Geometrical Properties of Sections of Buchsbaum-Rim Sheaves or How to Construct Gorenstein Schemes of Higher Codimension with SINGULAR

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Abstract

We present a method for constructing arithmetically Gorenstein subschemes of $\mathbb{P}^n$ of large codimension using sections of so called Buchsbaum-Rim sheaves and implement it in the computer algebra system SINGULAR.

1 Introduction

In this article we want to discuss one of the ways to construct arithmetically Gorenstein subvarieties of projective spaces. It is well-known that in codimension 2 an arithmetically Gorenstein subvariety is always a complete intersection. For codimension ≥ 4, however, the construction of Gorenstein subschemes, apart from complete intersections, is quite a complicated problem since no structure theorem is known as in codimension 3 ([2]). On the other hand, it is necessary to develop a technique for constructing such schemes for example in view of Gorenstein liaison.

Consider the reflexive kernel sheaves $\mathcal{B}_\phi$ of a sufficiently general, generically surjective morphisms ϕ between decomposable bundles over $\mathbb{P}^n$, so called Buchsbaum-Rim sheaves. The desired Gorenstein schemes appear quite unexpectedly as the top-dimensional part of the zero-locus of a regular section $s \in H^0(\mathbb{P}^n, \mathcal{B}_\phi)$ ([8], [9]). As an application one gets information about the geometrical properties of sections of certain non-split rank n vector bundles on $\mathbb{P}^n$.

We show a way how to implement the construction method in the computer algebra system SINGULAR and produce some examples of Gorenstein curves and threefolds in $\mathbb{P}^6$. This implementation should be helpful to test Schneider's conjecture (see [3, 4]), claiming that there are only finitely many families of smooth non-general type threefolds in $\mathbb{P}^6$, and than complete the list of such families.

Finally, we investigate a class of sheaves $\mathcal{B}_\phi$ where the degeneracy locus of ϕ does not have the expected codimension.

2 Gorenstein points in $\mathbb{P}^3$

Let $R = k[z_0, z_1, z_2, z_3]$, $k = \bar{k}$ and A a homogeneous $t \times (t+3)$-matrix over R such that ideal of all $t \times t$-minors of A is (z_0, z_1, z_2, z_3)-primary. A is given as a graded degree zero homomorphism between two free R-modules F and G of rank $t+3$ and t respectively:

$$0 \longrightarrow Q \longrightarrow F \xrightarrow{A} G \longrightarrow \mathrm{Coker}(A) \longrightarrow 0.$$

Because of our conditions on the minors, $\mathrm{Coker}(A)$ has finite length. Sheafifying this sequence, one gets:

$$0 \longrightarrow \mathcal{Q} \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow 0.$$

Then $\mathcal{Q}$ is a vector bundle of rank 3. After twisting, we can assume that $\mathcal{Q}$ has global sections. Choose a regular global section s, i.e. one with a zero-dimensional degeneracy locus.

Theorem 2.1 ([8]) *Let $X = Z(s)$ be the zero locus of s. Then X is arithmetically Gorenstein and its saturated ideal I_X has a free resolution*

$$0 \longrightarrow R(-c_1) \longrightarrow F(-c_1) \oplus G^* \longrightarrow G(-c_1) \oplus F^* \longrightarrow I_X \longrightarrow 0,$$

where $c_1 = c_1(\mathcal{Q})$ denotes the first Chern class of $\mathcal{Q}$.

Idea of the proof. The section s determines an exact sequence $0 \longrightarrow \mathcal{O}_{\mathbb{P}^3} \xrightarrow{s} \mathcal{Q} \longrightarrow \mathcal{C} \longrightarrow 0$. One can easily show that the cokernel $\mathcal{C}$ is reflexive of rank 2. Hence $\mathcal{C}^* \cong \mathcal{C}(-c_1)$, where $c_1(\mathcal{Q}) = c_1(\mathcal{C})$ is the first Chern class. Dualizing the above sequence, we get

$$0 \longrightarrow \mathcal{C}(-c_1) \longrightarrow \mathcal{Q}^* \xrightarrow{s^t} \mathcal{O}_{\mathbb{P}^3} \longrightarrow \mathcal{O}_X \longrightarrow 0$$

since $\mathcal{E}xt^1(\mathcal{C}, \mathcal{O}_{\mathbb{P}^3}) = \mathcal{O}_X$ and $\mathcal{Q}$ is locally free. Using $H^1_*(\mathbb{P}^3, \mathcal{C}) = H^1_*(\mathbb{P}^3, \mathcal{Q})$ one obtains a diagram

$$
\begin{tikzcd}
 & 0 & 0 & & 0 & \\
0 \arrow[r] & H^0_*(\mathcal{C}(-c_1)) \arrow[u] \arrow[r] & H^0_*(\mathcal{Q}^*) \arrow[u] \arrow[r] & I_X \arrow[r] & H^1_*(\mathcal{C}(-c_1)) \arrow[u] \arrow[r] & 0 \\
 & E_1(-c_1) \arrow[u] & F^* \arrow[u] & & G(-c_1) \arrow[u] & \\
 & R(-c_1) \oplus E_2(-c_1) \arrow[u] & G^* \arrow[u] & & F(-c_1) \arrow[u] & \\
 & E_3(-c_1) \arrow[u] & 0 \arrow[u] & & E_1(-c_1) \arrow[u] & \\
 & 0 \arrow[u] & & & E_2(-c_1) \arrow[u] & \\
 & & & & E_3(-c_1) \arrow[u] & \\
 & & & & 0 \arrow[u] &
\end{tikzcd}
$$

where $E_i = \Lambda^{t+i} F \otimes S^{\,i-1}(G)^* \otimes \Lambda^t G^*$ are the R-modules from the acyclic (!) Buchsbaum-Rim complex associated to $F \xrightarrow{A} G$. Now the application of a mapping cone, the horseshoe lemma and $\mathrm{hd}_R I_X = 2$ (X is arithmetically Cohen-Macaulay) yield an exact sequence

$$0 \longrightarrow R(-c_1) \oplus E_1(-c_1) \longrightarrow F(-c_1) \oplus G^* \oplus E_1(-c_1) \longrightarrow G(-c_1) \oplus F^* \longrightarrow I_X \longrightarrow 0.$$

Using this sequence, one proves immediately that the h-vector of R/I_X is symmetric. Furthermore, one shows that X has the generalized Cayley-Bacharach property with respect to the linear system $|\,K_{\mathbb{P}_3} \otimes \det \mathcal{Q}\,|$. But this implies that X is arithmetically Gorenstein and therefore we can delete $E_1(-c_1)$ in the above resolution.

