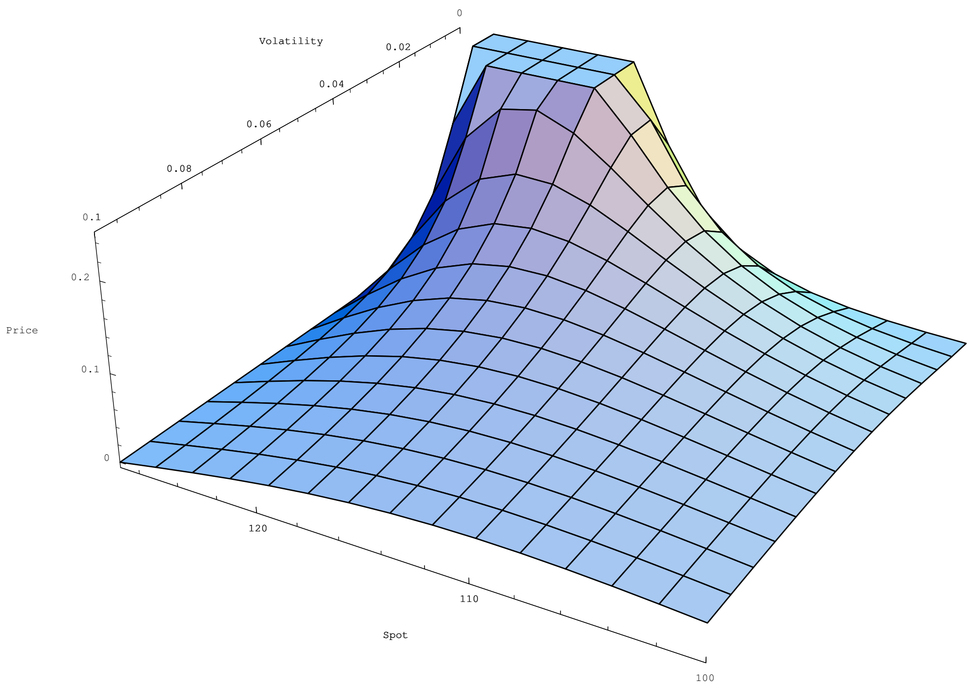
Oliver Faulhaber

Analytic Methods for Pricing Double Barrier Options in the Presence of Stochastic Volatility



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Supervised by Prof. Dr. Ralf Korn

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Preface

While there exist closed-form solutions for vanilla options in the presence of stochastic volatility for nearly a decade [Heston, 1993], practitioners still depend on numerical methods – in particular the Finite Difference and Monte Carlo methods – in the case of double barrier options. It was only recently that Lipton [2001] proposed (semi-)analytical solutions for this special class of path-dependent options.

Although he presents two different approaches to derive these solutions, he restricts himself in both cases to a less general model, namely one where the correlation and the interest rate differential are assumed to be zero. Naturally the question arises, if these methods are still applicable for the general stochastic volatility model without these restrictions.

In this paper we show that such a generalization fails for both methods. We will explain why this is the case and discuss the consequences of our results.

Acknowledgements

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Contents

Chapter 1

Heston's Stochastic Volatility Model

In this chapter we will recall the basic Black-Scholes model and point out its drawbacks. We then present Heston's Stochastic Volatility model as an alternative approach and derive a partial differential equation, which has to be satisfied by every contingent claim.

1.1 Introduction

One of the most fundamental problems in financial mathematics is the valuation of derivatives. By this we mean financial products whose value depends on the value of other underlying variables. Since we will put our focus on the currency market, we assume the underlying asset to be a foreign exchange (FX) rate.

One of the most simple derivatives is the European vanilla call, which gives the holder the right to purchase one unit of the foreign currency at a certain time in the future for a fixed exchange rate.

One can imagine two reasons, why someone should buy such an option. First he could use it to speculate, that is simply making a bet about future market movements. Since the call increases in value when the value of the underlying increases, such a call would be the right choice if one expects such a development. Due to the large variety of derivatives traded in today's markets, there exist custom-tailored products for all kind of expectations.

The second reason to buy the option is called *hedging*, i.e. a reduction of one's risk exposure. We can see the call as an "insurance" against a market upturn and the price of the derivative can therefore be seen as "premium".

Especially the latter reason is particular important for the currency market, where agreements of future payments are part of the daily business. Internationally operating companies have to carefully control the risk they are exposed to. But still the main question remains the same: How much should we pay for the option? Formally spoken we want to find today's fair price of such a derivative.

To achieve this, we basically have to calculate its expected value at maturity in the risk-neutral setting, using the information which is available today. We then discount this expectation to get today's value of the option.

It is clear, that we cannot expect to obtain *the* perfect price since we have to use simplified models of the real markets to apply our mathematical methods. But we want to have results, which capture the fundamental characteristics of the real prices and leave us enough freedom to calibrate them to the market.

1.2 Modelling the Asset Price

To do so, we first have to find a model for the underlying asset. The natural choice of an underlying process would be the Brownian Motion, which can be seen as the limit of a random walk as step sizes are reduced and step frequency increased. The links between Brownian Motion and finance can be traced back to Bachelier [1900], who used it as a model for French stock prices.

The increments of a Brownian Motion are independent normal random variables. As a consequence Brownian Motion can become negative which makes it an unsatisfactory model for FX rates and most other underlying assets. A more reasonable model was first suggested by Osborne [1959], who assumed the asset price to be a geometric Brownian Motion. Then the logarithm of the asset price is a Brownian Motion. To be more formal, the standard model for the price of a financial asset assumes that the price process S_t is the solution of a stochastic differential equation (SDE)

$$dS_t = S_t(\mu dt + \sigma dW_t) \tag{1.1}$$

where W_t is a Brownian Motion, and both the mean μ and the volatility σ are constant parameters. This SDE has solution

$$S_t = S_0 e^{\sigma W_t + (\mu - \frac{1}{2}\sigma)t}$$

1.3 Option Pricing in the Black-Scholes Model

The model introduced by Black and Scholes [Black and Scholes, 1973] is widely seen as *the* breakthrough in modern financial mathematics. They based all their calculations on a few ideal assumptions on the market, especially the "no-arbitrage principle". That denotes the assumption, that there should be no possibility for a risk free gain without initial capital. Put in other words, all investments with certain payoff should have the same yield.

Black and Scholes showed that one can then setup a risk free portfolio consisting of the underlying, a bond position and the option. Using the no-arbitrage principle and the risk free interest rate they obtained a unique (hence "fair") price for the option, which does *not* depend on the preferences of the market participants.

This has since then revolutionized financial markets, because the "personal opinion" of the trader was finally replaced by an objective criterion. Furthermore it gives the writer of the option the chance, to hedge the option and hence minimize his exposure to the market.

Black and Scholes use the before mentioned Geometric Brownian Motion (1.1) as a model for the underlying asset. They established the following pricing formula for vanilla calls:

$$C(\tau) = e^{-r_f\tau}SN(d_+) - e^{-r_d\tau}KN(d_-) \tag{1.2}$$

where

$$d_\pm = \frac{\ln(S/K) + (r_d - r_f)\tau}{\sigma\sqrt{\tau}} \pm \frac{1}{2}\sigma\sqrt{\tau}$$

One should note, that the pricing formula is independent of the mean μ (this indicates the before mentioned preference-independence), hence there is only a single unknown parameter in the BlackScholes formula, the volatility σ. Moreover the option price depends on the volatility only through the quantity $\sigma\sqrt{\tau}$, which is the integrated squared volatility over the remaining lifetime of the option. The model can therefore easily be adapted to allow for time varying parameter values for the volatility parameter, provided that the behavior is deterministic, and provided that the term $\sigma^2\tau$ is replaced by the integral $\int_0^T \sigma(t)^2 dt$.

The call option pricing function is an increasing function of the volatility. This observation can be verified by differentiation of (1.2). This means that not only can we calculate the price of an option given a value for the volatility parameter, but also that given the price of an option it is possible to deduce the unique value of the volatility which must be substituted into the BlackScholes formula to obtain the observed option price. We define the *implied volatility* $\hat{\sigma}$ to be the value of the volatility parameter σ, which is consistent with the BlackScholes formula and the observed call price. Thus we have a new measure of volatility. Implied volatility is a market assessment of the expected future volatility over the lifetime of the option. Implied volatility is a useful device because it provides a convenient shorthand for expressing the option price, and because it facilitates price comparisons of options with different characteristics.

1.4 Drawbacks of the Black-Scholes Model

There are several major drawbacks of the Black-Scholes model. Provided that the market prices options using the formula (1.2), then the implied volatility $\hat{\sigma}$ should be identical for all strikes K at a fixed maturity T. As seen in figure 1.1 this contradicts the market reality, where we can observe a so-called "smile effect". The characteristic shape of the curve indicates, that implied volatilities for out-of-the-money options are typically higher than those of at-the-money (ATM) options. Closely related to this phenomenon is the so-called "skew", an asymmetry which can also be observed in the market. Both effects cannot be modelled in the Black-Scholes model.

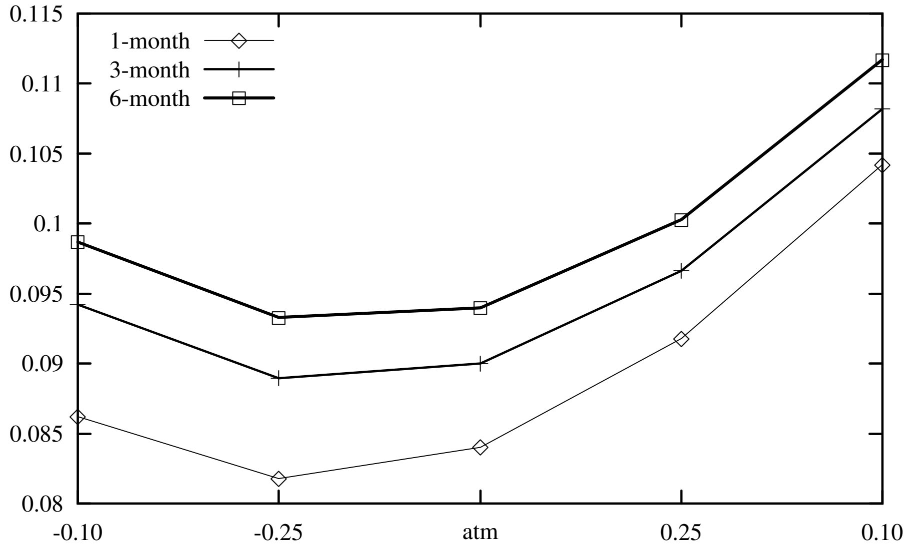

Figure 1.1: Implied volatilies of EUR/USD vanilla options for different maturities as of 29th April 2002. Liquid at-the-money options were chosen as well as options with a delta of -10%, -25%, 25% and 10%. (Source: *Commerzbank AG*)

The second drawback is the assumption, that the spot price follows a geometric Brownian Motion. This implies that the log returns should have a normal distribution. But the observed distribution is more leptokurtic than one would expect, as we can see in Figure 1.2. The Gaussian hypothesis can be rejected by means of statistical tests with almost certainty for the equity market [Faulhaber *et al.*, 2001] as well as for the foreign exchange market [Boothe and Glassman, 1987].

