} \nu_p[j, \infty) = 1 + \sum_{k=-1}^{\infty} k \nu_p(k)$$

tends to 1 as $p \to \infty$ and we succeeded in showing formula (4.14).

The next step is to relate $l_k^{p,\delta,u}$ and $\Lambda^p_{[pc_pt]}$. To do so, observe that, conditionally on $\{\tau_1^p < \infty\}$, the random variable $l_k^{p,\delta,u}$ is the sum of k independent Bernoulli variables B_i with parameter $\kappa_p(\delta,u)$. Fix any integer $A>0$ and define $A_p := c_pA+1$. Then, Doob's inequality implies that

$$
\begin{aligned}
& \mathbb{E}\left\{\sup_{0\le j\le \tau^p_{A_p}} \left|\frac{1}{c_p}(\Lambda_j^p - \frac{1}{\kappa_p(\delta,u)} l_j^{p,\delta,u})\right|^2\right\} && (4.15)\\
&= \mathbb{E}\left\{\sup_{0\le k\le A_p} 1_{\{\tau_k^p<\infty\}}\left|\frac{1}{c_p}\left(k - \frac{1}{\kappa_p(\delta,u)} l^{p,\delta,u}_{\tau_k^p}\right)\right|^2\right\}\\
&= \mathbb{E}\left\{\sup_{0\le k\le A_p} 1_{\{\tau_k^p<\infty\}}\left|\frac{1}{c_p\kappa_p(\delta,u)}\left(k\kappa_p(\delta,u) - \sum_{i=1}^{\tau_k^p} B_i\right)\right|^2\right\}\\
&\le \mathbb{E}\left\{\sup_{0\le k\le A_p}\left|\frac{1}{c_p\kappa_p(\delta,u)}\sum_{i=1}^{k}(\mathbb{E}B_i - B_i)\right|^2\right\}\\
&\le 4\left(\frac{1}{c_p\kappa_p(\delta,u)}\right)^2 \mathbb{E}\left\{\left|\sum_{i=1}^{A_p}(\mathbb{E}B_i - B_i)\right|^2\right\}\\
&\le \frac{8(A+1)}{c_p\kappa_p(\delta,u)}. && (4.16)
\end{aligned}
$$

This implies using (4.14) that

$$
\begin{aligned}
\limsup_{p\to\infty}\mathbb{E}\left\{\sup_{j\le\tau^p_{A_p}}\left|\frac{1}{c_p}\left(\Lambda_j^p - \frac{1}{\kappa_p(\delta,u)}l_j^{p,\delta,u}\right)\right|^2\right\} &\le \limsup_{p\to\infty}\left(8(A+1)\frac{1}{c_p\kappa_p(\delta,u)}\right)\\
&\le 8(A+1)\frac{1}{\kappa(\delta,u)}. \qquad (4.17)
\end{aligned}
$$

Now, let $\varepsilon>0$ and choose A large enough, such that

$$P\{L_t \ge A - 3\varepsilon\} < \varepsilon.$$

Fix $u>0$ and use (4.12) and (4.16) to find a $\delta>0$ small enough, and a p_0 (which depends on this δ), such that

$$P\left\{\left|\frac{1}{\kappa(\delta,u)}L_t^{\delta,u} - L_t\right| > \varepsilon\right\} < \varepsilon \qquad (4.18)$$

and

$$P\left\{\sup_{j\le\tau^p_{A_p}}\left|\frac{1}{c_p}\left(\Lambda_j^p - \frac{1}{\kappa_p(\delta,u)}l_j^{p,\delta,u}\right)\right| > \varepsilon\right\} < \varepsilon, \qquad (4.19)$$

for all $p \geq p_0$. Moreover, using (4.11) and (4.14), we can find a $p_1 \in \mathbb{N}$ (also depending on the δ chosen above), such that for all $p \geq p_1$

$$P\left\{\left|\frac{1}{c_p \kappa_p(\delta,u)} l^{p,\delta,u}_{[pc_pt]} - \frac{1}{\kappa(\delta,u)} L^{\delta,u}_t\right| > \varepsilon\right\} < \varepsilon. \tag{4.20}$$

Using these last three estimates (4.18),(4.19) and (4.20), we get for all $p \geq p_0 \vee p_1$

$$P\left\{\left|\frac{1}{c_p}\Lambda^p_{[pc_pt]} - L_t\right| > 3\varepsilon\right\} \leq 3\varepsilon + P\{[pc_pt] > \tau^p_{A_p}\}.$$

So to complete the proof, we just have to bound the second summand in the last formula, this can be done via

$$\begin{aligned} P\{\tau^p_{A_p} < [pc_pt]\} &\leq P\left\{\tau^p_{A_p} < \infty, l^{p,\delta,u}_{[pc_pt]} \geq l^{p,\delta,u}_{\tau^p_{A_p}}\right\} \\ &\leq \varepsilon + P\left\{\frac{1}{c_p \kappa_p(\delta,u)} l^{p,\delta,u}_{[pc_pt]} \geq A - \varepsilon\right\} \\ &\leq 3\varepsilon + P\{L_t \geq A - 3\varepsilon\} \leq 4\varepsilon. \end{aligned}$$

□

4.3 Tightness

In this section, we prove a functional limit theorem. As we have already seen in Chapter 2, the behaviour of the height process is very wild if it has *no* continuous sample paths, i.e. if the condition

$$\int_1^\infty \frac{1}{\psi(\lambda)} d\lambda < \infty \tag{4.21}$$

is not fulfilled. Hence, we have to assume that (4.21) holds to have any chance to get a functional convergence, as otherwise, the paths of the height process do *not* belong to any meaningful function space.