Corollary 2.2 **([8])** *Let s be a regular global section of a (non-split) rank three vector bundle $\mathcal{E}$ on $\mathbb{P}^3$ with $H^2_*(\mathbb{P}^3, \mathcal{E}) = 0$ but non-vanishing first cohomology. Then the zero scheme $X = Z(s)$ is arithmetically Gorenstein.*

Proof. The module $H^1_*(\mathbb{P}^3, \mathcal{E})$ has finite length by the Enriques-Zariski-Severi vanishing lemma and thus the Auslander-Buchsbaum theorem implies the existence of a minimal free resolution

$$0 \longrightarrow F_4 \longrightarrow F_3 \longrightarrow F_2 \longrightarrow F \longrightarrow G \longrightarrow H^1_*(\mathbb{P}^3, \mathcal{E}) \longrightarrow 0.$$

After sheafifying, we get $0 \longrightarrow \mathcal{F}_4 \longrightarrow \mathcal{F}_3 \longrightarrow \mathcal{F}_2 \longrightarrow \mathcal{F} \xrightarrow{\phi} \mathcal{G} \longrightarrow 0$. From the sequence $0 \longrightarrow Ker(\phi) \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow 0$ and the associated long exact sequence in cohomology we deduce that $Ker(\phi)$ is a vector bundle with $H^1_*(Ker(\phi)) \cong H^1_*(\mathcal{E})$, $H^2_*(Ker(\phi)) = 0$. Thus, $\mathcal{E} = Ker(\phi) \oplus \mathcal{L}$, where $\mathcal{L}$ is a decomposable bundle. But $\mathcal{E}$ does not split by Horrock's theorem and therefore $\mathcal{E} = Ker(\phi)$. Now apply theorem 2.1.

Example 2.3 Consider the 2×5-matrix

$$\phi = \begin{pmatrix} z_0^3 + z_1^3 & z_0^3 + z_2^3 & z_0^3 + z_3^3 & z_1^3 + z_2^3 & z_1^3 + z_3^3 \\ z_2^3 + z_3^3 & z_0^3 & z_1^3 & z_2^3 & z_3^3 \end{pmatrix}$$

as map in the exact sequence $0 \longrightarrow \mathcal{Q} \longrightarrow 5\,\mathcal{O}_{\mathbb{P}^3}(6) \xrightarrow{\phi} 2\,\mathcal{O}_{\mathbb{P}^3}(9) \longrightarrow 0$. Here $c_1(\mathcal{Q}) = 12$. The ideal J of the vanishing locus of a randomly chosen regular section $s \in H^0(\mathbb{P}^3, \mathcal{Q})$ is for example

$$\begin{aligned} J = \; & (\,103\,z_1^3\,z_2^3 + 66\,z_2^6 - 763\,z_0^3\,z_3^3 + 660\,z_1^3\,z_3^3 + 29\,z_2^3\,z_3^3 - 713\,z_3^6, \\ & 618\,z_0^3\,z_2^3 + 90\,z_2^6 - 5741\,z_0^3 z_3^3 + 5123\,z_1^3\,z_3^3 + 7067\,z_2^3 z_3^3 - 5935\,z_3^6, \\ & 103\,z_1^6 - 168\,z_2^6 + 959\,z_0^3 z_3^3 - 959\,z_1^3\,z_3^3 - 1230\,z_2^3\,z_3^3 + 1019\,z_3^6, \\ & 103\,z_0^3\,z_1^3 - 174\,z_2^6 + 710\,z_0^3\,z_3^3 - 710\,z_1^3\,z_3^3 - 987\,z_2^3\,z_3^3 + 934\,z_3^6, \\ & 4326\,z_0^6 - 8814\,z_2^6 + 12847\,z_0^3\,z_3^3 - 9139\,z_1^3\,z_3^3 - 13009\,z_2^3\,z_3^3 + 18923\,z_3^6\,). \end{aligned}$$

The non-saturated zero-scheme $Z(J)$ of degree 54 has h-vector

$$h_J = (1, 3, 6, 10, 15, 21, 23, 21, 15, 7, -3, -15, -20, -18, -9, -3)$$

and is consequently not Gorenstein. Computing the saturation I of J, we get the Gorenstein ideal

$$I = (\,13\,z_1^3{+}18\,z_2^3{+}31\,z_3^3,\ 546\,z_0^3{+}534\,z_2^3{+}2471\,z_3^3,\ 5976\,z_2^6{-}32430\,z_2^3 z_3^3{-}90965\,z_3^6\,)$$

with $\dim Z(I) = 0$ and $\deg Z(I) = 54$ – as expected in this case (cf. remark 2.5). The symmetric h-vector is $h_I = (1, 3, 6, 8, 9, 9, 8, 6, 3, 1)$ and a minimal free resolution is:

$$0 \longrightarrow R(-12) \longrightarrow R(-6) \oplus 2\,R(-9) \longrightarrow 2\,R(-3) \oplus R(-6) \longrightarrow I \longrightarrow 0.$$

Thus, we observe that the free resolution in theorem 2.1 is not necessarily minimal.

Remark 2.4 We want to mention that not *all* Gorenstein zero schemes in $\mathbb{P}^3$ can be obtained in this way. Indeed, the Buchsbaum-Eisenbud structure theorem [2] tells us that every codimension 3 arithmetically Gorenstein subscheme in $\mathbb{P}^n$ is precisely the zero set of the ideal I generated by the $2m \times 2m$ Pfaffians (= roots of the $2m \times 2m$ principal minors) of some $(2m+1) \times (2m+1)$ skew-symmetric matrix over $R = k[z_0, z_1, \ldots, z_n]$.

Now let $R = k[z_0, z_1, z_2, z_3]$ and

$$A = \begin{pmatrix} 0 & -z_2 & 0 & z_3 & 0 \\ z_2 & 0 & -z_1 & 0 & -z_3 \\ 0 & z_1 & 0 & 0 & -z_2 \\ -z_3 & 0 & 0 & 0 & -z_1 \\ 0 & z_3 & z_2 & z_1 & 0 \end{pmatrix}.$$

A *minimal* generating system of the ideal I is in this case for example $(z_1^2, z_2^2, z_1 z_2 + z_3^2, z_1 z_3, z_2 z_3)$. which defines a non-reduced Gorenstein point X of degree 5. It can be shown that X cannot be obtained via a regular section of a rank 3 Buchsbaum-Rim sheaf.

