Portfolio Optimization with Discrete Trading Strategies in a Continuous-time Setting

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Chapter 1

Introduction

Modern portfolio theory started with the seminal papers of Merton (1969,1971) where he considered the problem of utility maximisation from consumption and/or terminal wealth in a perfect financial market. Using standard results of stochastic control he was able to derive a Bellman equation for the value function which can be solved explicitely when utility functions of the so-called HARA-class are used. Later on, the so-called martingale method for portfolio optimization was developed by Cox and Huang (1989) and Karatzas et al. (1987). As the name suggests, this approach uses martingale techniques to circumvent the Bellman equation. Unfortunately, this approach needs the assumption of a perfect financial market thus excluding transaction costs. As the optimal strategies obtained from the two approaches above require continuous trading, they cannot be practised in the presence of (fixed) transaction costs. (For a general overview of portfolio optimization and an introduction to standard problems see Korn (1997).) By including a fixed cost component in the transaction costs, Eastham and Hastings (1988) applied an impulse control approach to portfolio optimization which led to optimal strategies consisting of finitely many actions on finite time intervals. This approach was taken up by Korn (1998) who refined it in certain senses and obtained an approximately optimal strategy in the case of exponential utility and positive transaction costs. In the second chapter of this thesis we will link up with exactly this result. Moreover, we will consider two types of discrete trading strategies in the case of logarithmic and power utility (Chpater 3 and 4), respectively.

1.1 Motivation

As a motivating example we consider a financial market consisting of a riskless bond with constant price $P_0(t) = 1$ and a risky stock with price $P_1(t)$, evolving according to the price dynamics

$$dP_1(t) \quad = \quad P_1(t)\,[\mu\,dt + \sigma\,dW_t]\,,$$

$$P_1(0) \quad = \quad p_1$$

where $\mu\,,\sigma$ are real constants with $\sigma > 0$ and where W_t is a one-dimensional standard Brownian motion with the usual right-continuous Brownian filtration $\{F_t\}_{t\in[0,T]}$. A classical problem in continuous-time finance is that of maximising the expected utility from wealth at the time horizon T, i.e.

$$\max_{\pi} E(U(X^{\pi}(T))) \tag{1.1}$$

where π is a portfolio process, $X^{\pi}(t)$ is the corresponding wealth process and U is a utility function (e.g. logarithmic or power utility). (For an exact definition of a portfolio process see Korn (1997).) In the case of logarithmic utility (and without transaction costs) a well-known result is that the optimal portfolio process is constant in time:

$$\pi^*(t) \equiv \pi^* = \frac{\mu}{\sigma^2} \quad \forall\, t \in [0,T]\,.$$

As it is optimal to invest the same constant fraction of wealth π^* in the stock at every time instant, the investor has to choose trading strategies $\varphi(t)$ with infinite variation on every finite time interval. In the presence of transaction costs (above all in the presence of fixed transaction costs) an investor following this strategy will therefore be ruined almost immediately. Hence, in the presence of (fixed) transaction costs a different class of strategies must be considered. Throughout this thesis, we will therefore examine various types of discrete trading strategies in a continuous-time setting.

1.2 Main results

In the following chapter, we will focus on the portfolio optimization problem with strictly positive transaction costs and exponential utility. We will continue the ansatz via impulse control methods made by Korn (1998), who derived an approximately optimal solution to this problem in the one-dimensional case, which we will analyse. Here the most outstanding result we will obtain is the following: Speaking in terms of the volume of money invested in the stock (later on often called the "risky money"), the portfolio is rebalanced whenever the risky money reaches the boundaries of an interval which is symmetrical to the optimal risky money of continuous trading. Just as the optimal risky money of continuous trading, these boundaries are constant in time and do not depend on the underlying stock price! Further we will also have a look at the optimization problem with transaction costs and exponential utility in the multi-dimensional case. Using the same techniques as in the one-dimensional case we will derive the approximately optimal solution for two uncorrelated stocks. Here, the regions where no rebalancing takes place are given by a rectangle (only proportional costs) and an ellipsoid (only fixed, fixed and proportional costs). Again, the boundaries of these regions with respect to the risky money will be independent of the stock prices.

Inspired by the results of Chapter 2 we will then (Chapter 3) focus on so-called bounded portfolio processes: Here we consider the case of logarithmic and power utility, respectively. In contrast to the second chapter, this time the optimal continuous portfolio process is constant in time and independent of the underlying stock price. As we are interested in discrete rebalancing we examine those trading strategies consisting of rebalancing whenever the portfolio process reaches the boundaries of an interval which is symmetrical around the optimal constant portfolio process of continuous trading. Here we will show that the wealth process belonging to the bounded portfolio process converges in probability to the one corresponding to the optimal constant portfolio process of continuous trading. Further we will prove that the expected utility from terminal wealth belonging to the bounded portfolio process is very near to the optimal expected utility obtained in the case without transaction costs.

Finally (Chapter 4) we will consider trading strategies which are characterized by equidistant rebalancing: Again we will focus on logarithmic and power utility, respectively. Here the portfolio is rebalanced at predetermined equidistant time points by shifting the portfolio process to the optimal one of continuous trading. Also in this case we can show that the expected utility from terminal wealth corresponding to this particular portfolio process is quite near to the one which is optimal in the absence of transaction costs.

The results of Chapter 2, 3 and 4 will show that any of the three mentioned types of discrete trading strategies (portfolio processes) can replace the continuous trading strategy in the presence of transaction costs. Above all in the presence of fixed transaction costs these strategies will prevent the investor from immediate ruin but will allow him to achieve an expected utility from terminal wealth which is very near to the optimal one (which could be achieved in the case without transaction costs). Hence, these strategies are acceptable solutions to the optimization problem (1.1) when (fixed) transaction costs are assumed.

Chapter 2

Portfolio optimization with transaction costs and exponential utility

The list of papers dealing with portfolio optimization under transaction costs is still not very long. Especially the highly realistic case of transaction costs including both a fixed and a variable cost component is not dealt with very often. In Korn (1998) an approach of Eastham and Hastings (1988) was taken up and refined where the portfolio problem under fixed and proportional transaction costs was transformed into an impulse control problem. As an explicit example the exponential utility case in a market with just one stock and bond was considered. However, as it seems to be impossible to solve the corresponding quasi-variational inequalities explicitly, an asymptotic solution method was presented.

In the following we take up this asymptotic approach, simplify it and examine some of its characteristics. Further, we consider the exponential utility portfolio problem in a market with two stocks and a bond and derive asymptotic solutions for it. Our findings will also indicate how to deal with the general n stock market setting.

2.1 The problem

We consider a financial market consisting of a riskless bond with constant price $P_0(t) = 1$ (i.e., we assume a zero interest rate) and a risky stock with price $P_1(t)$ given by

$$\begin{aligned} dP_1(t) &= P_1(t)\left[\mu dt + \sigma dW_t\right], \\ P_1(0) &= p_1, \end{aligned}$$

where μ, σ are real constants with $\sigma > 0$ and where W_t is a one-dimensional standard Brownian motion with the usual right-continuous Brownian filtration

$\{\mathcal{F}_t\}_{t\in[0,T]}$.

Our goal is the maximization of the expected utility from wealth $X(T)$ at the time horizon T. In the following, we specialize to the case of the exponential utility function, i.e., we are looking for a self-financing strategy $(\varphi_0(t), \varphi_1(t))$, $t \in [0,T]$, for bond and stock investments which solves

$$\max_{(\varphi_0,\varphi_1)} E\left(-e^{-\lambda X(T)}\right) \tag{2.1}$$

where $\lambda > 0$ is the coefficient of risk aversion. (Of course the utility function is given by $U(x) = 1 - e^{-\lambda x}$, but we omit the constant part for notational convenience.) Instead of looking at the evolution of the stock and bond prices it is also possible and it will be more convenient to consider the evolution of the amounts of money invested in both stock and bond, which we denote by $S(t)$ and $B(t)$, respectively. Due to the above assumed price evolution the evolution of the stock and bond holdings are given by

$$dB(t) = 0\,, \tag{2.2}$$

$$dS(t) = S(t)\left(\mu dt + \sigma dW_t\right)\,. \tag{2.3}$$

In 1986, Pliska solved problem (2.1) in the case without transaction costs: The optimal amount of money invested in the stock is given by

$$\varphi_1^*(t) = \frac{\mu}{\lambda\,\sigma^2\,P_1(t)}\,, \quad t \in [0,T]\,, \tag{2.4}$$

implying in particular that the optimal amount of money invested in the stock S^* is constant over time. Gains will be directly shifted into the bond component, losses will be directly paid out of it.

In the presence of transaction costs, given by a fixed cost component

$$K \geq 0 \tag{2.5}$$

and a proportional cost component

$$k\,|\Delta S|\,, \quad 0 \leq k < 1\,, \tag{2.6}$$

where ΔS is the volume of the transaction (i.e. the difference of the value of S after and before the transaction), this solution cannot be the optimal one. Since $(\varphi_0^*(t), \varphi_1^*(t))$ changes at each time instant (with infinite variation (see (2.4))), transaction costs will lead to immediate ruin, above all if fixed costs have to be paid. Therefore we will have to look for another class of strategies which do not cause this effect. One possibility is to use the **impulse control method** which will be introduced in the next section.

Remark 1. Without transaction costs it is not possible to obtain an explicit solution of the optimization problem under the constraint of a non-negative wealth process. Therefore this restriction is omitted (Pliska (1986)) and the resulting optimal wealth process has positive probability to attain negative values:

$$X^*(T) = X^*(t) + \frac{1}{\lambda}\frac{\mu^2}{\sigma^2}(T-t) + \frac{1}{\lambda}\frac{\mu}{\sigma}(W_T - W_t)\,. \tag{2.7}$$

Presenting our solution to (2.1) as we will do in the third section we will also not require a non-negative wealth process in order to obtain an explicit (approximate) solution. Nevertheless for convenience with regard to some technical details we require non-negativity constraints when we introduce the impulse control method in the next section.

2.2 Impulse control method

Definition 1. An impulse control strategy $\{(\theta_i, \Delta S_i), i \in I\!N\}$ is a sequence of intervention times θ_i and corresponding actions ΔS_i with

i) $0 \le \theta_i \le \theta_{i+1} \quad a.s.\,,$

ii) θ_i is a stopping time with respect to the filtration

$$\mathcal{G}_t := \sigma(\,(S(s-), B(s-)), s \le t\,)\,,$$

iii) ΔS_i is $\mathcal{G}_{\theta_i}$-measurable,

iv) $B(\theta_i) \ge 0\,, S(\theta_i) \ge 0$ for all $i \in I\!N$.

An impulse control will be called admissible if we have

v) $P\left(\lim_{i\to\infty} \theta_i \le T\right) = 0 \quad \forall T \ge 0\,.$

Using this definition we can rewrite the above optimization problem as

$$\max_{(\theta_i, \Delta S_i)\in Z} E(-e^{-\lambda X(T)}) \tag{2.8}$$

where Z denotes the set of all admissible impulse control strategies.

As mentioned above, the times θ_i denote the trading times of the investor. His trades then consist of changing the amount of money invested in the stock by ΔS_i and a corresponding adjustment of the bond component. Hence at the time of the ith transaction the investor has to pay transaction costs given by

$$K + k|\Delta S_i|, \quad 0 \le K\,, 0 \le k < 1\,, \tag{2.9}$$

which are subtracted from the bond holdings. We assume that there will be no transaction costs at the terminal time T (or equivalently: the investor is only interested in maximising his paper wealth). The analytical analogue to the Hamilton-Jacobi-Bellman equation of stochastic control in the above setting is to characterize the value function

$$v(t,b,s) := \sup_{(\theta_i,\Delta S_i)\in Z} E^{(t,b,s)}(-e^{-\lambda X(T)}) \tag{2.10}$$

for all $(t,b,s) \in [0,T]\times I\!R_0^+ \times I\!R_0^+$ as a (certain kind of) solution of the so-called quasi variational inequalities.

Before we present their exact definition we will give some motivation for these inequalities: At each time instant in $[0,T]$ we have two alternatives: Rebalancing the portfolio or not. If it is optimal to intervene immediately, the value function before and after the transaction must coincide. This relation is expressed by the equality $v(t,b,s) = Mv(t,b,s)$ where Mv is given by

$$Mv(t,b,s) := \max_{\Delta S\in A(b,s)} v(t, b-K-\Delta S - k|\Delta S|, s+\Delta S) \tag{2.11}$$

with

$$A(b,s) = \left[-s, \min\left(\frac{b-K}{1+k}, \frac{b-K}{1-k}\right)\right]. \tag{2.12}$$

Due to its definition, Mv is the expected utility which corresponds to "doing the best immediate transaction ΔS and behaving optimally afterwards". On the other hand, if it is not optimal to intervene immediately, the value of v which is obtained by choosing the best strategy of all strategies (including the ones without an immediate transaction) has to exceed the value of Mv. Summing up both relations we obtain

$$v(t,b,s) \geq Mv(t,b,s)\,.$$

To derive the next part of the qvi we take a look at the Hamilton-Jacobi-Bellman equation of stochastic control. If $\tilde{v}$ is the value function in the problem without transaction costs, it has to satisfy the Hamilton-Jacobi-Bellman equation

$$\sup_{(b,s)}\{L\tilde{v}(t,b,s)\} := \sup_{(b,s)}\left\{\frac{1}{2}\sigma^2 s^2 \tilde{v}_{ss}(t,b,s) + \mu s\,\tilde{v}_s(t,b,s) + \tilde{v}_t(t,b,s)\right\} = 0$$

with

$$\tilde{v}(T,b,s) = U(b+s)$$

in order to have

$$\tilde{v}(t,b,s) = E^{(t,b,s)}(U(b+s))\,.$$

In the presence of transaction costs (where we do not rebalance continuously) we can therefore only require

$$Lv(t,b,s) := \frac{1}{2}\sigma^2 s^2 v_{ss}(t,b,s) + \mu s\, v_s(t,b,s) + v_t(t,b,s) \leq 0\,.$$

As long as we do not intervene, the process S (which is in our case the one that is controlled) is a diffusion. Therefore the value function has to satisfy (as in the Hamilton-Jacobi-Bellman case but of course without the sup operator) $Lv(t,b,s) = 0$.

Combining the above relations we obtain

$$(v(t,b,s) - Mv(t,b,s))\, Lv(t,b,s) = 0$$

no matter whether it is optimal to intervene or not.

Of course the assumptions we made to derive the above relations which are summed up in the next definition have to be justified later on.

Definition 2. The following relations are called the **quasi-variational inequalities (qvi)** corresponding to the optimization problem (2.10).

i) $Lv(t,b,s) := \frac{1}{2}\sigma^2 s^2 v_{ss}(t,b,s) + \mu s\, v_s(t,b,s) + v_t(t,b,s) \leq 0\,,$

ii) $v(t,b,s) \geq M\, v(t,b,s)\,,$

iii) $(v(t,b,s) - Mv(t,b,s))\, Lv(t,b,s) = 0\,,$

iv) $v(T,b,s) = -e^{-\lambda(b+s)}$

for all $(t,b,s) \in [0,T) \times I\!R_0^+ \times I\!R_0^+$.

Using the quasi-variational inequalities, we can construct a special impulse control strategy from a solution of the qvi.

Definition 3. Let v be a continuous solution of the quasi-variational inequalities. Further, assume that the investor starts his portfolio optimization at $t > 0$. Then the following impulse control strategy is called a **qvi-control**:

i) $(\theta_0, \Delta S_0) := (t, 0)\,,$

ii) $\theta_i = \inf\{u \geq \theta_{i-1} \geq t : v(u, B(u), S(u)) = Mv(u, B(u), S(u))\}\,,$

iii) $\Delta S_i :=$
$\arg\max_{\Delta S \in A(B(\theta_i -), S(\theta_i -))} \{\, v(\theta_i -, B(\theta_i -) - K - k|\Delta S| - \Delta S, S(\theta_i -) + \Delta S)\,\}\,,$

i.e. at every instant of time where v and Mv coincide the investor intervenes. He then chooses the action that is the maximiser of the optimization problem corresponding to Mv. Further, we assume that no transaction costs are associated with the formal starting zero intervention $(t, 0)$.

The justification for considering quasi-variational inequalities and qvi-controls is given by the next theorem. Before we state this result we will introduce the following manner of speaking: "A function f satisfies Itô's formula (with respect to the process S)" if it is continuous and satisfies

$$\begin{aligned} f(l, B(l), S(l)) &= f(0, B(0), S(0)) \\ &+ \int_0^l (f_l(u, B(u), S(u)) + f_S(u, B(u), S(u))\, \mu\, S(u))\, du \\ &+ \int_0^l \frac{1}{2} f_{SS}(u, B(u), S(u))\, \sigma^2\, S(u)^2\, du \\ &+ \int_0^l f_S(u, B(u), S(u))\, \sigma\, S(u)\, dW_u \end{aligned}$$

for all $l \in [0, T]$ in "a certain sense". This could be the usual case where f is a $C^{1,2}$-function. It will also include the cases where the partial derivatives are assumed to be left derivatives or generalised derivatives (compare for example Krylov (1980) or Haussmann (1994)) or a version including a local time term (provided f will be concave in this case).

Theorem 1. *If there exists a solution v^* of the qvi given in Definition 2 that satisfies Itô's formula and the growth condition*

$$E\left(\int_0^T (\sigma\, S(u)\, v_S^*(u, B(u), S(u)))^2\, du\right) < \infty \tag{2.13}$$

for each process $(B(l), S(l))\, , l \in [0, T]$ corresponding to admissible impulse controls then

$$v(t, b, s) \leq v^*(t, b, s) \quad \forall\, (t, b, s) \in [0, T) \times I\!R_0^+ \times I\!R_0^+\, . \tag{2.14}$$

Further, if there exists an admissible qvi-control as given in Definition 3 then it is an optimal impulse control and v^* is identical to the value function v.

Proof: Let $\{(\theta_k, \Delta S_k)\}_{k \in I\!N}$ be an admissible impulse control for fixed $T > 0$. Set

$$\hat{\theta}_k := \theta_k \wedge T$$

and for notational convenience

$$v^*(l) := v^*(l, B(l), S(l))\, , \quad l \in [0, T]\, .$$

Then we have

$$v^*(\hat{\theta}_k) - v^*(\hat{\theta}_{k-1}) = v^*(\hat{\theta}_{k-}) - v^*(\hat{\theta}_{k-1}) + 1_{\{\theta_k < T\}}(v^*(\hat{\theta}_k) - v^*(\hat{\theta}_{k-}))$$

which implies

$$v^*(\hat\theta_j) - v^*(t) = \sum_{k=1}^{j} \Big(v^*(\hat\theta_{k-}) - v^*(\hat\theta_{k-1})\Big) + \sum_{k=1}^{j} 1_{\{\theta_k<T\}}(v^*(\hat\theta_k) - v^*(\hat\theta_{k-}))\,.$$

Since v^* solves the qvi we obtain from Definition 2 ii) that

$$v^*(\hat\theta_{k-}) \ge v^*(\hat\theta_k)\,.$$

Further, on intervals without intervention, v^* satisfies

$$\begin{aligned} v^*(\hat\theta_{k-}) - v^*(\hat\theta_{k-1}) \quad &= \quad \int_{\hat\theta_{k-1}}^{\hat\theta_k} Lv^*(u,B(u),S(u))\,du \\ &\qquad + \int_{\hat\theta_{k-1}}^{\hat\theta_k} \sigma\, S(u)\, v_S^*(u,B(u),S(u))\, dW_u \\ &\overset{Def.2i)}{\le} \quad \int_{\hat\theta_{k-1}}^{\hat\theta_k} \sigma\, S(u)\, v_S^*(u,B(u),S(u))\, dW_u\,. \end{aligned}$$

These two inequalities together with the assumption (2.13) imply

$$\begin{aligned} & v^*(t) - E^{(t,b,s)}(v^*(\hat\theta_j)) \\ \ge \quad & -\sum_{k=1}^{j} E^{(t,b,s)} \left(\int_{\hat\theta_{k-1}}^{\hat\theta_k} \sigma\, S(u)\, v_S^*(u,B(u),S(u))\, dW_u\right) + 0 \\ = \quad & 0\,. \end{aligned}$$

Since $\{(\theta_k, \Delta S_k)\}_{k\in I\!N}$ is an admissible impulse control, it is obvious that

$$\theta_k \le \theta_{k+1} \quad \text{a.s } \forall k \in I\!N$$

and that

$$\lim_{k\to\infty} \theta_k > T \quad \text{a.s.}$$

thus implying that

$$\lim_{k\to\infty} \hat\theta_k = T \quad \text{a.s.}\,.$$

Due to $S(\hat\theta_j) \ge 0\,, B(\hat\theta_j) \ge 0$ and the particular form of the utility function, $|v^*|$ is bounded by 1 on the interval $[0,T]$ for each admissible impulse control. Hence dominated convergence implies

$$\begin{aligned} v^*(t) \quad &\ge \quad E^{(t,b,s)}(v^*(\hat\theta_j)) \\ &\overset{j\to\infty}{\longrightarrow} \quad E^{(t,b,s)}(v^*(T)) \\ &= \quad E^{(t,b,s)}(v^*(T,B(T),S(T))) \\ &\overset{Def.2iv)}{=} \quad E^{(t,b,s)}\left(-e^{-\lambda\, X(T)}\right)\,. \end{aligned}$$

Since this inequality holds for all admissible impulse controls, we have proved (2.14). If there exists an admissible qvi-control, Definition 2 iii) and Definition 3 even give us

$$v^*(t) = v^*(t,b,s) = v(t,b,s)\,.$$

□

Remark 2. If we assume that v^* satisfying the assumptions of Theorem 1 also satisfies

$$\begin{aligned} &\lim_{j\to\infty} E^{(t,b,s)}(v^*(\theta_j \wedge T, B(\theta_j \wedge T), S(\theta_j \wedge T))) \\ &= E^{(t,b,s)}(\lim_{j\to\infty} v^*(\theta_j \wedge T, B(\theta_j \wedge T), S(\theta_j \wedge T))) \end{aligned} \tag{2.15}$$

for all admissible impulse strategies (where we do not require the non-negativity of S and B from Definition 1 iv)) then Theorem 1 holds for all impulse control strategies (which satisfy Definition 1 i) to iii) and v)). Later on we will give some numerical examples where negative values of S can indeed occur.

Due to the results of Theorem 1 the only thing which remains to do in order to obtain the value function (2.10) is to solve the corresponding quasi-variational inequalities. Unfortunately, this could not be done explicitly. Thus an asymptotic method was proposed in Korn (1998) based on a similar asymptotic method by Whalley and Wilmott (1994) in the context of option pricing.

Before we introduce and apply this method, we first make some simplifications. Due to the relation

$$v(t,b,s) = e^{-\lambda b}\, v(t,0,s)$$

we can compute the optimal stock strategy independently of the bond holdings to solve the problem! We thus only have to find $v(t,0,s)$. To this end, the function

$$q(t,p,y) := v(t,0,y\,p)$$

is introduced and it is assumed that the (p,y)-space (i.e., the space of number of stocks held y and stock prices p) can be divided into three regions:

- the no-transaction region **NT** (i.e., the region in the (p,y)-space where it is optimal to do nothing) containing Pliska's optimal solution (given by the solid bold line in Figure 1 below);
- the selling region **SS** (i.e., the region in the (p,y)-space where it is optimal to sell shares);
- the buying region **BS** (i.e., the region in the (p,y)-space where it is optimal to buy shares).

The boundaries of NT (solid lines) are denoted by

$$\hat{\mathbf{y}}^+(\mathbf{t}, \mathbf{P_1}(\mathbf{t})) \quad \text{(upper boundary)}$$

and

$$\hat{\mathbf{y}}^-(\mathbf{t}, \mathbf{P_1}(\mathbf{t})) \quad \text{(lower boundary)}\,.$$

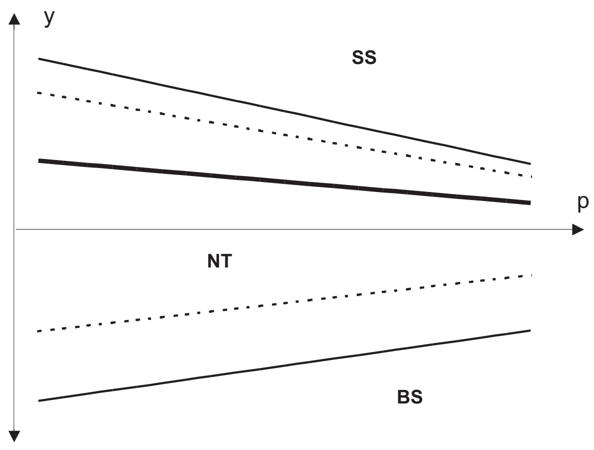

Fig. 1. Schematic diagram of (p,y)-space

Further, we assume that in NT there are so-called "restarting lines" to which the number of shares is shifted whenever it has reached the boundary of NT. These restarting lines (dashed lines) are denoted by

$$\mathbf{y}^{+}(\mathbf{t}, \mathbf{P_1}(\mathbf{t})) \quad \text{(upper restarting line)}$$

and

$$\mathbf{y}^{-}(\mathbf{t}, \mathbf{P_1}(\mathbf{t})) \quad \text{(lower restarting line)}\,.$$

To go on we have to set up the qvi in terms of q instead of v such that we obtain the following form of the qvi:

i) $Lq(t,p,y) := \frac{1}{2}\sigma^2\, p^2\, q_{pp}(t,p,y) + \mu\, p\, q_p(t,p,y) + q_t(t,p,y) \leq 0\,,$

ii) $q(t,p,y) \geq Mq(t,p,y) = \max\limits_{u \in A(0,py)} \left\{ e^{\lambda(K + k|u|p + up)} q(t,p,y+u) \right\},$

iii) $(\, q(t,p,y) - Mq(t,p,y)\,)\, Lq(t,p,y) = 0\,,$

iv) $q(T,p,y) = -e^{-\lambda\, p\, y}$

where $(t,p,y) \in [0,T) \times I\!R_0^+ \times I\!R_0^+$.

To solve these qvi, it is important to note that due to our remark upon the meaning of the qvi (before Definition 2), the classification of the three regions NT, SS and BS and the definition of q imply that

- $Lq = 0$ in NT,
- $Mq = q$ in BS and SS.

2.3 The asymptotic analysis approach

The basic idea behind this approach is that we assume that the value function (which in our case is replaced by q) can be viewed as a function of a small parameter ε that should indeed coincide with the value function of the problem without transaction costs for $\varepsilon = 0$. Hence, the transaction costs are also assumed to be a function of this parameter and must therefore vanish for $\varepsilon = 0$. In the no-transaction region (where we have $Lq = 0$ and $q > Mq$) the value function will (formally) be written as an expansion of powers of $\varepsilon^{1/4}$. We then substitute this expression into the equation $Lq = 0$ and order the terms by different powers of $\varepsilon^{1/4}$. This will give us an infinite set of equations for the coefficient functions of the expansion. However, believing that at least for small values of ε we could substitute the infinite expansion of the value function by a finite one to get a good approximation we will only consider the first three such equations which we can solve with the help of appropriate boundary conditions. (Since we only use the first three equations the approximation will be a quadratic one and the resulting no-transaction region will be symmetric.)

The natural point around which such an expansion should be made is the optimal number of shares of the stock of the problem without transaction costs, i.e. the Pliska solution

$$y^*(t,p) := \varphi_1^*(t) = \frac{\mu}{\lambda\,\sigma^2\,p}$$

where $p := P_1(t)$. As in Atkinson and Wilmott (1995) and in Whalley and Wilmott (1994) we first rescale the y-variable and the cost function by introducing Y and $K(.,.)$ given by

$$y = y^*(t,p) + \varepsilon^{1/4} Y \tag{2.16}$$

$$\varepsilon\, K(p,u) = k(\,p, |\varepsilon^{1/4} u|\,) \tag{2.17}$$

where $k(p,u)$ represents the transaction costs (the sum of fixed and proportional ones) if selling or buying of u shares of the stock takes place. It will prove to be very convenient to assume the following form of the cost function:

$$k(p,u) = A\,\varepsilon^{3/4}\,|u|p + B\,\varepsilon\,, \quad 0 < \varepsilon << 1\,, A, B > 0\,. \tag{2.18}$$

If we take

$$k = A\,\varepsilon^{3/4} \tag{2.19}$$

and

$$K = B\,\varepsilon \tag{2.20}$$

(2.18) coincides with our usual form of the cost function. As a consequence the scaled costs $K(p,u) = A|u|p + B$ are independent of ε. Certainly the above scaling seems to be unmotivated but it will prove to be the appropriate one to obtain proper results.

Since we have rescaled the variable of interest y, we have to define the value function in terms of the new variable Y. Thus we set

$$Q(t,p,Y) := q(t,p,y)\,. \tag{2.21}$$

Of course we also have to rescale the boundaries and restarting lines of the no-transaction region which we introduced in the previous section. According to the rescaling of y (2.16), we assume that the boundaries of the no-transaction region have the forms

$$\hat{y}^+(t,p) = y^*(t,p) + \varepsilon^{1/4}\,\hat{Y}^+ \quad \text{(upper boundary)}\,, \tag{2.22}$$

$$\hat{y}^-(t,p) = y^*(t,p) + \varepsilon^{1/4}\,\hat{Y}^- \quad \text{(lower boundary)}\,, \tag{2.23}$$

and that the restarting lines are of the forms

$$y^+(t,p) = y^*(t,p) + \varepsilon^{1/4}\,Y^+ \quad \text{(upper restarting line)}\,, \tag{2.24}$$

$$y^-(t,p) = y^*(t,p) + \varepsilon^{1/4}\,Y^- \quad \text{(lower restarting line)} \tag{2.25}$$

where the signs of $\hat{Y}^-$ and Y^- are negative. As explained above, we assume that the investor has to sell stocks if the pair $(P_1(t), y(t))$ of stock price and number of shares belongs to the selling region, thus shifting $(P_1(t), y(t))$ to the corresponding point of the upper restarting line and that he has to buy stocks to shift the pair $(P_1(t), y(t))$ from the buying region to the corresponding point of the lower restarting line. So the optimal action is to sell $\varepsilon^{1/4}\,(\hat{Y}^+ - Y^+)$ shares and to buy $\varepsilon^{1/4}\,(Y^- - \hat{Y}^-)$ shares, respectively. All these assumptions will be justified in the end when we obtain explicit expressions for $\hat{Y}^+, \hat{Y}^-, Y^+$ and Y^-.

In contrast to the previous section we allow the process $y(t)$ to attain negative values in order to get an explicit solution.

As we have stated at the end of the last section, we have $Mq = q$ in both BS and SS due to the definition of BS and SS. Hence, we obtain

$$\begin{aligned} Q(t,p,Y) &= q(t,p,y) \\ &= e^{\lambda(K + k\,\varepsilon^{1/4}\,|Y^+ - Y|\,p + \varepsilon^{1/4}\,(Y^+ - Y)\,p)}\, q(t,p,y^*(t,p) + \varepsilon^{1/4}\,Y^+) \\ &= e^{\lambda(\varepsilon\,K(p, Y - Y^+) + \varepsilon^{1/4}\,(Y^+ - Y)\,p)}\, Q(t,p,Y^+) \end{aligned} \tag{2.26}$$

for all $(p,y) \in SS$ and

$$Q(t,p,Y) = e^{\lambda(\varepsilon\,K(p, Y^- - Y) + \varepsilon^{1/4}\,(Y^- - Y)\,p)}\, Q(t,p,Y^-) \tag{2.27}$$

for all $(p,y) \in BS$.

In NT we (formally) expand $Q(t,p,Y)$ in powers of $\varepsilon^{1/4}$ around the Pliska solution $y^* = y^*(t,p)$ in the following way:

$$Q(t,p,Y) = -e^{-[\lambda\, p\,(y^* + \varepsilon^{1/4} Y) + H_0(t) + \varepsilon^{1/2}\, H_1(t,p) + \varepsilon\, H_2(t,p,Y) + \ldots]}\,. \tag{2.28}$$

Of course (2.28) is quite similar to Pliska's optimal expected utility

$$E^{(t,0,s)}(U(X^*(T))) = -e^{-\lambda\, p\, y^* - \frac{1}{2}\frac{\mu^2}{\sigma^2}(T-t)} \tag{2.29}$$

that can be derived from (2.7). In particular, for $\varepsilon = 0$ $Q(t,p,Y)$ would coincide with this for

$$H_0(t) = \frac{1}{2}\,\frac{\mu^2}{\sigma^2}\,(T-t)\,.$$

The next step is to use the equality $Lq = 0$ for $(p,y) \in NT$. Due to the above expansion of Q and due to $Q(t,p,Y) = q(t,p,y)$ we obtain

$$\begin{aligned}
q_t &= -Q\,[\,H_{0_t}(t) + \varepsilon^{1/2} H_{1_t}(t,p) + \varepsilon H_{2_t}(t,p,Y) + \cdots]\,,\\
q_p &= -Q\,[\,\lambda\, y^* + \lambda\,\varepsilon^{1/4} Y + \varepsilon^{1/2} H_{1_p}(t,p) - \varepsilon^{3/4} H_{2_Y}(t,p,Y)\, y_p^*\\
&\qquad + \varepsilon\, H_{2_p}(t,p,Y) + \cdots]\,,\\
q_{pp} &= Q\,\{\,\lambda^2 y^{*2} + 2\,\varepsilon^{1/4}\lambda^2 y^*\, Y + \varepsilon^{1/2}[\,\lambda^2 Y^2 + 2\,\lambda\, y^* H_{1_p}(t,p)\\
&\qquad - H_{1_{pp}}(t,p) - H_{2_{YY}}(t,p,Y)\, y_p^{*2}\,] + \varepsilon^{3/4}\cdots\}
\end{aligned}$$

where the (lowest) subscripts denote partial derivatives with respect to the corresponding variables and the dots indicate that all remaining terms include powers of ε of higher (or at least equal) order. To obtain the derivatives with respect to p we have to note that the variables y^* and Y both are functions of p where $Y_p = -\varepsilon^{-1/4}\, y_p^*$.

Plugging the derivatives into the formula for Lq given in the last section and rearranging by powers of $\varepsilon^{1/4}$ (where we neglect terms of higher order) we derive

$$\begin{aligned}
0 \;=\; & Q\,\Big\{\Big[-H_{0_t} - \lambda\, p\,\mu\, y^* + \frac{1}{2}\lambda^2\sigma^2\, p^2\, y^{*2}\Big] + \varepsilon^{1/4}\,\Big[-\lambda\, p\,\mu\, Y + \lambda^2\sigma^2\, p^2\, y^*\, Y\Big]\\
& + \varepsilon^{1/2}\Big[\,-H_{1_t} - p\,\mu\, H_{1_p}\\
&\qquad + \frac{1}{2}\sigma^2\, p^2\,\Big(\,\lambda^2\, Y^2 + 2\,\lambda\, y^*\, H_{1_p} - H_{1_{pp}} - H_{2_{YY}}\, y_p^{*2}\Big)\Big]\Big\}\,.
\end{aligned}$$

As we require this to hold similarily for all $\varepsilon \geq 0$, in NT (where $Lq = 0$) the three following equations have to hold:

$$\begin{aligned}
O(1):&\qquad 0 = H_{0_t} + \lambda\, y^*\, p\,\mu - \frac{1}{2}\,\lambda^2\,\sigma^2\, p^2\, y^{*2}\,,\\
O(\varepsilon^{1/4}):&\qquad 0 = \lambda\, Y\,\mu\, p - \lambda^2\sigma^2\, p^2\, Y\, y^*\,,\\
O(\varepsilon^{1/2}):&\qquad 0 = H_{1_t} + H_{1_p}\,\mu\, p\\
&\qquad\qquad - \frac{1}{2}\,\sigma^2\, p^2\,(\lambda^2 Y^2 + 2\,\lambda\, y^*\, H_{1_p} - H_{1_{pp}} - H_{2_{YY}}\, y_p^{*2})\,.
\end{aligned}$$

By definition of y^* the equation $O(\varepsilon^{1/4})$ is always satisfied. Further, for $\varepsilon = 0$, Q can only be equal to the (suitably transformed) value function of the problem without transaction costs if it satisfies the final condition

$$Q(T,p,Y) = -e^{-\lambda\, p\, y^*}$$

thus implying

$$H_0(T) = 0\,.$$

Together with $O(1)$ this condition leads to

$$H_0(t) = \frac{1}{2}\frac{\mu^2}{\sigma^2}\,(T-t)\,, \tag{2.30}$$

i.e. for $\varepsilon = 0$ we have agreement with the value function of the case without transaction costs. To solve $O(\varepsilon^{1/2})$ we define

$$\begin{aligned} G(H_1) &:= H_{1_t} + \frac{1}{2}\sigma^2\, p^2\, H_{1_{pp}} \qquad (2.31) \\ &= H_{1_t} + H_{1_p}\,\mu\, p - \frac{1}{2}\sigma^2\, p^2\,(2\,\lambda\, y^*\, H_{1_p} - H_{1_{pp}}) \end{aligned}$$

thus obtaining

$$H_{2_{YY}} = \frac{\lambda^2\, Y^2}{y_p^{*2}} - 2\,\frac{G(H_1)}{\sigma^2\, p^2\, y_p^{*2}} \tag{2.32}$$

which leads to

$$H_2 = \frac{1}{12}\frac{\lambda^2}{y_p^{*2}}\,Y^4 - \frac{G(H_1)}{\sigma^2\, p^2\, y_p^{*2}}\,Y^2 + C\,Y + D \tag{2.33}$$

where $C\,,D$ are real constants. We will determine them with the help of the boundary conditions at the border of NT. We will require the usual smooth-pasting conditions for the value function (i.e. we assume it to be as smooth as possible which in this case means that it should be continuous and continuously differentiable in p (for fixed (t,y)) at the boundary points of NT). It is shown in Korn (1992), Chapter 4, that then there exists a suitable version of Itô's formula that can be applied. (Hence, the assumptions of Theorem 1 would be satisfied.) Using all these assumptions, we will be able to determine the explicit form of the (approximate) value function via its behaviour at the borders between NT, SS, and BS.

At first, we will concentrate on the border between NT and SS. Using (2.26) with $Y = \hat{Y}^+$ and the expansion (2.28) of Q, we obtain from the requirement of continuity of the (approximate) value function at the border between NT and SS that

$$\begin{aligned} &-e^{-[\lambda\, p\,(y^* + \varepsilon^{1/4}\hat{Y}^+) + H_0(t) + \varepsilon^{1/2}\, H_1(t,p) + \varepsilon\, H_2(t,p,\hat{Y}^+) + \ldots]} \\ &= -e^{\lambda(\varepsilon\, K(p,\hat{Y}^+ - Y^+) + \varepsilon^{1/4}\,(Y^+ - \hat{Y}^+)\, p)} \\ &\qquad e^{-[\lambda\, p\,(y^* + \varepsilon^{1/4} Y^+) + H_0(t) + \varepsilon^{1/2}\, H_1(t,p) + \varepsilon\, H_2(t,p,Y^+) + \ldots]}\,. \end{aligned}$$

Now comparing the coefficients of ε we get

$$H_2(t,p,Y^+) - H_2(t,p,\hat{Y}^+) = \lambda\, K(p, \hat{Y}^+ - Y^+) \tag{2.34}$$

for all $(p, \hat{Y}^+)$ such that the corresponding pair (p,y) is an element of the border between NT and SS.

With $Y = \hat{Y}^+$ and $\Delta Y := Y^+ - \hat{Y}^+$, (2.26) reads

$$Q(t,p,\hat{Y}^+) = Q(t,p,\hat{Y}^+ + \Delta Y)\, e^{\lambda\left(\varepsilon\, K(p,-\Delta Y) + \varepsilon^{1/4}\,\Delta Y\, p\right)}$$

where the second part equals Mq. Since we assume the transaction ΔY to be the optimal one whenever the border between NT and SS is attained and since there are no restrictions on the transaction value, the first order necessary conditions imply that the right side of the previous equation has to satisfy

$$\frac{d}{dZ}\left[Q(t,p,\hat{Y}^+ + Z)\, e^{\lambda\left(\varepsilon\, K(p,-Z) + \varepsilon^{1/4}\, Z\, p\right)}\right]\Bigg|_{Z=\Delta Y} = 0\,.$$

Due to

$$\frac{d}{dZ} Q(t,p,\hat{Y}^+ + Z) = -Q(t,p,\hat{Y}^+ + Z)\,\left[\lambda\, p\, \varepsilon^{1/4} + \varepsilon\, H_{2_Y}(t,p,\hat{Y}^+ + Z)\right]$$

and

$$\begin{aligned} &\frac{d}{dZ}\left[e^{\lambda\left(\varepsilon\, K(p,-Z) + \varepsilon^{1/4}\, p\, Z\right)}\right] \\ &\quad = \; e^{\lambda\left(\varepsilon\, K(p,-Z) + \varepsilon^{1/4}\, p\, Z\right)}\, \lambda\, \left[-\varepsilon\, K_u(p,-Z) + \varepsilon^{1/4}\, p\right] \end{aligned}$$

where K_u denotes the partial derivative of K with respect to the second component, we obtain

$$H_{2_Y}(t,p,Y^+) = -\lambda\, K_u(p, \hat{Y}^+ - Y^+) \tag{2.35}$$

for all $(p, \hat{Y}^+)$ as above.

Further, at the border between NT and SS the value function is required to be continuously differentiable with respect to the number of stocks. Therefore (2.26) implies

$$Q_Y(t,p,Y)|_{Y=\hat{Y}^+} = \frac{d}{dY}\left[e^{\lambda\left(\varepsilon\, K(p, Y-Y^+) + \varepsilon^{1/4}\,(Y^+ - Y)\, p\right)}\, Q(t,p,Y^+)\right]\Bigg|_{Y=\hat{Y}^+}.$$

Due to

$$Q_Y(t,p,Y) = -Q(t,p,Y)\,\left[\lambda\, p\, \varepsilon^{1/4} + \varepsilon\, H_{2_Y}(t,p,Y)\right]$$

and

$$\begin{aligned} &\frac{d}{dY}\left[e^{\lambda\left(\varepsilon\, K(p, Y-Y^+) + \varepsilon^{1/4}\,(Y^+ - Y)\, p\right)}\, Q(t,p,Y^+)\right] \\ &\quad = \; \lambda\, Q(t,p,Y)\,\left[\varepsilon\, K_u(p, Y - Y^+) - \varepsilon^{1/4}\, p\right] \end{aligned}$$

we therefore obtain

$$H_{2_Y}(t,p,\hat{Y}^+) = -\lambda\, K_u(p, \hat{Y}^+ - Y^+)\,. \tag{2.36}$$

Now we will use the equalities (2.34)-(2.36) to determine the explicit form of H_2: Due to (2.33) the derivative of H_2 with respect to Y reads

$$H_{2_Y}(t,p,Y) = \frac{\lambda^2}{3\, y_p^{*2}}\, Y^3 - 2\, \frac{G(H_1)}{(\sigma\, p\, y_p^*)^2}\, Y + C\,. \tag{2.37}$$

Using (2.35) and (2.36) we thus obtain

$$0 = \frac{\lambda^2}{3\, y_p^{*2}}\, (Y^{+3} - \hat{Y}^{+3}) - 2\, \frac{G(H_1)}{(\sigma\, p\, y_p^*)^2}\, (Y^+ - \hat{Y}^+)$$

which is equivalent to

$$G(H_1) = \frac{(\sigma\, p\, \lambda)^2}{6}\, (Y^{+2} + Y^+\, \hat{Y}^+ + \hat{Y}^{+2}) \tag{2.38}$$

for all $(p, \hat{Y}^+)$ such that (p,y) is an element of the border between NT and SS at time t.

Due to (2.37) and (2.38) equation (2.36) implies the equality

$$\begin{aligned} -\lambda\, A\, p &= -\lambda\, K_u(p, \hat{Y}^+ - Y^+) \\ &= H_{2_Y}(t,p,\hat{Y}^+) \\ &= \frac{\lambda^2}{3\, y_p^{*2}}\, \hat{Y}^{+3} - 2\, \frac{\lambda^2}{6\, y_p^{*2}}\, \hat{Y}^+\, (Y^{+2} + Y^+\, \hat{Y}^+ + \hat{Y}^{+2}) + C \end{aligned}$$

leading to

$$-A\, p = -\frac{\lambda}{3\, y_p^{*2}}\, (Y^+ + \hat{Y}^+)\, \hat{Y}^+\, Y^+ + \frac{C}{\lambda}\,. \tag{2.39}$$

Further, replacing $G(H_1)$ in the general form (2.33) of H_2 we obtain

$$H_2(t,p,Y) = \frac{\lambda^2}{12\, y_p^{*2}}\, Y^4 - \frac{\lambda^2}{6\, y_p^{*2}}\, (Y^{+2} + Y^+\, \hat{Y}^+ + \hat{Y}^{+2})\, Y^2 + C\, Y + D\,.$$

Then (2.34) reads

$$\begin{aligned} &\lambda\, K(p, \hat{Y}^+ - Y^+) \\ &\quad = \frac{\lambda^2}{12\, y_p^{*2}}\, (Y^{+4} - \hat{Y}^{+4}) + C\, (Y^+ - \hat{Y}^+) \\ &\qquad - \frac{\lambda^2}{6\, y_p^{*2}}\, (Y^{+2} + Y^+\, \hat{Y}^+ + \hat{Y}^{+2})\, (Y^{+2} - \hat{Y}^{+2})\,. \end{aligned}$$

Since the function on the left side is an even function, the function on the right has to be even, too, for all $(p, \hat{Y}^+)$ at the border between NT and SS. This is only the case if $C = 0$. (If $k, K > 0$ the functions Y^+ and $\hat{Y}^+$ do not coincide.) Due to $C = 0$ we then obtain

$$\lambda K(p, \hat{Y}^+ - Y^+) = \frac{\lambda^2}{12\, y_p^{*2}} (\hat{Y}^+ + Y^+)^3 (\hat{Y}^+ - Y^+) \tag{2.40}$$

and (from (2.39))

$$A = \frac{\lambda}{3\, p\, y_p^{*2}} (Y^+ + \hat{Y}^+)\, \hat{Y}^+ Y^+ .$$

Using the relation

$$y_p^{*2} = \frac{\mu^2}{\sigma^4 \lambda^2 p^4}$$

we derive

$$(Y^+ + \hat{Y}^+)\, \hat{Y}^+ Y^+ = \frac{3\, A\, \mu^2}{\lambda^3 \sigma^4 p^3} \tag{2.41}$$

and (from (2.40))

$$\begin{aligned} \lambda\, B + \lambda \left[\frac{\lambda^3 \sigma^4 p^3}{3\, \mu^2} (Y^+ + \hat{Y}^+)\, \hat{Y}^+ Y^+ \right] p\, (\hat{Y}^+ - Y^+) \\ = \frac{\lambda^4 \sigma^4 p^4}{12\, \mu^2} (\hat{Y}^+ + Y^+)^3 (\hat{Y}^+ - Y^+) \end{aligned}$$

thus giving us

$$(\hat{Y}^+ - Y^+)^3 (\hat{Y}^+ + Y^+) = \frac{12\, B\, \mu^2}{\lambda^3 \sigma^4 p^4} . \tag{2.42}$$

We can perform all these computations for (p, Y) such that (p, y) is an element of BS (and the border between NT and BS, respectively). Then we obtain the equalities

$$\begin{aligned} \lambda K(p, Y^- - \hat{Y}^-) &= H_2(t, p, Y^-) - H_2(t, p, \hat{Y}^-) , \\ H_{2_Y}(t, p, Y^-) &= \lambda K_u(p, Y^- - \hat{Y}^-) , \\ H_{2_Y}(t, p, \hat{Y}^-) &= \lambda K_u(p, Y^- - \hat{Y}^-) \end{aligned}$$

as well as

$$G(H_1) = \frac{(\sigma\, p\, \lambda)^2}{6} (Y^{-2} + Y^- \hat{Y}^- + \hat{Y}^{-2})$$

for all $(p, \hat{Y}^-)$ with $(p, y) \in NT \bigcap BS$. Due to (2.38) and $\hat{Y}^- \leq Y^- \leq 0$ it is obvious that

$$Y^- = -Y^+ , \qquad \hat{Y}^- = -\hat{Y}^+ . \tag{2.43}$$

Therefore it is sufficient to determine Y^+ and $\hat{Y}^+$ which have to satisfy the equations (2.41) and (2.42). Of course it still remains to prove that these two equations have a unique solution. This will be done in:

Proposition 1. *Let the parameters* $\mu\,,\sigma\,,\lambda\,,p$ *be positive.*

i) For positive parameters $A\,,B$ *(i.e. the cost parameters* $k\,,K$ *are positive, too) there exists a unique solution* $(Y^+,\hat{Y}^+)$ *to the system (2.41)/(2.42) of non-linear equations satisfying*

$$0 < Y^+ < \hat{Y}^+ \,.$$

ii) For $A=0\,,B>0$ *(i.e.* $k=0$ *and* $K>0$*) the only non-negative solution to (2.41)/(2.42) is given by*

$$(Y^+,\hat{Y}^+) = \left(0, \sqrt[4]{\frac{12\,B\,\mu^2}{\lambda^3\,\sigma^4\,p^4}}\right)\,. \tag{2.44}$$

iii) For $A>0\,,B=0$ *(i.e.* $k>0\,,K=0$*) the only non-negative solution to (2.41)/(2.42) is given by*

$$(Y^+,\hat{Y}^+) = \left(\sqrt[3]{\frac{3\,A\,\mu^2}{2\,\sigma^4\,\lambda^3\,p^3}}, \sqrt[3]{\frac{3\,A\,\mu^2}{2\,\sigma^4\,\lambda^3\,p^3}}\right)\,. \tag{2.45}$$

Proof: The proof is given in Korn (1998). □

Figure 2 shows one possible position of the no-transaction region and the restarting lines for $k\,,K>0$:

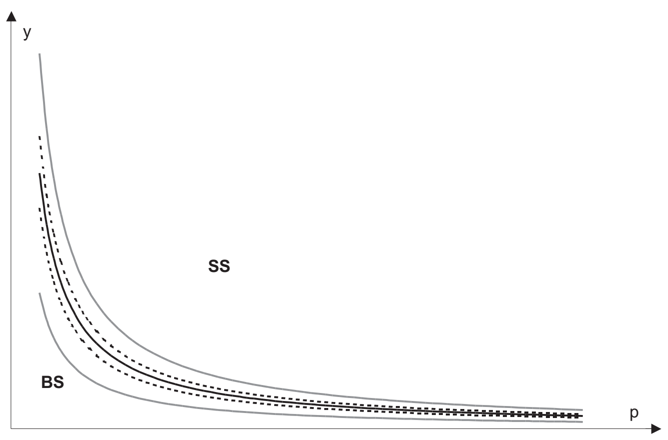

Fig. 2. (Asymptotically) optimal strategy

Remark 3. If $\sigma, \lambda, p > 0$ and $\mu = 0$ the only non-negative solution to the system (2.41)/(2.42) is given by $(Y^+, \hat{Y}^+) = (0, 0)$.

Although we have shown the existence and uniqueness of the solution to the system (2.41)/(2.42), we still have to compute the solution in the case $k, K > 0$ numerically. Doing this and analysing the behaviour and the properties of the solution in the three different cases will be the main aim of the next section.

2.4 Properties of the asymptotic solution

2.4.1 Constant boundaries and restarting points

In the case without transaction costs and with zero interest rate Pliska's optimal volume of stock holdings is constant in p. In this subsection we will show that in the case of strictly positive transaction costs the (approximately) optimal boundaries and restarting points for the stock holdings as presented in the previous section are also constant in p. Lemma 1 will serve to prove this fact for $k, K > 0$.

Lemma 1. *Let $k, K > 0$. Let $(Y^+(p), \hat{Y}^+(p)) = (Y^+, \hat{Y}^+)$ be the unique solution of (2.41)/(2.42) corresponding to Propsition 1 i). Then there exists exactly one real number $\bar{\gamma} > 1$ satisfying*

$$\hat{Y}^+(p) = Y^+(p)\,\bar{\gamma} \qquad \forall\, p > 0\,. \tag{2.46}$$

Proof: We consider the system

$$\begin{cases} (1+\gamma)\,\gamma\,\left(\varepsilon^{-1/4}\,z\right)^3 & = & \varepsilon^{-3/4}\,\frac{c}{p^3}\,, \\ (\gamma-1)^3\,(\gamma+1)\,\left(\varepsilon^{-1/4}\,z\right)^4 & = & \varepsilon^{-1}\,\frac{d}{p^4} \end{cases}$$

where

$$\begin{aligned} c &= \frac{3\,\mu^2\,k}{\sigma^4\,\lambda^3}\,, \\ d &= \frac{12\,\mu^2\,K}{\sigma^4\,\lambda^3}\,. \end{aligned}$$

This system has exactly one solution $(\bar{z}, \bar{\gamma})$ with $\bar{z} > 0$ and $\bar{\gamma} > 1$:
Solving the first equation for z and putting this into the second one leads to

$$\frac{c^4}{d^3}\,(\gamma-1)^9 = \gamma^4 + \gamma^5$$

which has exactly one solution $\bar{\gamma} > 1$. Then the first equation implies

$$\bar{z} = \left(\frac{c}{p^3\,\bar{\gamma}\,(1+\bar{\gamma})}\right)^{1/3} \tag{2.47}$$

with $\bar{z} > 0$.

Due to Proposition 1 i) the system (2.41)/(2.42) has exactly one solution $(Y^+(p), \hat{Y}^+(p))$ with $0 < Y^+(p) < \hat{Y}^+(p)$. Hence this solution must coincide with $(\varepsilon^{-1/4}\,\bar{z}, \varepsilon^{-1/4}\,\bar{z}\,\bar{\gamma})$. $\square$

Remark 4. The above proof shows that $\bar{\gamma}$ satisfying (2.46) only depends on the fixed parameters $k, K, \mu, \sigma, \lambda$ and that it is independent of p and ε. Since $\bar{z}$ as defined in the above proof by (2.47), too, is independent of ε we obtain that the boundary of NT

$$\hat{y}^+(p) = y^*(p) + \varepsilon^{1/4}\,\hat{Y}^+(p) = y^*(p) + \bar{z}\,\bar{\gamma} \tag{2.48}$$

and the restarting line

$$y^+(p) = y^*(p) + \varepsilon^{1/4}\,Y^+(p) = y^*(p) + \bar{z} \tag{2.49}$$

are also independent of ε!

Proposition 2. *Let $k\,,K \geq 0$. Let*

$$\hat{S}^+(p) := p\,y^*(p) + p\,\varepsilon^{1/4}\,\hat{Y}^+(p) \tag{2.50}$$

be the upper boundary of the no-transaction region with respect to the stock holdings, let

$$S^+(p) := p\,y^*(p) + p\,\varepsilon^{1/4}\,Y^+(p) \tag{2.51}$$

be the corresponding restarting line. Then the boundary of the no-transaction region and the restarting line are constant in p. Thus, $\hat{S}^+(p) \equiv \hat{S}^+$ and $S^+(p) \equiv S^+$ for all $p > 0$. The analogous results for $\hat{S}^-(p)$ and $S^-(p)$ are valid, too.

Proof: We consider the following four possible cases:

1. No transaction costs ($k = K = 0$). Since $\hat{Y}^+(p) = Y^+(p) = 0$ we obtain $p\,\hat{Y}^+(p) = p\,Y^+(p) = 0$.
2. Only fixed transaction costs ($k = 0\,, K > 0$). Then (2.44) implies $Y^+(p) = 0 = p\,Y^+(p)$ and $p\,\varepsilon^{1/4}\,\hat{Y}^+(p) = \left(\frac{12\,\mu^2\,K}{\lambda^3\,\sigma^4}\right)^{1/4}$.
3. Only proportional transaction costs ($k > 0\,, K = 0$). Then (2.45) gives us $p\,\varepsilon^{1/4}\,Y^+(p) = p\,\varepsilon^{1/4}\,\hat{Y}^+(p) = \left(\frac{3\,\mu^2\,k}{2\,\lambda^3\,\sigma^4}\right)^{1/3}$.
4. Fixed and proportional transaction costs ($k\,, K > 0$). From the proof of Lemma 1 we obtain that $p\,\varepsilon^{1/4}\,Y^+(p) = p\,\bar{z} = \left(\frac{c}{\bar{\gamma}(1+\bar{\gamma})}\right)^{1/3}$ is constant in p and, consequently, $p\,\varepsilon^{1/4}\,\hat{Y}^+(p) = p\,\varepsilon^{1/4}\,Y^+(p)\,\bar{\gamma}$ is also constant in p.

Since $p\,y^*(p)$, too, is constant in p, Proposition 2 is proved. □

Considering the stock holdings, we have to rebalance our portfolio whenever $S(t)$, the amount of money invested in the stock, equals $\hat{S}^+$ or $\hat{S}^-$. Then the stock holdings are shifted to S^+ and S^-, respectively.

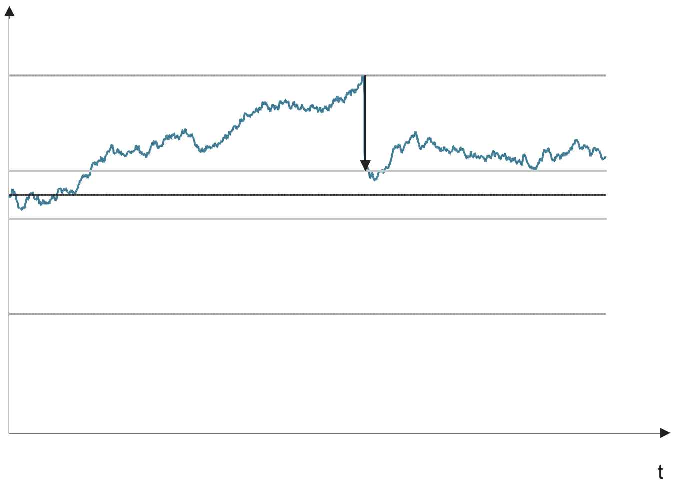

Fig. 3. (Asymptotically) optimal risky money (1)

Since the no-transaction region with respect to the stock holdings is given by one constant interval of real numbers we will in the following always focus on the stock holdings instead of the number of shares.

2.4.2 Optimal strategies

In the last subsection we found out that the boundaries of the no-transaction region and the restarting points (with respect to the volume of the stock holdings) are constant in p. In this subsection we will analyse the position of these boundaries and restarting points, respectively. We will show that it depends on the combination of the parameters $k, K, \mu, \sigma, \lambda$ whether the lower boundary and restarting point (in fact the boundary and the restarting line are single points), respectively, are above or below zero which then determines the strategy the investor has to follow. We will show that in some cases the investor always has to buy shares (if he starts with negative stock holdings) or always has to sell shares (if he starts with positive stock holdings) and that in other cases both transactions can occur.