Recall that we denote by $\{G^p_n : n \in \mathbb{N}_0\}$ the sequence of μ_p-Galton-Watson processes associated to the sequence of discrete μ_p-height processes $\{H^p_n : n \in \mathbb{N}_0\}$. The following theorem is due to Le Gall and Duquèsne (see [LGD]).

Theorem 4.4 (Functional Convergence)
Suppose that the convergence of the finite dimensional marginals (4.3) holds and that the height process $\{H_t : t \geq 0\}$ has continuous sample paths. Moreover, assume that for every $\delta > 0$

$$\liminf_{p\to\infty} P\{G^p_{[\delta c_p]} = 0\} > 0. \tag{4.22}$$

Then, the convergence

$$\left\{\frac{1}{c_p}H^p_{[pc_pt]} : t \geq 0\right\} \xrightarrow[p\to\infty]{(d)} \{H_t : t \geq 0\} \tag{4.23}$$

in distribution on the Skorokhod space $\mathcal{D}$.

One may wonder about the meaning of the technical condition (4.22). It is shown in [LGD] that:

- The condition (4.22) is really a *necessary* condition to get functional convergence.
- The condition (4.22) is *not* automatically satisfied.

Nevertheless, in the important special case of stable branching mechanisms $\psi(\lambda) = \lambda^{1+\beta}$ for $\beta \in (1, 2]$, the condition (4.22) *is* automatically fulfilled (see [LGD]).

Before we start to prove the theorem, note the following: As we already know the convergence of the finite dimensional marginal distributions, it is *enough* to show the *tightness* property of the laws of the processes $\{\frac{1}{c_p}H^p_{[pc_pt]} : t \geq 0\}$ in the set of probability measures $\mathcal{P}(\mathcal{D})$ on the Skorokhod space $\mathcal{D}$. In our case of interest, one can find a very useful (generally valid) lemma in the book of Ethier/Kurtz ([EK86], Corollary 3.7.4). We directly apply this lemma to our situation.

***Lemma* 4.5** *The laws of the processes $\{\frac{1}{c_p}H^p_{[pc_pt]} : t \geq 0\}$ are tight in the set of probability measures $\mathcal{P}(\mathcal{D})$ if the following two conditions are satisfied:*

(i) For all $t \geq 0$ and all $\eta > 0$, there exists a $K \geq 0$, such that

$$\liminf_{p\to\infty} P\left\{\frac{1}{c_p}H^p_{[pc_pt]} \leq K\right\} \geq 1 - \eta.$$

(ii) For every $T > 0$ and every $\delta > 0$,

$$\lim_{n\to\infty} \limsup_{p\to\infty} P\left\{\sup_{1\leq i\leq 2^n} \sup_{t\in[(i-1)2^{-n}T, i2^{-n}T]} \left|\frac{1}{c_p}H^p_{[pc_pt]} - \frac{1}{c_p}H^p_{[pc_p(i-1)2^{-n}T]}\right| > \delta\right\} = 0.$$

The first part of the lemma just means that the one-dimensional distributions are uniformly tight. The second part should be interpreted as a *regularity condition*, which is necessary to get the tightness of the laws of the whole processes from the uniform tightness of the one-dimensional marginals. Let us now use this lemma to prove the functional convergence theorem.

Proof of Theorem 4.4: Our aim is to show that the conditions (i) and (ii) of the previous lemma are satisfied. Note that (i) is clear by the finite-dimensional convergence.

To complete the proof, we *only* have to show the regularity condition (ii). Therefore, fix $\delta > 0$ and $T > 0$. Then, we can cut the problem into several parts by

$$\begin{aligned}&P\left\{\sup_{1\le i\le 2^n}\sup_{t\in[(i-1)2^{-n}T,i2^{-n}T]}\left|\frac{1}{c_p}H^p_{[pc_pt]}-\frac{1}{c_p}H^p_{[pc_p(i-1)2^{-n}T]}\right|>\delta\right\}\\&\le A_1(n,p)+A_2(n,p)+A_3(n,p),\end{aligned}$$

where summands of the right hand side are of the form

$$\begin{aligned}A_1(n,p) &:= P\left\{\sup_{1\le i\le 2^n}\left|\frac{1}{c_p}H^p_{[pc_pi2^{-n}T]}-\frac{1}{c_p}H^p_{[pc_p(i-1)2^{-n}T]}\right|>\frac{\delta}{5}\right\}\\A_2(n,p) &:= P\left\{\sup_{t\in[(i-1)2^{-n}T,i2^nT]}\frac{1}{c_p}H^p_{[pc_pt]}>\frac{1}{c_p}H^p_{[pc_p(i-1)2^{-n}T]}+\frac{4\delta}{5}\text{ for some }1\le i\le 2^n\right\}\\A_3(n,p) &:= P\left\{\inf_{t\in[(i-1)2^{-n}T,i2^nT]}\frac{1}{c_p}H^p_{[pc_pt]}<\frac{1}{c_p}H^p_{[pc_pi2^nT]}-\frac{4\delta}{5}\text{ for some }1\le i\le 2^n\right\}.\end{aligned}$$

So, the proof reduces to bound each of these summands. For the first one, this is easy, because the convergence of the finite dimensional distributions implies that

$$\limsup_{p\to\infty}A_1(n,p)\le P\left\{\sup_{1\le i\le 2^n}\left|H_{i2^{-n}T}-H_{(i-1)2^{-n}T}\right|>\frac{\delta}{5}\right\},$$

which tends to 0 as $n\to\infty$, because we assumed that the paths of the height process $\{H_t : t\ge 0\}$ are continuous almost surely.