Remark 2.5 Suppose that $\mathcal{F} = \bigoplus_{i=1}^{t+3} \mathcal{O}_{\mathbb{P}_3}(a_i)$, $\mathcal{G} = \bigoplus_{j=1}^{t} \mathcal{O}_{\mathbb{P}_3}(b_j)$ and let ω denote the hyperplane class. We want to compute the degree of X in terms of the integers a_i and b_j. Clearly, the Poincaré dual of fundamental class $[X]$ equals $c_3(\mathcal{Q})$. The Chern polynomial of the bundle $\mathcal{Q}$ is

$$c_\omega(\mathcal{Q}) = c_\omega(\mathcal{F}) c_\omega(\mathcal{G})^{-1} = \frac{\prod_{i=1}^{t+3}(1 + a_i\,\omega)}{\prod_{j=1}^{t}(1 + b_j\,\omega)} = \sum_{i=0}^{3} s_i(a)\,\omega^i \left(\sum_{j=0}^{\min(t,3)} s_j(b)\,\omega^j \right)^{-1},$$

where $s_i(a) = s_i(a_1, \ldots, a_{t+3})$ and $s_j(b) = s_j(b_1, \ldots, b_t)$ respectively denote the elementary symmetric functions. A straightforward computation gives:

$$\deg(X) = \begin{cases} s_3(a) - b_1\, s_2(a) + b_1^2\, s_1(a) - b_1^3 & , t = 1 \\ s_3(a) - (b_1 + b_2)\, s_2(a) + (b_1^2 + b_1 b_2 + b_2^2)\, s_1(a) - (b_1^3 + b_1^2 b_2 + b_1 b_2^2 + b_2^3) & , t = 2 \\ s_3(a) - s_3(b) - s_1(a)\, s_2(b) - s_2(a)\, s_1(b) + 2 s_1(b)\, s_2(b) + s_1(b)^2\, s_1(a) - s_1(b)^3 & , t \geq 3 \end{cases}$$

Therefore one can use the described method to construct Gorenstein codimension three subschemes of a *given degree*.

3 The General Case

Migliore, Nagel and Peterson extended the above result in [9] to larger, odd codimension. This provides a construction technique for Gorenstein subschemes of $\mathbb{P}^n$ with prescribed degree in cases where no structure theorem is known, as for example in codim $5, 7, 9, \ldots$ One can also arrange that the new subschemes contain a given equidimensional subscheme of the same codimension. This is very useful from the viewpoint of Gorenstein liaison.

The setup in the general situation is the following:

Let $Z = \mathrm{Proj}(R)$, where R is a graded Gorenstein k-algebra of $\dim(R) = n + 1$. Let $\phi : \mathcal{F} \longrightarrow \mathcal{G}$ be a morphism of vector bundles over Z of rank f and g respectively, $f > g$ such that

1. the degeneracy locus of ϕ has codimension $f - g + 1$.
2. $F := H^0_*(Z, \mathcal{F})$ and $G := H^0_*(Z, \mathcal{G})$ are free R-modules.

Consider the exact sequence $0 \longrightarrow \mathcal{B}_\phi \longrightarrow \mathcal{F} \xrightarrow{\phi} \mathcal{G} \longrightarrow \mathcal{M}_\phi \longrightarrow 0$. The kernel $\mathcal{B}_\phi$ is called a Buchsbaum-Rim sheaf. It is reflexive (as a second syzygy module) of rank $r = f - g$. Let $\mathcal{P}$ be a decomposable vector bundle of rank q, $1 \leq q < r$, and $\psi : \mathcal{P} \longrightarrow \mathcal{B}_\phi$ be a "generalized section". It induces a map $\Lambda^q \psi^* : \Lambda^q \mathcal{B}_\phi^* \longrightarrow \Lambda^q \mathcal{P}^*$, where $\Lambda^q \mathcal{P}^*$ is a line bundle. Let $\delta_q := \Lambda^q \psi^*$. Then we get a resolution of the degeneracy locus S

$$\Lambda^q \mathcal{B}_\phi^* \otimes (\Lambda^q \mathcal{P}^*)^{-1} \xrightarrow{\delta_q \otimes id} \mathcal{O}_Z \longrightarrow \mathcal{O}_S \longrightarrow 0.$$

We furthermore assume that the generalized section ψ is regular, i.e. S has the expected codimension $r - q + 1$. Let I_S denote the saturated ideal of S, X

the top-dimensional part of S, I_X its saturated ideal and P the graded module $H^0_*(\mathbb{P}^n, \mathcal{P})$.

The aim of the story : Study the geometrical properties of S and X.

We cite parts of the main result obtained in [9].

Theorem 3.1 *Using the notation from above, one has the following:*

- *If $r+q$ is odd then $X = S$ is arithmetically Cohen-Macaulay if and only if $q = 1$. In this case, X has Cohen-Macaulay type $\leq 1 + \binom{\frac{r}{2}+g-1}{g-1}$.*
- *If $r+q$ is even then X is arithmetically Cohen-Macaulay if and only if $q = 1, 2$. In the case $q = 1$, X is arithmetically Gorenstein. If moreover $r < n$ then components of S has either codimension $r-q+1$ or codimension $r+1$.*
- *Moreover, there is a resolution*

$$0 \longrightarrow A_r \oplus C_r \longrightarrow \ldots \longrightarrow A_1 \oplus C_1 \longrightarrow I_X \otimes \Lambda^q P^* \longrightarrow 0,$$

where

$$\begin{aligned} A_k &= \bigoplus_{\substack{i+2j=k+q-1 \\ q \leq i+j \leq \frac{r+q-1}{2}}} \Lambda^i F^* \otimes S^j(G)^* \otimes S^{i+j-q}(P), \\ C_k &= \bigoplus_{\substack{i+2j=r+1-q-k \\ i+j \leq \frac{r-q}{2}}} \Lambda^i F \otimes S^j(G) \otimes S^{r-q-i-j}(P) \otimes \Lambda^f F^* \otimes \Lambda^g G. \end{aligned}$$

Remark 3.2 In order to clear the fog in the jungle of wedges and symmetric powers in theorem 3.1, we explicitly write down the resolution of I_X for the case of a regular section in a rk-5 Buchsbaum-Rim sheaf $\mathcal{B}_\phi$ on $\mathbb{P}^6$ (i.e. $q = 1, r = 5$, $R = k[z_0, \ldots, z_6]$ and $c_1 := c_1(\mathcal{B}_\phi)$):

$$\begin{aligned} 0 \longrightarrow R(-c_1) \longrightarrow S^2(G)^* \oplus F(-c_1) \longrightarrow F^* \otimes G^* \oplus G(-c_1) \oplus (\Lambda^2 F)(-c_1) \longrightarrow \\ \longrightarrow (\Lambda^2 F)^* \oplus G^* \oplus F \otimes G(-c_1) \longrightarrow F^* \oplus S^2(G)(-c_1) \longrightarrow I_X \longrightarrow 0. \end{aligned}$$