Notation 1. (Pliska's optimal stock holdings.) Let

$$S^* := y^* p = \frac{\mu}{\lambda \sigma^2} \tag{2.52}$$

denote the optimal constant amount of money invested in the stock if no transaction costs have to be paid (in the zero interest case).

Proposition 3. *Let* $\mu, \sigma, \lambda > 0$.

i) Let $k = 0, K > 0$. *Then we have* $S^- \equiv S^* > 0$ *and*

$$\hat{S}^- < 0 \quad \Longleftrightarrow \quad \frac{\mu^2}{K \lambda \sigma^4} < 12\,. \tag{2.53}$$

ii) Let $k > 0, K = 0$. *Then we obtain*

$$\hat{S}^- \equiv S^- < 0 \quad \Longleftrightarrow \quad \frac{\mu}{k \sigma^2} < \frac{3}{2}\,. \tag{2.54}$$

iii) Let $k, K > 0$ *and define* $U := 3 \frac{k \sigma^2}{\mu}$ *and* $V := 12 \frac{K \lambda \sigma^4}{\mu^2}$. *Let* $\bar{\gamma} > 1$ *be given as in Lemma 1. Then we get*

$$\hat{S}^- < S^- < 0 \iff \begin{cases} U &>& \bar{\gamma}(1+\bar{\gamma}) \\ V &>& (\bar{\gamma}-1)^3(\bar{\gamma}+1) \end{cases} \tag{2.55}$$

$$\hat{S}^- < 0 \leq S^- \iff \begin{cases} \frac{1+\bar{\gamma}}{\bar{\gamma}^2} &<& U &\leq& \bar{\gamma}(1+\bar{\gamma}) \\ \frac{(\bar{\gamma}-1)^3(\bar{\gamma}+1)}{\bar{\gamma}^4} &<& V &\leq& (\bar{\gamma}-1)^3(\bar{\gamma}+1) \end{cases} \tag{2.56}$$

$$0 \leq \hat{S}^- < S^- \iff \begin{cases} U &\leq& \frac{1+\bar{\gamma}}{\bar{\gamma}^2} \\ V &\leq& \frac{(\bar{\gamma}-1)^3(\bar{\gamma}+1)}{\bar{\gamma}^4} \end{cases} . \tag{2.57}$$

Proof: Since the lower boundary and the restarting point are defined as $\hat{S}^- = S^* - p\,\varepsilon^{1/4}\,\hat{Y}^+$ and $S^- = S^* - p\,\varepsilon^{1/4}\,Y^+$ (compare with (2.50)/(2.51) and (2.52)) we only have to replace $p\,\varepsilon^{1/4}\,\hat{Y}^+$ and $p\,\varepsilon^{1/4}\,Y^+$ by the suitable expressions given in the proof of Proposition 2: In the case of i) we have

$$p\,\varepsilon^{1/4}\hat{Y}^+ = \sqrt[4]{\frac{12\,K\,\mu^2}{\lambda^3\,\sigma^4}}\,.$$

Therefore

$$\hat{S}^- < 0 \quad \Longleftrightarrow \quad \sqrt[4]{\frac{12\,K\,\mu^2}{\lambda^3\,\sigma^4}} > \frac{\mu}{\lambda\,\sigma^2} \quad \Longleftrightarrow \quad \frac{\mu^2}{\lambda\,K\,\sigma^4} < 12\,.$$

In the case of ii) we have

$$p\,\varepsilon^{1/4}\hat{Y}^+ = \sqrt[3]{\frac{3\,k}{2\,\lambda^3}\frac{\mu^2}{\sigma^4}}$$

which implies

$$\hat{S}^- < 0 \quad \Longleftrightarrow \quad \sqrt[3]{\frac{3\,k\,\mu^2}{2\,\lambda^3\,\sigma^4}} > \frac{\mu}{\lambda\,\sigma^2} \quad \Longleftrightarrow \quad \frac{\mu}{k\,\sigma^2} < \frac{3}{2}\,.$$

To prove iii) we use that

$$p\,\varepsilon^{1/4}\,Y^+ = \sqrt[3]{\frac{3\,k\,\mu^2}{\lambda^3\,\sigma^4}\frac{1}{\bar{\gamma}(1+\bar{\gamma})}}$$

which implies

$$S^- < 0 \quad \Longleftrightarrow \quad \bar{\gamma}(1+\bar{\gamma}) < 3\,k\,\frac{\sigma^2}{\mu}$$

and (due to $\hat{Y}^+ = \bar{\gamma}\,Y^+$)

$$\hat{S}^- < 0 \quad \Longleftrightarrow \quad \frac{1+\bar{\gamma}}{\bar{\gamma}^2} < 3\,k\,\frac{\sigma^2}{\mu}\,.$$

Since the proof of Lemma 1 also implies

$$p\,\varepsilon^{1/4}\,Y^+ = \sqrt[4]{\frac{12\,K\,\mu^2}{\lambda^3\,\sigma^4}\frac{1}{(\bar{\gamma}-1)^3(1+\bar{\gamma})}}$$

we obtain

$$S^- < 0 \quad \Longleftrightarrow \quad (\bar{\gamma}-1)^3\,(\bar{\gamma}+1) < 12\,K\,\lambda\frac{\sigma^4}{\mu^2}$$

and

$$\hat{S}^- < 0 \quad \Longleftrightarrow \quad \frac{(\bar{\gamma}-1)^3(\bar{\gamma}+1)}{\bar{\gamma}^4} < 12\,K\,\lambda\frac{\sigma^4}{\mu^2}\,.$$

Taking into consideration that $\bar{\gamma} > 1$ we immediately obtain the above statements. □

In the following we will show that both cases of positive and negative lower boundaries occur for suitable parameter sets. While this is obvious in the cases of only fixed or proportional costs, it is not in the case with fixed and proportional costs because we cannot derive a closed form solution there. Nevertheless, in this case we are able to find suitable examples numerically if we make no restrictions on the parameters. Examples 1 give survey of various possibilities of parameter sets.

Examples 1.

a) $\mu = 0.025\,,\ \sigma = 0.1\,,\ \lambda = 0.01 \implies S^* = 250\,.$

k	K	$\hat{S}^-$	S^-	S^+	$\hat{S}^+$
0.01	1	148.21	234.32	265.68	351.79
0	1	156.94	250	250	343.06
0.01	0	204.57	204.57	295.43	295.43
0.01	0.01	193.78	213.88	286.12	306.22

b) $\mu = 0.025\,,\ \sigma = 0.1\,,\ \lambda = 0.001 \implies S^* = 2500\,.$

k	K	$\hat{S}^-$	S^-	S^+	$\hat{S}^+$
0.01	0.01	1997.26	2090.96	2909.04	3002.74
0	0.01	2334.51	2500	2500	2665.49
0.01	0	2045.72	2045.72	2954.28	2954.28
0.001	0.01	2223.4	2343.49	2656.51	2776.6
0	0.01	2334.51	2500	2500	2665.49
0.001	0	2289.14	2289.14	2710.86	2710.86

c) $\mu = 0.025\,,\ \sigma = 0.2\,,\ \lambda = 0.01 \implies S^* = 62.5\,.$

k	K	$\hat{S}^-$	S^-	S^+	$\hat{S}^+$
0.01	0	44.47	44.47	80.53	80.53
0	0.1	36.33	62.5	62.5	88.67
0.01	0.1	30.84	53.42	71.58	94.16
0.01	1	13.58	58.02	66.98	111.42

d) $\mu = 0.03\,,\ \sigma = 0.2\,,\ \lambda = 0.1 \implies S^* = 7.5\,.$

k	K	$\hat{S}^-$	S^-	S^+	$\hat{S}^+$
0.001	1	-1.57	7.48	7.52	16.57
0	1	-1.56	7.5	7.5	16.56
0.001	0	6.56	6.56	8.44	8.44
0.01	1	-1.66	7.3	7.7	16.66
0.01	0	5.46	5.46	9.54	9.54
0.01	0.1	2.12	6.97	8.03	12.88
0	0.1	2.4	7.5	7.5	12.6
0.1	1	-2.4	6	9	17.4

e) $\mu = 0.01\,,\, \sigma = 0.2\,,\, \lambda = 0.01 \implies S^* = 25\,.$

k	K	$\hat{S}^-$	S^-	S^+	$\hat{S}^+$
0.5	1	-22.94	-1.33	51.33	72.94
0	1	-4.43	25	25	54.43
0.5	0	-11.06	-11.06	61.06	61.06
0.5	0.5	-20.35	-3.13	53.13	70.35
0.1	0.5	-7.81	12.41	37.59	57.81

f) $\mu = 0.02\,,\, \sigma = 0.5\,,\, \lambda = 0.0001 \implies S^* = 800\,.$

k	K	$\hat{S}^-$	S^-	S^+	$\hat{S}^+$
0.1	0.5	-326.79	-58.35	1658.35	1926.79
0	0.5	357.33	800	800	1242.67
0.1	0	-186.49	-186.49	1786.49	1786.49
0.1	1	-365.03	-27.22	1627.22	1965.03
0	1	273.57	800	800	1326.43
0.1	0	-186.49	-186.49	1786.49	1786.49

To close this subsection we will look at the different types of strategies that can appear:

a) Only fixed transaction costs $(k = 0)$: If the lower boundary of the no-transaction region is below zero, the strategy of the investor is as follows: Starting with a positive number of shares (positive volume of stock holdings) the investor has to sell shares whenever he has to intervene. Starting with a negative number of shares, his first transaction will be to buy shares whereas any following transaction will be to sell shares.

   If the lower boundary is above zero, the investor has to sell shares if the stock price increases sufficiently and to buy shares if the stock price decreases sufficiently.

   In the case of fixed transaction costs a transaction always shifts the number of shares (volume of stock holdings) to Pliska's optimal number of shares (volume of stock holdings).

b) Only proportional transaction costs $(K = 0)$: If the lower boundary of the no-transaction region is below zero the investor's strategy is to sell shares if he starts with positive stock holdings and to buy shares if he starts with negative stock holdings. In the case of the lower boundary being above zero, the investor's strategy is the same as in a).

   With proportional transaction costs intervening means doing the least transaction that keeps the number of shares (volume of stock holdings) inside the no-transaction region (boundary=restarting point), i.e., the action is of local time type.

c) Fixed and proportional transaction costs: If lower boundary and lower restarting point both are below zero the investor has to sell shares if he starts with positive stock holdings and to buy shares if he starts with negative stock holdings. Starting with positive stock holdings, he also has to sell shares in the case when the lower boundary is below zero but the lower restarting point is above zero. In this case, if he starts with negative stock holdings, his first transaction will be to buy shares whereas any following transaction is to sell shares. If lower boundary and restarting point both are above zero, transactions can be selling or buying depending on the movements of the stock price.

2.4.3 Economic justification

Since we computed an approximate solution to our optimization problem, we are interested in at least a reasonable behaviour (in an economic sense) of this solution. Varying the transaction cost parameters should lead to changes of the boundaries and restarting points that match economically. For example, increasing the fixed cost parameter we expect the upper boundary of the no-transaction region to increase, too. In this part we will show that our approximate solution does react in a reasonable way to variations of the parameters.

Proposition 4. *Let* $\mu\,,\sigma\,,\lambda > 0$ *be fixed.*

i) Let $k = 0, K > 0$. *Then*

$$K \uparrow \quad \Longrightarrow \quad \hat{S}^{+} \uparrow , \hat{S}^{-} \downarrow \ .$$

Further

$$\hat{S}^{+} \stackrel{K\to 0}{\longrightarrow} S^{*}\ , \quad \hat{S}^{-} \stackrel{K\to 0}{\longrightarrow} S^{*}\,.$$

ii) Let $k > 0, K = 0$. *Then*

$$k \uparrow \quad \Longrightarrow \quad S^{+} = \hat{S}^{+} \uparrow , S^{-} = \hat{S}^{-} \downarrow \ .$$

Further

$$S^{+} = \hat{S}^{+} \stackrel{k\to 0}{\longrightarrow} S^{*}\ , \quad S^{-} = \hat{S}^{-} \stackrel{k\to 0}{\longrightarrow} S^{*}\,.$$

iii) Let $k, K > 0$. *Then*

$$k \uparrow \quad \Longrightarrow \quad \hat{S}^{+} \uparrow , S^{+} \uparrow , \quad \hat{S}^{-} \downarrow , S^{-} \downarrow , \quad \hat{S}^{+} - S^{+} = -(\hat{S}^{-} - S^{-}) \downarrow ,$$

$$K \uparrow \quad \Longrightarrow \quad \hat{S}^{+} \uparrow , S^{+} \downarrow , \quad \hat{S}^{-} \downarrow , S^{-} \uparrow , \quad \hat{S}^{+} - S^{+} = -(\hat{S}^{-} - S^{-}) \uparrow \ .$$

Proof: Since S^* given in (2.52) is independent of the cost parameters we only have to differentiate the suitable expressions for $p\,\varepsilon^{1/4}\,\hat{Y}^+$ and $p\,\varepsilon^{1/4}\,Y^+$, given in the proofs of Proposition 2 and Lemma 1, with respect to k and K. Since the proofs of i) and ii) are straightforward we only prove iii). Here we start analysing the relation between $\bar{\gamma}$ given in Lemma 1 and the cost parameters. From the proof of Lemma 1 we know that $\bar{\gamma}$ is given as the intersection point of the functions

$$Z_1(\gamma) := \tilde{C}\,(\gamma-1)^9$$

and

$$Z_2(\gamma) := \gamma^4 + \gamma^5$$

where

$$\tilde{C} := \frac{3}{4^3}\,\frac{\mu^2}{\sigma^4\,\lambda^3}\,\frac{k^4}{K^3}\,.$$

Due to $Z_1(1)=0\,, Z_2(1)=2$ and $\lim\limits_{\gamma\to\infty}\frac{Z_1(\gamma)}{Z_2(\gamma)}=\infty$ it is obvious that the intersection point $\bar{\gamma}$ of Z_1 and Z_2 decreases with an increase in $\tilde{C}$. Since $\tilde{C}$ increases with k and decreases with K we obtain

$$\frac{\partial\bar{\gamma}}{\partial k} \le 0\,,\ \frac{\partial\bar{\gamma}}{\partial K} \ge 0\,.$$

From the proofs of Proposition 2 and Lemma 1 we also obtain

$$p\,\varepsilon^{1/4}\,Y^+ = \sqrt[3]{\frac{3\,k\,\mu^2}{\lambda^3\sigma^4}}\,\sqrt[3]{\frac{1}{\bar{\gamma}(1+\bar{\gamma})}} =: a$$

as well as

$$p\,\varepsilon^{1/4}\,Y^+ = \sqrt[4]{\frac{12K\mu^2}{\lambda^3\sigma^4}}\,\sqrt[4]{\frac{1}{(\bar{\gamma}-1)^3(1+\bar{\gamma})}} =: b\,.$$

Together with the above results this implies

$$\frac{\partial a}{\partial K} = -\frac{1}{3}\sqrt[3]{\frac{3\,k\,\mu^2}{\lambda^3\sigma^4}}\left(\frac{1}{\bar{\gamma}\,(1+\bar{\gamma})}\right)^{-\frac{2}{3}}\,(\bar{\gamma}+\bar{\gamma}^2)^{-2}\,(1+2\bar{\gamma})\,\bar{\gamma}' \le 0\,,$$

$$\begin{aligned}\frac{\partial(\bar{\gamma}\,a)}{\partial K} &= \frac{1}{3}\sqrt[3]{\frac{3\,k\,\mu^2}{\lambda^3\sigma^4}}\left(\frac{\bar{\gamma}^2}{1+\bar{\gamma}}\right)^{-\frac{2}{3}}\frac{2\bar{\gamma}\,\bar{\gamma}'(1+\bar{\gamma})-\bar{\gamma}^2\,\bar{\gamma}'}{(1+\bar{\gamma})^2}\\ &= \frac{1}{3}\sqrt[3]{\frac{3\,k\,\mu^2}{\lambda^3\,\sigma^4}}\left(\frac{\bar{\gamma}^2}{1+\bar{\gamma}}\right)^{-\frac{2}{3}}\frac{2\bar{\gamma}+\bar{\gamma}^2}{(1+\bar{\gamma})^2}\,\bar{\gamma}'\\ &\ge 0\,,\end{aligned}$$

$$
\begin{aligned}
\frac{\partial b}{\partial k} &= -\frac{1}{4}\sqrt[4]{\frac{12\,K\,\mu^2}{\lambda^3\,\sigma^4}}\left(\frac{1}{(\bar\gamma-1)^3\,(1+\bar\gamma)}\right)^{-\frac{3}{4}} \\
&\qquad \frac{3\,(\bar\gamma-1)^2\,\bar\gamma'(1+\bar\gamma)+(\bar\gamma-1)^3\,\bar\gamma'}{(\bar\gamma-1)^6\,(1+\bar\gamma)^2} \\
&= -\frac{1}{4}\sqrt[4]{\frac{12\,K\,\mu^2}{\lambda^3\,\sigma^4}}\left(\frac{1}{(\bar\gamma-1)^3\,(1+\bar\gamma)}\right)^{-\frac{3}{4}}\frac{2+4\,\bar\gamma}{(\bar\gamma-1)^4\,(1+\bar\gamma)^2}\,\bar\gamma' \\
&\geq 0
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{\partial(\bar\gamma\,b)}{\partial k} &= \frac{1}{4}\sqrt[4]{\frac{12\,K\,\mu^2}{\lambda^3\,\sigma^4}}\left(\frac{\bar\gamma^4}{(\bar\gamma-1)^3\,(1+\bar\gamma)}\right)^{-\frac{3}{4}} \\
&\qquad \frac{4\,\bar\gamma^3\,\bar\gamma'\,(\bar\gamma-1)^3\,(1+\bar\gamma)-\bar\gamma^4\,[3\,(\bar\gamma-1)^2\bar\gamma'\,(1+\bar\gamma)+(\bar\gamma-1)^3\,\bar\gamma']}{(\bar\gamma-1)^6\,(1+\bar\gamma)^2} \\
&= -\frac{1}{4}\sqrt[4]{\frac{12\,K\,\mu^2}{\lambda^3\,\sigma^4}}\left(\frac{\bar\gamma^4}{(\bar\gamma-1)^3\,(1+\bar\gamma)}\right)^{-\frac{3}{4}}\frac{4\,\bar\gamma^3+2\,\bar\gamma^4}{(\bar\gamma-1)^4\,(1+\bar\gamma)^2}\,\bar\gamma' \\
&\geq 0\,.
\end{aligned}
$$

So far we have already proved $\frac{\partial S^+}{\partial K}\leq 0\,,\ \frac{\partial \hat S^+}{\partial K}\geq 0\,,\ \frac{\partial S^+}{\partial k}\geq 0\,,\ \frac{\partial \hat S^+}{\partial k}\geq 0\,,$ which also implies $\frac{\partial(\hat S^+-S^+)}{\partial K}\geq 0\,.$ To show $\frac{\partial(\hat S^+-S^+)}{\partial k}\leq 0$ we use the above results:

$$
\begin{aligned}
\frac{\partial(\bar\gamma\,b)}{\partial k}-\frac{\partial b}{\partial k} &= \frac{1}{4}\sqrt[4]{\frac{12\,K\,\mu^2}{\lambda^3\,\sigma^4}}\left(\frac{1}{(\bar\gamma-1)^3\,(1+\bar\gamma)}\right)^{-\frac{3}{4}} \\
&\qquad \bar\gamma\,'\left\{-\bar\gamma^{-3}\,\frac{4\,\bar\gamma^3+2\,\bar\gamma^4}{(\bar\gamma-1)^4\,(1+\bar\gamma)^2}+\frac{2+4\,\bar\gamma}{(\bar\gamma-1)^4\,(1+\bar\gamma)^2}\right\} \\
&= \frac{1}{4}\sqrt[4]{\frac{12\,K\,\mu^2}{\lambda^3\,\sigma^4}}\left(\frac{1}{(\bar\gamma-1)^3\,(1+\bar\gamma)}\right)^{-\frac{3}{4}}\frac{2}{(\bar\gamma-1)^3\,(\bar\gamma+1)^2}\,\bar\gamma' \\
&\leq 0\,.
\end{aligned}
$$

Due to $Y^+=-Y^-$ we also obtain the results for S^- and $\hat S^-$. $\square$

Examples 2. $\mu = 0.03\,,\ \sigma = 0.2\,,\ \lambda = 0.001 \implies S^* = 750\,.$

a) $k = 0\,.$

K	$\hat{S}^+$
1E-12	750.287
1E-8	752.866
1E-4	778.663
0.001	800.971
0.01	840.641
0.1	911.185
1	1036.633
100	1656.413

$K = 0\,.$

k	$\hat{S}^+ = S^+$
1E-12	750.094
1E-8	752.036
1E-4	793.86
0.001	844.494
0.01	953.581
0.1	1188.603

b) $K = 0.01\,.$

k	$\hat{S}^+$	S^+	$\hat{S}^+ - S^+$	$\bar{\gamma}$
0	840.641	750	90.641	-
0.00001	841.641	751.967	89.674	46.5848
0.0001	848.958	764.968	83.99	6.6112
0.001	883.753	813.849	69.904	2.095
0.01	982.224	927.4	54.824	1.309
0.1	1210.208	1167.685	42.523	1.1018

c) $k = 0.001\,.$

K	$\hat{S}^+$	S^+	$\hat{S}^+ - S^+$	$\bar{\gamma}$
0	844.494	844.494	0	-
1E-12	844.511	844.478	0.033	1.0004
1E-8	844.849	844.14	0.709	1.0075
1E-4	852.337	837.062	15.275	1.1755
0.001	861.856	829.033	32.823	1.415
0.01	883.753	813.849	69.904	2.0949
0.1	934.72	790.553	144.167	4.555
1	1046.113	768.135	277.978	16.3284
100	1657.437	752.045	905.392	443.7953

Remark 5. In the case of $k, K > 0$ the Newton method only converges properly if k is not too small.

As we have seen, in the case of both fixed and proportional transaction costs, an increase of the proportional cost parameter leads to an increase of the upper boundary and restarting point and a decrease of the volume of transaction: The investor intervenes "later" than in the case of a lower proportional cost parameter in the sense that he allows his stock holdings to differ more from Pliska's optimal stock holdings without intervening. If the investor has to intervene, the volume of transaction is lower because the investor thus tries to compensate for the higher proportional cost parameter.

An increase of the fixed transaction costs leads to an increase of the upper boundary of the no-transaction region, a decrease of the upper restarting point and thus to an increase of the volume of transaction: Again the investor intervenes "later" since the transaction costs are higher. If he has to intervene, his transaction has a higher volume because he wants to benefit as good as possible from the higher fixed transaction costs, which he has to pay anyway.

2.4.4 Non-negative wealth process

As we mentioned at the beginning of this chapter (Remark 1 (2.7)) Pliska's optimal wealth process has a positive probability to attain negative values. In the case of strictly positive transaction costs we can guarantee - at least in some cases - a wealth process that will stay non-negative during the whole period of time. In those cases we have thus - at least approximately - solved the more complicated problem with the additional non-negativity requirements.

We consider the case $k = 0\,, K > 0\,.$ Then the proof of Proposition 2 implies that the profit of selling shares $\hat{S}^+ - S^*$ exceeds the transaction costs K if and only if $12 > \frac{\lambda^3\,\sigma^4\,K^3}{\mu^2}\,.$

Let us now assume that $12 > \frac{\lambda^3\,\sigma^4\,K^3}{\mu^2}$ is satisfied. In the case of $\hat{S}^- < 0$ the wealth process remains non-negative for all $t \in [\,0, T]$ if

1) $S(0) > 0\,, B(0) > 0\,,$

2) $S(0) < 0\,, B(0) > K + \,(\hat{S}^+ - S^*)\,.$

Proof: We only prove the second case since the first one is trivial. Let

$$\tau_0 := \inf\{t : S(t) = \hat{S}^-\}\,.$$

Then we obtain

1. $X(t) > 0$ for all $0 \le t < \tau_0$:

$$\begin{aligned} X(t) &= B(0) + S(t) \\ &> K + \,(\hat{S}^+ - S^*) + \hat{S}^- \\ &= K + \,(\hat{S}^+ - S^*) + (2\,S^* - \hat{S}^+) \\ &= K + S^* \\ &> 0\,. \end{aligned}$$

2. $X(\tau_0) > 0$ (immediately after the transaction (buying)):

$$X(\tau_0) = \underbrace{(B(0) - K - (\hat{S}^+ - S^*))}_{=B(\tau_0)} + S^* > 0\,.$$

3. $X(t) > 0$ for all $t > \tau_0$ without any further transaction:

$$X(t) = B(\tau_0) + S^{S^*,\tau_0}(t) > B(\tau_0) > 0$$

where $S^{S^*,\tau_0}(l)$ denotes the stock holdings at time l if $S(\tau_0) = S^*$.

4. $X(t) > 0$ for $t > \tau_0$ in the case of further transactions (selling): $S(t)$ stays above zero. Since the profit exceeds the transaction costs the bond holdings increase, thus $B(t) > 0$, too. □

2.5 Optimization under the assumption of constant boundaries

As we proved in subsection 2.4.1, the approximately optimal boundaries and restarting points for the stock holdings are constant in p and time independent. This finding gives us the idea of trying to solve our optimization problem by assuming that the boundaries and restarting points we are looking for are constant (in p). This then only seems to be a four parameter static (!) problem which at first sight should be easier to solve than the original impulse control problem. Furthermore, if we were able to solve the optimization problem in this way and the solution differed from the one we derived via asymptotic analysis, we knew that the solution from subsection 2.4.1 could not be the optimal one.

However, it will turn out that solving the optimization problem by assuming constant boundaries is much more complicated than solving it by the asymptotic analysis. Even more, obtaining an explicit solution seems to be impossible.

As above, S^* denotes Pliska's optimal volume of stock holdings given in (2.52). Then we assume the boundaries of the no-transaction region and the restarting points to be multiples of S^*:

$$\alpha\, S^* = \text{"upper restarting point"}\,, \tag{2.58}$$

$$\beta\, S^* = \text{"upper boundary of the no-transaction region"} \tag{2.59}$$

with $\alpha, \beta > 0$. For simplicity, we restrict ourselves to the case of a negative lower restarting point and $S(0) \in (0, \beta S^*]$, implying that only the upper boundary/upper restarting point are involved in our calculations.

To simplify our further computations, we approximate the exponential function via

$$e^{-\lambda x} \doteq 1 - \lambda\, x \tag{2.60}$$

thus obtaining the optimization problem

$$\max_{\alpha,\beta>0} E(\lambda\, X(T) - 1)\,. \tag{2.61}$$

2.5.1 Exact optimization

Although we only have to compute the expectation of $X(T)$ (depending on α and β) which seems to be very simple, we will see in the following that this is not possible since we will not be able to obtain a closed form solution for $E(X(T))$:

Due to the assumed form of the no-transaction region the wealth process now reads

$$X(T) \;=\; B(0) + S_T^N - N_T\,[K - (1-k)\,(\beta-\alpha)\,S^*] \tag{2.62}$$

where N_t is the number of transactions between 0 and t and S_t^N is the volume of stock holdings if N_t transactions have been made up to time t. By definition S_t^N and $S(t)$ coincide before the first intervention takes place, whereas S_t^N evolves completely differently from $S(t)$ afterwards. Above all S_t^N is no longer continuous then.

In order to obtain the expectation of the terminal wealth we start with the computation of $E(N_T) = E^{S(0)}(N_T)$ which is given by

$$E^{S(0)}(N_T) = \sum_{n=0}^{\infty} n\, P^{S(0)}(N_T = n)\,.$$

(In the course of the computations S_0^N will attain two different values: $S(0)$ before the first rebalancing and $\alpha\, S^*$ afterwards. Therefore we will always indicate the respective one by using the conditional distribution given S_0^N.)

For $n=0$ we have

$$\begin{aligned} P^{S_0^N}(N_T = 0) \;&=\; P^{S_0^N}\left(\max_{0\le t\le T} S_t^N = \max_{0\le t\le T} S(t) < \beta\, S^*\right) \\ &=\; P^{S_0^N}\left(\max_{0\le t\le T}\{W_t + \hat{\mu} t\} < Y(S_0^N)\right) \end{aligned}$$

with

$$\hat{\mu} \;:=\; \frac{\mu - \frac{1}{2}\sigma^2}{\sigma}\,, \tag{2.63}$$

$$Y(S_0^N) \;:=\; \frac{1}{\sigma}\ln\left(\frac{\beta\, S^*}{S_0^N}\right)\,. \tag{2.64}$$

Let Q be the equivalent martingale measure such that W_t is a Brownian motion with drift $\hat{\mu}$ with respect to Q. Then $W_t = \tilde{W}_t + \hat{\mu}t$ where $\tilde{W}_t$ is a Brownian motion with respect to Q. Due to Girsanov's theorem this implies

$$\begin{aligned}
P^{S_0^N}(N_T = 0) &= P^{S_0^N}\left(\max_{0\le t\le T}\{W_t + \hat{\mu}t\} < Y(S_0^N)\right) \\
&= Q^{S_0^N}\left(\max_{0\le t\le T}\{\tilde{W}_t + \hat{\mu}t\} < Y(S_0^N)\right) \\
&= E_Q^{S_0^N}\left(I_{\left\{\max\limits_{0\le t\le T}\{\tilde{W}_t+\hat{\mu}t\}<Y(S_0^N)\right\}}\right) \\
&= E_P^{S_0^N}\left(I_{\left\{\max\limits_{0\le t\le T} W_t<Y(S_0^N)\right\}} e^{\hat{\mu}W_T-\frac{1}{2}\hat{\mu}^2 t}\right) \\
&=: E_P^{S_0^N}\left(I_{\{M_T<Y(S_0^N)\}} e^{\hat{\mu}W_T-\frac{1}{2}\hat{\mu}^2 t}\right).
\end{aligned}$$

Using that the joint density of $(M_T, W_T)\,, T > 0$, is given by

$$f_{(M_T,W_T)}(x,s) = \sqrt{\frac{2}{\pi}}\,\frac{2x-s}{\sqrt{T^3}}\,e^{-\frac{(2x-s)^2}{2T}} \qquad , x \in [0,\infty)\,, s \in I\!R\,, s \le x$$

(see Karatzas and Shreve (1988), p.95) we obtain

$$\begin{aligned}
P^{S_0^N}(N_T = 0) &= \int_0^{Y(S_0^N)}\left(\int_{-\infty}^{x} e^{\hat{\mu}\,s-\frac{1}{2}\hat{\mu}^2\,T}\sqrt{\frac{2}{\pi}}\,\frac{2x-s}{\sqrt{T^3}}\,e^{-\frac{(2x-s)^2}{2\,T}}\,ds\right)dx \\
&= \int_{-\infty}^{Y(S_0^N)}\left(\int_{\max(s,0)}^{Y(S_0^N)} e^{\hat{\mu}\,s-\frac{1}{2}\hat{\mu}^2\,T}\sqrt{\frac{2}{\pi}}\,\frac{2x-s}{\sqrt{T^3}}\,e^{-\frac{(2x-s)^2}{2\,T}}\,dx\right)ds \\
&= \int_{-\infty}^{0}\left(\int_{0}^{Y(S_0^N)} e^{\hat{\mu}\,s-\frac{1}{2}\hat{\mu}^2\,T}\sqrt{\frac{2}{\pi}}\,\frac{2x-s}{\sqrt{T^3}}\,e^{-\frac{(2x-s)^2}{2\,T}}\,dx\right)ds \\
&\quad + \int_{0}^{Y(S_0^N)}\left(\int_{s}^{Y(S_0^N)} e^{\hat{\mu}\,s-\frac{1}{2}\hat{\mu}^2\,T}\sqrt{\frac{2}{\pi}}\,\frac{2x-s}{\sqrt{T^3}}\,e^{-\frac{(2x-s)^2}{2\,T}}\,dx\right)ds \\
&= \int_{-\infty}^{0}\frac{2}{\sqrt{2\,\pi\,T}}\,e^{\hat{\mu}\,s-\frac{1}{2}\hat{\mu}^2\,T}\left(\int_{0}^{Y(S_0^N)}\frac{2x-s}{T}\,e^{-\frac{(2x-s)^2}{2\,T}}\,dx\right)ds \\
&\quad + \int_{0}^{Y(S_0^N)}\frac{2}{\sqrt{2\,\pi\,T}}\,e^{\hat{\mu}\,s-\frac{1}{2}\hat{\mu}^2\,T}\left(\int_{s}^{Y(S_0^N)}\frac{2x-s}{T}\,e^{-\frac{(2x-s)^2}{2\,T}}\,dx\right)ds
\end{aligned}$$

$$\begin{aligned}
&= \int_{-\infty}^{0} \frac{1}{\sqrt{2\,\pi\,T}}\, e^{\hat\mu\, s - \frac{1}{2}\hat\mu^2\, T} \left(e^{-\frac{s^2}{2\,T}} - e^{-\frac{\left(2\,Y(S_0^N) - s\right)^2}{2\,T}} \right) ds \\
&\quad + \int_{0}^{Y(S_0^N)} \frac{1}{\sqrt{2\,\pi\,T}}\, e^{\hat\mu\, s - \frac{1}{2}\hat\mu^2\, T} \left(e^{-\frac{s^2}{2\,T}} - e^{-\frac{\left(2\,Y(S_0^N) - s\right)^2}{2\,T}} \right) ds \\
&= \int_{-\infty}^{Y(S_0^N)} \frac{1}{\sqrt{2\,\pi\,T}}\, e^{\hat\mu\, s - \frac{1}{2}\hat\mu^2\, T} \left(e^{-\frac{s^2}{2\,T}} - e^{-\frac{\left(2\,Y(S_0^N) - s\right)^2}{2\,T}} \right) ds \\
&= \int_{-\infty}^{Y(S_0^N)} \frac{1}{\sqrt{2\,\pi\,T}}\, e^{-\frac{1}{2\,T}(s - \hat\mu\, T)^2}\, ds \\
&\quad - e^{2\,\hat\mu\, Y(S_0^N)} \int_{-\infty}^{Y(S_0^N)} \frac{1}{\sqrt{2\,\pi\,T}}\, e^{-\frac{1}{2\,T}\left(s - \left(2\,Y(S_0^N) + \hat\mu\, T\right)\right)^2}\, ds \\
&= \int_{-\infty}^{\frac{Y(S_0^N) - \hat\mu\, T}{\sqrt{T}}} \frac{1}{\sqrt{2\,\pi}}\, e^{-\frac{1}{2}\tilde s^2}\, d\tilde s \\
&\quad - e^{2\,\hat\mu\, Y(S_0^N)} \int_{-\infty}^{\frac{Y(S_0^N) - \left(2\,Y(S_0^N) + \hat\mu\, T\right)}{\sqrt{T}}} \frac{1}{\sqrt{2\,\pi}}\, e^{-\frac{1}{2}\tilde s^2}\, d\tilde s \\
&= \Phi\left(\frac{Y(S_0^N) - \hat\mu\, T}{\sqrt{T}} \right) - e^{2\,\hat\mu\, Y(S_0^N)}\, \Phi\left(-\frac{Y(S_0^N) + \hat\mu\, T}{\sqrt{T}} \right) \qquad (2.65)
\end{aligned}$$

with $\hat\mu$ and $Y(S_0^N)$ as given in (2.63)/(2.64).

By the help of this formula we can now compute $P^{S(0)}(N_T = 1)$ where $\{N_T = 1\}$ means that the process S_t^N which starts in $S_0^N = S(0)$

- reaches $\beta\, S^*$ before T,
- is shifted to $\alpha\, S^*$
- and does not reach $\beta\, S^*$ again.

We define the stopping time

$$\tau_1 := \inf\{t \geq 0 : S_t^N \equiv S(t) = \beta\, S^*\} = \inf\{t \geq 0 : W_t + \hat\mu\, t = Y(S_0^N)\} \qquad (2.66)$$

and the corresponding stopping time (with respect to Q)

$$\tilde\tau_1 = \inf\{t \geq 0 : W_t = Y(S_0^N)\}$$

where $S_0^N \in \{S(0), \alpha\, S^*\}$. Then we obtain

$$\begin{aligned}
P^{S_0^N}(N_T = 1) &= E^{S_0^N}(\, P^{S_0^N}(N_T = 1) \,|\, \tau_1 \,) \\
&= \int_0^T P^{S_0^N}(N_T = 1 \,|\, \tau_1 = t)\, P^{S_0^N}(\tau_1 \in\, dt) \\
&= \int_0^T P^{\alpha S^*}(N_{T-t} = 0)\, P^{S_0^N}(\tau_1 \in\, dt)\,. \qquad (2.67)
\end{aligned}$$

Due to $\{N_T = 0\} = \{\tau_1 > T\}$ (2.65) implies

$$P^{S_0^N}(\tau_1 \le T) = 1 - \Phi\left(\frac{Y(S_0^N) - \hat{\mu}\, T}{\sqrt{T}}\right) + e^{\,2\,\hat{\mu}\, Y(S_0^N)}\, \Phi\left(-\frac{Y(S_0^N) + \hat{\mu}\, T}{\sqrt{T}}\right).$$

Hence the density of τ_1 reads

$$\begin{aligned}
P^{S_0^N}(\tau_1 \in\, dt) &= \varphi\left(\frac{Y(S_0^N) - \hat{\mu}t}{\sqrt{t}}\right)\left[\frac{Y(s_0^N)}{2\sqrt{t^3}} + \frac{\hat{\mu}}{2\sqrt{t}}\right] I_{(0,\infty)}(t) \\
&\quad + e^{2\,\hat{\mu}\, Y(S_0^N)}\, \varphi\left(-\frac{Y(S_0^N) + \hat{\mu}t}{\sqrt{t}}\right)\left[\frac{Y(S_0^N)}{2\sqrt{t^3}} - \frac{\hat{\mu}}{2\sqrt{t}}\right] I_{(0,\infty)}(t) \\
&= \frac{Y(S_0^N)}{\sqrt{t^3}}\, \varphi\left(\frac{Y(S_0^N) - \hat{\mu}t}{\sqrt{t}}\right) I_{(0,\infty)}(t) \qquad (2.68)
\end{aligned}$$

due to $Y(S_0^N) \ge 0$ and the equality

$$e^{2\,\hat{\mu}\, Y(S_0^N)}\, \varphi\left(-\frac{Y(S_0^N) + \hat{\mu}t}{\sqrt{t}}\right) = \varphi\left(\frac{Y(S_0^N) - \hat{\mu}t}{\sqrt{t}}\right).$$

Further, replacing T by $T - t$ and S_0^N by $\alpha\, S^*$ in (2.65) we obtain the value of $P^{\alpha S^*}(N_{T-t} = 0)\,, t \in [0, T]$.

Using the probabilities we have already calculated we gradually obtain the probabilities for $K \ge 2$:

$$\begin{aligned}
P^{S_0^N}(N_T = K) &= E^{S_0^N}(P^{S_0^N}(N_T = K)|\tau_1) \\
&= \int_0^T P^{\alpha S^*}(N_{T-t} = K - 1)\, P^{S_0^N}(\tau_1 \in\, dt)\,. \qquad (2.69)
\end{aligned}$$

Since these probabilities have no closed forms, $E^{S(0)}(N_T)$ has no closed form, either.

Although this result already implies that we cannot derive a closed formula for our optimization problem, we will - for completeness - take a brief look at the expectation of S_T^N. Defining

$$\tau_n := \inf\left\{t \ge 0 : S_t^N = \beta\, S^* \text{ for the n-th time }, \; S_0^N \in \{S(0), \alpha S^*\}\right\} \qquad (2.70)$$

we obtain

$$\begin{aligned}
E^{S(0)}(S_T^N) &= \sum_{n=1}^{\infty}\int_0^T E^{S(0)}\left(S_T^N|N_T=n,\tau_n=t\right)\,P^{S(0)}(N_T=n,\tau_n\in dt)\\
&\qquad +E^{S(0)}\left(S_T^N|N_T=0\right)P^{S(0)}(N_T=0)\\
&= \sum_{n=1}^{\infty}\int_0^T E^{S(0)}\left(S_T^N|N_T=n,\tau_n=t\right)\,P^{\alpha S^*}(N_{T-t}=0)\,P^{S(0)}(\tau_n\in dt)\\
&\qquad +E^{S(0)}\left(S_T^N|N_T=0\right)P^{S(0)}(N_T=0)\,.
\end{aligned}$$

Obviously the conditional expectations are given by

$$E^{S(0)}(S_T^N|N_T=0)=E^{S(0)}(S(T))=S(0)\,e^{\mu T}$$

and

$$\begin{aligned}
E^{S(0)}(S_T^N|N_T=n,\tau_n=t) &= E^{S(0)}\left(\frac{S_T^N}{S_t^N}\,S_t^N\,\middle|\,N_T=n,\tau_n=t\right)\\
&= E^{S(0)}\left(\frac{S(T)}{S(t)}\,\alpha\,S^*\right)\\
&= \alpha S^*\,e^{\mu(T-t)}\,.
\end{aligned}$$

Further, for $n>1$ we obtain

$$\begin{aligned}
&P^{S(0)}(\tau_n\le t)\\
&= E^{S(0)}(\,P^{S(0)}(\tau_n\le t)\,|\,\tau_1\,)\\
&= \int_0^t P^{S(0)}(\tau_n\le t\,|\,\tau_1=u_1)\,P^{S(0)}(\tau_1\in du_1)\\
&= \int_0^t P^{\alpha S^*}(\tau_{n-1}\le t-u_1)\,P^{S(0)}(\tau_1\in du_1)\\
&= \int_0^t E^{\alpha S^*}(\,P^{\alpha S^*}(\tau_{n-1}\le t-u_1)\,|\,\tau_1\,)\,P^{S(0)}(\tau_1\in du_1)\\
&= \int_0^t\int_0^{t-u_1} P^{\alpha S^*}(\tau_{n-1}\le t-u_1\,|\,\tau_1=u_2)\,P^{\alpha S^*}(\tau_1\in du_2)\,P^{S(0)}(\tau_1\in du_1)\\
&= \int_0^t\int_0^{t-u_1} P^{\alpha S^*}(\tau_{n-2}\le t-u_1-u_2)\,P^{\alpha S^*}(\tau_1\in du_2)\,P^{S(0)}(\tau_1\in du_1)\\
&\;\vdots\\
&= \int_0^t\int_0^{t-u_1}\cdots\int_0^{t-v_{n-1}} P^{\alpha S^*}(\tau_1\in du_n)\cdots P^{\alpha S^*}(\tau_1\in du_2)P^{S(0)}(\tau_1\in du_1)
\end{aligned}$$

with $v_k:=u_1+\cdots+u_k\,,k=1,\ldots,n-1\,.$

Since some of the formulae we need to compute the expectation of the wealth process have no closed forms, we see that - even in the case of the simplified utility function - it is impossible to find a closed form solution. This finding justifies our approach via asymptotic analysis.

2.5.2 Optimization using bounds

To close this section we will look at the optimization problem (2.61) using bounds for the expectations in question: Due to $N_t \geq 0$ and $S_t^N \geq 0$ their expectations have to be non-negative as well. Assuming that $E(N_T) \leq Z\,, Z \in I\!N$ and that

$$Q := K - (1-k)\,(\beta - \alpha)\,S^* \geq 0$$

which is satisfied if and only if

$$\beta - \alpha \leq \frac{K}{1-k}\,\frac{1}{S^*}$$

we obtain

$$E(X_T) \geq B(0) - Z\,Q\,. \tag{2.71}$$

Due to $Z\,Q \geq 0$ this lower bound is maximal if $Q = 0$ which is equivalent to

$$\beta - \alpha = \frac{K}{1-k}\,\frac{1}{S^*}\,. \tag{2.72}$$

Unfortunately, we are not given any information beyond the difference $\beta - \alpha$. Therefore we are free to choose one of the parameters $\alpha\,, \beta$.

Remark 6. In the next section we will show that the upper bound for $E(N_T)$ exists and we will compute it explicitely.

2.6 Intervention times

As we have shown in subsection 2.4.2 there are three different possibilities for the position of the no-transaction region $[\hat{S}^-, \hat{S}^+]$ and the restarting points $S^-\,, S^+$:

- $\hat{S}^- > 0\,,$
- $\hat{S}^- < 0 < S^-\,,$
- $S^- < 0$

where $\hat{S}^- \leq S^- \leq S^* \leq S^+ \leq \hat{S}^+$. Depending on the positions of $\hat{S}^-$ and $S(0)$ we obtain the following definitions of the first intervention time:

Definition 4. (Intervention time.) Let $S(0) \in [\hat{S}^-, \hat{S}^+]$ and

$$\tau_x := \inf\{t \geq 0 : S(t) = x\}\,. \tag{2.73}$$

If $\hat{S}^- > 0$ the first intervention time is given by

$$\begin{aligned}\tau &:= \inf\{t \geq 0 : S(t) = \hat{S}^- \text{ or } S(t) = \hat{S}^+\} \\ &= \min\{\tau_{\hat{S}^-}, \tau_{\hat{S}^+}\}\,. \end{aligned} \tag{2.74}$$

If $\hat{S}^- < 0$ the first intervention time is given by

$$\tau := \begin{cases} \inf\{t \geq 0 : S(t) = \hat{S}^+\} = \tau_{\hat{S}^+} & \text{if} \quad S(0) \in (0, \hat{S}^+]\,, \\ \inf\{t \geq 0 : S(t) = \hat{S}^-\} = \tau_{\hat{S}^-} & \text{if} \quad S(0) \in [\hat{S}^-, 0)\,. \end{cases} \tag{2.75}$$

Remark 7. For $\hat{S}^- = 0$ (which has not occured in our examples) the first intervention time is given by

$$\begin{aligned} \tau &:= \inf\{t \geq 0 : S(t) = \hat{S}^+\} \\ &= \tau_{\hat{S}^+} \end{aligned}$$

and can therefore be treated like the first intervention time in the case

$$\hat{S}^- < 0\,, S(0) > 0\,.$$

To start with we will analyse the case $\hat{S}^- > 0$. We will show that τ is almost surely finite. Further we will compute the expectation of τ and the probabilities that S reaches $\hat{S}^+$ and $\hat{S}^-$ first, respectively.

Lemma 2. *Let $\hat{S}^- > 0$ and $S(0) \in [\hat{S}^-, \hat{S}^+]$. Let τ be the first intervention time from Definition 4 (2.74). Then*

$$P(\tau < \infty) = 1\,. \tag{2.76}$$

Proof: By definition of τ we have

$$\begin{aligned} \{\tau < \infty\} &= \{\min\{\tau_{\hat{S}^-}, \tau_{\hat{S}^+}\} < \infty\} \\ &\supseteq \{\tau_{\hat{S}^+} < \infty\} \end{aligned}$$

as well as

$$\{\tau < \infty\} \supseteq \{\tau_{\hat{S}^-} < \infty\}\,.$$

This implies

$$P(\tau < \infty) \geq P(\tau_{\hat{S}^+} < \infty)$$

and

$$P(\tau < \infty) \geq P(\tau_{\hat{S}^-} < \infty)\,.$$

Therefore we only have to show that either

$$P(\tau_{\hat{S}^+} < \infty) = 1$$

or

$$P(\tau_{\hat{S}^-} < \infty) = 1\,.$$

In order to do this we use the definition of $S(t), t \geq 0$, which implies

$$\begin{aligned}\tau_{\hat{S}^+} &= \inf\{t \geq 0 : S(t) = \hat{S}^+\} \\ &= \inf\left\{t \geq 0 : W_t + \frac{\mu - \frac{1}{2}\sigma^2}{\sigma} t = \frac{1}{\sigma} \ln \frac{\hat{S}^+}{S(0)}\right\}.\end{aligned}$$

Analogously we obtain

$$\tau_{\hat{S}^-} = \inf\left\{t \geq 0 : W_t + \frac{\mu - \frac{1}{2}\sigma^2}{\sigma} t = \frac{1}{\sigma} \ln \frac{\hat{S}^-}{S(0)}\right\}.$$

Defining

$$\begin{aligned}d^{\hat{S}^+} &:= \frac{1}{\sigma} \ln \frac{\hat{S}^+}{S(0)}, \\ d^{\hat{S}^-} &:= \frac{1}{\sigma} \ln \frac{\hat{S}^-}{S(0)}\end{aligned}$$

and

$$\hat{\mu} := \frac{\mu - \frac{1}{2}\sigma^2}{\sigma}$$

we obtain

$$\tau_{\hat{S}^+} = \inf\{t \geq 0 : W_t + \hat{\mu} t = d^{\hat{S}^+}\}$$

and

$$\tau_{\hat{S}^-} = \inf\{t \geq 0 : W_t + \hat{\mu} t = d^{\hat{S}^-}\}.$$

Obviously $S(0) \in [\hat{S}^-, \hat{S}^+]$ implies $d^{\hat{S}^+} \geq 0$ and $d^{\hat{S}^-} \leq 0$. Setting $\tilde{\tau} := \tau_{\hat{S}^+}$ if $\hat{\mu} \geq 0$ and $\tilde{\tau} := \tau_{\hat{S}^-}$ otherwise, we obtain from Karatzas and Shreve (1988), p.197 that

$$P(\tilde{\tau} < \infty) = \begin{cases} e^{\hat{\mu}\, d^{\hat{S}^+} - |\hat{\mu}\, d^{\hat{S}^+}|} = 1 & \text{if } \hat{\mu} \geq 0, \\ e^{\hat{\mu}\, d^{\hat{S}^-} - |\hat{\mu}\, d^{\hat{S}^-}|} = 1 & \text{otherwise}, \end{cases}$$

which implies

$$P(\tau < \infty) = 1.$$

□

Knowing that the first intervention time is almost surely finite we will determine the expectation of τ and the probability that S reaches the upper boundary first (reaches the lower boundary first).