Another drawback is the so-called volatility clustering. One can observe, that big market moves are followed by another big move, while small moves are followed by small moves – a feature which obviously cannot be captured by a model assuming constant volatility.

In view of these drawbacks, researchers around the world proposed extensions, alterations and generalizations of the Black-Scholes model: some suggested a non-constant volatility (local volatility or constant elasticity of volatility), others considered so-called jump diffusion models, where discrete jumps of the stock price can occur. In recent years there can be found a large variety of more exotic suggestions, like generalized Levy processes and fractional Brownian Motion. In addition there exists the stochastic volatility models, which make the assumption that the volatility of the underlying is also driven by a stochastic process. This can be seen as a natural extension of the Black-Scholes model and we will later show, that we are able to resolve all the observed drawbacks while still remain in a model simple enough to try an analytical approach.

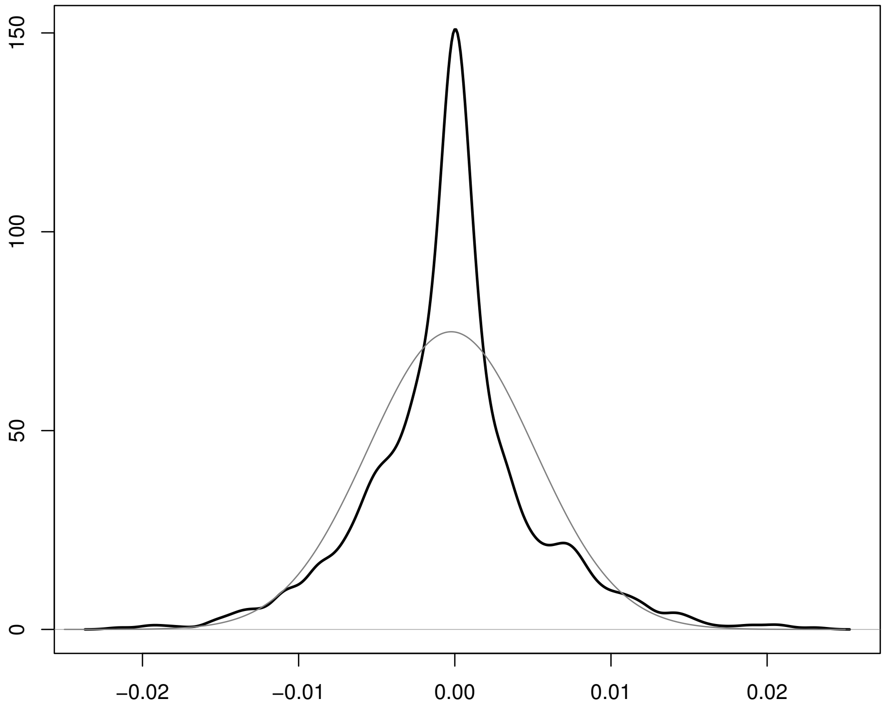

Figure 1.2: Distribution of the daily EUR/USD spot returns from January 1998 to December 2001 in comparison with a normal distribution with corresponding mean and variance. (Source: *www.oanda.com*)

Such models were first introduced by Hull and White [1987] who modelled the variance as a Geometric Brownian Motion. Scott [1987] and Stein and Stein [1991] considered models, where the logarithm of the volatility or the volatility itself are Ornstein-Uhlenbeck processes. Hull and White [1988] and Heston [1993] finally suggested a model, where the volatility is modelled as a square-root process. This is the model we will focus on in this paper.

But why is the Black-Scholes model still so widely used? The main reason is its easy analytical tractability, which results in simple formulas for most pricing problems. It is also quite accurate for at-the-money vanilla options, but one should be careful when using Black-Scholes prices for far out-of-the-money options or exotic options – in these cases market prices can show huge deviations from the theoretical BS prices.

1.5 The Heston Model

Heston [1993] suggested to model the volatility of the spot price as a square-root process, which was also considered by Cox, Ingersoll and Ross [1985] to model the short rate in the interest rate market. We can therefore write Heston's model explicitly as

$$dS_t = S_t\big(\mu dt + \sqrt{v(t)}dW_t^{(1)}\big)$$
$$dv_t = \tilde{\kappa}\big(\tilde{\theta} - v(t)\big)dt + \sigma\sqrt{v(t)}dW_t^{(2)}$$

where $W_t^{(1)}$ and $W_t^{(2)}$ are Brownian Motions with correlation ρ.

The special choice of the above square-root process has two particular advantages: First it is a mean-reverting process, i.e. the volatility tends towards a long-term value θ. The rate of this mean-reversion is determined by κ. Second, the use of $\sqrt{v(t)}$ guarantees that the volatility is always positive.

Using a stochastic process to model the volatility leads to a more flexible distribution of the spot returns than the log-normal distribution implied by the Black-Scholes Model. Heston showed that an increase of the correlation ρ generates an asymmetry in the distribution, while a change of the volatility of variance σ allows us to obtain a higher kurtosis. Hence both the before mentioned "smile" and "skew" effects can be achieved by the Heston Model. Furthermore the auto-correlation caused by the mean-reversion implies that our model features volatility clustering. This means we were able to resolve all the drawbacks of the Black-Scholes model mentioned in the previous section.

1.6 Derivation of the Partial Differential Equation

Theorem 1.1. *Every contingent claim $U(t,v,S)$ paying $g(S) = U(T,v,S)$ has to satisfy the partial differential equation*

$$U_t + (r_d - r_f)SU_S + \frac{1}{2}\sigma^2 vU_{vv} + \frac{1}{2}vS^2U_{SS} +$$
$$\rho\sigma vSU_{vS} - r_dU + \big(\tilde{\kappa}(\tilde{\theta} - v) - \lambda v\big)U_v = 0$$

Proof. We will set up a replicating portfolio, consisting of the money market, the underlying and another security with price function $Q(t,v,S)$. We start with initial wealth X_0 which evolves according to

$$dX = \Delta dS + \Gamma dQ + r_d\big(X - \Gamma Q\big)dt - \big(r_d - r_f\big)\Delta S dt$$

where Δ and Γ are the number of shares and the number of securities Q held at time t. We have to determine Γ and Δ such that $X(t) = U(t,v,S)$ for all $t \in [0,T]$. We therefore compare the differentials of U and X. Ito's Formula yields

$$dU = \mathbf{U}dt + \sigma\sqrt{v}U_v dW^{(2)} + \sqrt{v}U_S dW^{(1)}$$
$$dX = \Delta S\big(\mu - (r_d - r_f)\big)dt + \Delta\sqrt{v}SdW^{(1)} + r_dXdt + \Gamma dQ - r_d\Gamma Qdt$$

where

$$\mathbf{U} = U_t + \tilde{\kappa}(\tilde{\theta} - v)U_v + \mu S U_S + \frac{1}{2}\sigma v U_{vv} + \frac{1}{2} v S^2 U_{SS} + \rho\sigma v S U_{vS} \qquad (1.3)$$

Requiring $X = U$ and replacing dW by its representation of the form of (1.3) we get

$$\begin{aligned} dX = \Delta S\big(\mu - (r_d - r_f)\big)dt + \Delta\sqrt{v} S dW^{(1)} + r_d X dt - r_d \Gamma Q dt + \\ + \Gamma(\mathbf{Q} dt + \sigma\sqrt{v} Q_v dW^{(2)} + \sqrt{v} S Q_S dW^{(1)}) \end{aligned}$$

We can now compare the coefficients of $dW^{(1)}$ and $dW^{(2)}$, which yields

$$\begin{aligned} \Delta\sqrt{v}S + \Gamma\sqrt{v}SQ_S &= \sqrt{v}SU_S \\ \Gamma\sigma\sqrt{v}Q_v &= \sigma\sqrt{v}U_v \end{aligned}$$

which implies

$$\begin{aligned} \Gamma &= \frac{U_v}{Q_v} \\ \Delta &= U_S - \frac{U_v}{Q_v} Q_S \end{aligned}$$

We can now substitute these quantities into the original formulas and compare the dt-terms. For them to be equal we must have

$$\begin{aligned} &\frac{1}{Q_v}\big(Q_t + \tilde{\kappa}(\tilde{\theta} - v)Q_v + (r_d - r_f)SQ_S + \frac{1}{2}\sigma v Q_{vv} + \frac{1}{2} v S^2 Q_{SS} + \rho\sigma v S Q_{vS} - r_d Q\big) \\ &= \frac{1}{U_v}\big(U_t + \tilde{\kappa}(\tilde{\theta} - v)U_v + (r_d - r_f)SU_S + \frac{1}{2}\sigma v U_{vv} + \frac{1}{2} v S^2 U_{SS} + \rho\sigma v S U_{vS} - r_d U\big) \end{aligned}$$

Since the right-hand side depends only on Q, the left-hand side only on U and the choice of Q is arbitrary, both sides must be a function $\lambda(t, v, S)$. □

$\lambda(t, v, S)$ is called the *market price of volatility risk*. In practice we usually assume[1] that λ is independent of both the stock price S and the time t. We further assume that the market price of volatility risk is proportional to the volatility and hence we write λ simply as

$$\lambda(t, v, S) = \lambda v$$

After reverting the time, i.e. applying the transformation

$$\tau = T - t \qquad (1.4)$$

and elimination of the market price of volatility risk λ by replacing $\tilde{\kappa}$ and $\tilde{\theta}$ by $\kappa := \tilde{\kappa} + \lambda$ and $\theta := \frac{\tilde{\kappa}\tilde{\theta}}{\tilde{\kappa}+\lambda}$ one can therefore write the pde as

$$\begin{aligned} U_\tau - (r_d - r_f)SU_S - \frac{1}{2}\sigma^2 v U_{vv} - \frac{1}{2} v S^2 U_{SS} - \\ \rho\sigma v S U_{vS} + r_d U - \kappa(\theta - v)U_v = 0 \end{aligned} \qquad (1.5)$$

[1]The motivation for this choice can be found in Breeden [1979] and Cox, Ingersoll, Ross [1985]

In practice, we can determine the parameter λ by one volatility-dependent asset and then use it to price all the others. This is analogous to extracting the implied volatility in the Black-Scholes model.

For the sake of notational simplicity , we will assume $\lambda = 0$ in the following.

Chapter 2

Double Barrier Options

In this chapter we present various kinds of double-barrier options and derive some of their properties, in particular a modified put-call parity.

2.1 Introduction

Double barrier options are a special class among the so-called ”path-dependent options”, whose payoff depend not only on the spot price at maturity but also on its path throughout the option’s lifetime. This kind of options features two barriers A and B. As soon as the spot price reaches one of these barriers, a certain behavior of the option is triggered. We can write this as terminal and boundary conditions

$$\begin{aligned} U(T,v,S) &= u(S) \\ U(t,v,A) &= P(t) \\ U(t,v,B) &= Q(t) \end{aligned}$$

where P(t) and Q(t) are the *rebates* paid when the respective barrier is hit.

These options are very popular in practice, because they allow the holder to participate in an up-turn (respectively down-turn) of the spot price, while being considerably cheaper than the corresponding vanilla options.