Corollary 3.3 *Under the assumptions of the previous theorem, let $R = k[z_0, z_1, \ldots, z_n]$, $q = 1$ and r be an* **odd** *integer. Choose a homogeneous $t \times (t+r)$-matrix A over R, defining a morphism between free modules $F \xrightarrow{A} G$. Suppose that the ideal of all $t \times t$-minors of A has the expected codimension $r+1$. After sheafifying, we get an exact sequence*

$$\begin{array}{ccccccc} 0 \longrightarrow \mathcal{B}_A \longrightarrow & \mathcal{F} & \xrightarrow{A} & \mathcal{G} & \longrightarrow Coker(A) \longrightarrow 0. \\ & || & & || & \\ & \bigoplus_{i=1}^{t+r} \mathcal{O}_{\mathbb{P}^n}(a_i) & \xrightarrow{A} & \bigoplus_{j=1}^{t} \mathcal{O}_{\mathbb{P}^n}(b_j) & \end{array}$$

Let s be a regular section of $\mathcal{B}_A$, $S := Z(s)$ and X be the top-dimensional part of S. Then X is arithmetically Gorenstein of codimension r.

Using the Buchsbaum-Rim resolution of $\mathcal{B}_A$, one can compute $\deg(X) = c_r(\mathcal{B}_A)$ in terms of the twists a_i and b_j of $\mathcal{F}$ and $\mathcal{G}$ respectively in a similar fashion as for the Gorenstein points in $\mathbb{P}^3$, cf. remark 2.5 and e.g. [6].

Thus, one could perhaps use the method in order to produce non-general type smooth Gorenstein 3-folds in $\mathbb{P}^6$ (or 5-folds in $\mathbb{P}^{10}$) with prescribed degree. M. Schneider conjectured that families of non-general type smooth k-folds in $\mathbb{P}^n$ are bounded if $n \leq 2k$. Computer experiments would certainly help to classify these families once the conjecture has been established, see for example [4] for steps in this direction in the case $n = 4, k = 2$, where the boundedness has been proved by Ellingsrud and Peskine.

Remark 3.4 It can be shown that the Buchsbaum-Rim sheaves on $\mathbb{P}^n$ are exactly the reflexive Eilenberg-MacLane sheaves $\mathcal{B}$ of rank $r \leq n$ such that $H_*^{n-r+1}(\mathbb{P}^n, \mathcal{B})^\vee$ is a Cohen-Macaulay R-module of dimension $\leq n - r$.

This implies immediately:

Corollary 3.5 *Let $\mathcal{E}$ be a (non-split) vector bundle of odd rank $3 \leq r \leq n$ on $\mathbb{P}^n$ with vanishing intermediate cohomology, with the exception of $H_*^{n-r+1}(\mathcal{E}) \neq 0$. Let s be a regular section of $\mathcal{E}$. Then the top-dimensional component of the r-codimensional zero-locus $Z(s) \subset \mathbb{P}^n$ is arithmetically Gorenstein.*

4 Relation to Good Determinantal Subschemes

As an additional motivation for the study of Buchsbaum-Rim sheaves $\mathcal{B}_\phi$, we want to mention that the loci of regular sections of their duals $\mathcal{B}_\phi^*$ actually correspond to a certain class of determinantal subschemes of projective space.

Definition 4.1 *A subscheme $V = V_t(A) \subset \mathbb{P}^n$ is called determinantal if it is given as the zero set of all $t \times t$-minors of some homogeneous $g \times f$-matrix A over $R = k[z_0, \ldots, z_n]$. If V_t has the expected codimension $(g-t+1)(f-t+1)$ it is called a standard determinantal subscheme.*

In codimension 2 the situation is simple.

Theorem 4.2 (Hilbert-Burch) *A codimension 2 subscheme in $\mathbb{P}^n$ is standard determinantal if and only if it is arithmetically Cohen-Macaulay.*

Now let A be a $t \times (t+r)$-matrix such that $V_t(A)$ has the expected codimension $r+1$. Suppose that one can delete a generalized row from A such that the ideal of maximal minors of the resulting $(t-1) \times (t+r)$-matrix has the expected codimension $r+2$. Then V_t is called a *good determinantal* subscheme.

Every complete intersection in $\mathbb{P}^n$ is a good determinantal subscheme. In particular, all codimension 2 Gorenstein schemes are good determinantal subschemes.

Examples

The rational normal curve $V_2(A) \subset \mathbb{P}^n$ given by

$$A = \begin{pmatrix} z_0 & z_1 & z_2 & \dots & z_{n-1} \\ z_1 & z_2 & z_3 & \dots & z_n \end{pmatrix}$$

is a determinantal subscheme. Let

$$B = \begin{pmatrix} z_1 & z_2 & z_3 & 0 \\ 0 & z_1 & z_2 & z_3 \end{pmatrix}.$$

Then $V_2(B) \subset \mathbb{P}^3$ is standard determinantal but not good determinantal.

The main result in the context with Buchsbaum-Rim sheaves is:

Theorem 4.3 ([7]) *Let X be a subscheme of $\mathbb{P}^n$ with codim$(X) \geq 2$. The following statements are equivalent:*

(i) X is a good determinantal subscheme of codimension $r+1$.

(ii) X is a zero locus $Z(s)$ of a regular section $s \in H^0(\mathbb{P}^n, \mathcal{B}_\phi^*)$ of the **dual** *$\mathcal{B}_\phi^*$ of a Buchsbaum-Rim sheaf of rank $r+1$.*

(iii) X is standard determinantal and locally a complete intersection outside of some subscheme $Y \subset X$ of codimension $r+2$ in $\mathbb{P}^n$.

Corollary 4.4 *A zero-scheme in $\mathbb{P}^3$ is good determinantal if and only if it is standard determinantal and a local complete intersection.*

Let us mention that in the example above the ideal of $V_2(B)$ is $(z_1, z_2, z_3)^2$. It is easy to see that it is not a local complete intersection.

5 Implementation in SINGULAR and Examples

Now we want to describe how to put the construction method into practice. The second author of this survey article implemented it in the computer algebra system SINGULAR[1]. All the procedures in this and the following section will soon be available as the SINGULAR library `buchsrim.lib`.

The first question is: How can we compute a global section of a Buchsbaum-Rim sheaf $\mathcal{B}_A$ of rank r in an algorithmical way?