Theorem 2. (Expectation and Probabilities I.) *Assume that* $[\hat{S}^-, \hat{S}^+] \subseteq (0, \infty)$. *Further assume* $S(0) \in [\hat{S}^-, \hat{S}^+]$.

i) *Let* $\frac{\mu}{\sigma^2} \neq \frac{1}{2}$. *Then we have*

$$\begin{aligned} E^{S(0)}(\tau) &= -\frac{2}{\sigma^2}\left\{\frac{1}{2\frac{\mu}{\sigma^2}-1}\ln\frac{S(0)}{\hat{S}^-} + \frac{1}{(2\frac{\mu}{\sigma^2}-1)^2}\left[\left(\frac{S(0)}{\hat{S}^-}\right)^{1-2\frac{\mu}{\sigma^2}} - 1\right]\right\} \\ &+\frac{2}{\sigma^2}\left(S(0)^{1-2\frac{\mu}{\sigma^2}} - (\hat{S}^-)^{1-2\frac{\mu}{\sigma^2}}\right) \\ &\quad \frac{\ln\frac{\hat{S}^+}{\hat{S}^-} + \frac{1}{2\frac{\mu}{\sigma^2}-1}\left[\left(\frac{\hat{S}^+}{\hat{S}^-}\right)^{1-2\frac{\mu}{\sigma^2}} - 1\right]}{(2\frac{\mu}{\sigma^2}-1)\left((\hat{S}^+)^{1-2\frac{\mu}{\sigma^2}} - (\hat{S}^-)^{1-2\frac{\mu}{\sigma^2}}\right)}. \end{aligned} \tag{2.77}$$

Further we have

$$P^{S(0)}(S(\tau) = \hat{S}^+) = \frac{S(0)^{1-2\frac{\mu}{\sigma^2}} - (\hat{S}^-)^{1-2\frac{\mu}{\sigma^2}}}{(\hat{S}^+)^{1-2\frac{\mu}{\sigma^2}} - (\hat{S}^-)^{1-2\frac{\mu}{\sigma^2}}} \tag{2.78}$$

which implies

$$P^{S(0)}(S(\tau) = \hat{S}^-) = 1 - \frac{S(0)^{1-2\frac{\mu}{\sigma^2}} - (\hat{S}^-)^{1-2\frac{\mu}{\sigma^2}}}{(\hat{S}^+)^{1-2\frac{\mu}{\sigma^2}} - (\hat{S}^-)^{1-2\frac{\mu}{\sigma^2}}}. \tag{2.79}$$

ii) *Let* $\frac{\mu}{\sigma^2} = \frac{1}{2}$. *Then we obtain*

$$E^{S(0)}(\tau) = \frac{1}{\sigma^2}\ln\left(\frac{S(0)}{\hat{S}^-}\right)\ln\left(\frac{\hat{S}^+}{S(0)}\right). \tag{2.80}$$

Further we obtain

$$P^{S(0)}(S(\tau) = \hat{S}^+) = \frac{\ln(S(0)) - \ln(\hat{S}^-)}{\ln(\hat{S}^+) - \ln(\hat{S}^-)} \tag{2.81}$$

which implies

$$P^{S(0)}(S(\tau) = \hat{S}^-) = \frac{\ln(\hat{S}^+) - \ln(S(0))}{\ln(\hat{S}^+) - \ln(\hat{S}^-)}. \tag{2.82}$$

Proof: We will split up the proof into three steps. At first we derive a differential equation which is in the case of specific boundary conditions solved by $E(\tau), P(S(\tau) = \hat{S}^+)$ and $P(S(\tau) = \hat{S}^-)$. Then we solve this differential equation in general. Thirdly we derive the specific solutions which give us the expectation and the probabilities. Since $S(0)$ is fixed throughout this proof we

set $E^{S(0)} = E$ and $P^{S(0)} = P$.
Step 1: To simplify our notations we define

$$a := \hat{S}^-, \quad b := \hat{S}^+$$

which implies

$$\tau = \min\{\tau_a, \tau_b\}.$$

Further we define the bounded stopping time

$$\tau^{(n)} := \tau \wedge n.$$

Since τ is almost surely finite it is obvious that

$$\lim_{n\to\infty} \tau^{(n)} = \tau \quad a.s.$$

and that (due to monotone convergence)

$$\lim_{n\to\infty} E(\tau^{(n)}) = E(\tau).$$

To derive the requested differential equation in general we consider an arbitrary process Z which develops according to

$$dZ_t = \hat{\mu}(Z_t)\,dt + \hat{\sigma}(Z_t)\,dW_t, \quad Z_0 = z_0 \in [a,b]$$

where $\hat{\mu}(.)$ and $\hat{\sigma}(.)$ are progressively measurable, bounded stochastic processes. (Of course this includes the case of $Z_t = S(t)$.) Let $f \in C^2$. Then Itô's rule implies

$$\begin{aligned} f(Z_{\tau^{(n)}}) &= f(z_0) + \int_0^{\tau^{(n)}} \left[f'(Z_s)\hat{\mu}(Z_s) + \frac{1}{2} f''(Z_s)\hat{\sigma}^2(Z_s) \right] ds \\ &\quad + \int_0^{\tau^{(n)}} f'(Z_s)\hat{\sigma}(Z_s)dW_s. \end{aligned}$$

Since $f' \in C([a,b])$ and $\hat{\sigma}$ is bounded, the product $f'\,\hat{\sigma}$ is bounded on $[a,b]$. Therefore we obtain

$$E(f(Z_{\tau^{(n)}})) = f(z_0) + E\left(\int_0^{\tau^{(n)}} f'(Z_s)\hat{\mu}(Z_s) + \frac{1}{2} f''(Z_s)\hat{\sigma}^2(Z_s)ds \right).$$

To compute the expectation of τ we have to solve the differential equation

$$f'(z)\,\hat{\mu}(z) + \frac{1}{2} f''(z)\,\hat{\sigma}^2(z) = -1, \quad z \in (a,b),$$

$$f(a) = f(b) = 0$$

which implies

$$E(f(Z_{\tau^{(n)}})) = f(z_0) - E(\tau^{(n)})$$

and

$$E(f(Z_\tau)) = 0\,.$$

Since $Z_{\tau^{(n)}} \in [a, b]$ and f is continuous on $[a, b]$, dominated convergence implies

$$\lim_{n\to\infty} E(f(Z_{\tau^{(n)}})) \;=\; E(f(Z_\tau))$$

which leads to

$$0 = E(f(Z_\tau)) = f(z_0) - E(\tau)$$

and

$$E(\tau) = f(z_0)\,.$$

To compute the probability that Z reaches b first, we have to solve the differential equation

$$f'(z)\,\hat\mu(z) + \frac{1}{2} f''(z)\,\hat\sigma^2(z) = 0\,, \quad z \in (a, b)\,,$$

$$f(a) = 0, \quad f(b) = 1$$

which implies

$$E(f(Z_{\tau^{(n)}})) = f(z_0)$$

and

$$E(f(Z_\tau)) = P(Z_\tau = b)\,.$$

Dominated convergence then implies

$$P(Z_\tau = b) = f(z_0)\,.$$

Step 2: Consider the differential equation

$$\frac{1}{2} f''(z)\hat\sigma^2(z) + f'(z)\hat\mu(z) = K\,,$$

$$f(a) = \eta_1\,, \quad f(b) = \eta_2$$

where

$$\begin{aligned} \hat\mu(z) &= \mu\, z\,, \\ \hat\sigma(z) &= \sigma\, z\,. \end{aligned}$$

(These are the coefficients of S.) Then it is equivalent to solve

$$f''(z) + c(z)\, f'(z) = b(z)$$

where

$$\begin{aligned} c(z) &= 2\frac{\hat{\mu}(z)}{\hat{\sigma}^2(z)} \\ &= 2\frac{\mu}{\sigma^2 \, z}\,, \\ b(z) &= 2\frac{K}{\hat{\sigma}^2(z)} \\ &= 2\frac{K}{\sigma^2 \, z^2}\,. \end{aligned}$$

Defining

$$p(z) := e^{\int_a^z c(s)\,ds}$$

which satisfies

$$p'(z) = p(z)c(z)\,, \quad p(z) > 0$$

we obtain

$$f''(z) + c(z)\,f'(z) = b(z) \quad \Longleftrightarrow$$
$$p(z)[\,f''(z) + c(z)\,f'(z)\,] = p(z)\,b(z)$$

where

$$\begin{aligned} p(z)[\,f''(z) + c(z)\,f'(z)\,] &= p(z)f''(z) + f'(z)p(z)c(z) \\ &= p(z)f''(z) + f'(z)p'(z) \\ &= [\,p(z)f'(z)\,]'\,. \end{aligned}$$

Solving

$$f''(z) + c(z)\,f'(z) = b(z)$$

is therefore equivalent to solving

$$[\,p(z)f'(z)\,]' = p(z)b(z)\,.$$

This last equation can be solved by integrating two times: From the equality

$$\int_a^z [\,p(s)f'(s)\,]'\,ds = \int_a^z p(s)b(s)\,ds$$

we obtain

$$f'(z) = \frac{1}{p(z)}\left\{\int_a^z p(s)b(s)\,ds + p(a)\,f'(a)\right\}$$

which gives us

$$\begin{aligned} f(z) &= \int_a^z \frac{1}{p(t)}\left[\int_a^t p(s)b(s)\,ds\right]dt \\ &\quad + p(a)f'(a)\int_a^z \frac{1}{p(t)}\,dt + f(a)\,. \end{aligned}$$

Due to

$$p(a) = 1\,, \quad f(a) = \eta_1\,, \quad f'(a) =: c_0$$

we get

$$\begin{aligned} f(z) &= \int_a^z \frac{1}{p(t)} \left[\int_a^t p(s)b(s)\,ds \right] dt \\ &\quad + c_0 \int_a^z \frac{1}{p(t)}\,dt + \eta_1 \end{aligned}$$

where the paramter c_0 is uniquely determined by the second boundary condition $f(b) = \eta_2$ which is satisfied if and only if

$$c_0 = \frac{\eta_2 - \eta_1 - \int_a^b \frac{1}{p(t)} \left[\int_a^t p(s)b(s)\,ds \right] dt}{\int_a^b \frac{1}{p(t)}\,dt}\,.$$

To simplify our further calculations we define

$$\begin{aligned} L_1(z) &:= \int_a^z \frac{1}{p(t)}\,dt\,, \\ L_2(t) &:= \int_a^t p(s)b(s)\,ds\,, \\ L_3(z) &:= \int_a^z \frac{1}{p(t)}\,L_2(t)\,dt \end{aligned}$$

giving us

$$f(z) = L_3(z) + c_0 L_1(z) + \eta_1$$

and

$$c_0 = \frac{\eta_2 - \eta_1 - L_3(b)}{L_1(b)}\,.$$

Besides we define

$$q := \frac{\mu}{\sigma^2}\,.$$

At first we determine p:

$$\begin{aligned} p(s) &= e^{\int_a^s c(t)\,dt} \\ &= e^{\int_a^s 2\frac{\mu}{\sigma^2 t}\,dt} \\ &= e^{2\,q \int_a^s \frac{1}{t}\,dt} \\ &= e^{2\,q\,(\ln s - \ln a)} \\ &= \left(\frac{s}{a}\right)^{2\,q}\,. \end{aligned}$$

From this point on we have to consider two different cases: At first we consider the case $2q = 1$ and then $2q \neq 1$.
<u>Step 2</u> with <u>$2q = 1$</u>: Let $2q = 1$. Then we obtain

$$p(s) = \frac{s}{a}$$

which implies

$$\begin{aligned} L_1(z) &= \int_a^z \frac{1}{p(t)}\,dt \\ &= \int_a^z \frac{a}{t}\,dt \\ &= a\left[\ln z - \ln a\right], \end{aligned}$$

$$\begin{aligned} L_2(t) &= \int_a^t p(s)b(s)\,ds \\ &= \frac{2K}{\sigma^2 a}\int_a^t \frac{1}{s}\,ds \\ &= \frac{2K}{\sigma^2 a}[\ln t - \ln a] \end{aligned}$$

and

$$\begin{aligned} L_3(z) &= \int_a^z \frac{1}{p(t)} L_2(t)\,dt \\ &= \frac{2K}{\sigma^2 a}\int_a^z \frac{a}{t}\ln\left(\frac{t}{a}\right)\,dt \\ &= \frac{2K}{\sigma^2}\int_0^{\ln\frac{z}{a}} y\,dy \\ &= \frac{K}{\sigma^2}\ln^2\left(\frac{z}{a}\right) \end{aligned}$$

thus giving us

$$f(z) = \frac{K}{\sigma^2}\ln^2\left(\frac{z}{a}\right) + \frac{\ln\frac{z}{a}}{\ln\frac{b}{a}}\left[\eta_2 - \eta_1 - \frac{K}{\sigma^2}\ln^2\left(\frac{b}{a}\right)\right] + \eta_1\,.$$

Step 3 with $2q = 1$: Now we use the specific boundary conditions to derive the expectation and the desired probabilities. At first we assume

$$K = -1\,, \quad \eta_1 = \eta_2 = 0\,.$$

Then $f(z) = E(\tau)$ where

$$\begin{aligned} f(z) &= -\frac{1}{\sigma^2}\ln^2\left(\frac{z}{a}\right) + \frac{1}{\sigma^2}\ln\left(\frac{b}{a}\right)\ln\left(\frac{z}{a}\right) \\ &= \frac{1}{\sigma^2}\ln\left(\frac{z}{a}\right)\ln\left(\frac{b}{z}\right)\,. \end{aligned}$$

Secondly we assume

$$K = 0\,, \quad \eta_1 = 0\,, \quad \eta_2 = 1\,.$$

Then $f(z) = P(Z_\tau = b)$ where

$$f(z) = \frac{\ln \frac{z}{a}}{\ln \frac{b}{a}}.$$

Due to $a \le z \le b$ we have $0 \le P(Z_\tau = b) \le 1 \quad \forall\, z \in [a,b]$.
Step 2 with $2q \neq 1$: Let $2q \neq 1$. Then we obtain

$$\begin{aligned}
L_1(z) &= \int_a^z \frac{1}{p(t)}\, dt \\
&= \int_a^z \left(\frac{t}{a}\right)^{-2q} dt \\
&= a^{2q} \frac{1}{1-2q} \left(z^{1-2q} - a^{1-2q}\right),
\end{aligned}$$

$$\begin{aligned}
L_2(t) &= \int_a^t p(s)\, b(s)\, ds \\
&= \frac{2K}{\sigma^2} \int_a^t \left(\frac{s}{a}\right)^{2q} s^{-2}\, ds \\
&= \frac{2K}{\sigma^2} a^{-2q} \int_a^t s^{2q-2}\, ds \\
&= \frac{2K}{\sigma^2} a^{-2q} \frac{1}{2q-1} \left(t^{2q-1} - a^{2q-1}\right)
\end{aligned}$$

and

$$\begin{aligned}
L_3(z) &= \int_a^z \frac{1}{p(t)}\, L_2(t)\, dt \\
&= \frac{2K}{\sigma^2(2q-1)}\, a^{-2q} \int_a^z \left\{ a^{2q}\, t^{-1} - a^{2q-1} \left(\frac{t}{a}\right)^{-2q} \right\} dt \\
&= \frac{2K}{\sigma^2(2q-1)} \int_a^z \left\{ \frac{1}{t} - t^{-2q}\, a^{2q-1} \right\} dt \\
&= \frac{2K}{\sigma^2(2q-1)} \left(\ln \frac{z}{a} - \frac{a^{2q-1}}{1-2q} \left[z^{1-2q} - a^{1-2q} \right] \right) \\
&= \frac{2K}{\sigma^2} \left\{ \frac{1}{2q-1} \ln \frac{z}{a} + \frac{1}{(2q-1)^2} \left[\left(\frac{z}{a}\right)^{1-2q} - 1 \right] \right\}
\end{aligned}$$

thus implying

$$\begin{aligned}
f(z) &= \eta_1 + \frac{2K}{\sigma^2} \left\{ \frac{1}{2q-1} \ln \frac{z}{a} + \frac{1}{(2q-1)^2} \left[\left(\frac{z}{a}\right)^{1-2q} - 1 \right] \right\} \\
&\quad + \left(z^{1-2q} - a^{1-2q}\right) \\
&\quad \frac{\eta_2 - \eta_1 - \frac{2K}{\sigma^2} \left\{ \frac{1}{2q-1} \ln \frac{b}{a} + \frac{1}{(2q-1)^2} \left[\left(\frac{b}{a}\right)^{1-2q} - 1 \right] \right\}}{b^{1-2q} - a^{1-2q}}.
\end{aligned}$$

Step 3 with $2q \neq 1$: Again we use the specific boundary conditions to derive the expectation and the desired probabilities. At first we assume

$$K = -1\,, \quad \eta_1 = \eta_2 = 0\,.$$

Then $f(z) = E(\tau)$ where

$$\begin{aligned} f(z) &= -\frac{2}{\sigma^2}\left\{\frac{1}{2q-1}\ln\frac{z}{a} + \frac{1}{(2q-1)^2}\left[\left(\frac{z}{a}\right)^{1-2q} - 1\right]\right\} \\ &\quad + \frac{2}{\sigma^2}\left(z^{1-2q} - a^{1-2q}\right)\frac{\ln\frac{b}{a} + \frac{1}{2q-1}\left[\left(\frac{b}{a}\right)^{1-2q} - 1\right]}{(2q-1)\left(b^{1-2q} - a^{1-2q}\right)}\,. \end{aligned}$$

Secondly we assume

$$K = 0\,, \quad \eta_1 = 0\,, \quad \eta_2 = 1\,.$$

Then $f(z) = P(Z_\tau = b)$ where

$$f(z) = \frac{z^{1-2q} - a^{1-2q}}{b^{1-2q} - a^{1-2q}}\,.$$

Due to $a \leq z \leq b$ we have $P(Z_\tau = b) \in [0,1] \quad \forall\, z \in [a,b]$ in both cases $2q > 1$ and $2q < 1$.

Replacing now z_0 by $S(0)$, Z by S, a by $\hat{S}^-$ and b by $\hat{S}^+$ completes the proof. $\square$

Remark 8. From (2.41)/(2.42) we obtain that the values of $S^*, \hat{S}^+, \hat{S}^-$ are constant if $\frac{\mu}{\sigma^2}$ (and λ) are constant. Further, Theorem 2 shows that the probabilities (given that $S^*, \hat{S}^+, \hat{S}^-$ are fixed) only depend on the quotient $\frac{\mu}{\sigma^2}$ whereas the expectations depend on the single values of σ and μ (since they depend on σ^2 and $\frac{\mu}{\sigma^2}$). The latter effect can be explained as follows: The probability that the risky money reaches the upper boundary $\hat{S}^+$ first (reaches the lower boundary first, respectively) only depends on the direction of the stock movement whereas the expectation also depends on the velocity of the stock movement which is given by σ.

The following examples should illustrate the quantitative behaviour of the expectation and the probability that the upper boundary is reached first:

Examples 3.

a) $k = 0.01\,,\ K = 0\,,\ \mu = 0.025\,,\ \sigma = 0.1\,,\ \lambda = 0.01 \quad \Longrightarrow$
$S^* = 250\,,\ \hat{S}^- = S^- = 204.57\,,\ \hat{S}^+ = S^+ = 295.43\,.$

$S(0)$	$E(\tau)$	$P(S(\tau) = \hat{S}^+)$
295.43	0	1
280	1.36915	0.92855
265	2.44704	0.83739
250	3.13623	0.71635
235	3.22568	0.55286
220	2.38661	0.32773
204.57	0	0

b) $k = 0\,,\ K = 0.1\,,\ \mu = 0.025\,,\ \sigma = 0.2\,,\ \lambda = 0.01 \quad \Longrightarrow$
$S^* = S^+ = S^- = 62.5\,,\ \hat{S}^- = 36.33\,,\ \hat{S}^+ = 88.67\,.$

$S(0)$	$E(\tau)$	$P(S(\tau) = \hat{S}^+)$
88.67	0	1
70	3.80573	0.75636
62.5	4.70102	0.63436
50	4.6182	0.38382
36.33	0	0

c) $k = 0.01\,,\ K = 0.1\,,\ \mu = 0.03\,,\ \sigma = 0.2\,,\ \lambda = 0.001 \quad \Longrightarrow$
$S^* = 750\,,\ \hat{S}^- = 482.15\,,\ S^- = 599.43\,,\ S^+ = 900.57\,,\ \hat{S}^+ = 1017.85\,.$

$S(0)$	$E(\tau)$	$P(S(\tau) = \hat{S}^+)$
1017.85	0	1
900.57	1.82929	0.86064
850	2.46696	0.79183
750	3.32535	0.63581
650	3.3815	0.44504
599.43	2.94964	0.33087
482.15	0	0

d) $k = 0.01$, $K = 0.1$, $\mu = 0.01$, $\sigma = 0.2$, $\lambda = 0.001 \implies$
$S^* = 250$, $\hat{S}^- = 111.98$, $S^- = 183.56$, $S^+ = 316.44$, $\hat{S}^+ = 388.02$.

$S(0)$	$E(\tau)$	$P(S(\tau) = \hat{S}^+)$
388.02	0	1
350	3.18267	0.89141
316.44	5.63792	0.79054
280	7.79315	0.67475
250	9.02664	0.57363
220	9.58915	0.46624
183.56	8.98386	0.3254
150	6.52854	0.18269
111.98	0	0

Now we will look at the case $\hat{S}^- < 0$. Assume that $t \in [0, T]$. Before the first intervention takes place, we have $S(t) \in (0, \hat{S}^+]$ and $S(t) \in [\hat{S}^-, 0)$, respectively, depending on the value of $S(0)$. If $S(0) > 0$ the investor rebalances his portfolio whenever $S(t) = \hat{S}^+$. If $S(0) < 0$ his first intervention takes place when $S(t) = \hat{S}^-$. (The further interventions depend on the sign of the restarting line S^- .) Therefore the relevant intervention times are $\tau = \tau_{\hat{S}^+}$ if $S(0) > 0$ and $\tau = \tau_{\hat{S}^-}$ if $S(0) < 0$.

Remark 9. If $\hat{S}^- < 0$, it is obvious that

$$P(S(\tau) = \hat{S}^+) = \begin{cases} 1 & \text{if } S(0) > 0\,, \\ 0 & \text{if } S(0) < 0\,. \end{cases}$$

Theorem 3. (Expectation II.) *Assume that* $\hat{S}^- < 0$ *and* $\mu - \frac{1}{2}\sigma^2 \geq 0$. *Then we have*

$$E^{S(0)}(\tau) = \frac{1}{\sigma} \ln \frac{\hat{S}^+}{S(0)} \quad \int_0^\infty \frac{1}{\sqrt{2\pi t}}\, e^{-\frac{\left(\ln \frac{\hat{S}^+}{S(0)} - \left(\mu - \frac{1}{2}\sigma^2\right)t\right)^2}{2\sigma^2 t}}\, dt \tag{2.83}$$

if $S(0) > 0$ *and*

$$E^{S(0)}(\tau) = \frac{1}{\sigma} \ln \frac{\hat{S}^-}{S(0)} \quad \int_0^\infty \frac{1}{\sqrt{2\pi t}}\, e^{-\frac{\left(\ln \frac{\hat{S}^-}{S(0)} - \left(\mu - \frac{1}{2}\sigma^2\right)t\right)^2}{2\sigma^2 t}}\, dt \tag{2.84}$$

if $S(0) < 0$.

Proof: As in the foregoing proof we set $a := \hat{S}^-$, $b := \hat{S}^+$. Further we set

$z := S(0)$. At first let $z > 0$. Due to Definition 4 (2.75) we obtain

$$\begin{aligned}\{\tau \le t\} &= \{\tau_b \le t\} \\ &= \left\{\max_{u\in[0,t]} S(u) \ge b\right\} \\ &= \left\{\max_{u\in[0,t]} \{W_u + \hat{\mu}u\} \ge Y(z)\right\}\end{aligned}$$

where (in analogy to subsection 2.5.1)

$$\hat{\mu} := \frac{\mu - \frac{1}{2}\sigma^2}{\sigma},$$

$$Y(z) := \frac{1}{\sigma}\ln\frac{b}{z} \ge 0\,.$$

Replacing $Y(S_0^N)$ by $Y(z)$ we obtain the density of τ from the results of Subsection 2.5.1 (2.68):

$$f_\tau(t) = \frac{Y(z)}{\sqrt{t^3}}\,\varphi\left(\frac{Y(z) - \hat{\mu}t}{\sqrt{t}}\right)\,I_{(0,\infty)}(t)\,.$$

For $z < 0$ we have

$$\begin{aligned}\{\tau \le t\} &= \{\tau_a \le t\} \\ &= \left\{\min_{u\in[0,t]} S(u) \le a\right\} \\ &= \left\{\max_{u\in[0,t]}[-S(u)] \ge -a\right\} \\ &= \left\{\max_{u\in[0,t]}\{W_u + \hat{\mu}u\} \ge L(z)\right\}\end{aligned}$$

where

$$L(z) := \frac{1}{\sigma}\ln\frac{a}{z} \ge 0\,.$$

Similar to the case $z > 0$ we obtain the density of τ:

$$f_\tau(t) = \frac{L(z)}{\sqrt{t^3}}\,\varphi\left(\frac{L(z) - \hat{\mu}t}{\sqrt{t}}\right)\,I_{(0,\infty)}(t)\,.$$

Replacing now b by $\hat{S}^+$, a by $\hat{S}^-$ and z by $S(0)$ we obtain (2.83) and (2.84).□

Remark 10. If $\hat{S}^- < 0$ and $\mu - \frac{1}{2}\sigma^2 < 0$ we obtain

$$E(\tau) = \infty\,.$$

Proof: Again we set $Y(z) = \frac{1}{\sigma} \ln \frac{\hat{S}^+}{S(0)}$ and $L(z) = \frac{1}{\sigma} \ln \frac{\hat{S}^-}{S(0)}$. Further we define

$$\hat{\tau}_c := \inf\{t \geq 0 : W_t + \hat{\mu} t = c\}.$$

If $S(0) > 0$ the above proof gives us that

$$\{\tau \leq t\} = \left\{ \max_{u \in [0,t]} \{W_u + \hat{\mu} u\} \geq Y(z) \right\} = \{\hat{\tau}_{Y(z)} \leq t\}.$$

From the results of Karatzas and Shreve (1988), p.196/197 we therefore obtain

$$\begin{aligned} P(\tau < \infty) &= e^{\frac{1}{\sigma} \ln \frac{\hat{S}^+}{S(0)} (\hat{\mu} - |\hat{\mu}|)} \\ &= \left(\frac{\hat{S}^+}{S(0)} \right)^{-2 \frac{|\hat{\mu}|}{\sigma}} \\ &= \left(\frac{S(0)}{\hat{S}^+} \right)^{2 \frac{|\hat{\mu}|}{\sigma}} \\ &< 1 \end{aligned}$$

implying

$$E(\tau) = \infty.$$

We obtain the same results in the case $S(0) < 0$ with $\hat{S}^-$ instead of $\hat{S}^+$. □

Before we compute the expectations given in Theorem 3 numerically we will take a look at the shape of the product $t\, f_\tau(t)\,, t > 0$.

Proposition 5. *We define*

$$g(t) := t\, f_\tau(t) \quad , t > 0 \tag{2.85}$$

where f_τ *is given as in Theorem 3. Then we obtain that*

i) $\lim_{t \to 0} g(t) = 0$,

ii) $\lim_{t \to \infty} g(t) = 0$,

iii) g *has exactly one maximum which is attained in* $t = t^*$ *where*

$$t^* := \frac{\sqrt{1 + 2 \frac{w^2 \hat{\mu}^2}{\sigma^2}} - 1}{2 \hat{\mu}^2} \tag{2.86}$$

with

$$w := \begin{cases} \ln \frac{\hat{S}^+}{S(0)} & \text{if} \quad S(0) > 0, \\ \ln \frac{\hat{S}^-}{S(0)} & \text{if} \quad S(0) < 0 \end{cases}$$

and

$$\hat{\mu} := \frac{\mu - \frac{1}{2}\sigma^2}{\sigma}.$$

Proof: Since g equals the function which has to be integrated to obtain the expectations of Theorem 3 we have

$$\begin{aligned} g(t) &= \frac{w}{\sigma}\frac{1}{\sqrt{t}}\,e^{-\frac{1}{2t}\left(\frac{w}{\sigma}-\hat{\mu}\,t\right)^2} \\ &= \frac{w}{\sigma}\frac{1}{\sqrt{t}}\,e^{-\frac{w^2}{2\sigma^2 t}-\frac{\hat{\mu}^2}{2}t+\frac{\hat{\mu}\,w}{\sigma}}\,. \end{aligned}$$

Obviously, $g(t) \stackrel{t\to\infty}{\longrightarrow} 0$. To prove i) we set $x := \frac{1}{\sqrt{t}}$ where $x\to\infty \iff t\to 0$. Then we obtain

$$\begin{aligned} g(x) &= \frac{w}{\sigma}\,x\,e^{-\frac{w^2}{2\sigma^2}x^2-\frac{\hat{\mu}^2}{2x^2}+\frac{\hat{\mu}\,w}{\sigma}} \\ &= \frac{w}{\sigma}\,e^{\frac{\hat{\mu}\,w}{\sigma}-\frac{\hat{\mu}^2}{2x^2}}\,\frac{x}{e^{\frac{w^2}{2\sigma^2}x^2}} \\ &\stackrel{x\to\infty}{\longrightarrow} 0\,. \end{aligned}$$

To prove iii) we differentiate g with respect to t and show that the derivative possesses exactly one positive zero. Due to i),ii) and the non-negativity of g this must be the maximum of g.

$$\begin{aligned} \frac{\partial}{\partial t}g(t) &= \frac{w}{\sigma}\,e^{-\frac{1}{2t}\left(\frac{w}{\sigma}-\hat{\mu}\,t\right)^2}\left[-\frac{1}{2\,t^{3/2}}+\frac{1}{\sqrt{t}}\left(-\frac{\hat{\mu}^2}{2}+\frac{w^2}{2\,\sigma^2\,t^2}\right)\right]=0 \\ \iff &\quad t^2+\frac{t}{\hat{\mu}^2}-\frac{w^2}{2\,\sigma^2\,\hat{\mu}^2}=0 \\ \stackrel{t>0}{\iff} &\quad t=-\frac{\sqrt{1+2\,\frac{w^2\,\hat{\mu}^2}{\sigma^2}}-1}{2\,\hat{\mu}^2}\,. \end{aligned}$$

□

Corollary 1. *Let $g^* := g(t^*)$ be the maximum of g given in (2.85). For all R with $0<R<g^*$ there exist $t_1 = t_1(R) > 0$ and $t_2 = t_2(R) > 0$ such that*

$$\begin{aligned} g(t) &< R \quad \forall\, t\in[0,t_1)\,, \\ g(t) &\geq R \quad \forall\, t\in[t_1,t_2]\,, \\ g(t) &< R \quad \forall\, t\in(t_2,\infty)\,. \end{aligned}$$

Proof: Proposition 5 yields that g increases for $t\in[0,t^*]$ and decreases for $t\in[t^*,\infty)$. Together with $g(t)\stackrel{t\to 0}{\longrightarrow}0$, $g(t)\stackrel{t\to\infty}{\longrightarrow}0\,, g(t)\geq 0\ \forall\, t\in[0,\infty)$ and the continuity of g this implies the assertion. □

Due to the above results we can choose a small constant R between 0 and g^* and replace the interval $(0,\infty)$ by the corresponding interval $[t_1(R),t_2(R)]\subset(0,\infty)$.

Since the choice of the lower bound R affects the precision and the speed of the numerical integration it is important to choose it in a way such that the integration does not take too long but is sufficiently exact. To compute the following examples we have chosen $R = 1^{-10}$. Further we have chosen $\Delta t = 0.01$ as the integration step on the relevant interval $[t_1, t_2]$. This choice guarantees a numerical integration which is quite fast but also very exact.

Examples 4. $\mu = 0.03\,,\ \sigma = 0.2\,,\ \lambda = 0.1 \quad \Longrightarrow \quad S^* = 7.5\,,\ \mu - \frac{1}{2}\sigma^2 = 0.01\,.$

a) $k = 0\,,\ K = 1 \quad \Longrightarrow \quad \hat{S}^- = -1.56\,,\ S^- = S^+ = 7.5\,,\ \hat{S}^+ = 16.56\,.$

$S(0)$	$E(\tau)$
16.56	0
14	16.79328
11.5	36.46431
7.5	79.20871
3.5	155.42272
1	280.69902
-0.2	205.41237
-1	44.46858
-1.56	0

b) $k = 0.01\,,\ K = 1 \quad \Longrightarrow \quad \hat{S}^- = -1.66\,,\ S^- = 7.3\,,\ S^+ = 7.7\,,\ \hat{S}^+ = 16.66\,.$

$S(0)$	$E(\tau)$
16.66	0
14	17.39533
11.5	37.06636
7.7	77.17903
7.5	79.81076
7.3	82.51363
3.5	156.02477
1	281.30106
-0.2	211.62555
-1	50.68176
-1.66	0

c) $k = 0.6\,,\ K = 0 \quad\Longrightarrow\quad \hat{S}^- = S^- = -0.05\,,\ \hat{S}^+ = S^+ = 15.05\,.$

$S(0)$	$E(\tau)$
15.05	0
12	22.64713
10	40.87929
7.5	69.6475
5	110.19401
2.5	179.50873
1	271.1378
-0.01	160.94379
-0.05	0

d) $k = 1.3\,,\ K = 0.6 \Longrightarrow \hat{S}^- = -5.96\,,\ S^- = -0.2\,,\ S^+ = 15.2\,,\ \hat{S}^+ = 20.96\,.$

$S(0)$	$E(\tau)$
20.96	0
17	20.94025
15.2	32.13204
12	55.77092
10	74.00308
7.5	102.77128
5	143.3178
2.5	212.63251
1	304.26159
-0.1	408.76556
-0.2	339.45084
-1	178.50705
-3	68.64582
-5.96	0

Remark 11.

a) For $0 < S(0) \le \hat{S}^+$ the expectation of the first intervention time decreases with $S(0)$. If $S(0) \downarrow 0$ the expectation tends to ∞. For $\hat{S}^- \le S(0) < 0$ the expectation increases with $S(0)$ and for $S(0) \uparrow 0$ the expectation also tends to ∞.

b) As we have seen in the last two examples, the proportional cost parameter k has to be (unrealistically) high in order to obtain a negative restarting line S^-. This relation can be explained with the help of Proposition 3: (2.54) and the first inequality of (2.55) show that it is necessary (but in general not sufficient) to choose $k > \frac{2}{3}\frac{\mu}{\sigma^2}$ in order to have $S^- < 0$. Due to the assumption $\frac{\mu}{\sigma^2} \geq \frac{1}{2}$ we therefore obtain $k \geq \frac{1}{3}$, in the above examples even higher values of k are needed.

In the following we will analyse the impact of $E(\tau)$ on the expected number of interventions between 0 and T and thus on the expected transaction costs. In the presence of transaction costs each transaction reduces the wealth of the investor. Therefore the investor is interested in not intervening too often. On the other hand, the only reason to make interventions is to maximize the expected utility of his terminal wealth: To do this the investor has to keep his risky money near to the optimal constant one of an investor who rebalances continuously and therefore he has to intervene whenever his risky money differs "too much" from S^*. The described conflict causes the relation between the boundaries/restarting lines of the no-transaction region and the cost parameters as shown above.

To compute the expected number of interventions we need the following notations:

Notation 2. Let τ_0 be the first intervention time (as given in Definition 4) if the volume of risky money S starts in $S(0) \in [\hat{S}^-, \hat{S}^+]$. Let $\tau_1, \tau_2, \ldots$ be the periods of time after which the further interventions take place or would take place if the time horizon was not bounded (also as given in Definition 4 with $S(0) \in \{S^-, S^+\}$). According to our model after any intervention the process S starts in S^+ or S^- depending on the value of S right before the intervention. Therefore we have to distinguish between the following cases:

a) $k > 0$, $K = 0$: Since boundary and restarting line are equal, the investor has to make the minimal intervention to keep S at the boundary of the no-transaction region. This means that the stopping times accumulate and it is not clear what the stopping times $\tau_1, \tau_2, \ldots$ are.

b) $k = 0$, $K > 0$. In this case the restarting lines satisfy $S^* = S^+ = S^-$. Therefore the stopping times $\tau_1, \tau_2, \ldots$ are well-defined (for $\hat{S}^- > 0$ as well as for $\hat{S}^- < 0$) and they are identically distributed. Besides, $\tau_0, \tau_1, \ldots$ are independent.

c) $k > 0$, $K > 0$, $\hat{S}^- > 0$. Here the definition of the next stopping time always depends on the value of S before the foregoing intervention. Therefore we will omit this case.

d) $k > 0$, $K > 0$ and $S^- < 0$. Here we have to distinguish between $S(0) > 0$ and $S(0) < 0$. For $S(0) > 0$ the process S always starts in S^+ whereas it always starts in S^- for $S(0) < 0$. For fixed $S(0)$ the stopping times

$\tau_1, \tau_2, \ldots$ are therefore well-defined and they are identically distributed. $\tau_0, \tau_1, \ldots$ are independent.

e) $k > 0$, $K > 0$ and $\hat{S}^- < 0 < S^-$. After any intervention, the process S starts in S^+ if $S(0) > 0$. If $S(0) < 0$ the first intervention shifts S to S^-. After any further intervention S^+ is the relevant restarting line. Therefore $\tau_1, \tau_2 \ldots$ are identically distributed if $S(0) > 0$ whereas only $\tau_2, \tau_3, \ldots$ are identically distributed if $S(0) < 0$. Of course $\tau_0, \tau_1, \ldots$ are independent.

Theorem 4. (Expected Number of Interventions.) *Assume that* $S(0) \in [\hat{S}^-, \hat{S}^+]$. *Then the expected number of interventions in* $[0, T]$ *is bounded from above by*

$$\hat{N} := \frac{T}{E(\tau_1)} + 1 \tag{2.87}$$

if

i) $k = 0$, $K > 0$, $\hat{S}^- > 0$,

ii) $k = 0$, $K > 0$, $\hat{S}^- < 0$, $\mu \geq \frac{1}{2}\sigma^2$,

iii) $k > 0$, $K > 0$, $S^- < 0$, $\mu \geq \frac{1}{2}\sigma^2$,

iv) $k > 0$, $K > 0$, $\hat{S}^- < 0 < S^-$, $\mu \geq \frac{1}{2}\sigma^2$ *and* $S(0) > 0$.

The expected number of interventions in $[0, T]$ *is bounded from above by*

$$\hat{N} := \frac{T}{E(\tau_2)} + 2 \tag{2.88}$$

if

v) $k > 0$, $K > 0$, $\hat{S}^- < 0 < S^-$, $\mu \geq \frac{1}{2}\sigma^2$ *and* $S(0) < 0$.

Proof: Step 1: We start giving some definitions. Let T_0 be the sum of the stopping times which do not possess the same distribution as the following ones. This means

$$T_0 = \tau_0$$

in the cases i), ii), iii) and iv) and

$$T_0 = \tau_0 + \tau_1$$

in the last case. Further let $\rho_i, i \geq 1$ denote all the following identically distributed stopping times. (This means $\rho_1 = \tau_1$, $\rho_2 = \tau_2 \ldots$ in the cases i), ii), iii) and iv) and $\rho_1 = \tau_2\,, \rho_2 = \tau_3\,, \ldots$ in the last case.) Further we define N as the number of interventions that occur between 0 and T and $\tilde{N}$ as the number of all interventions after T_0. We also define

$$L_{\tilde{N}} := \sum_{i=1}^{\tilde{N}} \rho_i$$

which describes the length of time between T_0 and the last intervention and is therefore bounded by $T - T_0$ if $T_0 < T$ and equals 0 otherwise.
Step 2: Let T_0 be fixed and below T. Let Y be an arbitrary random variable with values in $[0, T-T_0]$ and density f_Y which is continuous on $[0, T-T_0]$. Then we define

$$Z_Y(u) := E(u^Y)$$

for $u \in (0, \infty)$. (This expectation exists due to $Y \leq T - T_0$.) For $Y = L_{\tilde{N}}$ we thus obtain

$$\begin{aligned} Z^{T_0}_{L_{\tilde{N}}}(u) &= E^{T_0}(u^{L_{\tilde{N}}}) \\ &= E^{T_0}(\, E^{T_0}(u^{L_{\tilde{N}}}) \,|\, \tilde{N}\,) \\ &= \sum_{k=0}^{\infty} E^{T_0}(u^{L_{\tilde{N}}} | \tilde{N} = k)\, P^{T_0}(\tilde{N} = k)\,. \end{aligned}$$

Let f_{L_k} denote the density of L_k. Then we have

$$\begin{aligned} &E^{T_0}(u^{L_{\tilde{N}}} | \tilde{N} = k) \\ &= E^{T_0}(u^{L_k} | L_k \leq T - T_0\,, L_{k+1} > T - T_0) \\ &= E^{T_0}(u^{L_k} | L_k \leq T - T_0\,, L_k + \rho_{k+1} > T - T_0) \\ &= E^{T_0}(\, E^{T_0}(u^{L_k} | L_k \leq T - T_0\,, \rho_{k+1} + L_k > T - T_0) |\, L_k\,) \\ &= \int_0^{T-T_0} E^{T_0}(u^x | x \leq T - T_0\,, \rho_{k+1} + x > T - T_0)\, f_{L_k}(x)\, dx \\ &= \int_0^{T-T_0} u^x\, E^{T_0}(1 | x \leq T - T_0\,, \rho_{k+1} + x > T - T_0)\, f_{L_k}(x)\, dx \\ &= \int_0^{T-T_0} u^x f_{L_k}(x)\, dx \\ &= E^{T_0}(u^{L_k}) \\ &= E^{T_0}\left(u^{\sum_{i=1}^{k} \rho_i}\right) \\ &= \left(E^{T_0}(u^{\rho_1})\right)^k \end{aligned}$$

since $(\rho_i)_{i\geq 1}$ are iid and ρ_{k+1} is independent of L_k. This then implies

$$Z^{T_0}_{L_{\tilde{N}}}(u) = \sum_{k=0}^{\infty} \left(E^{T_0}(u^{\rho_1})\right)^k\, P^{T_0}(\tilde{N} = k)$$

$$
\begin{aligned}
&= E^{T_0}\left[\left(E^{T_0}(u^{\rho_1})\right)^{\tilde{N}}\right] \\
&= Z_{\tilde{N}}^{T_0}\left(Z_{\rho_1}^{T_0}(u)\right).
\end{aligned}
$$

Step 3: Under the same assumptions on Y as above we derive

$$E(Y) = Z_Y'(1).$$

To prove this we define

$$g(y,u) := u^y\, f_Y(y)$$

on $[0, T-T_0] \times (0,\infty)$. Then g and $\frac{\partial}{\partial u} g$ are continuous on $[0, T-T_0] \times (0,\infty)$. Therefore we are able to interchange the integral and the differential operator in the following computation (see Bröcker (1992), p.28) which leads to

$$
\begin{aligned}
Z_Y'(u) &= \frac{\partial}{\partial u} E(u^Y) \\
&= \frac{\partial}{\partial u} \int_0^{T-T_0} u^y f_Y(y)\, dy \\
&= \int_0^{T-T_0} y\, u^{y-1}\, f_Y(y)\, dy
\end{aligned}
$$

thus giving us $E(Y) = Z_Y'(1)$. Since $Z_Y(1) = 1$ and since ρ_1 is independent of T_0 we obtain for $Y = L_{\tilde{N}}$

$$
\begin{aligned}
E^{T_0}(L_{\tilde{N}}) &= Z_{L_{\tilde{N}}}^{T_0\,\prime}(1) \\
&= \frac{d}{du}\left(Z_{\tilde{N}}^{T_0}(Z_{\rho_1}^{T_0}(u)\right)\Big|_{u=1} \\
&= Z_{\tilde{N}}^{T_0\,\prime}\left(Z_{\rho_1}^{T_0}(1)\right) Z_{\rho_1}^{T_0\,\prime}(1) \\
&= Z_{\tilde{N}}^{T_0\,\prime}(1)\, Z_{\rho_1}^{T_0\,\prime}(1) \\
&= E^{T_0}(\tilde{N})\, E^{T_0}(\rho_1) \\
&= E^{T_0}(\tilde{N})\, E(\rho_1).
\end{aligned}
$$

Step 4: After all these preparations we are finally able to calculate the expected number of interventions. At first we consider the case of $T_0 = \tau_0$. Let $S(0) \in [\hat{S}^-, \hat{S}^+]$ be fixed, let f_{T_0} denote the density of T_0 which depends on $S(0)$. Then we obtain

$$
\begin{aligned}
E(N) &= E(E(N|T_0)) \\
&= \int_0^\infty E(N|T_0 = s)\, f_{T_0}(s)\, ds \\
&= \int_0^T E(N|T_0 = s)\, f_{T_0}(s)\, ds + \int_T^\infty \underbrace{E(N|T_0 = s)}_{=0}\, f_{T_0}(s)\, ds
\end{aligned}
$$

$$\begin{aligned}
&= \int_0^T E(\tilde{N}+1|T_0=s)\, f_{T_0}(s)\, ds \\
&= \int_0^T f_{T_0}(s)\, ds + \int_0^T E(\tilde{N}|T_0=s)\, f_{T_0}(s)\, ds \\
&= P(T_0 \le T) + \int_0^T E(\tilde{N}|T_0=s)\, f_{T_0}(s)\, ds
\end{aligned}$$

where

$$\begin{aligned}
E(\tilde{N}|T_0=s) &= E^s(\tilde{N}) \\
&= \frac{E^s(L_{\tilde{N}})}{E(\rho_1)} \\
&\le \frac{T-s}{E(\rho_1)}\,.
\end{aligned}$$

This implies

$$\begin{aligned}
E(N) &\le P(T_0 \le T) + \frac{1}{E(\rho_1)} \Big\{T\, P(T_0 \le T) - E(T_0\, I_{\{T_0 \le T\}})\Big\} \\
&\le 1 + \frac{T}{E(\rho_1)}\,.
\end{aligned}$$

Obviously, the last inequality holds for all $S(0) \in [\hat{S}^-, \hat{S}^+]$, since $S(0)$ does not affect the distribution of ρ_1. Due to $\rho_1 = \tau_1$ we have proved the assertions i)-iv).

To prove the upper bound in the case of $T_0 = \tau_0 + \tau_1$, we have to take into account that in this case $N = \tilde{N} + 2$. Therefore we have to replace N by $\tilde{N}+2$ instead of $\tilde{N}+1$ in the above proof thus obtaining

$$E(N) \le 2 + \frac{T}{E(\rho_1)}$$

where $\rho_1 = \tau_2$. □

Remark 12. To compute $\hat{N}$ as given in (2.87)/(2.88) we have to replace $E(\tau_1)$ and $E(\tau_2)$ by the suitable expectations, respectively. Let τ be the first intervention time of Definition 4. Then we have

i) $E(\tau_1) = E^{S^*}(\tau)$,

ii) $E(\tau_1) = E^{S^*}(\tau)$,

iii) $E(\tau_1) = E^{S^-}(\tau)$ if $S(0) < 0$ and $E(\tau_1) = E^{S^+}(\tau)$ if $S(0) > 0$,

iv) $E(\tau_1) = E^{S^+}(\tau)$,

v) $E(\tau_2) = E^{S^+}(\tau)$

where the conditional expectation indicates the starting value of S after the foregoing intervention.

To illustrate the quantitative behaviour of the upper bound of the expectation we will give some examples. The corresponding data are taken from the various examples above, where we computed some expectations of the first intervention time, depending on the starting value of S. (Note that now this starting value is supposed to be the starting value after the foregoing intervention which is not equal to $S(0)$!) Choosing in each case the correct starting value (see Remark 12) we obtain the required bound of the expectation:

Examples 5. Let $S(0) \in [\hat{S}^-, \hat{S}^+]$.

i) $k = 0$, $K = 0.1$, $\mu = 0.025$, $\sigma = 0.2$, $\lambda = 0.01 \implies$
$\hat{S}^- = 36.33$, $S^- = S^* = S^+ = 62.5$, $\hat{S}^+ = 88.67$, $E^{S^*}(\tau) = 4.70102$.

T	$\hat{N}$
0.5	1.02262
1	1.21272
2	1.42544
5	2.0636

ii) $k = 0$, $K = 1$, $\mu = 0.03$, $\sigma = 0.2$, $\lambda = 0.1 \implies$
$\hat{S}^- = -1.56$, $S^- = S^* = S^+ = 7.5$, $\hat{S}^+ = 16.56$, $E^{S^*}(\tau) = 79.20871$.

T	$\hat{N}$
0.5	1.00631
1	1.01262
2	1.02525
5	1.06312

iii) $k = 1.3$, $K = 0.6$, $\mu = 0.03$, $\sigma = 0.2$, $\lambda = 0.1 \implies$
$S^* = 7.5$, $\hat{S}^- = -5.96$, $S^- = -0.2$, $S^+ = 15.2$, $\hat{S}^+ = 20.96$,
$E^{S^-}(\tau) = 339.45084$ if $S(0) < 0$ and $E^{S^+}(\tau) = 32.13204$ if $S(0) > 0$.

T	$\hat{N}\,(S(0) < 0)$	$\hat{N}\,(S(0) > 0)$
0.5	1.00147	1.01556
1	1.00295	1.03112
2	1.00589	1.06224
5	1.01473	1.15561

iv)+v) $k = 0.01\,,\ K = 1\,,\ \mu = 0.03\,,\ \sigma = 0.2\,,\ \lambda = 0.1 \quad \Longrightarrow$
$S^* = 7.5\,,\ \hat{S}^- = -1.66\,,\ S^- = 7.3\,,\ S^+ = 7.7\,,\ \hat{S}^+ = 16.66\,,$
$E^{S^+}(\tau) = 77.17903\,.$

T	$\hat{N}\,(S(0) > 0)$	$\hat{N}\,(S(0) < 0)$
0.5	1.00648	2.00648
1	1.01296	2.01296
2	1.02591	2.02591
5	1.06478	2.06478

Theorem 5. (Expected transaction costs.) *We assume that each transaction in* $[0,T]$ *causes fixed transaction costs* $K > 0$ *and proportional costs* $k|\Delta S| = k|\hat{S}^+ - S^+|\,, k > 0$*, which have to be paid at the moment of the intervention. Further we assume that the interest rate satisfies* $r = 0$*. Then the terminal value of the accumulated expected transaction costs is bounded from above by*

$$\hat{C}_T := (K + k|\Delta S|)\,\hat{N} \tag{2.89}$$

with $\hat{N}$ *as given in (2.87)/(2.88).*
Proof: Due to $r = 0$ the accumulated transaction costs are given by

$$C_T \ := \ (k|\Delta S| + K)\,N$$

where N denotes the number of interventions between 0 and T (see proof of Theorem 4). This then implies

$$\begin{aligned} E(C_T) &= (k|\Delta S| + K)\,E(N) \\ &\leq (k|\Delta S| + K)\,\hat{N}\,. \end{aligned}$$

□

To close this section we will give some examples illustrating the quantitative behaviour of (2.89):

Examples 6. Let $S(0) \in [\hat{S}^-, \hat{S}^+]\,.$

i) $k = 0\,,\ K = 0.1\,,\ \mu = 0.025\,,\ \sigma = 0.2\,,\ \lambda = 0.01 \quad \Longrightarrow$
$\hat{S}^- = 36.33\,,\ S^- = S^* = S^+ = 62.5\,,\ \hat{S}^+ = 88.67\,,\ |\Delta S| = 26.17\,.$

T	$\hat{C}_T$
0.5	26.76197
1	31.73688
2	37.30376
5	54.00441

ii) $k = 0\,,\ K = 1\,,\ \mu = 0.03\,,\ \sigma = 0.2\,,\ \lambda = 0.1 \quad \Longrightarrow$
$\hat{S}^- = -1.56\,,\ S^- = S^* = S^+ = 7.5\,,\ \hat{S}^+ = 16.56\,,\ |\Delta S| = 9.06\,.$

T	$\hat{C}_T$
0.5	9.11717
1	9.17434
2	9.28877
5	9.63187

iii) $k = 1.3\,,\ K = 0.6\,,\ \mu = 0.03\,,\ \sigma = 0.2\,,\ \lambda = 0.1 \quad \Longrightarrow$
$S^* = 7.5\,,\ \hat{S}^- = -5.96\,,\ S^- = -0.2\,,\ S^+ = 15.2\,,\ \hat{S}^+ = 20.96\,,\ |\Delta S| = 5.76\,.$

T	$\hat{C}_T\,(S(0) < 0)$	$\hat{C}_T\,(S(0) > 0)$
0.5	5.76847	5.84963
1	5.77699	5.93925
2	5.79393	6.1185
5	5.84484	6.65631

iv)+v) $k = 0.01\,,\ K = 1\,,\ \mu = 0.03\,,\ \sigma = 0.2\,,\ \lambda = 0.1 \quad \Longrightarrow$
$S^* = 7.5\,,\ \hat{S}^- = -1.66\,,\ S^- = 7.3\,,\ S^+ = 7.7\,,\ \hat{S}^+ = 16.66\,,\ |\Delta S| = 8.96\,.$

T	$\hat{C}_T\,(S(0) > 0)$	$\hat{C}_T\,(S(0) < 0)$
0.5	9.01806	17.97806
1	9.07612	18.03612
2	9.19215	18.15215
5	9.54043	18.50043

2.7 The multi-dimensional case

2.7.1 Asymptotic analysis

Now we will analyse the optimization problem in the case of a market with $n \geq 1$ stocks and one bond with interest rate $r = 0$. The stock price processes are given by

$$\begin{aligned} dP_i(t) &= P_i(t)\left[\mu_i\,dt + \sum_{j=1}^{n} \sigma_{ij}\,dW_t^j\right], \\ P_i(0) &= p_i\,, \end{aligned}$$

$i = 1, \ldots, n$, where W_t is an n-dimensional Brownian motion. For notational convenience we set $P_i := P_i(t)\,, i = 1, \ldots, n$, and define $p := (P_1, \ldots, P_n)$ and $1/p := (1/P_1, \ldots, 1/P_n)$.

Paralleling the computations of Example 3.31 in Korn (1997) we easily show that in the absence of transaction costs it is optimal to hold exactly

$$y^* = \frac{1}{\lambda}\,\mathrm{diag}\left(\frac{1}{p}\right)(\sigma\sigma')^{-1}\,\mu \tag{2.90}$$

shares at each time instant where $\mathrm{diag}(x)\,, x = (x_1, \ldots, x_n)\,,$ is defined as the diagonal matrix with $x_1, \ldots, x_n$ on its diagonal.

In order to find the optimal strategy in the presence of transaction costs we will now carry out the same asymptotic procedure as in the one-dimensional case. Again we set $B = 0$ and replace $v(t, 0, y\,p)$ by $q(t, p, y)$. With

$$\begin{aligned} Lq(t,p,y) \;&:=\; q_t(t,p,y) + q_p(t,p,y)(\mathrm{diag}(p)\,\mu) \\ &\quad + \frac{1}{2}\,\mathrm{trace}(q_{pp}(t,p,y)\,\mathrm{diag}(p)\,\sigma\sigma'\mathrm{diag}(p)\,) \end{aligned}$$

and

$$Mq(t,p,y) := \max_{u \in I\!R^n}\{e^{\lambda(K+k|u|'p+u'p)}q(t,p,y+u)\}$$

where $|u| = (|u_1|, \ldots, |u_n|)\,,$ the quasi variational inequalities with respect to q are given by

$$\begin{aligned} Lq(t,p,y) \;&\leq\; 0\,, \\ q(t,p,y) \;&\geq\; Mq(t,p,y)\,, \\ (q(t,p,y) - Mq(t,p,y))\,Lq(t,p,y) \;&=\; 0\,, \\ q(T,p,y) \;&=\; -e^{-\lambda\,p'y}\,. \end{aligned}$$

Again, we define $Q(t,p,Y) := q(t,p,y)$ where $y = y^*(t,p) + \varepsilon^{1/4}Y\,, Y \in I\!R^n\,,$ and expand Q in NT in powers of $\varepsilon^{1/2}$ such that we obtain

$$Q(t,p,Y) = -e^{-[\lambda\,p'(y^*+\varepsilon^{1/4}Y)+H_0(t)+\varepsilon^{1/2}H_1(t,p)+\varepsilon H_2(t,p,Y)+\cdots]}\,.$$

Due to this expansion of Q the derivatives of q read

$$\begin{aligned} q_t \;&=\; -Q\,[\,H_{0_t}(t) + \varepsilon^{1/2}H_{1_t}(t,p) + \varepsilon H_{2_t}(t,p,Y) + \cdots]\,, \\ q_{p_i} \;&=\; -Q\,[\,\lambda\,y_i^* + \lambda\,\varepsilon^{1/4}Y_i + \varepsilon^{1/2}H_{1_{p_i}}(t,p) - \varepsilon^{3/4}H_{2_{Y_i}}(t,p,Y)(y_i^*)_{p_i} \\ &\qquad + \varepsilon\,H_{2_{p_i}}(t,p,Y) + \cdots]\,, \\ q_{p_ip_j} \;&=\; Q\,\{\,\lambda^2 y_i^* y_j^* + \varepsilon^{1/4}\lambda^2(y_i^*Y_j + y_j^*Y_i) + \\ &\qquad \varepsilon^{1/2}[\,\lambda^2 Y_iY_j + \lambda(y_i^*H_{1_{p_j}}(t,p) + y_j^*H_{1_{p_i}}(t,p)) \\ &\qquad - H_{1_{p_ip_j}}(t,p) - H_{2_{Y_iY_j}}(t,p,Y)(y_i^*)_{p_i}(y_j^*)_{p_j}\,] + \varepsilon^{3/4}\cdots\} \end{aligned}$$

which we plug into the above formula for Lq and rearrange by powers of $\varepsilon^{1/4}$. Then the neglection of higher order terms leads to

$$0 \;=\; Q\left\{\left[-H_{0_t} - \lambda\sum_{i=1}^n p_i\,\mu_i\,y_i^* + \frac{1}{2}\lambda^2\sum_{i=1}^n\sum_{j=1}^n\sum_{k=1}^n \sigma_{ik}\,\sigma_{jk}\,p_i\,p_j\,y_i^*\,y_j^*\right]\right.$$

$$
\begin{aligned}
&+\varepsilon^{1/4}\left[-\lambda\sum_{i=1}^{n}p_i\,\mu_i\,Y_i+\frac{1}{2}\lambda^2\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\sigma_{ik}\,\sigma_{jk}\,p_i\,p_j\,(y_i^*\,Y_j+y_j^*\,Y_i)\right]\\
&+\varepsilon^{1/2}\left[\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\sigma_{ik}\,\sigma_{jk}\,p_i\,p_j\right.\\
&\qquad(\,\lambda^2 Y_i\,Y_j+\lambda\,y_i^* H_{1_{p_j}}+\lambda\,y_j^* H_{1_{p_i}}-H_{1_{p_ip_j}}-H_{2_{Y_iY_j}}\,(y_i^*)_{p_i}\,(y_j^*)_{p_j}\,)\\
&\qquad\left.\left.-H_{1_t}-\sum_{i=1}^{n}p_i\,\mu_i\,H_{1_{p_i}}\right]\right\}
\end{aligned}
$$

in NT, and we thus have to solve the following three equations:

$$
\begin{aligned}
O(1):\quad & 0=H_{0_t}+\lambda\,y^{*\prime}(\mathrm{diag}(p)\,\mu)\\
& \quad-\frac{1}{2}\,\lambda^2\,\mathrm{trace}[\,(\mathrm{diag}(p)\,\sigma\,\sigma'\mathrm{diag}(p))\,(y^*\,y^{*\prime})\,]\,,\\
O(\varepsilon^{1/4}):\quad & 0=\lambda\,Y'\,(\mathrm{diag}(p)\,\mu)\\
& \quad-\frac{1}{2}\,\lambda^2\,\mathrm{trace}[\,(\mathrm{diag}(p)\,\sigma\,\sigma'\mathrm{diag}(p))\,(\,y^*\,Y'+Y\,y^{*\prime}\,)\,]\,,\\
O(\varepsilon^{1/2}):\quad & 0=H_{1_t}+H_{1_p}(\mathrm{diag}(p)\,\mu)-\frac{1}{2}\,\mathrm{trace}[(\,\mathrm{diag}(p)\,\sigma\,\sigma'\mathrm{diag}(p)\,)\\
& \quad(\lambda^2 Y\,Y'+\lambda\,(y^*\,H_{1_p}+(y^*\,H_{1_p})')-H_{1_{pp}}-y_p^{*\prime}\,H_{2_{YY}}\,y_p^*)].
\end{aligned}
$$

If we define $U:=\mathrm{diag}(p)\,\sigma\sigma'\mathrm{diag}(p)$ and use $\mathrm{trace}(M\,X\,Z')=X'M'\,Z$ for any vectors $X,Z\in I\!R^n$ and any matrix $M\in I\!R^{n,n}$, equation $O(\varepsilon^{1/4})$ leads to

$$\lambda\,Y'\mathrm{diag}(p)\,\mu=\lambda^2 Y'U\,y^*\,.$$

This is correct due to the form of y^* which implies even more:

$$\mathrm{diag}(p)\,\mu=\lambda\,U\,y^*\,.$$

We use the last equality to solve equation $O(1)$:

$$
\begin{aligned}
0 &= H_{0_t}+\lambda^2\,y^{*\prime}\,U\,y^*-\frac{1}{2}\,\lambda^2\,\underbrace{\mathrm{trace}(U\,y^*\,y^{*\prime})}_{=y^{*\prime}\,U\,y^*}\\
&= H_{0_t}+\frac{1}{2}\,\lambda^2\,y^{*\prime}\,U\,y^*\,.
\end{aligned}
$$

Together with the boundary condition $H_0(T)=0$, which ensures that Q is equal to the value function without transaction costs in the case $\varepsilon=0$, this leads to

$$
\begin{aligned}
H_0(t) &= \frac{1}{2}\,\lambda^2\,y^{*\prime}\,U\,y^*\,(T-t)\\
&= \frac{1}{2}\,\mu'\,(\sigma\,\sigma')^{-1}\,\mu\,(T-t)\,.
\end{aligned}
$$

Unfortunately, we will not be able to solve $O(\varepsilon^{1/2})$ in general. In the following we will therefore concentrate on the case of two uncorrelated stocks where we will solve this equation and thus the whole optimization problem. We will give closed form solutions in the cases of only fixed and only proportional costs whereas we will give a numerical solution in the case of both fixed and proportional transaction costs.

2.7.2 Two uncorrelated stocks

As we consider the case of two uncorrelated stocks, we assume that $n = 2$ and $\sigma_{12} = \sigma_{21} = 0$. The optimal number of shares is then given by

$$y^* = \frac{1}{\lambda}\left(\frac{\mu_1}{p_1\,\sigma_{11}^2}, \frac{\mu_2}{p_2\,\sigma_{22}^2}\right)' \tag{2.91}$$

thus implying that the optimal constant wealth invested into the two stocks is given by

$$(S_1^*, S_2^*)' = \left(\frac{\mu_1}{\lambda\,\sigma_{11}^2}, \frac{\mu_2}{\lambda\,\sigma_{22}^2}\right)'. \tag{2.92}$$

Using the above definition of U we obtain for $O(\varepsilon^{1/2})$

$$\begin{aligned} 0 &= H_{1_t} + H_{1_p}(\mathrm{diag}(p)\,\mu) - \frac{1}{2}\mathrm{trace}[(\,\mathrm{diag}(p)\,\sigma\,\sigma'\mathrm{diag}(p)\,) \\ &\qquad (\,\lambda^2\,Y\,Y' + \lambda\,(\,y^*\,H_{1_p} + (y^*\,H_{1_p})'\,) - H_{1_{pp}} - y_p^{*\prime}\,H_{2_{YY}}\,y_p^*\,)\,] \\ &= H_{1_t} + \frac{1}{2}\mathrm{trace}\,(U\,H_{1_{pp}}) - \frac{1}{2}\lambda^2\,Y'\,U'\,Y + \frac{1}{2}\mathrm{trace}\,(U\,y_p^{*\prime}\,H_{2_{YY}}\,y_p^*) \\ &=: F(H_1) - \frac{1}{2}\lambda^2\,Y'\,U'\,Y + \frac{1}{2}\mathrm{trace}\,(U\,y_p^{*\prime}\,H_{2_{YY}}\,y_p^*) \\ &= F(H_1) - \frac{1}{2}\lambda^2(a_1\,Y_1^2 + a_2\,Y_2^2) + \frac{1}{2}\frac{1}{\lambda^2}\left(\frac{\mu_1^2}{a_1}H_{2_{Y_1Y_1}} + \frac{\mu_2^2}{a_2}H_{2_{Y_2Y_2}}\right) \end{aligned}$$

where $a_1 := p_1^2\,\sigma_{11}^2$ and $a_2 := p_2^2\,\sigma_{22}^2$. For notational convenience we set $H := H_2$, $x := Y_1$, $y := Y_2$ and $F_1 := F(H_1)$ which leads to

$$0 \;=\; F_1 - \frac{1}{2}\lambda^2(a_1\,x^2 + a_2\,y^2) + \frac{1}{2}\frac{1}{\lambda^2}\left(\frac{\mu_1^2}{a_1}H_{xx} + \frac{\mu_2^2}{a_2}H_{yy}\right). \tag{2.93}$$

Fixing (p_1, p_2) and t we will drop these variables in the following.

A simple integration to obtain H as in the one-stock case is no longer possible. We therefore determine H as the unique solution of an appropriate free boundary problem thereby also obtaining the optimal intervention and restarting points for our problem. As those solutions are unique (see Crank (1984) for standard results on free boundary problems), we only have to construct the relevant solution which we will do below.

Before we start this proceeding we will introduce the following notations: The pair

$$(\hat{x}^+, \hat{y}^+)$$

will always denote an element of $I\!R^2$ satisfying

$$y^* + \varepsilon^{1/4} (\hat{x}^+, \hat{y}^+)' \quad \in \quad \partial NT$$

whereas the pair

$$(x^+, y^+)$$

will always denote an element of $I\!R^2$ such that the corresponding pair

$$y^* + \varepsilon^{1/4} (x^+, y^+)'$$

is an element of the restarting line.