In the following we will concentrate mainly on a double-barrier knock-out call, which is principally a vanilla call that ceases to exist as soon as one of the barriers is hit. This can be written in terms of the above conditions as

$$\begin{aligned} U^{DOC}(T,v,S) &= (S-K)^{+} \\ U^{DOC}(t,v,A) = U^{DOC}(t,v,B) &= 0 \end{aligned}$$

and after performing the time-reverting transformation (1.4) we obtain

$$U^{DOC}(0,v,S) = (S-K)^{+} \tag{2.1}$$

$$U^{DOC}(\tau,v,A) = U^{DOC}(\tau,v,B) = 0 \tag{2.2}$$

The conditions for the corresponding double-barrier knock-out put are derived in complete analogy as

$$U^{DOP}(0,v,S) = (K-S)^+$$
$$U^{DOP}(\tau,v,A) = U^{DOC}(\tau,v,B) = 0$$

In chapter 6 we will also look at double-one-touch (double-no-touch) options, which pay 1 unit if the spot price (never) hits one of the barriers, and which can be written as

$$U^{DOT}(0,v,S) = 0$$
$$U^{DOT}(\tau,v,A) = U^{DOT}(\tau,v,B) = e^{r_d \tau}$$

respectively

$$U^{DNT}(0,v,S) = 1$$
$$U^{DNT}(\tau,v,A) = U^{DNT}(\tau,v,B) = 0$$

2.2 Properties

We will now derive several identities, which illustrate the relationships between different kind of double-barrier options. This will allow us to concentrate on a special case of double-barrier calls and then price more general options by means of the theorems given below.

Similar to the well-known put-call parity for vanillas we will first prove the following generalized theorem by standard arbitrage arguments:

Theorem 2.1 (Put-Call-Parity for Double Barrier Options). *Let U^{DOC}, U^{DOP} and U^{DNT} the double-barrier options as defined before, with barriers A and B. Then we have for all t and S*

1. $U^{DOC}(t,S,K) - U^{DOP}(t,S,K) = U^{DOC}(t,S,A) + (A-K)U^{DNT}(t,S)$
2. $U^{DOP}(t,S,K) - U^{DOC}(t,S,K) = U^{DOP}(t,S,B) + (K-B)U^{DNT}(t,S)$

Proof. We will only proof the first equation, the second one is done in complete analogy.

Let us therefore consider a portfolio P_A consisting of a long double-barrier call and a short double-barrier put with strike K and barriers A and B. Its payout is given by

$$P_A(T) = \begin{cases} (S-K)^+ - (K-S)^+ = (S-K), & \text{if the barriers were not hit} \\ 0, & \text{else} \end{cases}$$

We compare this with the payoff of a second portfolio P_B consisting of a long double-barrier call with strike A and $(A-K)$ long double-no-touch options. It is given by

$$P_B(T) = \begin{cases} (S-A)^+ + (A-K) = (S-K), & \text{if the barriers were not hit} \\ 0, & \text{else} \end{cases}$$

As both portfolios yield the same payoff at maturity time T, its values must be equal at any time $t < T$ in order to prevent arbitrage. $\square$

Since a double-barrier call (put) is worthless when the strike K is above the upper barrier B (below the lower barrier A) we obtain the following lemma as a direct consequence of this theorem.

Lemma 2.2. *Let $K \notin [A,B]$. Then*

$$U^{DOC}(t,S,K) = \begin{cases} U^{DOC}(t,S,A) + (A-K)U^{DNT}(t,S), & \text{if} \quad K < A \\ 0, & \text{if} \quad K > B \end{cases}$$

$$U^{DOP}(t,S,K) = \begin{cases} U^{DOC}(t,S,B) + (K-B)U^{DNT}(t,S), & \text{if} \quad K > B \\ 0, & \text{if} \quad K < B \end{cases}$$

We can also prove a put-call symmetry similar to the vanilla case and express a double-barrier call in the domestic market as a put in the foreign market (and vice versa):

Theorem 2.3 (Put-Call-Symmetry for Double Barrier Options).

$$U^{DOC}(t,S,K,r_d,r_f,A,B) = U^{DOP}(t,K,S,r_f,r_d,\frac{K^2}{B},\frac{K^2}{A})$$

and

$$U^{DOP}(t,S,K,r_d,r_f,A,B) = U^{DOC}(t,K,S,r_f,r_d,\frac{K^2}{B},\frac{K^2}{A})$$

Proof. This property is a direct consequence of the symmetry inherent in the currency market. The holder of a call has the right to buy one unit of the foreign currency at a rate of K, which is equivalent to sell K units of the home currency at a rate of 1/K. Hence the prices for the two options have to coincide once they are expressed in the same currency, i.e.

$$U^{DOC}(t,S,K,r_d,r_f,A,B) = SK \cdot U^{DOP}(t,1/S,1/K,r_f,r_d,\frac{K}{BS},\frac{K}{AS})$$

Due to the homogeneity of option prices (which is a consequence of their representation as discounted expectations) we have

$$SK \cdot U^{DOP}(t,1/S,1/K,r_f,r_d,\frac{K}{BS},\frac{K}{AS}) = U^{DOP}(t,K,S,r_f,r_d,\frac{K^2}{B},\frac{K^2}{A})$$

and obtain the claim. The proof of the second equation is done in complete analogy. $\square$

To conclude this chapter we will now present an identity which links the knock-out barriers with their complements, the knock-in barriers. These options yield a payoff only, if the price of the underlying hit one of the barriers within the options' lifetime. It is natural to state the following theorem:

Theorem 2.4 (In-Out-Parity for Double Barrier Options). *Let U^{DIC}, U^{DIP} denote a double-barrier knock-in call and put respectively. Let U^{Call} and U^{Put} denote a vanilla call and put. Then*

$$U^{DOC}(t,S,K,r_d,r_f,A,B)+U^{DIC}(t,S,K,r_d,r_f,A,B)=U^{Call}(t,S,K,r_d,r_f)$$

and

$$U^{DOP}(t,S,K,r_d,r_f,A,B)+U^{DIP}(t,S,K,r_d,r_f,A,B)=U^{Put}(t,S,K,r_d,r_f)$$

Proof. Let us consider a portfolio consisting of a knock-out call and a knock-in call. The payoff at maturity is the same as that of a vanilla call. To prevent arbitrage, the two portfolios must therefore have the same price at any time $t \leq T$. □

With the help of these properties we can from now on concentrate on the case of a double-barrier knock-out call whose strike price is within the two barriers. The lemma 2.2 allows us then to extend our formulas to knock-out calls with an arbitrary strike price. We can use theorem 2.1 or theorem 2.3 to obtain prices for knock-out puts and finally get results for the knock-in variants by means of theorem 2.4.

Chapter 3

The unrestricted Problem

In order to price a double-barrier call we will first transform the pricing problem into a more tractable form. Then we will derive the transition probability density function (also called *Green's function*) $p(t,X,v,t',X',v')$ for the resulting problem.

3.1 Transformations

In a first step we will write the derived pricing problem for double-barrier knock-out calls, namely

$$\begin{aligned} U_\tau - (r_d - r_f)SU_S - \frac{1}{2}\sigma^2 vU_{vv} - \frac{1}{2}vS^2U_{SS} - \rho\sigma vSU_{vS} + r_dU - \kappa(\theta - v)U_v = 0 \\ U^{DOC}(0,v,S) = (S-K)^+ \\ U^{DOC}(\tau,v,A) = U^{DOC}(\tau,v,B) = 0 \end{aligned}$$

in forward terms to remove the interest rates. We therefore apply the transformation

$$\hat{U}(\tau,v,F) = \hat{U}(\tau,v,e^{(r_d-r_f)\tau}S) = U(\tau,v,S)e^{r_d\tau}$$

which yields the following form for the process $\hat{U}(\tau,v,F)$:

$$\begin{aligned} \hat{U}_\tau - \frac{1}{2}vF^2\hat{U}_{FF} - \rho\sigma vF\hat{U}_{vF} - \frac{1}{2}\sigma^2 v\hat{U}_{vv} - \kappa(\theta - v)\hat{U}_v = 0 \\ \hat{U}(0,v,F) = (F-K)^+ \\ \hat{U}(\tau,v,\hat{A}) = \hat{U}(\tau,v,\hat{B}) = 0 \\ \hat{U}_\tau(\tau,0,F) - \kappa\theta\hat{U}_v(\tau,0,F) = 0 \end{aligned}$$

with transformed barriers

$$\hat{A} = e^{(r_d-r_f)\tau}A, \quad \hat{B} = e^{(r_d-r_f)\tau}B$$

We now nondimensionalize the problem by representing $\hat{U}$ as

$$\hat{U}(\tau,v,F) = K\Phi(\tau,v,\xi)$$

with $\xi = F/K$ and then further reduce the problem via the transformation

$$\begin{aligned} \tau \to \tau, \quad \xi \to X = \ln(\xi) \\ \Phi(\tau, v, \xi) \to W(\tau, v, X) = e^{-X/2}\Phi(\tau, v, \xi) \end{aligned}$$

This results in the following pricing problem:

$$\begin{aligned} W_\tau - \frac{1}{2} v W_{XX} - \frac{1}{2}\sigma^2 v W_{vv} - \rho\sigma v W_{vX} - \hat{\kappa}(\hat{\theta} - v) W_v + \frac{1}{8} v W = 0 \\ W(0, v, X) = (e^{X/2} - e^{-X/2})^+ \\ W(\tau, v, \hat{a}) = W(\tau, v, \hat{b}) = 0 \end{aligned} \tag{3.1}$$

where

$$\begin{aligned} \hat{a} = \ln(\hat{A}/K), \quad \hat{b} = \ln(\hat{B}/K) \\ \hat{\kappa} = \kappa - \sigma\rho/2, \quad \hat{\theta} = \kappa\theta/\hat{\kappa} \end{aligned}$$

One can show that this is the backward Kolmogorov equation [Karatzas and Shreve, 1991] with killing at a rate of $\frac{v}{8}$ for the following system of stochastic differential equations

$$\begin{aligned} dX = \sqrt{v} dW_t^{(1)} \\ dv = \hat{\kappa}(\hat{\theta} - v) dt + \sigma\sqrt{v} dW_t^{(2)} \end{aligned} \tag{3.2}$$

3.2 Free Green's Function

Definition 3.1 (Green's Function). *The* Green's function $p(t, X, v, t', X', v')$ *of the stochastic process* (X_t, v_t) *expresses the probability for the process to be in the state* (X', v') *at time* t' *when it is known that it was in the state* (X, v) *at time* t*. We therefore have*

$$p(t, X, v, t', X', v') = P\Big(X_{t'} = X', v_{t'} = v' \mid X_t = X, v_t = v\Big) \tag{3.3}$$

Once we found the Green's function of our pricing problem, we can use the following theorem to obtain a solution:

Theorem 3.2. *Let* $p(t, X, v, t', X', v')$ *be the Green's function of a process* (X_t, v_t) *and let* $w(X, v)$ *be the payoff function. The pricing formula* $W(t, X, v)$ *can then be expressed as*

$$W(t, v, X) = \int\int p(t, X, v, T, X', v') w(X', v') dX' dv' \tag{3.4}$$

In our case we can make use of the fact, that the coefficients and the killing rate are independent of t, X and can hence concentrate on the differences $\tau = t' - t$ and $Y = X' - X$ rather than on the arguments themselves:

$$p(t,X,v,t',X',v') = p(\tau,Y,v,v')$$

p as function of (t,X,v) solves the backward Kolmogorov equation (note the change of sign in the p_{vY}-term)

$$p_\tau - \frac{1}{2}vp_{YY} - \frac{1}{2}\sigma^2 vp_{vv} + \rho\sigma vp_{vY} - \hat{\kappa}(\hat{\theta} - v)p_v + \frac{1}{8}vp = 0 \tag{3.5}$$

with corresponding initial condition

$$p(\tau \to 0,Y,v,v') \to \delta(Y)\delta(v' - v)$$

where δ is a Dirac δ-function as defined below.