We start with a homogeneous $t \times (t+r)$ matrix A. Recall that A is a block matrix with blocks A_{ij} consisting of homogeneous polynomials in $R = k[z_0, \dots, z_n]$. It can be considered as a part of the exact sequence

$$0 \longrightarrow \mathcal{B}_A \longrightarrow \mathcal{F} \xrightarrow{A} \mathcal{G}.$$

[1] SINGULAR is available at `http://www.singular.uni-kl.de/`

Twist this sequence until we get global sections of $\mathcal{F}$. Now apply $H^0_*(\bullet)$. We get $0 \longrightarrow H^0_*(\mathcal{B}_A) \longrightarrow F \xrightarrow{A} G$. Let $H \xrightarrow{B} F \xrightarrow{A} G$ be a syzygy sequence of A. The space of global sections of $\mathcal{B}_A$ is just the degree zero component of $Im(B)$. So, we have to compute a matrix representation B of the first syzygy module of A, say via the command `syz(A)`. Next, we take a linear combination $s = \sum_i f_i B_i$ of the columns B_i, where the coefficients f_i are *randomly chosen* homogeneous forms of some high degree d. The ideal of the vanishing locus $Z(s)$ of the section is of course the transpose of the column vector s.
Given a homogeneous matrix A and the degree d of the forms f_i, the SINGULAR procedure `section(module A, int d)` generates such a random section s and returns the ideal of $Z(s)$. One should check whether the section s is regular, e.g. by testing `dim(std(section(A,d)))`$= n - r + 1$.

The most expensive part is the isolation of the top-dimensional components X of $Z(s)$. There are several ways to perform this computation:

$$I_X = Ann_R Ext^r_R(\, R/I_{Z(s)}, R\,)$$

This can be computed using the standard scripts for Ext and Ann in SINGULAR, but especially for high codimension r, this method is quite time-consuming: One has to compute about r Gröbner basis in order to get I_X.
Instead, we use the following trick from liaison theory: Choose a regular sequence J in $I_{Z(s)}$ of length r (a randomly chosen sequence of homogeneous elements of high degree is generically regular). Then the double ideal quotient $(J : (J : I_{Z(s)}))$ is the saturated ideal I_X (SINGULAR procedure: `top(ideal i)`). This method is much faster, but one should definitely try to find another, more effective way to compute I_X.

The procedure `br(...)` includes everything described above and returns the ideal of the top-dimensional part of a regular section of the Buchsbaum-Rim sheaf $\mathcal{B}_A$ associated to a homogeneous matrix A. This matrix can either be chosen randomly or be a specific one. As an option, one can obtain a detailed protocol of all the computations.

Example 5.1 We give an annotated and slightly edited SINGULAR session which produces an arithmetically Gorenstein curve of degree 21 in $\mathbb{P}^6$.

Consider a 1×6-matrix A of randomly chosen linear forms on $\mathbb{P}^6$ such that

$$0 \longrightarrow \mathcal{B}_A \longrightarrow 6\,\mathcal{O}_{\mathbb{P}^6}(2) \xrightarrow{A} \mathcal{O}_{\mathbb{P}^6}(3) \longrightarrow \mathcal{M}_A \longrightarrow 0 \qquad (1)$$

is exact with $\mathrm{Supp}(\mathcal{M}_A) = Z$, $\dim Z = 0$. The first Chern class of the reflexive rk-5 sheaf $\mathcal{B}_A$ equals 9. Using the Buchsbaum-Rim resolution of $\mathcal{B}_A$, we obtain $\deg S = \deg X = 21$.

```
>ring r=32003,z(0..6),dp;
>LIB ¨buchsrim.lib¨;        (load the library buchsrim.lib)
>ideal i=br(1,5,1,2);
```

```
// We start with a random 1 x 6 matrix A of degree 1 forms:
```

```
// A[1,1],A[1,2],A[1,3],A[1,4],A[1,5],A[1,6]

// Check the codimension of Supp(Coker(A))...
// The codimension is 6 as expected.

// The vanishing locus Z(j) of a randomly chosen section of the
// kernel sheaf is given by:

// j[1]=z(0)*z(3)-12625*z(1)*z(3)+68*z(2)*z(3)+11333*z(3)^2+ ···
// j[2]=z(2)^2+15226*z(1)*z(3)+8747*z(2)*z(3)-12086*z(3)^2+ ···
⋮
// j[6]=z(0)*z(1)-3269*z(1)*z(3)+9019*z(2)*z(3)+1243*z(3)^2- ···

// Its top-dimensional part Z(i) is given by:

// i[1]=z(0)*z(3)-12625*z(1)*z(3)+68*z(2)*z(3)+11333*z(3)^2+ ···
// i[2]=z(2)^2+15226*z(1)*z(3)+8747*z(2)*z(3)-12086*z(3)^2+ ···
// i[3]=z(1)*z(2)-7394*z(1)*z(3)+2944*z(2)*z(3)-4713*z(3)^2- ···
// i[4]=z(0)*z(2)+309*z(1)*z(3)+2913*z(2)*z(3)+2392*z(3)^2- ···
// i[5]=z(1)^2-14366*z(1)*z(3)+11492*z(2)*z(3)+15380*z(3)^2+ ···
// i[6]=z(0)*z(1)-3269*z(1)*z(3)+9019*z(2)*z(3)+1243*z(3)^2- ···
// i[7]=z(0)*z(4)^2-11212*z(1)*z(4)^2+9631*z(2)*z(4)^2- ···

> hilb(std(i));   ( compute the Hilbert functions of a standard basis of i )

// 1    t^0
// -6   t^2
// 21   t^4
// -21  t^5
// 6    t^7
// -1   t^9

// 1 t^0   ( The coefficients of the 2nd Hilbert function form the h-vector )
// 5 t^1
// 9 t^2
// 5 t^3
// 1 t^4

// codimension = 5 ( SINGULAR computes codim_A^7 Z(I) and dim_A^7 Z(I) )
// dimension = 2
// degree = 21     ( ... as expected )
```

From the h-vector $h_{I_X} = (1, 5, 9, 5, 1)$ we deduce the arithmetic genus $p_a(X) = 20$, the Castelnuovo-Mumford regularity $\mathrm{reg}(\mathcal{I}_X) = 5$ and $\mathcal{O}_X \cong \omega_X(-2)$. Using the command `mres(i,0)`, we compute a *minimal* free resolution of the ideal I_X:

$$\begin{aligned} 0 \longrightarrow R(-9) \longrightarrow R(-6) \oplus 6\,R(-7) \longrightarrow 21\,R(-5) \oplus R(-6) \longrightarrow \\ \longrightarrow R(-3) \oplus 21\,R(-4) \longrightarrow 6\,R(-2) \oplus R(-3) \longrightarrow I_X \longrightarrow 0 \end{aligned}$$

Convince yourself that this is exactly the resolution from remark 3.2 which is therefore minimal in this case.