Further we assume that for each pair $(\hat{x}^+, \hat{y}^+)$ $((x^+, y^+))$ there exists a corresponding pair $(\hat{x}^-, \hat{y}^-) = (-\hat{x}^+, -\hat{y}^+)$ $((x^-, y^-) = (-x^+, -y^+))$ as it is in the one-dimensional case. Due to this symmetry we will concentrate on the case $\hat{x}^+ \geq x^+ \geq 0\,, \hat{y}^+ \geq y^+ \geq 0\,.$

We start analysing the case of only proportional transaction costs:

2.7.3 Proportional transaction costs

Let $k > 0\,, K = 0\,.$ Further let (p_1, p_2) and t be fixed. We now try the following ansatz for both the function H and the NT region: For suitable constants $A, \ldots, E$ the function H is of the form

$$H(x,y) = A\,x^4 + B\,x^2 + C\,y^4 + D\,y^2 + E \quad \forall\,(x,y) \in NT \tag{2.94}$$

with

$$NT = \{(x^*, y^*)' + \varepsilon^{1/4}\,(x,y)' \,:\, |x| \leq \alpha, |y| \leq \beta\}\,. \tag{2.95}$$

The boundary of NT is thus given by a rectangle and also coincides with the restarting line, i.e. $(\hat{x}^+, \hat{y}^+)$ and the corresponding pair (x^+, y^+) are equal. Relation (2.93) and the assumed form of H and NT determine α and β as

$$\alpha \;=\; \varepsilon^{-\frac{1}{4}} \left(\frac{3}{2} \frac{k\,\mu_1^2}{\sigma_{11}^4\,p_1^3\,\lambda^3} \right)^{\frac{1}{3}}\,, \tag{2.96}$$

$$\beta \;=\; \varepsilon^{-\frac{1}{4}} \left(\frac{3}{2} \frac{k\,\mu_2^2}{\sigma_{22}^4\,p_2^3\,\lambda^3} \right)^{\frac{1}{3}}\,. \tag{2.97}$$

Remark: As the two stocks are independent and as there are no fixed transaction costs, there will be no bargaining effect of combining actions in both stocks. Therefore we conjecture that the no-transaction region is given by a rectangle.

Determination of the constants: We will now show that indeed there exist suitable constants $A, \ldots, E$ such that $H(x,y)$ and NT as given in (2.94)/(2.95) satisfy the transformed qvi in the usual approximative sense: While solving the one-dimensional case, we have derived some conditions on H which have to be satisfied in the two-dimensional case, too. Taking into consideration the form of NT these conditions read

(1) $H(x^+,y^+) - H(\hat{x}^+,\hat{y}^+) = \lambda\, K(\hat{x}^+ - x^+, \hat{y}^+ - y^+)$
for all $(x^+,y^+) = (\hat{x}^+,\hat{y}^+) \in \partial NT$,

(2) $H_x(\alpha,\hat{y}^+) = -\lambda\, K_x(\alpha - \alpha, \hat{y}^+ - y^+)$ for all $0 \le y^+ = \hat{y}^+ \le \beta$,

(3) $H_y(\hat{x}^+,\beta) = -\lambda\, K_y(\hat{x}^+ - x^+, \beta - \beta)$ for all $0 \le x^+ = \hat{x}^+ \le \alpha$.

Here the conditions (1)-(3) will not be sufficient to determine the (approximate) value function uniquely. Therefore we will also require that the second derivative with respect to the number of stocks does exist for some particular elements of ∂NT and that it is continuous there. With respect to x we thus obtain from (2.26) that

$$Q_{xx}(x,\hat{y}^+)|_{x=\alpha} = \frac{d^2}{dx\,dx}\left[e^{\lambda\left(\varepsilon\, K(x-x^+,\hat{y}^+-y^+)+\varepsilon^{1/4}\,(x^+-x,y^+-\hat{y}^+)'\,p\right)}\,Q(x^+,y^+)\right]\Big|_{x=\alpha}$$

for all $0 \le y^+ = \hat{y}^+ \le \beta$. Due to (2.28) this is true if and only if for all $y^+ = \hat{y}^+ \in [0,\beta]$ the equality

$$\begin{aligned} &Q(\alpha,\hat{y}^+)\,\left\{\left[\lambda\, p_1\,\varepsilon^{1/4} + \varepsilon\, H_x(\alpha,\hat{y}^+)\right]^2 - \varepsilon\, H_{xx}(\alpha,\hat{y}^+)\right\} \\ =\;\; &Q(\alpha,\hat{y}^+)\,\left[\lambda\,\varepsilon\, K_x(\alpha - \alpha, \hat{y}^+ - y^+) - \lambda\,\varepsilon^{1/4}\, p_1\right]^2 \end{aligned}$$

is satisfied. With the help of (2.36) we thus obtain that

$$H_{xx}(\alpha,\hat{y}^+) = 0\,, \quad 0 \le \hat{y}^+ \le \beta\,. \tag{2.98}$$

Since we can perform the same derivation with respect to y we obtain the two additional conditions

(4) $H_{xx}(\alpha,\hat{y}^+) = 0$ for all $0 \le \hat{y}^+ \le \beta$,

(5) $H_{yy}(\hat{x}^+,\beta) = 0$ for all $0 \le \hat{x}^+ \le \alpha$.

Now we determine the constants $A, \ldots, E$ such that the above conditions as well as relation (2.93) are satisfied: Since $\lambda\, K(0,0) = 0$, equality (1) is always satisfied. Further, the equalities (4) and (5) read

$$H_{xx}(\alpha,\hat{y}^+) = 12\, A\, \alpha^2 + 2\, B = 0\,, \quad 0 \le \hat{y}^+ \le \beta$$

and

$$H_{yy}(\hat{x}^+, \beta) = 12\, C\, \beta^2 + 2\, D = 0\,, \quad 0 \leq \hat{x}^+ \leq \alpha\,.$$

Hence, we obtain

$$A = -\frac{B}{6\,\alpha^2}$$

and

$$C = -\frac{D}{6\,\beta^2}$$

which leads to

$$H(x,y) = -\frac{B}{6\,\alpha^2}\, x^4 + B\, x^2 - \frac{D}{6\,\beta^2}\, y^4 + D\, y^2 + E\,. \tag{2.99}$$

Hence, relation (2.93) is satisfied if and only if

$$\frac{1}{2}\lambda^2\, a_1 = -\frac{1}{\lambda^2}\, \frac{\mu_1^2}{a_1\, \alpha^2}\, B$$

and

$$\frac{1}{2}\lambda^2\, a_2 = -\frac{1}{\lambda^2}\, \frac{\mu_2^2}{a_2\, \beta^2}\, D$$

giving us

$$B = -\frac{1}{2}\, \lambda^4\, \frac{a_1^2}{\mu_1^2}\, \alpha^2$$

and

$$D = -\frac{1}{2}\, \lambda^4\, \frac{a_2^2}{\mu_2^2}\, \beta^2\,.$$

From (2.94) we then obtain that

$$H_x(x,y) = -\frac{1}{2}\, \lambda^4\, \alpha^2\, \frac{a_1^2}{\mu_1^2}\, \left(-\frac{2}{3\,\alpha^2}\, x^3 + 2\, x\right)$$

and

$$H_y(x,y) = -\frac{1}{2}\, \lambda^4\, \beta^2\, \frac{a_2^2}{\mu_2^2}\, \left(-\frac{2}{3\,\beta^2}\, y^3 + 2\, y\right)\,.$$

Using the remaining conditions (2) and (3) we can finally determine α and β:

Equalities (2) and (3) are satisfied if and only if

$$-\lambda\, k\, \varepsilon^{-\frac{3}{4}}\, p_1 = -\frac{2}{3}\, \lambda^4\, \frac{a_1^2}{\mu_1^2}\, \alpha^3$$

and

$$-\lambda\, k\, \varepsilon^{-\frac{3}{4}}\, p_2 = -\frac{2}{3}\, \lambda^4\, \frac{a_2^2}{\mu_2^2}\, \beta^3$$

which imply

$$\alpha = \varepsilon^{-\frac{1}{4}} \left(\frac{3}{2} \frac{k\, \mu_1^2}{\sigma_{11}^4\, p_1^3\, \lambda^3} \right)^{\frac{1}{3}}$$

and

$$\beta = \varepsilon^{-\frac{1}{4}} \left(\frac{3}{2} \frac{k\, \mu_2^2}{\sigma_{22}^4\, p_2^3\, \lambda^3} \right)^{\frac{1}{3}} .$$

□

Considering the wealth invested in the first and the second stock, respectively, we have to rebalance our portfolio whenever our wealth-combination reaches the boundary of the set

$$\{(S_1^*, S_2^*)' + (S_1, S_2)' : |S_1| \leq \tilde{S}_1, |S_2| \leq \tilde{S}_2\}$$

where

$$\tilde{S}_1 = \left(\frac{3}{2} \frac{k\, \mu_1^2}{\sigma_{11}^4\, \lambda^3} \right)^{\frac{1}{3}} , \tag{2.100}$$

$$\tilde{S}_2 = \left(\frac{3}{2} \frac{k\, \mu_2^2}{\sigma_{22}^4\, \lambda^3} \right)^{\frac{1}{3}} . \tag{2.101}$$

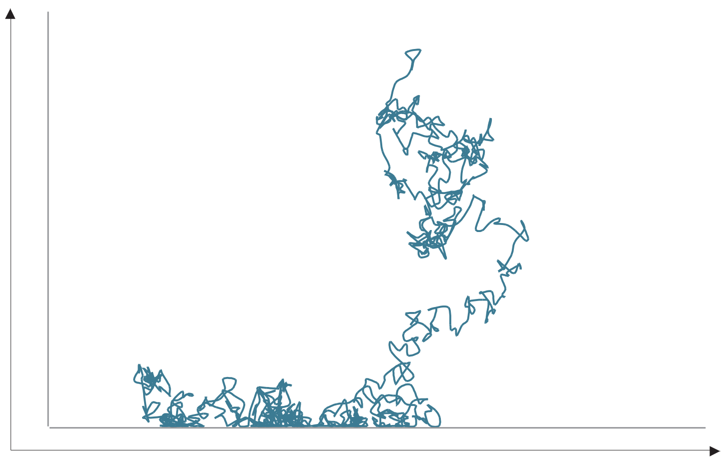

Fig. 4. (Asymptotically) optimal risky money (2)

Obviously, the no-transaction region with respect to the risky money is (as in the one-dimensional case) independent of the prices (compare with subsection 2.4.1). Further our successful computations confirm the assumptions we made on the form of H and NT. (Fig. 4 shows the lower left quadrant of the no-transaction region corresponding to one particular parameter set.)

Remark 13. From the last formulae ((2.100),(2.101)) we immediately obtain that the no-transaction region increases with an increase of k and converges to the single point $(S_1^*, S_2^*)'$ if k tends to 0. This is what we expect because of economic reasons: The higher the proportional cost parameter is, the less often the investor wants to rebalance his portfolio to compensate for the higher transaction costs. Further the no-transaction region increases in $\frac{\mu}{\sigma^2}$ whereas it is linear in $1/\lambda$. ((2.91) implies that these relations also hold for the optimal risky money S^*.)

The following examples will show this quantitative behaviour:

Examples 7.

a) $\mu_1 = \mu_2 = 0.03\,,\ \sigma_{11} = 0.3\,,\ \sigma_{22} = 0.4\,,\ \lambda = 0.001 \quad \Longrightarrow$
$S_1^* = 333.33\,,\ S_2^* = 187.5\,.$

k	$\tilde{S}_1$	$\tilde{S}_2$
1E-12	0.055	0.037
1E-10	0.255	0.174
1E-8	1.186	0.808
1E-6	5.503	3.75
0.0001	25.544	17.406
0.001	55.032	37.5
0.01	118.563	80.791
0.1	255.437	174.06

b) $\mu_1 = 0.02\,,\ \mu_2 = 0.03\,,\ \sigma_{11} = 0.3\,,\ \sigma_{22} = 0.4\,,\ \lambda = 0.001 \quad \Longrightarrow$
$S_1^* = 222.22\,,\ S_2^* = 187.5\,.$

k	$\tilde{S}_1$	$\tilde{S}_2$
1E-12	0.042	0.037
1E-10	0.195	0.174
1E-8	0.905	0.808
1E-6	4.2	3.75
0.0001	19.493	17.406
0.001	41.997	37.5
0.01	90.481	80.791
0.1	194.935	174.06

2.7.4 Fixed transaction costs

Let $k = 0$, $K > 0$. Further let (p_1, p_2) and t be fixed. We try the following ansatz for both the function H and the NT region: For suitable constants A, B the function H is of the form

$$H(x,y) = A\left(\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} - 1\right)^2 + B \quad \forall\,(x,y) \in NT \tag{2.102}$$

where NT is given by the ellipsoid

$$\left\{(x^*, y^*)' + \varepsilon^{1/4}\,(x,y)' \,:\, \frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} \le 1\right\}. \tag{2.103}$$

The boundary of NT consists of all points

$$(x^*, y^*)' + \varepsilon^{1/4}\,(x,y)'$$

with

$$\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} = 1 \tag{2.104}$$

whereas the restarting line only consists of Pliska's optimal number of shares given in (2.91), i.e., $(x^+, y^+) = (0,0)$. Since the investor has to pay fixed transaction costs there will be bargaining effects of rebalancing in both stocks. Therefore we choose an ellipsoid to model the no-transaction region. Relation (2.93) and the assumed form of H and NT determine α and β as

$$\alpha = \varepsilon^{-\frac{1}{4}}\left(\frac{K}{\lambda^3}\right)^{\frac{1}{4}} \frac{\mu_1}{p_1\,\sigma_{11}}\left(\frac{192}{17\,\mu_1^2 + \mu_2^2 - \sqrt{\mu_1^4 + \mu_2^4 + 34\,\mu_1^2\,\mu_2^2}}\right)^{\frac{1}{4}}, \tag{2.105}$$

$$\beta = \varepsilon^{-\frac{1}{4}}\left(\frac{K}{\lambda^3}\right)^{\frac{1}{4}} \frac{\mu_2}{p_2\,\sigma_{22}}\left(\frac{192}{17\,\mu_2^2 + \mu_1^2 - \sqrt{\mu_1^4 + \mu_2^4 + 34\,\mu_1^2\,\mu_2^2}}\right)^{\frac{1}{4}}. \tag{2.106}$$

Determination of the constants: As in the case of proportional costs we will now show that there exist constants A and B such that H and NT as given in (2.102)/(2.103) satisfy the transformed qvi in the usual approximative sense: The conditions on H now read

(1) $H(0,0) - H(\hat{x}^+, \hat{y}^+) = \lambda\,K(\hat{x}^+, \hat{y}^+)$ for all $(\hat{x}^+, \hat{y}^+) \in \partial NT$,

(2) $H_x(0,0) = -\lambda\,K_x(\alpha, 0)$,

(3) $H_x(\alpha, 0) = -\lambda\,K_x(\alpha, 0)$,

(4) $H_y(0,0) = -\lambda\,K_y(0, \beta)$,

(5) $H_y(0,\beta) = -\lambda\, K_y(0,\beta)$.

We determine the constants A and B such that the conditions above as well as relation (2.93) are satisfied: Due to $k=0$ equality (1) implies that

$$A = H(0,0) - H(\hat{x}^+,\hat{y}^+) = \lambda\, K(\hat{x}^+,\hat{y}^+) = \frac{\lambda\, K}{\varepsilon}\,.$$

Equalities (2) to (5) are always satisfied since $k=0$ and (2.104) holds. Before we use relation (2.93) we transform H. We set

$$\tilde{\alpha} := \sqrt{\frac{\lambda\, K}{\varepsilon}\,\frac{1}{\alpha^2}}\,, \tag{2.107}$$

and

$$\tilde{\beta} := \sqrt{\frac{\lambda\, K}{\varepsilon}\,\frac{1}{\beta^2}} \tag{2.108}$$

thus obtaining

$$H(x,y) = \left(x^2\,\tilde{\alpha} + y^2\,\tilde{\beta} - \sqrt{\frac{\lambda\, K}{\varepsilon}}\right)^2 + B\,.$$

Using the second derivatives of H with respect to x and y, respectively, we see that relation (2.93) is satisfied if and only if

$$\lambda^4\, a_1 = 12\,\frac{\mu_1^2}{a_1}\,\tilde{\alpha}^2 + 4\,\frac{\mu_2^2}{a_2}\,\tilde{\alpha}\,\tilde{\beta}$$

and

$$\lambda^4\, a_2 = 12\,\frac{\mu_2^2}{a_2}\,\tilde{\beta}^2 + 4\,\frac{\mu_1^2}{a_1}\,\tilde{\alpha}\,\tilde{\beta}\,.$$

The first equation is equivalent to

$$\tilde{\beta} = \frac{\lambda^4\, a_1 - 12\,\frac{\mu_1^2}{a_1}\,\tilde{\alpha}^2}{4\,\mu_2^2\,\tilde{\alpha}}\,a_2$$

which we insert into the second equation. Thus we get

$$\lambda^4\, a_2 = \frac{\mu_1^2\, a_2}{\mu_2^2\, a_1}\left(a_1\,\lambda^4 - 12\,\frac{\mu_1^2}{a_1}\,\tilde{\alpha}^2\right) + \frac{3}{4}\,\frac{a_2}{\mu_2^2\,\tilde{\alpha}^2}\left(a_1\,\lambda^4 - 12\,\frac{\mu_1^2}{a_1}\,\tilde{\alpha}^2\right)^2$$

which is equivalent to

$$\tilde{\alpha}^4 - (17\,\mu_1^2 + \mu_2^2)\,\frac{\lambda^4\, a_1^2}{96\,\mu_1^4}\,\tilde{\alpha}^2 + \frac{3}{4}\,\frac{\lambda^8\, a_1^4}{96\,\mu_1^4} = 0\,.$$

Setting $\hat{\alpha} := \tilde{\alpha}^2$ we get the corresponding solutions

$$\hat{\alpha}_1 = \frac{\lambda^4\, a_1^2}{192\,\mu_1^4}\left(17\,\mu_1^2 + \mu_2^2 + \sqrt{\mu_1^4 + \mu_2^4 + 34\,\mu_1^2\,\mu_2^2}\right)$$

and

$$\hat{\alpha}_2 = \frac{\lambda^4\, a_1^2}{192\,\mu_1^4}\left(17\,\mu_1^2 + \mu_2^2 - \sqrt{\mu_1^4 + \mu_2^4 + 34\,\mu_1^2\,\mu_2^2}\right)$$

which are positive. Since $\tilde{\alpha}$ has to be positive, too, we find

$$\tilde{\alpha}_1 = \frac{\lambda^2\, a_1}{\sqrt{192}\,\mu_1^2}\sqrt{17\,\mu_1^2 + \mu_2^2 + \sqrt{\mu_1^4 + \mu_2^4 + 34\,\mu_1^2\,\mu_2^2}}$$

and

$$\tilde{\alpha}_2 = \frac{\lambda^2\, a_1}{\sqrt{192}\,\mu_1^2}\sqrt{17\,\mu_1^2 + \mu_2^2 - \sqrt{\mu_1^4 + \mu_2^4 + 34\,\mu_1^2\,\mu_2^2}}\,.$$

In order to decide which is the right $\tilde{\alpha}$ we look at the corresponding values of $\tilde{\beta}$:

$$\tilde{\beta}_i = \underbrace{\frac{a_2}{4\,\tilde{\alpha}_i\,\mu_2^2}}_{>0}\,\underbrace{\left(a_1\,\lambda^4 - 12\,\frac{\mu_1^2}{a_1}\,\tilde{\alpha}_i^2\right)}_{=:\,C_i}$$

where

$$\begin{aligned}
C_1 &= a_1\,\lambda^4 - \frac{a_1\,\lambda^4}{16\,\mu_1^2}\underbrace{\left(17\,\mu_1^2 + \mu_2^2 + \sqrt{\mu_1^4 + \mu_2^4 + 34\,\mu_1^2, \mu_2^2}\right)}_{>\,16\,\mu_1^2} < 0\,,\\
C_2 &= a_1\,\lambda^4 - \frac{a_1\,\lambda^4}{16\,\mu_1^2}\underbrace{\left(17\,\mu_1^2 + \mu_2^2 - \underbrace{\sqrt{\mu_1^4 + \mu_2^4 + 34\,\mu_1^2, \mu_2^2}}_{>\,\mu_1^2+\mu_2^2}\right)}_{<\,16\,\mu_1^2} > 0\,.
\end{aligned}$$

Therefore, there is only one positive pair $(\tilde{\alpha}, \tilde{\beta})$ which reads

$$\begin{aligned}
\tilde{\alpha} &= \frac{\lambda^2\, a_1}{\sqrt{192}\,\mu_1^2}\, z\,,\\
\tilde{\beta} &= \frac{a_2}{4\,\mu_2^2\,\tilde{\alpha}}\left(a_1\,\lambda^4 - 12\,\frac{\mu_1^2}{a_1}\,\tilde{\alpha}^2\right)
\end{aligned}$$

with

$$z := \sqrt{17\,\mu_1^2 + \mu_2^2 - \sqrt{\mu_1^4 + \mu_2^4 + 34\,\mu_1^2\,\mu_2^2}}\,.$$

From (2.107) and (2.108) we obtain

$$\alpha = \left(\frac{\lambda\, K}{\varepsilon}\right)^{\frac{1}{4}} \frac{1}{\sqrt{\tilde{\alpha}}}$$

and

$$\beta = \left(\frac{\lambda\, K}{\varepsilon}\right)^{\frac{1}{4}} \frac{1}{\sqrt{\tilde{\beta}}}$$

which then imply

$$\begin{aligned}
\alpha &= \left(\frac{K}{\lambda^3}\right)^{\frac{1}{4}} \varepsilon^{-\frac{1}{4}} \frac{\mu_1}{p_1\,\sigma_{11}} \sqrt[4]{192}\, \frac{1}{\sqrt{z}}\,, \\
\beta &= \left(\frac{K}{\lambda^3}\right)^{\frac{1}{4}} \varepsilon^{-\frac{1}{4}} \frac{\mu_2}{p_2\,\sigma_{22}} \sqrt[4]{192}\, \frac{\sqrt{z}}{\sqrt{3}\,\sqrt{16\,\mu_1^2 - z^2}}\,.
\end{aligned}$$

Further transformations even show that (as we expected it due to symmetry)

$$\beta = \left(\frac{K}{\lambda^3}\right)^{\frac{1}{4}} \varepsilon^{-\frac{1}{4}} \frac{\mu_2}{p_2\,\sigma_{22}} \sqrt[4]{192}\, \frac{1}{\sqrt{z_2}}$$

where

$$z_2 := \sqrt{17\,\mu_2^2 + \mu_1^2 - \sqrt{\mu_1^4 + \mu_2^4 + 34\,\mu_1^2\,\mu_2^2}}$$

which completes the proof. □

Considering again the wealth invested in the two stocks we have to rebalance our portfolio whenever our wealth-combination equals one of the elements of

$$\left\{ (S_1, S_2)' = (S_1^*, S_2^*)' + (\tilde{S}_1, \tilde{S}_2)' : \frac{\tilde{S}_1^2}{a^2} + \frac{\tilde{S}_2^2}{b^2} = 1 \right\} \tag{2.109}$$

where

$$a := \left(\frac{K}{\lambda^3}\right)^{\frac{1}{4}} \frac{\mu_1}{\sigma_{11}} \left(\frac{192}{17\,\mu_1^2 + \mu_2^2 - \sqrt{\mu_1^4 + \mu_2^4 + 34\,\mu_1^2\,\mu_2^2}} \right)^{\frac{1}{4}}, \tag{2.110}$$

$$b := \left(\frac{K}{\lambda^3}\right)^{\frac{1}{4}} \frac{\mu_2}{\sigma_{22}} \left(\frac{192}{17\,\mu_2^2 + \mu_1^2 - \sqrt{\mu_1^4 + \mu_2^4 + 34\,\mu_1^2\,\mu_2^2}} \right)^{\frac{1}{4}}. \tag{2.111}$$

In contrast to the purely proportional cost problem, we now do the biggest possible transaction towards the optimal point without transaction costs. This is clear as the transaction costs are not size dependent. On the other hand, we realize that the form of NT clearly points out that a transaction now typically involves rebalancing in both stocks. Hence the expected bargaining effect occurs.

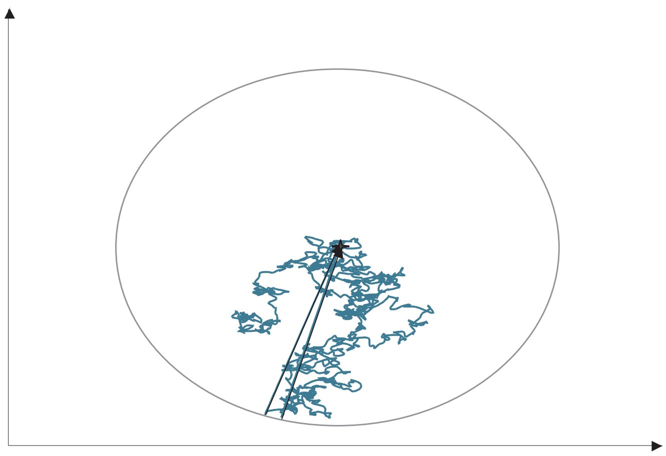

Fig. 5. (Asymptotically) optimal risky money (3)

Remark 14. From the last formulae we obtain some insights into the relation between the fixed costs and the size of the no-transaction region: The no-transaction region increases with an increase of K and converges to the single point $(S_1^*, S_2^*)'$ if K tends to 0. This is what we expect because of economic reasons: The higher the fixed costs are, the less often the investor wants to rebalance his portfolio to compensate for the higher transaction costs.

Further (2.110)/(2.111) imply that the no-transaction region decreases with λ (which is also true for S^*) and that the axes of the ellipsoid in (2.109) only depend on the corresponding σ but depend on both drift terms μ_1, μ_2. (a is linear in $1/\sigma_{11}$ whereas b is linear in $1/\sigma_{22}$).

The following examples will illustrate the quantitative relation between the size of the no-transaction region and the cost parameters.

Examples 8.

a) $\mu_1 = \mu_2 = 0.03\,,\ \sigma_{11} = 0.3\,,\ \sigma_{22} = 0.4\,,\ \lambda = 0.001 \quad \Longrightarrow$
$S_1^* = 333.33\,,\ S_2^* = 187.5\,.$

K	a	b
1E-16	0.021	0.015
1E-12	0.205	0.154
1E-8	2.053	1.54
0.0001	20.534	15.4
0.001	36.515	27.386
0.01	64.934	48.7
0.1	115.47	86.603
1	205.338	154.004
100	649.336	487.002

b) $\mu_1 = 0.02\,,\ \mu_2 = 0.03\,,\ \sigma_{11} = 0.3\,,\ \sigma_{22} = 0.4\ ,\lambda = 0.001 \quad \Longrightarrow$
$S_1^* = 222.22\,,\ S_2^* = 187.5\,.$

K	a	b
1E-16	0.017	0.015
1E-12	0.175	0.15
1E-8	1.748	1.498
0.0001	17.477	14.98
0.001	31.078	26.638
0.01	55.266	47.37
0.1	98.278	84.237
1	174.766	149.796
100	552.659	473.697

2.7.5 Proportional and fixed transaction costs

Let $k > 0\,, K > 0$. Further let (p_1, p_2) and t be fixed. For suitable constants $A, \ldots, F$ we assume the function H to be of the form

$$H(x,y) = A\,x^4 + B\,y^4 + C\,x^2\,y^2 + D\,x^2 + E\,y^2 + F \quad \forall\,(x,y) \in NT \qquad (2.112)$$

with NT given by the ellipsoid

$$\left\{ (x^*, y^*)' + \varepsilon^{1/4}\,(x,y)' \;:\; \frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} \leq 1 \right\}\,. \qquad (2.113)$$

The boundary of NT consists of all points

$$(x^*, y^*)' + \varepsilon^{1/4}\,(x,y)'$$

with

$$\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} = 1 \tag{2.114}$$

whereas the restarting line (which is also assumed to be an ellipsoid) consists of all points

$$(x^*, y^*)' + \varepsilon^{1/4}\,(x, y)'$$

with

$$\frac{x^2}{(\alpha - d_1)^2} + \frac{y^2}{(\beta - d_2)^2} = 1 \tag{2.115}$$

where $0 < d_1 < \alpha$, $0 < d_2 < \beta$.

Relation (2.93) and the assumed form of H and NT determine α and β as

$$\begin{aligned} \alpha &= \frac{d_1}{2}\left(1 + \sqrt{1 + \frac{k}{K}\,\varepsilon^{1/4}\,p_1\,d_1}\right), & (2.116)\\ \beta &= \frac{d_2}{2}\left(1 + \sqrt{1 + \frac{k}{K}\,\varepsilon^{1/4}\,p_2\,d_2}\right) & (2.117) \end{aligned}$$

where (d_1, d_2) is the positive solution (which has to be determined numerically) of the non-linear system

$$f_1(d_1, d_2) = 0, \quad f_2(d_1, d_2) = 0 \tag{2.118}$$

with

$$\begin{aligned} f_1(d_1, d_2) &:= d_1^9\,\frac{\tilde{k}}{\tilde{K}}\,p_1 + d_1^8 - q(d_2)\,, & (2.119)\\ f_2(d_1, d_2) &:= \tilde{K} + \tilde{k}\,\frac{1 - m(d_1, d_2)}{2}\,\Big(p_1\,g_1(d_1) + \sqrt{3}\,p_2\,h_1(d_2)\Big) & \\ &\quad -\frac{m(d_1, d_2)^4 - 1}{16}\,\Big(g_2(d_1)\,g_1(d_1)^4 + 9\,h_2(d_2)\,h_1(d_2)^4 & \\ &\qquad +3\,l(d_2)\,g_1(d_1)^2\,h_1(d_2)^2\Big) & \\ &\quad -\frac{m(d_1, d_2)^2 - 1}{4}\,\Big(g_3(d_1)\,g_1(d_1)^2 + 3\,h_3(d_2)\,h_1(d_2)^2\Big) & (2.120) \end{aligned}$$

where

$$\begin{aligned} \tilde{K} &:= \lambda\,\frac{K}{\varepsilon}\,, & (2.121)\\ \tilde{k} &:= \lambda\,\frac{k}{\varepsilon^{3/4}}\,, & (2.122)\\ C_1 &:= \frac{\lambda^4}{12}\,a_1^2\,\frac{\mu_1^2 - \mu_2^2}{\mu_1^4}\,, & (2.123) \end{aligned}$$

$$C_2 := \left(\frac{\mu_2}{\mu_1}\right)^4 \left(\frac{a_1}{a_2}\right)^2 , \tag{2.124}$$

$$u(d_1) := \sqrt{1 + \frac{\tilde{k}}{\tilde{K}} d_1 \, p_1} , \tag{2.125}$$

$$v(d_2) := \sqrt{1 + \frac{\tilde{k}}{\tilde{K}} d_2 \, p_2} , \tag{2.126}$$

$$q(d_2) := \left(\tilde{K} \frac{d_2^4 \, v(d_2)}{C_2 \, \tilde{K} + C_1 \, d_2^4 \, v(d_2)}\right)^2 , \tag{2.127}$$

$$g_1(d_1) := \frac{d_1}{2} \, (1 + u(d_1)) , \tag{2.128}$$

$$h_1(d_2) := \frac{d_2}{2} \, (1 + v(d_2)) , \tag{2.129}$$

$$g_2(d_1) := \frac{\tilde{K}}{d_1^4 \, u(d_1)} , \tag{2.130}$$

$$h_2(d_2) := \frac{\tilde{K}}{d_2^4 \, v(d_2)} , \tag{2.131}$$

$$l(d_2) := \frac{\lambda^4 \, a_1 \, a_2}{2 \, \mu_1^2} - 6 \, \frac{\tilde{K}}{d_2^4 \, v(d_2)} \, \frac{\mu_2^2 \, a_1}{\mu_1^2 \, a_2} , \tag{2.132}$$

$$g_3(d_1) := -\frac{2 \, \tilde{K} + \frac{3}{2} \, \tilde{k} \, p_1 \, d_1}{d_1^2 \, u(d_1)} , \tag{2.133}$$

$$h_3(d_2) := -\frac{2 \, \tilde{K} + \frac{3}{2} \, \tilde{k} \, p_2 \, d_2}{d_2^2 \, v(d_2)} , \tag{2.134}$$

$$m(d_1, d_2) := \frac{2 \, (g_1(d_1) - d_1) \, (h_1(d_2) - d_2)}{\sqrt{g_1(d_1)^2 \, (h_1(d_2) - d_2)^2 + 3 \, h_1(d_2)^2 \, (g_1(d_1) - d_1)^2}} . \tag{2.135}$$

Determination of the constants: We will show that there exist suitable constants $A, \ldots, F$ such that H and NT as given above satisfy the transformed qvi in the usual approximative sense: The conditions on H we derived in the one-dimensional case now read

(1) $H(x^+, y^+) - H(\hat{x}^+, \hat{y}^+) = \lambda \, K(\hat{x}^+ - x^+, \hat{y}^+ - y^+)$ for all pairs $(\hat{x}^+, \hat{y}^+)$ and the corresponding pairs (x^+, y^+) which satisfy (2.114) and (2.115), respectively,

(2) $H_x(\alpha - d_1, 0) = -\lambda \, K_x(d_1, 0)$,

(3) $H_x(\alpha, 0) = -\lambda \, K_x(d_1, 0)$,

(4) $H_y(0, \beta - d_2) = -\lambda \, K_y(0, d_2)$,

(5) $H_y(0,\beta) = -\lambda\, K_y(0,d_2)\,.$

We determine the constants $A,\ldots,F$ such that the conditions above as well as relation (2.93) are satisfied: Due to the above definitions of $\tilde{k}$ and $\tilde{K}$ we have that

$$\lambda\, K(x,y) = \tilde{K} + \tilde{k}\,(|x|\,p_1 + |y|\,p_2)\,.$$

The equalities (3) and (2), respectively, then show that

$$-\tilde{k}\,p_1 = 4\,A\,\alpha^3 + 2\,D\,\alpha$$

and

$$-\tilde{k}\,p_1 = 4\,A\,(\alpha-d_1)^3 + 2\,D\,(\alpha-d_1)$$

thus implying

$$D = -\frac{\tilde{k}\,p_1 + 4\,A\,\alpha^3}{2\,\alpha}$$

and

$$D = -\frac{\tilde{k}\,p_1 + 4\,A\,(\alpha-d_1)^3}{2\,(\alpha-d_1)}\,.$$

These two equations for D give us

$$\tilde{k}\,p_1\,(\alpha-d_1) + 4\,A\,\alpha^3\,(\alpha-d_1) = \tilde{k}\,p_1\,\alpha + 4\,A\,(\alpha-d_1)^3\,\alpha$$

which is equivalent to

$$A = \frac{\tilde{k}\,p_1}{4\,(\alpha-d_1)\,(2\,\alpha-d_1)\,\alpha}\,.$$

This then implies

$$D = -\frac{\tilde{k}\,p_1\,[(\alpha-d_1)\,(2\,\alpha-d_1)+\alpha^2]}{2\,\alpha\,(\alpha-d_1)\,(2\,\alpha-d_1)}\,.$$

Simultaneously we obtain from the equalities (4) and (5) that

$$B = \frac{\tilde{k}\,p_2}{4\,(\beta-d_2)\,(2\,\beta-d_2)\,\beta}$$

and

$$E = -\frac{\tilde{k}\,p_2\,[(\beta-d_2)\,(2\,\beta-d_2)+\beta^2]}{2\,\beta\,(\beta-d_2)\,(2\,\beta-d_2)}\,.$$

Due to these four equalities the equality (1) then reads for the special choice of the corresponding points $(\hat{x}^+, \hat{y}^+) = (\alpha, 0)$ and $(x^+, y^+) = (\alpha - d_1, 0)$

$$\begin{aligned}
&\tilde{K} + \tilde{k}\, p_1\, d_1 \\
&\quad = A\,[(\alpha - d_1)^4 - \alpha^4] + D\,[(\alpha - d_1)^2 - \alpha^2] \\
&\quad = \frac{\tilde{k}\, p_1}{\alpha(\alpha - d_1)(2\alpha - d_1)} \left\{ \frac{1}{4} \left[(\alpha - d_1)^4 - \alpha^4 \right] \right. \\
&\qquad\qquad \left. - \frac{1}{2} \left[(\alpha - d_1)^2 - \alpha^2 \right] \left[(\alpha - d_1)(2\alpha - d_1) + \alpha^2 \right] \right\} \\
&\quad = -\frac{\tilde{k}\, p_1\, d_1}{\alpha\,(\alpha - d_1)} \underbrace{\left\{ \frac{1}{4} \left(2\,\alpha^2 - 2\,\alpha\, d_1 + d_1^2 \right) - \frac{1}{2} \left(3\,\alpha^2 - 3\, d_1\, \alpha + d_1^2 \right) \right\}}_{-\left(\frac{1}{2}\, d_1 - \alpha\right)^2} \\
&\quad = \tilde{k}\, p_1\, \frac{d_1 \left(\frac{1}{2} d_1 - \alpha \right)^2}{\alpha\,(\alpha - d_1)}
\end{aligned}$$

$$\Longleftrightarrow \quad \tilde{K}\, \alpha\, (\alpha - d_1) = \tilde{k}\, p_1\, d_1 \left\{ \left(\frac{1}{2} d_1 - \alpha \right)^2 - \alpha\,(\alpha - d_1) \right\} = \frac{1}{4} \tilde{k}\, p_1\, d_1^3$$

$$\Longleftrightarrow \quad \alpha^2 - d_1\, \alpha - \frac{1}{4} \frac{\tilde{k}}{\tilde{K}}\, p_1\, d_1^3 = 0$$

$$\Longleftrightarrow \quad \alpha = \frac{d_1}{2} \left(1 + \sqrt{1 + \frac{\tilde{k}}{\tilde{K}}\, p_1\, d_1} \right) .$$

This is the only solution since α has to be bigger than d_1. In analogy we derive

$$\beta = \frac{d_2}{2} \left(1 + \sqrt{1 + \frac{\tilde{k}}{\tilde{K}}\, p_2\, d_2} \right) .$$

Using this relation between α and d_1 and β and d_2, respectively, we obtain

$$\begin{aligned}
A &= \frac{\tilde{k}\, p_1}{4\,(\alpha - d_1)\,(2\,\alpha - d_1)\,\alpha} \\
&= \frac{\tilde{K}}{d_1^4 \sqrt{1 + \frac{\tilde{k}}{\tilde{K}}\, p_1\, d_1}}
\end{aligned}$$

and

$$B = \frac{\tilde{K}}{d_2^4 \sqrt{1 + \frac{\tilde{k}}{\tilde{K}}\, p_2\, d_2}} .$$

Now we will use relation (2.93) to determine the remaining parameters d_1, d_2 and C. Due to the form of H we find that relation (2.93) holds if and only if

$$\frac{1}{2}\lambda^2 a_1 = \frac{1}{2\lambda^2}\left(12\,A\,\frac{\mu_1^2}{a_1} + 2\,C\,\frac{\mu_2^2}{a_2}\right)$$

and

$$\frac{1}{2}\lambda^2 a_2 = \frac{1}{2\lambda^2}\left(12\,B\,\frac{\mu_2^2}{a_2} + 2\,C\,\frac{\mu_1^2}{a_1}\right).$$

These equalities imply

$$C = \lambda^4\,\frac{a_1\,a_2}{2\,\mu_2^2} - 6\,A\,\frac{\mu_1^2\,a_2}{\mu_2^2\,a_1}$$

and

$$C = \lambda^4\,\frac{a_1\,a_2}{2\,\mu_1^2} - 6\,B\,\frac{\mu_2^2\,a_1}{\mu_1^2\,a_2}$$

and determine C in dependence on d_1 or d_2. Furthermore we also obtain

$$\begin{aligned} A &= \frac{\lambda^4}{12}\,a_1^2\,\frac{\mu_1^2-\mu_2^2}{\mu_1^4} + \left(\frac{\mu_2}{\mu_1}\right)^4\left(\frac{a_1}{a_2}\right)^2 B \\ &= C_1 + C_2\,B \end{aligned}$$

with positive constants C_1, C_2 as given in (2.123) and (2.124). Comparing this equation with the upper form of A (using also the upper form of B) we find that

$$C_1 + C_2\,\frac{\tilde{K}}{d_2^4\sqrt{1+\frac{\tilde{k}}{\tilde{K}}\,p_2\,d_2}} = \frac{\tilde{K}}{d_1^4\sqrt{1+\frac{\tilde{k}}{\tilde{K}}\,p_1\,d_1}}$$

$$\Longleftrightarrow \quad f_1(d_1,d_2) = \frac{\tilde{k}}{\tilde{K}}\,p_1\,d_1^9 + d_1^8 - \underbrace{\left(\tilde{K}\,\frac{d_2^4\sqrt{1+\frac{\tilde{k}}{\tilde{K}}\,p_2\,d_2}}{C_2\,\tilde{K} + C_1\,d_2^4\sqrt{1+\frac{\tilde{k}}{\tilde{K}}\,p_2\,d_2}}\right)^2}_{=\,q(d_2)} = 0\,.$$

In order to find a second relation between d_1 and d_2 we use the equality (1) for another time. Since equality (1) has to hold for all points $(\hat{x}^+, \hat{y}^+)$ satisfying (2.114) and the corresponding points (x^+, y^+) satisfying (2.115), we now choose a pair satisfying (2.114) which is different from the above chosen ones $(\hat{x}^+, \hat{y}^+) = (\alpha, 0)$ and $(\hat{x}^+, \hat{y}^+) = (0, \beta)$. This time we take

$$(\hat{x}^+, \hat{y}^+) = \left(\frac{\alpha}{2}, \beta\,\frac{\sqrt{3}}{2}\right).$$

To obtain the corresponding pair (x^+, y^+) we intersect the straight line which joins $(0,0)$ and $(\alpha/2, \beta\sqrt{3}/2)$ with the ellipsoid given in (2.115): For $r \in (0,1)$ we obtain that

$$(x^+, y^+)' = r\left(\frac{\alpha}{2}, \beta\,\frac{\sqrt{3}}{2}\right)'$$

with

$$1 = \frac{x^{+2}}{(\alpha - d_1)^2} + \frac{y^{+2}}{(\beta - d_2)^2}$$

$$\begin{aligned}\iff \quad 1 &= \frac{r^2\,\alpha^2}{4}\frac{1}{(\alpha-d_1)^2} + \frac{3\,r^2\,\beta^2}{4}\frac{1}{(\beta-d_2)^2} \\ &= \frac{r^2}{4}\,\frac{\alpha^2\,(\beta-d_2)^2 + 3\,\beta^2\,(\alpha-d_1)^2}{(\alpha-d_1)^2\,(\beta-d_2)^2}\end{aligned}$$

$$\iff \quad r = \frac{2\,(\alpha-d_1)\,(\beta-d_2)}{\sqrt{\alpha^2\,(\beta-d_2)^2 + 3\,\beta^2\,(\alpha-d_1)^2}}\,.$$

For this particular r and the corresponding pairs $(\hat{x}^+, \hat{y}^+)$ and (x^+, y^+) we have that

$$\begin{aligned}\lambda\,K(\hat{x}^+ - x^+, \hat{y}^+ - y^-) &= \tilde{K} + \tilde{k}\,(\,p_1(\hat{x}^+ - x^+) + p_2(\hat{y}^+ - y^+)\,) \\ &= \tilde{K} + \tilde{k}\,(1-r)\,(p_1\,\hat{x}^+ + p_2\,\hat{y}^+) \\ &= \tilde{K} + \tilde{k}\,\frac{1-r}{2}\,\left(p_1\,\alpha + \sqrt{3}\,p_2\,\beta\right)\end{aligned}$$

and that

$$\begin{aligned}&H(x^+, y^+) - H(\hat{x}^+, \hat{y}^+) \\ &\quad = A\left(x^{+4} - \hat{x}^{+4}\right) + B\left(y^{+4} - \hat{y}^{+4}\right) + D\left(x^{+2} - \hat{x}^{+2}\right) + E\left(y^{+2} - \hat{y}^{+2}\right) \\ &\qquad + C\left(x^{+2}y^{+2} - \hat{x}^{+2}\hat{y}^{+2}\right) \\ &\quad = \left(r^4 - 1\right)\left(A\,\hat{x}^{+4} + B\,\hat{y}^{+4} + C\,\hat{x}^{+2}\hat{y}^{+2}\right) + \left(r^2 - 1\right)\left(D\,\hat{x}^{+2} + E\,\hat{y}^{+2}\right) \\ &\quad = \frac{r^4 - 1}{16}\left(A\,\alpha^4 + 9\,B\,\beta^4 + 3\,C\,\alpha^2\,\beta^2\right) + \frac{r^2 - 1}{4}\left(D\,\alpha^2 + 3\,E\,\beta^2\right)\end{aligned}$$

which must be equal in order to satisfy the equality (1).

Furthermore, expressing the constants C, D, E with the help of d_1 and d_2 (we have already done this for A and B) we obtain

$$C \;=\; \lambda^4\,\frac{a_1\,a_2}{2\,\mu_1^2} - 6\,\frac{\tilde{K}}{d_2^4\,\sqrt{1 + \frac{\tilde{k}}{\tilde{K}}\,p_2\,d_2}}\,\frac{\mu_2^2\,a_1}{\mu_1^2\,a_2}\,,$$

$$
\begin{aligned}
D &= -\frac{2\,\tilde{K} + \frac{3}{2}\,\tilde{k}\,p_1\,d_1}{d_1^2\sqrt{1 + \frac{\tilde{k}}{\tilde{K}}\,p_1\,d_1}}\,, \\
E &= -\frac{2\,\tilde{K} + \frac{3}{2}\,\tilde{k}\,p_2\,d_2}{d_2^2\sqrt{1 + \frac{\tilde{k}}{\tilde{K}}\,p_2\,d_2}}\,.
\end{aligned}
$$

Using the above definitions of $g_1, g_2, g_3, h_1, h_2, h_3, l$ and m we see that

$$
\begin{aligned}
\alpha &= g_1(d_1)\,, \\
\beta &= h_1(d_2)\,, \\
r &= m(d_1, d_2)\,, \\
A &= g_2(d_1)\,, \\
B &= h_2(d_2)\,, \\
C &= l(d_2)\,, \\
D &= g_3(d_1)\,, \\
E &= h_3(d_2)\,,
\end{aligned}
$$

which completes the derivation of the non-linear system

$$
\begin{aligned}
f_1(d_1, d_2) &= 0\,, \\
f_2(d_1, d_2) &= 0
\end{aligned}
$$

with f_1 , f_2 as given above. □

Unfortunately we are not able to derive a closed form solution to the non-linear system (2.118). Hence we have to solve it numerically. The next proposition will show the existence of a positive solution (d_1, d_2) of (2.118).

Proposition 6. *Let f_1 , f_2 be defined as in (2.118)-(2.135).*

i) Let $d_2 \geq 0$ be fixed. Let $\hat{d}_1$ be the unique finite positive solution of the equation $\frac{\tilde{k}}{\tilde{K}}\,p_1\,d_1^9 + d_1^8 - \left(\frac{\tilde{K}}{C_1}\right)^2 = 0$ if $\mu_1^2 \neq \mu_2^2$.

a) Let $\mu_1^2 \geq \mu_2^2$. Then $f_1(d_1, d_2) = 0$ has a unique finite solution $d_1(d_2) \geq 0$ where

$$d_1(0) = \lim_{d_2 \to 0} d_1(d_2) = 0$$

and

$$d_1(d_2) > 0 \quad \forall\, d_2 > 0\,.$$

Further, $d_1(d_2)$ is continuous and strictly monotonously increasing in $d_2 > 0$ with

$$\lim_{d_2 \to \infty} d_1(d_2) = \begin{cases} \hat{d}_1 & \text{if} \quad \mu_1^2 > \mu_2^2\,, \\ \infty & \text{if} \quad \mu_1^2 = \mu_2^2\,. \end{cases}$$

b) Let $\mu_1^2 < \mu_2^2$. Let $\hat{d}_2$ be the unique positive finite solution of the equation $\frac{\tilde{k}}{\tilde{K}}\, p_2\, d_2^9 + d_2^8 - \left(\frac{C_2}{C_1}\tilde{K}\right)^2 = 0$. Then $f_1(d_1,d_2)$ has a unique finite solution $d_1(d_2) \geq 0$ for all $0 \leq d_2 \neq \hat{d}_2$ where

$$d_1(0) = \lim_{d_2 \to 0} d_1(d_2) = 0$$

and

$$d_1(d_2) > 0 \quad \forall\, 0 < d_2 \neq \hat{d}_2\,.$$

Further $d_1(d_2)$ is continuous and strictly monotonously increasing in $0 < d_2 < \hat{d}_2$ with

$$\lim_{d_2 \to \hat{d}_2} d_1(d_2) = \infty$$

and continuous and strictly monotonously decreasing in $d_2 > \hat{d}_2$ with

$$\lim_{d_2 \to \infty} d_1(d_2) = \hat{d}_1\,.$$

ii) $f_2(0,0) = \lim_{d_2 \to 0} f_2(d_1(d_2), d_2) = -\frac{3}{2}\,\tilde{K} < 0\,.$

iii) $\lim_{d_2 \to \infty} f_2(d_1(d_2), d_2) = \infty\,.$

iv) Let $\mu_1^2 < \mu_2^2$. Then $\lim_{d_2 \to \hat{d}_2} f_2(d_1(d_2), d_2) = \infty$ where $\hat{d}_2$ is given in i)b).

v) Let $\mu_1^2 \geq \mu_2^2$. Then f_2 is continuous for all $d_2 \geq 0$.

vi) Let $\mu_1^2 < \mu_2^2$. Then f_2 is continuous for all $0 \leq d_2 \neq \hat{d}_2$.

Proof: i) Let $d_2 \geq 0$ be fixed. Let either $\mu_1^2 \geq \mu_2^2$ or $\mu_1^2 < \mu_2^2$ where $d_2 \neq \hat{d}_2$. Then we have

$$f_1(0, d_2) = -q(d_2) < 0 \text{ with } |q(d_2)| < \infty\,,$$

$$f_1(d_1, d_2) \overset{d_1 \to \infty}{\longrightarrow} \infty$$

and

$$\frac{\partial}{\partial d_1} f_1(d_1, d_2) = 9\, \frac{\tilde{k}}{\tilde{K}}\, p_1\, d_1^8 + 8\, d_1^7 > 0 \quad \forall\, d_1 > 0\,,$$

implying that f_1 possesses exactly one finite zero $d_1(d_2) \geq 0$. (Analogously, we obtain the existence and uniqueness of $\hat{d}_1$ and $\hat{d}_2$.) Since $C_2\, \tilde{K} > 0$, we obtain $q(0) = 0$ and thus $d_1(0) = 0$ whereas $d_1(d_2) > 0$ for $d_2 > 0$ (for $0 < d_2 \neq \hat{d}_2$ in the case $\mu_1^2 < \mu_2^2$).

From (2.127) which reads

$$q(d_2) = \left(\frac{\tilde{K}\, d_2^4\, v(d_2)}{C_2\, \tilde{K} + C_1\, d_2^4\, v(d_2)} \right)^2 =: \left(\frac{\tilde{K}\, Z(d_2)}{C_2\, \tilde{K} + C_1\, Z(d_2)} \right)^2$$

we obtain

$$q'(d_2) \;=\; 2\,\frac{\tilde{K}^3\, C_2\, Z(d_2)\, Z'(d_2)}{[\, C_2\, \tilde{K} + C_1\, Z(d_2)\,]^3}$$

where

$$Z'(d_2) = 4\, d_2^3\, v(d_2) + \frac{\tilde{k}}{2\,\tilde{K}}\, p_2\, \frac{d_2^4}{v(d_2)}\,.$$

Due to $Z(d_2) > 0$ and $Z'(d_2) > 0$ for $d_2 > 0$ we obtain

$$q'(d_2) > 0 \quad \Longleftrightarrow \quad C_2\, \tilde{K} + C_1\, Z(d_2) > 0\,,\ d_2 > 0\,.$$

a) Let $\mu_1^2 \geq \mu_2^2$. Then we obtain $C_1 \geq 0$ and

$$C_2\, \tilde{K} + C_1\, Z(d_2) > 0 \quad \forall\, d_2 \geq 0\,.$$

This implies $q'(d_2) > 0$ for all $d_2 > 0$. Since $d_1(d_2)$ increases with an increase in $q(d_2)$, $d_1(d_2)$ is strictly monotonously increasing in $d_2 > 0$. Due to the continuity of $q(d_2)$, $d_1(d_2)$ is also continuous . Furthermore

$$Z(d_2) = d_2^4 \sqrt{1 + \frac{\tilde{k}}{\tilde{K}}\, p_2\, d_2} \stackrel{d_2 \to \infty}{\longrightarrow} \infty\,.$$

Hence we obtain

$$q(d_2) \stackrel{d_2 \to \infty}{\longrightarrow} \begin{cases} \left(\frac{\tilde{K}}{C_1}\right)^2 & \text{if} \quad \mu_1^2 > \mu_2^2\,, \\ \infty & \text{if} \quad \mu_1^2 = \mu_2^2 \end{cases}$$

which implies

$$d_1(d_2) \stackrel{d_2 \to \infty}{\longrightarrow} \begin{cases} \hat{d}_1 & \text{if} \quad \mu_1^2 > \mu_2^2\,, \\ \infty & \text{if} \quad \mu_1^2 = \mu_2^2\,. \end{cases}$$

b) Let $\mu_1^2 < \mu_2^2$. Then $C_1 < 0$. This implies

$$\begin{aligned} q'(d_2) > 0 \quad &\Longleftrightarrow \quad C_1\, Z(d_2) > -C_2\, \tilde{K}\,,\ d_2 > 0 \\ &\Longleftrightarrow \quad Z(d_2) < \tilde{K}\, \left|\frac{C_2}{C_1}\right|\,,\ d_2 > 0 \\ &\Longleftrightarrow \quad d_2^4 \sqrt{1 + \frac{\tilde{k}}{\tilde{K}} p_2\, d_2} < \tilde{K}\, \left|\frac{C_2}{C_1}\right|\,,\ d_2 > 0 \\ &\Longleftrightarrow \quad Y(d_2) := \frac{\tilde{k}}{\tilde{K}} p_2\, d_2^9 + d_2^8 - \left(\frac{C_2}{C_1}\, \tilde{K}\right)^2 < 0\,,\ d_2 > 0\,. \end{aligned}$$

By definition of $\hat{d}_2$ we have

$$Y(d_2) < 0 \quad \forall\, d_2 \in [0, \hat{d}_2)$$

whereas

$$Y(d_2) \geq 0 \quad \forall\, d_2 \in [\hat{d}_2, \infty)\,.$$

Therefore we obtain

$$q'(d_2) \begin{cases} > 0 & \text{if} \quad d_2 \in (0, \hat{d}_2)\,, \\ < 0 & \text{if} \quad d_2 \in (\hat{d}_2, \infty) \end{cases}$$

giving us that $d_1(d_2)$ increases strictly monotonously in $0 < d_2 < \hat{d}_2$ and decreases strictly monotonously in $d_2 > \hat{d}_2$.

Since $Y(\hat{d}_2) = 0$ holds if and only if $C_1\, Z(d_2) = -C_2\, \tilde{K}$, we obtain

$$q'(d_2) \stackrel{d_2 \to \hat{d}_2}{\longrightarrow} \begin{cases} \infty & \text{if} \quad d_2 < \hat{d}_2\,, \\ -\infty & \text{if} \quad d_2 > \hat{d}_2 \end{cases}$$

as well as

$$q(d_2) \stackrel{d_2 \to \hat{d}_2}{\longrightarrow} \infty\,.$$

This then implies

$$d_1(d_2) \stackrel{d_2 \to \hat{d}_2}{\longrightarrow} \infty\,.$$

Further we have

$$q(d_2) \stackrel{d_2 \to \infty}{\longrightarrow} \left(\frac{\tilde{K}}{C_1}\right)^2$$

which implies

$$d_1(d_2) \stackrel{d_2 \to \infty}{\longrightarrow} \hat{d}_1\,.$$

Therefore $d_1(d_2)$ is continuous in $0 \leq d_2 \neq \hat{d}_2$.
ii) We define

$$\begin{aligned} Z_1(d_1) &:= g_2(d_1)\, g_1(d_1)^4\,, \\ Z_2(d_2) &:= h_2(d_2)\, h_1(d_2)^4\,, \\ Z_3(d_1, d_2) &:= l(d_2)\, g_1(d_1)^2\, h_1(d_2)^2\,, \\ Z_4(d_1) &:= g_3(d_1)\, g_1(d_1)^2\,, \\ Z_5(d_2) &:= h_3(d_2)\, h_1(d_2)^2\,, \end{aligned}$$

thus getting

$$\begin{aligned} f_2(d_1, d_2) \;=\; & \tilde{K} + \frac{\tilde{k}}{2}\,[1 - m(d_1, d_2)]\,[p_1\, g_1(d_1) + \sqrt{3}\, p_2\, h_1(d_2)] \\ & -\frac{m(d_1, d_2)^4 - 1}{16}\,[Z_1(d_1) + 9\, Z_2(d_2) + 3\, Z_3(d_1, d_2)] \\ & -\frac{m(d_1, d_2)^2 - 1}{4}\,[Z_4(d_1) + 3\, Z_5(d_2)]\,. \end{aligned} \tag{2.136}$$

Using (2.128)-(2.131) and (2.133)/(2.134) we obtain

$$g_1(d_1) = \frac{d_1}{2}\,(1+u) \overset{d_1\to 0}{\longrightarrow} 0\,,$$

$$Z_1(d_1) = \frac{\tilde{K}}{16}\,\frac{(1+u)^4}{u} \overset{d_1\to 0}{\longrightarrow} \tilde{K}\,,$$

$$Z_4(d_1) = -\frac{2\tilde{K}+\frac{3}{2}\tilde{k}\,p_1\,d_1}{4}\,\frac{(1+u)^2}{u} \overset{d_1\to 0}{\longrightarrow} -2\,\tilde{K}$$

and, simultaneously,

$$h_1(d_2) \overset{d_2\to 0}{\longrightarrow} 0\,,$$

$$Z_2(d_2) \overset{d_2\to 0}{\longrightarrow} \tilde{K}\,,$$

$$Z_5(d_2) \overset{d_2\to 0}{\longrightarrow} -2\,\tilde{K}\,.$$

(For notational convenience we will always write the functions $u\,,v$ suppressing their dependence on d_1 and d_2, respectively.) Further (2.135) gives us

$$\begin{aligned} m(d_1,d_2) &= 2\frac{\frac{d_1}{2}\,(u-1)\,\frac{d_2}{2}\,(v-1)}{\sqrt{\frac{d_1^2}{4}\,(1+u)^2\,\frac{d_2^2}{4}\,(v-1)^2+3\,\frac{d_2^2}{4}\,(1+v)^2\,\frac{d_1^2}{4}\,(u-1)^2}} \\ &= 2\,\frac{(u-1)\,(v-1)}{\sqrt{(u+1)^2\,(v-1)^2+3\,(v+1)^2\,(u-1)^2}} \\ &= 2\,\frac{1}{\sqrt{\left(\frac{u+1}{u-1}\right)^2+3\,\left(\frac{v+1}{v-1}\right)^2}}\,. \end{aligned} \qquad (2.137)$$

Due to $v \overset{d_2\to 0}{\longrightarrow} 1$ and $u=\sqrt{1+\frac{\tilde{k}}{\tilde{K}}\,p_1\,d_1(d_2)} \overset{d_2\to 0}{\longrightarrow} 1$ we obtain $m(d_1,d_2) \overset{d_2\to 0}{\longrightarrow} 0$.