As one could expect, the proofs for $A_2(n,p)$ and $A_3(n,p)$ are very similar. So we only show the one for $A_2(n,p)$. Therefore, let $i\in\{1,\dots,2^n\}$, such that

$$\sup_{t\in[(i-1)2^{-n}T,i2^{-n}T]}\frac{1}{c_p}H^p_{[pc_pt]}>\frac{1}{c_p}H^p_{[pc_p(i-1)2^{-n}T]}+\frac{4\delta}{5}. \tag{4.24}$$

Let us introduce a sequence of stopping times (τ^p_k) by $\tau^p_0=0$ and

$$\tau^p_{k+1}:=\inf\left\{t\ge\tau^p_k:\frac{1}{c_p}H^p_{[pc_pt]}>\inf_{\tau^p_k\le r\le t}\frac{1}{c_p}H^p_{[pc_pr]}+\frac{\delta}{5}\right\},\qquad\text{for }k\ge 1.$$

Hence, the interval $[(i-1)2^{-n}T, i2^{-n}T]$ must contain at least one of the random times $\tau^p_k, k\ge 0$. Denote by τ^p_j the first such time. Then,

$$\sup_{t\in[(i-1)2^{-n}T,\tau^p_j)}\frac{1}{c_p}H^p_{[pc_pt]}\le\frac{1}{c_p}H^p_{[pc_p(i-1)2^{-n}T]}+\frac{\delta}{5}.$$

Note that the positive jumps of $\{\frac{1}{c_p}H^p_{[pc_pt]} : t\ge 0\}$ are of size $\frac{1}{c_p}$, hence,

$$\frac{1}{c_p}H^p_{[pc_p\tau^p_j]}\le\frac{1}{c_p}H^p_{[pc_p(i-1)2^{-n}T]}+\frac{\delta}{5}+\frac{1}{c_p}<\frac{1}{c_p}H^p_{[pc_p(i-1)2^{-n}T]}+\frac{2\delta}{5},$$

as we can choose p so large, that $c_p > \frac{5}{\delta}$. Then, (4.24) implies that

$$\sup_{t\in[\tau_j^p, i2^{-n}T]} \frac{1}{c_p} H^p_{[pc_pt]} > \frac{1}{c_p} H^p_{[pc_p\tau_j^p]} + \frac{\delta}{5},$$

and therefore, $\tau_{j+1}^p \leq i2^{-n}T$. This yields that for such large p,

$$A_2(n,p) \leq P\left\{\tau_k^p < T \text{ and } \tau_{k+1}^p - \tau_k^p < 2^{-n}T \text{ some } k \geq 0\right\}. \tag{4.25}$$

With very similar arguments, one gets the same bound for $A_3(n,p)$. To complete the proof, we have to study the limit behaviour of the right hand side in formula (4.25). This can be achieved via the following lemmas:

***Lemma* 4.6** *The random variables $\tau_{k+1}^p - \tau_k^p$, $(k \geq 1)$ are independent identically distributed. Moreover, under the assumptions of the theorem,*

$$\lim_{x\downarrow 0}(\limsup_{p\to\infty} P\{\tau_1^p \leq x\}) = 0.$$

***Lemma* 4.7** *For all $x > 0$ and integers $p \geq 1$, set*

$$G_p(x) := P\left\{\tau_k^p < T,\, \tau_{k+1}^p - \tau_k^p < x \textit{ for some } k \geq 0\right\} \qquad \textit{and}$$

$$F_p(x) := \sup_{k\geq 0} P\left\{\tau_k^p < T \textit{ and } \tau_{k+1}^p - \tau_k^p < x\right\}.$$

Then, for all integers $L \geq 1$, we have

$$G_p(x) \leq LF_p(x) + Le^T \int_0^\infty e^{-Ly} F_p(y)\, dy.$$

We do not prove the second lemma, this is in fact Lemma 8.2 of [EK86] directly applied to our setting. Before we prove the first one, we already show how to use it to complete the proof of the functional convergence theorem.

Recall that we are interested in the limit behaviour of $x \mapsto G_p(x)$. Hence, define

$$F(x) := \limsup_{p\to\infty} F_p(x) \qquad \text{and} \qquad G(x) := \limsup_{p\to\infty} G_p(x).$$

By the first lemma, we have that $F(x) \downarrow 0$ as $x \downarrow 0$. By the second one and the monotonicity of the limit superior, we have for all integers $L \geq 1$,

$$G(x) \leq LF(x) + Le^T \int_0^\infty e^{-Ly} F(y) dy. \tag{4.26}$$

Hence, $G(x)$ tends also to 0 as $x \downarrow 0$ and $L \uparrow \infty$, as the second summand in equation (4.26) becomes arbitrarily small if L is large enough. Therefore, (4.25) implies that

$$\lim_{n\to\infty}(\limsup_{p\to\infty} A_2(n,p)) = 0$$

and the same for $A_3(n,p)$, which completes the proof of the theorem. $\square$

To be rigorus, we still have to show Lemma 4.7. This can be done via the following arguments:

Proof of Lemma 4.7: Recall that we denote by $\{W_n^p : n \in \mathbb{N}_0\}$ the sequence of random walks with increment distribution $\nu_p(k) = \mu_p(k+1)$ for $k = -1, 0, 1, \ldots$. Fix a $k \geq 1$ and let $\tilde{W}_t^p$ be the shifted random walk

$$\tilde{W}_t^p = W^p_{\tau_k^p + t} - W^p_{\tau_k^p},$$

and denote by $\{\tilde{H}_n^p : n \in \mathbb{N}_0\}$ the associated discrete height process. By the Markov property of W^p, this shifted random walk is independent of the past of W^p up to time τ_k^p and has the same distribution as W^p. Note, that for every $t \geq 0$,

$$\begin{aligned}
\frac{1}{c_p} H^p_{[pc_p(\tau_k^p+t)]} &= \frac{1}{c_p} \# \left\{ j < [pc_p(\tau_k^p + t)] : W_j^p = \inf_{j \leq s \leq [pc_p(\tau_k^p+t)]} W_s^p \right\} \\
&= \frac{1}{c_p} \# \left\{ j < [pc_p \tau_k^p] : W_j^p = \inf_{j \leq s \leq [pc_p(\tau_k^p+t)]} W_s^p \right\} \\
&\quad + \frac{1}{c_p} \# \left\{ [pc_p \tau_k^p] \leq j \leq [pc_p(\tau_k^p + t)] : W_j^p = \inf_{j \leq s \leq [pc_p(\tau_k^p+t)]} W_s^p \right\} \\
&= \inf_{\tau_k^p \leq r \leq \tau_k^p + t} \frac{1}{c_p} H^p_{[pc_p r]} + \frac{1}{c_p} \tilde{H}^p_{[pc_p t]}. \qquad (4.27)
\end{aligned}$$

Hence,

$$\tau_{k+1}^p - \tau_k^p = \inf\left\{ t \geq 0 : \frac{1}{c_p} \tilde{H}^p_{[pc_p t]} > \frac{\delta}{5} \right\}, \qquad (4.28)$$

and the random variables $\tau_{k+1}^p - \tau_k^p, k \geq 1$ are indentically distributed, because the right hand side of the last displayed formula does *not* depend on k. As the shifted random walk $\tilde{W}^p$ is independent of the past of the random walk W^p up to time τ_k^p (4.28) also implies that the random variables $\tau_{k+1}^p - \tau_k^p, k \geq 1$ are independent.

Let us now examine the limit behaviour of $P\{\tau_1^p \leq x\}$. For $\eta > 0$, define

$$T_\eta^p := \inf\left\{ t \geq 0 : \frac{1}{c_p} W^p_{[pc_p t]} = -\frac{[p\eta]}{p} \right\}.$$

Note that

$$\begin{aligned}
P\{\tau_1^p \leq x\} &= P\left\{ \sup_{s \leq x} \frac{1}{c_p} H^p_{[pc_p s]} > \frac{\delta}{5} \right\} \\
&\leq P\left\{ \sup_{s \leq T_\eta^p} \frac{1}{c_p} H^p_{[pc_p \eta]} > \frac{\delta}{5} \right\} + P\{T_s^p < x\}. \qquad (4.29)
\end{aligned}$$

We deal both summands separately. Firstly, recall that we denote by $\{T_x : x \leq 0\}$ the first passage time process of $-x$ by the underlying Lévy process X. Moreover, recall the assumption that the suitable rescaled random walk converges in distribution towards this Lévy process. Hence, we also have that

$$\limsup_{p\to\infty} P\{T_\eta^p < x\} \leq P\{T_\eta \leq x\},$$

for any $\eta > 0$. Nevertheless, the right hand side tends to 0 as $x \downarrow 0$ just using the continuity of P.

So, we only have to deal with the first summand in formula (4.29). Recall that we can construct a Galton-Watson process from the discrete height process, by counting the numbers of particles in each level of the discrete height process. Hence, the maximum of the discrete height process corresponds to the extinction time of the associated Galton-Watson process. Moreover, recall that we can construct the discrete height process from the random walk W stopped at its first hitting time of some $-s$, $s \in \mathbb{N}$. Then, s corresponds to the number of *excursions* of the discrete height process or equivalently to the *starting mass* of the associated Galton-Watson process. Hence, due to our scaling,

$$\sup_{s\leq T_\eta^p} \frac{1}{c_p} H^p_{[pc_p s]}$$

is distributed like $c_p^{-1}(M_p - 1)$, where M_p is the extinction time of a μ_p-Galton-Watson process started at $[p\eta]$. Therefore,

$$\begin{aligned} P\left\{\sup_{s\leq T_\eta^p} \frac{1}{c_p} H^p_{[pc_p s]} > \frac{\delta}{5}\right\} &= P\left\{M_p > \frac{\delta}{5}c_p + 1\right\} \\ &= 1 - P\left\{\tilde{G}^{p,\eta}_{[\frac{\delta}{5}c_p]+1}\right\}, \end{aligned}$$

where we denote by $\{\tilde{G}^{p,\eta}_n : n \in \mathbb{N}_0\}$ a μ_p-Galton-Watson process with start in $[p\eta]$. As for all $\eta \leq 1$

$$P\left\{G^p_{[\frac{\delta}{5}c_p]} = 0\right\} \leq P\left\{\tilde{G}^{p,\eta}_{[\frac{\delta}{5}c_p]} = 0\right\},$$

our assumption (4.22) implies that for small η

$$\liminf_{p\to\infty} P\left\{\tilde{G}^{p,\eta}_{[\frac{\delta}{5}c_p]} = 0\right\} > 0.$$