Definition 3.3 (Dirac δ-function). *A function*[1] δ *is called* Dirac δ-function, *when it satisfies the following conditions:*

1. $\delta(x) = 0 \quad \forall \quad x \neq 0$

2. $\int_{-\varepsilon}^{+\varepsilon} \delta(x)dx = 1 \quad \forall \quad \varepsilon > 0$

One important property of such δ-functions, of which we will make use in later chapters, is the following.

Corollary 3.4. *Let δ be a Dirac δ-function and f an arbitrary function. Then*

$$\int_a^b \delta(x - x')f(x)dx = f(x') \quad \forall x' \in [a,b] \tag{3.6}$$

Another simplification can be done, when we observe that the payoff of our double barrier call is independent of the volatility. Hence we only need to find its integral over v', i.e.

$$q(\tau,Y,v) = \int_0^\infty p(\tau,Y,v,v')dv'$$

q solves the Kolmogorov equation with initial condition

$$q(0,Y,v) = \int_0^\infty \delta(Y)\delta(v' - v)dv' = \delta(Y)\int_0^\infty \delta(v' - v)dv' = \delta(Y)$$

[1] Formally the δ-function can be represented as the limit of strongly peaked functions with unit integral and should therefore seen as a ”generalized function”.

Due to the fact that the coefficients of q are linear functions of v, we can now guess the following form:

$$q(\tau,Y,v) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikY+2(\mathbf{A}(\tau,k)-\mathbf{B}(\tau,k)v)/\sigma^2}dk \tag{3.7}$$

We are now left with the task to determine the functions $\mathbf{A}(\tau,k)$ and $\mathbf{B}(\tau,k)$. This can be achieved by substituting (3.7) into (3.5), which leads to the following system of ODEs:

$$\mathbf{B}_\tau + \mathbf{B}^2 + (ik\sigma\rho+\hat{\kappa})\mathbf{B} - \frac{1}{4}\sigma^2(k^2+\frac{1}{4}) = 0, \quad \mathbf{B}(0,k) = 0 \tag{3.8}$$

$$\mathbf{A}_\tau + \kappa\theta\mathbf{B} = 0, \quad \mathbf{A}(0,k) = 0 \tag{3.9}$$

The first ODE is a Riccati equation and hence we can use the standard substitution

$$\mathbf{B} = \frac{\mathbf{C}_\tau}{\mathbf{C}} \tag{3.10}$$

to linearize it. Writing **A** in terms of **C** yields

$$\mathbf{A} = -\kappa\theta\ln(\sigma) \tag{3.11}$$

and performing the substitution in (3.8) results in

$$\mathbf{C}_{\tau\tau} + (ik\sigma\rho+\hat{\kappa})\mathbf{C}_\tau - \frac{1}{4}\sigma^2(k^2+\frac{1}{4})\mathbf{C} = 0 \tag{3.12}$$

$$\mathbf{C}(0,k) = 1, \mathbf{C}_\tau(0,k) = 0 \tag{3.13}$$

The solution for **C** is therefore

$$\mathbf{C} = \frac{e^{(\mu+\zeta)\tau}\big(-\mu+\zeta+(\mu+\zeta)e^{-2\zeta\tau}\big)}{2\zeta} \tag{3.14}$$

We can substitute this back into the formulas for **A** and **B**, which results in

$$\mathbf{A}(\tau,k) = -\kappa\theta(\mu+\zeta)\tau - \kappa\theta\ln\Big(\frac{-\mu+\zeta+(\mu+\zeta)e^{-2\zeta\tau}}{2\zeta}\Big)$$

$$\mathbf{B}(\tau,k) = \frac{\sigma^2(k^2+1/4)(1-e^{-2\zeta\tau})}{4\big(-\mu+\zeta+(\mu+\zeta)e^{-2\zeta\tau}\big)}$$

where

$$\mu(k) = -\frac{1}{2}(ik\sigma\rho+\hat{\kappa}) \tag{3.15}$$

$$\zeta(k) = \frac{1}{2}\sqrt{k^2\sigma^2(1-\rho^2)+2ik\sigma\rho\hat{\kappa}+\hat{\kappa}^2+\frac{\sigma^2}{4}} \tag{3.16}$$

One should note, that the formulas above are defined in the complex plane and hence we have to deal with discontinuities of the logarithm along the negative semi-axis. To rectify this problem we count the number of times (with signs) the curve

$$k \to Z_R + iZ_I = \frac{-\mu + \zeta + (\mu + \zeta)e^{-2\zeta\tau}}{2\zeta}$$

intersects with the negative semi-axis and add or subtract $2\pi i$ every time it does so. Therefore we define the set

$$\{k_j\} := \{k \mid Z_I(\tau, k) = 0 \wedge Z_R(\tau, k) < 0\}$$

and a function $N(\tau, k)$ via

$$N(\tau, k) = \begin{cases} -\sum_{0 \le k_j \le k} sign[dZ_I(\tau, k_j)/dk] & \text{if } k \le 0, \\ \sum_{0 \le k_j \le k} sign[dZ_I(\tau, k_j)/dk] & \text{if } k > 0 \end{cases}$$

and get the adjusted function $\mathbf{A}$, defined as

$$\mathbf{A}(\tau, k) = -\kappa\theta(\mu + \zeta)\tau - \kappa\theta \ln\left(\frac{-\mu + \zeta + (\mu + \zeta)e^{-2\zeta\tau}}{2\zeta} + 2i\pi N(\tau, k)\right) \tag{3.17}$$

Chapter 4

Method of Images

We are now left with the task to incorporate the barriers into the free Green's function we derived in the previous chapter. The first approach Lipton considers is the so-called method of images. We derive a generalized reflection principle and use it to express the restricted problem as an infinite sum of the unrestricted one.

The method was first used by Dewynne, Howison and Wilmott [1993]. Andreasen [2001] applied it to single barrier options in the presence of stochastic volatility, restricting himself to the uncorrelated case with zero interest rate.

In the following we will describe the method in more detail and show where the extension to the general case fails.

4.1 Bounded Green's Function

In order to incorporate the barriers into the Green's function we will first present the reflection principle for the one-barrier and the two-barrier case and then apply the latter one to derive the bounded Green's function as a lemma.

Theorem 4.1 (Reflection Principle). *Let $b > 0$ and W_t a Brownian Motion. Let T_b be the first passage time with respect to level b defined as*

$$T_b := \inf_{t \geq 0} \big(W_t = b\big)$$

Then we can express the joint distribution of first passage time and Brownian Motion directly with the distribution of the Brownian Motion. In particular we have

$$P\big(T_b < T, W(T) < x\big) = P\big(W(T) > 2b - x\big) \quad \forall x < b \tag{4.1}$$

Proof. Let $x < b$.We can interpret the left hand side of equation (4.1) as a new Brownian Motion B_t starting at $B_{T_b} = b$ and ending at $B_T < x$. For each path of this Brownian Motion exists a corresponding "shadow path"[1] as pictured in figure 4.1,

[1]This proof was chosen because of its illustrative nature. A more formal proof based on the strong Markov property of the Brownian Motion can be found in Karatzas and Shreve [1991].

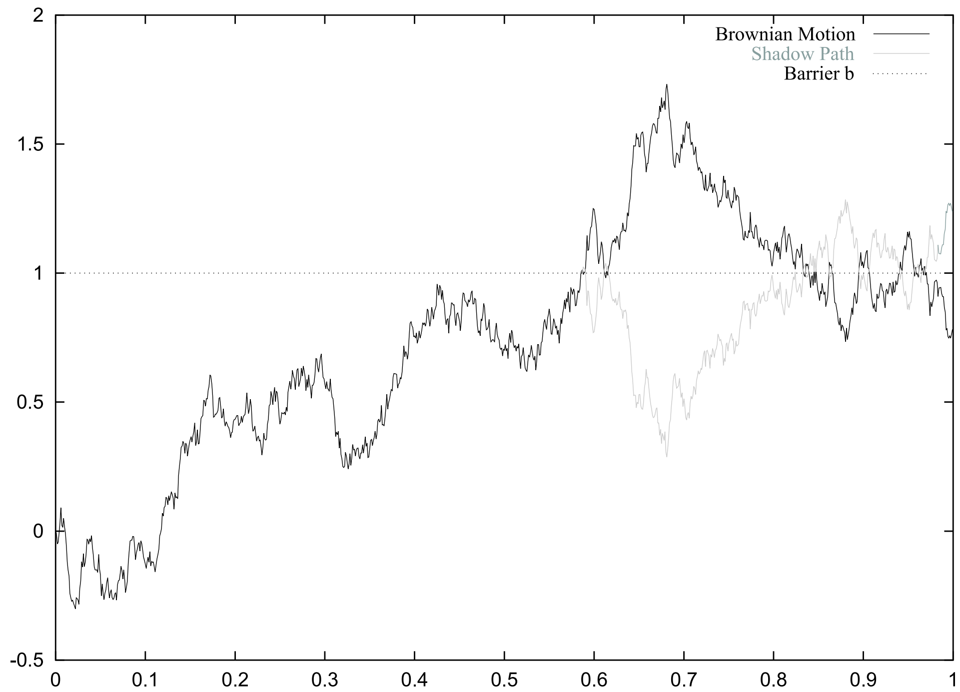

Figure 4.1: Visualization of the Reflection Principle. There is a correspondence between paths from 0 to $W(T)$ which cross b, and paths from 0 to $2b - W(T)$.

which is obtained by reflecting B_t at the barrier b. The shadow path therefore ends at $(2b - B_T) > (2b - x)$.

Since we know that a Brownian Motion starting at 0 is symmetric in the sense

$$P(W_t = w) = P(W_t = -w) \quad \forall\, w, t$$

we can deduce

$$P\big(B_T < x\big) = P\big(B_T > 2b - x\big)$$

We assumed $x < b$, therefore $W(T) > 2b - x$ implies $T_b < T$ and we obtain the claim:

$$P\big(T_b < T, W(T) < x\big) = P\big(T_b < T, W(T) > 2b - x\big) = P\big(W(T) > 2b - x\big)$$

$\square$

In order to generalize this theorem to the case of two barriers we have to consider the iterated reflections at the two barriers which result in an infinite series:

Theorem 4.2 (Generalized Reflection Principle). *Let $a < 0 < b$ and W_t a Brownian Motion with transition density $p_T(x,y)$. Let $T_{a,b}$ be the first passage time with respect to levels a and b defined as*

$$T_{a,b} := \inf_{t \geq 0} \big(W_t = b \vee W_t = a\big)$$

Then we have

$$P\big(W_T \in dy, T_{a,b} < T\big) = \tag{4.2}$$
$$= \sum_{n=-\infty}^{\infty} \Big(p_T(2n(a-b), y) - p_T(2n(b-a), y-2a)\Big)$$

As we are going to concentrate in the following on the transition probability density functions rather than on the distributions we adjusted the representation accordingly. A proof can be found in Karatzas and Shreve 1991 for the case $a = 0$.