To determine the value of $\lim\limits_{d_2\to 0} Z_3(d_1,d_2)$ we write $Z_3(d_1,d_2)$ as follows:

$$\begin{aligned} Z_3(d_1,d_2) &= \frac{d_1^2}{4}\,(1+u)^2\,\frac{d_2^2}{4}\,(1+v)^2\left[\frac{\lambda^4\,a_1\,a_2}{2\mu_1^2}-6\,\tilde{K}\,\frac{\mu_2^2\,a_1}{\mu_1^2\,a_2}\,\frac{1}{d_2^4\,v}\right] \\ &= \frac{\lambda^4\,a_1\,a_2}{32\,\mu_1^2}\,(1+u)^2\,(1+v)^2\,d_1^2\,d_2^2-\frac{3}{8}\,\tilde{K}\,\frac{\mu_2^2\,a_1}{\mu_1^2\,a_2}\,(1+u)^2\,\frac{(1+v)^2}{v}\,\frac{d_1^2}{d_2^2} \\ &=: Z_{31}+Z_{32}\,. \end{aligned}$$

Obviously, $Z_{31} \overset{d_2\to 0}{\longrightarrow} 0$. To determine the limit of Z_{32} we replace d_1^2 with the help of the relation $f_1(d_1,d_2)=0$:

$$0 = f_1(d_1,d_2) = d_1^9\,\frac{\tilde{k}}{\tilde{K}}\,p_1+d_1^8-\left(\frac{\tilde{K}\,d_2^4\,v}{C_2\,\tilde{K}+C_1\,d_2^4\,v}\right)^2$$

$$\iff \quad (d_1^4\, u)^2 = \left(\frac{\tilde{K}\, d_2^4\, v}{C_2\, \tilde{K} + C_1\, d_2^4\, v} \right)^2$$

$$\iff \quad d_1^8 = \left(\frac{\tilde{K}\, d_2^4\, v}{C_2\, \tilde{K} + C_1\, d_2^4\, v} \right)^2 \frac{1}{u^2}$$

$$\iff \quad d_1^2 = \sqrt{\frac{\tilde{K}\, d_2^4\, v}{|C_2\, \tilde{K} + C_1\, d_2^4\, v|}}\, \frac{1}{\sqrt{u}} = d_2^2 \sqrt{\frac{\tilde{K}\, v}{u\, |C_2\, \tilde{K} + C_1\, d_2^4\, v|}}\,.$$

Therefore we have

$$\frac{d_1^2}{d_2^2} = \sqrt{\frac{\tilde{K}\, v}{u\, |C_2\, \tilde{K} + C_1\, d_2^4\, v|}} \xrightarrow{d_2 \to 0} \frac{1}{\sqrt{C_2}} = \frac{\mu_1^2\, a_2}{\mu_2^2\, a_1}$$

implying

$$Z_3(d_1, d_2) \xrightarrow{d_2 \to 0} -6\, \tilde{K}\,.$$

Together with (2.136) this leads to

$$\begin{aligned} f_2(0,0) &= \lim_{d_2 \to 0} f_2(d_1(d_2), d_1) \\ &= \tilde{K} + \frac{1}{16}\, [\tilde{K} + 9\, \tilde{K} - 18\, \tilde{K}] + \frac{1}{4}\, [-2\, \tilde{K} - 6\, \tilde{K}] \\ &= -\frac{3}{2} \tilde{K}\,. \end{aligned}$$

iii) Let $\hat{u} := u(\hat{d}_1)$ with $\hat{d}_1$ as defined in i) and $Z_1, \ldots, Z_5$ as in the proof of ii). Again we will write $u\,, v$ suppressing $d_1\,, d_2$ and we will write d_1 instead of $d_1(d_2)$. Due to

$$d_1 \xrightarrow{d_2 \to \infty} \begin{cases} \hat{d}_1 & \text{if} \quad \mu_1^2 \neq \mu_2^2\,, \\ \infty & \text{if} \quad \mu_1^2 = \mu_2^2 \end{cases}$$

we obtain

$$\begin{aligned} m(d_1, d_2) &\overset{(2.137)}{=} \frac{2}{\sqrt{\left(\frac{u+1}{u-1}\right)^2 + 3\, \left(\frac{v+1}{v-1}\right)^2}} \\ &\xrightarrow{d_2 \to \infty} \begin{cases} \frac{2}{\sqrt{\left(\frac{\hat{u}+1}{\hat{u}-1}\right)^2 + 3}} < 1 & \text{if} \quad \mu_1^2 \neq \mu_2^2\,, \\ \qquad 1 & \text{if} \quad \mu_1^2 = \mu_2^2\,. \end{cases} \end{aligned}$$

Further we have

$$g_1(d_1) = \frac{d_1}{2}\, (1 + u) \xrightarrow{d_2 \to \infty} \begin{cases} \frac{\hat{d}_1}{2}\, (1 + \hat{u}) & \text{if} \quad \mu_1^2 \neq \mu_2^2\,, \\ \qquad \infty & \text{if} \quad \mu_1^2 = \mu_2^2\,, \end{cases}$$

$$h_1(d_2) = \frac{d_2}{2}\, (1 + v) \xrightarrow{d_2 \to \infty} \infty\,,$$

$$Z_1(d_1) = \frac{\tilde{K}}{16}\,\frac{(1+u)^4}{u} \overset{d_2\to\infty}{\longrightarrow} \begin{cases} \frac{\tilde{K}}{16}\,\frac{(1+\hat{u})^4}{\hat{u}} & \text{if} \quad \mu_1^2 \neq \mu_2^2\,, \\ \infty & \text{if} \quad \mu_1^2 = \mu_2^2\,, \end{cases}$$

$$Z_2(d_2) = \frac{\tilde{K}}{16}\,\frac{(1+v)^4}{v} \overset{d_2\to\infty}{\longrightarrow} \infty\,,$$

$$l(d_2) = \frac{\lambda^4\,a_1\,a_2}{2\mu_1^2} - 6\tilde{K}\,\frac{\mu_2^2\,a_1}{\mu_1^2\,a_2\,d_2^4\,v} \overset{d_2\to\infty}{\longrightarrow} \frac{\lambda^4\,a_1\,a_2}{2\mu_1^2}\,,$$

$$Z_3(d_1,d_2) = l(d_2)\,\frac{d_1^2}{4}\,(1+u)^2\,\frac{d_2^2}{4}\,(1+v)^2 \overset{d_2\to\infty}{\longrightarrow} \infty\,,$$

$$\begin{aligned} Z_4(d_1) \quad &= \quad -\frac{2\,\tilde{K} + \frac{3}{2}\,\tilde{k}\,p_1\,d_1}{4}\,\frac{(1+u)^2}{u} \\ &\overset{d_2\to\infty}{\longrightarrow} \quad \begin{cases} -\frac{2\,\tilde{K} + \frac{3}{2}\,\tilde{k}\,p_1\,\hat{d}_1}{4}\,\frac{(1+\hat{u})^2}{\hat{u}} & \text{if} \quad \mu_1^2 \neq \mu_2^2\,, \\ -\infty & \text{if} \quad \mu_1^2 = \mu_2^2\,, \end{cases} \end{aligned}$$

$$Z_5(d_2) = -\frac{2\,\tilde{K} + \frac{3}{2}\,\tilde{k}\,p_2\,d_2}{4}\,\frac{(1+v)^2}{v} \overset{d_2\to\infty}{\longrightarrow} -\infty\,.$$

Let $\underline{\mu_1^2 \neq \mu_2^2}$. Since these limits imply

$$\tilde{k}\,\frac{1-m(d_1,d_2)}{2}\,[p_1\,g_1(d_1) + \sqrt{3}\,p_2\,h_1(d_2)] \overset{d_2\to\infty}{\longrightarrow} \infty\,,$$

$$-\frac{m(d_1,d_2)^4 - 1}{16}\,[Z_1(d_1) + 9\,Z_2(d_2) + 3\,Z_3(d_1,d_2)] \overset{d_2\to\infty}{\longrightarrow} \infty$$

and

$$-\frac{m(d_1,d_2)^2 - 1}{4}\,[Z_4(d_1) + 3\,Z_5(d_2)] \overset{d_2\to\infty}{\longrightarrow} -\infty\,,$$

we have to take into consideration the order of convergence of the different parts of f_2. Then we can write f_2 as follows: Let $X_1, \ldots, X_6$ be real functions with finite limits where $\lim\limits_{d_2\to\infty} X_2, \ldots, \lim\limits_{d_2\to\infty} X_4 > 0$ and $\lim\limits_{d_2\to\infty} X_5, \lim\limits_{d_2\to\infty} X_6 < 0$. Then

$$\begin{aligned} f_2(d_1(d_2), d_2) \quad &= \quad X_1 + X_2\,d_2\,(1+v) + X_3\,\frac{(1+v)^4}{v} + X_4\,d_2^2\,(1+v)^2 \\ &\quad\quad +X_5\,\frac{(1+v)^2}{v} + X_6\,\frac{d_2\,(1+v)^2}{v} \\ &= \quad d_2^2\,(1+v)^2\,\left(\frac{X_1}{d_2^2\,(1+v)^2} + \frac{X_2}{d_2\,(1+v)} + X_3\,\frac{(1+v)^2}{d_2^2\,v} \right. \\ &\quad\quad \left. +X_4 + \frac{X_5}{d_2^2\,v} + X_6\,\frac{1}{d_2\,v} \right) \\ &\overset{d_2\to\infty}{\longrightarrow} \quad \infty\,. \end{aligned}$$

Let now $\underline{\mu_1^2 = \mu_2^2}$. Here we write f_2 as follows:

$$\begin{aligned}
f_2(d_1, d_2) &= \tilde{K} + \tilde{k}\,\frac{1 - m(d_1, d_2)}{2}\,[p_1\,g_1(d_1) + \sqrt{3}\,p_2\,h_1(d_2)] \\
&\quad + \frac{1}{16}\,(m(d_1, d_2) + 1)\,(m(d_1, d_2)^2 + 1)\,(1 - m(d_1, d_2)) \\
&\qquad [Z_1(d_1) + 9\,Z_2(d_2) + 3\,Z_3(d_1, d_2)] \\
&\quad + \frac{1}{4}\,(m(d_1, d_2) + 1)\,(1 - m(d_1, d_2))\,[Z_4(d_1) + 3\,Z_5(d_2)] \\
&= \tilde{K} + (1 - m(d_1, d_2))\,\Big\{ \frac{\tilde{k}}{2}\,[p_1\,g_1(d_1) + \sqrt{3}\,p_2\,h_1(d_2)] \\
&\qquad + \frac{1}{16}\,(m(d_1, d_2) + 1)\,(m(d_1, d_2)^2 + 1) \\
&\qquad\quad [Z_1(d_1) + 9\,Z_2(d_2) + 3\,Z_3(d_1, d_2)] \\
&\qquad + \frac{1}{4}\,(m(d_1, d_2) + 1)\,[Z_4(d_1) + 3\,Z_5(d_2)] \Big\} \\
&=: \tilde{K} + (1 - m(d_1, d_2))\,R(d_1, d_2)\,.
\end{aligned}$$

With the same argumentation as above we can write $R(d_1, d_2)$ as

$$\begin{aligned}
R(d_1, d_2) &= Y_1\,d_1\,(1+u) + Y_2\,d_2\,(1+v) + Y_3\,\frac{(1+u)^4}{u} \\
&\quad + Y_4\,\frac{(1+v)^4}{v} + Y_5\,d_1^2\,d_2^2\,(1+u)^2\,(1+v)^2 + Y_6\,\frac{(1+u)^2}{u} \\
&\quad + Y_7\frac{(1+v)^2}{v} + Y_8\,\frac{d_1\,(1+u)^2}{u} + Y_9\,\frac{d_2\,(1+v)^2}{v} \\
&= d_1^2\,d_2^2\,(1+u)^2\,(1+v)^2\,\Big\{ \frac{Y_1}{d_1\,(1+u)\,d_2^2\,(1+v)^2} \\
&\quad + \frac{Y_2}{d_1^2\,(1+u)^2\,d_2\,(1+v)} + \frac{Y_3\,(1+u)^2}{u\,d_1^2\,d_2^2\,(1+v)^2} \\
&\quad + \frac{Y_4\,(1+v)^2}{v\,d_1^2\,d_2^2\,(1+u)^2} + Y_5 + \frac{Y_6}{u\,d_1^2\,d_2^2\,(1+v)^2} \\
&\quad + \frac{Y_7}{v\,d_1^2\,d_2^2\,(1+u)^2} + \frac{Y_8}{u\,d_1\,d_2^2\,(1+v)^2} + \frac{Y_9}{v\,d_1^2\,d_2\,(1+u)^2} \Big\} \\
&=: d_1^2\,d_2^2\,(1+u)^2\,(1+v)^2\,\hat{Y}
\end{aligned}$$

where $Y_1, \ldots, Y_9\,, \hat{Y}$ are real functions with finite limits and with

$$\lim_{d_2\to\infty} Y_1, \ldots, \lim_{d_2\to\infty} Y_5 > 0\,, \ \lim_{d_2\to\infty} Y_6, \ldots, \lim_{d_2\to\infty} Y_9 < 0 \ \text{ and } \ \lim_{d_2\to\infty} \hat{Y} > 0\,.$$

Then we obtain

$$
\begin{aligned}
&(1 - m(d_1, d_2))\, R(d_1, d_2) \\
&= d_1^2\, d_2^2\, (1+u)^2\, (1+v)^2\, \hat{Y}\, \frac{\sqrt{\left(\frac{u+1}{u-1}\right)^2 + 3\, \left(\frac{v+1}{v-1}\right)^2} - 2}{\sqrt{\left(\frac{u+1}{u-1}\right)^2 + 3\, \left(\frac{v+1}{v-1}\right)^2}}
\end{aligned}
$$

where

$$d_2^2\, (1+v)^2\, d_1^2\, (1+u)^2\, \hat{Y} \stackrel{d_2 \to \infty}{\longrightarrow} \infty$$

faster than

$$\sqrt{\left(\frac{u+1}{u-1}\right)^2 + 3\, \left(\frac{v+1}{v-1}\right)^2} \stackrel{d_2 \to \infty}{\longrightarrow} 2\,.$$

Since $\sqrt{\left(\frac{u+1}{u-1}\right)^2 + 3\, \left(\frac{v+1}{v-1}\right)^2} - 2 \geq 0$ we therefore obtain

$$f_2(d_1(d_2), d_2) \stackrel{d_2 \to \infty}{\longrightarrow} \infty\,.$$

iv) Let $\mu_1^2 < \mu_2^2$. Then i)b) delivers

$$\lim_{d_2 \to \hat{d}_2} d_1(d_2) = \infty\,.$$

This implies

$$
\begin{aligned}
m(d_1(d_2), d_2) &\stackrel{(2.137)}{=} \frac{2}{\sqrt{\left(\frac{u+1}{u-1}\right)^2 + 3\, \left(\frac{v+1}{v-1}\right)^2}} \\
&\stackrel{d_2 \to \hat{d}_2}{\longrightarrow} \frac{2}{\sqrt{1 + 3\, \left(\frac{\hat{v}+1}{\hat{v}-1}\right)^2}} < 1
\end{aligned}
$$

where $\hat{v} := v(\hat{d}_2)$. Further we have that

$$
\begin{aligned}
g_1(d_1) &= \frac{d_1}{2}\, (1+u) \stackrel{d_2 \to \hat{d}_2}{\longrightarrow} \infty\,, \\
h_1(d_2) &= \frac{d_2}{2}\, (1+v) \stackrel{d_2 \to \hat{d}_2}{\longrightarrow} \frac{\hat{d}_2}{2}\, (1+\hat{v})\,, \\
Z_1(d_1) &= \frac{\tilde{K}}{16}\, \frac{(1+u)^4}{u} \stackrel{d_2 \to \hat{d}_2}{\longrightarrow} \infty\,, \\
Z_2(d_2) &= \frac{\tilde{K}}{16}\, \frac{(1+v)^4}{v} \stackrel{d_2 \to \hat{d}_2}{\longrightarrow} \frac{\tilde{K}}{16}\, \frac{(1+\hat{v})^4}{\hat{v}}\,, \\
Z_4(d_1) &= -\frac{2\, \tilde{K} + \frac{3}{2}\, \tilde{k}\, p_1\, d_1}{4}\, \frac{(1+u)^2}{u} \stackrel{d_2 \to \hat{d}_2}{\longrightarrow} -\infty
\end{aligned}
$$

and

$$\begin{aligned} Z_5(d_2) &= -\frac{2\,\tilde K + \frac{3}{2}\,\tilde k\, p_2\, d_2}{4}\,\frac{(1+v)^2}{v} \\ &\overset{d_2\to\hat d_2}{\longrightarrow} -\frac{2\,\tilde K + \frac{3}{2}\,\tilde k\, p_2\, \hat d_2}{4}\,\frac{(1+\hat v)^2}{\hat v}\,. \end{aligned}$$

By definition of $\hat d_2$ we also have

$$\frac{\tilde k}{\tilde K}\,p_2\,\hat d_2^9 + \hat d_2^8 - \left(\frac{C_2}{C_1}\,\tilde K\right)^2 = 0$$

$$\iff \quad \hat d_2^8\,\hat v^2 = \left(\frac{C_2}{C_1}\,\tilde K\right)^2\,.$$

Due to $C_1 < 0$ we therefore obtain

$$\begin{aligned} \hat d_2^4\,\hat v &= -\frac{C_2}{C_1}\,\tilde K \\ &= \frac{12\,\tilde K}{\lambda^4\,a_2^2}\,\frac{\mu_2^4}{\mu_2^2-\mu_1^2}\,, \end{aligned}$$

thus implying

$$\begin{aligned} \lim_{d_2\to\hat d_2} l(d_2) &= \lim_{d_2\to\hat d_2}\left[\frac{\lambda^4\,a_1\,a_2}{2\,\mu_1^2} - 6\,\frac{\tilde K\,\mu_2^2\,a_1}{d_2^4\,v\,\mu_1^2\,a_2}\right] \\ &= \frac{\lambda^4\,a_1\,a_2}{2\,\mu_2^2} > 0\,. \end{aligned}$$

This then gives us

$$Z_3(d_1,d_2) \overset{d_2\to\hat d_2}{\longrightarrow} \infty\,.$$

In analogy to the proof of iii) ($\mu_1^2 \neq \mu_2^2$) we obtain

$$\lim_{d_2\to\hat d_2} f_2(d_1(d_2),d_2) = \infty\,.$$

v)+ vi) All functions which f_2 consists of are continuous in $d_1 \geq 0$ and $d_2 \geq 0$, respectively. Since d_1 is continuous if $\mu_1^2 \geq \mu_2^2$ and discontinuous only in $d_2 = \hat d_2$ if $\mu_1^2 < \mu_2^2$ we obtain the above assertion. □

Corollary 2. (Existence of a zero of (2.118).) *The results of Proposition 6 guarantee the existence of a positive solution* (d_1,d_2) *of the system*

$$f_1(d_1,d_2) = 0$$

$$f_2(d_1,d_2) = 0\,.$$

If $\mu_1^2 < \mu_2^2$, *we even know that at least one zero must be smaller than* $\hat{d}_2$.

Remark 15. The above derivation of $f_1(d_1, d_2) = 0$ offers two different formulae for the parameter C which coincide inside the no-transaction region. To define f_2 (which is derived from a condition on the boundary of the no-transaction region) we have chosen the one that depends on d_2: This choice delivers one (numerical) solution of (2.118) whereas a different choice of C (as a function of d_1) delivers two solutions of (2.118) if $2\,\mu_1^2 < \mu_2^2$. To avoid restrictions on the parameters we therefore choose $C = l(d_2)$.
Proof: Let $\bar{C} = l(d_1(d_2))$ be given by

$$\bar{C} = l(d_1(d_2)) := \frac{\lambda^4\, a_1\, a_2}{2\,\mu_2^2} - 6\,\frac{\tilde{K}}{d_1^4\, u}\,\frac{\mu_1^2\, a_2}{\mu_2^2\, a_1}.$$

Let $\bar{f}_2$ be the same as f_2 with C replaced by $\bar{C}$. From Proposition 6 i) we obtain

$$\bar{l} := \lim_{d_2 \to \infty} l(d_1(d_2)) = \begin{cases} \frac{\lambda^4\, a_1\, a_2}{2\,\mu_2^2} - 6\,\frac{\tilde{K}}{\hat{d}_1^4\, \hat{u}}\,\frac{\mu_1^2\, a_2}{\mu_2^2\, a_1} & \text{if} \quad \mu_1^2 \neq \mu_2^2\,, \\ \frac{\lambda^4\, a_1\, a_2}{2\,\mu_2^2} & \text{if} \quad \mu_1^2 = \mu_2^2\,. \end{cases}$$

Let us now focus on the case $\mu_1^2 \neq \mu_2^2$. Here we have

$$\bar{l} = \frac{\lambda^4\, a_1\, a_2}{2\,\mu_2^2} - 6\,\frac{\tilde{K}}{\hat{d}_1^4\, \hat{u}}\,\frac{\mu_1^2\, a_2}{\mu_2^2\, a_1}$$

$$\iff \quad \hat{d}_1^4\, \hat{u} = 12\,\tilde{K}\,\frac{\mu_1^2\, a_2}{a_1\,(\lambda^4\, a_1\, a_2 - 2\,\bar{l}\,\mu_2^2)}$$

where the left hand side is by definition of $\hat{d}_1$ equal to

$$\sqrt{\left(\frac{\tilde{K}}{C_1}\right)^2}.$$

1. Let $\mu_1^2 > \mu_2^2$. Since $C_1 > 0$ we obtain

$$\frac{\tilde{K}}{C_1} = 12\,\tilde{K}\,\frac{\mu_1^2\, a_2}{a_1\,(\lambda^4\, a_1\, a_2 - 2\,\bar{l}\,\mu_2^2)} \qquad \iff$$

$$\begin{aligned} \lambda^4\, a_1\, a_2 - 2\,\bar{l}\,\mu_2^2 \quad &= \quad C_1\,\frac{12\,\mu_1^2\, a_2}{a_1} \\ &\overset{(2.123)}{=} \quad \lambda^4\, a_1\, a_2\,\frac{\mu_1^2 - \mu_2^2}{\mu_1^2} \end{aligned}$$

$$\iff \quad \bar{l} = \frac{\lambda^4 \, a_1 \, a_2}{2 \, \mu_1^2} \, .$$

2. Let $\mu_1^2 < \mu_2^2$. Since $C_1 < 0$ we obtain

$$-\frac{\tilde{K}}{C_1} = 12 \, \tilde{K} \, \frac{\mu_1^2 \, a_2}{a_1 \, (\lambda^4 \, a_1 \, a_2 - 2 \, \bar{l} \, \mu_2^2)} \qquad \iff$$

$$\begin{aligned} -(\lambda^4 \, a_1 \, a_2 - 2 \, \bar{l} \, \mu_2^2) &= C_1 \, \frac{12 \, \mu_1^2 \, a_2}{a_1} \\ &= \lambda^4 \, a_1 \, a_2 \, \frac{\mu_1^2 - \mu_2^2}{\mu_1^2} \end{aligned}$$

$$\iff \quad \bar{l} = \frac{\lambda^4 \, a_1 \, a_2}{2 \, \mu_2^2} \, \frac{2 \, \mu_1^2 - \mu_2^2}{\mu_1^2} \, .$$

Here we have to distinguish between three different cases:
a) $2 \, \mu_1^2 > \mu_2^2$: Then $\bar{l} > 0$.
b) $2 \, \mu_1^2 < \mu_2^2$: Then $\bar{l} < 0$.
c) $2 \, \mu_1^2 = \mu_2^2$: Then $\bar{l} = 0$.

Summing up all the results we have found out we get

$$l(d_1(d_2)) \stackrel{d_2 \to \infty}{\longrightarrow} \begin{cases} \frac{\lambda^4 \, a_1 \, a_2}{2 \, \mu_1^2} & \text{if} \quad \mu_1^2 \geq \mu_2^2 \, , \\ \frac{\lambda^4 \, a_1 \, a_2}{2 \, \mu_2^2} \left(2 - \frac{\mu_2^2}{\mu_1^2}\right) > 0 & \text{if} \quad \mu_2^2 < 2 \, \mu_1^2 < 2 \, \mu_2^2 \, , \\ \frac{\lambda^4 \, a_1 \, a_2}{2 \, \mu_2^2} \left(2 - \frac{\mu_2^2}{\mu_1^2}\right) < 0 & \text{if} \quad 2 \, \mu_1^2 < \mu_2^2 \, . \end{cases}$$

Using the definition of Z_3 given in the proof of Proposition 6 ii) we therefore obtain

$$Z_3(d_1(d_2), d_2) \stackrel{d_2 \to \infty}{\longrightarrow} \begin{cases} \infty & \text{if} \quad 2 \, \mu_1^2 > \mu_2^2 \, , \\ -\infty & \text{if} \quad 2 \, \mu_1^2 < \mu_2^2 \, . \end{cases}$$

Since Z_3 is the term which determines the limit of $\bar{f}_2$ (compare with the proof of Proposition 6 iii)) we get that

$$\lim_{d_2 \to \infty} \bar{f}_2(d_1(d_2), d_2) = \begin{cases} \infty & \text{if} \quad 2 \, \mu_1^2 > \mu_2^2 \, , \\ -\infty & \text{if} \quad 2 \, \mu_1^2 < \mu_2^2 \, . \end{cases}$$

Further, if $2 \, \mu_1^2 < \mu_2^2$, Proposition 6 i)b) gives us that

$$d_1(d_2) \stackrel{d_2 \to \hat{d}_2}{\longrightarrow} \infty$$

thus implying

$$l(d_1(d_2)) \stackrel{d_2 \to \hat{d}_2}{\longrightarrow} \frac{\lambda^4 \, a_1 \, a_2}{2 \, \mu_2^2} > 0$$

and

$$Z_3(d_1(d_2), d_2) \overset{d_2 \to \hat{d}_2}{\longrightarrow} \infty\,.$$

Since the other parts of $\bar{f}_2$ (which are the same as in f_2) are not affected by the choice of C, we obtain in anlogy to the proof of Proposition 6 iv) that

$$\bar{f}_2(d_1(d_2), d_2) \overset{d_2 \to \hat{d}_2}{\longrightarrow} \infty\,.$$

Also in analogy to the proof of Proposition 6 ii) we obtain that $f_2(0,0) = -\frac{3}{2}\tilde{K}$. Therefore the equation $\bar{f}_2(d_1(d_2), d_2) = 0$ possesses at least two solutions $d_2 > 0$ if $2\,\mu_1^2 < \mu_2^2$. □

The solution (d_1, d_2) of (2.118) still depends on the prices (p_1, p_2) (and on ε). As in the cases of only proportional or only fixed transaction costs we are interested in a solution which is independent of the prices (and of ε). To find this solution we again consider the volume of stock holdings. The next proposition will give us boundaries and restarting lines which are independent of the prices (and of ε).

Proposition 7. (Independence of the prices.) *We define*

$$\begin{aligned} \tilde{d}_1 &:= \varepsilon^{1/4}\, p_1\, d_1\,, & (2.138)\\ \tilde{d}_2 &:= \varepsilon^{1/4}\, p_2\, d_2\,, & (2.139)\\ \tilde{\alpha} &:= \varepsilon^{1/4}\, p_1\, \alpha\,, & (2.140)\\ \tilde{\beta} &:= \varepsilon^{1/4}\, p_2\, \beta & (2.141) \end{aligned}$$

with d_1, d_2, α, β as above. Then both the no-transaction region and the restarting line corresponding to the wealth-combination $(S_1, S_2)'$ are given by ellipsoids which are independent of the prices (and ε): The boundary of NT consists of all points

$$(S_1, S_2)' = (S_1^*, S_2^*)' + (\tilde{S}_1, \tilde{S}_2)'$$

with

$$\frac{\tilde{S}_1^2}{\tilde{\alpha}^2} + \frac{\tilde{S}_2^2}{\tilde{\beta}^2} = 1 \tag{2.142}$$

whereas the restarting line consists of all points

$$(S_1, S_2)' = (S_1^*, S_2^*)' + (\tilde{S}_1, \tilde{S}_2)'$$

with

$$\frac{\tilde{S}_1^2}{(\tilde{\alpha} - \tilde{d}_1)^2} + \frac{\tilde{S}_2^2}{(\tilde{\beta} - \tilde{d}_2)^2} = 1 \tag{2.143}$$

where $(\tilde{d}_1, \tilde{d}_2)$ is the solution of

$$\tilde{f}_1(\tilde{d}_1, \tilde{d}_2) = 0\,, \quad \tilde{f}_2(\tilde{d}_1, \tilde{d}_2) = 0 \tag{2.144}$$

with

$$\tilde{f}_1(\tilde{d}_1,\tilde{d}_2) \;:=\; \frac{k}{K}\,\tilde{d}_1^9 + \tilde{d}_1^8 - \tilde{q}(\tilde{d}_2)\,, \tag{2.145}$$

$$\begin{aligned}\tilde{f}_2(\tilde{d}_1,\tilde{d}_2) \;:=\;& \lambda\,K + \lambda\,k\frac{1-\tilde{m}}{2}\left(\tilde{g}_1(\tilde{d}_1) + \sqrt{3}\tilde{h}_1(\tilde{d}_2)\right)\\ &-\frac{\tilde{m}^4-1}{16}\left(\tilde{g}_2(\tilde{d}_1)\tilde{g}_1(\tilde{d}_1)^4 + 9\,\tilde{h}_2(\tilde{d}_2)\tilde{h}_1(\tilde{d}_2)^4 + 3\,\tilde{l}(\tilde{d}_2)\tilde{g}_1(\tilde{d}_1)^2\tilde{h}_1(\tilde{d}_2)^2\right)\\ &-\frac{\tilde{m}^2-1}{4}\left(\tilde{g}_3(\tilde{d}_1)\tilde{g}_1(\tilde{d}_1)^2 + 3\,\tilde{h}_3(\tilde{d}_2)\tilde{h}_1(\tilde{d}_2)^2\right)\end{aligned} \tag{2.146}$$

where

$$\tilde{C}_1 \;:=\; \frac{C_1}{p_1^4} = \frac{\lambda^4}{12}\sigma_{11}^4\frac{\mu_1^2-\mu_2^2}{\mu_1^4}\,, \tag{2.147}$$

$$\tilde{C}_2 \;:=\; \frac{C_2\,p_2^4}{p_1^4} = \left(\frac{\mu_2\,\sigma_{11}}{\mu_1\,\sigma_{22}}\right)^4\,, \tag{2.148}$$

$$\tilde{q}(\tilde{d}_2) \;:=\; q(d_2)\,p_1^8\,\varepsilon^2 = \left(\frac{\tilde{d}_2^4\sqrt{1+\frac{k}{K}\tilde{d}_2}}{\tilde{C}_2 + \frac{\tilde{C}_1}{\lambda\,K}\tilde{d}_2^4\sqrt{1+\frac{k}{K}\tilde{d}_2}}\right)^2\,, \tag{2.149}$$

$$\tilde{g}_1(\tilde{d}_1) \;:=\; \tilde{\alpha} = \frac{\tilde{d}_1}{2}\left(1+\sqrt{1+\frac{k}{K}\,\tilde{d}_1}\right)\,, \tag{2.150}$$

$$\tilde{h}_1(\tilde{d}_2) \;:=\; \tilde{\beta} = \frac{\tilde{d}_2}{2}\left(1+\sqrt{1+\frac{k}{K}\,\tilde{d}_2}\right)\,, \tag{2.151}$$

$$\tilde{g}_2(\tilde{d}_1) \;:=\; \tilde{A} := \frac{A}{p_1^4} = \frac{\lambda\,K}{\tilde{d}_1^4\sqrt{1+\frac{k}{K}\tilde{d}_1}}\,, \tag{2.152}$$

$$\tilde{h}_2(\tilde{d}_2) \;:=\; \tilde{B} := \frac{B}{p_2^4} = \frac{\lambda\,K}{\tilde{d}_2^4\sqrt{1+\frac{k}{K}\tilde{d}_2}}\,, \tag{2.153}$$

$$\tilde{l}(\tilde{d}_2) \;:=\; \tilde{C} := \frac{C}{p_1^2\,p_2^2} = \frac{\lambda^4\,\sigma_{11}^2\,\sigma_{22}^2}{2\,\mu_1^2} - 6\frac{\lambda\,K}{\tilde{d}_2^4\sqrt{1+\frac{k}{K}\tilde{d}_2}}\left(\frac{\mu_2\,\sigma_{11}}{\mu_1\,\sigma_{22}}\right)^2\,, \tag{2.154}$$

$$\tilde{g}_3(\tilde{d}_1) \;:=\; \tilde{D} := \sqrt{\varepsilon}\,\frac{D}{p_1^2} = -\lambda\frac{2\,K + \frac{3}{2}k\,\tilde{d}_1}{\tilde{d}_1^2\sqrt{1+\frac{k}{K}\tilde{d}_1}}\,, \tag{2.155}$$

$$\tilde{h}_3(\tilde{d}_2) \;:=\; \tilde{E} := \sqrt{\varepsilon}\,\frac{E}{p_2^2} = -\lambda\frac{2\,K + \frac{3}{2}k\,\tilde{d}_2}{\tilde{d}_2^2\sqrt{1+\frac{k}{K}\tilde{d}_2}}\,, \tag{2.156}$$

$$\tilde{m} \;:=\; r = \frac{2\,(\tilde{\alpha}-\tilde{d}_1)\,(\tilde{\beta}-\tilde{d}_2)}{\sqrt{\tilde{\alpha}^2\,(\tilde{\beta}-\tilde{d}_2)^2 + 3\,\tilde{\beta}^2\,(\tilde{\alpha}-\tilde{d}_1)^2}}\,. \tag{2.157}$$

Obviously (2.142)-(2.157) are independent of the prices (p_1,p_2) *and of* ε *.*

Proof: Using the above definitions as well as (2.113)-(2.135) we obtain that

$$0 = f_1(d_1, d_2) \quad \Longleftrightarrow$$

$$0 \;=\; f_1(d_1, d_2)\, \varepsilon^2\, p_1^8 = \frac{k}{K}\,\tilde{d}_1^9 + \tilde{d}_1^8 - \tilde{q}(\tilde{d}_2) = \tilde{f}_1(\tilde{d}_1, \tilde{d}_2)$$

and

$$0 = f_2(d_1, d_2) \quad \Longleftrightarrow$$

$$\begin{aligned} 0 &= f_2(d_1, d_2)\, \varepsilon \\ &= \lambda\, K + \lambda\, k \frac{1-r}{2}(\tilde{\alpha} + \sqrt{3}\tilde{\beta}) \\ &\quad - \frac{r^4 - 1}{16}\left(\tilde{A}\tilde{\alpha}^4 + 9\,\tilde{B}\tilde{\beta}^4 + 3\,\tilde{C}\,\tilde{\alpha}^2\,\tilde{\beta}^2\right) \\ &\quad - \frac{r^2 - 1}{4}\left(\tilde{D}\,\tilde{\alpha}^2 + 3\,\tilde{E}\,\tilde{\beta}^2\right) \\ &= \tilde{f}_2(\tilde{d}_1, \tilde{d}_2)\,. \end{aligned}$$

We also obtain the shape of NT and of the restarting line with respect to the risky money $(S_1, S_2)'$. □

Remark 16. As in the case with only fixed transaction costs, the form of NT points out that a transaction typically involves rebalancing in both stocks. However, we do not do the biggest possible transaction in the case of both fixed and proportional transaction costs but we shift the pair $(S_1, S_2)' \in \partial NT$ to the corresponding pair of the restarting line.

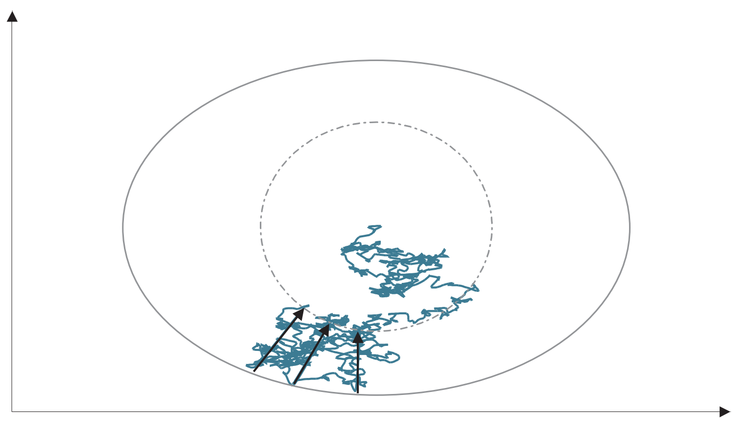

Fig. 6. **(Asymptotically) optimal risky money (4)**

As we stated in Remark 15, we always obtain a unique positive solution $(\tilde{d}_1, \tilde{d}_2)$ to the system $\tilde{f}_1(\tilde{d}_1, \tilde{d}_2) = 0$, $\tilde{f}_2(\tilde{d}_1, \tilde{d}_2) = 0$ if we use $\tilde{C} = \tilde{l}(\tilde{d}_2)$ to define $\tilde{f}_2$ (at least all the examples we calculated possess a unique solution). (Of course the results of Remark 15 also hold with respect to (S_1, S_2).) In the following we will describe how we proceed to obtain this solution:
To solve (2.144) numerically we determine an approximation of the vector $(\tilde{d}_1^*, \tilde{d}_2^*)$ solving $\tilde{f}_1(\tilde{d}_1^*, \tilde{d}_2^*) = 0$ and $\tilde{f}_2(\tilde{d}_1^*, \tilde{d}_2^*) = 0$. We fix N values of $\tilde{d}_2$ in the interval $[\delta, Z] \subseteq (0, \infty)$ as follows: $\tilde{d}_2^{i+1} = \tilde{d}_2^i + \Delta\tilde{d}_2$, $\Delta\tilde{d}_2 > 0$ fixed , $i = 1, \ldots, N$, where $\tilde{d}_2^1 = \delta$ and $\tilde{d}_2^N = Z$. For each fixed $\tilde{d}_2$ we solve $\tilde{f}_1(\tilde{d}_1) = 0$. Then we compute $\tilde{f}_2(\tilde{d}_1(\tilde{d}_2), \tilde{d}_2)$ for each $\tilde{d}_2$. If $[\delta, Z]$ contains the zero of $\tilde{f}_2$ and if this is unique, this procedure delivers the biggest $\tilde{d}_2$ with $\tilde{f}_2(\tilde{d}_1(\tilde{d}_2), \tilde{d}_2) < 0$ and the smallest $\tilde{d}_2$ with $\tilde{f}_2(\tilde{d}_1(\tilde{d}_2), \tilde{d}_2) \geq 0$. The mean of these two values then gives us an approximation of $\tilde{d}_2^*$. To obtain good approximations and to be sure that $\tilde{f}_2$ has exactly one zero in $(0, \infty)$ we start with $\delta \approx 0$ and $Z \approx 10000$. Knowing the rough position of the zero given by the first computation, we then take smaller intervals with $\delta \to \tilde{d}_2^*$ and $Z \to \tilde{d}_2^*$. The results given below to illustrate the quantitative behaviour of the approximative solution are calculated with $\Delta\tilde{d}_2 \leq 0.0001$. In each case $\tilde{f}_2(\tilde{d}_1(\tilde{d}_2), \tilde{d}_2)$ always possesses exactly one zero.

Examples 9.

a) $\mu_1 = \mu_2 = 0.02$, $\lambda = 0.001$, $k = 0.01$, $K = 0.1$.

σ_{11}	σ_{22}	S_1^*	S_2^*	$\tilde{\alpha}$	$\tilde{\beta}$	$\tilde{\alpha} - \tilde{d}_1$	$\tilde{\beta} - \tilde{d}_2$
0.4	0.5	125	80	87.263	67.239	36.805	26.026
0.4	0.4	125	125	87.762	87.762	37.082	37.082
0.5	0.4	80	125	68.424	88.817	26.645	37.667

The first and the last cases are exactly the same, only the roles of the stocks are interchanged. Due to the fact, that we compute the solution $(\tilde{d}_1^*, \tilde{d}_2^*)$ approximately, the obtained results differ slightly.

b) $\mu_1 = 0.03$, $\mu_2 = 0.02$, $\lambda = 0.001$, $k = 0.01$, $K = 0.1$.

σ_{11}	σ_{22}	S_1^*	S_2^*	$\tilde{\alpha}$	$\tilde{\beta}$	$\tilde{\alpha} - \tilde{d}_1$	$\tilde{\beta} - \tilde{d}_2$
0.3	0.4	333.33	125	153.996	80.892	76.26	33.305
0.4	0.3	187.5	222.22	112.396	119.7	51.136	55.435
0.3	0.3	333.33	222.22	155.902	77.445	116.122	53.322

c) Some examples with negative boundaries of the no-transaction region for at least one component of the risky money:
$\mu_1 = 0.04$, $\mu_2 = 0.02$, $\sigma_{11} = 0.4$, $\sigma_{22} = 0.15$, $\lambda = 0.1 \implies S_1^* = 2.5$, $S_2^* = 8.889$.

k	K	$\tilde{\alpha}$	$\tilde{\beta}$	$\tilde{\alpha} - \tilde{d}_1$	$\tilde{\beta} - \tilde{d}_2$
0.1	0.1	3.598	7.358	1.093	2.937
0.1	1	5.642	11.876	0.576	2.017

In the following we will analyse how the no-transaction region and the restarting line depend on the cost parameters. We will give the results for k fixed and various K and vice versa. Additionally, we will give the corresponding results for $k = 0$ and $K = 0$, respectively.

Examples 10.

a) $\mu_1 = 0.04$, $\mu_2 = 0.02$, $\sigma_{11} = 0.4$, $\sigma_{22} = 0.35$, $\lambda = 0.001 \implies S_1^* = 250$, $S_2^* = 163.265$. Let $K = 0.1$ be fixed.

k	$\tilde{\alpha}$	$\tilde{\beta}$	$\tilde{\alpha} - \tilde{d}_1$	$\tilde{\beta} - \tilde{d}_2$
0	95.951	87.753	0	0
0.00001	96.046	87.735	0.229	0.191
0.0001	96.88	87.589	2.191	1.802
0.001	103.452	87.076	16.105	12.1
0.01	132.962	94.44	63.37	40.817
0.1	220.55	142.046	167.665	103.197

Let $k = 0.001$ be fixed.

K	$\tilde{\alpha}$	$\tilde{\beta}$	$\tilde{\alpha} - \tilde{d}_1$	$\tilde{\beta} - \tilde{d}_2$
0	45.428	34.195	45.428	34.195
1E-12	41.202	25.673	41.184	25.659
1E-6	42.13	26.339	40.237	24.958
0.001	52.557	35.044	33.407	20.766
0.01	68.589	50.818	26.731	17.765
0.1	103.452	87.076	16.105	12.1
1	173.497	155.571	6.688	5.438
100	539.873	493.413	0.726	0.606

b) $\mu_1 = 0.04$, $\mu_2 = 0.02$, $\sigma_{11} = \sigma_{22} = 0.3$, $\lambda = 0.001$ $\Longrightarrow$
$S_1^* = 444.44$, $S_2^* = 222.22$. Let $K = 0.1$ be fixed.

k	$\tilde{\alpha}$	$\tilde{\beta}$	$\tilde{\alpha} - \tilde{d}_1$	$\tilde{\beta} - \tilde{d}_2$
0	127.934	102.378	0	0
0.00001	128.105	102.338	0.406	0.26
0.0001	129.58	102.015	3.836	2.421
0.001	140.527	100.681	26.427	15.401
0.01	185.279	109.199	96.005	49.272
0.1	313.873	166.966	246.254	123.443

Let $k = 0.001$ be fixed.

K	$\tilde{\alpha}$	$\tilde{\beta}$	$\tilde{\alpha} - \tilde{d}_1$	$\tilde{\beta} - \tilde{d}_2$
0	66.667	41.997	66.667	41.997
1E-12	59.684	30.637	59.66	30.621
1E-6	60.838	31.352	58.416	29.8
0.001	74.086	40.851	49.592	24.889
0.01	94.991	58.62	41.128	21.613
0.1	140.527	100.681	26.427	15.401
1	232.549	180.992	11.596	7.245
100	719.968	575.589	1.289	0.825

c) $\mu_1 = 0.04$, $\mu_2 = 0.02$, $\sigma_{11} = 0.35$, $\sigma_{22} = 0.4$, $\lambda = 0.001$ $\Longrightarrow$
$S_1^* = 326.531$, $S_2^* = 125$. Let $K = 0.1$ be fixed.

k	$\tilde{\alpha}$	$\tilde{\beta}$	$\tilde{\alpha} - \tilde{d}_1$	$\tilde{\beta} - \tilde{d}_2$
0	109.658	76.784	0	0
0.00001	109.784	76.751	0.299	0.146
0.0001	110.881	76.479	2.843	1.384
0.001	119.231	74.973	20.302	9.401
0.01	153.812	77.673	76.146	31.56
0.1	253.961	112.481	195.607	79.529

Let $k = 0.001$ be fixed.

K	$\tilde{\alpha}$	$\tilde{\beta}$	$\tilde{\alpha} - \tilde{d}_1$	$\tilde{\beta} - \tilde{d}_2$
0	54.281	28.618	54.281	28.618
1E-12	47.825	20.185	47.804	20.174
1E-6	48.785	20.702	46.696	19.528
0.001	60.591	28.136	39.379	15.976
0.01	79.416	42.374	32.503	13.785
0.1	119.231	74.973	20.302	9.401
1	198.767	135.631	8.644	4.186
100	617.052	431.683	0.948	0.464

d) $\mu_1 = \mu_2 = 0.02\,,\ \sigma_{11} = 0.3\,,\ \sigma_{22} = 0.4\,,\ \lambda = 0.001 \quad \Longrightarrow$
$S_1^* = 222.22\,,\ S_2^* = 125\,.$ Let $K = 0.1$ be fixed.

k	$\tilde{\alpha}$	$\tilde{\beta}$	$\tilde{\alpha} - \tilde{d}_1$	$\tilde{\beta} - \tilde{d}_2$
0	94.281	70.711	0	0
0.00001	94.342	70.736	0.221	0.124
0.0001	94.873	70.958	2.104	1.196
0.001	99.101	72.969	15.004	8.978
0.01	120.742	85.962	56.052	36.085
0.1	194.762	135.184	146.28	97.666

Let $k = 0.001$ be fixed.

K	$\tilde{\alpha}$	$\tilde{\beta}$	$\tilde{\alpha} - \tilde{d}_1$	$\tilde{\beta} - \tilde{d}_2$
0	41.997	28.618	41.997	28.618
1E-12	36.026	24.55	36.009	24.537
1E-6	36.879	25.212	35.149	23.872
0.001	46.861	32.875	29.232	19.247
0.01	63.096	45.307	23.886	15.144
0.1	99.101	72.969	15.004	8.978
1	169.485	126.519	6.398	3.664
100	530.427	397.754	0.701	0.394

The next remark will sum up the observations we obtain from the above examples:

Remark 17.

a) If the quotient $\frac{k}{K}$ is big, the parameters $\tilde{\alpha}\,,\tilde{\beta}\,,\tilde{\alpha} - \tilde{d}_1\,,\tilde{\beta} - \tilde{d}_2$ can vary extremely if $\Delta\tilde{d}_2$ is not small enough. In this case it is necessary to check whether the results are stable and if this is not the case to compute the results with $\Delta\tilde{d}_2$ chosen sufficiently small. The following example shows this instability: Let $\mu_1 = \mu_2 = 0.02$, $\sigma_{11} = 0.3$, $\sigma_{22} = 0.4$, $\lambda = 0.001$. If $k = 0.001$ and $K = 1E - 14$ we obtain in dependence on $\Delta\tilde{d}_2$ quite different results:

$\Delta\tilde{d}_2$	$\tilde{\alpha}$	$\tilde{\beta}$	$\tilde{\alpha} - \tilde{d}_1$	$\tilde{\beta} - \tilde{d}_2$
0.001	30.764	20.963	30.76	20.961
1E-6	36.024	24.547	36.02	24.544

b) If $K > 0$ is fixed, the results corresponding to $k = 0$ are the limits of the results corresponding to $k \to 0$. This is clear since in both cases the no-transaction region is of ellipsoid shape. In contrast to this, if $k > 0$ is fixed, the results corresponding to $K = 0$ are not the limits of the results corresponding to $K \to 0$. In the first case the no-transaction region is a rectangle whereas in the second case the no-transaction region is always an ellipsoid. Therefore we cannot obtain the first one as the limit of the second one. This shows that the cases $K = 0$ and $K > 0$ have to be treated with completely different mathematical tools. (As stated in section 2.1 the occurrence of fixed transaction costs implies that the optimal strategy in the case without fixed transaction costs leads to immediate ruin. Therefore we introduced the impulse control method to solve the optimization problem in the presence of fixed transaction costs.)

c) Let $K > 0$ be fixed, $k \geq 0$. Then we observe the following relations:

$$k \uparrow \Longrightarrow \tilde{\alpha} \uparrow , \tilde{\alpha} - \tilde{d}_1 \uparrow , \tilde{d}_1 \downarrow , \tilde{\beta} - \tilde{d}_2 \uparrow , \tilde{d}_2 \downarrow \ .$$

These observations correspond to those we have made in the one-dimensional case. If $\mu_1 > \mu_2$, an increase in k causes a decrease in $\tilde{\beta}$ if k is small and an increase in $\tilde{\beta}$ if k is big. If $\mu_1 = \mu_2$ an increase in k causes an increase in $\tilde{\beta}$.

d) Let $k > 0$ be fixed, $K > 0$. Then we observe the following relations:

$$K \uparrow \Longrightarrow \tilde{\alpha} \uparrow , \tilde{\alpha} - \tilde{d}_1 \downarrow , \tilde{d}_1 \uparrow , \tilde{\beta} \uparrow , \tilde{\beta} - \tilde{d}_2 \downarrow , \tilde{d}_2 \uparrow \ .$$

Here all observations correspond to those we have made in the one-dimensional case.

2.8 Conclusion

We have solved the optimization problem with transaction costs and exponential utility approximately. Since we assumed the presence of a fixed cost component we had to use the impulse control method which led to a discrete trading strategy which is of course much closer to reality than the one of the classical Merton approach (which leads to continuous strategies). Using the asymptotic analysis approach we obtained an approximative solution which consists of trading whenever the amount of money invested in the stock(s) reaches the boundary of the so-called no-transaction region which is in our cases given by an interval, a rectangle and an ellipsoid and which is independent of the stock price(s) and the time.

Driven by this result we also tried to solve the optimization problem under the assumption of constant boundaries. As this ansatz showed up to be much more complicated than the ansatz based on the impulse control method and as we were not able to compute a closed form solution, this unsuccessful attempt justified the ansatz via impulse control and asymptotic analysis.

Chapter 3

Bounded portfolio processes

Throughout the second chapter we restricted ourselves to the case of exponential utility. The outcome of this particular choice is that the optimal amount of money invested in the stock S^* is constant over time. Using the asymptotic analysis approach we then obtained an approximately optimal strategy which is characterized by rebalancing whenever the risky money $S(t)\,, t \in [0,T]$ reaches the boundary of the symmetrical interval $[\hat{S}^-, \hat{S}^+] = [S^* - \Delta S, S^* + \Delta S]\,, \Delta S > 0\,.$

Taking up this result we will now focus on those strategies which are characterized by rebalancing whenever the portfolio process (instead of the risky money) reaches the boundary of the symmetrical interval $[\pi^* - \varepsilon, \pi^* + \varepsilon] \subset [0,1]$ with $\pi^* \in (0,1)\,, \varepsilon > 0$ where π^* can be chosen as the optimal constant portfolio process of an investor with logarithmic or power utility. (Of course we would have liked to derive an at least approximative optimal strategy in the case of these utility functions and in the presence of transaction costs as we did in the second chapter for exponential utility. Unfortunately, we are not able to do that. Therefore we take the results of the second chapter as an inspiration but we will not claim any optimality of the type of strategy dealt with in this third chapter!)

In the following we will analyse the behaviour and the properties of the wealth process corresponding to a strategy with a bounded portfolio process: We will analyse its expected deviation from the optimal wealth process, its limit behaviour and the so-called efficiency which describes its quality compared with the optimal wealth process. Further we will analyse the intervention times and the transaction costs caused by this particular type of strategy.

3.1 The model

Again we consider a financial market consisting of a riskless bond with price B_t and a risky stock with price P_t given by

$$\begin{aligned} dB_t &= r\, B_t\, dt\,, \\ B_0 &= 1 \end{aligned}$$

and

$$\begin{aligned} dP_t &= P_t\,[\mu\,dt + \sigma\,dW_t]\,, \\ P_0 &= p_0 > 0 \end{aligned}$$

with real numbers $r\,,\mu$ and $\sigma > 0$ where W_t is a one-dimensional standard Brownian motion with the usual right-continuous Brownian filtration $\{F_t\}_{t\in[0,T]}$.

Throughout this chapter we will consider the case of an investor who does not want to trade continuously (for example in the presence of transaction costs or for any other reason). Since he tries to achieve a similar expected utility of his terminal wealth in $t = T$ as an investor with a constant portfolio process

$$\pi^* \in (0,1) \tag{3.1}$$

who rebalances his portfolio continuously, it should be reasonable for him to keep his portfolio process within a small interval around π^*. More precisely, we assume that he rebalances his portfolio whenever his portfolio process reaches the boundary of the interval

$$D := [\pi^* - \varepsilon, \pi^* + \varepsilon] \tag{3.2}$$

with $\varepsilon > 0$ such that $D \subset [0,1]$, and that he does not trade in between. Any transaction simply consists of shifting his portfolio process to π^*. Further we assume that the investor is allowed to start at $t = 0$ with any fraction of wealth which is an element of D and that he possesses an initial capital of 1.

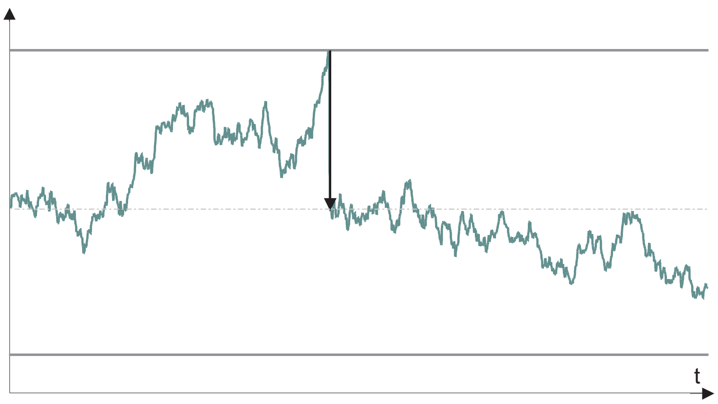

Fig. 7. Bounded portfolio process

To formalize these assumptions we will introduce the following notations:

Notation 1. (Portfolio process.) Let $\hat{\pi}_t\,, t \in [0,T]\,,\ \hat{\pi}_0 \in D$ denote the portfolio process of the investor who rebalances his portfolio whenever his portfolio process reaches the boundaries of D by shifting his portfolio process to π^* and who does not trade in between.

Notation 2. (Wealth process.) Let $(V_t^{\hat{\pi}})_{t\in[0,T]}$ denote the wealth process which corresponds to the portfolio process $(\hat{\pi}_t)_{t\in[0,T]}$ as given in Notation 1. Since it evolves according to the differential equation

$$\begin{aligned} dV_t^{\hat{\pi}} &= V_t^{\hat{\pi}}\left[(r + \hat{\pi}_t\,(\mu - r))\,dt + \hat{\pi}_t\,\sigma\,dW_t\right], \\ V_0^{\hat{\pi}} &= v = 1\,, \end{aligned}$$

(see Korn and Korn (1999), p. 71) it reads

$$V_t^{\hat{\pi}} = e^{rt+(\mu-r)\int_0^t \hat{\pi}_s\,ds - \frac{1}{2}\sigma^2\int_0^t \hat{\pi}_s^2\,ds + \sigma\int_0^t \hat{\pi}_s\,dW_s}\,. \tag{3.3}$$

Remark 1. For any portfolio process $(\pi_t)_{t\in[0,T]}$ the corresponding wealth process $(X_t^{\pi})_{t\in[0,T]}$ with $X_0^{\pi} = 1$ is given by

$$X_t^{\pi} = e^{rt+(\mu-r)\int_0^t \pi_s\,ds - \frac{1}{2}\sigma^2\int_0^t \pi_s^2\,ds + \sigma\int_0^t \pi_s\,dW_s}\,. \tag{3.4}$$

For the special choice of $\pi_t \equiv \pi^*\ \forall\ t \in [0,T]$ the corresponding wealth process reads

$$X_T^{\pi^*} = e^{\left[r+(\mu-r)\pi^* - \frac{1}{2}\sigma^2\pi^{*2}\right]T + \sigma\,\pi^*\,W_T}\,. \tag{3.5}$$

Remark 2. Usually π^* will be chosen as the optimal constant fraction of wealth invested in the stock that solves the optimization problem

$$\max_{\pi} E(U(X_T^{\pi}))$$

where X_T^{π} denotes the terminal wealth corresponding to π and U denotes the logarithmic or power utility function which are given by

$$U_{\ln}(x) = \ln x \quad , x > 0\,, \tag{3.6}$$

$$U_{\gamma}(x) = \frac{1}{\gamma}x^{\gamma} \quad , x > 0\,, \gamma \in (0,1)\,. \tag{3.7}$$

To close this section we will derive the differential equation describing the evolution of $\hat{\pi}_t$ in periods without interventions. Then, $\hat{\pi}_t$ evolves continuously since it is only influenced by the movements of the Brownian motion that drives the stock price.