Example 5.2 A Gorenstein threefold of degree 13 in $\mathbb{P}^6$.

Choose the matrix $A = (a_{11}, a_{12}, a_{13}, a_{14})$ of quadratic forms in the exact sequence

$$0 \longrightarrow \mathcal{B}_A \longrightarrow 4\,\mathcal{O}_{\mathbb{P}^6}(3) \xrightarrow{A} \mathcal{O}_{\mathbb{P}^6}(5) \longrightarrow \mathcal{M}_A \longrightarrow 0$$

randomly and in such a way that the vanishing locus has the expected codimension 4. Note that $c_1(\mathcal{B}_A) = 7$ and $\deg(X) = c_3(\mathcal{B}_A) = 13$. Again, we present an edited Singular session:

```
>ring r=23,z(0..6),dp;
>LIB ¨buchsrim.lib¨;
>ideal i=br(1,3,2,3);
```

We compute the two Hilbert functions of a Gröbner basis of the ideal i= I_X.

```
>hilb(std(i));

// 1  t^0
// -1 t^2
// -4 t^3
// 4  t^4
// 1  t^5
// -1 t^7

// 1 t^0
// 3 t^1
// 5 t^2
// 3 t^3
// 1 t^4

// codimension = 3
// dimension = 4     (affine...)
// degree = 13
```

As expected, the arithmetically Gorenstein threefold X has degree 13. Its h-vector is $h_{I_X} = (1,3,5,3,1)$. Furthermore, we get $\mathcal{O}_X \cong \omega_X$ and $\operatorname{reg}(\mathcal{I}_X) = 5$. A minimal free resolution of I_X obtained via `mres(i,0)` is

$$0 \longrightarrow R(-7) \longrightarrow 4\,R(-4) \oplus R(-5) \longrightarrow R(-2) \oplus 4\,R(-3) \longrightarrow I_X \longrightarrow 0,$$

as stated in theorem 3.1.

6 Application to Gorenstein Liaison

We shall apply the construction method also to deal with the following problem: let V be equidimensional scheme in $\mathbb{P}^n$ of odd codimension c. We want to find an arithmetically Gorenstein subscheme X of the same codimension c, which contains V. Thus, we get a direct G-link $V \overset{X}{\sim} W$, where W denotes the residue, i.e. the scheme associated to the saturated ideal $(\,I_X : I_V\,)$. For sure we can find a complete intersection with the desired property. But from the point of view of Gorenstein liaison we are interested in arithmetically Gorenstein schemes which are not complete intersections.

Algorithm

1. Choose a rk-c Buchsbaum-Rim sheaf $\mathcal{B}_\phi$ on $\mathbb{P}^n$.

$$0 \longrightarrow \mathcal{B}_\phi \longrightarrow \mathcal{F} \xrightarrow{\phi} \mathcal{G} \longrightarrow \mathcal{M}_\phi \longrightarrow 0.$$

2. Choose a regular section s of $\mathcal{B}_\phi(j)$ (for some shift $j \in \mathbb{Z}$), which is also in $H^0_*(\mathbb{P}^n, \mathcal{F} \otimes \mathcal{I}_V)$. Then the zero set $Z(s)$ of the section contains V.

3. Compute the top-dimensional part X of the $Z(s)$. Then due to corollary 3.3, X is arithmetically Gorenstein of codimension c **containing** V.

Remark 6.1 *We can always find a $j \in \mathbb{Z}$ such that $H^0(\mathbb{P}^n, \mathcal{B}_\phi(j)) \cap H^0(\mathbb{P}^n, \mathcal{F} \otimes \mathcal{I}_V) \neq 0$.*

Let us consider an example.

Example 6.2 Let F be the Veronese surface in $\mathbb{P}^5$ given by the 2×2-minors of the matrix

$$\begin{pmatrix} z_0 & z_3 & z_4 \\ z_3 & z_1 & z_5 \\ z_4 & z_5 & z_2 \end{pmatrix}$$

Therefore F is arithmetically Cohen-Macaulay (cf. [1], p. 84). Consider the exact sequence

$$0 \longrightarrow \mathcal{B}_\phi \longrightarrow 4\,\mathcal{O}_{\mathbb{P}^5}(2) \xrightarrow{\phi} \mathcal{O}_{\mathbb{P}^5}(3) \longrightarrow \mathcal{M}_\phi \longrightarrow 0,$$

where $\phi := (z_1, z_2, z_3, z_4)$ and the cokernel $\mathcal{M}_\phi$ is supported on the line $L = Z(z_1, z_2, z_3, z_4) \subset \mathbb{P}^5$. The first syzygy module of ϕ is

$$B = \begin{pmatrix} 0 & 0 & 0 & -z_2 & -z_3 & -z_4 \\ 0 & -z_3 & -z_4 & z_1 & 0 & 0 \\ -z_4 & z_2 & 0 & 0 & z_1 & 0 \\ z_3 & 0 & z_2 & 0 & 0 & z_1 \end{pmatrix}.$$

View the saturated ideal of the surface F

$$I_F = (\, z_3z_4 - z_0z_5,\, z_1z_4 - z_3z_5,\, z_2z_3 - z_4z_5,\, z_1z_2 - z_5^2,\, z_0z_2 - z_4^2,\, z_0z_1 - z_3^2\,)$$

as a 1×6-matrix. The global sections of $4\,\mathcal{O}_{\mathbb{P}^5}(2) \otimes \mathcal{I}_F$ are linear combinations of columns of the matrix $I_F \otimes \mathrm{Id}_4$. To find a common section s of $\mathcal{B}_\phi(d)$ and $4\,\mathcal{O}_{\mathbb{P}^5}(2) \otimes \mathcal{I}_F$, we consider the intersection of the modules B and $I_F \otimes \mathrm{Id}_4$. Then s is just a linear combination of the columns of the matrix $B \cap (I_F \otimes \mathrm{Id}_4)$ where the coefficients are homogeneous d-forms in $z_0, \ldots, z_5$ whose degree d has to be chosen large enough.