As we assumed the measures μ_p to be (sub)critical, we get

$$\lim_{\eta\downarrow 0}\liminf_{p\to\infty} P\left\{\tilde{G}^{p,\eta}_{[\frac{\delta}{5}c_p]} = 0\right\} = 1,$$

and we infer

$$\lim_{\eta\downarrow 0}\left(\limsup_{p\to\infty} P\left\{\sup_{s\leq T^p_\eta}\frac{1}{c_p}H^p_{[pc_ps]}>\frac{\delta}{5}\right\}\right) = \lim_{\eta\downarrow 0}\left(1-\liminf_{p\to\infty} P\left\{\tilde{G}^{p,\eta}_{[\frac{\delta}{5}c_p]}=0\right\}\right) = 0,$$

which completes the proof of Lemma 4.7. □

Let us denote by $T := \inf\{t > 0 : Z_t = 0\}$ the *extinction* time of the ψ-CSBP $\{Z_t : t \geq 0\}$. Under our assumption of finite biodiversity, we know that $T < \infty$ almost surely. The following corollary in an interesting consequence of the last proof.

***Corollary* 4.8** *Denote by T_p the extinction time of the μ_p-Galton-Watson processes $\{G^p_n : n \in \mathbb{N}_0\}$ with start in $G^p_0 = p$ and assume that the functional convergence (4.23) holds. Then, we also have that $\frac{1}{c_p}T_p \to T$ in distribution, as p tends to infinity.*

As we already pointed out, in the case of stable branching mechanisms $\psi(\lambda) = \lambda^{1+\beta}$ for $\beta \in (1, 2]$, the technical condition (4.22) is automatically fulfilled and the statement of the functional convergence theorem holds true.

***Corollary* 4.9** *Assume that $\mu_p = \mu$ for every $p \in \mathbb{N}$ and that either one of the conditions (4.1) or (4.2) holds. Then, there is a $\beta \in (1, 2]$, such that for the height process $\{H_t : t \geq 0\}$, associated with a β-stable branching mechanism, the convergence*

$$\left\{\frac{1}{c_p}H_{[pc_pt]} : t \geq 0\right\} \xrightarrow[p\to\infty]{(d)} \{H_t : t \geq 0\} \tag{4.30}$$

holds in distribution on the Skorokhod space $\mathcal{D}$.

We do not prove this corollary here and refer to [LGD]. Nevertheless, it is interesting to summarize some of the results for ψ-CSBP with *stable* branching mechanisms.

So assume that $\psi(\lambda) = \lambda^{1+\beta}$ for $\beta \in (1, 2]$. Then, the following facts hold true:

- the ψ-CSBP die out in finite time (Theorem 1.29)
- the associated ψ-height process has continuous sample paths (Theorem 2.31)
- we can compute the Hausdorff dimension of the zeroset of their height processes to $\frac{1}{1+\beta}$
- and last but not least, the convergence of suitable discrete height processes always holds in a functional sense (Corollary 4.9).

In the next chapter, we discuss a generalization of a classical result by *Zubkov* for CSBP. As we also see, in this context, the *special* role played by the stable branching mechanisms.

Chapter 5

Zubkov's Theorem for CSBP

Consider a critical μ-Galton-Watson tree τ and let γ_n be the generation of the *most recent common ancestor* of all particles in the tree alive at generation n. Moreover, denote by $\{G_n : n \in \mathbb{N}\}$ the μ-Galton-Watson branching process associated with τ that counts the particles alive in each generation.

As we know that (sub)critical Galton-Watson trees are finite almost surely, it is in interesting question to look at the long term behaviour of γ_n conditioned on the survival of the tree. In 1975, A. Zubkov proved his celebrated theorem saying that

$$\lim_{n\to\infty} P\{n^{-1}\gamma_n \leq u \,|\, G_n > 0\} = u \tag{5.1}$$

for all $0 \leq u \leq 1$, i.e. the suitable rescaled limit of the last common ancestor, conditioned on the survival of the tree, is uniformly distributed.

It is very natural to ask, if we have a similar behaviour also for the continuous setting where the genealogy is given by the height process. Jean-François Le Gall and Thomas Duquèsne answered this question positively ([LGD]). But before we state the main results, we have to fix some notation.

Throughout this chapter, let $\{X_t : t \geq 0\}$ a ψ-Lévy process from the class $\mathcal{L}$ and assume moreover, that the associated ψ-height process is continuous. This means, speaking in terms of genealogy, that the corresponding ψ-CSBP is of finite biodiversity.