Proposition 1. (Portfolio process on the intervals without interventions.) *As long as the investor does not intervene, his portfolio process evolves according to the differential equation*

$$\begin{aligned} d\pi_t &= \pi_t\,[1-\pi_t]\,[(\mu - r - \pi_t\,\sigma^2)\,dt + \sigma\,dW_t]\,, & (3.8)\\ \pi_0 &\in (\pi^* - \varepsilon, \pi^* + \varepsilon)\,. & (3.9)\end{aligned}$$

Proof: Let X_t denote the wealth process belonging to the portfolio process π_t, let $S_t := \pi_t\,X_t$ be the amount of money invested in the stock which develops according to (2.3). Then we have

$$\pi_t = \frac{S_t}{X_t} =: f(S_t, X_t)$$

with

$$dS_t = S_t\,(\mu\,dt + \sigma\,dW_t)$$

and

$$dX_t = dS_t + dB_t = dS_t + r\,B_t\,dt\,.$$

Using Itô's rule we obtain

$$\begin{aligned} d\pi_t &= f_S\,dS_t + f_X\,dX_t \\ &\quad + \frac{1}{2}(f_{SS}\,d{<}S{>}_t + f_{XX}\,d{<}X{>}_t + 2f_{SX}\,d{<}X,S{>}_t)\end{aligned}$$

where

$$\begin{aligned} f_S &= \frac{1}{X}\,,\\ f_X &= -\frac{S}{X^2}\,,\\ f_{SS} &= 0\,,\\ f_{XX} &= \frac{2S}{X^3}\,,\\ f_{SX} &= -\frac{1}{X^2}\,.\end{aligned}$$

Due to

$$d{<}X{>}_t = d{<}S{>}_t = d{<}S,X{>}_t$$

we derive

$$\begin{aligned} d\pi_t &= \frac{1}{X_t}\, dS_t - \frac{S_t}{X_t^2}\,(dS_t + r\, B_t\, dt) + \frac{1}{2}\left(\frac{2S_t}{X_t^3}\, d{<}S{>}_t - \frac{2}{X_t^2}\, d{<}S{>}_t\right) \\ &= \frac{1}{X_t}\left(1 - \frac{S_t}{X_t}\right) dS_t - r\,\frac{S_t}{X_t}\,\frac{B_t}{X_t}\, dt + \frac{1}{X_t^2}\left(\frac{S_t}{X_t} - 1\right) S_t^2\, \sigma^2\, dt \\ &= (1-\pi_t)\,\pi_t\,(\mu\, dt + \sigma\, dW_t) - r\,\pi_t\,(1-\pi_t)\, dt + \pi_t^2\,(\pi_t - 1)\,\sigma^2\, dt \\ &= \pi_t\,(1-\pi_t)\,[\,(\mu - r - \pi_t\,\sigma^2)\, dt + \sigma\, dW_t\,]\,. \end{aligned}$$

□

It is easy to derive a multi-dimensional version of (3.8)/(3.9), but we will always restrict ourselves to the one-dimensional market model.

3.2 Comparison of wealth processes

Throughout this section we will compare the wealth process $V_t^{\hat{\pi}}$ given in (3.3) with the wealth process $X_t^{\pi^*}$ as given in (3.5). Additionally we will also compare $V_t^{\hat{\pi}}$ with the wealth process corresponding to a pure bond investment.

3.2.1 Convergence towards and deviations from a wealth process with constant portfolio process

By assumption the portfolio process $\hat{\pi}_t$ satisfies $\hat{\pi}_t \in [\pi^* - \varepsilon, \pi^* + \varepsilon]\ \forall\, t \in [0,T]$. Hence, the corresponding wealth process $V_t^{\hat{\pi}}$ depends on the choice of ε. Of course we expect the wealth process to converge somehow to the wealth process $X_t^{\pi^*}$ of an investor with constant portfolio process π^* if ε tends to 0. The next proposition shows that this is true with regard to convergence in probability. Later on we will show that this is also true with respect to the expectation and the expected utilities in the case of logarithmic and power utility as given by (3.6)/(3.7). Deriving the latter results we will also obtain some upper bounds for the deviations of $V_T^{\hat{\pi}}$ from $X_T^{\pi^*}$ with regard to the expectation and the expected utilities.

Proposition 2. *Let $(\hat{\pi}_t)_{t\in[0,T]}$ be given as in Notation 1. Then the corresponding wealth process $V_t^{\hat{\pi}} = V_t^{\hat{\pi}}(\varepsilon)\,, t \in [0,T]$ satisfies*

$$V_T^{\hat{\pi}}(\varepsilon) \xrightarrow{\varepsilon \downarrow 0} X_T^{\pi^*}$$

in probability.

Proof: We start defining

$$A_\varepsilon \;:=\; \int_0^T \hat{\pi}_s\, ds\,,$$

$$\begin{aligned} B_\varepsilon &:= \int_0^T \hat{\pi}_s^2 \, ds \,, \\ C_\varepsilon &:= \int_0^T \hat{\pi}_s \, dW_s \,. \end{aligned}$$

Due to $\hat{\pi}_s \in [\pi^* - \varepsilon, \pi^* + \varepsilon] \quad \forall s \in [0,T]$, we obtain

$$A_\varepsilon \in [(\pi^* - \varepsilon)T, (\pi^* + \varepsilon)T]$$

and

$$B_\varepsilon \in [(\pi^* - \varepsilon)^2 T, (\pi^* + \varepsilon)^2 T]$$

which imply the almost sure convergences

$$A_\varepsilon \xrightarrow{\varepsilon \downarrow 0} \pi^* T$$

and

$$B_\varepsilon \xrightarrow{\varepsilon \downarrow 0} \pi^{*2} T \,.$$

Furthermore we can show the L^2-convergence of C_ε to $\pi^* W_T$: Due to the Itô isometry we obtain

$$\begin{aligned} E((C_\varepsilon - \pi^* W_T)^2) &= E\left(\left(\int_0^T (\hat{\pi}_s - \pi^*) \, dW_s\right)^2\right) \\ &= E\left(\int_0^T (\hat{\pi}_s - \pi^*)^2 \, ds\right) \in [0, \varepsilon^2 T] \,. \end{aligned}$$

Therefore we have $E\left((C_\varepsilon - \pi^* W_T)^2\right) \xrightarrow{\varepsilon \downarrow 0} 0$ and $C_\varepsilon \xrightarrow{\varepsilon \downarrow 0} \pi^* W_T$ in L^2 .

Since almost-sure-convergence and L^2-convergence imply convergence in probability, we get

$$\begin{aligned} A_\varepsilon &\xrightarrow{\varepsilon \downarrow 0} \pi^* T \,, \\ B_\varepsilon &\xrightarrow{\varepsilon \downarrow 0} \pi^{*2} T \,, \\ C_\varepsilon &\xrightarrow{\varepsilon \downarrow 0} \pi^* W_T \end{aligned}$$

in probability. This gives us

$$\begin{aligned} (\mu - r) \, A_\varepsilon &\xrightarrow{\varepsilon \downarrow 0} (\mu - r) \, \pi^* T \,, \\ -\frac{1}{2}\sigma^2 \, B_\varepsilon &\xrightarrow{\varepsilon \downarrow 0} -\frac{1}{2}\sigma^2 \, \pi^{*2} T \,, \\ \sigma \, C_\varepsilon &\xrightarrow{\varepsilon \downarrow 0} \sigma \, \pi^* W_T \end{aligned}$$

in probability as well as

$$(\mu - r)A_\varepsilon - \frac{1}{2}\sigma^2 B_\varepsilon + \sigma C_\varepsilon \xrightarrow{\varepsilon \downarrow 0} \left[(\mu - r)\pi^* - \frac{1}{2}\sigma^2 \pi^{*2}\right] T + \sigma \pi^* W_T$$

in probability. Due to the continuity of the exponential function we finally obtain

$$V_T^{\hat{\pi}} \xrightarrow{\varepsilon\downarrow 0} X_T^{\pi^*}$$

in probability. □

In order to show the convergence of the expectation and the expected utilities of the wealth process $V_T^{\hat{\pi}}$ towards those of the wealth process $X_T^{\pi^*}$ we will at first compute each of these values individually. From Proposition 3 we will obtain the results in question in the case of a constant portfolio process.

Proposition 3. *The wealth process corresponding to* $\pi_s = \pi^* \quad \forall\, s \in [0,T]$ *satisfies*

i)

$$E(X_T^{\pi^*}) \;=\; e^{(r+(\mu-r)\pi^*)\,T}\,, \tag{3.10}$$

ii)

$$E(\ln X_T^{\pi^*}) \;=\; \left[r+(\mu-r)\pi^* - \frac{1}{2}\sigma^2\pi^{*2}\right] T\,, \tag{3.11}$$

iii)

$$E\left(\frac{1}{\gamma}\left(X_T^{\pi^*}\right)^{\gamma}\right) \;=\; \frac{1}{\gamma}e^{\gamma\left[r+(\mu-r)\pi^*-\frac{1}{2}\sigma^2\pi^{*2}\,(1-\gamma)\right]\,T} \quad ,\gamma \in (0,1)\,. \tag{3.12}$$

Proof: Whereas the first two expectations are straight forward we obtain the last expectation due to

$$\begin{aligned}(X_T^{\pi^*})^{\gamma} \;&=\; e^{\gamma\left[r+(\mu-r)\pi^*-\frac{1}{2}\sigma^2\pi^{*2}\right]\,T+\frac{1}{2}\gamma^2\sigma^2\pi^{*2}\,T} \\ &\quad\; e^{-\frac{1}{2}\gamma^2\sigma^2\pi^{*2}\,T+\gamma\,\sigma\,\pi^*W_T}\end{aligned}$$

where

$$e^{-\frac{1}{2}\gamma^2\sigma^2\pi^{*2}\,T+\gamma\,\sigma\,\pi^*W_T}$$

is a martingale with expectation 1. □

As mentioned above we are interested in the results of Proposition 3 with regard to the optimal constant portfolio process belonging to logarithmic and power utility, respectively. The next notation introduces the corresponding portfolio processes:

Notation 3. Let

$$\pi_1^* := \frac{\mu - r}{\sigma^2} \tag{3.13}$$

denote the optimal fraction of wealth with respect to the logarithmic utility function (3.6), let

$$\pi_2^* := \frac{\mu - r}{\sigma^2(1-\gamma)} \quad , \gamma \in (0,1) \tag{3.14}$$

denote the optimal fraction of wealth with respect to a power utility function as given in (3.7).

Using the definition of π_1^* and π_2^* we obtain the expectations in question:

Corollary 1. *Let π_1^*, π_2^* be given as in (3.13)/(3.14). Then we have*

i)

$$\begin{aligned} E(X_T^{\pi_1^*}) &= e^{(r+\sigma^2 \pi_1^{*2})\,T} && (3.15)\\ &= e^{\left(r+\frac{(\mu-r)^2}{\sigma^2}\right)T}, && (3.16) \end{aligned}$$

ii)

$$\begin{aligned} E(X_T^{\pi_2^*}) &= e^{(r+(1-\gamma)\sigma^2 \pi_2^{*2})\,T} && (3.17)\\ &= e^{\left(r+\frac{(\mu-r)^2}{\sigma^2(1-\gamma)}\right)T}, && (3.18) \end{aligned}$$

iii)

$$\begin{aligned} E(\ln X_T^{\pi_1^*}) &= \left(r + \frac{1}{2}\sigma^2 \pi_1^{*2}\right) T && (3.19)\\ &= \left(r + \frac{1}{2}\frac{(\mu-r)^2}{\sigma^2}\right) T\,, && (3.20) \end{aligned}$$

iv)

$$\begin{aligned} E\left(\frac{1}{\gamma}\left(X_T^{\pi_2^*}\right)^\gamma\right) &= \frac{1}{\gamma} e^{\gamma\left(r+\frac{1}{2}(1-\gamma)\sigma^2\pi_2^{*2}\right)T} && (3.21)\\ &= \frac{1}{\gamma} e^{\gamma\left(r+\frac{1}{2}\frac{(\mu-r)^2}{\sigma^2(1-\gamma)}\right)T}. && (3.22) \end{aligned}$$

Remark 3. Due to the above definitions of π_1^* and π_2^* the last two expectations given by (3.19)/(3.20) and (3.21)/(3.22), respectively, are the optimal ones with respect to the particular utility function.

For the non-constant portfolio strategy $\hat{\pi}_t$ it is not so easy to have closed form expressions such as the ones given in Proposition 3 and Corollary 1, respectively. We can however give bounds in the forms of intervals:

Proposition 4. *Let $V_t^{\hat{\pi}}, t \in [0,T]$ be given as in Notation 2. Then $V_t^{\hat{\pi}}$ satisfies*

i)
$$E(V_T^{\hat{\pi}}) \in \left[e^{(r+(\mu-r)(\pi^*-\varepsilon))T}, e^{(r+(\mu-r)(\pi^*+\varepsilon))T}\right], \tag{3.23}$$

ii)
$$E(\ln V_T^{\hat{\pi}}) \in [(r+g^-)T, (r+g^+)T] \tag{3.24}$$

where
$$g^- = \left[(\mu-r)\pi^* - \frac{1}{2}\sigma^2(\pi^{*2}+\varepsilon^2)\right] - \varepsilon\left|\pi^*\sigma^2 - (\mu-r)\right| \tag{3.25}$$

and
$$g^+ = \frac{1}{2}\frac{(\mu-r)^2}{\sigma^2}, \tag{3.26}$$

iii)
$$E\left(\frac{1}{\gamma}\left(V_T^{\hat{\pi}}\right)^\gamma\right) \in \left[\frac{1}{\gamma}e^{\gamma(r+h^-)T}, \frac{1}{\gamma}e^{\gamma(r+h^+)T}\right] \quad , \gamma \in (0,1) \tag{3.27}$$

where
$$h^+ = \frac{1}{2}\frac{(\mu-r)^2}{(1-\gamma)\sigma^2} \tag{3.28}$$

and
$$h^- = (\mu-r)\pi^* - \frac{1}{2}(1-\gamma)\sigma^2(\pi^{*2}+\varepsilon^2) - \varepsilon\left|(\mu-r) - (1-\gamma)\sigma^2\pi^*\right|. \tag{3.29}$$

Proof: To prove i) we use that

$$e^{-\frac{1}{2}\sigma^2\int_0^T \hat{\pi}_s^2\,ds + \sigma\int_0^T \hat{\pi}_s\,dW_s}$$

is a martingale with expectation 1 since $\hat{\pi}$ is bounded. Immediately this implies

$$E(V_T^{\hat{\pi}}) \in \left[e^{(r+(\mu-r)(\pi^*-\varepsilon))T}, e^{(r+(\mu-r)(\pi^*+\varepsilon))T}\right].$$

Proof of ii): As $\hat{\pi}$ is bounded we have that

$$\begin{aligned} E(\ln V_T^{\hat{\pi}}) &= rT + E\left(\int_0^T \left[(\mu-r)\hat{\pi}_s - \frac{1}{2}\sigma^2\hat{\pi}_s^2\right] ds\right) \\ &=: rT + E\left(\int_0^T g(\hat{\pi}_s)\,ds\right). \end{aligned}$$

Since

$$g'(\hat{\pi}) = (\mu - r) - \sigma^2 \hat{\pi} = 0 \quad \Longleftrightarrow \quad \hat{\pi} = \frac{\mu - r}{\sigma^2}$$

where

$$g''(\hat{\pi}) = -\sigma^2 < 0\,,$$

we obtain

$$g(\hat{\pi}) \leq g\left(\frac{\mu - r}{\sigma^2}\right) = \frac{1}{2}\,\frac{(\mu - r)^2}{\sigma^2} =: g^+\,.$$

Further it is clear that

$$g(\hat{\pi}) \geq \min\left(g(\pi^* - \varepsilon), g(\pi^* + \varepsilon)\right) =: g^-$$

with

$$\begin{aligned} g(\pi^* - \varepsilon) &= (\mu - r)(\pi^* - \varepsilon) - \frac{1}{2}\sigma^2(\pi^* - \varepsilon)^2 \\ &= \left[(\mu - r)\pi^* - \frac{1}{2}\sigma^2(\pi^{*2} + \varepsilon^2)\right] + \varepsilon\left[\pi^*\sigma^2 - (\mu - r)\right] \\ &=: \hat{g} + \varepsilon\left[\pi^*\sigma^2 - (\mu - r)\right] \end{aligned}$$

and analogously

$$g(\pi^* + \varepsilon) = \hat{g} - \varepsilon\left[\pi^*\sigma^2 - (\mu - r)\right]\,.$$

Therefore we obtain

$$\begin{aligned} g^- &= \hat{g} - \varepsilon\left|\,\pi^*\sigma^2 - (\mu - r)\,\right| \\ &= \left[(\mu - r)\pi^* - \frac{1}{2}\sigma^2(\pi^{*2} + \varepsilon^2)\right] - \varepsilon\left|\,\pi^*\sigma^2 - (\mu - r)\,\right| \end{aligned}$$

which implies

$$E(\ln V_T^{\hat{\pi}}) \in \left[(r + g^-)T, (r + g^+)T\right]$$

with g^-, g^+ as calculated above.
Proof of iii): Here we use the equality

$$\begin{aligned} \left(V_T^{\hat{\pi}}\right)^\gamma &= e^{\gamma\left(rT + \int_0^T \left[\,(\mu - r)\hat{\pi}_s - \frac{1}{2}\sigma^2\hat{\pi}_s^2\,\right] ds\right)} e^{\frac{1}{2}\gamma^2\sigma^2\int_0^T \hat{\pi}_s^2\, ds} \\ &\quad e^{\gamma\,\sigma\int_0^T \hat{\pi}_s\, dW_s - \frac{1}{2}\gamma^2\sigma^2\int_0^T \hat{\pi}_s^2\, ds} \\ &= e^{\gamma\, rT + \gamma\int_0^T \left[\,(\mu - r)\hat{\pi}_s - \frac{1}{2}\sigma^2\hat{\pi}_s^2(1 - \gamma)\,\right] ds} \\ &\quad e^{\gamma\,\sigma\int_0^T \hat{\pi}_s\, dW_s - \frac{1}{2}\gamma^2\sigma^2\int_0^T \hat{\pi}_s^2\, ds} \\ &=: e^{\gamma\, rT + \gamma\int_0^T h(\hat{\pi}_s)\, ds} e^{\gamma\sigma\int_0^T \hat{\pi}_s\, dW_s - \frac{1}{2}\gamma^2\sigma^2\int_0^T \hat{\pi}_s^2\, ds}\,. \end{aligned}$$

Again

$$e^{\gamma\sigma\int_0^T \hat{\pi}_s\, dW_s - \frac{1}{2}\gamma^2\sigma^2\int_0^T \hat{\pi}_s^2\, ds}$$

is a martingale with expectation 1 due to the fact that $\hat{\pi}$ is bounded. Analogously to the results in the proof of ii) we obtain that

$$h(\hat{\pi}) \in [h^-, h^+]$$

where

$$h^+ := h\left(\frac{\mu - r}{(1-\gamma)\sigma^2}\right) = \frac{1}{2}\frac{(\mu - r)^2}{(1-\gamma)\sigma^2}$$

and

$$\begin{aligned} h^- &:= \min(h(\pi^* - \varepsilon), h(\pi^* + \varepsilon)) \\ &= (\mu - r)\pi^* - \frac{1}{2}(1-\gamma)\sigma^2(\pi^{*2} + \varepsilon^2) - \varepsilon \left| (\mu - r) - (1-\gamma)\sigma^2\pi^* \right| . \end{aligned}$$

Therefore we have that

$$E\left(\frac{1}{\gamma}\left(V_T^{\hat{\pi}}\right)^\gamma\right) \in \left[\frac{1}{\gamma}e^{\gamma(r+h^-)T}, \frac{1}{\gamma}e^{\gamma(r+h^+)T}\right]$$

with h^- and h^+ as above. $\square$

Corollary 2. *Let π_1^* and π_2^*, respectively, be defined as above ((3.13)/(3.14)). Then we obtain*

i)

$$E(V_T^{\hat{\pi}}) \in \left[e^{(r+\sigma^2\pi_1^*(\pi_1^*-\varepsilon))T}, e^{(r+\sigma^2\pi_1^*(\pi_1^*+\varepsilon))T}\right] \tag{3.30}$$

and

$$E(V_T^{\hat{\pi}}) \in \left[e^{(r+(1-\gamma)\sigma^2\pi_2^*(\pi_2^*-\varepsilon))T}, e^{(r+(1-\gamma)\sigma^2\pi_2^*(\pi_2^*+\varepsilon))T}\right], \tag{3.31}$$

ii)

$$E(\ln V_T^{\hat{\pi}}) \in \left[\left(r + \frac{1}{2}\sigma^2(\pi_1^{*2} - \varepsilon^2)\right)T, \left(r + \frac{1}{2}\sigma^2\pi_1^{*2}\right)T\right], \tag{3.32}$$

iii)

$$E\left(\frac{1}{\gamma}\left(V_T^{\hat{\pi}}\right)^\gamma\right) \in \left[\frac{1}{\gamma}e^{\gamma\left(r+\frac{1}{2}(1-\gamma)\sigma^2(\pi_2^{*2}-\varepsilon^2)\right)T}, \frac{1}{\gamma}e^{\gamma\left(r+\frac{1}{2}(1-\gamma)\sigma^2\pi_2^{*2}\right)T}\right]. \tag{3.33}$$

Using the results of Propositions 3 and 4 we can easily compute upper bounds for the deviations of $V_T^{\hat{\pi}}$ from the wealth process $X_T^{\pi^*}$ with regard to expectation and expected utility:

Corollary 3. *Considering the two wealth processes $V_T^{\hat{\pi}}$ and $X_T^{\pi^*}$ we obtain that the deviations of their expectations and those of their expected utilities are bounded from above as follows:*

i)
$$E(X_T^{\pi^*} - V_T^{\hat{\pi}}) \leq e^{(r+(\mu-r)\pi^*)T} \left(1 - e^{-(\mu-r)\,\varepsilon\, T}\right), \tag{3.34}$$

ii)
$$E(\ln X_T^{\pi^*} - \ln V_T^{\hat{\pi}}) \leq \left(\frac{1}{2}\sigma^2\varepsilon^2 + \varepsilon|\,\pi^*\sigma^2 - (\mu - r)\,|\right)T, \tag{3.35}$$

iii)
$$E\left(\frac{1}{\gamma}\left(X_T^{\pi^*}\right)^\gamma - \frac{1}{\gamma}\left(V_T^{\hat{\pi}}\right)^\gamma\right) \leq \tag{3.36}$$
$$\frac{1}{\gamma}e^{\gamma\left[r+(\mu-r)\pi^* - \frac{1}{2}(1-\gamma)\sigma^2\pi^{*2}\right]T}\left(1 - e^{-\gamma\left[\,\frac{1}{2}(1-\gamma)\sigma^2\varepsilon^2 + \varepsilon\,|(\mu-r)-(1-\gamma)\sigma^2\pi^*|\,\right]T}\right).$$

With the results of Corollary 1 as well as the bounds given in Corollary 2 we also obtain upper bounds for the deviations in question for the special cases of $\pi^* = \pi_1^*$ and $\pi^* = \pi_2^*$, respectively:

Corollary 4. *Let $\pi^* = \pi_1^*$ and $\pi^* = \pi_2^*$, respectively. Then we obtain*

i)
$$E(X_T^{\pi_1^*} - V_T^{\hat{\pi}}) \leq e^{(r+\sigma^2\pi_1^{*2})T}\left(1 - e^{-\sigma^2\,\pi_1^*\,\varepsilon\,T}\right) \tag{3.37}$$

and

$$E(X_T^{\pi_2^*} - V_T^{\hat{\pi}}) \leq e^{(r+(1-\gamma)\,\sigma^2\pi_2^{*2})T}\left(1 - e^{-(1-\gamma)\,\sigma^2\,\pi_2^*\,\varepsilon\,T}\right), \tag{3.38}$$

ii)
$$E(\ln X_T^{\pi_1^*} - \ln V_T^{\hat{\pi}}) \leq \frac{1}{2}\sigma^2\varepsilon^2 T, \tag{3.39}$$

iii)
$$E\left(\frac{1}{\gamma}\left(X_T^{\pi_2^*}\right)^\gamma - \frac{1}{\gamma}\left(V_T^{\hat{\pi}}\right)^\gamma\right) \leq$$
$$\frac{1}{\gamma}e^{\gamma\left[r+\frac{1}{2}(1-\gamma)\sigma^2\pi_2^{*2}\right]T}\left(1 - e^{-\frac{1}{2}\,\gamma(1-\gamma)\sigma^2\varepsilon^2 T}\right). \tag{3.40}$$

Using some of the foregoing results we will now show the convergence of the expectations corresponding to the discrete strategy $\hat{\pi}_t$ towards the expectations corresponding to the continuous strategy π^*.

Proposition 5. *Let $\hat{\pi}_t\,, t \in [0,T]$ be given as in Notation 1. Let $\pi_1^*\,, \pi_2^*$ be given as in (3.13)/(3.14). Then we obtain that*

i)

$$E(V_T^{\hat{\pi}}) \overset{\varepsilon\downarrow 0}{\longrightarrow} E(X_T^{\pi^*})\,,$$

ii)

$$E(\ln(V_T^{\hat{\pi}})) \overset{\varepsilon\downarrow 0}{\longrightarrow} E(\ln(X_T^{\pi_1^*}))\,,$$

iii)

$$E\left(\frac{1}{\gamma}(V_T^{\hat{\pi}})^\gamma\right) \overset{\varepsilon\downarrow 0}{\longrightarrow} E\left(\frac{1}{\gamma}(X_T^{\pi_2^*})^\gamma\right)\,.$$

Proof: The above statements follow directly from (3.10)/(3.23), (3.19)/(3.32), (3.21)/(3.33). □

In the following subsection we will use the results on the expected utilities ((3.24)-(3.29)) to derive the so-called efficiency.

3.2.2 Efficiency

The aim of this subsection is to compare the expected utility of the terminal wealth $V_T^{\hat{\pi}}$ corresponding to the bounded portfolio process $\hat{\pi}_t$ with that one of $X_T^{\pi^*}$, the terminal wealth corresponding to the constant portfolio process π^*. As above, we examine the cases of the logarithmic and the power utility functions. Therefore the constant portfolio processes which are the optimal ones in each case are given by π_1^* (3.13) and π_2^* (3.14), respectively. Now our aim is to compute the initial capital $X_0^{\pi^*}$ of the optimally rebalancing investor which is needed to achieve exactly the same expected utility as the discretely trading investor who starts with an initial capital of 1.

The quotient of the initial capitals measures how good the discrete rebalancing is and is therefore called the efficiency of the discrete strategy. As we do not know the distribution of the portfolio process $\hat{\pi}_t$ we will use the lower bounds for the corresponding expected utilities we have derived in 3.2.1. Hence we will only get lower bounds for the efficiency, too.

Definition 1. (Efficiency.) Let $V_0^{\hat{\pi}} = v = 1$ be the initial capital of the discretely trading investor with portfolio process $\hat{\pi}$ given in Notation 1, let $X_0^{\pi^*} = x > 0$ be the initial capital of the optimally rebalancing investor with constant portfolio process π^*. Then the efficiency θ of the discrete strategy is defined as the fraction of the initial capital of the discretely rebalancing investor which is needed by the continuously rebalancing investor to achieve exactly the same expected utility at T:

$$\theta := \frac{x}{v} = x \quad \text{such that} \quad E(U(X_T^{x,\pi^*})) = E(U(V_T^{1,\hat{\pi}}))\,.$$

Remark 4. Usually the efficiency is defined for the wealth process at the end of the first period if the interval $[0,T]$ is divided into intervals of equal length (Rogers (2001)). In our case we define it for the wealth process at T since the first intervention time is a random time.

Logarithmic utility

Let π_1^* denote the optimal constant portfolio process corresponding to the logarithmic utility function. Due to Corollary 1 iii) we obtain

$$E(\ln(X_T^{\pi_1^*})) \;=\; \left(r + \frac{1}{2}\sigma^2\,\pi_1^{*2}\right)T\,.$$

Proposition 6. *Let $\hat{\pi}_t\,, t \in [0,T]$ be given as in Notation 1. Then the efficiency of the discrete trading strategy is bounded from below by*

$$\theta_b^{\ln} = e^{-\frac{1}{2}\sigma^2\,\varepsilon^2 T}\,. \tag{3.41}$$

Proof: From Proposition 4 ii) we obtain that

$$\begin{aligned} E(\ln(V_T^{\hat{\pi}})) \;\geq\; & rT + \Big[(\mu - r)\pi_1^* - \frac{1}{2}\,\sigma^2\,(\pi_1^{*2} + \varepsilon^2) \\ & -\varepsilon\,|\pi_1^*\,\sigma^2 - (\mu - r)|\Big]\,T\,. \end{aligned}$$

Due to $\pi_1^* = \frac{\mu - r}{\sigma^2}$ this implies

$$E(\ln(V_T^{\hat{\pi}})) \geq \left[r + \frac{1}{2}\sigma^2\,(\pi_1^{*2} - \varepsilon^2)\right]T\,.$$

Therefore the condition

$$E(\ln(X_T^{\pi_1^*})) = E(\ln(V_T^{\hat{\pi}}))$$

implies

$$\begin{aligned} \ln x + rT + \frac{1}{2}\sigma^2\,\pi_1^{*2}T \geq rT + \frac{1}{2}\sigma^2\,(\pi_1^{*2} - \varepsilon^2)T \quad &\Longleftrightarrow \\ x \geq e^{-\frac{1}{2}\sigma^2\,\varepsilon^2 T}\,. & \end{aligned}$$

□

Power utility

Let π_2^* be the optimal constant portfolio process corresponding to a power utility function. Due to Corollary 1 iv) we obtain

$$E\left(\frac{1}{\gamma}(X_T^{\pi_2^*})^\gamma\right) \;=\; \frac{1}{\gamma}e^{\gamma\left[r+\frac{1}{2}\sigma^2\pi_2^{*2}(1-\gamma)\right]T}\,.$$

Proposition 7. *Let $\hat{\pi}_t, t \in [0,T]$ be given as in Notation 1. Then the efficiency of the discrete trading strategy is bounded from below by*

$$\theta_b^\gamma = e^{-\frac{1}{2}\sigma^2(1-\gamma)\,\varepsilon^2 T}\,. \tag{3.42}$$

Proof: From Proposition 4 iii) we obtain that

$$E\left(\frac{1}{\gamma}(V_T^{\hat{\pi}})^\gamma\right) \;\geq\; \frac{1}{\gamma}\,e^{\gamma r T}\,e^{\gamma\left[(\mu-r)\pi_2^*-\frac{1}{2}(1-\gamma)\,\sigma^2\,(\pi_2^{*2}+\varepsilon^2)\right]T}$$
$$e^{-\gamma\,\varepsilon\,|(\mu-r)-(1-\gamma)\,\sigma^2\,\pi_2^*|\,T}\,.$$

Due to $\pi_2^* = \frac{\mu-r}{\sigma^2\,(1-\gamma)}$ this implies

$$E\left(\frac{1}{\gamma}(V_T^{\hat{\pi}})^\gamma\right) \;\geq\; \frac{1}{\gamma}e^{\gamma\left[r+\frac{1}{2}\,(1-\gamma)\,\sigma^2(\pi_2^{*2}-\varepsilon^2)\right]T}\,.$$

From the condition

$$E\left(\frac{1}{\gamma}(X_T^{\pi_2^*})^\gamma\right) = E\left(\frac{1}{\gamma}(V_T^{\hat{\pi}})^\gamma\right)$$

we then obtain

$$\frac{1}{\gamma}x^\gamma e^{\gamma\,r\,T+\frac{1}{2}\gamma\,\sigma^2\pi_2^{*2}(1-\gamma)T} \geq \frac{1}{\gamma}e^{\gamma\,r\,T+\frac{1}{2}\gamma\,\sigma^2(1-\gamma)(\pi_2^{*2}-\varepsilon^2)T}$$
$$\iff \quad x \geq e^{-\frac{1}{2}\sigma^2(1-\gamma)\varepsilon^2\,T}\,.$$

□

Remark 5.

a) The lower bounds of the efficiency are independent of μ and r: The lower bounds of the expected utility contain in each case exactly the same parts depending on μ and r as the optimal expected utilities of the continuous trading problem. Therefore, the parameters μ and r do not have any impact on the lower bounds of the efficiency. (Of course this observation only holds if the constant portfolio process equals π_1^* and π_2^*, respectively. For an arbitrary choice of $\pi^* \in (0,1)$ the lower bounds additionally depend on $\mu - r$ and $\pi^*\sigma^2$.)

b) In the case of power utility with $0 < \gamma < 1$ the lower bound is always higher than in the case of logarithmic utility. Further the lower bound θ_b^γ increases with γ. Of course π_2^* also increases with γ (for fixed values of σ and $\mu - r$) and the interval $[\pi_2^* - \varepsilon, \pi_2^* + \varepsilon]$ therefore contains higher values than for lower values of γ.

c) The lower bounds decrease with increasing T and ε, respectively: For increasing T and ε, respectively, the interval containing $\hat{\pi}$ becomes larger. Therefore the admissible deviations of $\hat{\pi}$ from π^* become larger, too, thus causing a loss of efficiency.

d) The lower bounds decrease with an increase in σ: A higher σ strenghtens the influence of the Brownian motion, thus causing higher fluctuations of $\hat{\pi}$ and higher deviations of $\hat{\pi}$ from the optimal portfolio process π^*.

e) For $\varepsilon \to 0$ (as well as for $\sigma \to 0$ and $T \to 0$) the lower bounds converge to 1 which is of course well expected.

To find appropriate examples, we have to take into consideration the assumptions we made at the beginning: $[\pi_1^* - \varepsilon, \pi_1^* + \varepsilon] \subset [0,1]$ and $[\pi_2^* - \varepsilon, \pi_2^* + \varepsilon] \subset [0,1]$, respectively. These assumptions are synchronously satisfied if and only if

$$\frac{\mu - r}{\sigma^2} \in (\varepsilon, (1-\gamma)(1-\varepsilon)) \quad \text{with } \gamma \in (0,1), \varepsilon > 0$$

where μ, r do not influence the efficiency bounds. Therefore we have to choose the parameters γ and ε such that the above interval for $\frac{\mu-r}{\sigma^2}$ is not empty.

In the following examples illustrating the quantitative behaviour of $\theta_b^{\ln}$ and θ_b^γ we will also give the lower bound of $\frac{E(V_T^{\hat{\pi}})}{E(X_T^{\pi^*})}$ which is equal to $e^{-(\mu-r)\varepsilon T}$ (see (3.10)/(3.23)). Hence we also have to fix $\mu - r$ here.

Examples 1.

σ	T	ε	γ	$\mu - r$	$\theta_b^{\ln}$	θ_b^γ	$\frac{E(V_T^{\hat{\pi}})}{E(X_T^{\pi^*})}$
0.2	1	0.1	0.1	0.03	0.99980	0.99982	0.997
0.2	1	0.2	0.1	0.02	0.99920	0.99928	0.99601
0.2	1	0.4	0.1	0.02	0.99681	0.99712	0.99203
0.2	0.5	0.1	0.1	0.03	0.99990	0.99991	0.9985
0.2	0.5	0.1	0.3	0.02	0.99990	0.99993	0.999
0.2	0.5	0.1	0.7	0.01	0.99990	0.99997	0.9995
0.1	1	0.1	0.5	0.002	0.99995	0.999975	0.9998
0.2	1	0.1	0.5	0.01	0.9998	0.9999	0.999
0.4	1	0.1	0.5	0.05	0.9992	0.9996	0.99501

Remark 6.

a) The above examples show that even the lower bounds of the efficiency attain amazingly high values: In the case of $\sigma = 0.2\,, T = 1\,, \varepsilon = 0.4$ an efficiency of 0.99681 and 0.99712, respectively, is guaranteed, although the portfolio process is allowed to attain values within a range of 0.8 which is close to the whole interval $(0, 1)$.

b) The lower bounds of the efficiency always exceed the lower bound of $\frac{E(V_T^{\hat{\pi}})}{E(X_T^{\pi^*})}$ although also this bound attains quite high values.

3.2.3 Comparison to a pure bond investment

To close this section we will take a look at a pure bond investment and compare its expected terminal wealth and its expected utilities with those corresponding to the bounded portfolio process $\hat{\pi}$.

Proposition 8. *Let $\hat{\pi}_t\,, t \in [0,T]$ be given as in Notation 1. If $\mu \geq r$ we obtain the following relations:*

i)

$$E\left(V_T^{\hat{\pi}}\right) \geq e^{r\,T}\,.$$

ii) *Let $0 < \pi^* < 2\frac{\mu - r}{\sigma^2}$ and $0 < \varepsilon < \frac{1}{\sigma^2}\{\mu - r - |\pi^*\sigma^2 - (\mu - r)|\}$. Then*

$$E(\ln V_T^{\hat{\pi}}) > rT\,.$$

iii) *Let $0 < \pi^* < 2\frac{\mu - r}{\sigma^2(1-\gamma)}$ and $0 < \varepsilon < \frac{1}{(1-\gamma)\sigma^2}\{\mu - r - |(1-\gamma)\sigma^2\pi^* - (\mu - r)|\}$. Then*

$$E\left(\frac{1}{\gamma}(V_T^{\hat{\pi}})^\gamma\right) > \frac{1}{\gamma}e^{\gamma\,r\,T}\,.$$

Proof: i) Due to Proposition 4 i) and $\mu \geq r$ we obtain that

$$E(V_T^{\hat{\pi}}) \geq e^{(r + (\mu - r)(\pi^* - \varepsilon))T} \geq e^{r\,T}$$

if $\pi^* > \varepsilon$ which holds by assumption.
ii) Proposition 4 ii) implies that

$$E(\ln V_T^{\hat{\pi}}) \geq rT + g^- T > r\,T$$

if $g^- > 0$. Due to

$$g^- = -\frac{1}{2}\sigma^2\varepsilon^2 - |\pi^*\sigma^2 - (\mu - r)|\varepsilon + \left[(\mu - r)\pi^* - \frac{1}{2}\sigma^2\pi^{*2}\right]$$

we obtain that $g^- = 0$ if and only if

$$\varepsilon^2 + \frac{2|\pi^*\sigma^2 - (\mu - r)|}{\sigma^2}\varepsilon + \pi^{*2} - \frac{2(\mu - r)\pi^*}{\sigma^2} = 0\,.$$

Because of $\mu \geq r$ this is the case if and only if

$$\begin{aligned}
\varepsilon &= -\frac{|\pi^*\sigma^2 - (\mu - r)|}{\sigma^2} + \sqrt{\frac{(\pi^*\sigma^2 - (\mu - r))^2}{\sigma^4} - \pi^{*2} + \frac{2(\mu - r)\sigma^2\pi^*}{\sigma^4}} \\
&= -\frac{|\pi^*\sigma^2 - (\mu - r)|}{\sigma^2} + \sqrt{\frac{\pi^{*2}\sigma^4 + (\mu - r)^2}{\sigma^4} - \pi^{*2}} \\
&= -\frac{|\pi^*\sigma^2 - (\mu - r)|}{\sigma^2} + \sqrt{\frac{(\mu - r)^2}{\sigma^4}} \\
&= \frac{-|\pi^*\sigma^2 - (\mu - r)| + \mu - r}{\sigma^2}\,.
\end{aligned}$$

(The second solution can be neglected since ε has to be positive.) Therefore $0 < \varepsilon < \frac{1}{\sigma^2}\{\mu - r - |\pi^*\sigma^2 - (\mu - r)|\}(\leq \pi^*)$ implies that $g^- > 0$. Since the upper bound for ε is only positive if $\pi^* < 2\frac{\mu - r}{\sigma^2}$ this condition on π^* has to be satisfied, too.
iii) Due to the results of Proposition 4 iii) the proof of iii) is analogouos to the proof of ii) if we replace g^- by h^- and σ^2 by $\sigma^2(1-\gamma)$. □

The results of Proposition 8 imply that for the special choice of $\pi^* = \pi_1^*$ in ii) and $\pi^* = \pi_2^*$ in iii), respectively, the corresponding expected utility of $V_T^{\hat{\pi}}$ exceeds the one of the bond investment:

Corollary 5.

i) Let $\pi^* = \pi_1^* = \frac{\mu - r}{\sigma^2}$. *Then*

$$E(\ln V_T^{\hat{\pi}}) > rT\,.$$

ii) Let $\pi^* = \pi_2^* = \frac{\mu - r}{\sigma^2(1-\gamma)}$. *Then*

$$E\left(\frac{1}{\gamma}(V_T^{\hat{\pi}})^\gamma\right) > \frac{1}{\gamma}e^{\gamma r T}\,.$$

Proof: Due to the assumption $[\pi^* - \varepsilon, \pi^* + \varepsilon] \subset [0, 1]$ the constraints of Proposition 8 ii), iii) are satisfied. □

Of course we will also derive the efficiency of the pure bond investment. This will be done in the following proposition:

Proposition 9. (Efficiency of the pure bond strategy.) *Let* $\theta^{\ln}_{bond}\,,\theta^{\gamma}_{bond}$ *denote the efficiencies of the pure bond strategy (compared with the optimal strategy corresponding to* π^**). Then we have*

i)

$$\theta^{\ln}_{bond} = e^{-\frac{1}{2}\,\sigma^2\,\pi_1^{*\,2}\,T}\,,$$

ii)

$$\theta^{\gamma}_{bond} = e^{-\frac{1}{2}\,(1-\gamma)\,\sigma^2\,\pi_2^{*\,2}\,T}\,.$$

Proof: From (3.19) we have that

$$E(\ln(X_T^{\pi_1^*})) = \ln x + \left(r + \frac{1}{2}\,\sigma^2\,\pi_1^{*2}\right)\,T\,.$$

Due to

$$E\left(\ln\left(e^{r\,T}\right)\right) = r\,T$$

this leads to i). Analogously we obtain ii) from (3.21). □

The following examples will illustrate the quantitative behaviour of $\theta^{\ln}_{bond}$ and θ^{γ}_{bond} with regard to the data of Examples 1:

Examples 2.

σ	T	ε	γ	$\mu - r$	$\theta^{\ln}_{bond}$	θ^{γ}_{bond}
0.2	1	0.1	0.1	0.03	0.98881	0.98758
0.2	1	0.2	0.1	0.02	0.99501	0.99446
0.2	1	0.4	0.1	0.02	0.99501	0.99446
0.2	0.5	0.1	0.1	0.03	0.99439	0.99377
0.2	0.5	0.1	0.3	0.02	0.9975	0.99643
0.2	0.5	0.1	0.7	0.01	0.99938	0.99792
0.1	1	0.1	0.5	0.002	0.9998	0.9996
0.2	1	0.1	0.5	0.01	0.99875	0.93941
0.4	1	0.1	0.5	0.05	0.99222	0.9845

More extreme values of $\mu - r$ and a longer time horizon T will certainly increase the difference between $\theta^{\ln}_b$ and $\theta^{\ln}_{bond}$ (θ^{γ}_b and θ^{γ}_{bond}):

Examples 3.

σ	T	ε	$\mu - r$	$\theta_b^{\ln}$	$\theta_{bond}^{\ln}$
0.2	10	0.1	0.03	0.998	0.8936
0.4	1	0.1	0.14	0.9992	0.94059
0.4	5	0.2	0.08	0.98413	0.90484

Here we have to point out that $\theta_b^{\ln}$ is a lower bound whereas $\theta_{bond}^{\ln}$ is the exact efficiency!

Remark 7. The results of Subsections 3.2.2 and 3.2.3 show that (with respect to the expected utility) it is profitable to choose the bounded portfolio process as given in Notation 1. Although the bounded portfolio process will not lead to the optimal expected utility of a continuously rebalancing investor, it will lead to an expected utility that is quite near to the optimal one and that can be much higher than the one corresponding to the bond strategy.

3.3 Properties of the discrete strategy

As we stated at the beginning of this chapter, the investor rebalances his portfolio whenever his portfolio process $\hat{\pi}$ reaches one of the boundaries of the interval $[\pi^* - \varepsilon, \pi^* + \varepsilon]$. In the meantime, i.e. as long as $\hat{\pi}$ stays inside of $(\pi^* - \varepsilon, \pi^* + \varepsilon)$, $\hat{\pi}$ is driven by a Brownian motion (Proposition 1). This implies that the intervention times are random times. Throughout this section we will focus on them: We will derive their expectation and we will give an upper bound for the expected number of interventions. Further, these results will imply bounds for the expected transaction costs (if transaction costs have to be paid) and for the expected terminal wealth (also in the presence of transaction costs).

To start with we will take a look at the relation between the movements of the portfolio process $\hat{\pi}$ and those of the stock price. (As we will see in subsection 3.3.2 this relation is very helpful to derive the requested results on the intervention times).

3.3.1 Movements of the portfolio process versus movements of the stock price

The price of the stock $P_t, t \in [0,T]$, affects the value of the risky money and thus the fraction of wealth which is invested into the stock (at least if the investor does not intervene). Hence there is a certain relation between a relative change in the portfolio process and the underlying relative change in the stock price which we will determine in the following. More precisely, we will determine the relation between a relative change in the portfolio process and the underlying relative change in the risky money. (Without transactions the risky money S equals the stock price multiplied by a constant number of stocks φ.) The following proposition states this relation:

Proposition 10. *Let*

$$u_{\Delta t} := \frac{\hat{\pi}_{t+\Delta t}}{\hat{\pi}_t} \tag{3.43}$$

with $\hat{\pi}_t, \hat{\pi}_{t+\Delta t} \in D \subset [0,1]$ *denote the relative increase of the portfolio process between* t *and* $t+\Delta t$ *and let*

$$q_{\Delta t} := \frac{S_{t+\Delta t}}{S_t} \tag{3.44}$$

denote the relative increase of the stock price on the same interval. Then the equality

$$q_{\Delta t} = \frac{u_{\Delta t}(1-\hat{\pi}_t)}{1-u_{\Delta t}\,\hat{\pi}_t}\, e^{r\Delta t} \tag{3.45}$$

holds.

Proof: Due to the definition of $u_{\Delta t}$ and $q_{\Delta t}$ we obtain

$$\begin{aligned}
u_{\Delta t}\,\hat{\pi}_t &= \hat{\pi}_{t+\Delta t} \\
&= \frac{S_{t+\Delta t}}{V^{\hat{\pi}}_{t+\Delta t}} \\
&= \frac{q_{\Delta t} S_t}{q_{\Delta t} S_t + (V^{\hat{\pi}}_t - \hat{\pi}_t V^{\hat{\pi}}_t) e^{r\Delta t}} \\
&= \frac{S_t}{V^{\hat{\pi}}_t}\, \frac{q_{\Delta t}}{q_{\Delta t}\hat{\pi}_t + (1-\hat{\pi}_t)e^{r\Delta t}} \\
&= \hat{\pi}_t\, \frac{q_{\Delta t}}{q_{\Delta t}\hat{\pi}_t + (1-\hat{\pi}_t)e^{r\Delta t}}\,.
\end{aligned}$$

This implies

$$\begin{aligned}
u_{\Delta t} = \frac{q_{\Delta t}}{q_{\Delta t}\hat{\pi}_t + (1-\hat{\pi}_t)e^{r\Delta t}} \quad &\Longleftrightarrow \\
q_{\Delta t} = \frac{u_{\Delta t}(1-\hat{\pi}_t)}{1-u_{\Delta t}\hat{\pi}_t}\, e^{r\Delta t}\,.
\end{aligned}$$

□

Obviously, $q_{\Delta t}$ increases with $u_{\Delta t}$.

Corollary 6. *Let* $[t, t+\Delta t]$ *be an interval with* $\hat{\pi}_s \in (\pi^* - \varepsilon, \pi^* + \varepsilon)$ $\forall s \in [t, t+\Delta t)$ *and* $\hat{\pi}_{t+\Delta t} \in \{\pi^* - \varepsilon, \pi^* + \varepsilon\}$. *For* $u^+_{\Delta t} := \frac{\pi^*+\varepsilon}{\hat{\pi}_t}$ *and* $u^-_{\Delta t} := \frac{\pi^*-\varepsilon}{\hat{\pi}_t}$ *the corresponding values* $q^+_{\Delta t}$ *and* $q^-_{\Delta t}$ *are given by*

$$q^+_{\Delta t} = \frac{\pi^*+\varepsilon}{1-(\pi^*+\varepsilon)}\,\frac{1-\hat{\pi}_t}{\hat{\pi}_t}\, e^{r\Delta t} \tag{3.46}$$

and

$$q^-_{\Delta t} = \frac{\pi^* - \varepsilon}{1 - (\pi^* - \varepsilon)} \frac{1 - \hat{\pi}_t}{\hat{\pi}_t} \, e^{r\Delta t} \,. \tag{3.47}$$

Corollary 7. *Let* $t\,, \Delta t \geq 0\,,\ S^* := \pi^* V_t^{\hat{\pi}}\,.$ *If* $\hat{\pi}_t = \pi^*$ *the investor does not intervene as long as*

$$\frac{S_{t+\Delta t}}{S^* \, e^{r\Delta t}} \in \left(\frac{1 - \pi^*}{\pi^*} \, \frac{\pi^* - \varepsilon}{1 - (\pi^* - \varepsilon)}, \frac{1 - \pi^*}{\pi^*} \, \frac{\pi^* + \varepsilon}{1 - (\pi^* + \varepsilon)} \right) =: (q^*_-, q^*_+)\,. \tag{3.48}$$

Remark 8. Since S^* is fixed on the interval $[t, t + \Delta t)\,,$ $S_{t+\Delta t}$ is the only random component of $\frac{S_{t+\Delta t}}{S^* \, e^{r\Delta t}}$ there. For $\Delta t = 0$ (which implies $S_{t+\Delta t} = S_t = S^*$) we obtain $\frac{S_{t+\Delta t}}{S^* e^{r\Delta t}} = 1\,.$

By definition, the interval D is symmetrical around π^*, the restarting line of the portfolio process. Considering the process $\frac{S_{t+\Delta t}}{S^* \, e^{r\Delta t}}$, we see that the interval (q^*_-, q^*_+) is not symmetrical around 1:

Proposition 11. *The interval* (q^*_-, q^*_+) *is not symmetrical around 1.*
Proof: From (3.47) we obtain that

$$\begin{aligned} d_- &:= 1 - q^*_- \\ &= \frac{\pi^*[1 - (\pi^* - \varepsilon)] - (1 - \pi^*)(\pi^* - \varepsilon)}{\pi^*[1 - (\pi^* - \varepsilon)]} \\ &= \frac{\varepsilon}{\pi^*[1 - (\pi^* - \varepsilon)]} \end{aligned}$$

and

$$d_+ := q^*_+ - 1 = \frac{\varepsilon}{\pi^*[1 - (\pi^* + \varepsilon)]}\,.$$

Due to $[\pi^* - \varepsilon, \pi^* + \varepsilon] \subseteq (0, 1)$ this implies

$$\begin{aligned} d_+ - d_- &= \frac{\varepsilon}{\pi^*} \left[\frac{1}{1 - (\pi^* + \varepsilon)} - \frac{1}{1 - (\pi^* - \varepsilon)} \right] \\ &= \frac{\varepsilon}{\pi^*} \frac{1 - \pi^* + \varepsilon - (1 - \pi^*) + \varepsilon}{(1 - \pi^*)^2 - \varepsilon^2} \\ &= \frac{2\,\varepsilon^2}{\pi^*[(1 - \pi^*)^2 - \varepsilon^2]} \\ &> 0\,. \end{aligned}$$

□

Corollary 8. *As long as*

$$\frac{P_{t+\Delta t}}{P_t\, e^{r\Delta t}} \in (q_-^*, q_+^*)\,,$$

the investor does not intervene.
Proof: The statement follows from Corollary 7 and the definition of S_t which implies the equality

$$\frac{S_{t+\Delta t}}{S^* e^{r\Delta t}} = \frac{\varphi\, P_{t+\Delta t}}{\varphi\, P_t\, e^{r\Delta t}} = \frac{P_{t+\Delta t}}{P_t\, e^{r\Delta t}}\,.$$

(Here P_t denotes the stock price after the foregoing transaction.) □

As we have seen the investor can either watch his portfolio process $\hat{\pi}_t$ or the quotient $\frac{S_{t+\Delta t}}{S^* e^{r\Delta t}}$ or the quotient $\frac{P_{t+\Delta t}}{P_t e^{r\Delta t}}$ in order to find out whether he has to intervene. All these processes have to stay within constant intervals which only depend on π^* and ε. The first process has to stay within the interval $[\pi^* - \varepsilon, \pi^* + \varepsilon]$, the latter two within the interval $\left[\frac{1-\pi^*}{\pi^*}\,\frac{\pi^*-\varepsilon}{1-(\pi^*-\varepsilon)}, \frac{1-\pi^*}{\pi^*}\,\frac{\pi^*+\varepsilon}{1-(\pi^*+\varepsilon)}\right]$. Whereas the first alternative also tells the investor what he has to do (shifting $\hat{\pi}$ to π^*), the latter alternatives only tell him when to intervene. In these two cases the observed processes automatically return to 1 after any intervention. Hence it is more useful to observe the evolution of the portfolio process (as we have done it until now).

As it is not possible to "buy" or "sell" a part of the portfolio process which would be necessary to shift it to π^* whenever it has reached one of the boundaries of D, we have to find out what rebalancing the portfolio means in terms of the numbers of shares. The next proposition tells us how many shares the investor needs to shift his portfolio process to π^* when it has reached one of the boundaries of D:

Proposition 12. *Let $\hat{\pi}_t \in \{\pi^* - \varepsilon, \pi^* + \varepsilon\}$, let φ_{t+} denote the number of shares right after the rebalancing at time t. Then this number is given by*

$$\varphi_{t+} = \frac{\pi^*}{\hat{\pi}_t}\,\varphi_t\,. \tag{3.49}$$

Proof: By definition we have that

$$\hat{\pi}_t = \frac{\varphi_t\, P_t}{V_t^{\hat{\pi}}}$$

and

$$\pi^* = \frac{\varphi_{t+}\, P_t}{V_t^{\hat{\pi}}}$$

giving us the equality

$$\varphi_{t+} = \frac{\pi^*}{\hat{\pi}_t}\,\varphi_t\,.$$

□

Corollary 9. *If the portfolio process $\hat{\pi}_t$ equals one of the boundaries of D, the investor has to buy*

$$\Delta_t = \varphi_t \left(\frac{\pi^*}{\hat{\pi}_t} - 1 \right) \tag{3.50}$$

shares.

3.3.2 Intervention times

During this subsection we will determine the expectation of the rebalancing times. Further we will compute the probability that the portfolio process $\hat{\pi}$ reaches the upper/lower boundary of D first and we will determine an upper bound for the expected number of interventions between 0 and T. To start with we will give the definition of the rebalancing/intervention times:

Definition 2. (Intervention times.) The first intervention time is given by

$$\begin{aligned} \tau &:= \inf\{t \geq 0 : \hat{\pi}_t = \pi^* - \varepsilon \text{ or } \hat{\pi}_t = \pi^* + \varepsilon\} \\ &= \min\{\tau_{\pi^*+\varepsilon}, \tau_{\pi^*-\varepsilon}\} \end{aligned}$$

where

$$\hat{\pi}_0 \in [\pi^* - \varepsilon, \pi^* + \varepsilon]$$

and

$$\tau_x := \inf\{t \geq 0 : \hat{\pi}_t = x\} .$$

Before we can go on with deriving the expectation of τ we have to show that τ is almost surely finite. (Note that τ is not bounded by T!)

Lemma 1. *Let τ be the first intervention time as given in Definition 2, let $\hat{\pi}_0 \in [\pi^* - \varepsilon, \pi^* + \varepsilon] \subseteq (0,1)$. Then*

$$P(\tau < \infty) = 1 .$$

Proof: By definition of τ we have that

$$\begin{aligned} \{\tau < \infty\} &= \{\min\{\tau_{\pi^*-\varepsilon}, \tau_{\pi^*+\varepsilon}\} < \infty\} \\ &\supseteq \{\tau_{\pi^*+\varepsilon} < \infty\} \end{aligned}$$

as well as

$$\{\tau < \infty\} \supseteq \{\tau_{\pi^*-\varepsilon} < \infty\} .$$

This implies

$$P(\tau < \infty) \geq P(\tau_{\pi^*+\varepsilon} < \infty)$$

and

$$P(\tau<\infty)\geq P(\tau_{\pi^*-\varepsilon}<\infty)\,.$$

Therefore we only have to show that either

$$P(\tau_{\pi^*+\varepsilon}<\infty)=1$$

or

$$P(\tau_{\pi^*-\varepsilon}<\infty)=1\,.$$

In order to do that we use the results of Corollary 6: There we have to set $t:=0$ and $\Delta t:=t$, such that u_t denotes the relative increase of the portfolio process between 0 and t. Then we obtain

$$\begin{aligned}
\tau_{\pi^*+\varepsilon} &= \inf\{t\geq 0:\hat{\pi}_t=\pi^*+\varepsilon\}\\
&= \inf\{t\geq 0:\hat{\pi}_0\,u_t=\pi^*+\varepsilon\}\\
&= \inf\{t\geq 0:\hat{\pi}_0\,u_t=\hat{\pi}_0\,u_t^+\}\\
&= \inf\{t\geq 0:q_t=q_t^+\}\\
&= \inf\left\{t\geq 0:\frac{S_t}{S_0\,e^{r\,t}}=\frac{1-\hat{\pi}_0}{\hat{\pi}_0}\frac{\pi^*+\varepsilon}{1-(\pi^*+\varepsilon)}\right\}\\
&= \inf\left\{t\geq 0:e^{\left(\mu-r-\frac{1}{2}\sigma^2\right)t+\sigma W_t}=\frac{1-\hat{\pi}_0}{\hat{\pi}_0}\frac{\pi^*+\varepsilon}{1-(\pi^*+\varepsilon)}\right\}\\
&= \inf\left\{t\geq 0:W_t+\frac{\mu-r-\frac{1}{2}\sigma^2}{\sigma}t=\frac{1}{\sigma}\ln\left[\frac{1-\hat{\pi}_0}{\hat{\pi}_0}\frac{\pi^*+\varepsilon}{1-(\pi^*+\varepsilon)}\right]\right\}\,.
\end{aligned}$$

Analogously we obtain

$$\tau_{\pi^*-\varepsilon} = \inf\left\{t\geq 0:W_t+\frac{\mu-r-\frac{1}{2}\sigma^2}{\sigma}t=\frac{1}{\sigma}\ln\left[\frac{1-\hat{\pi}_0}{\hat{\pi}_0}\frac{\pi^*-\varepsilon}{1-(\pi^*-\varepsilon)}\right]\right\}\,.$$

Defining

$$d^{\pi^*+\varepsilon}:=\frac{1}{\sigma}\ln\left[\frac{1-\hat{\pi}_0}{1-(\pi^*+\varepsilon)}\frac{\pi^*+\varepsilon}{\hat{\pi}_0}\right]\,,$$

$$d^{\pi^*-\varepsilon}:=\frac{1}{\sigma}\ln\left[\frac{1-\hat{\pi}_0}{1-(\pi^*-\varepsilon)}\frac{\pi^*-\varepsilon}{\hat{\pi}_0}\right]$$

and

$$\hat{\mu}:=\frac{\mu-r-\frac{1}{2}\sigma^2}{\sigma}$$

we obtain

$$\tau_{\pi^*+\varepsilon}=\inf\{t\geq 0:W_t+\hat{\mu}\,t=d^{\pi^*+\varepsilon}\}$$

and

$$\tau_{\pi^*-\varepsilon}=\inf\{t\geq 0:W_t+\hat{\mu}\,t=d^{\pi^*-\varepsilon}\}\,.$$

Due to $\hat{\pi}_0 \in [\pi^* - \varepsilon, \pi^* + \varepsilon]$ it is obvious that

$$d^{\pi^*+\varepsilon} \geq 0$$

and

$$d^{\pi^*-\varepsilon} \leq 0\,.$$

Setting $\tilde{\tau} := \tau_{\pi^*+\varepsilon}$ if $\hat{\mu} \geq 0$ and $\tilde{\tau} := \tau_{\pi^*-\varepsilon}$ otherwise, we obtain from Karatzas and Shreve (1988), p.197 that

$$P(\tilde{\tau} < \infty) = \begin{cases} e^{\hat{\mu}\, d^{\pi^*+\varepsilon} - |\hat{\mu}\, d^{\pi^*+\varepsilon}|} = 1 & \text{if } \hat{\mu} \geq 0\,, \\ e^{\hat{\mu}\, d^{\pi^*-\varepsilon} - |\hat{\mu}\, d^{\pi^*-\varepsilon}|} = 1 & \text{otherwise}\,, \end{cases}$$

which implies

$$P(\tau < \infty) = 1\,.$$

□

Knowing that the first stopping time is almost surely finite, we will determine its expectation and the probability that the the upper (lower) boundary of the interval $[\pi^* - \varepsilon, \pi^* + \varepsilon]$ is reached first by the portfolio process $\hat{\pi}$. The results are given below.

