Regularization of Inverse Problems in Satellite Geodesy by Wavelet Methods with Orbital Data Given on Closed Surfaces

Diploma Thesis

by

Petra Baumann

University of Kaiserslautern

Laboratory of Technomathematics

Geomathematics Group

December 21, 2001

Contents

Introduction

The earth is a pulsating body with moving tectonic plates, melting ice caps, rising ocean levels, a variable gravitational field due to the postglacial adjustment processes, variations in angular velocity due to the exchange in angular momentum between solid earth, oceans and atmosphere, to make it short, our planet is a complex system for which modern geodesy tries to set up reliable models. In this context a precise and detailed determination of the earth's gravitational potential on the earth's surface is indispensable, and satellite techniques provide a globally dense set of well-distributed observations of the gravitational potential.

This thesis is concerned with the reconstruction of the earth's gravitational potential from satellite measurements. In this context, two measurement principles are of major interest: satellite-to-satellite-tracking (SST) and satellite gravity gradiometry (SGG), and among the two variants of satellite-to-satellite tracking which are under investigation in the geodetic community, we are here interested in the so-called high-low satellite-to-satellite tracking (SST high-low). In what follows, we will give a simplified explanation of the measurement principles. Here we assume that the movement of the satellite is a consequence exclusively of the gravitational potential of the earth.

In high-low satellite-to-satellite tracking, a low flying satellite (low earth orbiter) travels on a nearly circular and nearly polar orbit. The low earth orbiter is equipped with a GPS-receiver and uses the global positioning system (GPS) to determine its position as follows: The 24 GPS-satellites fly at an altitude of approximately 20000 km on orbits which are chosen in such a way that from every point on the earth's surface at least four of these GPS-satellites can be simultaneously seen. Due to the high altitude of the GPS-satellites, the earth's gravitational potential, which is attenuated exponentially with increasing distance from the earth's surface, is known at the orbit altitude of the GPS-satellites with negligible error, and thus the positions of the GPS-satellites can be determined with high accuracy. Each GPS-satellite continuously sends a signal which includes the sending time and its position at sending time, With this information the distances of the low earth orbiter from the GPS-satellites, and hence its position, can be calculated from the travelling time of the GPS-signals. By numerical differentiation of the orbit the acceleration of the satellite can be determined, and thus (according to Newton's law of motion) the gradient of the gravitational potential. This yields in particular the radial derivative of the potential, from which the potential can be uniquely recovered.

In satellite gravity gradiometry, a low earth orbiter, this time equipped with a gradiometer in addition to the GPS-receiver, travels on a nearly circular and nearly polar orbit. As in case of SST, the GPS-receiver admits an accurate determination of the satellite's position. The gradiometer on board the satellite measures the relative motion of two test masses inside the satellite, and with the help of these measurement data it is possible to determine the full gravity tensor (Hesse matrix) of the earth's gravitational potential in points on the orbit. In particular this yields knowledge of the second order radial derivative of the gravitational potential in the measurement points, which suffices to recover the earth's gravitational potential.

The principle of high-low satellite-to-satellite tracking is already realized in the German satellite mission CHAMP (CHAllenging Minisatellite Payload). The satellite CHAMP was launched on July 15th, 2000 from the cosmodrome Plesetsk in Russia. It travels on an orbit with an initial altitude of 454 km, an inclination of approximately $i = 87°$ and an eccentricity of approximately $e = 0.004$. During the five years lifetime of CHAMP the orbit altitude will decrease to 300 km at the end of the mission due to atmospheric drag.

The future ESA satellite mission GOCE (Gravity and steady-state Ocean Circulation Explorer), which will start in 2005/2006 and will last approximately two years, will realize the SGG-scenario. The satellite GOCE will fly at the low altitude of approximately 250 km. GOCE will fly on a nearly circluar orbit with eccentricity between $e = 0.001$ and $e = 0.0045$ and an inclination of approximately $i = 96.5°$.

For more precise and detailed information on the satellite missions CHAMP and GOCE, the reader is referred to the internet addresses

- `http://op.gpz-potsdam.de/champ/` (CHAMP)
- `http://www.goce-projektbuero.de` (GOCE)
- `http://www.esa.int/export/esaLP/goce.html` (GOCE)
- `http://goce.tu-graz.at` (GOCE)

Both missions are expected to yield an improvement of the present NASA model EGM96, a (global) model of the earth's gravitational potential which is presented as a (finite) set of Fourier coefficients (with respect to outer harmonics), as it is common in the geodetic community. Due to the non-space-localizing character of the outer harmonics, the calculation of a model of this type requires global data. The EGM96 model includes outer harmonic contributions complete up to degree 360, and was computed with the help of various kinds of data (terrestrial and satellite data), which were collected during the last decades. Due to the heterogeneity and the varying quality of the data (note that terrestrial measurements are expensive and currently only available in sufficiently high density for certain regions, whereas former satellite missions provided no high resolution due to a too large altitude), the EGM96 model is considered to be not very accurate with respect to the high outer harmonic degrees. The data material collected during the CHAMP mission is believed to yield a reliable model up to outer harmonic degree 80, whereas the Hesse tensor data measured during the GOCE mission are expected to give a further improved model which is accurate up to an outer harmonic degree of 220.

In this thesis we model the earth's gravitational potential in a quite different way. In contrast to the outer harmonics, which are not space-localizing, we use linear combinations of functions which are strongly space-localizing to model the earth's gravitational potential. This has the advantage, that high-frequency structures can be resolved, and furthermore, it is possible to compute a local model of the gravitational potential also from only locally given data. This aspect is in particular interesting for the GOCE mission, since due to the inclination of the orbit of the GOCE satellite so-called 'polar gaps', i.e., regions close to the poles where no measurements can be taken, occur. This problem cannot be easily handled, if the gravitational potential is represented in terms of functions which are not space-localizing. However, our approach does not intend to replace models of the earth's gravitational potential in terms of outer harmonic expansions completely, but it can be well combined with those models by taking an outer harmonic model of the earth's gravitational potential up to a certain degree as a 'basic approximation', which is subtracted from the data, and then using a linear combination of space-localizing functions to model the remaining high-frequeny part.

The thesis is organized in three parts. Part I covers the mathematical tools which are needed for the modelling of the earth's gravitational potential. Chapter 1 introduces the basic notation and the relevant background material about special functions which are used in gravitational potential approximation. After that the potential theoretic foundations behind the approximation of potentials by harmonic functions are given and the relevant spaces of harmonic functions are introduced. We conclude the chapter with the presentation of the Runge-Walsh approximation theorem, which is the crucial result behind the approximation procedures used in the thesis. It states that under certain circumstances a harmonic function can be approximated in uniform sense by a sequence of harmonic functions which have a larger domain of harmonicity. This larger domain of harmonicity can in case of the earth's gravitational potential in particular be chosen as the outer space of a sphere which is entirely inscribed in the earth's interior. Since the mathematical models of satellite-to-satellite tracking and satellite gravity gradiometry involve the first and second order radial derivatives of harmonic functions, we define function spaces in which harmonic functions and their first and second order radial derivatives can be represented in Chapter 2. Furthermore, we introduce pseudodifferential operators which operate on these spaces, and we define the SST- and the SGG-operator as pseudodifferential operators which map a harmonic function onto its first and second order radial derivative at satellite altitude. In Chapter 3 the SST-problem and the SGG-problem are classified as ill-posed pseudodifferential operator equations $\Lambda F = G$. The operator equations are ill-posed for the following two reasons: On the one hand, the operator equations cannot be solved for all right-hand sides G, and on the other hand the inverse of the SST-operator and the inverse of the SGG-operator are not bounded. The unboundedness means that data errors can be arbitrarily amplified if the gravitational potential is reconstructed from noisy SST- or SGG-measurements. In order to cope with the ill-posedness, so-called regularization techniques have to be applied, and their construction principles are briefly discussed. A regularization for an ill-posed operator equation $\Lambda F = G$ is a sequence of bounded linear operators which approximates the unbounded inverse operator Λ^{-1} pointwise. The discretization of these regularization schemes will be performed in Part II with the help of the spline functions that are introduced in Chapter 4. In Chapter 4 we define interpolating splines and so-called smoothing splines, which are appropriate in the presence of noisy data. Scaling functions and wavelets are introduced in Chapter 5. After that the construction of a regularization for the unbounded inverse Λ^{-1} is explained. If this regularization is realized with a so-called regularization scaling function it leads to a sequence of approximations of the solution of the ill-posed operator equation $\Lambda F = G$ at different scales of space-frequency resolution.

In Part II we give the details of the numerical realization of the proposed regularization techniques. Approximation of the right-hand-side G will in our case always be done by computing either an interpolating or smoothing spline from the SST- or SGG-data. In case of exact data an interpolating spline is calculated, whereas in case of noisy data a smoothing spline is computed. For the practice this means that large linear equation systems with a positive definite symmetric matrix have to be solved. In Chapter 6 we propose a domain decomposition method, namely the Schwarz alternating algorithm, which allows it to split these large linear equation systems into several smaller linear equation systems, which are then alternatingly solved in an iterative procedure. The splitting of the large linear equation system corresponds to a splitting of the pointset (on which the SST- or SGG-data is given) into a number of smaller, possibly overlapping, pointsets. In our implementation of the Schwarz alternating procedure these smaller pointsets can be associated to certain subdomains on a sphere. This is the reason why the Schwarz alternating procedure is called a domain decomposition algorithm. In Chapter 7 the numerical discretization of our regularization of the SST- and SGG-problem is explained. In the regularization the right-hand side G of our operator equation is replaced by the spline which was calculated with the domain decomposition method of Chapter 6 and which approximates G. Due to the properties of the splines this immediately leads to a discretization.

In Part III the results of our numerical experiments are presented. Chapter 8 contains an extensive study of the performance of the Schwarz alternating algorithm. Here the question how the number of subdomains (which correspond to the subdivision of the data pointset) and the overlap of the subdomains influence the convergence, the runtime and the memory requirement is addressed. The numerical studies show that (even without the use of fast summation techniques) the Schwarz alternating procedure allows the solution of much larger linear equation systems than those which can be solved with a direct solver. In addition the runtime is considerably reduced. In Chapter 9 the results for a local reconstruction of the gravitational potential from simulated SGG-data are presented. As mentioned above, an interpolating spline was calculated in case of 'exact' data and a smoothing spline was used in case of noisy data. For a good tuning of the smoothing parameter the potential reconstructed from noisy data has approximately the same accuracy as the potential reconstructed from 'exact' data.

Part I

Mathematical Modelling

Chapter 1

General Notation and Potential Theoretic Foundations

This chapter starts with the general notation used throughout the thesis and summarizes the relevant background material on special functions used in gravitational potential approximation, i.e., Legendre polynomials, spherical and outer harmonics, in Section 1.1. The presentation there is kept rather concise and omits proofs. For a detailed introduction into the theory of spherical harmonics and Legendre polynomials the reader is referred to [FrGeSchr1998] and [Mu1966]. After that the potential theoretic foundations of gravitational potential approximation by harmonic functions are given in Section 1.2 and the relevant function spaces are introduced. The crucial result behind the approximation procedures in this work are the Runge-Walsh approximation theorem and an extension of Helly's theorem. The Runge-Walsh approximation theorem states that under certain circumstances a harmonic function can be approximated in uniform sense by harmonic functions which have a larger domain of harmonicity. In gravitational potential approximation this domain can in particular be chosen to be the outer space of a sphere which is entirely inscribed in the earth's interior. The extension of Helly's theorem yields the additional information that it is even possible to choose such a so-called Runge-Walsh approximation in such a way that it 'interpolates' with respect to a finite set of bounded linear functionals. The presentation of the concepts in Section 1.2 essentially follows the one given in [Fr1999] and [He2002].

1.1 General Definitions and Notation

The symbols $\mathbb{N}$ and $\mathbb{R}$ denote the set of positive integers and real numbers, respectively. As usual, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, $\mathbb{R}^+ := \{s \in \mathbb{R} | s > 0\}$ and $\mathbb{R}_0^+ := \{s \in \mathbb{R} | s \geq 0\}$. $\mathbb{R}^3$ denotes the three-dimensional Euclidean space. For $x, y \in \mathbb{R}^3$, where $x = (x_1, x_2, x_3)^T$, $y = (y_1, y_2, y_3)^T$, the Euclidean inner product is defined by $x \cdot y := \sum_{i=1}^3 x_i y_i$, and the Euclidean norm of $x \in \mathbb{R}^3$ is accordingly given by $|x| := \sqrt{x \cdot x} = \sqrt{\sum_{i=1}^3 x_i^2}$.

Let $\Omega_r := \{x \in \mathbb{R}^3 : |x| = r\}$, $r \in \mathbb{R}^+$, denote the sphere in $\mathbb{R}^3$ with radius r and centre in the origin, and let $\Omega := \Omega_1$ denote the unit sphere in $\mathbb{R}^3$. Each $x \in \mathbb{R}^3 \backslash \{0\}$ may be uniquely represented in the form $x = \rho\xi$, where $\rho := |x|$, and $\xi := x/|x| = (\xi_1, \xi_2, \xi_3)^T \in \Omega$ is the directional unit vector of x. (Note that we usually denote directional unit vectors in $\mathbb{R}^3$ by small Greek letters, e.g. for $x, y \in \mathbb{R}^3\backslash\{0\}$, $\xi := x/|x|$ and $\eta := y/|y|$.)
With respect to the canonical orthonormal basis $\{\varepsilon^1, \varepsilon^2, \varepsilon^3\}$ in $\mathbb{R}^3$, each $\xi \in \Omega$ may be represented in spherical polar coordinates (φ, t), $\varphi \in [0, 2\pi)$, $t \in [-1, 1]$ according to

$$\xi = \sqrt{1-t^2}\left(\cos(\varphi)\varepsilon^1 + \sin(\varphi)\varepsilon^2\right) + t\varepsilon^3. \tag{1.1}$$

The coordinate transformation $t := \cos(\vartheta)$, $\vartheta \in [0, \pi]$ yields the representation of ξ in the usual spherical polar coordinates (φ, ϑ).
The surface element on the sphere Ω_r is denoted by $d\omega_r$. Its representation in spherical polar coordinates (φ, t) is given by $d\omega_r(x) = r^2\, d\varphi\, dt$.
In the presentation of numerical results we generally make use of spherical polar coordinates $(\varphi_g, \vartheta_g) \in [-\pi, \pi) \times [-\pi/2, \pi/2]$, as they are common in geodesy. These coordinates are obtained from the usual spherical polar coordinates $(\varphi, \vartheta) \in [0, 2\pi) \times [0, \pi]$ via the transformation $\varphi_g := \varphi - \pi$ and $\vartheta_g := \pi/2 - \vartheta$.

All function spaces in this thesis are spaces of real-valued functions. Let $\mathcal{U} \subset \mathbb{R}^3$ be an open or closed set. $\mathcal{F}(\mathcal{U})$ denotes the set of all measurable real-valued functions on $\mathcal{U}$. The set of all k-times continuously differentiable functions on $\mathcal{U}$ is denoted by $\mathcal{C}^{(k)}(\mathcal{U})$, where $k \in \mathbb{N}_0$, furthermore, $\mathcal{C}^{(\infty)}(\mathcal{U}) := \bigcap_{k=0}^{\infty} \mathcal{C}^{(k)}(\mathcal{U})$ and $\mathcal{C}(\mathcal{U}) := \mathcal{C}^{(0)}(\mathcal{U})$. As usual, the space $\mathcal{C}(\mathcal{U})$ is endowed with the supremum norm $\|F\|_{\mathcal{C}(\mathcal{U})} := \sup_{x\in\mathcal{U}} |F(x)|$.
For $F \in \mathcal{F}(\mathcal{U})$ and $1 \leq p \leq \infty$, define

$$\|F\|_{\mathcal{L}^p(\mathcal{U})} := \left(\int_{\mathcal{U}} |F(x)|^p\, dx\right)^{1/p}.$$

The space $\mathcal{L}^p(\mathcal{U}) := \{F \in \mathcal{F}(\mathcal{U}) | \|F\|_{\mathcal{L}^p(\mathcal{U})} < \infty\}$ with $\mathcal{L}^p$-norm $\|\cdot\|_{\mathcal{L}^p(\mathcal{U})}$ is a Banach space. The space $\mathcal{L}^2(\mathcal{U})$ of square-integrable functions on $\mathcal{U}$ is even a Hilbert space with inner product

$$(F, G)_{\mathcal{L}^2(\mathcal{U})} := \int_{\mathcal{U}} F(x)G(x)\, dx,$$

where $F, G \in \mathcal{L}^2(\mathcal{U})$.