Recall that we denote by N the excursion measure of $X - I$ away from 0 and that we can define the height process H under N. Then, for every $T > 0$, denote $N_{(T)}$ to be the conditional law

$$N\left[\cdot \,\middle|\, \sup_{0\leq s\leq\sigma} H_s \geq T\right],$$

where σ denotes the length of an excursion. We have to note that $N_{(T)}$ introduces a probability measure on the set of excursions that reach level T. Moreover, recall from

Chapter 2 (especially the remark after Theorem 2.28) that $u_t(\lambda) = N[1 - e^{-\lambda L_\sigma^t}]$ is the unique solution of the equation

$$u_t(\lambda) + \int_0^t \psi(u_s(\lambda))ds = \lambda. \tag{5.2}$$

Let us define

$$\begin{aligned} v(t) &:= u_t(\infty) \\ &= N[L_\sigma^t > 0] \\ &= N\left[\sup_{0\le s\le\sigma} H_s > t\right], \end{aligned} \tag{5.3}$$

which is due to Lemma 2.33 uniquely determined by

$$\int_{v(t)}^\infty \frac{1}{\psi(x)}dx = t.$$

For all $t \in [0,T)$, let $\mathcal{A}_t^T$ be the number of excursions of H above level t that hit level T. Speaking in terms of genealogy, this means that $\mathcal{A}_t^T$ is equal to the number of particles in generation t that have descendants in generation T. By our assumption of *finite biodiversity*, we clearly have that $\mathcal{A}_t^T < \infty$ almost surely. It is time to state now the main theorem of this chapter.

Theorem* 5.1 *(Zubkov's theorem)
Let γ_T be generation of the most recent common ancestor of all particles alive at some time $T > 0$. Then, we can characterize its distribution under $N_{(T)}$ by

$$N_{(T)}[\gamma_T > t] = \frac{\tilde\psi(v(T))}{\tilde\psi(v(T-t))} \tag{5.4}$$

for all $t \in [0,T)$, where $\tilde\psi(x) = x^{-1}\psi(x)$.

Later we see that γ_T is uniformly distributed under $N_{(T)}$ if and only if the branching mechanism ψ is *stable*, i.e. $\psi(\lambda) = \lambda^\alpha$, for some $\alpha \in (1,2]$. But first, let us prove Theorem 5.1.

The key to proof is to consider for some fixed $T > 0$ the numbers $\mathcal{A}_t^T$ as a *stochastic process* in t. Then we can characterize γ_T by the fact that $\gamma_T > t$ if and only if $\mathcal{A}_t^T = 1$. We will see the details later, at first consider the following theorem, which is of independent interest.

***Theorem* 5.2** *For every $T > 0$, we can characterize the one-dimensional distributions of the process $\{\mathcal{A}_t^T : 0 \le t < T\}$ under $N_{(T)}$ by*

$$N_{(T)}\left[\exp -\lambda\mathcal{A}_t^T\right] = 1 - \frac{u_t((1-e^{-\lambda})v(T-t))}{v(T)} \qquad \textit{for } \lambda > 0.$$

Proof of Theorem 5.2: Before we really start, we fix some notation and state the key lemma. Let us fix a $t \in (0,T]$. Then the definition of $\mathcal{A}_t^T$ makes also sense under the conditional probability $N_{(t)}$. We already know from Chapter 3 that $\mathcal{A}_t^T$ is Poisson distributed under P, conditioned on the survival of the CSBP up to generation t. The main ingrediant to prove Theorem 5.2 is to show that $\mathcal{A}_t^T$ is also Poisson distributed under $N_{(t)}$ and to compute the parameter in terms of v. Then, we can use this information to compute it's Laplace transform. Denote by

$$\mathcal{E} := \{e_i^t : i = 1, \dots, \mathcal{A}_t^T\}$$

the successive excursions of the height process H above level t that hit level $T - t$, shifted in time and space, such that they start at time 0 in the point 0. Note, that by our assumption of finite biodiversity, the set $\mathcal{E}$ is finite almost surely. Moreover, recall that even if the height process is in general *not* a Markov process, we have shown in Chapter 2 that we can construct a local time of H at any level $a \geq 0$, denoted by L_t^a. Let us use $L_{(i)}^t$ to denote the local time of H at the beginning of the excursion e_i^t. The key to the proof of Theorem 5.2 is the following lemma:

***Lemma* 5.3** *Under $N_{(t)}$, conditionally on L_σ^t, i.e. the local time at level t at the end of an excursion of the height process, the point measure*

$$\sum_{i=1}^{\mathcal{A}_t^T} \delta_{(L_{(i)}^t, e_i^t)}$$

is a Poisson point measure on the space $\mathbb{R}_+ \times \mathcal{D}$ with intensity measure

$$1_{[0,L_\sigma^t]}(l) 1_{\{\sup H_s > T-t\}}\, dl\, dN.$$

Proof: Denote by f_i^t the successive excursions of the height process H above level t that hit level T under the original probability measure P. Moreover, let l_i^t be the corresponding local times at the beginning of such an excursion.
Moreover, let us denote by λ_1 the local time of the height process at level t at the end of the first excursion that hits level T. Note that the distribution of

$$\left(\lambda_1, \sum_{\{i: l_i^t \leq \lambda_1\}} \delta_{(l_i^t, f_i^t)}\right)$$

under P is the same as the one of

$$\left(L_\sigma^t, \sum_{i=1}^{\mathcal{A}_t^T} \delta_{(L_i^t, e_i^t)}\right)$$

under the conditional probability $N_{(t)}$. Hence, to prove the assertion, it is enough to show that, conditionally on λ_1, the point measure

$$\sum_{\{i: l_i^t \leq \lambda_1\}} \delta_{(l_i^t, f_i^t)} \tag{5.5}$$

is a Poisson point measure with intensity measure

$$1_{[0,\lambda_1]}(l)1_{\{\sup H_s > T-t\}}\, dl\, dN. \tag{5.6}$$

Recall that $\{H^t_s : s \geq 0\}$ is the process obtained by glueing together the positive excursions of the height process above level t. Then by our construction, the f^t_i are the successive excursions of $\{H^t_s : s \geq 0\}$ that hit level $T-t$ and the l^t_i are the corresponding local times at level 0. As we know by Theorem 2.23 that $\{H^t_s : s \geq 0\}$ has the same distribution as $\{H_s : s \geq 0\}$, we get the assertion. □