Theorem 1. (Expectation and probabilities.) *Assume that* $[\pi^* - \varepsilon, \pi^* + \varepsilon] \subseteq (0,1)$ *and that* $\hat{\pi}_0 \in [\pi^* - \varepsilon, \pi^* + \varepsilon]$.

i) *Let* $\frac{\mu - r}{\sigma^2} \neq \frac{1}{2}$. *Then the expectation of* τ *is given by*

$$\begin{aligned} &E^{\hat{\pi}_0}(\tau) \\ &= \frac{2}{2(\mu - r) - \sigma^2} \left\{ -\ln\left(\frac{\hat{\pi}_0}{1 - \hat{\pi}_0} \frac{1 - (\pi^* - \varepsilon)}{\pi^* - \varepsilon} \right) \right. \\ &\quad \left. + \frac{1 - \left(\frac{\pi^* - \varepsilon}{1 - (\pi^* - \varepsilon)} \frac{1 - \hat{\pi}_0}{\hat{\pi}_0} \right)^{2\frac{\mu - r}{\sigma^2} - 1}}{1 - \left(\frac{\pi^* - \varepsilon}{1 - (\pi^* - \varepsilon)} \frac{1 - (\pi^* + \varepsilon)}{\pi^* + \varepsilon} \right)^{2\frac{\mu - r}{\sigma^2} - 1}} \ln\left(\frac{\pi^* + \varepsilon}{1 - (\pi^* + \varepsilon)} \frac{1 - (\pi^* - \varepsilon)}{\pi^* - \varepsilon} \right) \right\}. \end{aligned} \tag{3.51}$$

Further we have that

$$P^{\hat{\pi}_0}(\hat{\pi}_\tau = \pi^* + \varepsilon) = \frac{1 - \left(\frac{\pi^* - \varepsilon}{1 - (\pi^* - \varepsilon)} \frac{1 - \hat{\pi}_0}{\hat{\pi}_0} \right)^{2\frac{\mu - r}{\sigma^2} - 1}}{1 - \left(\frac{\pi^* - \varepsilon}{1 - (\pi^* - \varepsilon)} \frac{1 - (\pi^* + \varepsilon)}{\pi^* + \varepsilon} \right)^{2\frac{\mu - r}{\sigma^2} - 1}} \tag{3.52}$$

implying

$$P^{\hat{\pi}_0}(\hat{\pi}_\tau = \pi^* - \varepsilon) = 1 - \frac{1 - \left(\frac{\pi^* - \varepsilon}{1 - (\pi^* - \varepsilon)} \frac{1 - \hat{\pi}_0}{\hat{\pi}_0} \right)^{2\frac{\mu - r}{\sigma^2} - 1}}{1 - \left(\frac{\pi^* - \varepsilon}{1 - (\pi^* - \varepsilon)} \frac{1 - (\pi^* + \varepsilon)}{\pi^* + \varepsilon} \right)^{2\frac{\mu - r}{\sigma^2} - 1}}\,. \tag{3.53}$$

ii) *Let* $\frac{\mu-r}{\sigma^2}=\frac{1}{2}$. *Then the expectation of* τ *is given by*

$$E^{\hat{\pi}_0}(\tau) \;=\; \frac{1}{\sigma^2}\ln\left(\frac{\hat{\pi}_0}{1-\hat{\pi}_0}\frac{1-(\pi^*-\varepsilon)}{\pi^*-\varepsilon}\right)\ln\left(\frac{\pi^*+\varepsilon}{1-(\pi^*+\varepsilon)}\frac{1-\hat{\pi}_0}{\hat{\pi}_0}\right). \tag{3.54}$$

Further we have that

$$P^{\hat{\pi}_0}(\hat{\pi}_\tau=\pi^*+\varepsilon) \;=\; \frac{\ln\left(\frac{\hat{\pi}_0}{1-\hat{\pi}_0}\frac{1-(\pi^*-\varepsilon)}{\pi^*-\varepsilon}\right)}{\ln\left(\frac{\pi^*+\varepsilon}{1-(\pi^*+\varepsilon)}\frac{1-(\pi^*-\varepsilon)}{\pi^*-\varepsilon}\right)} \tag{3.55}$$

which implies

$$P^{\hat{\pi}_0}(\hat{\pi}_\tau=\pi^*-\varepsilon) \;=\; 1-\frac{\ln\left(\frac{\hat{\pi}_0}{1-\hat{\pi}_0}\frac{1-(\pi^*-\varepsilon)}{\pi^*-\varepsilon}\right)}{\ln\left(\frac{\pi^*+\varepsilon}{1-(\pi^*+\varepsilon)}\frac{1-(\pi^*-\varepsilon)}{\pi^*-\varepsilon}\right)}. \tag{3.56}$$

Proof: We will split up the proof into three steps as we did it in the proof of Theorem 2 of Chapter 2. At first we derive a differential equation which is in the case of specific boundary conditions solved by $E(\tau), P(\hat{\pi}_\tau=\pi^*-\varepsilon)$ and $P(\hat{\pi}_\tau=\pi^*+\varepsilon)$. Then we solve this differential equation in general. Thirdly we derive the specific solutions which give us the expectation and the probabilities. To simplify our computations we set $E^{\hat{\pi}_0}=E$ and $P^{\hat{\pi}_0}=P$.
Step 1: As this step is equal to the first step of the proof of Theorem 2, Chapter 2 we will only recall the results: Let

$$a:=\pi^*-\varepsilon\,,\quad b:=\pi^*+\varepsilon$$

and

$$dZ_t=\hat{\mu}(Z_t)\,dt+\hat{\sigma}(Z_t)\,dW_t\,,\quad Z_0=z_0\in[a,b]$$

where $Z_t=\hat{\pi}_t$. Then the expectation satisfies $E(\tau)=f(z_0)$ where f solves the differential equation

$$f'(z)\,\hat{\mu}(z)+\frac{1}{2}f''(z)\,\hat{\sigma}^2(z)=-1\,,\quad z\in(a,b)\,,$$

$$f(a)=f(b)=0\,.$$

For different boundary conditions we have that $P(Z_\tau=b)=f(z_0)$ where f solves the differential equation

$$f'(z)\,\hat{\mu}(z)+\frac{1}{2}f''(z)\,\hat{\sigma}^2(z)=0\,,\quad z\in(a,b)\,,$$

$$f(a)=0,\quad f(b)=1\,.$$

Step 2: Consider the differential equation

$$\frac{1}{2}f''(z)\hat{\sigma}^2(z) + f'(z)\hat{\mu}(z) = K\,,$$
$$f(a) = \eta_1\,, \quad f(b) = \eta_2$$

where

$$\begin{aligned}\hat{\mu}(z) &:= z(1-z)(\mu - r - z\sigma^2)\,,\\ \hat{\sigma}(z) &:= z(1-z)\sigma\,.\end{aligned}$$

(These are the coefficients of π, i.e. of $\hat{\pi}$ before the first intervention takes place.) Then it is equivalent to solve

$$f''(z) + c(z)\,f'(z) = b(z)$$

where

$$\begin{aligned}c(z) &= 2\frac{\hat{\mu}(z)}{\hat{\sigma}^2(z)}\\ &= 2\frac{\mu - r - z\sigma^2}{z(1-z)\sigma^2}\,,\\ b(z) &= 2\frac{K}{\hat{\sigma}^2(z)}\\ &= 2\frac{K}{z^2(1-z)^2\sigma^2}\,.\end{aligned}$$

Introducing

$$p(z) := e^{\int_a^z c(s)\,ds}$$

as we did in the second step of the proof of Theorem 2, Chapter 2 we can use the results obtained there:

$$\begin{aligned}f(z) &= \int_a^z \frac{1}{p(t)}\left[\int_a^t p(s)b(s)\,ds\right]dt\\ &\quad +c_0\int_a^z \frac{1}{p(t)}\,dt + \eta_1\end{aligned}$$

with

$$c_0 = \frac{\eta_2 - \eta_1 - \int_a^b \frac{1}{p(t)}\left[\int_a^t p(s)b(s)\,ds\right]dt}{\int_a^b \frac{1}{p(t)}\,dt}\,.$$

As in the proof of Theorem 2, Chapter 2 we then define

$$\begin{aligned}L_1(z) &:= \int_a^z \frac{1}{p(t)}\,dt\,,\\ L_2(t) &:= \int_a^t p(s)b(s)\,ds\,,\\ L_3(z) &:= \int_a^z \frac{1}{p(t)}\,L_2(t)\,dt\end{aligned}$$

which implies

$$f(z) = L_3(z) + c_0 L_1(z) + \eta_1$$

where

$$c_0 = \frac{\eta_2 - \eta_1 - L_3(b)}{L_1(b)} .$$

Besides we define

$$q := \frac{\mu - r}{\sigma^2} .$$

At first we determine p. Hence we compute the integral

$$\begin{aligned}\int_a^s c(t)\,dt &= \int_a^s 2\frac{\mu - r - t\sigma^2}{t(1-t)\sigma^2}\,dt \\ &= 2\,q \int_a^s \frac{1}{t(1-t)}\,dt - 2\int_a^s \frac{1}{1-t}\,dt\end{aligned}$$

where

$$\begin{aligned}\int_a^s \frac{1}{1-t}\,dt &= [-\ln(1-t)]_a^s \\ &= -\ln\left(\frac{1-s}{1-a}\right)\end{aligned}$$

and

$$\begin{aligned}\int_a^s \frac{1}{t(1-t)}\,dt &= \left[\ln\left(\frac{t}{1-t}\right)\right]_a^s \\ &= \ln\left(\frac{s}{1-s}\frac{1-a}{a}\right) .\end{aligned}$$

This implies

$$\int_a^s c(t)\,dt = 2\,q\,\ln\left(\frac{s}{1-s}\frac{1-a}{a}\right) + 2\ln\left(\frac{1-s}{1-a}\right)$$

and

$$p(s) = \left(\frac{s}{1-s}\right)^{2q}\left(\frac{1-a}{a}\right)^{2q}\left(\frac{1-s}{1-a}\right)^{2} .$$

From this point on we have to consider two different cases: At first we consider the case $2q = 1$ and then the other cases where $2q \neq 1$.
<u>Step 2</u> with <u>$2q = 1$</u>: Let $2q = 1$. Then p is given by

$$p(s) = \frac{(1-s)s}{(1-a)a}$$

which implies

$$\begin{aligned} L_1(z) &= \int_a^z \frac{1}{p(t)}\,dt \\ &= a(1-a)\int_a^z \frac{1}{t(1-t)}\,dt \\ &= a(1-a)\left[\ln\left(\frac{z}{1-z}\right) - \ln\left(\frac{a}{1-a}\right)\right], \end{aligned}$$

$$\begin{aligned} L_2(t) &= \int_a^t p(s)b(s)\,ds \\ &= \frac{2K}{\sigma^2 a(1-a)}\int_a^t \frac{1}{s(1-s)}\,ds \\ &= \frac{2K}{\sigma^2 a(1-a)}\left[\ln\left(\frac{t}{1-t}\right) - \ln\left(\frac{a}{1-a}\right)\right] \end{aligned}$$

and

$$\begin{aligned} L_3(z) &= \int_a^z \frac{1}{p(t)} L_2(t)\,dt \\ &= \frac{2K}{\sigma^2}\left\{\int_a^z \frac{1}{t(1-t)}\ln\left(\frac{t}{1-t}\right)\,dt - \ln\left(\frac{a}{1-a}\right)\int_a^z \frac{1}{t(1-t)}\,dt\right\}. \end{aligned}$$

Due to

$$\int_a^z \frac{1}{t(1-t)}\,dt = \ln\left(\frac{z}{1-z}\right) - \ln\left(\frac{a}{1-a}\right)$$

and (obtained with the help of the substitution $g := \ln\frac{t}{1-t}$)

$$\begin{aligned} \int_a^z \frac{1}{t(1-t)}\ln\left(\frac{t}{1-t}\right)\,dt &= \int\limits_{\ln\frac{a}{1-a}}^{\ln\frac{z}{1-z}} g\,dg \\ &= \frac{1}{2}\left[\left(\ln\left(\frac{z}{1-z}\right)\right)^2 - \left(\ln\left(\frac{a}{1-a}\right)\right)^2\right] \end{aligned}$$

we get

$$\begin{aligned} L_3(z) &= \frac{K}{\sigma^2}\left\{\left(\ln\left(\frac{z}{1-z}\right)\right)^2 - \left(\ln\left(\frac{a}{1-a}\right)\right)^2 \right. \\ &\qquad \left. -2\ln\left(\frac{a}{1-a}\right)\left(\ln\left(\frac{z}{1-z}\right) - \ln\left(\frac{a}{1-a}\right)\right)\right\} \\ &= \frac{K}{\sigma^2}\left\{\left(\ln\left(\frac{z}{1-z}\right)\right)^2 + \left(\ln\left(\frac{a}{1-a}\right)\right)^2 \right. \\ &\qquad \left. -2\ln\left(\frac{a}{1-a}\right)\ln\left(\frac{z}{1-z}\right)\right\} \end{aligned}$$

$$\begin{aligned} &= \frac{K}{\sigma^2}\left(\ln\left(\frac{z}{1-z}\right)-\ln\left(\frac{a}{1-a}\right)\right)^2 \\ &= \frac{K}{\sigma^2}\left(\ln\left(\frac{z}{1-z}\frac{1-a}{a}\right)\right)^2 \end{aligned}$$

which implies

$$f(z)=\frac{K}{\sigma^2}\left(\ln\left(\frac{z}{1-z}\frac{1-a}{a}\right)\right)^2+c_0\,a(1-a)\,\ln\left(\frac{z}{1-z}\frac{1-a}{a}\right)+\eta_1$$

where

$$c_0=\frac{\eta_2-\eta_1-\frac{K}{\sigma^2}\left(\ln\left(\frac{b}{1-b}\frac{1-a}{a}\right)\right)^2}{a(1-a)\ln\left(\frac{b}{1-b}\frac{1-a}{a}\right)}\,.$$

Setting

$$\hat{c}_0:=c_0\,a(1-a)$$

we derive the simpler formula

$$f(z)=\frac{K}{\sigma^2}\left(\ln\left(\frac{z}{1-z}\frac{1-a}{a}\right)\right)^2+\hat{c}_0\,\ln\left(\frac{z}{1-z}\frac{1-a}{a}\right)+\eta_1$$

with

$$\hat{c}_0=\frac{\eta_2-\eta_1-\frac{K}{\sigma^2}\left(\ln\left(\frac{b}{1-b}\frac{1-a}{a}\right)\right)^2}{\ln\left(\frac{b}{1-b}\frac{1-a}{a}\right)}\,.$$

<u>Step 3</u> with <u>$2q=1$</u>: Now we use the specific boundary conditions to derive the expectation and the probabilities. At first we assume the conditions

$$K=-1\,,\quad \eta_1=\eta_2=0\,.$$

Then $f(z)=E(\tau)$ holds where

$$\begin{aligned} f(z) &= \frac{1}{\sigma^2}\ln\left(\frac{z}{1-z}\frac{1-a}{a}\right)\left\{\ln\left(\frac{b}{1-b}\frac{1-a}{a}\right)-\ln\left(\frac{z}{1-z}\frac{1-a}{a}\right)\right\} \\ &= \frac{1}{\sigma^2}\ln\left(\frac{z}{1-z}\frac{1-a}{a}\right)\ln\left(\frac{b}{1-b}\frac{1-z}{z}\right)\,. \end{aligned}$$

Secondly we assume that

$$K=0\,,\quad \eta_1=0\,,\quad \eta_2=1\,.$$

Then $f(z)=P(Z_\tau=b)$ holds where

$$f(z)=\frac{\ln\left(\frac{z}{1-z}\frac{1-a}{a}\right)}{\ln\left(\frac{b}{1-b}\frac{1-a}{a}\right)}\,.$$

Due to $\frac{b}{1-b} \geq \frac{z}{1-z}$ and $\ln\left(\frac{z}{1-z}\frac{1-a}{a}\right) \geq 0$ we have $0 \leq P(Z_\tau = b) \leq 1 \quad \forall z \in [a,b]$.
Step 2 with $2q \neq 1$: Let $2q \neq 1$. Then we have

$$\begin{aligned} L_1(z) &= \int_a^z \frac{1}{p(t)}\,dt \\ &= \int_a^z \left(\frac{1-s}{s}\right)^{2q} \left(\frac{a}{1-a}\right)^{2q} \left(\frac{1-a}{1-s}\right)^2 ds \\ &= \left(\frac{a}{1-a}\right)^{2q} (1-a)^2 \int_a^z \left(\frac{1-s}{s}\right)^{2q} (1-s)^{-2}\,ds\,. \end{aligned}$$

With the substitution $y = \frac{1-s}{s}$ we obtain

$$\begin{aligned} L_1(z) &= -\left(\frac{a}{1-a}\right)^{2q} (1-a)^2 \int_{\frac{1-a}{a}}^{\frac{1-z}{z}} y^{2q-2}\,dy \\ &= -\frac{1}{2q-1}\left(\frac{a}{1-a}\right)^{2q} (1-a)^2 \left\{ \left(\frac{1-z}{z}\right)^{2q-1} - \left(\frac{1-a}{a}\right)^{2q-1} \right\} \\ &= -\frac{1}{2q-1}\left\{ \left(\frac{a}{1-a}\right)^{2q} (1-a)^2 \left(\frac{1-z}{z}\right)^{2q-1} - a\,(1-a) \right\}. \end{aligned}$$

Further L_2 is given by

$$\begin{aligned} L_2(t) &= \int_a^t p(s)\,b(s)\,ds \\ &= 2\,\frac{K}{\sigma^2}\left(\frac{1-a}{a}\right)^{2q} (1-a)^{-2} \int_a^t \left(\frac{s}{1-s}\right)^{2q} s^{-2}\,ds\,. \end{aligned}$$

In this case the substitution $y = \frac{s}{1-s}$ delivers

$$\begin{aligned} L_2(t) &= 2\,\frac{K}{\sigma^2}\left(\frac{1-a}{a}\right)^{2q} (1-a)^{-2} \int_{\frac{a}{1-a}}^{\frac{t}{1-t}} y^{2q-2}\,dy \\ &= \frac{2}{2q-1}\,\frac{K}{\sigma^2}\left(\frac{1-a}{a}\right)^{2q} (1-a)^{-2} \left\{ \left(\frac{t}{1-t}\right)^{2q-1} - \left(\frac{a}{1-a}\right)^{2q-1} \right\} \\ &= \frac{2}{2q-1}\,\frac{K}{\sigma^2}\left\{ \left(\frac{1-a}{a}\right)^{2q} (1-a)^{-2} \left(\frac{t}{1-t}\right)^{2q-1} - a^{-1}\,(1-a)^{-1} \right\}. \end{aligned}$$

This implies

$$\begin{aligned} L_3(z) &= \int_a^z \frac{1}{p(t)}\,L_2(t)\,dt \\ &= \frac{2K}{\sigma^2(2q-1)}\left\{ \int_a^z \frac{1}{t(1-t)}\,dt - \left(\frac{a}{1-a}\right)^{2q-1} \int_a^z \left(\frac{1-t}{t}\right)^{2q} (1-t)^{-2}\,dt \right\} \end{aligned}$$

with (as above)

$$\int_a^z \frac{1}{t(1-t)}\,dt = \ln\frac{z}{1-z} - \ln\frac{a}{1-a}$$

and

$$\int_a^z \left(\frac{1-t}{t}\right)^{2q}(1-t)^{-2}\,dt = \frac{-1}{2q-1}\left\{\left(\frac{1-z}{z}\right)^{2q-1} - \left(\frac{1-a}{a}\right)^{2q-1}\right\}$$

giving us

$$L_3(z) = \frac{2K}{\sigma^2(2q-1)}\left\{\ln\left(\frac{z}{1-z}\frac{1-a}{a}\right) + \frac{1}{2q-1}\left[\left(\frac{a}{1-a}\frac{1-z}{z}\right)^{2q-1} - 1\right]\right\}.$$

Finally, we obtain

$$\begin{aligned} f(z) &= \frac{2K}{\sigma^2(2q-1)}\left\{\frac{1}{2q-1}\left[\left(\frac{1-z}{z}\frac{a}{1-a}\right)^{2q-1} - 1\right]\right. \\ &\qquad\left. + \ln\left(\frac{z}{1-z}\frac{1-a}{a}\right)\right\} \\ &\quad + c_0\,\frac{a(1-a)}{2q-1}\left\{1 - \left(\frac{1-z}{z}\frac{a}{1-a}\right)^{2q-1}\right\} + \eta_1 \end{aligned}$$

where

$$c_0 = \frac{2q-1}{a(1-a)}\,\frac{\eta_2 - \eta_1 - \frac{2K}{\sigma^2(2q-1)}\left\{\frac{1}{2q-1}\left[\left(\frac{a}{1-a}\frac{1-b}{b}\right)^{2q-1} - 1\right] + \ln\left(\frac{b}{1-b}\frac{1-a}{a}\right)\right\}}{1 - \left(\frac{a}{1-a}\frac{1-b}{b}\right)^{2q-1}}.$$

Setting

$$\hat{c}_0 := c_0\frac{a(1-a)}{2q-1}$$

we obtain

$$\begin{aligned} f(z) &= \frac{2K}{\sigma^2(2q-1)}\left\{\frac{1}{2q-1}\left[\left(\frac{1-z}{z}\frac{a}{1-a}\right)^{2q-1} - 1\right]\right. \\ &\qquad\left. + \ln\left(\frac{z}{1-z}\frac{1-a}{a}\right)\right\} \\ &\quad + \hat{c}_0\left\{1 - \left(\frac{1-z}{z}\frac{a}{1-a}\right)^{2q-1}\right\} + \eta_1 \end{aligned}$$

where

$$\hat{c}_0 = \frac{\eta_2 - \eta_1 - \frac{2K}{\sigma^2(2q-1)}\left\{\frac{1}{2q-1}\left[\left(\frac{a}{1-a}\frac{1-b}{b}\right)^{2q-1} - 1\right] + \ln\left(\frac{b}{1-b}\frac{1-a}{a}\right)\right\}}{1 - \left(\frac{a}{1-a}\frac{1-b}{b}\right)^{2q-1}}.$$

Step 3 with $2q \neq 1$: Again we use the specific boundary conditions to derive the requested expectation and the probabilities. At first we assume that

$$K = -1\,, \quad \eta_1 = \eta_2 = 0\,.$$

Then $f(z) = E(\tau)$ holds where

$$\begin{aligned}
f(z) &= -\frac{2}{\sigma^2(2q-1)}\left\{\frac{1}{2q-1}\left[\left(\frac{a}{1-a}\frac{1-z}{z}\right)^{2q-1}-1\right]+\ln\left(\frac{z}{1-z}\frac{1-a}{a}\right)\right\} \\
&\quad +\frac{1-\left(\frac{a}{1-a}\frac{1-z}{z}\right)^{2q-1}}{1-\left(\frac{a}{1-a}\frac{1-b}{b}\right)^{2q-1}}\frac{2}{\sigma^2(2q-1)} \\
&\qquad\qquad \left\{\frac{1}{2q-1}\left[\left(\frac{a}{1-a}\frac{1-b}{b}\right)^{2q-1}-1\right]+\ln\left(\frac{b}{1-b}\frac{1-a}{a}\right)\right\} \\
&= \frac{2}{\sigma^2(2q-1)} \\
&\qquad \left\{-\ln\left(\frac{z}{1-z}\frac{1-a}{a}\right)+\frac{1-\left(\frac{a}{1-a}\frac{1-z}{z}\right)^{2q-1}}{1-\left(\frac{a}{1-a}\frac{1-b}{b}\right)^{2q-1}}\ln\left(\frac{b}{1-b}\frac{1-a}{a}\right)\right\}.
\end{aligned}$$

Secondly we assume that

$$K = 0\,, \quad \eta_1 = 0\,, \quad \eta_2 = 1\,.$$

Here $f(z) = P(Z_\tau = b)$ holds where

$$f(z) = \frac{1-\left(\frac{a}{1-a}\frac{1-z}{z}\right)^{2q-1}}{1-\left(\frac{a}{1-a}\frac{1-b}{b}\right)^{2q-1}}\,.$$

As above (case $2q = 1$) we have the inequalities $\frac{1-z}{z} \geq \frac{1-b}{b}$ and $\frac{a}{1-a}\frac{1-z}{z} \leq 1$.

1. Let $2q-1 > 0$. Then

   $$1-\left(\frac{a}{1-a}\frac{1-z}{z}\right)^{2q-1} \geq 0$$

   which is also valid for $z = b$ and which therefore implies

   $$f(z) \geq 0 \quad \forall z \in [a,b]\,.$$

   Together with the first inequality above this also gives us

   $$f(z) \leq 1 \quad \forall z \in [a,b]\,.$$

2. Let $2q-1<0$. Then

$$1-\left(\frac{a}{1-a}\frac{1-z}{z}\right)^{2q-1}\leq 0$$

which is also valid for $z=b$ and which therefore implies

$$f(z)\geq 0 \quad \forall z\in[a,b].$$

Since

$$1\leq\left(\frac{a}{1-a}\frac{1-z}{z}\right)^{2q-1}\leq\left(\frac{a}{1-a}\frac{1-b}{b}\right)^{2q-1}$$

we also obtain

$$f(z)\leq 1 \quad \forall z\in[a,b].$$

In any case we derive $0\leq P(Z_\tau=b)\leq 1 \quad \forall z\in[a,b]$.

Replacing now z by $\hat{\pi}_0$, Z by $\hat{\pi}$, a by $\pi^*-\varepsilon$ and b by $\pi^*+\varepsilon$ completes the proof. □

Remark 9. Considering the formulae of Theorem 1 we see that the values do not depend on the single parameters μ, r and σ: The expectations only depend on the parameters σ and $q=\frac{\mu-r}{\sigma^2}$. If these parameters are fixed the value of the expectation is constant for all μ, r with $\mu-r=q\,\sigma^2$. Since the probabilities only depend on q they are constant for all μ, r and σ with the same value of $\frac{\mu-r}{\sigma^2}$. To compute some examples it is therefore sufficient to choose the parameters σ and q.

The next examples will illustrate the quantitative behaviour of the expectation and the probabilities for various choices of π^*, ε and $\hat{\pi}_0$.

Examples 4.

a) $\sigma = 0.3\,,\ \frac{\mu - r}{\sigma^2} = 0.1667 \neq 0.5\,.\quad (\mu = 0.025\,,\ r = 0.01\,.)$

π^*	ε	$\hat{\pi}_0$	$E(\tau)$	$P(\hat{\pi}_\tau = \pi^* + \varepsilon)$	$P(\hat{\pi}_\tau = \pi^* - \varepsilon)$
0.4	0.2	0.3	6.74835	0.18783	0.81217
0.4	0.2	0.4	8.74636	0.40097	0.59903
0.4	0.2	0.5	6.77631	0.66025	0.33975
0.1667	0.1	0.1167	6.46597	0.2585	0.7415
0.1667	0.1	0.1667	7.00534	0.50352	0.49648
0.1667	0.1	0.2167	4.54124	0.74818	0.25182
0.4	0.1	0.37	1.80721	0.30788	0.69212
0.4	0.1	0.4	1.9854	0.45117	0.54883
0.4	0.1	0.43	1.81573	0.60306	0.39694
0.5	0.1	0.45	1.31192	0.20401	0.79599
0.5	0.1	0.5	1.81565	0.43283	0.56717
0.5	0.1	0.55	1.434	0.69441	0.30559
0.6	0.1	0.55	1.36403	0.18854	0.81146
0.6	0.1	0.6	1.96946	0.40881	0.59119
0.6	0.1	0.65	1.42165	0.7325	0.2675

b) $\sigma = 0.15\,,\ \frac{\mu - r}{\sigma^2} = 0.889 \neq 0.5\,.\quad (\mu = 0.03\,,\ r = 0.01\,.)$

π^*	ε	$\hat{\pi}_0$	$E(\tau)$	$P(\hat{\pi}_\tau = \pi^* + \varepsilon)$	$P(\hat{\pi}_\tau = \pi^* - \varepsilon)$
0.3	0.2	0.2	50.7608	0.57121	0.42879
0.3	0.2	0.3	45.0439	0.79376	0.20624
0.3	0.2	0.4	25.7558	0.91803	0.08197
0.4	0.1	0.35	6.55026	0.33704	0.66296
0.4	0.1	0.4	7.85392	0.60257	0.39743
0.4	0.1	0.45	5.4	0.81893	0.18107
0.5	0.1	0.45	5.7669	0.31477	0.68523
0.5	0.1	0.5	7.2468	0.57819	0.42181
0.5	0.1	0.55	5.19874	0.80355	0.19645
0.6	0.1	0.55	6.05942	0.29941	0.70059
0.6	0.1	0.6	7.92805	0.56041	0.43959
0.6	0.1	0.65	5.92125	0.79175	0.20825
0.889	0.1	0.8389	52.137	0.24853	0.75147
0.889	0.1	0.889	89.9236	0.48828	0.51172
0.889	0.1	0.9389	102.531	0.7289	0.2711

c) $\frac{\mu-r}{\sigma^2} = 0.5$. $\quad(\mu = 0.03\,,\ \sigma = 0.2\,,\ r = 0.01\,.)$

π^*	ε	$\hat{\pi}_0$	$E(\tau)$	$P(\hat{\pi}_\tau = \pi^* + \varepsilon)$	$P(\hat{\pi}_\tau = \pi^* - \varepsilon)$
0.3	0.1	0.25	4.98515	0.2933	0.7067
0.3	0.1	0.3	5.95366	0.54953	0.45047
0.3	0.1	0.35	4.09665	0.78225	0.21775
0.4	0.1	0.35	3.53253	0.2694	0.7306
0.4	0.1	0.4	4.47869	0.52146	0.47854
0.4	0.1	0.45	3.24398	0.76316	0.23684
0.5	0.1	0.45	3.10333	0.25254	0.74746
0.5	0.1	0.5	4.11005	0.5	0.5
0.5	0.1	0.55	3.10333	0.74746	0.25254
0.6	0.1	0.52	1.53533	0.09447	0.90553
0.6	0.1	0.6	4.47869	0.47854	0.52146
0.6	0.1	0.68	1.76243	0.88962	0.11038
0.7	0.1	0.65	4.09665	0.21775	0.78225
0.7	0.1	0.7	5.95366	0.45047	0.54953
0.7	0.1	0.75	4.98515	0.7067	0.2933
0.3	0.2	0.2	28.1047	0.36907	0.63093
0.3	0.2	0.3	28.5948	0.61438	0.38562
0.3	0.2	0.4	18.1624	0.81546	0.18454
0.6	0.2	0.5	14.0524	0.22629	0.77371
0.6	0.2	0.6	19.8846	0.45259	0.54741
0.6	0.2	0.7	16.8809	0.69918	0.30082

Remark 10.

a) If $\hat{\pi}_0$ increases, then the probability that the process $\hat{\pi}$ reaches the upper boundary $\pi^* + \varepsilon$ before it reaches the lower boundary increases, too. Of course we expected such a relation but we will also prove it:
Let $\frac{\mu-r}{\sigma^2} = 0.5$. Then

$$\frac{\partial}{\partial \hat{\pi}_0} P(\hat{\pi}_\tau = \pi^* + \varepsilon) = \frac{1}{\hat{\pi}_0(1-\hat{\pi}_0)} \frac{1}{\ln\left(\frac{\pi^*+\varepsilon}{1-(\pi^*+\varepsilon)} \frac{1-(\pi^*-\varepsilon)}{\pi^*-\varepsilon}\right)} > 0\,.$$

If $q := \frac{\mu-r}{\sigma^2} \neq 0.5$ then

$$\begin{aligned} &\frac{\partial}{\partial \hat{\pi}_0} P(\hat{\pi}_\tau = \pi^* + \varepsilon) \\ &= \frac{1}{\hat{\pi}_0^2} \left(\frac{1-\hat{\pi}_0}{\hat{\pi}_0}\right)^{2q-2} \left(\frac{\pi^*-\varepsilon}{1-(\pi^*-\varepsilon)}\right)^{2q-1} \frac{2q-1}{1-\left(\frac{\pi^*-\varepsilon}{1-(\pi^*-\varepsilon)} \frac{1-(\pi^*+\varepsilon)}{\pi^*+\varepsilon}\right)^{2q-1}} > 0 \end{aligned}$$

in both cases $q < 0.5$ and $q > 0.5$.

b) As a consequence we have that $P(\hat{\pi}_\tau = \pi^* - \varepsilon)$ decreases if $\hat{\pi}_0$ increases.

c) For $E(\tau)$ it is not possible to find such a relation to $\hat{\pi}_0$: Part a) of the example delivers the highest value of $E(\tau)$ for $\hat{\pi}_0 = \pi^*$ whereas part b) delivers various possibilities: There are also cases where the smallest or biggest value of $\hat{\pi}_0$ attains the highest value of $E(\tau)$. Therefore there is no common relation between $\hat{\pi}_0$ and $E(\tau)$ in the case of $\frac{\mu-r}{\sigma^2} \neq 0.5$. In the case of $\frac{\mu-r}{\sigma^2} = 0.5$ we observe the highest value of $E(\tau)$ in the case of $\hat{\pi}_0 = \pi^*$. However, a simple computation shows that it is not exactly π^* which delivers the highest value of $E(\tau)$:

$$\begin{aligned}
\frac{\partial}{\partial \hat{\pi}_0} E(\tau) &= \frac{1}{\sigma^2\, \hat{\pi}_0(1-\hat{\pi}_0)} \\
&\quad \left\{ \ln\left(\frac{\pi^*+\varepsilon}{1-(\pi^*+\varepsilon)} \frac{1-\hat{\pi}_0}{\hat{\pi}_0} \right) - \ln\left(\frac{\hat{\pi}_0}{1-\hat{\pi}_0} \frac{1-(\pi^*-\varepsilon)}{\pi^*-\varepsilon} \right) \right\} = 0 \\
&\iff \quad \frac{\pi^*+\varepsilon}{1-(\pi^*+\varepsilon)} \frac{1-\hat{\pi}_0}{\hat{\pi}_0} = \frac{\hat{\pi}_0}{1-\hat{\pi}_0} \frac{1-(\pi^*-\varepsilon)}{\pi^*-\varepsilon} \\
&\iff \quad \left(\frac{1-\hat{\pi}_0}{\hat{\pi}_0} \right)^2 = \frac{1-(\pi^*-\varepsilon)}{\pi^*-\varepsilon} \frac{1-(\pi^*+\varepsilon)}{\pi^*+\varepsilon} \\
&\iff \quad \frac{1}{\hat{\pi}_0} - 1 = \sqrt{\frac{1-(\pi^*-\varepsilon)}{\pi^*-\varepsilon} \frac{1-(\pi^*+\varepsilon)}{\pi^*+\varepsilon}} \\
&\iff \quad \hat{\pi}_0 = \frac{1}{1+\sqrt{\frac{1-(\pi^*-\varepsilon)}{\pi^*-\varepsilon} \frac{1-(\pi^*+\varepsilon)}{\pi^*+\varepsilon}}}.
\end{aligned}$$

From this result we obtain that in the case of $\pi^* = 0.6, \varepsilon = 0.1$ the expectation is maximal for $\hat{\pi}_0 = 0.604356 \neq \pi^*$. Also in the case of $\pi^* = 0.3, \varepsilon = 0.2$ the expectation attains its highest value for $\hat{\pi}_0 = 0.25 \neq \pi^*$. Only for $\pi^* = 0.5, \varepsilon = 0.1$ π^* delivers the highest value of $E(\tau)$.

d) If all parameters except $\hat{\pi}_0 = \pi^*$ remain constant, then, according to the figures, an increase in π^* causes a decrease in $E(\tau)$ until $\pi^* = 0.5$ and an increase in $E(\tau)$ if π^* is bigger than 0.5. We also see that the probability to reach the upper boundary first decreases with an increase in π^*.

e) Of course an increase in ε causes an increase in $E(\tau)$.

In the following we will analyse the impact of $E(\tau)$ on the expected number of interventions between 0 and T. This number is very important in the presence of transaction costs since each transaction reduces the wealth of the investor. Therefore the investor is interested in not intervening too often. On the other hand, the only reason to make interventions is to keep the portfolio process $\hat{\pi}$ not too far away from the constant portfolio process π^*. In the presence of transaction costs the investor therefore has to find a suitable ε which allows him to keep his portfolio process near to π^* without intervening too often. To compute the expected number of interventions we need the following notations which are quite similar to those of Notation 2, Chapter 2:

Notation 4. Let τ_0 be the first intervention time if the portfolio process starts at $\hat{\pi}_0 \in [\pi^* - \varepsilon, \pi^* + \varepsilon]$. Let $\tau_1, \tau_2, \ldots$ be the periods of time after which the further interventions take place or would take place if the time horizon was not bounded. According to our model after any intervention the portfolio process starts at π^*. Therefore the stopping times $(\tau_i)_{i \geq 0}$ are independent and even identically distributed for $i \neq 0$. Further we define N as the number of interventions that occur between 0 and T and $\tilde{N} = N - 1$ as the number of interventions without the first one τ_0. We also define

$$S_{\tilde{N}} := \sum_{i=1}^{\tilde{N}} \tau_i$$

which describes the length of time between τ_0 and the last intervention and is therefore bounded by $T - \tau_0$ if $\tau_0 < T$ and equals 0 otherwise.

Theorem 2. (Expected number of interventions.)

i) For all $\hat{\pi}_0 \in [\pi^* - \varepsilon, \pi^* + \varepsilon]$ the expected number of interventions in $[0, T]$ is bounded from above by

$$\hat{N} := \frac{T}{E(\tau_1)} + 1 . \tag{3.57}$$

ii) In the case of $\hat{\pi}_0 = \pi^*$ the expected number of interventions is even bounded from above by

$$\hat{N}^{\pi^*} := \frac{T}{E(\tau_1)} . \tag{3.58}$$

Proof: The proof is absolutely identical to the proof of the first case of Theorem 4, Chapter 2 ($T_0 = \tau_0$ and $\tau_1 = \rho_1, \tau_2 = \rho_2, \ldots$) if we replace $L_{\tilde{N}}$ by $S_{\tilde{N}}$.

In the case that the investor starts with $\hat{\pi}_0 = \pi^*$ the first stopping time τ_0 is not defined and therefore $N = \tilde{N}$. (It is important to note that $\tau_0 = 0$ means that the investor starts with $\hat{\pi}_0 = \pi^* - \varepsilon$ or $\hat{\pi}_0 = \pi^* + \varepsilon$.) This implies that the last equation of step 3 of the proof of Thoerem 4, Chapter 2 reads

$$E(S_N) = E(N)\, E(\tau_1) ,$$

implying that

$$E(N) = \frac{E(S_N)}{E(\tau_1)} \leq \frac{T}{E(\tau_1)}.$$

□

The following examples will illustrate the quantitative behaviour of $\hat{N}^{\pi^*}$ in dependence on π^*, ε and T.

Examples 5.

a) $\sigma = 0.3$, $\frac{\mu - r}{\sigma^2} = 0.1667$.

π^*	ε	$E(\tau_1)$	T	$\hat{N}^{\pi^*}$
0.4	0.2	8.74636	1	0.11433
0.4	0.2	8.74636	5	0.57167
0.1667	0.1	7.00534	1	0.14275
0.1667	0.1	7.00534	10	1.42748
0.4	0.1	1.9854	1	0.50368
0.4	0.1	1.9854	5	2.51838

b) $\sigma = 0.15$, $\frac{\mu - r}{\sigma^2} = 0.889$.

π^*	ε	$E(\tau_1)$	T	$\hat{N}^{\pi^*}$
0.3	0.2	45.0439	1	0.0222
0.3	0.2	45.0439	5	0.111
0.4	0.1	7.85392	1	0.12732
0.4	0.1	7.85392	10	1.27324
0.889	0.1	89.9236	1	0.01112
0.889	0.1	89.9236	5	0.0556

c) $\frac{\mu - r}{\sigma^2} = 0.5$.

π^*	ε	$E(\tau_1)$	T	$\hat{N}^{\pi^*}$
0.6	0.2	19.8846	1	0.05029
0.6	0.2	19.8846	5	0.25145
0.4	0.1	4.47869	1	0.22328
0.4	0.1	4.47869	10	2.2328
0.5	0.1	4.11005	1	0.24331
0.5	0.1	4.11005	10	2.43306
0.7	0.1	5.95366	1	0.16796
0.7	0.1	5.95366	5	0.83982

(The data also deliver $\hat{N} = \hat{N}^{\pi^*} + 1$.)

Remark 11. Depending on the parameters σ and $\frac{\mu - r}{\sigma^2}$ we obtain various bounds for the expected number of interventions for the same underlying interval $[\pi^* - \varepsilon, \pi^* + \varepsilon]$. (The above examples contain three different values of $\hat{N}^{\pi^*}$ for the interval $[0.3, 0.5]$.)

3.3.3 Fixed transaction costs

Throughout this last subsection we assume that each transaction causes fixed transaction costs which have to be paid at the time horizon. Under this assumption we will have a look at the expected transaction costs which have to be paid and at the expected terminal wealth.

Proposition 13. *We assume that each transaction in $[0, T]$ causes fixed transaction costs $K > 0$. Further we assume that the investor has to pay the accumulated transaction costs at time T . (Of course he also has to pay interest where the interest rate is denoted by r .) Then the expected transaction costs are bounded from above by*

$$\hat{C} := K\, e^{r\,T} \left(\frac{T}{E(\tau_1)} + 1 \right) = K\, e^{r\,T}\, \hat{N} \tag{3.59}$$

and by

$$\hat{C}^{\pi^*} := K\, e^{r\,T}\, \frac{T}{E(\tau_1)} = K\, e^{r\,T}\, \hat{N}^{\pi^*} , \tag{3.60}$$

respectively, if the portfolio process starts at $\hat{\pi}_0 = \pi^*$. Further the expected terminal wealth is bounded from below by

$$\hat{W} := V_0^{\hat{\pi}}\, e^{(r + (\mu - r)(\pi^* - \varepsilon))T} - \hat{C} \tag{3.61}$$

and by

$$\hat{W}^{\pi^*} := V_0^{\hat{\pi}}\, e^{(r + (\mu - r)(\pi^* - \varepsilon))T} - \hat{C}^{\pi^*} , \tag{3.62}$$

respectively.

Proof: The upper bounds for the expected transaction costs follow directly from Theorem 2 whereas the lower bound for the expected terminal wealth is obtained from the lower bound in the case without transaction costs ((3.23)) and from (3.59)/(3.60). □

The next examples will illustrate the quantitative behaviour of $\hat{C}^{\pi^*}$ and $\hat{W}^{\pi^*}$. For comparison we will also give the terminal wealth of the bond strategy ($V_0^{\hat{\pi}}\, e^{r\,T}$).

Examples 6.

a) $\mu = 0.025\,,\ \sigma = 0.3\,,\ r = 0.01\,,\ V_0^{\hat{\pi}} = 100\,,\ K = 0.1\,.$

π^*	ε	$E(\tau_1)$	T	$\hat{C}^{\pi^*}$	$\hat{W}^{\pi^*}$	$V_0^{\hat{\pi}}\, e^{r\,T}$
0.1667	0.1	7.00534	10	0.1578	111.471	110.517
0.4	0.1	1.9854	15	0.8778	123.419	116.183
0.4	0.1	1.9854	10	0.5566	115.047	110.517
0.4	0.1	1.9854	5	0.2648	107.255	105.127

b) $\mu = 0.03\,,\ \sigma = 0.15\,,\ r = 0.01\,,\ V_0^{\hat{\pi}} = 100\,,\ K = 0.1\,.$

π^*	ε	$E(\tau_1)$	T	$\hat{C}^{\pi^*}$	$\hat{W}^{\pi^*}$	$V_0^{\hat{\pi}}\, e^{r\,T}$
0.4	0.1	7.85392	10	0.1407	117.21	110.517
0.889	0.1	89.9236	10	0.0123	129.393	110.517
0.889	0.1	89.9236	1	0.0011	102.61	101.005

c) $\mu = 0.03\,,\ \sigma = 0.2\,,\ r = 0.01\,,\ V_0^{\hat{\pi}} = 100\,,\ K = 0.1\,.$

π^*	ε	$E(\tau_1)$	T	$\hat{C}^{\pi^*}$	$\hat{W}^{\pi^*}$	$V_0^{\hat{\pi}}\, e^{r\,T}$
0.4	0.1	4.47869	10	0.2468	117.104	110.517
0.5	0.1	4.11005	10	0.2689	119.453	110.517
0.7	0.1	5.95366	5	0.0883	111.54	105.127

(The data also deliver $\hat{C} = \hat{C}^{\pi^*} + K\, e^{r\,T}$ and $\hat{W} = \hat{W}^{\pi^*} - K\, e^{r\,T}$.)

Remark 12.

a) If π^* and T are not too small the lower bound of the expected terminal wealth ($\hat{W}^{\pi^*}$) is much higher than the terminal wealth corresponding to the bond.

b) The relative advantage of the bounded portfolio process over the pure bond strategy ($\hat{W}^{\pi^*}/(V_0^{\hat{\pi}} e^{r\,T})$) is roughly linear in time (Examples 6 a)).

c) For the three different cases of $\pi^* = 0.4\,, \varepsilon = 0.1\,, T = 10$ we obtain roughly the same value of $\hat{W}^{\pi^*}$. (A higher value of $\frac{\mu - r}{\sigma^2}$ causes a higher value of $\hat{W}^{\pi^*}$.)

d) For fixed parameters $\mu\,, \sigma\,, r$ each value of π^* can be realized as the optimal portfolio process corresponding to logarithmic or power utility (with $\gamma < 1$). Depending on π^* the lower bound of the expected terminal wealth can attain quite various values (Examples 6 b)).

Corollary 10. *Let $\tilde{V}_T^{\hat{\pi}}$ be the terminal wealth after the transaction costs have been paid, let $\mu > r$.*

i) Let $\hat{\pi}_0 \in [\pi^* - \varepsilon, \pi^* + \varepsilon]$. Then $E(\tilde{V}_T^{\hat{\pi}}) > 0$ if

$$\varepsilon < \pi^* - \frac{\ln\left(\frac{K}{V_0^{\hat{\pi}}}\left(\frac{T}{E(\tau_1)} + 1\right)\right)}{(\mu - r)\,T}\,.$$

ii) Let $\hat{\pi}_0 = \pi^*$. Then $E(\tilde{V}_T^{\hat{\pi}}) > 0$ if

$$\varepsilon < \pi^* - \frac{\ln\left(\frac{K}{V_0^{\hat{\pi}}}\frac{T}{E(\tau_1)}\right)}{(\mu - r)\,T}\,.$$

Remark 13. Since K is assumed to be very small, the upper bounds for ε usually attain high values ($>> 1$), thus guaranteeing a positive expected terminal wealth.

3.4 Conclusion

Throughout this chapter we have analysed the situation of an investor who rebalances his portfolio by shifting his portfolio process $\hat{\pi}$ to a constant number π^* whenever $\hat{\pi}$ reaches the boundary of the interval $[\pi^* - \varepsilon, \pi^* + \varepsilon] \subset [0, 1]$. We have shown that the corresponding wealth process $V_T^{\hat{\pi}}$ converges in probability towards that one of an investor with constant portfolio process π^* . Further we have derived bounds for the expected terminal wealth and the expected utility the investor obtains from the strategy corresponding to a bounded portfolio process $\hat{\pi}$. These bounds imply bounds for the so-called "efficiency" which describes how good the expected utility of $V_T^{\hat{\pi}}$ is, compared with that one corresponding to π^* (which is in the case of logarithmic and power utility, respectively, the optimal one). Here the bounded portfolio process $\hat{\pi}$ achieves amazingly high values. We have also shown that $V_t^{\hat{\pi}}$ exceeds the pure bond investment in terms of expectation and expected utility. Further we have analysed the random times when the investor has to intervene and have derived an upper bound for the expected number of interventions. This bound has then been used to compute bounds for the expected (fixed) transaction costs and the expected terminal wealth in the presence of fixed transaction costs.

Due to all these results investing corresponding to a bounded portfolio process $\hat{\pi}$ seems to be a good alternative to that one corresponding to an (optimal) constant portfolio process π^* (above all in the presence of fixed transaction costs): Although the investor using $\hat{\pi}$ only intervenes very rarely, he is able to achieve results which are very close to those of the optimally rebalancing investor.

Chapter 4

Strategies with equidistant rebalancing

In the previous chapter we analysed the situation of an investor with a bounded portfolio process: This investor rebalanced his portfolio whenever his portfolio process reached the boundary of the symmetrical interval $[\pi^* - \varepsilon, \pi^* + \varepsilon]$ by shifting the portfolio process to π^*. As π^* was (most of the time) assumed to be the optimal constant portfolio process of a continuously rebalancing investor with logarithmic or power utility, the aim of the discretely rebalancing investor was to keep his portfolio process near to the optimal constant one but not to intervene too often. Throughout this chapter we will look at discrete strategies which are quite similar to those corresponding to the bounded portfolio process: Again the investor tries to keep his portfolio process near to the optimal one π^* but this time he rebalances his portfolio in equidistant time points which are determined at $t = 0$ and which do not depend on the value of the portfolio process at the time of intervention. Naturally, this will lead to a portfolio process which is not bounded at all but which has deterministic intervention times.

In the following we will determine the expected terminal wealth, the expected volume of the single transactions as well as that of the accumulated transactions. Further we will derive lower bounds for the expected utility of the terminal wealth with respect to logarithmic and power utility, respectively, which will then imply lower bounds for the efficiency of the discrete strategy (where the efficiency will be defined as the one of Chapter 3). Additionally we will give an optimal but approximative efficiency via Taylor-approximation for logarithmic utility. Lastly, we will compare the strategies of Chapter 3 and this chapter with respect to their efficiency.

4.1 The model and basic results

As in the two previous chapters the financial market consists of a riskless bond with price B_t and a risky stock with price P_t given by

$$\begin{aligned} dB_t &= r\, B_t\, dt\,, \\ B_0 &= 1 \end{aligned}$$

and

$$\begin{aligned} dP_t &= P_t\,[\mu\, dt + \sigma\, dW_t]\,, \\ P_0 &= p_0 > 0 \end{aligned}$$

with real numbers $r\,,\mu$ and $\sigma > 0$ where W_t is a one-dimensional standard Brownian motion with the usual right-continuous Brownian filtration $\{F_t\}_{t\in[0,T]}$.

Throughout this chapter we will analyse the situation of an investor who is restricted to rebalancing his portfolio at the equidistant time points

$$t \in \{\Delta t,\, 2\Delta t, \ldots, (N-1)\Delta t\} =: E \tag{4.1}$$

where $N\Delta t = T$ with $\Delta t > 0$. Of course Δt has to be specified at $t = 0$ which makes all the intervention times deterministic ones.

Again the aim of the investor is to attain a similar expected utility of his terminal wealth at $t = T$ as an investor with a constant portfolio process $\pi^* \in (0,1)$ who rebalances continuously. Therefore he is assumed to shift his portfolio process to π^* whenever $t \in E$. Further we assume that at $t = 0$ the investor starts with the initial capital of 1 and that his portfolio process equals π^*, the portfolio process of the continuously rebalancing investor.

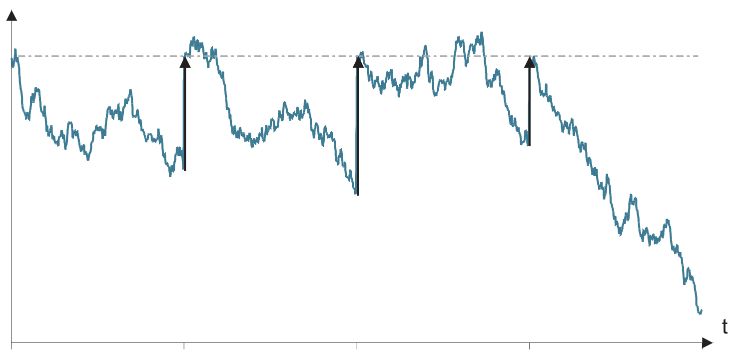

Fig. 8. Portfolio process with equidistant rebalancing

To formalize these assumptions we introduce the following notation:

Notation 1. (Portfolio process.) Let $\bar{\pi}_t\,, t \in [0,T]\,, \bar{\pi}_0 = \pi^* \in (0,1)$ denote the portfolio process of the investor who rebalances his portfolio at the discrete time points $\Delta t\,, 2\Delta t\,, \ldots, (N-1)\Delta t$ by shifting his portfolio process to π^* and who does not intervene in between.

Remark 1. Due to its definition $\bar{\pi}$ is continuous on the intervals

$$[k\Delta t, (k+1)\Delta t)\,, k = 0, \ldots, N-2$$

and

$$[(N-1)\Delta t, N\Delta t]\,.$$

On these intervals $\bar{\pi}$ evolves according to the differential equation

$$\begin{aligned} d\pi_t &= \pi_t\,[1-\pi_t]\,[(\mu - r - \pi_t\sigma^2)dt + \sigma dW_t]\,, \\ \pi_0 &= \pi^* \in (0,1) \end{aligned}$$

(see Proposition 1, Chapter 3). Hence $\bar{\pi}$ is not bounded.

Remark 2. Usually π^* is the optimal portfolio process of a continuously rebalancing investor who maximizes the expected utility of his terminal wealth and whose utility function coincides with the one of the discretely rebalancing investor. Since we will only consider the cases of logarithmic or power utility functions, the optimal portfolio process will always be constant over time (which we already assumed above).

Notation 2. (Wealth process.) Let $(V_t^{\bar{\pi}})_{t\in[0,T]}$ with $V_0^{\bar{\pi}} = 1$ denote the wealth process corresponding to the portfolio process $(\bar{\pi}_t)_{t\in[0,T]}$ as given in Notation 1.

Since the portfolio process $\bar{\pi}$ is unbounded we will not find simple bounds for the expected terminal wealth $E(V_T^{\bar{\pi}})$ as we did in Chapter 3 for $E(V_T^{\hat{\pi}})$. Nevertheless, we will write the wealth process in such a way that it is more convenient to deal with: To do that we introduce the random variable

$$Z_k = e^{\left(\mu - \frac{1}{2}\sigma^2\right)\Delta t + \sigma(W_{(k+1)\Delta t} - W_{k\Delta t})} \tag{4.2}$$

which is the return of the stock on the interval

$$[k\Delta t, (k+1)\Delta t]\,, k = 0, \ldots, N-1\,.$$

Obviously, $(Z_k)_{k=0,\ldots,N-1}$ are i.i.d. random variables. With their help we can write the terminal wealth as follows:

$$V_T^{\bar{\pi}} \;=\; V_{N\Delta t}^{\bar{\pi}}$$

$$
\begin{aligned}
&= V^{\bar{\pi}}_{(N-1)\Delta t}\left(\pi^* Z_{N-1} + (1-\pi^*)\, e^{r\Delta t}\right) \\
&= V^{\bar{\pi}}_{(N-2)\Delta t}\left(\pi^* Z_{N-2} + (1-\pi^*)\, e^{r\Delta t}\right)\left(\pi^* Z_{N-1} + (1-\pi^*)\, e^{r\Delta t}\right) \\
&= V^{\bar{\pi}}_{(N-2)\Delta t}\prod_{k=N-2}^{N-1}\left(\pi^* Z_k + (1-\pi^*)\, e^{r\Delta t}\right) \\
&\vdots \\
&= V^{\bar{\pi}}_0\prod_{k=0}^{N-1}\left(\pi^* Z_k + (1-\pi^*)\, e^{r\Delta t}\right) \\
&= \prod_{k=0}^{N-1}\left(\pi^*\left(Z_k - e^{r\Delta t}\right) + e^{r\Delta t}\right) . \qquad (4.3)
\end{aligned}
$$

Remark 3. Due to $\pi^* \in (0,1)$ the terminal wealth is non-negative.