We now turn our attention to Legendre polynomials, spherical and outer harmonics and their interdependence. Spherical harmonics and Legendre polynomials are eigenfunctions of certain differential operators, which are closely related to each other.
Let $\mathcal{U} \subset \mathbb{R}^3$ be an open set. A function $F \in C^{(2)}(\mathcal{U})$ is called harmonic in $\mathcal{U}$ if it satisfies the Laplace equation $\Delta F = 0$ in $\mathcal{U}$, where Δ denotes the Laplace operator in $\mathbb{R}^3$, whose representation in terms of the spherical polar coordinates introduced in (1.1) is

$$\Delta_{\rho\xi} = \left(\frac{\partial}{\partial\rho}\right)^2 + \frac{2}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\Delta^*_\xi.$$

Δ^* denotes the Beltrami operator of the unit sphere, whose representation in terms of spherical polar coordinates is given by

$$\Delta^*_\xi = (1-t^2)\left(\frac{\partial}{\partial t}\right)^2 - 2t\frac{\partial}{\partial t} + \frac{1}{1-t^2}\left(\frac{\partial}{\partial \varphi}\right)^2.$$

A well-known result of potential theory (see for example [Mik1970], [Mir1970],) states that a harmonic function $F \in \mathcal{C}^{(2)}(\mathcal{U})$ is even in $\mathcal{C}^{(\infty)}(\mathcal{U})$. A harmonic function defined on an unbounded set is called regular at infinity if it satisfies the conditions $|F(x)| = \mathcal{O}(|x|^{-1})$ and $|\nabla F(x)| = \mathcal{O}(|x|^{-2})$ for $|x| \to \infty$ (uniformly with respect to all directions).

The set of all polynomials in $\mathbb{R}^3$ of degree $n \in \mathbb{N}_0$ is denoted by $\mathrm{Pol}_n(\mathbb{R}^3)$. A polynomial $H \in \mathrm{Pol}_n(\mathbb{R}^3)$ is called homogeneous, if $H(\lambda x) = \lambda^n H(x)$ for all $\lambda \in \mathbb{R}$ and all $x \in \mathbb{R}^3$. $\mathrm{Harm}_n(\mathbb{R}^3)$ denotes the space of all homogeneous harmonic polynomials in $\mathbb{R}^3$ of degree n.

The spherical harmonics $Y_n : \Omega \to \mathbb{R}$ of degree $n \in \mathbb{N}_0$ are defined as the everywhere on Ω infinitely differentiable eigenfunctions of the Beltrami operator corresponding to the eigenvalues $(\Delta^*)^\wedge(n) = -n(n+1)$, $n \in \mathbb{N}_0$. The functions $H_n : \mathbb{R}^3 \to \mathbb{R}$ defined by $H_n(x) := \rho^n Y_n(\xi)$, where $x = \rho\xi$ and $\xi \in \Omega$, are elements of $\mathrm{Harm}_n(\mathbb{R}^3)$. Conversely, for each homogeneous harmonic polynomial $H \in \mathrm{Harm}_n(\mathbb{R}^3)$ the restriction $H|_\Omega$ is a spherical harmonic of degree n and thus the spherical harmonics of degree n can equivalently be introduced as restrictions of homogeneous harmonic polynomials in $\mathbb{R}^3$ of degree n to the unit sphere. We denote the space of all spherical harmonics of degree n by $\mathrm{Harm}_n(\Omega)$. It is a linear space of dimension $2n+1$. Furthermore, spherical harmonics of different degrees are $\mathcal{L}^2(\Omega)$-orthogonal, i.e., for $Y_n \in \mathrm{Harm}_n(\Omega)$ and $Y_m \in \mathrm{Harm}_m(\Omega)$ with $m, n \in \mathbb{N}_0$, $n \neq m$, we have

$$(Y_n, Y_m)_{\mathcal{L}^2(\Omega)} = 0. \tag{1.2}$$

Let $p, q \in \mathbb{N}_0$ with $p \leq q$. The space of all spherical harmonics of degrees $n \in \{p, \ldots, q\}$ is denoted by $\mathrm{Harm}_{p,\ldots,q}(\Omega)$. Due to the orthogonality relation (1.2) the relations $\mathrm{Harm}_{p,\ldots,q}(\Omega) = \bigoplus_{n=p}^{q} \mathrm{Harm}_n(\Omega)$ and $\mathrm{Harm}(\Omega) = \bigoplus_{n=0}^{\infty} \mathrm{Harm}_n(\Omega)$ hold true.

Let $\{Y_{n,k}\}_{1\leq k\leq 2n+1}$ be an $\mathcal{L}^2(\Omega)$-orthonormal basis of $\mathrm{Harm}_n(\Omega)$, $n \in \mathbb{N}_0$. Then the set $\{Y_{n,k}\}_{n\in\mathbb{N}_0,\ 1\leq k\leq 2n+1} := \bigcup_{n=0}^{\infty}\{Y_{n,k}\}_{1\leq k\leq 2n+1}$ is a complete $\mathcal{L}^2(\Omega)$-orthonormal system for $\mathcal{L}^2(\Omega)$. $\{Y_{n,k}\}_{n\in\mathbb{N}_0,\ 1\leq k\leq 2n+1}$ induces a complete $\mathcal{L}^2(\Omega)$-orthonormal system for $\mathcal{L}^2(\Omega_r)$ via

$$Y^r_{n,k}(x) := \frac{1}{r} Y_{n,k}(x/|x|),\ x \in \Omega_r.$$

Every function $F \in \mathcal{L}^2(\Omega_r)$ can be expanded into its Fourier series with respect to the complete orthonormal system $\{Y^r_{n,k}\}_{n\in\mathbb{N}_0,\ 1\leq k\leq 2n+1}$:

$$F = \sum_{n=0}^{\infty}\sum_{k=1}^{2n+1} F^r_{n,k} Y^r_{n,k},$$

where the Fourier coefficients $\{F^r_{n,k}\}_{n\in\mathbb{N}_0,\ 1\le k\le 2n+1}$ are given by

$$F^r_{n,k} := \int\limits_{\Omega_r} F(x) Y^r_{n,k}(x)\, d\omega_r(x).$$

Furthermore, the span of $\{Y^r_{n,k}\}_{n\in\mathbb{N}_0,\ 1\le k\le 2n+1}$ is dense in $\mathcal{C}(\Omega_r)$ with respect to the supremum norm.

The Legendre polynomials $P_n : [-1,1] \to [-1,1]$ of degree $n \in \mathbb{N}_0$ are the only everywhere on $[-1,1]$ infinitely differentiable eigenfunctions of the Legendre operator

$$L_t = (1-t^2)\left(\frac{d}{dt}\right)^2 - 2t\frac{d}{dt}$$

corresponding to the eigenvalues $L^\wedge(n) = -n(n+1)$, which satisfy $P_n(1) = 1$. Apart from a multiplicative constant, the function $P_n(\varepsilon^3\cdot\ ) : \Omega \to [-1,1], \xi \mapsto P_n(\varepsilon\cdot\xi), \xi\in\Omega$, is the only spherical harmonic of degree n which is invariant under orthogonal transformations which leave ε^3 fixed. Legendre polynomials of different degrees are $\mathcal{L}^2([-1,1])$-orthogonal, to be more spcific,

$$\int\limits_{-1}^{1} P_n(t)P_m(t)\, dt = \frac{2}{2n+1}\delta_{n,m} \tag{1.3}$$

for $m,n\in\mathbb{N}_0$, where $\delta_{n,m}$ denotes the Kronecker symbol.
As a well-known consequence of the orthogonality relation (1.3) (see e.g. [DeHo1993]), the Legendre polynomials satisfy the three-term reconstruction formula

$$(n+1)P_{n+1}(t) = (2n+1)tP_n(t) - nP_{n-1}(t), \tag{1.4}$$

where $n\in\mathbb{N}$, $t\in[-1,1]$ and $P_0(t) = 1$, $P_1(t) = t$, which permits a stable numerical computation of P_n, $n\in\mathbb{N}_0$.

One outstanding result of the theory of spherical harmonics is the addition theorem, which relates the spherical harmonics of degree n to the Legendre polynomial of degree n:

Theorem 1.1 (Addition Theorem of Spherical Harmonics) *Let $n\in\mathbb{N}_0$ and let $\{Y_{n,k}\}_{1\le k\le 2n+1}$ be an $\mathcal{L}^2(\Omega)$-orthonormal basis of* $\mathrm{Harm}_n(\Omega)$. *Then*

$$\sum_{k=1}^{2n+1} Y_{n,k}(\xi)Y_{n,k}(\eta) = \frac{2n+1}{4\pi}P_n(\xi\cdot\eta)$$

for all $\xi,\eta\in\Omega$.

Let $\{Y_{n,k}\}_{n\in\mathbb{N}_0,\ 1\le k\le 2n+1} \subset \mathrm{Harm}(\Omega)$ be a complete $\mathcal{L}^2(\Omega)$-orthonormal system for $\mathcal{L}^2(\Omega)$. The outer harmonics for the sphere Ω_r (corresponding to $\{Y_{n,k}\}_{n\in\mathbb{N}_0,\ 1\le k\le 2n+1}$) are defined by

$$H_{n,k}(r;x) := \frac{1}{r}\left(\frac{r}{|x|}\right)^{n+1} Y_{n,k}(x/|x|),\ x\in\mathbb{R}^3\backslash\{0\},\ n\in\mathbb{N}_0,\ 1\le k\le 2n+1.$$

The function $H_{n,k}(r;\cdot)$ is called an outer harmonic of degree n and order k. For all $n\in\mathbb{N}_0,\ k\in\{1,\ldots,2n+1\}$, the function $H_{n,k}(r;\cdot)$ is an element of $\mathcal{C}^{(\infty)}(\mathbb{R}^3\backslash\{0\})$, which is harmonic in Ω_r^{ext} and regular at infinity. Furthermore, $H_{n,k}(r;\cdot)|_{\Omega_r} = Y^r_{n,k}$, and thus $H_{n,k}(r;\cdot)$ is nothing else than the uniquely determined solution of the exterior Dirichlet boundary value problem for the homogeneous Laplace equation for the sphere Ω_r with boundary value $Y^r_{n,k}$. As a direct consequence of Theorem 1.1, the outer harmonics satisfy the addition theorem

$$\sum_{k=1}^{2n+1} H_{n,k}(r;x)H_{n,k}(r;y) = \frac{2n+1}{4\pi r^2}\left(\frac{r^2}{|x||y|}\right)^{n+1} P_n\left(\frac{x}{|x|}\cdot\frac{y}{|y|}\right),\ x,y\in\overline{\Omega_r^{ext}}.$$

1.2 Principles of Constructive Approximation

In this section we formulate the theoretical foundations behind the approximation procedures in this thesis.

We start with the definition of the spaces $\mathrm{Pot}^{(k)}(\overline{\Sigma^{ext}})$ for $k\in\mathbb{N}_0$. These are solution spaces to the exterior Dirichlet problem for the homogeneous Laplace equation with boundary data given on a $C^{(2)}$-regular surface $\Sigma\subset\mathbb{R}^3$. We then introduce the Sobolev-like Hilbert spaces $\mathcal{H}(\{A_n\};\overline{\Omega_r^{ext}})$, where $\{A_n\}_{n\in\mathbb{N}_0}\subset\mathbb{R}_0^+$ denotes a sequence of non-negative real numbers and Ω_r is a sphere whose radius $r\in\mathbb{R}^+$ satisfies the condition $r<\inf_{x\in\Sigma}|x|$ and analyse their mathematical properties. The Runge-Walsh approximation theorem tells us that in case that $A_n\ge 1$ for almost all $n\in\mathbb{N}_0$ every potential $F\in\mathrm{Pot}^{(0)}(\overline{\Sigma^{ext}})$ can be approximated in uniform sense in $\overline{\Sigma^{ext}}$ by a sequence of functions in $\mathcal{H}(\{A_n\};\overline{\Omega_r^{ext}})$, which are harmonic in Ω_r^{ext}, and the extension of Helly's theorem finally ensures that $F\in\mathrm{Pot}^{(0)}(\overline{\Sigma^{ext}})$ can be approximated uniformly by 'interpolating' functions in $\mathcal{H}(\{A_n\};\overline{\Omega_r^{ext}})$ with respect to a set of bounded linear functionals on $\mathcal{H}(\{A_n\};\overline{\Omega_r^{ext}})$.

We start with the definition of a $\mathcal{C}^{(2)}$-regular surface in $\mathbb{R}^3$.

Definition 1.2 *A surface $\Sigma\subset\mathbb{R}^3$ is called a $\mathcal{C}^{(2)}$-regular surface, if it satisfies the following properties*
(i) Σ divides the three-dimensional Euclidean space $\mathbb{R}^3$ into the bounded region Σ^{int} (inner space) and the unbounded region Σ^{ext} (outer space).
(ii) Σ^{int} contains the origin.
(iii) Σ is closed and compact and free of double points.
(iv) Σ is locally of class $\mathcal{C}^{(2)}$, i.e., to every $x\in\Sigma$ there exists an open neighbourhood $U\in\mathbb{R}^3$ such that $\Sigma\cap U$ has a $\mathcal{C}^{(2)}$-parameterization.

Note that in our terminology a region is an open and connected set. A sphere $\Omega_r \subset \mathbb{R}^3$ with radius $r \in \mathbb{R}^+$ is always $\mathcal{C}^{(2)}$-regular. Furthermore, the conditions given in Definition 1.2 imply that on Σ there exists an outer unit normal field $\nu : \Sigma \to \mathbb{R}^3$, which is continuously differentiable.

We introduce function spaces which are related to the exterior Dirichlet problem for the homogeneous Laplace equation:

Problem 1.3 (Exterior Dirichlet Problem) *Let $\Sigma \subset \mathbb{R}^3$ be a $\mathcal{C}^{(2)}$-regular surface. Given $F \in \mathcal{C}^{(0)}(\Sigma)$. Find a function $V : \overline{\Sigma^{ext}} \to \mathbb{R}$ with the properties:*
(i) $V \in \mathcal{C}^{(2)}(\Sigma^{ext}) \cap \mathcal{C}^{(0)}(\overline{\Sigma^{ext}})$.
(ii) $\Delta V = 0$ on Σ^{ext}.
(iii) V is regular at infinity.
(iv) $\lim\limits_{\substack{\tau \to 0 \\ \tau > 0}} V(x + \tau\nu(x)) = F(x)$ for $x \in \Sigma$.

Definition 1.4 *Let $\Sigma \subset \mathbb{R}^3$ be a $\mathcal{C}^{(2)}$-regular surface. Define the space* $\mathrm{Pot}(\Sigma^{ext})$ *by*

$$\mathrm{Pot}(\Sigma^{ext}) := \left\{ U \in \mathcal{C}^{(2)}(\Sigma^{ext}) | \Delta U = 0 \text{ in } \Sigma^{ext} \text{ and } U \text{ regular at infinity} \right\},$$

and the spaces $\mathrm{Pot}^{(k)}(\overline{\Sigma^{ext}})$*, where $0 \leq k \leq \infty$, by*

$$\mathrm{Pot}^{(k)}(\overline{\Sigma^{ext}}) := \{ U \in \mathcal{C}^{(k)}(\overline{\Sigma^{ext}}) | U|_{\Sigma^{ext}} \in Pot(\Sigma^{ext}) \}.$$

$\mathrm{Pot}^{(k)}(\overline{\Sigma^{ext}})$ is the space of all solutions to the exterior Dirichlet boundary value problem for the homogeneous Laplace equation with $\mathcal{C}^{(k)}$-smooth boundary data on Σ.

In particular, the outer harmonics $\{H_{n,k}(r;\cdot)\}_{n\in\mathbb{N}_0,\ 1\leq k\leq 2n+1}$ introduced in Section 1.1 are elements of $\mathrm{Pot}^{(\infty)}(\overline{\Omega_r^{ext}})$, since for all $n \in \mathbb{N}_0$ and $1 \leq k \leq 2n+1$ the function $H_{n,k}(r;\cdot)$ is the solution of the exterior Dirichlet boundary value problem of potential theory for the sphere Ω_r with boundary value given by $Y_{n,k}^r \in \mathcal{C}^{(\infty)}(\Omega_r)$.