Now, we come back to the proof of Theorem 5.2. By the previous lemma, it follows immediately that, conditionally on L^t_σ, the random variables $\mathcal{A}^T_t$ are Poisson distributed under $N_{(t)}$ with parameter

$$\begin{aligned}\int_{\mathbb{R}_+\times\mathcal{D}} 1_{[0,L^t_\sigma]}1_{\{\sup H_s > T-t\}}\, dl\, N(de) &= N\left[\sup_{0\leq s\leq\sigma} H_s > T-t\right] L^t_\sigma \\ &= v(T-t)L^t_\sigma.\end{aligned}$$

We therefore get, using that $u_t(\lambda) = N[1-\exp(-\lambda L^t_\sigma)]$,

$$\begin{aligned}N_{(t)}\left[e^{-\lambda\mathcal{A}^T_t}\right] &= N_{(t)}\left[\exp\left(-L^t_\sigma v(T-t)(1-e^{-\lambda})\right)\right] \\ &= 1-\frac{1}{v(t)}N\left[1-\exp -L^t_\sigma v(T-t)(1-e^{-\lambda})\right] \\ &= 1-\frac{1}{v(t)}u_t((1-e^{-\lambda})v(T-t)).\end{aligned} \tag{5.7}$$

Moreover, we can observe that

$$\begin{aligned}N_{(t)}\left[1-e^{-\lambda\mathcal{A}^T_t}\right] &= \frac{1}{v(t)}N\left[1-e^{-\lambda\mathcal{A}^T_t}\right] \\ &= \frac{v(T)}{v(t)}N_{(T)}\left[1-e^{-\lambda\mathcal{A}^T_t}\right].\end{aligned} \tag{5.8}$$

Hence, we get the assertion using (5.7). □

Now we are ready to start with the proof of the main Theorem 5.1.

Proof of Theorem 5.1: As $\gamma_T > t$ if and only if $\mathcal{A}^T_t = 1$, we have by Theorem 5.2 that

$$\begin{aligned}N_{(T)}[\gamma_T > t] &= \lim_{\lambda\to\infty} e^\lambda N_{(T)}\left[e^{-\lambda\mathcal{A}^T_t}\right] \\ &= \lim_{\lambda\to\infty} e^\lambda\left(1-\frac{u_t((1-e^{-\lambda})v(T-t))}{v(T)}\right).\end{aligned} \tag{5.9}$$

As $v(t) = u_t(\infty)$ and $u_{t+r} = u_t \circ u_r$, we also have that $u_t(v(r)) = v(t+r)$ which implies that $u_t(v(T-t)) = v(T)$. Using a first order Taylor expansion of the function $u_t(\cdot)$, we get that

$$\begin{aligned}&u_t((1-e^{-\lambda})v(T-t)) \\ =\ &u_t(v(T-t)) + \left[(1-e^{-\lambda})v(T-t) - v(T-t)\right]\frac{\partial}{\partial\lambda}u_t(v(T-t)) + o(\varepsilon^2).\end{aligned}$$

Plugging this into (5.9) yields

$$\begin{aligned} N_{(T)}[\gamma_T > t] &= \lim_{\lambda\to\infty}\left[e^{\lambda}\left(e^{-\lambda}\frac{v(T-t)}{v(T)}\frac{\partial}{\partial\lambda}u_t(\lambda)v(T-t) + o(e^{-\lambda})\right)\right] \\ &= \frac{v(T-t)}{v(T)}\frac{\partial}{\partial\lambda}u_t(v(T-t)). \end{aligned}$$

To complete the proof, we have to verify that

$$\frac{\partial}{\partial\lambda}u_t(\lambda) = \frac{\psi(u_t(\lambda))}{\psi(\lambda)}, \tag{5.10}$$

because then, using $u_t(v(T-t)) = v(T)$ we get

$$\begin{aligned} \frac{\partial}{\partial\lambda}u_t(v(T-t)) &= \frac{\psi(u_t(v(T-t)))}{\psi(v(T-t))} \\ &= \frac{\psi(v(T))}{\psi(v(T-t))}, \end{aligned}$$

which yields the assertion

$$N_{(T)}[\gamma_T > t] = \frac{v(T-t)}{v(T)}\frac{\psi(v(T))}{\psi(v(T-t))} = \frac{\tilde{\psi}(v(T))}{\tilde{\psi}(v(T-t))}.$$

To prove (5.10), recall from (5.2) that

$$u_t(\lambda) + \int_0^t \psi(u_s(\lambda))\,ds = \lambda.$$

Differentiating this integral equation yields

$$\frac{\partial}{\partial\lambda}u_t(\lambda) + \int_0^t \psi'(u_s(\lambda))\frac{\partial}{\partial\lambda}u_s(\lambda)\,ds = 1.$$