We can now apply the theorem to our problem and obtain

Lemma 4.3 (Bounded Green's Function). *Let* $\rho = 0$. *Then*

$$G(\tau, X, v, X') \tag{4.3}$$
$$= \sum_{n=-\infty}^{\infty} \Big(q(\tau, X' - X_n, v) - q(\tau, X' + X_n - 2\tilde{a}, v)\Big)$$

where $X_n = X + 2n(\tilde{b} - \tilde{a})$.

One should note, that the theorems 4.1 and 4.2 rely heavily on the symmetry of the Brownian Motion.

As lemma 4.3 is a direct consequence of the general reflection principle, we have to check if the free Green's function in our problem is indeed even in Y. This is true for the case $\rho = 0$: Then k appears only as k^2 in formulas (3.15, 3.16). This implies, that $q(\tau, Y, v)$ is invariant with respect to the reflections $Y, k \to -Y, -k$.

But let us recall section 1.5, where we stated that the presence of the correlation causes the distribution to be skewed. This asymmetry prevents us from using lemma 4.3 in the general setting. Mathematically it can be seen in formulas (3.15) and (3.16), where we then have to deal with terms linear in k and hence cannot assure the invariance with respect to the above mentioned reflections.

One way to resolve this problem may be the use of Girsanov's Theorem. We could make a change of measure to remove the asymmetry[2] and then use theorem 4.2.

4.2 The uncorrelated case

In the special case with $\rho = 0$ we can nevertheless use this method in a straightforward manner. First of all we know that $N \equiv 0$, so we do not need to worry about the correction term in (3.17).[3]

[2] as it can be done for a Brownian Motion with drift [Karatzas and Shreve, 1991]

[3] In all observed cases, even with correlation, we have $N \equiv 0$. One could try to prove this, but since the method doesn't work in the correlated case anyway, it would be of no practical interest.

The Green's function keeps it basic form, only the functions μ and ζ reduce to

$$\mu(k) = -\frac{1}{2}\kappa \tag{4.4}$$

$$\zeta(k) = \frac{1}{2}\sqrt{k^2\sigma^2 + \kappa^2 + \frac{\sigma^2}{4}} \tag{4.5}$$

We can now write the solution of (3.1) as

$$W(\tau,v,X) = \int_{\hat{a}}^{\hat{b}} G(\tau,X,v,X')W(0,v,X')dX'$$

which results in the case of a double barrier call without rebate in

$$W(\tau,v,X) = \int_{\hat{a}}^{\hat{b}} G(\tau,X,v,X')(e^{X'/2} - e^{-X'/2})^+ dX' \tag{4.6}$$

Performing the necessary substitutions yields the following result:

$$W(\tau,v,X) = \sum_{n=-\infty}^{\infty} W_n(\tau,v,X)$$

where

$$\begin{aligned} W_n(\tau,v,X) \quad = \quad & e^{\hat{b}/2}\mathbf{H}^+(\tau,X_n-\hat{b},v) - \mathbf{H}^+(\tau,X_n,v) \\ & - e^{-\hat{b}/2}\mathbf{H}^-(\tau,X_n-\hat{b},v) + \mathbf{H}^-(\tau,X_n,v) \\ & - e^{-\hat{b}/2}\mathbf{H}^+(\tau,X_n-2\hat{a}+\hat{b},v) + \mathbf{H}^+(\tau,X_n-2\hat{a},v) \\ & + e^{\hat{b}/2}\mathbf{H}^-(\tau,X_n-2\hat{a}+\hat{b},v) - \mathbf{H}^-(\tau,X_n-2\hat{a},v) \end{aligned}$$

and

$$\mathbf{H}^{\pm}(\tau,X,v) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{-ikX+2(\mathbf{A}(\tau,k)-\mathbf{B}(\tau,k)v)/\sigma^2}}{ik \pm 1/2}$$

Finally we undo the transformations of section 3.1 and get the final pricing formula:

$$U(\tau,v,X) = e^{-r_d\tau}\sum_{n=-\infty}^{\infty} U_n(\tau,v,X)$$

where

$$\begin{aligned} U_n(\tau,v,X) \quad = \quad & \sqrt{F\hat{B}}\,\mathbf{H}^+\big(\tau,\ln(\frac{F_n}{\hat{B}}),v\big) - \sqrt{FK}\,\mathbf{H}^+\big(\tau,\ln(\frac{F_n}{K}),v\big) \\ & - K\sqrt{F\hat{B}}\,\mathbf{H}^-\big(\tau,\ln(\frac{F_n}{\hat{B}}),v\big) + \sqrt{FK}\,\mathbf{H}^-\big(\tau,\ln(\frac{F_n}{K}),v\big) \\ & - K\sqrt{F\hat{B}}\,\mathbf{H}^+\big(\tau,\ln(\frac{F_n\hat{B}}{A^2}),v\big) + \mathbf{H}^+\big(\tau,\ln(\frac{F_nK}{A^2}),v\big) \\ & + \sqrt{F\hat{B}}\,\mathbf{H}^-\big(\tau,\ln(\frac{F_n\hat{B}}{A^2}),v\big) - \mathbf{H}^-\big(\tau,\ln(\frac{F_nK}{A^2}),v\big) \end{aligned}$$

and

$$F_n = \frac{F\hat{B}^{2n}}{\hat{A}^{2n}}$$

These formulas are not only computationally expensive, but also somehow inelegant. Let us therefore consider an alternative method in the next chapter.

Chapter 5

Eigenfunction Expansion Method

The second approach under Lipton's consideration is solving the problem via an eigenfunction expansion. This method is widely used in physics and engineering to solve partial differential equations. Its application to probabilistic problems was first considered by Karlin and McGregor [1960], but it was only recently that the method was applied directly to financial problems [Beaglehole, 1991]. It was Lewis [2000] who first extended the approach to stochastic volatility models and Lipton [2001] finally considered explicitly the Heston model in the Foreign Exchange setting.

The Eigenfunction Expansion method can be seen as a generalized Fourier series expansion. It uses the separation of variables to write the solution of the partial differential equation (3.5) as infinite series of its eigenfunctions. Replacing the final payoff with a Dirac δ-function results then in the bounded Green's function.

In the following we will describe this method in more detail and study its adaptability for the general case. To keep the arguments simple we will first concentrate on the constant volatility case and illustrate the procedure. We then show how to get the results for the general stochastic volatility model directly out of the results we obtained in chapter 3.

5.1 Constant Volatility

5.1.1 Spectrum and Eigenfunctions

Let us consider for the moment the classical Black-Scholes setting as presented in section 1.3. At first we have to transform the basic pricing problem

$$V_t + \frac{1}{2}\sigma^2 S^2 V_{SS} + (r_f - r_d) S V_S - r_d V = 0$$

in its nondimensional form, first by substituting

$$\chi = \sigma^2 \tau \tag{5.1}$$

to get the form

$$V_\chi - \frac{1}{2}S^2 V_{SS} - \frac{(r_f - r_d)}{\sigma^2} S V_S + \frac{r_d}{\sigma^2} V = 0$$

second by the transformation

$$V(\chi, S, K) \to K\Theta(\chi, \zeta), \quad \zeta = S/K \tag{5.2}$$

to get the form

$$\Theta_\chi - \frac{1}{2}\zeta^2 \Theta_{\zeta\zeta} - \frac{(r_f - r_d)}{\sigma^2} \zeta \Theta_\zeta + \frac{r_d}{\sigma^2}\Theta = 0$$

followed by introducing $X = \ln(\zeta)$ to obtain

$$\Theta_\chi - \frac{1}{2}\Theta_{XX} + \big(\frac{1}{2} - \frac{1}{\sigma^2}(r_f - r_d)\big)\Theta_X + \frac{r_d}{\sigma^2}\Theta = 0$$

and finally by a change of the dependent variable[1]

$$U(\chi, X) = e^{(\frac{r}{\sigma^2} + \gamma_-^2/2)\chi + \gamma_- X}\Theta(\chi, X) \tag{5.3}$$

where

$$\gamma_\pm = \pm\frac{1}{2} + \frac{r_f - r_d}{\sigma^2}$$

The transformed problem for the constant volatility case is then just the heat equation

$$U_\chi - \frac{1}{2}U_{XX} = 0 \tag{5.4}$$

with boundary conditions

$$U(\chi, a) = 0 = U(\chi, b)$$

We now write $U(\chi, X)$ in the form

$$U(\chi, X) = e^{-\mu\chi} g(X) \tag{5.5}$$

to separate of variables. Substituting this into (5.4) we end up with the so-called "eigenfunction problem"

$$0 = \big(e^{-\mu\chi} g(X)\big)_\chi - \frac{1}{2}\big(e^{-\mu\chi} g(X)\big)_{XX} = -\mu g - \frac{1}{2} g_{XX} \tag{5.6}$$

or equivalently

$$2\mu g + g_{XX} = 0$$

with adjusted boundary conditions

$$g(a) = 0 \tag{5.7}$$

$$g(b) = 0 \tag{5.8}$$

[1] This step can be interpreted as a Girsanov transformation from a probabilistic viewpoint.

After replacing 2μ by k^2 we can write the well-known general solution [Polyanin and Zaitsev, 1996] to this simple harmonic Motion equation as

$$g(X) = \begin{cases} \Phi\cosh(kX) + \Psi\sinh(kX) & ,k^2 < 0 \\ \Phi + \Psi X & ,k^2 = 0 \\ \Phi\cos(kX) + \Psi\sin(kX) & ,k^2 > 0 \end{cases}$$

We can ignore the first two cases, since the two boundary conditions require the function to have to roots at a and b. In the (linear) case of $k^2 = 0$ this implies the trivial solution $g(x) \equiv 0$. In the case of $k^2 < 0$ we can shift the function by performing the transformation $X \to X - a$ and then use boundary condition (5.7) to deduce that Φ has to be zero (because $\cosh(0) \neq 0$). The second condition (5.8) implies that Ψ is also zero, so that we obtain again only the trivial solution.

Let us therefore consider the third case, namely

$$g(X) = \Phi\cos(kX) + \Psi\sin(kX) \tag{5.9}$$

Using the same arguments as before we can deduce that Φ has to be 0. The second condition (5.8) gives us a condition on k to get a nontrivial solution:

$$\sin\Big(k(b-a)\Big) = 0 \qquad \Rightarrow \qquad k = k_n = \frac{\pi n}{b-a}$$

Now we can write the eigenvalues μ_n simply as

$$\mu_n = \frac{1}{2}k_n^2 = \frac{\pi^2 n^2}{2(b-a)^2}$$

Using this result on equation (5.9) we get the following form of the eigenfunctions:

$$g_n(X) = \Psi_n \sin\big(k_n(X-a)\big) \tag{5.10}$$

One should emphasize, that only the presence of the two barriers ensure the spectrum to be discrete and have this rather simple form. The discreteness is especially useful in practice and is one of the main reasons for the application of the Eigenfunction Expansion method in the present situation.