Computing this with SINGULAR, one observes that the vanishing locus $Z(s)$ of a random section s is for example given by the ideal:

$$\begin{aligned} I &= (z_1z_4 - z_3z_5,\, z_2z_3 - z_4z_5,\, z_0z_2 - z_4^2,\, z_0z_1 - z_3^2,\, z_3z_4^2 - z_0z_4z_5,\, z_3^2z_4 - z_0z_3z_5) \\ &= I_F \cap (z_1, z_2, z_3, z_4) \cap (z_0, z_3, z_4). \end{aligned}$$

$Z(s)$ is consequently the union of F, containing the embedded line L, and the plane $H = Z(z_0, z_3, z_4)$. The regularity of the section s can be easily checked by computing the Hilbert function of a standard basis of I. We get $\dim Z(s) = 2$, $\deg Z(s) = 5$ and $h_I = (1, 3, 2, -1)$ as h-vector.
After isolating the top-dimensional part $X \subset Z(s)$, we get a Gorenstein surface containing the Veronese surface F. The saturated ideal of X is

$$I_X = (\, z_3z_4 - z_0z_5,\, z_1z_4 - z_3z_5,\, z_2z_3 - z_4z_5,\, z_0z_2 - z_4^2,\, z_0z_1 - z_3^2\,) = I_F \cap I_H.$$

The symmetric h-vector $h_{I_X} = (1, 2, 1)$ and the minimal free resolution

$$0 \longrightarrow R(-5) \longrightarrow 5\,R(-3) \longrightarrow 5\,R(-2) \longrightarrow I_X \longrightarrow 0$$

confirm that X is Gorenstein. The Veronese surface F is therefore G-linked to the plane H.

7 Generalized Buchsbaum-Rim Sheaves

It is a natural question to ask what happens if the degeneracy locus of $\phi : \mathcal{F} \longrightarrow \mathcal{G}$ does not have the expected codimension $f - g + 1$. Then the associated Buchsbaum-Rim complex is no longer acyclic and the sections of the kernel sheaves often lose the nice geometrical properties mentioned in theorem 3.1.

We restrict our attention to a particular class of morphisms ϕ, where the degeneracy locus is an *almost complete intersection*. A convenient way to obtain such examples is the following:

Let $G \subset \mathbb{P}^n$ be a codimension 3 arithmetically Gorenstein subscheme with $\mathcal{O}_G \cong \omega_G(l)$. Choose a complete intersection X of type (d_1, d_2, d_3) containing G. Let V denote the residue of G under the CI-link, i.e. the variety associated to the ideal $(I_X : I_G)$, and let $\alpha := d_1 + d_2 + d_3$. Using the exact sequence $0 \to \mathcal{I}_X \to \mathcal{I}_G \to \omega_V(l) \to 0$, the two resolutions

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-\alpha) \longrightarrow \bigoplus_{i<j} \mathcal{O}_{\mathbb{P}^n}(-d_i - d_j) \longrightarrow \bigoplus_{i=1}^{3} \mathcal{O}_{\mathbb{P}^n}(-d_i) \longrightarrow \mathcal{I}_X \longrightarrow 0$$

and

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(l - n - 1) \longrightarrow \mathcal{E}_2 \longrightarrow \mathcal{E}_1 \longrightarrow \mathcal{I}_G \longrightarrow 0$$

($\mathcal{E}_1$ and $\mathcal{E}_2$ are decomposable bundles of the same rank m) and a mapping cone, we get morphisms of the form

$$\boxed{0 \longrightarrow \mathcal{K}_\phi \longrightarrow \mathcal{F} \xrightarrow{\phi} \mathcal{G} \longrightarrow \mathcal{O}_V(d) \longrightarrow 0,} \tag{2}$$

where

$$\left.\begin{array}{l} \mathcal{F} = \bigoplus_{i=1}^3 \mathcal{O}_{\mathbb{P}^n}(d-d_i) \oplus \mathcal{O}_{\mathbb{P}^n}(n+1+d-l-\alpha) \\ \mathcal{G} = \mathcal{O}_{\mathbb{P}^n}(d) \end{array}\right\} \quad \textit{for some } d > \max_i\{d_i\}.$$

The degeneracy locus V of those morphisms ϕ is an almost complete intersection and has codimension 3 and not 4 as expected. Note that the bundles $\mathcal{F}$ have global sections.

Definition 7.1 *We call the kernel sheaves $\mathcal{K}_\phi$ in (2) generalized Buchsbaum-Rim sheaves.*

They have a free resolution

$$0 \longrightarrow \mathcal{E}_1^*(d-\alpha) \longrightarrow \bigoplus_{i<j} \mathcal{O}_{\mathbb{P}^n}(d-d_i-d_j) \oplus \mathcal{E}_2^*(d-\alpha) \longrightarrow \mathcal{K}_\phi \longrightarrow 0 \tag{3}$$

and are reflexive rk-3 sheaves (as 2nd syzygy sheaves).

Our interest now focuses on properties of regular sections s of generalized Buchsbaum-Rim sheaves $\mathcal{K}_\phi$. Unfortunately, their zero-loci $Z(s)$ are no longer arithmetically Gorenstein. However, one can still determine a resolution of $I_{Z(s)}$.

Theorem 7.2 *Let s be a regular section of a generalized Buchsbaum-Rim sheaf $\mathcal{K}_\phi$ on $\mathbb{P}^3$. Then the zero locus $Z(s)$ is an almost complete intersection and its saturated ideal $I_{Z(s)} \subset R = k[z_0, z_1, z_2, z_3]$ has a free resolution*

$$0 \longrightarrow R(\alpha-d-b) \oplus E_1^*(-b) \longrightarrow \bigoplus_{i=1}^3 R(d_i-b) \oplus R(-d) \oplus E_2^*(-b) \longrightarrow$$

$$\longrightarrow \bigoplus_{i=1}^3 R(d_i-d) \oplus R(d-b) \longrightarrow I_{Z(s)} \longrightarrow 0,$$

where $E_i := H^0_*(\mathbb{P}^3, \mathcal{E}_i)$ and $b := 2d-\alpha-l+4$.