To get a solvable equation, we differentiate now with respect to t

$$\frac{\partial}{\partial t}\frac{\partial}{\partial\lambda}u_t(\lambda) = -\frac{\partial}{\partial\lambda}u_t(\lambda)\psi'(u_t(\lambda)),$$

which is solved by $\{u_t(\lambda) : t, \lambda \geq 0\}$ with

$$\frac{\partial}{\partial\lambda}u_t(\lambda) = \exp\left(-\int_0^t \psi'(u_s(\lambda))\,ds\right). \tag{5.11}$$

Recall that $\frac{\partial}{\partial t}u_t(\lambda) = -\psi(u_t(\lambda))$, hence,

$$\begin{aligned} \frac{\partial}{\partial t}\log\psi(u_t(\lambda)) &= \frac{1}{\psi(u_t(\lambda))}\psi'(u_t(\lambda))\frac{\partial}{\partial t}u_t(\lambda) \\ &= -\psi'(u_t(\lambda)). \end{aligned}$$

Hence, by the fundamental theorem of calculus,

$$\begin{aligned} -\int_0^t \psi'(u_s(\lambda))\,ds &= \int_0^t \frac{\partial}{\partial s}\log\psi(u_s(\lambda))\,ds \\ &= \log\psi(u_t(\lambda)) - \log\psi(u_0(\lambda)) \\ &= \log\frac{\psi(u_t(\lambda))}{\psi(\lambda)}. \end{aligned}$$

Finally, we get using (5.11)

$$\frac{\partial}{\partial\lambda}u_t(\lambda) = \exp\left(\log\frac{\psi(u_t(\lambda))}{\psi(\lambda)}\right) = \frac{\psi(u_t(\lambda))}{\psi(\lambda)}$$

which leads to formula (5.10) and finishes the proof of the theorem. □

As promised, we now present the interesting case of *stable* branching mechanisms. This special case *is* in fact the analogue to the classical theorem in the discrete setting.

***Corollary* 5.4** *For every $T > 0$, the random variable γ_T is uniformly distributed over $[0,T]$ under $N_{(T)}$ if and only if the branching mechanisms are stable, i.e. $\psi(\lambda) = \lambda^\alpha$ for some $\alpha \in (1,2]$.*

Proof: Let us first assume that the branching mechanism is stable, i.e. $\psi(\lambda) = c\lambda^\alpha$, for some $c > 0$ and some $\alpha \in (1,2]$. Recall that $v(t)$ is determined by

$$\int_{v(t)}^\infty \frac{1}{\psi(\lambda)}\,d\lambda = t, \tag{5.12}$$

and it is elementary to check that

$$\int_{(c(\alpha-1)t)^{-\frac{1}{\alpha-1}}}^\infty \frac{1}{c\lambda^\alpha}\,d\lambda = t,$$

such that we have $v(t) = (c(\alpha-1)t)^{-\frac{1}{\alpha-1}}$ in that case. Using Theorem 5.1, it is easy to compute

$$\begin{aligned} N_{(T)}[\gamma_T > t] &= \frac{\tilde\psi(v(T))}{\tilde\psi(v(T-t))} \\ &= \frac{c[(c(\alpha-1)T)^{-\frac{1}{\alpha-1}}]^\alpha}{(c(\alpha-1)T)^{-\frac{1}{\alpha-1}}}\cdot\frac{[c(\alpha-1)(T-t)]^{-\frac{1}{\alpha-1}}}{c[(c(\alpha-1)(T-t))^{-\frac{1}{\alpha-1}}]^\alpha} \\ &= 1-\frac{t}{T}. \end{aligned}$$

To view the opposite direction, assume that γ_T is uniformly distributed over $[0,T]$ under $N_{(T)}$. Again, by Theorem 5.1, we see that

$$\begin{aligned} \frac{\tilde\psi(v(T))}{\tilde\psi(v(T-t))} &= 1-\frac{t}{T} \\ &= \frac{C}{T}\cdot\frac{T-t}{C}, \end{aligned}$$

such that $\tilde{\psi}(v(t)) = \frac{C}{t}$ for some $C > 0$. Moreover, differentiating $\log v(t)$ yields

$$\begin{aligned} (\log v(t))' &= \frac{1}{v(t)} v'(t) \\ &= \frac{C}{t\psi(v(t))} v'(t). \end{aligned}$$

Now, formula (5.12) and the substitution rule imply that

$$\begin{aligned} t &= -\int_0^t \frac{1}{\psi(v(x))} v'(x)\, dx \\ &= -\int_0^t \frac{x}{C} (\log v(x))'\, dx \end{aligned}$$

and we get that $v(t) = ct^{-C}$, which implies that

$$\psi(\lambda) = c\lambda^{1+\frac{1}{C}}.$$

Hence, ψ has the desired form. $\square$

We have to remark that one could certainly give a different approach to these results using approximation by the discrete setting, as it is provided by Chapter 4. Nevertheless, the *nicest* way to answer questions concerning the genealogy, is to treat the height process directly and use its properties - *exactly* what we have done. Moreover, this way also includes (using the functional convergence theorem) a new proof of the classical Zubkov theorem for discrete case.

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Erklärung

Hiermit erkläre ich, daß ich diese Arbeit selbständig verfaßt und keine anderen als die angegebenen Hilfsmittel verwendet habe.

Kaiserslautern, den 25. April 2001