5.1.2 Properties of the Eigenfunctions

In order to understand the next steps, we should first have a look at the theoretical background of this approach. Let us start with a formal definition of an eigenfunction:

Definition 5.1 (Eigenvalue and Eigenfunction). *Let L be a differential operator. Then $\lambda \in \mathbb{R}$ is called* eigenvalue *of the operator, if there exist a function $0 \not\equiv u \in C^1([a,b])$ such that*

$$Lu + \lambda u = 0 \tag{5.11}$$

u is then called the eigenfunction *corresponding to λ.*

The theoretical properties of eigenfunctions of a general differential operator are hardly known. But our particular eigenfunction problem (5.6) is a so-called Sturm-Liouville boundary problem, for which exists an extensive theory [Walter, 1996]. In particular it is known that the eigenvalues are non-negative real numbers (as shown in the previous section) and that the eigenfunctions are orthogonal. In particular we can state the following theorem.

Theorem 5.2 (Orthogonality of the Eigenfunctions). *Let* $\aleph = \mathsf{L}^2([a,b])$ *be the Hilbert space of square-integrable functions on* $[a,b]$*, where the inner product is defined as*

$$\langle f,g\rangle = \int_a^b f(X)g(X)dX$$

Then the eigenfunctions g_n *are orthogonal and we have*

$$\int_a^b g_n(X)g_m(X)dX = \delta_{nm}\Big(\frac{1}{2}\Psi_n^2(b-a)\Big)$$

Proof. Using the trigonometric equality

$$\sin(a)\sin(b) = \frac{1}{2}\big(\cos(a-b)-\cos(a+b)\big)$$

we can easily calculate

$$\begin{aligned}
\int_a^b g_n(X)g_m(X)dX &= \int_a^b \Psi_n\Psi_m \sin\Big(\frac{\pi n(X-a)}{b-a}\Big)\sin\Big(\frac{\pi m(X-a)}{b-a}\Big)dX \\
&= \frac{1}{2}\Psi_n\Psi_m\int_a^b \cos\Big(\frac{\pi(X-a)}{b-a}(n-m)\Big)-\cos\Big(\frac{\pi(X-a)}{b-a}(n+m)\Big)dX
\end{aligned}$$

which in the case of $n=m$ reduces to

$$\begin{aligned}
&= \frac{1}{2}\Psi_n^2\Big(\int_a^b 1dX - \int_a^b \cos\big(\frac{\pi(X-a)}{b-a}(2n)\big)dX\Big) \\
&= \frac{1}{2}\Psi_n^2\Big(b-a) - \Big[\frac{\sin(\frac{\pi(X-a)}{b-a}(2n)}{2n\pi/(b-a)}\Big]_a^b\Big) \\
&= \frac{1}{2}\Psi_n^2(b-a)
\end{aligned}$$

and in the case of $n\neq m$ yields

$$\begin{aligned}
&= \frac{1}{2}\Psi_n\Psi_m\Big(\Big[\frac{\sin(\frac{\pi(X-a)}{b-a}(n-m)}{(n-m)\pi/(b-a)}\Big]_a^b - \Big[\frac{\sin(\frac{\pi(X-a)}{b-a}(n+m)}{(n+m)\pi/(b-a)}\Big]_a^b\Big) \\
&= \frac{1}{2}\Psi_n\Psi_m\Big(\sin\big(\pi(n-m)\big)-\sin\big(0\big)-\sin\big(\pi(n+m)\big)+\sin\big(0\big)\Big) \\
&= 0
\end{aligned}$$

□

Furthermore, the eigenfunctions form a complete orthogonal set of functions defined on the interval $[a,b]$. Therefore we can express each continuous function $f(x)$ in terms of the eigenfunctions $g_n(x)$, i.e.

$$f(x) = \sum_{n=1}^{\infty} \Psi_n g_n(x) \tag{5.12}$$

with coefficients $\Psi_n \in \mathbb{R}$. This expansion is unique and it can be shown that the series is absolutely and uniformly convergent.[2]

5.1.3 Solution of the time-dependent Problem

Combining formulas (5.5) and (5.10) we get the solutions of the time-dependent problem

$$U_n(\chi,X) = e^{-\mu\chi} g_n(X) = e^{-\frac{1}{2}k_n^2\chi} \Psi_n \sin\big(k_n(X-a)\big)$$

and hence the general solution as linear combination of the $U_n(\chi,X)$ has the form

$$U(\chi,X) = \sum_{n=1}^{\infty} e^{-\frac{1}{2}k_n^2\chi} \Psi_n \sin\big(k_n(X-a)\big)$$

For the remaining task of determining the coefficients we can make use of the fact that

$$u(X) = U(0,X) = \sum_{n=1}^{\infty} \Psi_n \sin\big(k_n(X-a)\big)$$

in combination with the orthogonality conditions from subsection 5.1.2 to get

$$\begin{aligned} \int_a^b g_n(X)u(X)dX &= \int_a^b g_n(X) \sum_{j=1}^{\infty} \Psi_j \sin\big(k_j(X-a)\big) dX \\ &= \sum_{j=1}^{\infty} \int_a^b g_n(X) g_j(X) dX \\ &= \Psi_n^2 \frac{b-a}{2} \end{aligned}$$

Therefore the coefficients Ψ_n can be written as

$$\Psi_n = \frac{2\int_a^b u(X)\sin\big(k_n(X-a)\big)dX}{b-a} \tag{5.13}$$

[2] Since these properties generalize the Fourier series to an arbitrary orthogonal set we also denote this also as the *generalized Fourier series*.

5.1.4 Green's Function

We can now get the Green's Function by means of choosing a Dirac δ-function (as defined in chapter 3) as payoff function $u(X)$ which reduces formula (5.13) to

$$\begin{aligned}\Psi_n &= \frac{2\int_a^b \delta(X-X')\sin\big(k_n(X-a)\big)dX}{b-a} \\ &= \frac{2}{b-a}\sin\big(k_n(X'-a)\big)\end{aligned}$$

Hence the Green's function has the form

$$G(\chi,X,X') = \frac{2}{b-a}\sum_{n=1}^{\infty} e^{-\frac{1}{2}k_n^2\chi}\sin\big(k_n(X-a)\big)\sin\big(k_n(X'-a)\big)$$

and the solution of the pricing problem (5.4) can now again be written as convolution of the Green's function with the payoff function

$$\begin{aligned}U(\chi,X) &= \int_a^b G(\chi,X,X')u(X')dX' \\ &= \frac{2}{b-a}\sum_{n=1}^{\infty} e^{-\frac{1}{2}k_n^2\chi}\sin\big(k_n(X-a)\big)\int_a^b \sin\big(k_n(X'-a)\big)u(X')dX'\end{aligned}$$

5.1.5 Pricing of a Double-Barrier Call

Now we can easily find the price for all kinds of double barrier options. To illustrate this we will do the calculations explicitly for a double barrier call without rebate, i.e. a contingent claim paying

$$V^{DOC}(T,S) = (S-K)^+$$

if the underlying stayed within the range (A,B) throughout its lifetime. Applying the transformations of section 3.1 to the payoff function yields

$$\begin{aligned}\Theta^{DOC}(0,\zeta) &= (\zeta-1)^+ \\ \Theta^{DOC}(0,X) &= (e^X-1)^+\end{aligned}$$

and finally

$$U^{DOC}(0,X) = u^{DOC}(X) = e^{\gamma_- X}(e^X-1)^+ = (e^{\gamma_+ X} - e^{\gamma_- X})^+$$

Therefore the pricing formula has the form

$$\begin{aligned} U^{DOC}(\chi, X) &= \int_a^b G(\chi, X, X')(e^{\gamma_+ X'} - e^{\gamma_- X'})^+ dX' \\ &= \int_0^b G(\chi, X, X')(e^{\gamma_+ X'} - e^{\gamma_- X'}) dX' \\ &= \frac{2}{b-a} \sum_{n=1}^{\infty} e^{-\frac{1}{2}k_n^2 \chi} \sin\big(k_n(X-a)\big)\big(\varphi_n^+ - \varphi_n^-\big) \end{aligned}$$

where

$$\varphi_n^{\pm} = \int_0^b e^{\gamma_\pm X'} \sin\big(k_n(X'-a)\big) dX'$$

We can now use integration by parts twice on the $\varphi_n^{\pm}$ to get

$$\begin{aligned} \varphi_n^{\pm} &= \Big[\frac{1}{\gamma_\pm} e^{\gamma_\pm X'} \sin\big(k_n(X'-a)\big)\Big]_0^b - \int_0^b e^{\gamma_\pm X'} \cos\big(k_n(X'-a)\big) dX' \\ &= \Big[\frac{1}{\gamma_\pm} e^{\gamma_\pm X'} \sin\big(k_n(X'-a)\big)\Big]_0^b - \frac{k_n}{\gamma_\pm}\Big(\Big[\frac{1}{\gamma_\pm} e^{\gamma_\pm X'} \cos\big(k_n(X'-a)\big)\Big]_0^b + \frac{k_n}{\gamma_\pm}\varphi_n^{\pm}\Big) \end{aligned}$$

and therefore

$$\begin{aligned} \varphi_n^{\pm} &= \frac{1}{1+\frac{k_n^2}{(\gamma_\pm)^2}}\Big(-\frac{1}{\gamma_\pm}\sin(-k_n a) - \frac{k_n}{(\gamma_\pm)^2} e^{\gamma_\pm b}\cos(\pi n) + \frac{k_n}{(\gamma_\pm)^2}\cos(-k_n a)\Big) \\ &= \frac{\gamma_\pm \sin(k_n a) + k_n(-1)^{n+1} e^{\gamma_\pm b} + k_n \cos(k_n a)}{k_n^2 + (\gamma_\pm)^2} \end{aligned} \tag{5.14}$$

Finally we reverse the transformation (5.1) and end up with the pricing formula

$$V^{DOC}(\tau, S) = \frac{2Ke^{-(r_d/\sigma^2 + \gamma_-^2/2)\sigma^2\tau}(K/S)^{\gamma_-}}{\ln(B/A)} \sum_{n=1}^{\infty} (\varphi_n^+ - \varphi_n^-) e^{-k_n^2\sigma^2\tau/2} \sin\big(k_n \ln(S/A)\big)$$

5.2 Application to Stochastic Volatility Model

We can now try to proceed in analogy with the previous section to obtain a solution for the stochastic volatility case. Since transformation (5.1) is not applicable due to the non-constant volatility, we would nevertheless need to find a suitable transformation.

To avoid unnecessary calculations we will therefore look directly at the free Green's function obtained in chapter 4 and derive the Eigenfunction Expansion in a straightforward manner.