The idea behind the introduction of the spaces $\mathcal{H}(\{A_n\}; \overline{\Omega_r^{ext}})$ is to define a mathematical structure on $\mathrm{Pot}^{(\infty)}(\overline{\Omega_r^{ext}})$ which enables us to treat the approximation of a harmonic function in $\mathrm{Pot}^{(\infty)}(\overline{\Omega_r^{ext}})$ within the framework of the theory of Sobolev spaces. Moreover, we will see in Chapter 2 that a slight generalization of the theory leads us to the construction of special spaces for the approximation of first and second order radial derivatives of harmonic functions, which are needed for the formulation of the SST- and the SGG-problem as ill-posed pseudodifferential operator equations in Chapter 3.

Definition 1.5 *Let $\{A_{n,k}\}_{n\in\mathbb{N}_0,\ k=1,\ldots,2n+1} \subset \mathbb{R}_0^+$ be a sequence of non-negative real numbers and define $\mathcal{N} = \mathcal{N}(\{A_{n,k}\}) := \{(n,k) | A_{n,k} \neq 0\}$. The elements of the sequence $\{A_{n,k}^{-1}\}_{n\in\mathbb{N}_0,\ k=1,\ldots,2n+1}$ are defined to be $A_{n,k}^{-1}$ in case $A_{n,k} \neq 0$ and zero otherwise. Consequently, $\mathcal{N}(\{A_{n,k}^{-1}\}) = \mathcal{N}(\{A_{n,k}\})$. For two sequences $\{A_{n,k}\}_{n\in\mathbb{N}_0,\ k=1,\ldots,2n+1} \subset \mathbb{R}$ and $\{B_{n,k}\}_{n\in\mathbb{N}_0,\ k=1,\ldots,2n+1} \subset \mathbb{R}$ the elements of the sequence $\{A_{n,k}B_{n,k}^{-1}\}_{n\in\mathbb{N}_0,\ k=1,\ldots,2n+1}$ are* *analogously defined to be* $A_{n,k}B_{n,k}^{-1}$ *whenever* $A_{n,k} \neq 0$ *and* $B_{n,k} \neq 0$ *and zero otherwise. Hence,* $\mathcal{N}(\{A_{n,k}B_{n,k}^{-1}\}) = \mathcal{N}(\{A_{n,k}\}) \cap \mathcal{N}(\{B_{n,k}\})$. *In case* $\{A_{n,k}\}$ *depends only on* n, *i.e.* $A_{n,k} = A_n$ *for all* $k = 1, \ldots, 2n+1$ *and all* $n \in \mathbb{N}_0$, *we write* $\{A_n\}_{n\in\mathbb{N}_0}$ *and* $\mathcal{N} = \mathcal{N}(\{A_n\})$.

Since we are later on only interested in pseudodifferential operator equations involving rotation invariant operators, we may restrict ourselves to the discussion of sequences $\{A_n\}_{n\in\mathbb{N}_0} \subset \mathbb{R}_0^+$. We thus introduce the following spaces:

Definition 1.6 *Let* $\{A_n\}_{n\in\mathbb{N}_0} \subset \mathbb{R}_0^+$ *be a sequence of non-negative real numbers and denote* $\mathcal{N} = \mathcal{N}(A_n)$. *Let* $F \in Pot^{(\infty)}(\overline{\Omega_r^{ext}})$. *The Fourier coefficients of* F *exist and are given by*

$$F_{n,k}^r := (F, H_{n,k}(r;\cdot))_{\mathcal{L}^2(\Omega_r)} = \int\limits_{\Omega_r} F(x) Y_{n,k}^r(x)\, d\omega_r(x).$$

Define for $F \in \mathrm{Pot}(\overline{\Omega_r^{ext}})$

$$\|F\|_{\mathcal{H}(\{A_n\};\overline{\Omega_r^{ext}})} := \left(\sum_{n\in\mathcal{N}} \sum_{k=1}^{2n+1} A_n^2 (F_{n,k}^r)^2 \right)^{1/2}$$

and

$$\mathcal{H}(\{A_n\};\overline{\Omega_r^{ext}}) := \overline{\left\{ \begin{array}{l} F \in Pot^{(\infty)}(\overline{\Omega_r^{ext}}) \,|\, F_{n,k}^r = 0 \text{ for } n \in \mathbb{N}_0 \backslash \mathcal{N} \\ \text{and } \sum\limits_{n\in\mathcal{N}} \sum\limits_{k=1}^{2n+1} A_n^2 (F_{n,k}^r)^2 < \infty \end{array} \right\}}^{\|\cdot\|_{\mathcal{H}(\{A_n\};\overline{\Omega_r^{ext}})}}.$$

The space $\mathcal{H}(\{A_n\};\overline{\Omega_r^{ext}})$, *endowed with the inner product*

$$(F,G)_{\mathcal{H}(\{A_n\};\overline{\Omega_r^{ext}})} = F *_{\mathcal{H}(\{A_n\};\overline{\Omega_r^{ext}})} G := \sum_{n\in\mathcal{N}} \sum_{k=1}^{2n+1} A_n^2 F_{n,k}^r G_{n,k}^r$$

for $F, G \in \mathcal{H}(\{A_n\};\overline{\Omega_r^{ext}})$, *is a Sobolev-like Hilbert space (Sobolev space). The inner product of two functions* F *and* G *in* $\mathcal{H}(\{A_n\};\overline{\Omega_r^{ext}})$ *is also called the convolution of* F *and* G.

Each $F \in \mathcal{H}(\{A_n\};\overline{\Omega_r^{ext}})$ is uniquely determined by its coefficients $\{F_{n,k}^r\}_{n\in\mathcal{N},\ k=1,\ldots,2n+1}$ which satisfy

$$\sum_{n\in\mathcal{N}} \sum_{k=1}^{2n+1} A_n^2 (F_{n,k}^r)^2 < \infty,$$

and thus the sequence $\{A_n\}_{n\in\mathbb{N}_0}$ controls the admissible growth behaviour of the coefficients $\{F_{n,k}^r\}_{n\in\mathcal{N},\ 1\le k\le 2n+1}$. The set of functions

$$\left\{ A_n^{-1} H_{n,k}(r;\cdot) \,|\, n \in \mathcal{N},\ 1 \le k \le 2n+1 \right\}$$

is a complete orthonormal system in $\mathcal{H}(\{A_n\};\overline{\Omega_r^{ext}})$, and thus F can be represented by its orthogonal expansion in terms of the functions $A_n^{-1}H_{n,k}(r;\cdot)$

$$\begin{aligned} F &= \sum_{n\in\mathcal{N}}\sum_{k=1}^{2n+1} \left(F, A_n^{-1}H_{n,k}(r;\cdot)\right)_{\mathcal{H}(\{A_n\};\overline{\Omega_r^{ext}})} A_n^{-1}H_{n,k}(r;\cdot) \\ &= \sum_{n\in\mathcal{N}}\sum_{k=1}^{2n+1} A_n^2A_n^{-2}F_{n,k}^r\, H_{n,k}(r;\cdot). \end{aligned}$$

This shows, that, by construction of the space $\mathcal{H}(\{A_n\};\overline{\Omega_r^{ext}})$, a function $F\in\mathcal{H}(\{A_n\};\overline{\Omega_r^{ext}})$ can formally be represented by its series expansion in terms of outer harmonics,

$$F = \sum_{n\in\mathcal{N}}\sum_{k=1}^{2n+1} F_{n,k}^r H_{n,k}(r;\cdot), \tag{1.5}$$

which in general has to be understood in distributional sense. In our applications, however, it makes sense to consider only those spaces $\mathcal{H}(\{A_n\};\overline{\Omega_r^{ext}})$, whose elements are harmonic in Ω_r^{ext} and regular at infinity, i.e., $F\in \mathrm{Pot}(\Omega_r^{ext})$. Theorem 1.9 will show that this requirement is fulfilled, if $A_n\geq 1$ for almost all $n\in\mathcal{N}$. In this case, $F\in\mathcal{H}(\{A_n\};\overline{\Omega_r^{ext}})$ satisfies $F|_{\Omega_r}\in\mathcal{L}^2(\Omega_r)$, and the Fourier coefficients $F_{n,k}^r,\ n\in\mathcal{N},\ 1\leq k\leq 2n+1$ are given by the Lebesgue integrals

$$F_{n,k}^r = \int_{\Omega_r} F(x)Y_{n,k}^r(x)\, d\omega_r(x),\ n\in\mathcal{N},\ 1\leq k\leq 2n+1.$$

As a preparation to Theorem 1.9 we need the following convergence result for sequences of harmonic functions, which is known from the theory of elliptic partial differential equations as Harnack's theorem.

Theorem 1.7 *Let $\mathcal{U}$ be a finite domain in $\mathbb{R}^3$. Suppose that $\{F_n\}_{n\in\mathbb{N}_0}$ is a sequence of harmonic functions $F_n:\mathcal{U}\to\mathbb{R}$. Furthermore, let the functions F_n, $n\in\mathbb{N}_0$, be continuous in the closed domain $\overline{\mathcal{U}}:=\mathcal{U}\cup\partial\mathcal{U}$, where $\partial\mathcal{U}$ denotes the boundary of $\mathcal{U}$. If the sequence $\{F_n\}_{n\in\mathbb{N}_0}$ converges uniformly on $\partial\mathcal{U}$, then the following statement holds true: The sequence $\{F_n\}_{n\in\mathbb{N}_0}$ converges uniformly in the closed domain $\overline{\mathcal{U}}$, and the limit function is harmonic in $\mathcal{U}$. Furthermore, the sequence of derivatives of any order of the functions F_n, $n\in\mathbb{N}_0$, converge uniformly to the corresponding derivatives of the limit function in any closed subdomain $\mathcal{U}'$ of $\mathcal{U}$.*

Proof: A proof of this theorem can for example be found in [Mik1970]. □

Corollary 1.8 *Let $\{A_n\}_{n\in\mathbb{N}_0}\subset\mathbb{R}_0^+$ be a sequence of non-negative real numbers. Let $r\in\mathbb{R}^+$ and let $F\in\mathcal{H}(\{A_n\};\overline{\Omega_r^{ext}})$. Suppose that the series expansion of F in terms of outer harmonics*

$$F = \sum_{n\in\mathcal{N}}\sum_{k=1}^{2n+1} F_{n,k}^r H_{n,k}(r;\cdot)$$

converges uniformly on every compact subset of a domain $\mathcal{U} \subset \overline{\Omega_r^{ext}}$. *Then the function* F *is infinitely differentiable and harmonic in* $\mathcal{U}$ *and for* $\alpha \in \mathbb{N}_0^3$

$$\sum_{n\in\mathcal{N}} \sum_{k=1}^{2n+1} F_{n,k}^r \nabla^\alpha H_{n,k}(r;\cdot)$$

converges uniformly to $\nabla^\alpha F$ *on every compact subset of* $\mathcal{U}$.

Theorem 1.9 *Let* $\{A_n\}_{n\in\mathbb{N}_0} \subset \mathbb{R}_0^+$ *be a sequence of non-negative real numbers which satisfy* $A_n \geq 1$ *for almost all* $n \in \mathcal{N}$ *and let* $r \in \mathbb{R}^+$ *and* $F \in \mathcal{H}(\{A_n\};\overline{\Omega_r^{ext}})$. *Then the following statements hold true: The series expansion of* $F \in \mathcal{H}(\{A_n\};\overline{\Omega_r^{ext}})$, *given by*

$$F = \sum_{n\in\mathcal{N}} \sum_{k=1}^{2n+1} F_{n,k}^r \, H_{n,k}(r;\cdot), \tag{1.6}$$

is convergent in $\mathcal{L}^2(\Omega_r)$*-sense on* Ω_r *and uniformly convergent on every subset* $\overline{\Omega_{r+\delta}^{ext}}$, *with* $\delta > 0$, *of* $\overline{\Omega_r^{ext}}$. *The space* $\mathcal{H}(\{A_n\};\overline{\Omega_r^{ext}})$ *is a subspace of* $\mathrm{Pot}(\Omega_r^{ext})$. *Moreover, the series expansion (1.6) of* $F \in \mathcal{H}(\{A_n\};\overline{\Omega_r^{ext}})$ *may be differentiated in* $x \in \Omega_r^{ext}$ *term by term (for partial derivatives of arbitrary order), and the series*

$$\sum_{n\in\mathcal{N}} \sum_{k=1}^{2n+1} F_{n,k}^r \nabla^\alpha H_{n,k}(r;\cdot),$$

where $\alpha \in \mathbb{N}_0^3$ *denotes a three-dimensional multiindex, converges locally uniformly on* Ω_r^{ext} *every compact subset of* Ω_r^{ext} *to* $\nabla^\alpha F$, *i.e., it converges uniformly on every subset* $\overline{\Omega_{r+\delta}^{ext}}$, *with* $\delta > 0$, *of* $\overline{\Omega_r^{ext}}$.

Proof: The property $A_n \geq 1$ for almost all $n \in \mathcal{N}$ implies that $\mathcal{H}(\{A_n\};\overline{\Omega_r^{ext}})$ is a subspace of $\mathcal{H}(\{1\};\overline{\Omega_r^{ext}}) = \overline{\mathrm{Pot}^{(\infty)}(\overline{\Omega_r^{ext}})}^{\|\cdot\|_{\mathcal{L}^2(\Omega_r)}}$. Hence, $F \in \mathcal{H}(\{A_n\};\overline{\Omega_r^{ext}})$ satisfies

$$\|F\|^2_{\mathcal{H}(\{1\};\overline{\Omega_r^{ext}})} = \sum_{n\in\mathcal{N}} \sum_{k=1}^{2n+1} (F_{n,k}^r)^2 = \|F\|_{\mathcal{L}^2(\Omega_r)} < \infty,$$

and the series expansion (1.6) of F converges on Ω_r in $\mathcal{L}^2(\Omega_r)$-sense. Let $x \in \overline{\Omega_{r+\delta}^{ext}}$, where $\delta > 0$. Because of the Cauchy Schwarz inequality and the addition theorem

$$\begin{aligned}
\left| \sum_{\substack{n\in\mathcal{N},\\ n\geq N}} \sum_{k=1}^{2n+1} F_{n,k}^r \, H_{n,k}(r;x) \right| &\leq \left(\sum_{\substack{n\in\mathcal{N},\\ n\geq N}} \sum_{k=1}^{2n+1} A_n^2 (F_{n,k}^r)^2 \right)^{1/2} \left(\sum_{\substack{n\in\mathcal{N},\\ n\geq N}} \sum_{k=1}^{2n+1} A_n^{-2} \left(H_{n,k}(r;x)\right)^2 \right)^{1/2} \\
&\leq \|F\|_{\mathcal{H}(\{A_n\};\overline{\Omega_r^{ext}})} \left(\sum_{\substack{n\in\mathcal{N},\\ n\geq N}} \sum_{k=1}^{2n+1} \frac{2n+1}{4\pi r^2 A_n^2} \left(\frac{r}{r+\delta}\right)^{2(n+1)} \right)^{1/2} .
\end{aligned}$$

The sum in the second term is independent of $x \in \overline{\Omega_{r+\delta}^{ext}}$, is convergent by the quotient criterion and becomes arbitrary small, if N is chosen large enough. Hence the series expansion (1.6) of F converges locally uniformly on Ω_r^{ext}. The uniform convergence of (1.6) and Corollary 1.8 imply that $F \in \mathcal{C}^{(\infty)}(\Omega_r^{ext})$ and that F is harmonic on Ω_r^{ext}. Due to Corollary 1.8 the series expansion (1.6) of F may be differentiated term by term for derivatives of arbitrary order. It remains to show that F is regular at infinity. Let $x \in \Omega_r^{ext}$ with $|x| > 2r$. Using the series expansion (1.6), the definition of the outer harmonics, and the addition theorem, we can estimate

$$\begin{aligned} |F(x)| &= \frac{1}{|x|}\left|\sum_{n\in\mathcal{N}}\sum_{k=1}^{2n+1} F_{n,k}^r \left(\frac{r}{|x|}\right)^n Y_{n,k}(x/|x|)\right| \\ &\leq \frac{1}{|x|}\|F\|_{\mathcal{H}(\{A_n\};\overline{\Omega_r^{ext}})}\left(\sum_{n\in\mathcal{N}}\frac{2n+1}{4\pi A_n^2}\left(\frac{1}{2}\right)^{2(n+1)}\right)^{1/2}. \end{aligned}$$