Proof. Let $\mathcal{A}$ denote the image of ϕ in (2). Splitting the sequence into two short ones and applying $\mathcal{H}om(\bullet, \mathcal{O}_{\mathbb{P}^3})$, we get

$$0 \longrightarrow \mathcal{H}om(\mathcal{O}_V, \mathcal{O}_{\mathbb{P}^3}(-d)) \longrightarrow \mathcal{G}^* \longrightarrow \mathcal{A}^* \longrightarrow \mathcal{E}xt^1(\mathcal{O}_V, \mathcal{O}_{\mathbb{P}^3}(-d)) \longrightarrow 0$$

and

$$0 \longrightarrow \mathcal{A}^* \longrightarrow \mathcal{F}^* \longrightarrow \mathcal{K}_\phi^* \longrightarrow \mathcal{E}xt^1(\mathcal{A}, \mathcal{O}_{\mathbb{P}^3}) \longrightarrow 0.$$

$\mathcal{E}xt^i_{\mathcal{O}_{\mathbb{P}_3}}(\mathcal{O}_V, \omega_{\mathbb{P}^3})$ vanishes for $0 \le i < 3$ and therefore $\mathcal{G}^* \cong \mathcal{A}^*$ implies $\mathcal{A} \cong \mathcal{O}_{\mathbb{P}^3}(d)$. Hence the short exact sequence $0 \longrightarrow \mathcal{K}_\phi \longrightarrow \mathcal{F} \longrightarrow \mathcal{A} \longrightarrow 0$ shows that $\mathcal{K}_\phi$ is even locally free. Furthermore, we obtain a free resolution $0 \longrightarrow G^* \longrightarrow F^* \longrightarrow H^0_*(\mathbb{P}^3, \mathcal{K}^*_\phi) \longrightarrow 0$ since $\mathcal{E}xt^1(\mathcal{A}, \mathcal{O}_{\mathbb{P}^3}) = 0$ and $H^1_*\mathcal{G} = 0$. Let

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^3} \xrightarrow{s} \mathcal{K}_\phi \longrightarrow \mathcal{C} \longrightarrow 0 \tag{4}$$

be the exact sequence induced by the regular section s. Dualizing it and using $\mathcal{E}xt^1(\mathcal{K}_\phi, \mathcal{O}_{\mathbb{P}^3}) = 0$, we get

$$0 \longrightarrow \mathcal{C}^* \longrightarrow \mathcal{K}^*_\phi \longrightarrow \mathcal{I}_{Z(s)} \longrightarrow 0.$$

Dualizing a second time, we realize that $\mathcal{C}$ is a reflexive rank 2 sheaf. Therefore, $\mathcal{C}^* \cong \mathcal{C}(-c_1)$, where $c_1 = c_1(\mathcal{C})$. Now use (3) and (4) in order to check that $H^1_*(\mathbb{P}^3, \mathcal{K}_\phi) \cong H^1_*(\mathbb{P}^3, \mathcal{C}) = 0$ and to get a free resolution of $H^0_*\mathcal{C}(-c_1)$ via a mapping cone:

$$0 \to R(-c_1) \oplus E_1^*(d-\alpha-c_1) \to \bigoplus_{i=1}^{3} R(d+d_i-\alpha-c_1) \oplus E_2^*(d-\alpha-c_1) \to H^0_*\mathcal{C}(-c_1) \to 0$$

Consequently, we obtain the following diagram:

$$\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
0 \longrightarrow & \begin{matrix} R(-c_1) \\ \oplus \\ E_1^*(d-\alpha-c_1) \end{matrix} & \longrightarrow & \begin{matrix} \bigoplus_{i=1}^3 R(d+d_i-\alpha-c_1) \\ \oplus \\ E_2^*(d-\alpha-c_1) \end{matrix} & \longrightarrow & & H^0_*\mathcal{C}(-c_1) & \longrightarrow 0 & \\
 & & & & & & \downarrow & & \\
0 \longrightarrow & G^* & \longrightarrow & F^* & \longrightarrow & & H^0_*(\mathcal{K}^*_\phi) & \longrightarrow 0 & \\
 & & & & & & \downarrow & & \\
 & & & & & & I_{Z(s)} & & \\
 & & & & & & \downarrow & & \\
 & & & & & & 0. & &
\end{array}$$

Another application of the mapping cone lemma and the fact that $c_1(\mathcal{C}) = c_1(\mathcal{K}_\phi) = \deg(\mathcal{F}) - \deg(\mathcal{G})$ is equal to $3d - 2\alpha - l + 4 = d - \alpha + b$ imply the claim. Note that F is a free R-module of rank 4. Thus, $Z(s)$ is an almost complete intersection of type $(d-d_1, d-d_2, d-d_3, d-\alpha-l+4)$.

Example 7.3 As Gorenstein scheme G, let us take 5 points in $\mathbb{P}^3$ in general position. Their saturated ideal I_G can be obtained using a regular section s of the Buchsbaum-Rim sheaf

$$0 \longrightarrow \mathcal{B}_\psi \longrightarrow 4\,\mathcal{O}_{\mathbb{P}^3}(2) \xrightarrow{\psi} \mathcal{O}_{\mathbb{P}^3}(3) \longrightarrow 0,$$

where $\psi = (z_0, z_1, z_2, z_3)$. Note that $\mathcal{B}_\psi = \Omega^1_{\mathbb{P}^3}(3)$ and $c_3(\mathcal{B}_\psi) = 5$. According to theorem 2.1, there exists a free resolution

$$0 \longrightarrow R(-5) \longrightarrow 5\,R(-3) \longrightarrow 5\,R(-2) \longrightarrow I_G \longrightarrow 0.$$

Now we choose a complete intersection X of three cubic forms containing G. The residue V are 22 points in $\mathbb{P}^3$. Their saturated ideal I_V is an almost complete intersection and has the following minimal free resolution:

$$0 \longrightarrow 5\,R(\,-7\,) \longrightarrow 8\,R(\,-6\,) \longrightarrow 3\,R(\,-3\,) \oplus R(\,-4\,) \xrightarrow{\phi} I_V \longrightarrow 0.$$

Compare it with (2) and (3). Thus, the "data" for this example is

$$\mathcal{E}_1 = 5\,\mathcal{O}_{\mathbb{P}^3}(-2),\ \mathcal{E}_2 = 5\,\mathcal{O}_{\mathbb{P}^3}(-3),\ d_1 = d_2 = d_3 = 3,\ \alpha = 9 \ \text{ and } \ l = -1.$$

We choose $d = 6$. Using the SINGULAR command `mres(section(syz(`ϕ`),3),0)` we get the a minimal free resolution of the degree 13 zero-locus $Z(s)$ of a regular section $s \in H^0(\,\mathbb{P}^3, \mathcal{K}_\phi\,)$:

$$0 \longrightarrow 4\,R(\,-6\,) \longrightarrow 7\,R(\,-5\,) \longrightarrow 3\,R(\,-3\,) \oplus R(\,-2\,) \longrightarrow I_{Z(s)} \longrightarrow 0$$

Note that we recover exactly this sequence after deleting the "ghost-summand" $R(\,-5\,) \oplus R(\,-6\,)$ in the (non-minimal) free resolution from theorem 7.2.

Remark 7.4 The authors believe that it is straightforward to show that an analogon of theorem 7.2 holds for regular sections s of generalized Buchsbaum-Rim sheaves $\mathcal{K}_\phi$ on $\mathbb{P}^n$, $n \geq 4$. One should use the fact that the Cohen-Macaulay type and the resolution are preserved under general hyperplane sections.

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