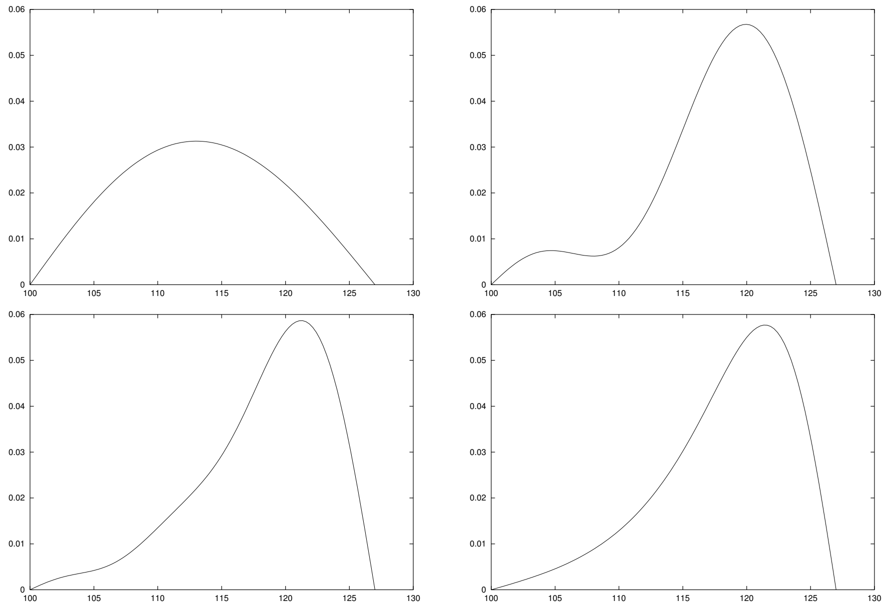

Figure 5.1: Convergence of the Eigenfunction Expansion for a Double Barrier Call. The curves show the summation of the first 1, 3, 5 and 10 sine terms.

We can rewrite formula (3.7) as

$$\begin{aligned} G(\tau,X,v,X') &= \frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\Big(\Psi\sin\big(k(X-X')\big)+\Theta\cos\big(k(X-X')\big)\Big)e^{2(\mathbf{A}(\tau,k)-\mathbf{B}(\tau,k)v)/\sigma^2}dk \\ &= \frac{1}{2\pi}\int\limits_{-\infty}^{\infty}Q(X-X')R(\tau,v)dk \end{aligned}$$

Now we can use the same arguments as in section 5.1:

- The boundary condition $G(\tau,X,v,\hat{a})=0$ yields $Q(X-\hat{a})=0$ and therefore $\Theta=0$.
- The second boundary condition $G(\tau,X,v,\hat{b})=0$ yields $k=k_n=\frac{\pi n}{\hat{b}-\hat{a}}$
- Using the orthogonality properties to obtain the coefficients Ψ_n results in $\Psi_n=\frac{2}{\hat{b}-\hat{a}}\int\limits_{\hat{a}}^{\hat{b}}u(X)\sin\big(k_n(X-\hat{a})\big)dX$

We then obtain the bounded Green's function by setting $u(X)=\delta(X-X')$:

$$G(\tau,X,v,X')=\frac{2}{\hat{b}-\hat{a}}\sum_{n=1}^{\infty}e^{2(\mathbf{A}(\tau,k)-\mathbf{B}(\tau,k)v)/\sigma^2}\sin\big(k_n(X-\hat{a})\big)\sin\big(k_n(X'-\hat{a})\big)$$

and hence get the solution of problem (3.1) as

$$W^{DOC}(\tau,v,X) = \int_{\hat{a}}^{\hat{b}} G(\tau,X,v,X')(e^{X'/2} - e^{-X'/2})^{+} dX' \tag{5.15}$$

which can, after performing the required integration in analogy to (5.14), be written as

$$W^{DOC}(\tau,v,X) = \sum_{n=1}^{\infty} e^{2(\mathbf{A}(\tau,k)-\mathbf{B}(\tau,k)v)\sigma^2} \varphi_n^{DOC} \sin\left(k_n \ln(F/\hat{A})\right) dX'$$

where

$$\varphi_n^{DOC} = \frac{2\Big((-1)^{n+1} k_n(\sqrt{\hat{B}/K} - \sqrt{K/\hat{B}}) + \sin\left(k_n \ln(\hat{A}/K)\right)\Big)}{(k_n^2 + 1/4)\ln(\hat{B}/\hat{A})}$$

We are now only left with the task of undoing the previous transformations. Let us therefore apply the back-transformation

$$U^{DOC}(\tau,S,v) = e^{-r_d\tau + X/2} K W(\tau,X,v), \quad X = \ln(F/K)$$

and obtain as final pricing formula

$$U^{DOC}(\tau,S,v) = e^{-r_d\tau}\sqrt{FK}\sum_{n=1}^{\infty} e^{2(\mathbf{A}(\tau,k_n)-\mathbf{B}(\tau,k_n)v)/\sigma^2} \varphi_n^{DOC} \sin(k_n \ln(F/\hat{A})) \tag{5.16}$$

Now we can see, why a generalization of Lipton's special case cannot lead to equations of the form (5.16): If we introduce a correlation $\rho \neq 0$ we would obtain a W_{vX} term in the partial differential equation. When we substitute the derived eigenfunction expansion into (3.1), we therefore end up with cosine terms. The other derivatives all have an even order with respect to X, so they yield only sine terms. Since these cannot be expressed by the cosine terms, the overall sum cannot be constant 0.

Even a choice of $r_d \neq r_f$ would cause problems, since this results in time-dependent barriers. As a consequence of this, we cannot perform a separation of variables as in (5.5) to obtain a time-independent problem.

In the general case we can still find approximate solutions if we assume that σT is sufficiently small [Lipton and McGhee, 2002]. We can then expand U in powers of σ and solve the resulting equation system for the terms up to the second order. But even this approximate method rests ultimately on numerical methods.

5.3 The uncorrelated case

Under the above mentioned restrictions, i.e. a situation where $r_d = r_f$ and $\rho = 0$, we can nevertheless use pricing formula (5.16). We then have $F = S$, $\hat{A} = A$, $\hat{B} = B$ and can rewrite the formula therefore as

$$U^{DOC}(\tau,S,v) = e^{-r_d\tau}\sqrt{SK}\sum_{n=1}^{\infty} e^{2(\mathbf{A}(\tau,k_n)-\mathbf{B}(\tau,k_n)v)/\sigma^2} \varphi_n \sin(k_n \ln(S/A)) \tag{5.17}$$

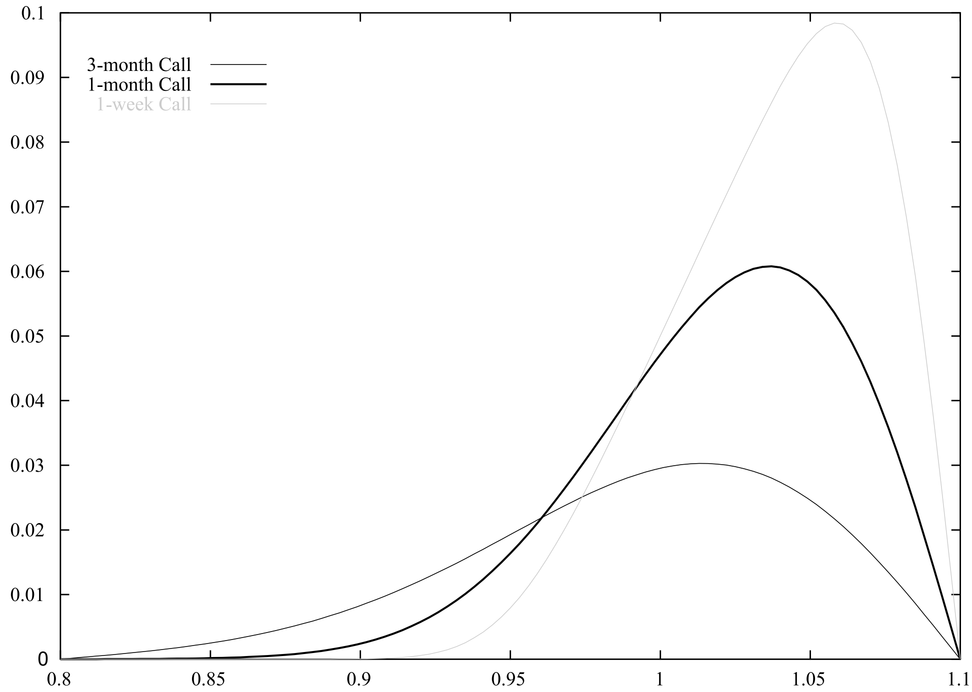

Figure 5.2: The price of double-barrier calls with different maturities under Stochastic Volatility. The parameters are $v = 0.0225$, $\theta = 0.04$, $\varepsilon = 0.40$, $\kappa = 3$, $K = 0.95$, $A = 0.8$, $B = 1.10$, $r_d = r_f = 0.03$ and $\rho = 0$.

where

$$\varphi_n = \varphi_n^{DOC} = \frac{2\Big((-1)^{n+1}k_n(\sqrt{B/K} - \sqrt{K/B}) + \sin\big(k_n \ln(A/K)\big)\Big)}{(k_n^2 + 1/4)\ln(B/A)}$$

To conclude this chapter we will also present the pricing formula for a double barrier put, which can be derived in complete analogy by simply performing the integration (5.15) with the corresponding payoff. The price is then given by formula (5.16) with

$$\varphi_n = \varphi_n^{DOP} = \frac{2\Big(k_n(\sqrt{K/A} - \sqrt{A/K}) + \sin\big(k_n \ln(A/K)\big)\Big)}{(k_n^2 + 1/4)\ln(B/A)}$$

Chapter 6

Application to other options

In this chapter we will show, how the obtained formulas can be easily adopted to options with other payoff schemes. At first we will price a Double-No-Touch by simply solving formula (4.6) with the corresponding payoff function. Then we will show how we can price a Double-One-Touch by using arbitrage arguments. And finally we will use remote artificial barriers to find asymptotic prices for single barrier and vanilla options.

6.1 Solving the Integral

We will now derive results for a double-no-touch option in analogy to the previous chapter. Its payoff was given in chapter 2 as

$$U^{DNT}(0,v,S) = 1$$

We can now transform the payoff, following exactly the steps of section 3.1, which results in

$$W^{DNT}(0,v,X) = e^{-X/2}\frac{1}{K}$$

In analogy with section 5.2 we can now solve the pricing problem. The solution is identical with (5.16) except that

$$\varphi_n = \varphi_n^{DNT} = \frac{2k_n\big((-1)^{n+1}\sqrt{K/\hat{B}} + \sqrt{K/\hat{A}}\big)}{(k_n^2+1/4)K\ln(\hat{B}/\hat{A})}$$

It may seem a bit surprising that the function depends on the strike K, but we can easily show that this cancels out once we substitute the coefficient into the pricing formula.

6.2 Replication Portfolios

We can also price other options by using standard arbitrage arguments as we did before in section 2.2. We only need to construct a replicating portfolio, consisting of options for which we know the price and then use this portfolio to price the option.

Let us therefore look at a Double-One-Touch to illustrate the method. This option is presented in section 2.1. We can replicate its final payoff by a double-no-touch option and some fixed cash investment. In particular we have

$$V^{DO)}(t,v,S) = e^{-r_d\tau} - V^{DNT}(t,v,S)$$

since the portfolio consisting of one Double-No-Touch and one Double-One-Touch must be worth exactly 1 unit at maturity.