The sum in the last term is convergent, which implies $|F(x)| = \mathcal{O}(\||x|^{-1})$ for $|x| \to \infty$. The second regularity condition $|\nabla F(x)| = \mathcal{O}(\||x|^{-2})$ for $|x| \to \infty$ follows in a similar way with the help of the vectorial addition theorem for spherical harmonics. □

Under the stronger assumption that $\{A_n\}_{n\in\mathbb{N}_0} \subset \mathbb{R}_0^+$ is a so-called summable sequence, the series expansion (1.5) of $F \in \mathcal{H}(\{A_n\};\overline{\Omega_r^{ext}})$ is even uniformly convergent in $\overline{\Omega_r^{ext}}$, and thus $F \in \mathrm{Pot}^{(0)}(\overline{\Omega_r^{ext}})$:

Definition 1.10 *A sequence $\{A_n\}_{n\in\mathbb{N}_0} \subset \mathbb{R}_0^+$ of non-negative real numbers is called summable if*

$$\sum_{n\in\mathcal{N}}\frac{2n+1}{A_n^2} < \infty. \tag{1.7}$$

Lemma 1.11 *Let $\{A_n\} \subset \mathbb{R}_0^+$ be a sequence of non-negative real numbers which is summable, and let $r \in \mathbb{R}^+$. Then the space $\mathcal{H}(\{A_n\};\overline{\Omega_r^{ext}})$ is a subspace of $\mathrm{Pot}^{(0)}(\overline{\Omega_r^{ext}})$ and the truncated series expansion $\{F_N\}_{N\in\mathcal{N}}$ of $F \in \mathcal{H}(\{A_n\};\overline{\Omega_r^{ext}})$ in terms of outer harmonics,*

$$F_N := \sum_{\substack{n\in\mathcal{N}\\ n\leq N}}\sum_{k=1}^{2n+1} F_{n,k}^r H_{n,k}(r;\cdot) \tag{1.8}$$

is uniformly convergent in $\overline{\Omega_r^{ext}}$. Moreover, the error estimate

$$|F_N(x) - F(x)| \leq \|F\|_{\mathcal{H}(\{A_n\};\overline{\Omega_r^{ext}})}\left(\sum_{\substack{n\in\mathcal{N}\\ n>N}}\frac{2n+1}{4\pi r^2 A_n^2}\left(\frac{r}{|x|}\right)^{2(n+1)}\right)^{1/2}, \tag{1.9}$$

holds true.

Proof: The statement follows by elementary calculations. □

In gravitational potential approximation we are interested in the approximation of a harmonic function in $\mathrm{Pot}^{(0)}(\overline{\Sigma^{ext}})$ by functions in $\mathcal{H}(\{A_n\};\overline{\Omega_r^{ext}})$, where $\Omega_r \subset \Sigma^{int}$. If $\{A_n\}_{n\in\mathbb{N}_0} \subset \mathbb{R}^+$ is a sequence of positive real numbers which satisfy $A_n \geq 1$ for almost all $n \in \mathbb{N}_0$, the connection between the spaces $\mathrm{Pot}^{(0)}(\overline{\Sigma^{ext}})$ and $\mathcal{H}(\{A_n\};\overline{\Omega_r^{ext}})$ is given by the Runge-Walsh approximation theorem, which has its roots in a paper published by Runge in 1885 [Ru1885]. The formulation here is according to [Fr1980].

Theorem 1.12 (Runge-Walsh Approximation) *Let $\Sigma \subset \mathbb{R}^3$ be a $\mathcal{C}^{(2)}$-regular surface and let Ω_r with $r \in \mathbb{R}^+$ be a Bjerhammar sphere for Σ, i.e., a sphere which is contained in Σ^{int}. Let $\{H_{n,k}(r;\cdot)\}_{n\in\mathbb{N}_0,\ 1\leq k\leq 2n+1}$ be a complete orthonormal system of outer harmonics for $\mathcal{H}(\{1\};\overline{\Omega_r^{ext}}) = \overline{\mathrm{Pot}^{(\infty)}(\overline{\Omega_r^{ext}})}^{\|\cdot\|_{\mathcal{L}^2(\Omega_r)}}$. Then the span of $\{H_{n,k}(r;\cdot)|_{\overline{\Sigma^{ext}}}\}_{n\in\mathbb{N}_0,\ 1\leq k\leq 2n+1}$ is dense in $\mathrm{Pot}^{(0)}(\overline{\Sigma^{ext}})$ with respect to the supremum norm $\|\cdot\|_{\mathcal{C}(\overline{\Sigma^{ext}})}$, i.e.*

$$\overline{\mathrm{span}_{n\in\mathbb{N}_0,1\leq k\leq 2n+1}\{H_{n,k}(r;\cdot)|_{\overline{\Sigma^{ext}}}\}}^{\|\cdot\|_{\mathcal{C}(\overline{\Sigma^{ext}})}}.$$

Proof: A proof can be found in [Fr1980]. □

Corollary 1.13 *Let $\Sigma \subset \mathbb{R}^3$ be a $\mathcal{C}^{(2)}$-regular surface and let Ω_r with $r \in \mathbb{R}^+$ be a Bjerhammar sphere for Σ. Let $\{A_n\}_{n\in\mathbb{N}_0} \subset \mathbb{R}^+$ be a sequence of positive real numbers with $A_n \geq 1$ for almost all $n \in \mathbb{N}_0$. Then*

$$\mathcal{H}(\{A_n\};\overline{\Omega_r^{ext}})|_{\overline{\Sigma^{ext}}} := \left\{F : \overline{\Sigma^{ext}} \to \mathbb{R} \middle|\, F = G|_{\overline{\Sigma^{ext}}} \text{ for some } G \in \mathcal{H}(\{A_n\};\overline{\Omega_r^{ext}})\right\}$$

satisfies

$$\overline{\mathcal{H}(\{A_n\};\overline{\Omega_r^{ext}})|_{\overline{\Sigma^{ext}}}}^{\|\cdot\|_{\mathcal{C}(\overline{\Sigma^{ext}})}} = \mathrm{Pot}^{(0)}(\overline{\Sigma^{ext}}).$$

Proof: From Theorem 1.9 (iii) it is clear that $\mathcal{H}(\{A_n\};\overline{\Omega_r^{ext}}) \in \mathrm{Pot}^{(0)}(\overline{\Sigma^{ext}})$. Moreover, since $A_n > 0$ for all $n \in \mathbb{N}_0$, the outer harmonics $H_{n,k}(r;\cdot)$ are contained in $\mathrm{Pot}^{(0)}(\overline{\Sigma^{ext}})$ for all $n \in \mathbb{N}_0$ and $1 \leq k \leq 2n+1$, and hence the statement holds true. □

Corollary 1.13 guarantees that a function $F \in \mathrm{Pot}^{(0)}(\overline{\Sigma^{ext}})$ can be approximated with arbitrary accuracy by functions in $\mathcal{H}(\{A_n\};\overline{\Omega_r^{ext}})$, i.e., given $\varepsilon > 0$ there exists $G \in \mathcal{H}(\{A_n\};\overline{\Omega_r^{ext}})$ such that $\|F-G\|_{\mathcal{C}(\overline{\Sigma^{ext}})} \leq \varepsilon$. The extension of Helly's theorem (Theorem 1.14) tells us that it is even possible to choose this function $G \in \mathcal{H}(\{A_n\};\overline{\Omega_r^{ext}})$ in such a way that it also 'interpolates' with respect to a finite set of bounded linear functionals on $\mathcal{H}(\{A_n\};\overline{\Omega_r^{ext}})$.

Theorem 1.14 (Extension of Helly's Theorem) *Let $\mathcal{K}$ be a dense and convex subset in a normed linear space $\mathcal{X}$ with norm $\|\cdot\|_{\mathcal{X}}$, and let $\mathcal{L}_1,\ldots,\mathcal{L}_N$ be N bounded linear functionals on $\mathcal{X}$. Then for an element $F \in \mathcal{X}$ and given $\varepsilon > 0$, there exists an element $G \in \mathcal{K}$ with the following properties:*
(i) $\|F-G\|_{\mathcal{X}} < \varepsilon$, and
(ii) $\mathcal{L}_iF = \mathcal{L}_iG$ for all $i = 1,\ldots,N$.

Proof: A proof can be found in [Ya1950]. □

For our applications we need the following specialization of the extension of Helly's theorem:

Theorem 1.15 *Let $\Sigma \subset \mathbb{R}^3$ be a $\mathcal{C}^{(2)}$-regular surface and let Ω_r be a Bjerhammar sphere for Σ, i.e., $\Omega_r \subset \Sigma^{int}$. Assume that $X_N = \{x_1^N, \dots, x_N^N\}$ is a set of points in Σ^{ext} such that $\sup_{x\in\Sigma} |x| < \min_{i=1,\dots,N} |x_i^N|$ and let $\{A_n\}_{n\in\mathbb{N}_0} \subset \mathbb{R}^+$ be a sequence of positive real numbers which satisfy $A_n \geq 1$ for almost all $n \in \mathbb{N}_0$. Then $U \in \mathrm{Pot}^{(0)}(\overline{\Sigma^{ext}})$ can be approximated uniformly on $\overline{\Sigma^{ext}}$ by 'interpolating' functions in $\mathcal{H}(\{A_n\}; \overline{\Omega_r^{ext}})$ with respect to the first and second order radial derivatives in the points x_i^N, $i = 1, \dots, N$, i.e., given $\varepsilon > 0$, there exists a function $F \in \mathcal{H}(\{A_n\}; \overline{\Omega_r^{ext}})$ such that*

$$\sup_{x\in\overline{\Sigma^{ext}}} |U(x) - F(x)| \leq \varepsilon \ \textit{and}\ \frac{\partial^h F(x_i^N)}{\partial\rho^h} = \frac{\partial^h U(x_i^N)}{\partial\rho^h} \ \textit{for}\ i = 1, \dots, N,$$

where $h = 1$ in case of the first order radial derivative and $h = 2$ in case of the second order radial derivative.

Proof: A detailed proof of the statement can be found in [He2002]. Due to Corollary 1.13 the space $\mathcal{H}(\{A_n\}; \overline{\Omega_r^{ext}})|_{\overline{\Sigma^{ext}}}$ is a dense and convex subspace of $\mathrm{Pot}^{(0)}(\overline{\Sigma^{ext}})$ with respect to $\|\cdot\|_{\mathcal{C}(\overline{\Sigma^{ext}})}$, and it is shown that the linear functionals $\mathcal{L}_i^N, \dots, \mathcal{L}_N^N$ given by $\mathcal{L}_i^N : \mathrm{Pot}^{(0)}(\overline{\Sigma^{ext}}) \to \mathbb{R}, G \mapsto \mathcal{L}_i^N G := \frac{\partial^h G(x_i^N)}{\partial r^h}$, $i = 1, \dots, N$, are bounded. □

An 'interpolating' and approximating function $F \in \mathcal{H}(\{A_n\}; \overline{\Omega_r^{ext}})$ for a potential $U \in \mathrm{Pot}^{(0)}(\overline{\Sigma^{ext}})$ in the sense of Corollary 1.13 and Theorem 1.15 will in the sequel be called an 'interpolating' Runge-Walsh approximation of U. It should be noted that Theorem 1.15 only guarantees the existence of an 'interpolating' Runge-Walsh approximation for a potential $U \in \mathrm{Pot}^{(0)}(\overline{\Sigma^{ext}})$ and gives no hint on how to construct it. Yet it is the theoretical motivation for the approximation procedures which are used in this thesis.

Chapter 2

Sobolev Spaces and Pseudodifferential Operators

In Chapter 1 we introduced the spaces $\mathcal{H}(\{A_n\}; \overline{\Omega_r^{ext}})$ as reference spaces in which we treat the approximation of a harmonic function in $\mathrm{Pot}^{(\infty)}(\overline{\Omega_r^{ext}})$. The mathematical model of SST and SGG assumes that the observational data are given in form of the first and second order radial derivatives of the gravitational potential at satellite altitude, respectively. Hence the theory developed in Section 1.2 has to be slightly generalized to cover also first and second order radial derivatives of functions of class $\mathcal{H}(\{A_n\}; \overline{\Omega_r^{ext}})$, which leads to the introduction of the Sobolev-like Hilbert spaces $\mathcal{H}(\{A_n\}; h; \overline{\Omega_r^{ext}})$ in Section 2.1, where $\{A_n\}_{n\in\mathbb{N}_0} \subset \mathbb{R}_0^+$ is a series of non-negative real numbers and $h \in \{0, 1, 2\}$.

Section 2.2 introduces pseudodifferential operators between the spaces $\mathcal{H}(\{A_n\}; h_1; \overline{\Omega_{r_1}^{ext}})$ and $\mathcal{H}(\{A_n\}; h_2; \overline{\Omega_{r_2}^{ext}})$, where $r_1, r_2 \in \mathbb{R}^+$ and $h_1, h_2 \in \{0, 1, 2\}$. These operators are defined by the way they act on the Fourier coefficients of $F \in \mathcal{H}(\{A_n\}; h_1; \overline{\Omega_{r_1}^{ext}})$ in its formal series expansion $\sum_{n\in\mathcal{N}} \sum_{k=1}^{2n+1} F_{n,k}^{r_1} H_{n,k}(r_1; h_1; \cdot)$ and provide an adequate means to formulate the SST- and the SGG-problem, respectively, within the framework of the Sobolev space theory presented here.

2.1 Sobolev Spaces

Let $\{A_n\}_{n\in\mathbb{N}_0} \subset \mathbb{R}_0^+$ be a sequence of non-negative real numbers which satisfy $A_n \geq 1$ for almost all $n \in \mathcal{N}$. Let $r \in \mathbb{R}^+$. From Theorem 1.9 we know that any function $F \in \mathcal{H}(\{A_n\}; \overline{\Omega_r^{ext}})$ is infinitely differentiable and harmonic in Ω_r^{ext}, and the partial derivative of F of order $\alpha \in \mathbb{N}_0^3$ is given by

$$\nabla^\alpha F = \sum_{n\in\mathcal{N}} \sum_{k=1}^{2n+1} F_{n,k}^r \nabla^\alpha H_{n,k}(r; \cdot),$$

where the series on the right-hand side converges uniformly on every subset $\overline{\Omega_{r+\delta}^{ext}}$, where $\delta > 0$, of $\overline{\Omega_r^{ext}}$ to $\nabla^\alpha F$.