Before we go on with computing the expectation of the terminal wealth we will give some helpful results on the random variables Z_k:

Lemma 1. *Let $y > 0$. Then the definition of $Z_k, k = 0, \ldots, N-1$, implies that*

i)

$$P(Z_k \leq y) = \Phi\left(\frac{\ln y - (\mu - \frac{1}{2}\sigma^2)\Delta t}{\sigma\sqrt{\Delta t}}\right) , \qquad (4.4)$$

ii)

$$E(Z_k) = e^{\mu\Delta t} , \qquad (4.5)$$

iii)

$$E(Z_k^2) = e^{(2\mu+\sigma^2)\Delta t} . \qquad (4.6)$$

Proof: Since the proofs of i) and ii) are straight forward, they are omitted. We only prove iii):

$$E(Z_k^2) = e^{2\left(\mu - \frac{1}{2}\sigma^2\right)\Delta t}\underbrace{E\left(e^{2\,\sigma W_{\Delta t}}\right)}_{=e^{\frac{4}{2}\sigma^2\,\Delta t}} = e^{(2\mu+\sigma^2)\Delta t} .$$

□

Proposition 1. *The expectation of the wealth process $V^{\bar{\pi}}_T$ with $V^{\bar{\pi}}_0 = 1$ is given by*

$$E(V^{\bar{\pi}}_T) = \left[\pi^*\left(e^{\mu\Delta t} - e^{r\Delta t}\right) + e^{r\Delta t}\right]^N . \qquad (4.7)$$

Proof: Since $Z_k\,, k = 0, \ldots, N-1$, are i.i.d., we obtain from (4.3) that

$$\begin{aligned}
E(V_T^{\bar{\pi}}) &= E\left\{\prod_{k=0}^{N-1}\left[\pi^*\left(Z_k - e^{r\Delta t}\right) + e^{r\Delta t}\right]\right\} \\
&= \prod_{k=0}^{N-1} E\left[\pi^*\left(Z_k - e^{r\Delta t}\right) + e^{r\Delta t}\right] \\
&= \prod_{k=0}^{N-1}\left[\pi^*\left(E(Z_k) - e^{r\Delta t}\right) + e^{r\Delta t}\right] \\
&= \prod_{k=0}^{N-1}\left[\pi^*\left(e^{\mu\Delta t} - e^{r\Delta t}\right) + e^{r\Delta t}\right] \\
&= \left[\pi^*\left(e^{\mu\Delta t} - e^{r\Delta t}\right) + e^{r\Delta t}\right]^N .
\end{aligned}$$

□

Remark 4.

a) If $\mu \geq r$ (which is usually the case), the expected terminal wealth exceeds the final wealth of the pure bond investment. (This is also true if $V_0^{\bar{\pi}} \neq 1$.)

b) If $\mu = r$, which is the case in the risk-neutral market, the expectation equals the value of the bond:

$$E(V_T^{\bar{\pi}}) = e^{r\Delta t N} = e^{rT} .$$

c) In the case $\mu > r$ the expected terminal wealth increases with $\pi^* \in (0,1)$.

From (4.7) we also obtain the limit of $E(V_T^{\bar{\pi}})$ for $N \to \infty$:

Corollary 1.

$$\lim_{N\to\infty} E(V_T^{\bar{\pi}}) = e^{(r+\pi^*(\mu-r))\,T} . \tag{4.8}$$

Proof: For each $\varepsilon > 0$ we have

$$\begin{aligned}
E(V_T^{\bar{\pi}}) &= \left[\pi^*\left(e^{\mu\Delta t} - e^{r\Delta t}\right) + e^{r\Delta t}\right]^N \\
&= \left[\pi^* e^{\mu\frac{T}{N}} + e^{r\frac{T}{N}}(1-\pi^*)\right]^N \\
&= \left[\pi^*\left(1 + \mu\frac{T}{N}\right) + \left(1 + r\frac{T}{N}\right)(1-\pi^*) + o\left(\frac{1}{N}\right)\right]^N \\
&= \left[1 + \frac{T}{N}(\pi^*(\mu - r) + r) + o\left(\frac{1}{N}\right)\right]^N \\
&\begin{cases} \geq & \left[1 + \frac{T}{N}(r + \pi^*(\mu-r)) - \frac{\varepsilon}{N}\right]^N , \\ \leq & \left[1 + \frac{T}{N}(r + \pi^*(\mu-r)) + \frac{\varepsilon}{N}\right]^N \end{cases}
\end{aligned}$$

which implies

$$e^{(r+\pi^*(\mu-r))T-\varepsilon} \le \lim_{N\to\infty} E(V_T^{\bar{\pi}}) \le e^{(r+\pi^*(\mu-r))T+\varepsilon}\,.$$

Since ε is arbitrariliy small we obtain (4.8). □

Remark 5. As we have seen in Proposition 3 i) of Chapter 3 the above limit

$$e^{(r+\pi^*\,(\mu-r))\,T}$$

is the expected terminal wealth of the investor with a constant portfolio process π^*.

By definition, the portfolio process $\bar{\pi}_t\,, t\in[0,T]$, is continuous except at the time points $k\Delta t\,, k=1,\dots,N-1$ when it is shifted to π^*. In the following we will examine the value of $\bar{\pi}$ right before the rebalancing at $k\Delta t\,, k=1,\dots,N-1$ takes place.

Notations 3. Let $\bar{\pi}_{k\Delta t-}\,, k=1,\dots,N-1$, denote the value of the portfolio process just before the rebalancing in $k\Delta t$ takes place, let $S_{k\Delta t-}\,, k=1,\dots,N-1$, denote the corresponding value of the risky money and let $\varphi_{k\Delta t}\,, k=0,\dots,N-1$, denote the number of stocks right after the rebalancing at $k\Delta t$.

Proposition 2. *The value of $\bar{\pi}_{k\Delta t-}\,, k=1,\dots,N-1$, is given by*

$$\bar{\pi}_{k\Delta t-} = \frac{\pi^*\,Z_{k-1}}{\pi^*\,Z_{k-1}+(1-\pi^*)\,e^{r\,\Delta t}} \tag{4.9}$$

with Z_{k-1} as given in (4.2).

Proof: Due to $V_{k\Delta t}^{\bar{\pi}} = V_{k\Delta t-}^{\bar{\pi}}$ (which is the case if the investor doesn't have to pay transaction costs at the moment of rebalancing) we obtain

$$\begin{aligned}
\bar{\pi}_{k\Delta t-} &= \frac{S_{k\Delta t-}}{V_{k\Delta t-}^{\bar{\pi}}}\\
&= \frac{\varphi_{(k-1)\Delta t}\,P_{k\Delta t}}{V_{k\Delta t}^{\bar{\pi}}}\\
&= \frac{\pi^*\,V_{(k-1)\Delta t}^{\bar{\pi}}}{P_{(k-1)\Delta t}}\,\frac{P_{k\Delta t}}{V_{k\Delta t}^{\bar{\pi}}}\\
&= \pi^*\,\frac{P_{k\Delta t}}{P_{(k-1)\Delta t}}\,\frac{V_{(k-1)\Delta t}^{\bar{\pi}}}{V_{k\Delta t}^{\bar{\pi}}}\\
&= \pi^*\,Z_{k-1}\,\frac{1}{\pi^*\,Z_{k-1}+(1-\pi^*)\,e^{r\,\Delta t}}\,.
\end{aligned}$$

□

Corollary 2. *The value of the portfolio process has to be reduced at* $k\Delta t\,, k = 1, \ldots, N-1$ *if and only if the corresponding relative increment of the stock,* Z_{k-1}*, exceeds the increment of the bond,* $e^{r\Delta t}$*.*

Proof: Due to Proposition 2 and due to $\pi^* \in (0,1)$ we get that

$$\begin{aligned} \bar{\pi}_{k\Delta t-} > \pi^* \quad &\Longleftrightarrow \quad \frac{Z_{k-1}}{\pi^* Z_{k-1} + (1-\pi^*)\, e^{r\,\Delta t}} > 1 \\ &\Longleftrightarrow \quad Z_{k-1} > \pi^* Z_{k-1} + (1-\pi^*)\, e^{r\,\Delta t} \\ &\Longleftrightarrow \quad Z_{k-1}\,(1-\pi^*) > e^{r\,\Delta t}\,(1-\pi^*) \\ &\Longleftrightarrow \quad Z_{k-1} > e^{r\,\Delta t}\,. \end{aligned}$$

□

Remark 6. If the relative increment of the stock is greater than 1 (which implies an absolute growth) but lower than $e^{r\Delta t}$, the portfolio process satisfies $\bar{\pi}_{k\,\Delta t-} < \pi^*$. Therefore the investor has to increase the value of his portfolio process, although the stock has increased in the foregoing period.

To close this section we will focus on the expected volume of the single transactions $S_{k\Delta t} - S_{k\Delta t-}\,, k = 1, \ldots, N-1$ as well as on that one of the accumulated transactions. To determine these values we will use the proof of Proposition 2.

Proposition 3. *The expected volume of the transaction taking place at* $k\Delta t\,, k = 1, \ldots, N-1$*, is given by*

$$E(S_{k\Delta t} - S_{k\Delta t-}) \;=\; \pi^*\,(1-\pi^*)\,\left(e^{r\Delta t} - e^{\mu\Delta t}\right)\,\left[\pi^*\, e^{\mu\Delta t} + (1-\pi^*)\, e^{r\Delta t}\right]^{k-1}. \tag{4.10}$$

Proof: From (4.3) and the proof of Proposition 2 we get that

$$\begin{aligned} S_{k\Delta t} &= \pi^*\, V^{\bar{\pi}}_{k\Delta t} \\ &= \pi^*\, V^{\bar{\pi}}_{(k-1)\Delta t}\,\left[\pi^* Z_{k-1} + (1-\pi^*)\, e^{r\,\Delta t}\right] \end{aligned}$$

and that

$$S_{k\Delta t-} \;=\; \bar{\pi}_{k\Delta t-}\, V^{\bar{\pi}}_{k\Delta t} = \pi^* Z_{k-1}\, V^{\bar{\pi}}_{(k-1)\Delta t}\,.$$

Since Z_{k-1} and $V^{\bar{\pi}}_{(k-1)\Delta t}$ are independent we then obtain

$$\begin{aligned} E(S_{k\Delta t} - S_{k\Delta t-}) &= E\left(\pi^*\, V^{\bar{\pi}}_{(k-1)\Delta t}\,\left[\pi^* Z_{k-1} + (1-\pi^*)\, e^{r\,\Delta t} - Z_{k-1}\right]\right) \\ &= \pi^*\, E(V^{\bar{\pi}}_{(k-1)\Delta t})\, E\left[Z_{k-1}\,(\pi^*-1) + (1-\pi^*)\, e^{r\,\Delta t}\right] \\ &= \pi^*\,(1-\pi^*)\, E(V^{\bar{\pi}}_{(k-1)\Delta t})\, E\left(e^{r\Delta t} - Z_{k-1}\right) \\ &= \pi^*\,(1-\pi^*)\,\left[\pi^*\,\left(e^{\mu\,\Delta t} - e^{r\,\Delta t}\right) + e^{r\,\Delta t}\right]^{k-1}\,\left[e^{r\,\Delta t} - e^{\mu\,\Delta t}\right] \end{aligned}$$

□

Remark 7. Of course we assume that $\mu > r$ which implies that in the mean the investor has to reduce the volume of his risky money.

Proposition 4. *The expectation of the accumulated terminal value of all transactions is given by*

$$\begin{aligned} & E\left(\sum_{k=1}^{N-1} (S_{k\Delta t} - S_{k\Delta t-})\, e^{r\,\Delta t\,(N-k)}\right) \\ = & \; (1-\pi^*)\, e^{r\,T} \left\{1 - \left[\pi^*\, e^{(\mu-r)\Delta t} + (1-\pi^*)\right]^{N-1}\right\}. \end{aligned} \tag{4.11}$$

Proof: From (4.10) we obtain that

$$\begin{aligned} & \sum_{k=1}^{N-1} E(S_{k\Delta t} - S_{k\Delta t-})\, e^{r\,\Delta t\,(N-k)} \\ = & \sum_{k=1}^{N-1} \pi^*\,(1-\pi^*)\, \left[\pi^* \left(e^{\mu\,\Delta t} - e^{r\,\Delta t}\right) + e^{r\,\Delta t}\right]^{k-1} \left[e^{r\,\Delta t} - e^{\mu\,\Delta t}\right]\, e^{r\,\Delta t\,(N-k)} \\ = & \; \pi^*\,(1-\pi^*)\, \left[e^{r\,\Delta t} - e^{\mu\,\Delta t}\right]\, e^{r\,\Delta t\,(N-1)} \\ & \sum_{k=1}^{N-1} \left[\pi^* \left(e^{\mu\,\Delta t} - e^{r\,\Delta t}\right) + e^{r\,\Delta t}\right]^{k-1}\, e^{r\,\Delta t\,(1-k)} \end{aligned}$$

where

$$\begin{aligned} & \sum_{k=1}^{N-1} \left[\pi^* \left(e^{\mu\,\Delta t} - e^{r\,\Delta t}\right) + e^{r\,\Delta t}\right]^{k-1}\, e^{r\,\Delta t\,(1-k)} \\ = & \sum_{k=0}^{N-2} \left[\pi^*\, e^{(\mu-r)\Delta t} + (1-\pi^*)\right]^{k} \\ = & \frac{1 - \left[\pi^*\, e^{(\mu-r)\Delta t} + (1-\pi^*)\right]^{N-1}}{1 - \left[\pi^*\, e^{(\mu-r)\Delta t} + (1-\pi^*)\right]} \\ = & \frac{1 - \left[\pi^*\, e^{(\mu-r)\Delta t} + (1-\pi^*)\right]^{N-1}}{\pi^*\,\left[1 - e^{(\mu-r)\Delta t}\right]} \\ = & \frac{e^{r\,\Delta t}}{\pi^*}\, \frac{1 - \left[\pi^*\, e^{(\mu-r)\Delta t} + (1-\pi^*)\right]^{N-1}}{\left[e^{r\,\Delta t} - e^{\mu\Delta t}\right]}. \end{aligned}$$

This implies

$$\begin{aligned} & \sum_{k=1}^{N-1} E(S_{k\Delta t} - S_{k\Delta t-})\, e^{r\,\Delta t\,(N-k)} \\ = & \; (1-\pi^*)\, e^{r\,\Delta t\,N} \left\{1 - \left[\pi^*\, e^{(\mu-r)\Delta t} + (1-\pi^*)\right]^{N-1}\right\}. \end{aligned}$$

$\square$

Corollary 3. *The accumulated terminal value of all transactions satisfies*

$$\lim_{N\to\infty} E\left(\sum_{k=1}^{N-1} (S_{k\Delta t} - S_{k\Delta t-})\, e^{r\,\Delta t\,(N-k)}\right) \;=\; (1-\pi^*)\,\left[e^{r\,T} - e^{\mu\,\pi^*\,T}\right]. \quad (4.12)$$

Proof: For large N and small Δt, respectively, we use that

$$\begin{aligned}\left[\pi^*\, e^{(\mu-r)\Delta t} + (1-\pi^*)\right]^N &= \left[\pi^*\,(1+(\mu-r)\,\Delta t) + (1-\pi^*) + \pi^*\, o\left(\frac{1}{N}\right)\right]^N \\ &= \left[1 + \pi^*\,(\mu-r)\,\frac{T}{N} + \pi^*\, o\left(\frac{1}{N}\right)\right]^N \\ &\overset{N\to\infty}{\longrightarrow} e^{(\mu-r)\pi^*\,T}\end{aligned}$$

(see proof of Corollary 1). This implies

$$\begin{aligned}\lim_{N\to\infty} E\left(\sum_{k=1}^{N-1} (S_{k\Delta t} - S_{k\Delta t-})\, e^{r\,\Delta t\,(N-k)}\right) &= (1-\pi^*)\, e^{r\,T}\,\left\{1 - e^{(\mu-r)\pi^*\,T}\right\} \\ &= (1-\pi^*)\,\left\{e^{r\,T} - e^{\mu\,\pi^*\,T}\right\}.\end{aligned}$$

□

4.2 Efficiency

In analogy to Subsection 3.2.2 we would like to value the goodness of the portfolio process $\bar{\pi}$, i.e. we would like to compare the corresponding expected utility in the case of logarithmic or power utility with that one corresponding to the (optimal) constant portfolio process π^*. Hence we will determine the efficiency θ which is defined as the amount of money the continuously rebalancing investor needs at $t=0$ to achieve the same expected utility at $t=T$ as the discretely trading investor who starts with $V_0^{\bar{\pi}} = 1$. The higher the efficiency is, the better is the discrete trading strategy compared with the continuous one.

The next definition will formalize this:

Definition 1. Let $V_0^{\bar{\pi}} = v = 1$ be the initial capital of the discretely rebalancing investor, let $X_0^{\pi^*} = x > 0$ be the initial capital of the optimally rebalancing investor with constant portfolio process π^*. Then the efficiency θ of the discrete trading strategy is defined as the fraction of the initial capital of the discretely rebalancing investor which is needed by the continuously rebalancing investor to achieve exactly the same expected utility at T:

$$\theta := \frac{x}{v} = x \quad \text{such that} \quad E(U(X_T^{x,\pi^*})) = E(U(V_T^{1,\bar{\pi}}))\,.$$

Remark 8. Although it would be possible to define the efficiency with respect to $t = \Delta t$ as it is usually done (see Chapter 3, Remark 4), we again choose $t = T$ as we did in Chapter 3. Thus we will be able to compare the two types of discrete strategies with regard to their efficiency (Subsection 4.2.3).

Before we try to derive the expected utilities corresponding to $\bar{\pi}$ we recall the ones corresponding to the optimal constant portfolio processes π_1^* and π_2^*, respectively:

Remark 9. Let $X_t^{\pi^*}, t \in [0,T]$ be the wealth process corresponding to the portfolio process π^* as given in (3.5). For π_1^*, π_2^* as given in (3.13)/(3.14), the optimal constant portfolio processes with respect to logarithmic and power utility, respectively, we obtain from Corollary 1 of Chapter 3 that

$$E(\ln X_T^{\pi_1^*}) = \ln x + \left(r + \frac{1}{2}\frac{(\mu-r)^2}{\sigma^2}\right) T\,, \tag{4.13}$$

$$E\left(\frac{1}{\gamma}\left(X_T^{\pi_2^*}\right)^\gamma\right) = \frac{1}{\gamma} x^\gamma\, e^{\gamma\left(r + \frac{1}{2}\frac{(\mu-r)^2}{\sigma^2(1-\gamma)}\right)T}\,. \tag{4.14}$$

Notation 4. We set

$$K_{\ln}(\Delta t) \ := \ \left(r + \frac{1}{2}\frac{(\mu-r)^2}{\sigma^2}\right)\Delta t \tag{4.15}$$

and

$$K_\gamma(\Delta t) \ := \ e^{\gamma\left(r + \frac{1}{2}\frac{(\mu-r)^2}{\sigma^2\,(1-\gamma)}\right)\Delta t}\,. \tag{4.16}$$

Then the above expectations read

$$E\left(\ln X_T^{\pi_1^*}\right) = \ln x + N\, K_{\ln}(\Delta t) \tag{4.17}$$

and

$$E\left(\frac{1}{\gamma}\left(X_T^{\pi_2^*}\right)^\gamma\right) = \frac{1}{\gamma} x^\gamma \left(K_\gamma(\Delta t)\right)^N\,. \tag{4.18}$$

In the following subsection we will give lower bounds for the efficiency in the case of logarithmic and power utility, respectively.

4.2.1 Lower efficiency bounds for logarithmic and power utility

To obtain the efficiency, it is necessary to compute the expected utility of the terminal wealth $V_T^{\bar{\pi}}$. Although we are able to determine the distribution of the random variables Z_k, $k = 0, \ldots, N-1$, which are the only random ingredients of $V_T^{\bar{\pi}}$, it is not possible to compute the exact expected utility of $V_T^{\bar{\pi}}$ in the cases of logarithmic and power utility functions, analytically. Nevertheless, we will derive lower bounds for $E(U(V_T^{\bar{\pi}}))$ in these two cases:

From (4.3) we obtain that

$$\ln(V_T^{\bar{\pi}}) \quad = \quad \sum_{k=0}^{N-1} \ln\left[\pi^*(Z_k - e^{r\Delta t}) + e^{r\Delta t}\right]$$

and that

$$\frac{1}{\gamma}(V_T^{\bar{\pi}})^\gamma \quad = \quad \frac{1}{\gamma}\prod_{k=0}^{N-1}\left[\pi^*(Z_k - e^{r\Delta t}) + e^{r\Delta t}\right]^\gamma .$$

Since the random variables Z_k, $k = 0, \ldots, N-1$, are independent and identically distributed these equalities imply

$$E(\ln(V_T^{\bar{\pi}})) \quad = \quad NE\left\{\ln\left[\pi^*(Z_0 - e^{r\Delta t}) + e^{r\Delta t}\right]\right\}$$

and

$$E\left(\frac{1}{\gamma}(V_T^{\bar{\pi}})^\gamma\right) \quad = \quad \frac{1}{\gamma}\left(E\left\{\left[\pi^*(Z_0 - e^{r\Delta t}) + e^{r\Delta t}\right]^\gamma\right\}\right)^N ,$$

respectively. Setting

$$V_{\Delta t} := \pi^*(Z_0 - e^{r\Delta t}) + e^{r\Delta t} \tag{4.19}$$

we therefore obtain

$$E(\ln(V_T^{\bar{\pi}})) = NE(\ln(V_{\Delta t})) \tag{4.20}$$

as well as

$$E\left(\frac{1}{\gamma}(V_T^{\bar{\pi}})^\gamma\right) = \frac{1}{\gamma}\left[E\left((V_{\Delta t})^\gamma\right)\right]^N . \tag{4.21}$$

Using now the equalities (4.17) and (4.18) we immediately obtain that the efficiency is given by

$$\theta = e^{N\,[E(\ln V_{\Delta t}) - K_{\ln}(\Delta t)]} \tag{4.22}$$

in the case of logarithmic utility and that it is given by

$$\theta = \left(\frac{E((V_{\Delta t})^\gamma)}{K_\gamma(\Delta t)}\right)^{\frac{N}{\gamma}} \tag{4.23}$$

in the case of power utility. Hence it is sufficient to derive lower bounds for

$$E(\ln(V_{\Delta t}))$$

as well as for

$$E\left((V_{\Delta t})^{\gamma}\right) .$$

Obviously there is a natural lower bound for $V_{\Delta t}$:

$$V_{\Delta t} \geq (1-\pi^*)e^{r\Delta t} =: L \tag{4.24}$$

where $L > 0$. For an arbitrary $q > L$ we therefore subdivide the interval $[L, \infty)$ into two intervals $[L, q]$ and (q, ∞) . Since utility functions are increasing and concave we obtain that $U(x) > U(q)$ if $x > q$, whereas the connecting line between $U(L)$ and $U(q)$ is a lower bound for $U(x)$ if $x \in [L, q]$.

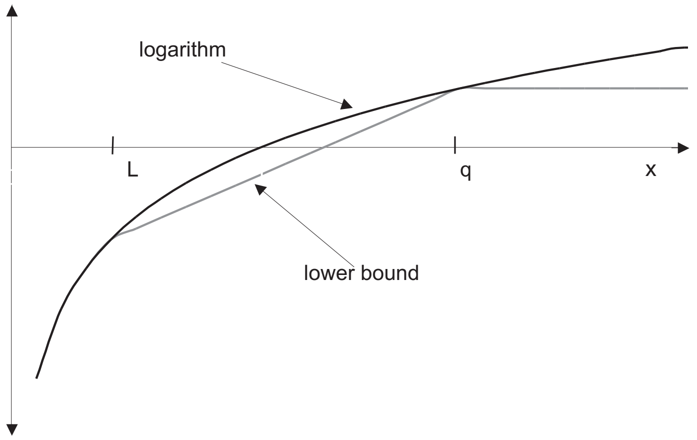

Fig. 9. Lower bound for logarithmic utility

Thus we immediately obtain lower bounds for $E(\ln(V_{\Delta t}))$ and $E((V_{\Delta t})^{\gamma})$, respectively:

Proposition 5. *Let $\pi^* \in (0,1)$, let L be given as in (4.24) for fixed $\Delta t > 0$. For each $q > L$ we then obtain*

$$E(\ln(V_{\Delta t})) \geq G_{\ln}(\Delta t, q) \tag{4.25}$$

where

$$G_{\ln}(\Delta t, q) \;=\; \ln(q) + \ln\left(\frac{L}{q}\right)\left[\frac{1}{L-q}\,\pi^*\,e^{\mu\Delta t}\,\Phi(v-\sigma\sqrt{\Delta t}) + \Phi(v)\right] \tag{4.26}$$

with

$$v = \frac{\ln\left(\frac{q-L}{\pi^*}\right) - \left(\mu - \frac{1}{2}\sigma^2\right)\Delta t}{\sigma\sqrt{\Delta t}}\,. \tag{4.27}$$

Proof: Let $q > L\,, x \geq L\,.$ Then we obtain

$$\begin{aligned} \ln(x) \;&\geq\; \ln(q)\, I_{\{x>q\}} \\ &\quad + \frac{1}{L-q}\left(\ln\left(\frac{L}{q}\right)\, x + L\,\ln(q) - q\,\ln(L)\right)\, I_{\{L\leq x\leq q\}}\,. \end{aligned}$$

Due to $V_{\Delta t} \geq L$ this implies

$$\begin{aligned} E(\ln(V_{\Delta t})) \;&\geq\; \ln(q)\, P(V_{\Delta t} > q) + \frac{1}{L-q}\ln\left(\frac{L}{q}\right)\, E(V_{\Delta t}\, I_{\{L\leq V_{\Delta t}\leq q\}}) \\ &\quad + \frac{1}{L-q}(L\,\ln(q) - q\,\ln(L))\, P(L \leq V_{\Delta t} \leq q)\,. \end{aligned}$$

Since we have that

$$\begin{aligned} & L \leq V_{\Delta t} \leq q \\ \iff \quad & 0 \leq Z_0 \leq \frac{q-L}{\pi^*} =: l \\ \iff \quad & -\infty \leq \frac{W_{\Delta t}}{\sqrt{\Delta t}} \leq \frac{\ln(l) - (\mu - \frac{1}{2}\sigma^2)\Delta t}{\sigma\sqrt{\Delta t}} = v \end{aligned}$$

we obtain

$$P(V_{\Delta t} > q) = 1 - \Phi(v)$$

and

$$P(L \leq V_{\Delta t} \leq q) = \Phi(v)\,.$$

Due to

$$E(V_{\Delta t}\, I_{\{L\leq V_{\Delta t}\leq q\}}) = \pi^* E(Z_0\, I_{\{Z_0\leq l\}}) + L\,\Phi(v)$$

with

$$\begin{aligned} E(Z_0\, I_{\{Z_0\leq l\}}) \;&=\; \int_{-\infty}^{v} \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}y^2}\, e^{\sigma\sqrt{\Delta t}y}\, dy\, e^{(\mu-\frac{1}{2}\sigma^2)\Delta t} \\ &=\; \int_{-\infty}^{v} \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}(y-2\sigma y\sqrt{\Delta t}+\sigma^2\Delta t)}\, dy\, e^{\mu\Delta t} \\ &=\; e^{\mu\Delta t}\,\Phi(v - \sigma\sqrt{\Delta t}) \end{aligned}$$

we finally derive the following lower bound for the expected utility of the wealth at time Δt:

$$E(\ln(V_{\Delta t})) \;\geq\; \ln(q)\,(1 - \Phi(v)) + \frac{1}{L-q}\,\ln\left(\frac{L}{q}\right)[\pi^*\, e^{\mu\Delta t}\,\Phi(v - \sigma\sqrt{\Delta t}) + L\,\Phi(v)]$$

$$
\begin{aligned}
& +\frac{1}{L-q}\,(L\,\ln(q) - q\,\ln(L))\,\Phi(v) \\
= \;& \ln(q) + \frac{1}{L-q}\,\ln\left(\frac{L}{q}\right)\pi^*\,e^{\mu\Delta t}\,\Phi(v-\sigma\sqrt{\Delta t}) - \ln(q)\,\Phi(v) \\
& +\Phi(v)\,\ln(L) \\
= \;& \ln(q) + \ln\left(\frac{L}{q}\right)\left[\frac{1}{L-q}\,\pi^*\,e^{\mu\Delta t}\,\Phi(v-\sigma\sqrt{\Delta t}) + \Phi(v)\right] \\
= \;& G_{\ln}(\Delta t, q)\,.
\end{aligned}
$$

□

Using the same technique we also derive the lower bound in the case of power utility functions:

Proposition 6. *Let* $\pi^* \in (0,1)$ *, let* L *be given as in (4.24) for fixed* $\Delta t > 0$ *. For each* $q > L$ *we then obtain*

$$E\left((V_{\Delta t})^\gamma\right) \geq G_\gamma(\Delta t, q) \quad , \gamma \in (0,1) \tag{4.28}$$

where

$$G_\gamma(\Delta t, q) \;=\; q^\gamma + (L^\gamma - q^\gamma)\left[\frac{1}{L-q}\,\pi^*\,e^{\mu\Delta t}\,\Phi(v-\sigma\sqrt{\Delta t}) + \Phi(v)\right] \tag{4.29}$$

with v *as given in Proposition 5 ((4.27)).*
Proof: Let $q > L\,, x \geq L$. Then we obtain

$$x^\gamma \geq q^\gamma\,I_{\{x>q\}} + \frac{1}{L-q}[(L^\gamma - q^\gamma)\,x + L\,q^\gamma - q\,L^\gamma]\,I_{\{L\leq x\leq q\}}\,.$$

Due to $V_{\Delta t} \geq L$ and due to the results in the proof of Proposition 5 this implies

$$
\begin{aligned}
E((V_{\Delta t})^\gamma) \;\geq\; & q^\gamma\,P(V_{\Delta t} > q) + \frac{1}{L-q}(L^\gamma - q^\gamma)\,E(V_{\Delta t}\,I_{\{L\leq V_{\Delta t}\leq q\}}) \\
& +\frac{1}{L-q}\,(L\,q^\gamma - q\,L^\gamma)\,P(L \leq V_{\Delta t} \leq q) \\
= \;& q^\gamma\,(1-\Phi(v)) + \frac{1}{L-q}(L^\gamma - q^\gamma)\,[\pi^*\,e^{\mu\Delta t}\,\Phi(v-\sigma\sqrt{\Delta t}) + L\,\Phi(v)] \\
& +\frac{1}{L-q}\,(L\,q^\gamma - q\,L^\gamma)\,\Phi(v) \\
= \;& q^\gamma - q^\gamma\,\Phi(v) + \frac{1}{L-q}(L^\gamma - q^\gamma)\,\pi^*\,e^{\mu\Delta t}\,\Phi(v-\sigma\sqrt{\Delta t}) + L^\gamma\,\Phi(v) \\
= \;& q^\gamma + (L^\gamma - q^\gamma)\left[\frac{1}{L-q}\,\pi^*\,e^{\mu\Delta t}\,\Phi(v-\sigma\sqrt{\Delta t}) + \Phi(v)\right] \\
= \;& G_\gamma(\Delta t, q)\,.
\end{aligned}
$$

□

Having determined the lower bounds with respect to $V_{\Delta t}$ we immediately obtain the lower bounds for the efficiency as defined in Definition 1.

Theorem 1. *Let $\Delta t > 0$ be fixed, let L be given as in (4.24). Further let $q > L$.*

i) In the case of logarithmic utility the efficiency of the discrete strategy corresponding to $\bar{\pi}$ is bounded from below by

$$\theta_e^{\ln} := e^{N\,[G_{\ln}(\Delta t,q)-K_{\ln}(\Delta t)]} \tag{4.30}$$

where

$$G_{\ln}(\Delta t, q) = \ln(q) + \ln\left(\frac{L}{q}\right)\left[\frac{1}{L-q}\,\pi_1^*\,e^{\mu\Delta t}\,\Phi(v-\sigma\sqrt{\Delta t}) + \Phi(v)\right], \tag{4.31}$$

$$K_{\ln}(\Delta t) = \left(r + \frac{1}{2}\frac{(\mu-r)^2}{\sigma^2}\right)\Delta t\,, \tag{4.32}$$

$$v = \frac{\ln\left(\frac{q-L}{\pi_1^*}\right) - \left(\mu - \frac{1}{2}\sigma^2\right)\Delta t}{\sigma\sqrt{\Delta t}}. \tag{4.33}$$

ii) In the case of power utility the efficiency of the discrete strategy corresponding to $\bar{\pi}$ is bounded from below by

$$\theta_e^{\gamma} := \left(\frac{G_\gamma(\Delta t, q)}{K_\gamma(\Delta t)}\right)^{\frac{N}{\gamma}} \quad ,\gamma \in (0,1) \tag{4.34}$$

where

$$G_\gamma(\Delta t, q) = q^\gamma + (L^\gamma - q^\gamma)\left[\frac{1}{L-q}\,\pi_2^*\,e^{\mu\Delta t}\,\Phi(v-\sigma\sqrt{\Delta t}) + \Phi(v)\right], \tag{4.35}$$

$$K_\gamma(\Delta t) = e^{\gamma\left(r+\frac{1}{2}\frac{(\mu-r)^2}{\sigma^2(1-\gamma)}\right)\Delta t}, \tag{4.36}$$

$$v = \frac{\ln\left(\frac{q-L}{\pi_2^*}\right) - \left(\mu - \frac{1}{2}\sigma^2\right)\Delta t}{\sigma\sqrt{\Delta t}}. \tag{4.37}$$

Proof: The results follow directly from Propositions 5 and 6, the definition of the efficiency and (4.22)/(4.23). □

Remark 10. Theorem 1 gives us lower bounds which depend on the parameter q. Since this parameter can be any real number larger than L, it remains to determine $q_{\max}$, the value of q such that the lower bounds attain the highest value. Of course $q_{\max}$ depends on the particular utility function, the choice of Δt and the values of the market coefficients. It can be determined numerically by computing the lower bound for various values of $q > L$ and choosing the one which delivers the highest bound. (Our numerical analysis has shown that the lower bound as a function of the parameter q always has exactly one maximum.) Of course this procedure only gives us an approximation of $q_{\max}$, but this approximation can be as accurate as desired. (To determine the approximate values of q_{max} for the examples below we have used $\Delta q = 0.01$.)

The following examples illustrate the quantitative behaviour of $\theta_e^{\ln}$ and θ_e^{γ}. To obtain deeper insights and for further use we will also give the corresponding lower bounds for the efficiency achieved after the first period Δt which are given by $\theta_{\Delta t}^{\ln} = (\theta_e^{\ln})^{1/N}$ and $\theta_{\Delta t}^{\gamma} = (\theta_e^{\gamma})^{1/N}$, respectively.

Examples 1.

a) $\mu = 0.03$, $\sigma = 0.2$, $r = 0.02$, $T = 1$, $\gamma_1 = 0.2$, $\gamma_2 = 0.7$ $\Longrightarrow$
$\pi_1^* = 0.25$, $\pi_{2\gamma_1}^* = 0.3125$, $\pi_{2\gamma_2}^* = 0.833$.

Δt	$\theta_{\Delta t}^{\ln}$	$\theta_e^{\ln}$	$\theta_{\Delta t}^{\gamma_1}$	$\theta_e^{\gamma_1}$	$\theta_{\Delta t}^{\gamma_2}$	$\theta_e^{\gamma_2}$
1	0.98799	0.98799	0.98474	0.98474	0.9475	0.9475
0.5	0.99146	0.98299	0.98914	0.9784	0.96305	0.92747
1/12	0.99648	0.95857	0.9955	0.94732	0.98455	0.82957
1/50	0.99824	0.91569	0.9978	0.89573	0.99235	0.68115
1/360	0.99877	0.64206	0.99847	0.57625	0.99708	0.34898

b) $\mu = 0.04$, $\sigma_{\ln} = 0.145$, $\sigma_{\gamma} = 0.29$, $r = 0.02$, $T = 0.5$, $\gamma = 0.75$ $\Longrightarrow$
$\pi_1^* = \pi_2^* = 0.951248514$.

Δt	$\theta_{\Delta t}^{\ln}$	$\theta_e^{\ln}$	$\theta_{\Delta t}^{\gamma}$	θ_e^{γ}
0.5	0.89942	0.89942	0.93329	0.93329
1/50	0.97834	0.57842	0.98622	0.70688
1/360	0.99092	0.19362	0.99479	0.39053

c) $\mu = 0.04$, $\sigma = 0.4$, $r = 0.02$, $T = 1$, $\gamma = 0.000000001$ $\Longrightarrow$
$\pi_1^* = \pi_2^* = 0.125$.

Δt	$\theta_{\Delta t}^{\ln}$	$\theta_e^{\ln}$	$\theta_{\Delta t}^{\gamma}$	θ_e^{γ}
1	0.992573721	0.992573721	0.992573787	0.992573787
0.5	0.994931105	0.989887903	0.994931012	0.989887718
1/12	0.997952377	0.975703366	0.997952378	0.975703377
1/360	0.999329866	0.78558304	0.999329797	0.785563513

d) $\mu = 0.04\,,\ \sigma = 0.2\,,\ r = 0.02\,,\ T = 0.1\,,\ \gamma = 0.2 \quad \Longrightarrow$
$\pi_1^* = 0.5\,,\ \pi_2^* = 0.625\,.$

Δt	L	$q_{\max}$	$\theta_{\Delta t}^{\ln}$	$\theta_e^{\ln}$
0.1	0.501	1.02	0.985735	0.985735
0.01	0.5001	1.01	0.995091	0.95198
0.001	0.50001	1	0.998248	0.83916
0.0001	0.500001	1	0.999447	0.575134
0.00001	0.5	1	0.999825	0.173747
0.000001	0.5	1	0.999945	0.004086

Δt	L	$q_{\max}$	$\theta_{\Delta t}^{\gamma}$	θ_e^{γ}
0.1	0.375751	1.02	0.981	0.981
0.01	0.375075	1.01	0.993721	0.938955
0.001	0.375008	1	0.997752	0.798474
0.0001	0.375001	1	0.999289	0.491029
0.00001	0.375	1	0.999775	0.105373
0.000001	0.375	1	0.999929	0.000825

Remark 11.

a) If Δt decreases to 0, the values of $\theta_{\Delta t}^{\ln}$ and $\theta_{\Delta t}^{\gamma}$ increase to 1 whereas the values of $\theta_e^{\ln}$ and θ_e^{γ} decrease to 0 (showing that these lower bounds are not very good). Obviously, the effects caused by Δt tending to 0 are not able to compensate for the effects caused by N tending to infinity. (The increasing values $\theta_{\Delta t}^{\ln} < 1$ and $\theta_{\Delta t}^{\gamma} < 1$, respectively, have to be multiplied more often with themselves.) Further L and $q_{\max}$ decrease to a constant number. (In both cases $q_{\max}$ seems to stabilise at 1.) Considering $G_{\ln}(\Delta t, q)$ and $G_{\gamma}(\Delta t, q)$ as functions of q, we obtain from the numerical analysis that the shapes of these functions become more and more angular if Δt tends to 0.

b) If $\mu\,, r\,, \sigma$ are fixed, $\gamma_1 < \gamma_2$ implies $\theta_e^{\gamma_1} > \theta_e^{\gamma_2}$ (Examples 1 a)). (In contrast to this, the lower bounds of the efficiency corresponding to the bounded portfolio process of Chapter 3 show a different behaviour: $\theta_b^{\gamma_1} < \theta_b^{\gamma_2}$ if $\gamma_1 < \gamma_2$.)

c) For fixed parameters $\mu\,, \sigma\,, r$ we always have $\theta_e^{\ln} > \theta_e^{\gamma}$ (Examples 1 a) and d)).

d) For fixed parameters $\mu\,, \sigma\,, r$ and $\gamma \approx 0$ we certainly have $\theta_e^{\ln} \approx \theta_e^{\gamma}$ (Examples 1 c)).

e) If $\mu\,,r$ are fixed and $\sigma_{\ln} < \sigma_\gamma$ are chosen such that $\pi_1^* = \pi_2^*$ holds we obtain the relation $\theta_e^{\ln} < \theta_e^\gamma$ (Examples 1 b)).

4.2.2 Approximative efficiency via Taylor-approximation for logarithmic utility

In the following we will approximate the efficiency in the case of logarithmic utility. As we will use Taylor-approximation, the periods Δt must be very short. In contrast to the foregoing subsection the investor's portfolio process at $t = 0$ and $t \in E$ will not be equal to π_1^*, the optimal portfolio process if the portfolio is rebalanced continuously but it will be chosen as the optimal fraction of wealth which maximizes the efficiency (and thus the expected utility) if the portfolio is rebalanced in equidistant time points. This optimal fraction of wealth will be called $\tilde{\pi} = \tilde{\pi}(\Delta t)$ and will be derived in the following theorem:

Theorem 2. *Let $U(x) = \ln(x)\,, x > 0$ and let $\Delta t > 0$ be small. Then the (approximately) optimal fraction of wealth invested into the stock at $t = 0$ and $t \in E$ is given by*

$$\tilde{\pi}(\Delta t) \quad = \quad \frac{f_2(\Delta t)}{f_1(\Delta t)} \tag{4.38}$$

with $f_1\,, f_2$ as given below. The corresponding expected utility is approximately given by

$$N\,G(\Delta t) \quad := \quad N\,\left[\frac{1}{2}\,\tilde{\pi}(\Delta t)\,f_2(\Delta t) + f_3(\Delta t)\right] \tag{4.39}$$

with

$$f_1(\Delta t) \quad = \quad e^{(2\mu+\sigma^2)\Delta t} - 2e^{\mu\Delta t} + 1 + \left(e^{r\Delta t} - 1\right)\left[e^{r\Delta t} - 1 - 2\left(e^{\mu\Delta t} - 1\right)\right]\,, \tag{4.40}$$

$$f_2(\Delta t) \quad = \quad \left(e^{r\Delta t} - 1\right)^2 - \left(e^{r\Delta t} - 1\right)\left(e^{\mu\Delta t} - 1\right) + e^{\mu\Delta t} - e^{r\Delta t}\,, \tag{4.41}$$

$$f_3(\Delta t) \quad = \quad \left(e^{r\Delta t} - 1\right)\left[1 - \frac{1}{2}\left(e^{r\Delta t} - 1\right)\right]\,. \tag{4.42}$$

Further, the approximate efficiency is given by

$$\theta_a \quad := \quad e^{N\,[G(\Delta t) - K_{\ln}(\Delta t)]} \tag{4.43}$$

with $K_{\ln}(\Delta t)$ as above ((4.15)).

Proof: Let $\pi \in (0,1)$ be an arbitrary fraction of wealth invested in the stock at $t = 0$ and $t \in E$ and let the corresponding portfolio process evolve continuously in between. From (4.3) we then obtain the equality

$$\begin{aligned} E(\ln(V_T^\pi)) \quad &= \quad N\,E(\ln(\pi Z_0 + (1-\pi)\,e^{r\Delta t})) \\ &= \quad N\,E(\ln(V_{\Delta t}(\pi)))\,. \end{aligned}$$

Since Δt is very small, Z_0 and $e^{r\Delta t}$ are near to 1. In analogy to Rogers (2001) we set

$$y := \pi\,(Z_0 - 1) + (1-\pi)(e^{r\,\Delta t} - 1)$$

which implies

$$1 + y = \pi\, Z_0 + (1-\pi)\, e^{r\,\Delta t}$$

and

$$y \approx 0\,.$$

For $-1 < y \le 1$ and $\xi \in (-1,1)$ we therefore obtain

$$\begin{aligned} \ln(1+y) &= \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k}\, y^k \\ &= y - \frac{y^2}{2} + \frac{1}{3}\,\xi^3 \\ &\approx y - \frac{y^2}{2}\,. \end{aligned}$$

To compute the approximative expected utility we thus have to compute

$$\begin{aligned} E(y) &= E(\pi\,(Z_0 - 1) + (1-\pi)(e^{r\Delta t} - 1)) \\ &= \pi\left(e^{\mu\Delta t} - 1\right) + (1-\pi)\left(e^{r\Delta t} - 1\right) \\ &= \pi\left(e^{\mu\Delta t} - e^{r\Delta t}\right) + e^{r\Delta t} - 1 \end{aligned}$$

and

$$\begin{aligned} E(y^2) &= E\Big(\pi^2(Z_0-1)^2 + (1-\pi)^2\left(e^{r\Delta t}-1\right)^2 \\ &\qquad +2\pi(Z_0-1)(1-\pi)\left(e^{r\Delta t}-1\right)\Big) \\ &= \pi^2 E(Z_0^2 - 2Z_0 + 1) + (1-\pi)^2\left(e^{r\Delta t}-1\right)^2 \\ &\quad +2\pi(1-\pi)\left(e^{r\Delta t}-1\right)E(Z_0-1) \\ &= \pi^2\left(e^{(2\mu+\sigma^2)\Delta t} - 2e^{\mu\Delta t} + 1\right) + (1-\pi)^2\left(e^{r\Delta t}-1\right)^2 \\ &\quad +2\pi(1-\pi)\left(e^{r\Delta t}-1\right)\left(e^{\mu\Delta t}-1\right) \\ &= \pi^2\left\{e^{(2\mu+\sigma^2)\Delta t} - 2e^{\mu\Delta t} + 1 + \left(e^{r\Delta t}-1\right)^2 - 2\left(e^{r\Delta t}-1\right)\left(e^{\mu\Delta t}-1\right)\right\} \\ &\quad +\pi\left\{-2\left(e^{r\Delta t}-1\right)^2 + 2\left(e^{r\Delta t}-1\right)\left(e^{\mu\Delta t}-1\right)\right\} + \left(e^{r\Delta t}-1\right)^2 \end{aligned}$$

which then imply

$$\begin{aligned} &E(\ln(V_{\Delta t}(\pi))) \\ &\approx\ -\frac{\pi^2}{2}\left\{e^{(2\mu+\sigma^2)\Delta t} - 2e^{\mu\Delta t} + 1 + \left(e^{r\Delta t}-1\right)\left[e^{r\Delta t} - 1 - 2\left(e^{\mu\Delta t}-1\right)\right]\right\} \end{aligned}$$

$$
\begin{aligned}
& +\pi\left\{\left(e^{r\Delta t}-1\right)^2-\left(e^{r\Delta t}-1\right)\left(e^{\mu\Delta t}-1\right)+e^{\mu\Delta t}-e^{r\Delta t}\right\} \\
& +\left(e^{r\Delta t}-1\right)\left[1-\frac{1}{2}\left(e^{r\Delta t}-1\right)\right] \\
= \; & -\frac{\pi^2}{2}\left(e^{(2\mu+\sigma^2)\Delta t}+e^{2r\Delta t}-2\,e^{(\mu+r)\Delta t}\right) \\
& +\pi\left(e^{2r\Delta t}-2\,e^{r\Delta t}+2\,e^{\mu\Delta t}-e^{(\mu+r)\Delta t}\right) \\
& +2\,e^{r\Delta t}-\frac{1}{2}\,e^{2r\Delta t}-\frac{3}{2} \\
= \; & \pi^2\frac{-f_1(\Delta t)}{2}+\pi f_2(\Delta t)+f_3(\Delta t)\,.
\end{aligned}
$$

To maximize the expected utility we compute the derivative with respect to π and determine its zero:

$$\frac{\partial}{\partial\pi}E(\ln(V_{\Delta t}(\pi)))\approx-\pi f_1(\Delta t)+f_2(\Delta t)=0$$

$$\iff \qquad \pi=\frac{f_2(\Delta t)}{f_1(\Delta t)}=\tilde{\pi}(\Delta t)\,.$$

Replacing π by $\tilde{\pi}$ in the above formula we obtain

$$E(\ln(V_{\Delta t}(\tilde{\pi})))\approx\frac{1}{2}\frac{f_2^2(\Delta t)}{f_1(\Delta t)}+f_3(\Delta t)=G(\Delta t)$$

and

$$E(\ln(V_T^{\tilde{\pi}}))\approx N\,G(\Delta t)\,.$$

From

$$E(\ln(X_T^{\pi_1^*}))=\ln(x)+N\,K_{\ln}(\Delta t)$$

we then obtain

$$\theta=x\approx e^{N\,[G(\Delta t)-K_{\ln}(\Delta t)]}$$

□

The next examples will illustrate the quantitative behaviour of $\tilde{\pi}$ and θ_a.

Examples 2.

a) $\mu=0.03$, $\sigma=0.2$, $r=0.02 \quad\Longrightarrow\quad \pi_1^*=0.25$,
$\Delta t=1/360 \quad\Longrightarrow\quad \tilde{\pi}=0.249946$.

T	θ_a
1	0.999999748
2	0.999999497
10	0.99999748

b) $\mu = 0.04\,,\ \sigma = 0.145\,,\ r = 0.02 \quad\Longrightarrow\quad \pi_1^* = 0.951249\,,$
$\Delta t = 1/360 \quad\Longrightarrow\quad \tilde{\pi} = 0.9509855\,.$

T	θ_a
0.5	0.9999988
1	0.9999976
5	0.99998817

c) $\mu = 0.04\,,\ \sigma = 0.4\,,\ r = 0.02 \quad\Longrightarrow\quad \pi_1^* = 0.125\,,$
$\Delta t = 1/360 \quad\Longrightarrow\quad \tilde{\pi} = 0.124947\,.$

T	θ_a
1	0.99999951
10	0.99999505

d) $\mu = 0.04\,,\ \sigma = 0.2\,,\ r = 0.02 \quad\Longrightarrow\quad \pi_1^* = 0.5\,,$
$\Delta t = 1/360 \quad\Longrightarrow\quad \tilde{\pi} = 0.499861\,.$

T	θ_a
0.1	0.999999875
1	0.99999875

Remark 12. The above data show that $\tilde{\pi}$ is always quite near to π_1^*.

In the next proposition we will show that $\tilde{\pi}(\Delta t)$ and $N\,G(\Delta t)$ can be expanded in Δt around the corresponding values π_1^* and $E(\ln(X_T^{\pi_1^*}))$, respectively.

Proposition 7. *For small Δt the approximately optimal portfolio process $\tilde{\pi}$ can be written as*

$$\tilde{\pi}(\Delta t) = \pi_1^* + C\Delta t + O(\Delta t^2) \tag{4.44}$$

with

$$C = \frac{\pi_1^*}{2}\{\sigma^2(\pi_1^* - 2\pi_1^{*2} - 1) - 4\mu\}\,. \tag{4.45}$$

Further, the approximate expected utility is given by

$$N\,G(\Delta t) = E(\ln(X_T^{\pi_1^*})) - \ln(x) + O(\Delta t) \tag{4.46}$$

implying that

$$\theta_a = e^{O(\Delta t)} \approx 1\,. \tag{4.47}$$

Proof: We define $g(\Delta t) := f_2(\Delta t)/f_1(\Delta t)$ with $f_1\,, f_2$ from (4.40)/(4.41) and expand it around 0 such that

$$g(\Delta t) = g(0) + g'(0)\Delta t + \frac{1}{2}g''(\xi)\Delta t^2$$

for some function $\xi = \xi(\Delta t) > 0$. Since $f_1(0) = 0$ and $f_2(0) = 0$ we consider the derivatives

$$\begin{aligned} f_1'(\Delta t) &= (2\mu+\sigma^2)e^{(2\mu+\sigma^2)\Delta t} + 2re^{2r\Delta t} - 2(\mu+r)e^{(\mu+r)\Delta t}\,, \\ f_2'(\Delta t) &= 2re^{2r\Delta t} - 2re^{r\Delta t} + 2\mu e^{\mu\Delta t} - (\mu+r)e^{(\mu+r)\Delta t} \end{aligned}$$

which imply

$$\begin{aligned} f_1'(0) &= \sigma^2\,, \\ f_2'(0) &= \mu - r \end{aligned}$$

thus delivering

$$g(0) = \frac{\mu - r}{\sigma^2} = \pi_1^*\,.$$

Further we have to determine

$$g'(\Delta t) = \frac{f_2'f_1 - f_2f_1'}{f_1^2}(\Delta t) =: \frac{A}{B}(\Delta t)$$

where $A(0) = 0$ and $B(0) = 0$. Using again the rule of de l'Hospital we obtain

$$\begin{aligned} A'(\Delta t) &= (f_2''f_1 - f_2f_1'')(\Delta t)\,, \\ B'(\Delta t) &= 2(f_1f_1')(\Delta t) \end{aligned}$$

where $A'(0) = 0$ and $B'(0) = 0$ due to $f_1(0) = 0$ and $f_2(0) = 0$. Therefore we have to look at $A''(\Delta t)/B''(\Delta t)$ where

$$\begin{aligned} A''(\Delta t) &= (f_2'''f_1 + f_2''f_1' - f_2'f_1'' - f_2f_1''')(\Delta t)\,, \\ B''(\Delta t) &= 2(f_1'^2 + f_1f_1'')(\Delta t) \end{aligned}$$

with

$$\begin{aligned} f_1''(\Delta t) &= (2\mu+\sigma^2)^2e^{(2\mu+\sigma^2)\Delta t} + 4r^2e^{2r\Delta t} - 2(\mu+r)^2e^{(\mu+r)\Delta t}\,, \\ f_2''(\Delta t) &= 4r^2e^{2r\Delta t} - 2r^2e^{r\Delta t} + 2\mu^2e^{\mu\Delta t} - (\mu+r)^2e^{(\mu+r)\Delta t}\,. \end{aligned}$$

With

$$\begin{aligned} f_1''(0) &= 2(\mu - r)^2 + \sigma^4 + 4\mu\sigma^2\,, \\ f_2''(0) &= (\mu - r)^2 \end{aligned}$$

we then obtain

$$g'(0) = \frac{A''(0)}{B''(0)} = \frac{\pi_1^*}{2}\{\sigma^2(\pi_1^* - 2\pi_1^{*2} - 1) - 4\mu\}\,.$$

Analogously, we can determine $g''(\Delta t)$ which is also finite but which needs a lot more steps to be computed. As we are not interested in the exact value of $g''(\xi)$ we omit its computation here.

To show (4.46) we use (4.41)/(4.42) which imply that

$$f_3(\Delta t) = r\Delta t + O(\Delta t^2)$$

and

$$f_2(\Delta t) = \pi_1^* \sigma^2 \Delta t + O(\Delta t^2) .$$

Using (4.44) we thus immediately get that

$$\begin{aligned} G(\Delta t) &= \frac{1}{2}\tilde{\pi} f_2 + f_3 \\ &= \left[r + \frac{1}{2}\pi_1^{*2}\sigma^2\right] \Delta t + O(\Delta t^2) \end{aligned}$$

implying that

$$\begin{aligned} N\, G(\Delta t) &= \left[r + \frac{1}{2}\pi_1^{*2}\sigma^2\right] T + O(\Delta t) \\ &\overset{(4.13)}{=} E(\ln(X_T^{\pi_1^*})) - \ln(x) + O(\Delta t) . \end{aligned}$$

From (4.17) and (4.43) we finally obtain that $\theta_a = e^{O(\Delta t)}$. □

4.2.3 Comparison

In Chapter 3 and the previous subsections of this chapter we have analysed different types of strategies which are all characterized by discrete rebalancing and which were used to achieve an expected utility at T similar to the one belonging to the optimal continuous portfolio process π^*.

Logarithmic utility

In the case of the equidistant trading from Subsection 4.2.1 (which belongs to the portfolio process $\bar{\pi}$) we give the maximum value of $\theta_e^{\ln}$ which is achieved for $\Delta t = T$ and we give the maximum value of $\theta_{\Delta t}^{\ln}$ which is achieved for $\Delta t = 1/360$. The Taylor-approximation of the efficiency belonging to the portfolio process $\tilde{\pi}$ from Subsection 4.2.2 is done with $\Delta t = 1/360$.

Let $\mu = 0.04$, $r = 0.02$.

σ	T	θ_a	$\theta_e^{\ln}$	$\theta_{\Delta t}^{\ln}$
0.145	0.5	0.9999988	0.89942	0.99092
0.4	1	0.99999951	0.992573721	0.999329866

These examples show that the lower bound $\theta_e^{\ln}$ is not very good. As the approximate efficiency θ_a is much higher we know that the efficiency corresponding to $\bar{\pi}$ must also be much higher. (Of course θ_a is obtained by using $\tilde{\pi}$ instead of π_1^*, but these values are quite close to each other (Examples 2, Chapter 4).) In particular, the efficiency must increase if the periods Δt are decreased, what is unfortunately not satisfied by $\theta_e^{\ln}$ (see Examples 1, Chapter 4). Hence the maximum value of $\theta_e^{\ln}$ for $\Delta t = T$ is only a very rough lower bound for the efficiency.

On the other hand, the above values of θ_a indicate that the equidistant rebalancing of the portfolio can be very successful if Δt is chosen sufficiently small.

To compare these results with the one from Chapter 3 belonging to the portfolio pocess $\hat{\pi}$ we have to choose suitable values of ε which can be used with regard to the above examples: The assumption

$$[\pi_1^* - \varepsilon, \pi_1^* + \varepsilon] \subset [0,1]$$

implies that the condition

$$\varepsilon < \min\{\pi_1^*, 1 - \pi_1^*\}$$

has to be satisfied:
In the case of the first example we therefore obtain the condition

$$\varepsilon < \min\{0.95125, 0.04875\} = 0.04875\,.$$

Choosing $\varepsilon = 0.04$ we get

$$\theta_b^{\ln} = 0.99999159\,.$$

Clearly this lower bound will even increase if ε decreases to 0.
In the second case the condition

$$\varepsilon < 0.125$$

has to be satisfied. For $\varepsilon = 0.12$ we obtain the lower bound

$$\theta_b^{\ln} = 0.998849\,.$$

Again this bound can be increased choosing smaller values for ε.

Power utility

As above we give the maximum value of θ_e^{γ} achieved for $\Delta t = T$ and the maximum value of $\theta_{\Delta t}^{\gamma}$ achieved for $\Delta t = 1/360$:

Let $\mu = 0.03\,, \sigma = 0.2\,, r = 0.02\,, T = 1\,.$

γ	π_2^*	θ_e^{γ}	$\theta_{\Delta t}^{\gamma}$	ε	θ_b^{γ}
0.2	0.3125	0.98474	0.99847	0.2	0.99936
0.7	0.833	0.9475	0.99708	0.1	0.99994

Again θ_b^γ would even increase if Δt was decreased and θ_e^γ is only a very bad lower bound of the efficiency belonging to the equidistant trading.

4.3 Conclusion

Throughout this chapter we have (most of the time) analysed the situation of an investor who rebalances his portfolio by shifting his portfolio process $\bar{\pi}$ to a constant number π^*. In contrast to the previous chapter, this time the investor only rebalances his portfolio at predetermined equidistant time points.

For the strategy corresponding to the induced portfolio process we have determined the expected terminal wealth, we have determined the expected volume of single and accumulated transactions and we have derived lower efficiency bounds, whereas we have determined an approximate efficiency for a slightly different portfolio process $\tilde{\pi}$. Later on we have analysed all these results on the efficiency as well as those obtained in Chapter 3. The final conclusion we draw from our analysis is that in the case of logarithmic and power utility functions, respectively, all kinds of discrete strategies (belonging to the portfolio processes $\hat{\pi}$, $\bar{\pi}$ and $\tilde{\pi}$) lead to an expected utility which is quite close to the optimal one achieved by continuous rebalancing. Above all in the presence of fixed transaction costs and with regard to the usual portfolio optimization problem it will therefore be useful to choose one of these discrete strategies, which will provide the investor with an expected utility that is nearly optimal but will prevent him from getting ruined by extensive trading as in the continuous rebalancing case.

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