In the same manner we can use the in-out parity 2.4 to price knock-in Barriers.

6.3 Asymptotic Approximation

Another approach is simply approximating the price of a single barrier or even vanilla option by assuming artificial barriers which are far away from the spot price.

This method is as easy to implement as it is powerful, since it allows us to apply our pricing formula to a wide variety of options. We can show that prices for up-and-out calls with a upper limit B – denoted by $V_B^{(SUOC)}$ – can be obtained by shifting the lower barrier of a double barrier call $V_{A,B}^{DOC}$ towards zero, i.e.

$$V_B^{(SUOC)} = \lim_{A\to 0} V_{A,B}^{DOC}$$

Analogously we can get formulas for down-and-out and vanilla calls

$$V_A^{SDOC} = \lim_{B\to\infty} V_{A,B}^{DOC}$$
$$V_A^{Call} = \lim_{A\to 0, B\to\infty} V_{A,B}^{DOC}$$

Corresponding formulas can be derived for puts in analogy.

Chapter 7

Summary

We saw in chapter 4 that the method of images is not a suitable method for solving the pricing problem of a double barrier option in the model under our consideration, namely the general stochastic volatility model with correlation. It was not possible to incorporate a correlation between the spot and volatility processes. This would have caused an asymmetry of the Green's function and therefore prevented the use of the reflection principle.

Another drawback of this method is the fact, that its practical implementation involves relatively many calculations and its performance is in most cases far inferior to the eigenfunction expansion method discussed in chapter 5. This method showed to be the more elegant approach and yields equations, which are easy to implement and converge comparatively fast. But again the generalization to the correlated model failed, due to the presence of the W_{vX}-term.

Both methods can therefore only be used in the non-correlated case, which is of little interest for those, who want to apply Heston's Stochastic Volatility Model to a realistic market. Further more we have to restrict ourselves to the case $r_d = r_f$, since the presence of an interest rate differential implies time-dependent barriers. We saw that this prevents us from using the eigenfunction expansion method as well as the method of images.

We therefore showed that Lipton's results can not be directly generalized.

Nevertheless, for situations where these restrictions don't constitute any problems (for example as benchmark for other methods) we can use the Eigenfunction Expansion approach as a stable and fast method to price double barrier options. As we saw in chapter 6 we can easily modify the method to price other barrier options with a different payoff function, e.g. a Double-No-Touch, and even vanillas. Further examples are given in appendix A as reference for the reader. Examples of implementation are presented in appendix B to show the efficiency of the methods we developed.

Several suggestions for future research in that field were made throughout this paper and the research being done in the next years will certainly yield some interesting results in this field. Until then we have to rely on the traditional numerical methods, despite their inefficiencies[1]. We may use the Monte Carlo method to directly simulate the paths of the spot price or the Finite Difference method, which approximates the partial differential equation by discretizing the underlying continuum.

[1] First of all this means a huge computational effort, which makes is impossible to update market prices fast enough.

Appendix A

Pricing Formulas for other Options

In the following we will derive pricing formulas for a number of additional options. This will illustrate the use of methods we presented in chapter 6, which can be directly applied to these examples.

A.1 Digital Options

Digital options pay 1 unit of money, if the spot price at maturity is above a strike K. Its payoff is therefore given by

$$U^{DIG}(0,v,S) = 1_{\{S \geq K\}}$$

The derivation of the pricing formula is similar to the case of a double-no-touch, we only have to set the lower limit of the integral to 0. This will result in prices for a kind of "double-barrier digital", which are given by our usual pricing formula,

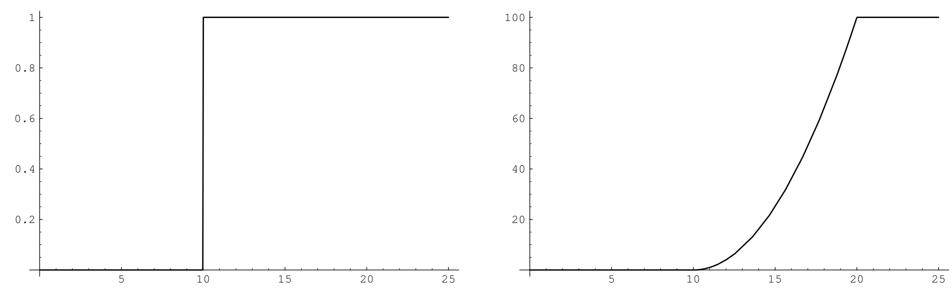

Figure A.1: Payoff functions of a (i) digital and (ii) power option, both with strike $K = 10$. The power option is capped at $C = 100$.

where the coefficients are

$$\varphi_n = \varphi_n^{DIG} = \frac{-\sin\left(k_n \ln(\hat{A}/K)\right) + 2k_n\left((-1)^{n+1}\sqrt{K/\hat{B}} + \cos\left(k_n \ln(\hat{A}/K)\right)\right)}{(k_n^2 + 1/4)K \ln(\hat{B}/\hat{A})}$$

In order to remove the influence of the barriers we have to shift them to 0 and ∞ as shown in section 6.3.

A.2 Power Options

A power option can be described as a vanilla call with squared, but capped payoff. More precisely the payoff of a power option with cap c and strike K is given by

$$U^{POW}(0,v,S) = \min\left(C, [(S-K)^+]^2\right)$$

We can easily decompose this function in three pieces, where the first and third one is constant, whereas the second is of quadratic form:

$$U^{POW}(0,v,S) = \begin{cases} 0, & S \leq K \\ (S-K)^2, & K < S \leq K + \sqrt{C} \\ C, & K + \sqrt{C} < S \end{cases}$$

Performing transformations yields

$$W^{POW}(0,v,X) = \begin{cases} 0, & X \leq 0 \\ K(e^{3X/2} - 2e^{X/2} + e^{-X/2}), & 0 < X \leq \ln(\tilde{C}) \\ (C/K)e^{-X/2}, & \ln(\tilde{C}) < X \end{cases}$$

where $\tilde{C} = 1 + \sqrt{C}/K$. We can now split the required integration (4.6) into smaller tasks, namely

$$\begin{aligned} \varphi_n = \varphi_n^{POW} &= \frac{2}{\hat{b}-\hat{a}} \int_{\hat{a}}^{\hat{b}} \sin\left(k_n(X'-\hat{a})\right) W(0,v,X')dX' \\ &= \frac{2}{\hat{b}-\hat{a}} (\varphi_n^{(Q,\frac{3}{2})} - 2\varphi_n^{(Q,\frac{1}{2})} + \varphi_n^{(Q,-\frac{1}{2})} + \varphi_n^{(C)}) \end{aligned}$$

where

$$\begin{aligned} \varphi_n^{(C)} &= \frac{C}{K} \int_{\ln(\tilde{C})}^{\hat{b}} \sin\left(k_n(X'-\hat{a})\right) e^{-X'/2} dX' \\ &= \frac{C}{K} \frac{\frac{1}{2}e^{-\ln(\tilde{C})/2} \sin\left(k_n\left(\ln(\tilde{C})-a\right)\right) + k_n(-1)^{n+1}e^{-b/2} + k_n e^{-ln(\tilde{C})/2} \cos\left(k_n\left(\ln(\tilde{C})-a\right)\right)}{k_n^2 + 1/4} \end{aligned}$$

and

$$
\begin{aligned}
\varphi_n^{(Q,\gamma)} &= \int\limits_{0}^{\ln(\tilde{C})} \sin(k_n(X'-\hat{a}))e^{\gamma X'}dX' \\
&= K\frac{\gamma\tilde{C}^{\gamma}\sin\left(k_n\left(\ln(\tilde{C})-a\right)\right)+\gamma\sin\left(k_n a\right)-k_n\tilde{C}^{\gamma}\cos\left(k_n\left(\ln(\tilde{C})-a\right)\right)+k_n\cos\left(k_n a\right)}{k_n^2+\gamma^2}
\end{aligned}
$$

The constant payoff once the cap is reached is represented by the $\varphi_n^{(C)}$-term, while the quadratic increase is represented by the sum of the $\varphi_n^{(Q,\gamma)}$-terms.

Again we have to shift the barriers in order to obtain prices for the unrestricted power option.

Appendix B

Implementation

The Eigenfunction Expansion Method was implemented in Mathematica 3.0 and the results were compared with the relevant benchmarks: for vanillas we used the analytic solutions described by Hakala and Wystup [2001], for barriers we used the finite-difference-method. For all options we stopped the summation, when the "gain"

$$G(n) = e^{-r_d \tau} \sqrt{FK} e^{2(\mathbf{A}(\tau,k_n) - \mathbf{B}(\tau,k_n) v)/\sigma^2} \varphi_n \sin(k_n \ln(F/\hat{A}))$$

was less than 10^{-9}. Lower and upper artificial barriers at 1 and 500 were introduced, if the need arose.

In table B we give an overview over the used market parameters. Tables B and B then list the deviation to the number of necessary iterations and the time it took to compute the price of the option on a dual-processor PC with two Pentium III 1 GHz.

Parameter	Value
κ	1.98937
r_d	0.036814
r_f	0.036814
σ	0.33147
S_0	123.4
K	120
v_0	0.014328
τ	0.50137
θ	0.011876
ρ	0

Table B.1: Market parameters.

One can see that the series converges much faster, when the barriers are close to each other (as it is the case for a double barrier call). Introducing remote artificial barriers to price single barrier or vanilla options causes the convergence to slow

	Vanilla Call	Vanilla Put	Digital	Power Call
V_{Ana}	5.56965	2.23072		
V_{Eigen}	5.56755	2.22973	0.639612	33.2701
Deviation (in %)	0.04	0.04		
Number of Iterations	302	342	164	302
Elapsed Time (in seconds)	0.40	0.47	0.22	1.172

Table B.2: Results of the Eigenfunction Expansion method for vanilla options in comparison with the analytical method. Results also for a digital option and a power option capped at 100.

	Up and Out Call	Double Barrier Call	DNT
Lower Barrier		120	120
Upper Barrier	127	127	127
V_{FD}	0.327446	n.a.	n.a.
V_{Eigen}	0.325321	0.109482	0.0317398
Number of Iterations	168	4	4
Elapsed Time (in seconds)	0.23	0.02	< 0.001

Table B.3: Results of the Eigenfunction Expansion method for barrier options. For the up-and-out call the Finite Difference result are given as a benchmark.

down, but the method seems to be stable enough to deal even with these extreme cases.

Another important factor for the rate of convergence is the time to maturity τ. The shorter the maturity is chosen, the more terms have to be summed up: The valuation of a 1-day put showed to take approximately five times longer than the valuation of a corresponding 1-year option. One should note, that this increase in computational effort had no effect on the precision of the results. We could approximate the exact price of a vanilla call at $t = T$ by choosing $\tau = 10^{-6}$ and still got the correct result – although the stopping criteria had to be changed to $G(n) < 10^{-10}$ and more than 50000 terms had to be summed up.

One should note, that this convergence behavior is complementing most alternative methods, which usually show a fast convergence when τ is small. As a thumb of rule we could therefore use our method, if the time to maturity is rather long, and change to an alternative method if τ is approximating 0. This will then lead to faster results than using just a single method for valuation.

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Ehrenwörtliche Erklärung

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