Thus the negative first order radial derivative of $F \in \mathcal{H}(\{A_n\}; \overline{\Omega_r^{ext}})$ in $x = \rho\xi$, where $\rho \geq r$ and $\xi \in \Omega$ is given by

$$\begin{aligned}
\left(-\frac{x}{|x|} \cdot \nabla_x\right) F(x) &= -\frac{\partial}{\partial \rho} F(\rho\xi) \\
&= -\sum_{n \in \mathcal{N}} \sum_{k=1}^{2n+1} F_{n,k}^r \frac{\partial}{\partial \rho} H_{n,k}(r; \rho\xi) \\
&= -\sum_{n \in \mathcal{N}} \sum_{k=1}^{2n+1} F_{n,k}^r \frac{\partial}{\partial \rho} \left(\frac{1}{r} \left(\frac{r}{\rho}\right)^{n+1} Y_{n,k}(\xi)\right) \\
&= \sum_{n \in \mathcal{N}} \sum_{k=1}^{2n+1} F_{n,k}^r \frac{(n+1)}{r} \frac{1}{r} \left(\frac{r}{\rho}\right)^{n+2} Y_{n,k}(\xi) \\
&= \sum_{n \in \mathcal{N}} \sum_{k=1}^{2n+1} \frac{n+1}{r} F_{n,k}^r H_{n,k}(r; 1; x),
\end{aligned}$$

where we have introduced the functions

$$H_{n,k}(r; 1; x) := \frac{1}{r} \left(\frac{r}{|x|}\right)^{n+2} Y_{n,k}(x/|x|),\ x \in \overline{\Omega_r^{ext}},\ n \in \mathcal{N},\ 1 \leq k \leq 2n+1.$$

Analogously, we obtain for the second order radial derivative in $x = \rho\xi$

$$\begin{aligned}
\left(-\frac{x}{|x|} \cdot \nabla_x\right) \left(-\frac{x}{|x|} \cdot \nabla_x\right) F(x) &= \frac{\partial^2}{\partial \rho^2} F(\rho\xi) \\
&= \sum_{n \in \mathcal{N}} \sum_{k=1}^{2n+1} F_{n,k}^r \frac{\partial^2}{\partial \rho^2} H_{n,k}(r; \rho\xi) \\
&= \sum_{n \in \mathcal{N}} \sum_{k=1}^{2n+1} F_{n,k}^r \frac{\partial^2}{\partial \rho^2} \left(\frac{1}{r} \left(\frac{r}{\rho}\right)^{n+1} Y_{n,k}(\xi)\right) \\
&= \sum_{n \in \mathcal{N}} \sum_{k=1}^{2n+1} F_{n,k}^r \frac{(n+1)(n+2)}{r^2} \frac{1}{r} \left(\frac{r}{\rho}\right)^{n+3} Y_{n,k}(\xi) \\
&= \sum_{n \in \mathcal{N}} \sum_{k=1}^{2n+1} \frac{(n+1)(n+2)}{r^2} F_{n,k}^r H_{n,k}(r; 2; x),
\end{aligned}$$

where

$$H_{n,k}(r; 2; x) := \frac{1}{r} \left(\frac{r}{|x|}\right)^{n+3} Y_{n,k}(x/|x|),\ x \in \overline{\Omega_r^{ext}},\ n \in \mathcal{N},\ 1 \leq k \leq 2n+1.$$

The introduction of the new function systems $\{H_{n,k}(r;1;\cdot)\}_{n\in\mathcal{N},\ 1\le k\le 2n+1}$ and $\{H_{n,k}(r;2;\cdot)\}_{n\in\mathcal{N},\ 1\le k\le 2n+1}$ is necessary because the SST-problem and SGG-problem involve first and second order radial derivatives of functions in $\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})$, which are not harmonic. Thus, we are led to the following generalization of the theory developed in Section 1.2:

Definition 2.1 *Let $h \in \{0,1,2\}$ and define the functions $H_{n,k}(r;h;\cdot) : \overline{\Omega_r^{ext}} \to \mathbb{R}$ for $n \in \mathbb{N}_0$ and $1 \le k \le 2n+1$ by*

$$H_{n,k}(r;h;x) := \frac{1}{r}\left(\frac{r}{|x|}\right)^{n+1+h} Y_{n,k}(x/|x|);\ x \in \overline{\Omega_r^{ext}}.$$

This definition includes the outer harmonics, since $H_{n,k}(r;0;\cdot) = H_{n,k}(r;\cdot)$ for $n \in \mathbb{N}_0$, $1 \le k \le 2n+1$. Moreover, $H_{n,k}(r;h;\cdot)|_{\Omega_r} = Y_{n,k}^r$ for $h \in \{0,1,2\}$ and $H_{n,k}(r;h;x) = (r/|x|)^h H_{n,k}(r;x)$, $x \in \overline{\Omega_r^{ext}}$. The functions $H_{n,k}(r;h;\cdot)$ are in $\mathcal{C}^{(\infty)}(\mathbb{R}^3\backslash\{0\})$, and they satisfy the addition theorem

$$\sum_{k=1}^{2n+1} H_{n,k}(r;h;x)H_{n,k}(r;h;y) = \frac{2n+1}{4\pi r^2}\left(\frac{r^2}{|x||y|}\right)^{n+h+1} P_n\left(\frac{x}{|x|}\cdot\frac{y}{|y|}\right),\ x,y \in \overline{\Omega_r^{ext}}.$$

Definition 2.2 *Let $\{A_n\}_{n\in\mathbb{N}_0} \subset \mathbb{R}_0^+$ be a sequence of non-negative real numbers and let $h \in \{0,1,2\}$. Let $F \in \operatorname{span}\{H_{n,k}(r;h;\cdot)\,|\, n \in \mathcal{N},\ 1 \le k \le 2n+1\}$. The Fourier coefficients of F with respect to the $\mathcal{L}^2(\Omega_r)$-norm exist and are given by*

$$F_{n,k}^r := (F, H_{n,k}(r;h;\cdot))_{\mathcal{L}^2(\Omega_r)} = \int_{\Omega_r} F(x)Y_{n,k}^r(x)\,d\omega_r(x).$$

On $\operatorname{span}\{H_{n,k}(r;h;\cdot)\,|\, n \in \mathcal{N},\ 1 \le k \le 2n+1\}$ define the inner product

$$(F,G)_{\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})} = F *_{\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})} G := \sum_{n\in\mathcal{N}}\sum_{k=1}^{2n+1} A_n^2 F_{n,k}^r G_{n,k}^r$$

and corresponding norm

$$\|F\|_{\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})} := \left(\sum_{n\in\mathcal{N}}\sum_{k=1}^{2n+1} A_n^2 (F_{n,k}^r)^2\right)^{1/2}.$$

The space

$$\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}}) := \overline{\operatorname{span}\{H_{n,k}(r;h;\cdot)\,|\, n \in \mathcal{N},\ 1 \le k \le 2n+1\}}^{\|\cdot\|_{\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})}},$$

endowed with the inner product $(\cdot,\cdot)_{\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})}$, is a Sobolev-like Hilbert space. The inner product of two functions F and G in $\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})$ is also called the convolution of F and G.

The set of functions

$$\left\{A_n^{-1} H_{n,k}(r;h;\cdot) \,|\, n \in \mathcal{N},\ 1 \le k \le 2n+1\right\}$$

is a complete orthonormal system in $\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})$, and thus each $F \in \mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})$ can formally be represented by the series expansion

$$F = \sum_{n\in\mathcal{N}} \sum_{k=1}^{2n+1} F_{n,k}^r H_{n,k}(r;h;\cdot), \tag{2.1}$$

which in general has to be understood in distributional sense.

Definition 2.3 *Let $\{A_n\}_{n\in\mathbb{N}_0} \subset \mathbb{R}_0^+$ be a sequence of non-negative real numbers, and let $h \in \{0,1,2\}$. A function $F \in \mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})$ is called bandlimited with bandlimit $m \in \mathbb{N}_0$ if the coefficients $F_{n,k}^r$ in its formal series representation (2.1) satisfy the following two conditions:*
(i) there exists a $k_0 \in \{1,\ldots,2m+1\}$ with $F_{m,k_0}^r \neq 0$ and
(ii) $F_{n,k}^r = 0$ for all $n > m$, $n \in \mathcal{N}$, $1 \le k \le 2n+1$.

The relation between the spaces $\mathcal{H}(\{A_n\};\overline{\Omega_r^{ext}})$ and $\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})$ can be characterized as follows: Obviously, $\mathcal{H}(\{A_n\};0;\overline{\Omega_r^{ext}}) = \mathcal{H}(\{A_n\};\overline{\Omega_r^{ext}})$, and in case of $h \in \{1,2\}$

$$\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}}) = \left\{ F : \overline{\Omega_r^{ext}} \to \mathbb{R} \,\middle|\, \begin{array}{l} F(x) := (r/|x|)^h G(x) \\ \text{for a } G \in \mathcal{H}(\{A_n\};\overline{\Omega_r^{ext}}) \end{array} \right\}. \tag{2.2}$$

As we are only interested in function spaces with a certain ‘smoothness’, it suffices to restrict the discussion of the spaces $\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})$ to the case that $\{A_n\}_{n\in\mathbb{N}_0} \subset \mathbb{R}_0^+$ is a sequence of non-negative real numbers which satisfy $A_n \ge 1$ for almost all $n \in \mathcal{N}$. Relation (2.2) and the fact that the inner product and norm in $\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})$, respectively, is the same for $h \in \{0,1,2\}$, are the reasons, why the convergence results and error estimates in Theorem 1.9 can be easily transferred to $\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})$.

Theorem 2.4 *Let $\{A_n\}_{n\in\mathbb{N}_0} \subset \mathbb{R}_0^+$ be a sequence of non-negative real numbers which satisfy $A_n \ge 1$ for almost all $n \in \mathcal{N}$ and let $r \in \mathbb{R}^+$. Then the spaces $\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})$, $h \in \{0,1,2\}$, have the following properties: The series expansion of $F \in \mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})$, given by*

$$F = \sum_{n\in\mathcal{N}} \sum_{k=1}^{2n+1} F_{n,k}^r\, H_{n,k}(r;h;\cdot), \tag{2.3}$$

is convergent in $\mathcal{L}^2(\Omega_r)$-sense on Ω_r and converges uniformly on every subset $\overline{\Omega_{r+\delta}^{ext}}$, with $\delta > 0$, of $\overline{\Omega_r^{ext}}$. The function $F \in \mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})$ is in $\mathcal{C}^{(\infty)}(\Omega_r^{ext})$, and its series expansion (2.3) may be differentiated on Ω_r^{ext} term by term (for partial derivatives of arbitrary order). The series

$$\sum_{n\in\mathcal{N}} \sum_{k=1}^{2n+1} F_{n,k}^r \nabla^\alpha H_{n,k}(r;h;\cdot),$$

where $\alpha \in \mathbb{N}_0^3$ denotes a three-dimensional multiindex, converges locally uniformly on Ω_r^{ext} to $\nabla^\alpha F$, i.e., it converges uniformly on every subset $\overline{\Omega_{r+\delta}^{ext}}$, with $\delta > 0$, of $\overline{\Omega_r^{ext}}$.

Proof: The proof is technical and its details can be found in [He2002]. □

2.2 Pseudodifferential Operators

In this section we introduce pseudodifferential operators between two spaces $\mathcal{H}(\{A_n\}; h_1; \overline{\Omega_{r_1}^{ext}})$ and $\mathcal{H}(\{B_n\}; h_2; \overline{\Omega_{r_2}^{ext}})$, where $\{A_n\}_{n\in\mathbb{N}_0}, \{B_n\}_{n\in\mathbb{N}_0} \subset \mathbb{R}_0^+$ are sequences of non-negative real numbers, $r_1, r_2 \in \mathbb{R}^+$ and $h_1, h_2 \in \{0, 1, 2\}$. Pseudodifferential operators are characterized by the way they act on the coefficients of a function of class $\mathcal{H}(\{A_n\}; h_1; \overline{\Omega_{r_1}^{ext}})$ in its series expansion in terms of the functions $H_{n,k}(r_1; h_1; \cdot)$, $n \in \mathcal{N}(\{A_n\})$, $1 \leq k \leq 2n+1$. As special examples we will then introduce the SST- and SGG- operator as pseudodifferential operators between the spaces $\mathcal{H}(\{A_n\}; \overline{\Omega_R^{ext}})$ and $\mathcal{H}(\{A_n\}; h; \overline{\Omega_r^{ext}})$, where $\{A_n\}_{n\in\mathbb{N}_0}$ is a sequence of positive real numbers with $A_n \geq 1$ for almost all $n \in \mathbb{N}_0$, $R, r \in \mathbb{R}^+$ with $R < r$, and $h = 1$ in case of SST and $h = 2$ in case of SGG.

Definition 2.5 *Let $r_1, r_2 \in \mathbb{R}^+$, $h_1, h_2 \in \{0, 1, 2\}$ and let $\{A_n\}_{n\in\mathbb{N}_0} \subset \mathbb{R}_0^r$ be a sequence of non-negative real numbers. Suppose that $\{\Lambda^\wedge(n)\}_{n\in\mathbb{N}_0} \subset \mathbb{R}$ is a sequence of real numbers which satisfies*

$$\lim_{n\to\infty} \frac{|\Lambda^\wedge(n)|}{(n+1/2)^t} = C > 0 \text{ for some } t \in \mathbb{R}. \tag{2.4}$$

Let $\{B_n\}_{n\in\mathbb{N}_0} \subset \mathbb{R}_0^+$ be a sequence of non-negative real numbers which satisfies

$$0 < B_n \leq \tilde{C} |\Lambda^\wedge(n)|^{-1} A_n \text{ for all } n \in \mathcal{N}(\{A_n |\Lambda^\wedge(n))|^{-1}\}) \tag{2.5}$$

with some constant $\tilde{C} > 0$. Then the operator $\Lambda : \mathcal{H}(\{A_n\}; h_1; \overline{\Omega_{r_1}^{ext}}) \to \mathcal{H}(\{B_n\}; h_2; \overline{\Omega_{r_2}^{ext}})$, $F \mapsto \Lambda F$, defined by

$$\Lambda F := \sum_{n\in\mathcal{N}(\{A_n|\Lambda^\wedge(n)|^{-1}\})} \sum_{k=1}^{2n+1} \Lambda^\wedge(n) F_{n,k}^{r_1} H_{n,k}(r_2; h_2; \cdot) \tag{2.6}$$

is called a (linear) pseudodifferential operator (PDO) of order t. $\{\Lambda^\wedge(n)\}_{n\in\mathbb{N}_0}$, is called the symbol of Λ, and the convergence of the series (2.6) has to be understood in $\mathcal{H}(\{B_n\}; h_2; \overline{\Omega_{r_2}})$-sense. Moreover, if

$$\lim_{n\to\infty} \frac{|\Lambda^\wedge(n)|}{(n+1/2)^t} = 0 \tag{2.7}$$

for all $t \in \mathbb{R}$, then $\Lambda : \mathcal{H}(\{A_n\}; h_1; \overline{\Omega_{r_1}^{ext}}) \to \mathcal{H}(\{B_n\}; h_2; \overline{\Omega_{r_2}^{ext}})$, defined by (2.6), is called a PDO of order $-\infty$.

Remark 2.6 *Condition (2.5) guarantees that* $\mathrm{im}(\Lambda) \subset \mathcal{H}(\{B_n\}; h_2; \overline{\Omega_{r_2}^{ext}})$, *and thus the PDO* Λ *is well-defined.*

Remark 2.7 *The order of a pseudodifferential operator* $\Lambda : \mathcal{H}(\{A_n\}; h_1; \overline{\Omega_{r_1}^{ext}}) \to \mathcal{H}(\{B_n\}; h_2; \overline{\Omega_{r_2}^{ext}})$ *tells us at which rate the information contained in the Fourier coefficients* $\{F^r_{n,k}\}_{n\in\mathcal{N}(\{A_n\}),\ 1\leq k\leq 2n+1}$ *of* $F \in \mathcal{H}(\{A_n\}; h_1; \overline{\Omega_{r_1}^{ext}})$ *is either attenuated or amplified under the impact of* Λ. *If the elements of the sequence* $\{A_n\}_{n\in\mathbb{N}_0}$ *are in particular chosen to be* $A_n := (n+1/2)^s$, $n \in \mathbb{N}_0$, *with* $s \in \mathbb{R}$ *arbitrary, (2.4) can be interpreted in the following way: A result given in [Fr1999] states that for the class of Sobolev spaces* $\mathcal{H}(\{(n+1/2)^s\}; \overline{\Omega_{r_1}^{ext}})$ *an analogon to the Sobolev embedding theorem known from classical Sobolev space theory is valid. To be more specific, whenever* F *is a function of class* $\mathcal{H}(\{(n+1/2)^s\}; \overline{\Omega_{r_1}^{ext}})$, *where* $s > k+1$, *then* F *is an element of* $\mathrm{Pot}^{(k)}(\overline{\Omega_{r_1}^{ext}})$. *Thus the parameter* s *gives information on the smoothness of the elements of* $\mathcal{H}(\{(n+1/2)^s\}; \overline{\Omega_{r}^{ext}})$ *at the boundary* $\overline{\Omega_{r_1}^{ext}}$. *It can be easily verified that if* $\Lambda : \mathcal{H}(\{(n+1/2)^s\}; \overline{\Omega_{r_1}^{ext}}) \to \mathcal{H}(\{B_n\}; \overline{\Omega_{r_2}^{ext}})$ *is a PDO of order* $t \in \mathbb{R}$, *then* $\mathrm{im}(\Lambda) = \mathcal{H}(\{(n+1/2)^s|\Lambda^\wedge(n)|^{-1}; \overline{\Omega_{r_2}^{ext}}) \subset \mathcal{H}(\{(n+1/2)^{s-t}\}; \overline{\Omega_{r_2}^{ext}})$. *Loosely spoken, this means that for* $t < 0$ *the image of* $F \in \mathcal{H}(\{(n+1/2)^s\}; \overline{\Omega_{r_1}^{ext}})$ *under the operator* Λ *is* $\lceil t \rceil$*-times smoother than* F *at the boundary, and vice-versa in case that* $t > 0$. *From the definition of the spaces* $\mathcal{H}(\{A_n\}; h; \overline{\Omega_{r}^{ext}})$ *it is obvious, that this interpretation can be transferred to the case* $\Lambda : \mathcal{H}(\{(n+1/2)^s\}; h_1; \overline{\Omega_{r_1}^{ext}}) \to \mathcal{H}(\{B_n\}; h_2; \overline{\Omega_{r_2}^{ext}})$. *Furthermore, in case* $\Lambda : \mathcal{H}(\{A_n\}; h_1; \overline{\Omega_{r_1}^{ext}}) \to \mathcal{H}(\{B_n\}; h_2; \overline{\Omega_{r_2}^{ext}})$ *is a PDO of order* $-\infty$, *relation (2.7) tells us that the information contained in the Fourier coefficients* $\{F^r_{n,k}\}_{n\in\mathcal{N},\ 1\leq k\leq 2n+1}$ *is attenuated at a rate which is stronger than polynomial.*

For the remainder of this thesis, we will, however, restrict ourselves to the discussion of pseudodifferential operators $\Lambda : \mathcal{H}(\{A_n\}; h_1; \overline{\Omega_{r_1}^{ext}}) \to \mathcal{H}(\{A_n\}; h_2; \overline{\Omega_{r_2}^{ext}})$ under the additional assumption that the symbol $\{\Lambda^\wedge(n)\}_{n\in\mathbb{N}_0}$ satisfies the condition $\lim_{n\in\mathcal{N},\ n\to\infty} |\Lambda^\wedge(n)| = 0$. This condition ensures that Λ is well-defined. According to Remark 2.7, this restriction implies that we are dealing with pseudodifferential operators with smoothing effect.

Theorem 2.8 *Let* $r_1, r_2 \in \mathbb{R}^+$, *let* $h_1, h_2 \in \{0,1,2\}$ *and* $\{A_n\}_{n\in\mathbb{N}_0} \subset \mathbb{R}_0^+$ *be a sequence of non-negative real numbers. Suppose that* $\Lambda : \mathcal{H}(\{A_n\}; h_1; \overline{\Omega_{r_1}^{ext}}) \to \mathcal{H}(\{A_n\}; h_2; \overline{\Omega_{r_2}^{ext}})$ *is a PDO whose symbol* $\{\Lambda^\wedge(n)\}_{n\in\mathbb{N}_0}$ *satisfies the limit relation*

$$\lim_{n\to\infty} |\Lambda^\wedge(n)| = 0. \tag{2.8}$$

Then the following statements hold true:
(i) The norm of Λ *is given by*

$$\|\Lambda\| = \max_{n\in\mathcal{N}(\{A_n(\Lambda^\wedge(n))^{-1}\})} |\Lambda^\wedge(n)|.$$

(ii) If $\Lambda^\wedge(n) \neq 0$ *for all* $n \in \mathcal{N}(\{A_n\})$, *then the operator* Λ *is injective and maps* $\mathcal{H}(\{A_n\}; h_1; \overline{\Omega_{r_1}^{ext}})$ *onto* $\mathcal{H}(\{A_n|\Lambda^\wedge(n)|^{-1}\}; h_2; \overline{\Omega_{r_2}^{ext}})$, *i.e.,*

$$\mathrm{im}(\Lambda) = \mathcal{H}(\{A_n|\Lambda^\wedge(n)|^{-1}\}; h_2; \overline{\Omega_{r_2}^{ext}})$$

and

$$\overline{\mathrm{im}(\Lambda)}^{\|\cdot\|_{\mathcal{H}(\{A_n\};h_2;\overline{\Omega_{r_2}^{ext}})}} = \mathcal{H}(\{A_n\};h_2;\overline{\Omega_{r_2}^{ext}}).$$

(iii) Λ *is compact.*

Proof: Ad (i): Due to condition (2.8), we can estimate

$$\begin{aligned}
\|\Lambda\| &:= \sup_{\substack{F\in\mathcal{H}(\{A_n\};h_1;\overline{\Omega_{r_1}^{ext}})\\ F\neq 0}} \frac{\|\Lambda F\|_{\mathcal{H}(\{A_n\};h_2;\overline{\Omega_{r_2}^{ext}})}}{\|F\|_{\mathcal{H}(\{A_n\};h_1;\overline{\Omega_{r_1}^{ext}})}} \\
&= \sup_{\substack{F\in\mathcal{H}(\{A_n\};h_1;\overline{\Omega_{r_1}^{ext}})\\ F\neq 0}} \frac{\left(\sum\limits_{n\in\mathcal{N}(A_n\Lambda^\wedge(n)^{-1})}\sum\limits_{k=1}^{2n+1} A_n^2\Lambda^\wedge(n)^2(F_{n,k}^{r_1})^2\right)^{1/2}}{\|F\|_{\mathcal{H}(\{A_n\};h_1;\overline{\Omega_{r_1}^{ext}})}} \\
&\leq \max_{n\in\mathcal{N}(\{A_n(\Lambda^\wedge(n))^{-1}\})} |\Lambda^\wedge(n)| \sup_{F\in\mathcal{H}(\{A_n\};h_1;\overline{\Omega_{r_1}^{ext}})} \frac{\|F\|_{\mathcal{H}(\{A_n\};h_1;\overline{\Omega_{r_1}^{ext}})}}{\|F\|_{\mathcal{H}(\{A_n\};h_1;\overline{\Omega_{r_1}^{ext}})}} \\
&= \max_{n\in\mathcal{N}(\{A_n(\Lambda^\wedge(n))^{-1}\})} |\Lambda^\wedge(n)|.
\end{aligned}$$

Let $\Lambda^\wedge(n_0) := \max_{n\in\mathcal{N}(\{A_n(\Lambda^\wedge(n))^{-1}\})} |\Lambda^\wedge(n)|$. In order to verify that the above estimate is sharp, choose $F := H_{n_0,k}(r_1;h_1;\cdot)$.

Ad (ii): Let $G \in \mathrm{im}(\Lambda)$, i.e., $G = \Lambda F$ for some $F \in \mathcal{H}(\{A_n\};h_1,\overline{\Omega_{r_1}^{ext}})$. Then

$$\begin{aligned}
\|G\|_{\mathcal{H}(\{A_n|\Lambda^\wedge(n)|^{-1}\};h_2,\overline{\Omega_{r_2}^{ext}})} &= \sum_{n\in\mathcal{N}}\sum_{k=1}^{2n+1}(A_n|\Lambda^\wedge(n)|)^2(\Lambda^\wedge(n)F_{n,k}^{r_1})^2 \\
&= \sum_{n\in\mathcal{N}}\sum_{k=1}^{2n+1} A_n^2(F_{n,k}^{r_1})^2 < \infty,
\end{aligned}$$

and thus $G \in \mathcal{H}(\{A_n|\Lambda^\wedge(n)|^{-1}\};h_2,\overline{\Omega_{r_2}^{ext}})$. Conversely, assume that $G \in \mathcal{H}(\{A_n|\Lambda^\wedge(n)|^{-1}\};h_2,\overline{\Omega_{r_2}^{ext}})$. Then the function

$$F = \sum_{n\in\mathcal{N}}\sum_{k=1}^{2n+1} G_{n,k}^{r_2}(\Lambda^\wedge(n))^{-1}H_{n,k}(r_1;h_1;\cdot)$$

is in $\mathcal{H}(\{A_n\};h_1;\overline{\Omega_{r_1}^{ext}})$ and $\Lambda F = G$. Consequently, $G \in \mathrm{im}(\Lambda)$. The second statement then follows from the fact that the system $\{H_{n,k}(r_2;h_2;\cdot)\}_{n\in\mathcal{N},\ 1\leq k\leq 2n+1}$ is complete in $\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})$.

Ad (iii): Let $\mathcal{K}\left(\mathcal{H}(\{A_n\};h_1;\overline{\Omega_{r_1}^{ext}}),\mathcal{H}(\{A_n\};h_2;\overline{\Omega_{r_2}^{ext}})\right)$ denote the set of all compact operators which map $\mathcal{H}(\{A_n\};h_1;\overline{\Omega_{r_1}^{ext}})$ into $\mathcal{H}(\{A_n\};h_2;\overline{\Omega_{r_2}^{ext}})$. Define the sequence of operators with finite dimensional range $\{\Lambda_N\}_{N\in\mathbb{N}_0}$ by

$$\Lambda_N := \sum_{\substack{n\in\mathcal{N}(\{A_n(\Lambda^\wedge(n))^{-1}\})\\ n\leq N}} \sum_{k=1}^{2n+1} \Lambda^\wedge(n)F_{n,k}^{r_1}H_{n,k}(r_2;h_2;\cdot).$$

For every $N \in \mathbb{N}_0$, Λ_N is compact. Furthermore, (i) and condition (2.8) imply that

$$\|\Lambda - \Lambda_N\| = \max_{\substack{n \in \mathcal{N}(\{A_n(\Lambda^\wedge(n))^{-1}\}) \\ n > N}} |\Lambda^\wedge(n)| \to 0 \quad \text{for} \quad N \to \infty,$$

and due to the completeness of $\mathcal{K}\left(\mathcal{H}(\{A_n\}; h_1; \overline{\Omega_{r_1}^{ext}}), \mathcal{H}(\{A_n\}; h_2; \overline{\Omega_{r_2}^{ext}})\right)$, the operator Λ is compact, too. □

It is a well-known fact from functional analysis that a compact operator between Hilbert spaces possesses a singular value decomposition. The formulation in the sequel is quoted from [Kr1989]:

Definition 2.9 *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces. Let $A : \mathcal{H}_1 \to \mathcal{H}_2$ be a compact linear operator and $A^* : \mathcal{H}_2 \to \mathcal{H}_1$ its adjoint. The non-negative square roots of the eigenvalues of the non-negative self adjoint compact operator $A^*A : \mathcal{H}_1 \to \mathcal{H}_1$ are called the singular values of A.*

Theorem 2.10 (Singular Value Decomposition) *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces. Let $A : \mathcal{H}_1 \to \mathcal{H}_2$ be a compact linear operator and $A^* : \mathcal{H}_2 \to \mathcal{H}_1$ its adjoint. Let $\{\sigma_n\}_{n \in \mathbb{N}_0}$ denote the sequence of the non-zero singular values of A repeated according to their multiplicity and ordered according to $\sigma_0 \geq \sigma_1 \geq \ldots$. Then there exist orthonormal systems $\{H_n^{(1)}\}_{n \in \mathcal{J}} \subset \mathcal{H}_1$ and $\{H_n^{(2)}\}_{n \in \mathcal{J}} \subset \mathcal{H}_2$, where $\mathcal{J} \subset \mathbb{N}_0$ can be either finite or $\mathcal{J} = \mathbb{N}_0$, such that*

$$AH_n^{(1)} = \sigma_n H_n^{(2)}, \; A^* H_n^{(2)} = \sigma_n H_n^{(1)}$$

for all $n \in \mathcal{J}$. For each $F \in \mathcal{H}_1$ there holds the singular value decomposition

$$F = \sum_{n \in \mathcal{J}} (F, H_n^{(1)})_{\mathcal{H}_1} H_n^{(1)} + QF$$

with the orthogonal projection $Q : \mathcal{H}_1 \to \ker(A)$ and

$$AF = \sum_{n \in \mathcal{J}} \sigma_n (F, H_n^{(1)})_{\mathcal{H}_1} H_n^{(2)}.$$

Each system $\{(\sigma_n, H_n^{(1)}, H_n^{(2)}),\, n \in \mathcal{J}\}$, with these properties is called a singular system of A.
If A is additionally injective, then $\{H_n^{(1)}\}_{n \in \mathcal{J}}$ is a complete orthonormal system in $\mathcal{H}_1$.

Proof: A proof can for instance be found in [Kr1989]. □.

Theorem 2.11 *Let $r_1, r_2 \in \mathbb{R}^+$, let $h_1, h_2 \in \{0, 1, 2\}$ and $\{A_n\}_{n \in \mathbb{N}_0} \subset \mathbb{R}_0^+$ be a sequence of non-negative real numbers. Suppose that $\Lambda : \mathcal{H}(\{A_n\}; h_1; \overline{\Omega_{r_1}^{ext}}) \to \mathcal{H}(\{A_n\}; h_2; \overline{\Omega_{r_2}^{ext}})$*

is a PDO whose symbol $\{\Lambda^\wedge(n)\}_{n\in\mathbb{N}_0} \subset \mathbb{R}_0^+$ *is a sequence of non-negative real numbers which satisfies the limit relation* $\lim_{n\in\mathcal{N},\ n\to\infty} \Lambda^\wedge(n) = 0$. *Then its adjoint operator* $\Lambda^* : \mathcal{H}(\{A_n\}; h_2; \overline{\Omega_{r_2}^{ext}}) \to \mathcal{H}(\{A_n\}; h_1; \overline{\Omega_{r_1}^{ext}})$ *is given by*

$$\Lambda^* G = \sum_{n\in\mathcal{N}(\{A_n(\Lambda^\wedge(n))^{-1}\})} \sum_{k=1}^{2n+1} \Lambda^\wedge(n) G_{n,k}^{r_2} H_{n,k}(r_1; h_1; \cdot).$$

Furthermore,

$$\left(\Lambda^\wedge(n), \frac{1}{A_n} H_{n,k}(r_1; h_1; \cdot), \frac{1}{A_n} H_{n,k}(r_2; h_2; \cdot)\right),$$

$n \in \mathcal{N}(\{A_n(\Lambda^\wedge(n))^{-1}\})$, $1 \le k \le 2n+1$, *is a singular system of* Λ.

Proof: All statements can be verified by straightforward computations. □.

Let $R \in \mathbb{R}^+$ and let $\{A_n\}_{n\in\mathbb{N}_0} \subset \mathbb{R}_0^+$ be a sequence of non-negative real numbers which additionally satisfy $A_n \ge 1$ for almost all $n \in \mathbb{N}_0$. The SST- and SGG-operator, respectively, maps a function of class $\mathcal{H}(\{A_n\}; \overline{\Omega_R^{ext}})$ onto its negative first and second order radial derivative on $\overline{\Omega_{R+\delta}^{ext}}$, $\delta > 0$, respectively:

Example 2.12 (SST-Operator) *Let* $\{A_n\}_{n\in\mathbb{N}_0} \subset \mathbb{R}_0^+$ *be a sequence of non-negative real numbers which satisfy* $A_n \ge 1$ *for almost all* $n \in \mathbb{N}_0$ *and let* $R, r \in \mathbb{R}^+$ *with* $R < r$. *The SST-operator* $\Lambda^{SST} : \mathcal{H}(\{A_n\}; \overline{\Omega_R^{ext}}) \to \mathcal{H}(\{A_n\}; 1; \overline{\Omega_r^{ext}})$ *is defined by*

$$\Lambda^{SST} F := \sum_{n\in\mathbb{N}_0} \sum_{k=1}^{2n+1} \frac{n+1}{r} \left(\frac{R}{r}\right)^n F_{n,k}^R \, H_{n,k}(r; 1; \cdot).$$

It is a pseudodifferential operator of order $-\infty$ *with symbol* $\{\frac{n+1}{r}\left(\frac{R}{r}\right)^n\}_{n\in\mathbb{N}_0}$.

Example 2.13 (SGG-Operator) *Let* $\{A_n\}_{n\in\mathbb{N}_0} \subset \mathbb{R}_0^+$ *be a sequence of non-negative real numbers which satisfy* $A_n \ge 1$ *for almost all* $n \in \mathbb{N}_0$ *and let* $R, r \in \mathbb{R}^+$ *with* $R < r$. *The SGG-operator* $\Lambda^{SGG} : \mathcal{H}(\{A_n\}; \overline{\Omega_R^{ext}}) \to \mathcal{H}(\{A_n\}; 2; \overline{\Omega_r^{ext}})$ *is defined by*

$$\Lambda^{SGG} F := \sum_{n\in\mathbb{N}_0} \sum_{k=1}^{2n+1} \frac{(n+1)(n+2)}{r^2} \left(\frac{R}{r}\right)^n F_{n,k}^R \, H_{n,k}(r; 2; \cdot).$$

It is a pseudodifferential operator of order $-\infty$ *with symbol* $\{\frac{(n+1)(n+2)}{r^2}\left(\frac{R}{r}\right)^n\}_{n\in\mathbb{N}_0}$.

Obviously, the SST-operator as well as the SGG-operator are injective and compact, but not surjective. Furthermore,

$$\left(\frac{n+1}{r}\left(\frac{R}{r}\right)^n, \frac{1}{A_n} H_{n,k}(R; \cdot), \frac{1}{A_n} H_{n,k}(r; 1; \cdot)\right), \ n \in \mathbb{N}_0,$$

and

$$\left(\frac{(n+1)(n+2)}{r^2}\left(\frac{R}{r}\right)^n, \frac{1}{A_n}H_{n,k}(R;\cdot), \frac{1}{A_n}H_{n,k}(r;2;\cdot)\right), \ n \in \mathbb{N}_0,$$

form a singular system of Λ^{SST} and Λ^{SGG}, respectively.

Due to the calculations on page 21 it is obvious that

$$\Lambda^{SST}F = -\left.\frac{\partial F}{\partial \rho}\right|_{\overline{\Omega_r^{ext}}} \quad \text{and} \quad \Lambda^{SST}F = \left.\frac{\partial^2 F}{\partial \rho^2}\right|_{\overline{\Omega_r^{ext}}}.$$

Chapter 3

Inverse Problems and Regularization

Section 3.1 starts with the introduction of the geometric concept on which we base the formulation of the SST- and SGG-problem. Both problems are then formulated as ill-posed pseudodifferential operator equations which relate the spaces $\mathcal{H}(\{A_n\};\overline{\Omega_R^{ext}})$ and $\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})$, where $h = 1$ in case of SST and $h = 2$ in case of SGG, and $R, r \in \mathbb{R}^+$, $R < r$. We then explain the concept of regularization in terms of a filtered singular value decomposition in Section 3.2. Since in reality measurements are taken in discrete points on the satellite orbit, we finally go over to the discrete formulation of the satellite problems in Section 3.3.

3.1 Inverse Problems in Satellite Geodesy and Regularization

Figure 3.1 illustrates the geometric concept on which the formulation is based. Let Σ_E be the earth's surface and Σ_S be the 'orbital surface', which are both supposed to be $\mathcal{C}^{(2)}$-regular surfaces. Furthermore, we choose a Bjerhammar sphere Ω_R in side the earth, and a Bjerhammar sphere Ω_r for the 'orbital surface' Σ_S, which lies closely under Σ_S. We assume that $\sup_{y\in\Sigma_E} |y| < \inf_{x\in\Sigma_S} |x|$, which is of course realistic for satellites flying at an altitude of more than 200 km. The earth's gravitational potential V is assumed to be of class $\mathrm{Pot}^{(0)}(\overline{\Sigma_E^{ext}})$.
As explained in the introduction to this thesis the observations in an SST- and SGG-satellite mission, respectively, yield knowledge of the first order radial derivative of V (SST), and second order radial derivative of V (SGG), respectively, on a point grid $X_N = \{x_1^N, \ldots, x_N^N\} \subset \Sigma_S$ on the satellite orbit. Thus, we have measured data $\{(x_i, \frac{\partial^h V(x_i)}{\partial \rho^h}) | \, i = 1, \ldots, N\}$, where $h = 1$ for SST and $h = 2$ for SGG. Due to Corollary 1.13 of the Runge-Walsh approximation theorem and the extension of Helly's theorem, the following holds true:
Let $\{A_n\}_{n\in\mathbb{N}_0} \subset \mathbb{R}^+$ be a sequence of positive real numbers with $A_n \geq 1$ for almost all

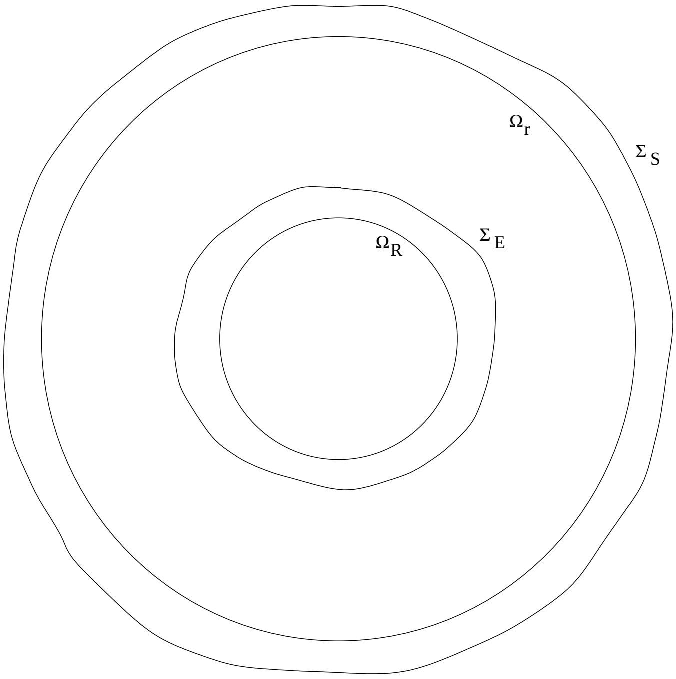


Figure 3.1: *Geometric concept:* Σ_E *and* Σ_S *denote the earth's surface and the satellite orbit, and* Ω_R *and* Ω_r *are Bjerhammar spheres for* Σ_E *and* Σ_S*, respectively.*

$n \in \mathbb{N}_0$ and $h \in \{1, 2\}$. Given $\varepsilon > 0$ there exists $U \in \mathcal{H}(\{A_n\}; \overline{\Omega_R^{ext}})$ with

$$\|U - V\|_{\mathcal{C}(\overline{\Sigma^{ext}})} \leq \varepsilon \quad \text{and} \quad \frac{\partial^h U(x_i^N)}{\partial \rho^h} = \frac{\partial^h V(x_i^N)}{\partial \rho^h}, \ i = 1, \ldots, N.$$

Therefore, it suffices to approximate the Runge-Walsh approximation U of V with the help of the measured SST-data and SGG-data.

This is the basic idea behind the approach to the SST- and SGG-problem in this work. We will first present a 'continuous' formulation of the SST- and SGG-problem as operator equations (see Problem 3.1 below), which assumes that $\frac{\partial^h U}{\partial \rho^h}|_{\overline{\Omega_r^{ext}}}$ is known. After investigating the properties of these operator equations with the help of the theory of inverse problems and motivating the need of regularization we present a 'discrete' formulation of the SST-problem and SGG-problem which takes into account that $\frac{\partial^h U}{\partial \rho^h}$ is only given on a point grid X_N on the 'orbital surface' Σ_S. How these 'discrete' problems are solved with a regularization scheme or, more precisely, how a suitable regularization of the 'continuous' operator equation is discretized with the help of the measured data is explained in detail in Chapter 7, where we present the concrete numerical realization.

It should be noted that the assumption that the satellite data is given on a closed $\mathcal{C}^{(2)}$-regular surface is for theoretical reasons; the numerical methods in Chapter 7 work also for a more 'realistic' data distribution.

For general literature on the theory of inverse problems we refer the reader to [EnHaNe1996] and [Ki1996]. Our presentation of general concepts, however, roughly follows the presentation in [Kr1989].

Problem 3.1 (SST-/SGG-Problem in Non-Discrete Formulation) *Let* $\Sigma_E, \Sigma_S \subset \mathbb{R}^3$ *be* $\mathcal{C}^{(2)}$*-regular surfaces with* $\sup_{y\in\Sigma_E}|y| < \inf_{x\in\Sigma_S}|x|$. *Let furthermore* Ω_R *and* Ω_r *be Bjerhammar spheres for* Σ_E *and* Σ_S*, respectively, where* $R < \inf_{y\in\Sigma_E}|y|$ *and* $\sup_{y\in\Sigma_E}|y| < r < \inf_{x\in\Sigma_S}|x|$. *Moreover, let* $\{A_n\}_{n\in\mathbb{N}_0} \subset \mathbb{R}^+$ *denote a sequence of positive real numbers which satisfy* $A_n \geq 1$ *for almost all* $n \in \mathbb{N}_0$*, and let* $\Lambda \in \{\Lambda^{SST}, \Lambda^{SGG}\}$ *denote the SST- operator and the SGG-operator in Example 2.12 and 2.13, respectively, i.e.,* $\Lambda : \mathcal{H}(\{A_n\}; \overline{\Omega_R^{ext}}) \to \mathcal{H}(\{A_n\}; h; \overline{\Omega_r^{ext}})$*, with* $h = 1$ *in case of SST and* $h = 2$ *in case of SGG is a PDO with symbol* $\{\Lambda^\wedge(n)\}_{n\in\mathbb{N}_0}$ *defined by* $\Lambda^\wedge(n) = ((n+1)/r)(R/r)^n$ *and* $\Lambda^\wedge(n) = ((n+1)(n+2)/r^2)(R/r)^n$*, respectively. Then the SST- and SGG- problem in non-discrete formulation reads as follows:*
Given $G \in \mathcal{H}(\{A_n\}; h; \overline{\Omega_r^{ext}})$*, find* $F \in \mathcal{H}(\{A_n\}; \overline{\Omega_R^{ext}})$ *such that*

$$\Lambda F = G. \tag{3.1}$$

Recalling Hadamard's definition of a well-posed problem, we can classify equation (3.1) as an ill-posed pseudodifferential operator equation:

Definition 3.2 *Let* $\mathcal{X}, \mathcal{Y}$ *be normed spaces and let* $A : \mathcal{X} \to \mathcal{Y}$ *be an operator from* $\mathcal{X}$ *into* $\mathcal{Y}$*. The equation*

$$AF = G,$$

where $F \in \mathcal{X}$ *and* $G \in \mathcal{Y}$ *is called well-posed if* A *is bijective and the inverse operator* $A^{-1} : \mathcal{Y} \to \mathcal{X}$ *is continuous. Otherwise the equation is called ill-posed.*

Theorem 3.3 *Problem 3.1 is ill-posed in the sense of Definition 3.2, since both existence of a solution for an arbitrary right-hand side and continuity of the inverse are violated.*

Proof: For $\Lambda \in \{\Lambda^{SST}, \Lambda^{SGG}\}$ we have $\Lambda^\wedge(n) > 0$ for all $n \in \mathbb{N}_0$. Thus Λ is injective, and according to Theorem 2.8, $\mathrm{im}(\Lambda) = \mathcal{H}(\{A_n(\Lambda^\wedge(n))^{-1}\}; h; \overline{\Omega_r^{ext}})$ and $\overline{\mathrm{im}(\Lambda)}^{\|\cdot\|_{\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})}} = \mathcal{H}(\{A_n\}; h; \overline{\Omega_r^{ext}})$. Since $\mathcal{H}(\{A_n(\Lambda^\wedge(n))^{-1}\}; h; \overline{\Omega_r^{ext}}) \neq \mathcal{H}(\{A_n\}; h; \overline{\Omega_r^{ext}})$, Λ is not surjective. Moreover, due to $\lim_{n\to\infty}\Lambda^\wedge(n) = 0$, Λ is compact, and since its range is not finite dimensional, its inverse $\Lambda^{-1} : \mathcal{H}(\{A_n\Lambda^\wedge(n)^{-1}\}; h; \overline{\Omega_r^{ext}}) \to \mathcal{H}(\{A_n\}; \overline{\Omega_R^{ext}})$ cannot be bounded. □

In our applications we wish to approximate the solution $F \in \mathcal{H}(\{A_n\}; \overline{\Omega_R^{ext}})$ to equation (3.1) from the knowledge of a perturbed right hand side $G^\delta \in \mathcal{H}(\{A_n\}; h; \overline{\Omega_r^{ext}})$ of $G = \Lambda F$ with a known error level

$$\|G - G^\delta\|_{\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})} \leq \delta.$$

For the error-affected right hand side, in general, we cannot expect $G^\delta \in \mathrm{im}(\Lambda)$. Using the error-affected data G^δ we want to construct a reasonable approximation F^δ to the exact solution F of the unperturbed equation $\Lambda F = G$. This approximation F^δ should of course depend continuously on the data G^δ. Methods for a stable approximate solution of ill-posed problems are called regularization methods. The basic idea behind these methods is to construct a pointwise approximation of the unbounded inverse operator $\Lambda^{-1} : \mathrm{im}(\Lambda) \to \mathcal{H}(\{A_n\};\overline{\Omega_R^{ext}})$ by a family of bounded linear operators $\{T_j\}_{j\in\mathbb{N}_0}$, $T_j : \mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}}) \to \mathcal{H}(\{A_n\};\overline{\Omega_R^{ext}})$. In a general setting, we introduce the notion of a regularization as follows:

Definition 3.4 *Let $\mathcal{X}$ and $\mathcal{Y}$ be normed spaces and let $A : \mathcal{X} \to \mathcal{Y}$ be an injective bounded linear operator. A family $\{T_j\}_{j\in\mathbb{N}_0}$ of bounded linear operators $T_j : \mathcal{Y} \to \mathcal{X}$ is called a regularization of $A^{-1} : \mathrm{im}(A) \to \mathcal{X}$ with discrete regularization parameter j, if for any $G \in \mathrm{im}(A)$ the limit relation*

$$\lim_{j\to\infty} \|T_j G - A^{-1}G\|_X = 0$$

is satisfied. $F_j := T_j G$ is called the j-level regularization of the problem $AF = G$ corresponding to the regularization $\{T_j\}_{j\in\mathbb{N}_0}$.

In Definition 3.4 we restrict ourselves to regularization schemes with a discrete regularization parameter $j \in \mathbb{N}_0$. This is due to the fact that in numerical calculations only a discrete variation of the regularization parameter is of interest.

Turning back to Problem 3.1, a j-level regularization $F_j^\delta := T_j G^\delta$ approximates the solution F of (3.1). Thus we have the following error estimate

$$\begin{aligned} \|F_j^\delta - F\|_{\mathcal{H}(\{A_n\};\overline{\Omega_R^{ext}})} &= \|T_j G^\delta - T_j G + T_j \Lambda F - F\|_{\mathcal{H}(\{A_n\};\overline{\Omega_R^{ext}})} \\ &\leq \delta\|T_j\| + \|T_j\Lambda F - F\|_{\mathcal{H}(\{A_n\};\overline{\Omega_R^{ext}})}, \end{aligned} \tag{3.2}$$

which shows that there are two conflicting contributions. The first term reflects the influence of the data error and the second term the approximation error. Due to the ill-posed nature of equation (3.1), $\|T_j\|$ cannot be uniformly bounded with respect to j, and the second term cannot be estimated uniformly with respect to F. Typically, the first term will be increasing as $j \to \infty$, since $\{T_j\}_{j\in\mathbb{N}_0}$ approximates the unbounded inverse Λ^{-1} pointwise, whereas the approximation error will be decreasing as $j \to \infty$. The art of treating an inverse problem is to find a strategy of how to choose the parameter j in dependence of the error level δ such that the error (3.2) is minimized. Obviously, a j-level regularization of equation (3.1) should converge to the exact solution when the error level tends to zero, if our strategy to choose the regularization parameter $j \in \mathbb{N}_0$ is reasonable.

Definition 3.5 *Let the notation and assumptions be the same as in Definition 3.4. A parameter choice strategy for a regularization scheme $\{T_j\}_{j\in\mathbb{N}_0}$, i.e., the choice of the regularization parameter $j = j(\delta)$ depending on the error level δ, is called regular if for all $G \in \mathrm{im}(A)$ and all $G^\delta \in Y$ with $\|G - G^\delta\|_Y \leq \delta$ there holds*

$$\lim_{\delta\to 0} \|T_{j(\delta)} G^\delta - A^{-1}G\|_X = 0.$$

3.2 Filtered Singular Value Decomposition

Starting from the ill-posed pseudodifferential operator equation (3.1), we use the singular value decomposition of $\Lambda : \mathcal{H}(\{A_n\};\overline{\Omega_R^{ext}}) \to \mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})$ as a basis for the construction of a regularization $\{T_j\}_{j\in\mathbb{N}_0}$ of $\Lambda^{-1} : \mathcal{H}(\{A_n(\Lambda^\wedge(n))^{-1}\};h;\overline{\Omega_r^{ext}}) \to \mathcal{H}(\{A_n\};\overline{\Omega_R^{ext}})$. We first mention the Picard Theorem (see [Kr1989]) which expresses the solution to an operator equation of the first kind with a compact operator in terms of the singular system:

Theorem 3.6 (Picard) *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces, let $A : \mathcal{H}_1 \to \mathcal{H}_2$ be a compact linear operator with singular system $(\sigma_n, H_n^{(1)}, H_n^{(2)})$, $n \in \mathbb{N}_0$, and let $A^* : \mathcal{H}_2 \to \mathcal{H}_1$ denote its adjoint operator. The equation*

$$AF = G$$

is solvable if and only if $G \in \ker(A^*)^\perp = \overline{\mathrm{im}(A)}^{\|\cdot\|_{\mathcal{H}_2}}$ and

$$\sum_{n\in\mathbb{N}_0} \sigma_n^{-2} |(G, H_n^{(2)})_{\mathcal{H}_2}|^2 < \infty. \tag{3.3}$$

Then a solution is given by

$$F = \sum_{n\in\mathbb{N}_0} \sigma_n^{-1} (G, H_n^{(2)})_{\mathcal{H}_2} H_n^{(1)}. \tag{3.4}$$

Proof: A proof can be found in [Kr1989]. □

Since both the SST-operator and the SGG-operator are injective, their range satisfies $\overline{\mathrm{im}(\Lambda)}^{\|\cdot\|_{\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})}} = \mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})$. Thus Theorem 3.6 tells us that a solution to Problem 3.1 exists if and only if $G \in \mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})$ satisfies the regularity condition

$$\sum_{n\in\mathcal{N}} \sum_{k=1}^{2n+1} A_n^2 \Lambda^\wedge(n)^{-2} (G_{n,k}^r)^2 < \infty. \tag{3.5}$$

Condition 3.3 shows how the decay of the sequence of singular values $\{\sigma_n\}_{n\in\mathbb{N}_0}$ determines the ill-posed nature of the operator equation $AF = G$ in Theorem 3.6. Since in the SST-/SGG-case $\{\Lambda^\wedge(n)\}_{n\in\mathbb{N}_0}$ decreases exponentially, we may classify Problem 3.1 as exponentially, or severely, ill-posed.

One way to obtain a regularization of A^{-1} in Theorem 3.6 is to construct a pointwise approximation to (3.4) by applying a so-called regularizing filter to the sequence of singular values $\{\sigma_n\}_{n\in\mathbb{N}_0}$.

Theorem 3.7 (Filtered Singular Value Decomposition) *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces, and let $A : \mathcal{H}_1 \to \mathcal{H}_2$ be an injective compact linear operator with singular system $(\sigma_n, H_n^{(1)}, H_n^{(2)})$, $n \in \mathbb{N}_0$. Let $f : \mathbb{N}_0 \times (0, \|A\|] \to \mathbb{R}$ be a function which satisfies the conditions* *(i)* $|f(j,\sigma)| \leq 1$ *for all* $j \in \mathbb{N}_0$ *and* $\sigma \in (0, \|A\|]$.

(ii) For each $j \in \mathbb{N}_0$ *there exists a positive constant* $C(j)$ *such that*

$$|f(j,\sigma)| \leq \sigma C(j) \text{ for all } 0 < \sigma \leq \|A\|$$

and
(iii) $\lim\limits_{j\to\infty} f(j,\sigma) = 1$ *for all* $0 < \sigma \leq \|A\|$.
Then the bounded linear operators $T_j : \mathcal{H}_2 \to \mathcal{H}_1$, $j \in \mathbb{N}_0$, *defined by*

$$T_j G := \sum_{n\in\mathbb{N}_0} \sigma_n^{-1} f(j,\sigma_n)(G, H_n^{(2)})_{\mathcal{H}_2} H_n^{(1)}$$

for all $G \in \mathcal{H}_2$, *describe a regularization scheme with*

$$\|T_j\| \leq C(j).$$

The function f *is called a regularizing filter to the ill-posed operator equation* $AF = G$, $F \in \mathcal{H}_1$, $G \in \mathcal{H}_2$.
Moreover, the parameter choice strategy $j = j(\delta)$ *for* $\{T_j\}_{j\in\mathbb{N}_0}$ *is regular, if* $\delta C(j(\delta)) \to 0$ *for* $j \to \infty$.

Proof: In order to verify $\|T_j\| \leq C(j)$, note that due to condition (ii),

$$\begin{aligned}
\|T_j G\|_{\mathcal{H}_1}^2 &= \sum_{n\in\mathbb{N}_0} \sigma_n^{-2} |f(j,\sigma_n)|^2 |(G, H_n^{(2)})_{\mathcal{H}_2}|^2 \\
&\leq |C(j)|^2 \sum_{n\in\mathbb{N}_0} |(G, H_n^{(2)})_{\mathcal{H}_2}|^2 \\
&\leq |C(j)|^2 \|G\|_{\mathcal{H}_2}^2 \text{ for all } G \in \mathcal{H}_2.
\end{aligned}$$

It remains to show that $\lim_{j\to\infty} \|T_j G - A^{-1}G\|_{\mathcal{H}_1} = 0$ for $G \in \mathrm{im}(A)$, i.e., to $G \in \mathcal{H}_2$ there exists an $F \in \mathcal{H}_1$ such that $AF = G$:

$$\begin{aligned}
(T_j G, H_n^{(1)})_{\mathcal{H}_1} &= \sigma_n^{-1} f(j,\sigma_n)(AF, H_n^{(2)})_{\mathcal{H}_2} \\
&= \sigma_n^{-1} f(j,\sigma_n)(F, A^* H_n^{(2)})_{\mathcal{H}_1} \\
&= f(j,\sigma_n)(F, H_n^{(1)})_{\mathcal{H}_1}.
\end{aligned}$$

Moreover, since A is injective, we have

$$\begin{aligned}
\|TjG - A^{-1}G\|_{\mathcal{H}_1} &= \|TjAF - F\|_{\mathcal{H}_1} \\
&= \sum_{n\in\mathbb{N}_0} |(TjAF - F, H_n^{(1)})_{\mathcal{H}_1}|^2 \\
&= \sum_{n\in\mathbb{N}_0} |f(j,\sigma_n) - 1|^2 |(F, H_n^{(1)})_{\mathcal{H}_1}|^2.
\end{aligned}$$

Due to condition (i),

$$\sum_{n\in\mathbb{N}_0} |f(j,\sigma_n)-1|^2 |(F,H_n^{(1)})_{\mathcal{H}_1}|^2 \leq 4\sum_{n\in\mathbb{N}_0} |(F,H_n^{(1)})_{\mathcal{H}_1}|^2 = 4\|F\|^2_{\mathcal{H}_1},$$

and hence for all $\varepsilon > 0$ there exists an $N = N(\varepsilon) \in \mathbb{N}_0$ such that

$$\sum_{\substack{n\in\mathbb{N}_0\\ n>N}} |(F,H_n^{(1)})_{\mathcal{H}_1}|^2 < \frac{\varepsilon}{2}.$$

Moreover, condition (iii) implies that to ε there exists a $j_0 = j_0(\varepsilon) \in \mathbb{N}_0$ such that

$$|f(j,\sigma_n)-1|^2 < \frac{\varepsilon}{2\|F\|^2_{\mathcal{H}_1}} \text{ for all } n \in \{0,\ldots,N\} \text{ and all } j \geq j_0.$$

Hence,

$$\|T_jG - F\|^2_{\mathcal{H}_1} < \frac{\varepsilon}{2\|F\|^2_{\mathcal{H}_1}} \sum_{\substack{n\in\mathbb{N}_0\\ n\leq N}} |(F,H_n^{(1)})_{\mathcal{H}_1}|^2 + 4\frac{\varepsilon}{8} < \varepsilon \text{ for all } j \geq j_0.$$

In order to verify the regularity condition for the parameter choice strategy, let $G \in \mathrm{im}(A)$ and $G^\delta \in \mathcal{H}_2$ with $\|G - G^\delta\|_{\mathcal{H}_2} \leq \delta$. Then the estimate

$$\begin{aligned} \|T_jG^\delta - A^{-1}G\|_{\mathcal{H}_1} &\leq \|T_j(G^\delta - G)\|_{\mathcal{H}_1} + \|T_jG - A^{-1}G\|_{\mathcal{H}_1} \\ &\leq \delta C(j) + \|T_jG - A^{-1}G\|_{\mathcal{H}_1} \end{aligned}$$

together with the fact that $\{T_j\}_{j\in\mathbb{N}_0}$ is a regularization for A^{-1} implies that $j = j(\delta)$ is regular if $\delta C(j(\delta)) \to 0$ for $j \to \infty$. □

3.3 Discrete Formulation of the SST-Problem and the SGG-Problem

Finally, we have to be aware of the fact that in our applications the function $G \in \mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})$ is only known in a set of $N \in \mathbb{N}$ points $X = \{x_1,\ldots,x_N\}$ on the satellite orbit Σ_S. Hence we have to reformulate Problem 3.1 in the following way:

Problem 3.8 (SST-/SGG-Problem in Discrete Formulation) *Let $\Sigma_E, \Sigma_S \subset \mathbb{R}^3$ be $\mathcal{C}^{(2)}$-regular surfaces with $\sup_{y\in\Sigma_E}|y| < \inf_{x\in\Sigma_S}|x|$. Let furthermore Ω_R and Ω_r be Bjerhammar spheres for Σ_E and Σ_S, respectively, where $R < \inf_{y\in\Sigma_E}|y|$ and $\sup_{y\in\Sigma_E}|y| < r < \inf_{x\in\Sigma_S}|x|$. Moreover, let $\{A_n\}_{n\in\mathbb{N}_0} \subset \mathbb{R}^+$ denote a sequence of positive real numbers which satisfy $A_n \geq 1$ for almost all $n \in \mathbb{N}_0$, and let $\Lambda \in \{\Lambda^{SST}, \Lambda^{SGG}\}$ denote the SST- and SGG-operator in Example 2.12 and 2.13, respectively, i.e., $\Lambda : \mathcal{H}(\{A_n\};\overline{\Omega_R^{ext}}) \to \mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})$, with $h = 1$ in case of SST and $h = 2$ in case of SGG is a pseudodifferential operator with symbol $\{\Lambda^\wedge(n)\}_{n\in\mathbb{N}_0}$ defined by $\Lambda^\wedge(n) = ((n+1)/r)(R/r)^n$ and*

$\Lambda^\wedge(n) = ((n+1)(n+2)/r^2)(R/r)^n$, respectively. Suppose that $X_N = \{x_1, \ldots, x_N\}$ is a set of $N \in \mathbb{N}$ points on Σ_S. Then the SST- and SGG- problem in discrete formulation reads as follows:
Reconstruct an approximation to the solution of $\Lambda F = G$ with the help of the given discrete values $\{(x_i, G(x_i) | \, i = 1, \ldots, N\}$ of $G \in \mathcal{H}(\{A_n\}; h; \overline{\Omega_r^{ext}})$.

The discrete SST-/SGG-problem can be treated by replacing the function $G \in \mathcal{H}(\{A_n\}; h; \overline{\Omega_r^{ext}})$ by a spline function which either interpolates or approximates the data. The necessary background material on spline functions in $\mathcal{H}(\{A_n\}; h; \overline{\Omega_r^{ext}})$ is presented in Chapter 4.

Chapter 4

Splines and Approximation of Bounded Linear Functionals

In Section 4.1 we introduce splines in Sobolev-like Hilbert spaces $\mathcal{H}(\{A_n\}; h; \overline{\Omega_r^{ext}})$. We start with the spline interpolation problem relative to a set of linearly independent bounded linear functionals on $\mathcal{H}(\{A_n\}; h; \overline{\Omega_r^{ext}})$.

The $\mathcal{H}(\{A_n\}; h; \overline{\Omega_r^{ext}})$-spline interpolation problem is uniquely solvable, and the interpolating spline of a function $G \in \mathcal{H}(\{A_n\}; h; \overline{\Omega_r^{ext}})$ relative to a set of linearly independent bounded linear functionals is characterized as the orthogonal projection of G onto the finite dimensional spline space spanned by the representers of the respective functionals. The bounded linear functionals will usually be measurement functionals and the interpolating spline of a function $G \in \mathcal{H}(\{A_n\}; h; \overline{\Omega_r^{ext}})$ assumes the same values with respect to these measurements. The fact that spline interpolation is nothing else but an orthogonal projection is the basis for the construction of the domain decomposition algorithm presented in Chapter 6, which can be exploited for an efficient numerical solution of the $\mathcal{H}(\{A_n\}; h; \overline{\Omega_r^{ext}})$-spline interpolation problem for a high number of measurements.

In the presence of error-affected data it is advisable to work with smoothing splines which approximate the data instead of interpolating them. Hence the $\mathcal{H}(\{A_n\}; h; \overline{\Omega_r^{ext}})$-spline smoothing problem is discussed in Section 4.2.

As a consequence of the Riesz representation theorem, spline interpolation in $\mathcal{H}(\{A_n\}; h; \overline{\Omega_r^{ext}})$ and best approximation in its dual space $\mathcal{H}(\{A_n\}; h; \overline{\Omega_r^{ext}})^*$ are equivalent problems. The significance of this equivalence for our applications lies in the fact that it leads to a simple way of how to discretize convolutions $F *_{\mathcal{H}(\{A_n\}; h; \overline{\Omega_r^{ext}})} G$ of two functions $F,\ G \in \mathcal{H}(\{A_n\}; h; \overline{\Omega_r^{ext}})$. In our applications we are only concerned with bounded evaluation functionals on $\mathcal{H}(\{A_n\}; h; \overline{\Omega_r^{ext}})$. Given $N \in \mathbb{N}$ samples of (without loss of generality) $G \in \mathcal{H}(\{A_n\}; h; \overline{\Omega_r^{ext}})$ in a set of points $\{x_1^N, \ldots, x_N^N\} \subset \overline{\Omega_r^{ext}}$, the idea is to view G as the representer of the bounded linear functional $\mathcal{L} : \mathcal{H}(\{A_n\}; h; \overline{\Omega_r^{ext}}) \to \mathbb{R}$, $\mathcal{L}(F) := F *_{\mathcal{H}(\{A_n\}; h; \overline{\Omega_r^{ext}})} G$, and to replace it by its interpolating $\mathcal{H}(\{A_n\}; h; \overline{\Omega_r^{ext}})$-spline with respect to the data $\{(x_i, G(x_i)\}_{1\leq i\leq N}$. Section 4.3 deals with this aspect. In the presence of error-affected data we can still do the approximation of $F *_{\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})} G$ by replacing G by a smoothing spline.

In the special case that we are working in a reproducing kernel Hilbert space, bounded evaluation functionals have a simple representation by means of the reproducing kernel which also leads to a simple representation of the interpolating spline. Hence reproducing kernel Hilbert spaces are discussed in Section 4.4.

In this thesis we focus on evaluation functionals on $\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})$, which are bounded if the evaluation point lies in Ω_r^{ext}. The representers of such bounded evaluation functionals in $\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})$ are strongly space localizing and the theory of reproducing kernel Hilbert spaces yields an idea how they can be easily represented. This will be investigated in Section 4.5. For applicability in numerical computations we are here particularly interested in choosing the spaces $\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})$ in such a way that the representers of bounded evaluation functionals have a representation as elementary functions. We present three classes of spaces with this property and write down the representers of (bounded) evaluation functionals in these spaces.

Section 4.6 contains the convergence result for $\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})$-spline interpolation in the case that the samples of the function which has to be approximated are taken exclusively on a $C^{(2)}$-regular surface which is contained in Ω_r^{ext}. This is of course one of the crucial points for the construction of an approximate solution to the discrete SST-problem and SGG-problem posed as Problem 3.8 in Chapter 3.

The material presented in the first four Sections is a generalization of the results given in [Fr1999] for the spaces $\mathcal{H}(\{A_n\};\overline{\Omega_r^{ext}})$ to the spaces $\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})$. The parameter choice strategy in Section 4.2, as well as the variant of the convergence proofs in Section 4.6 are quoted from [He2002].

4.1 Spline Interpolation

Definition 4.1 *Let $\{A_n\}_{n\in\mathbb{N}_0} \subset \mathbb{R}_0^+$ be a sequence of non-negative real numbers, let $r \in \mathbb{R}^+$ and $h \in \{0,1,2\}$ and suppose that $\{\mathcal{L}_1^N, \ldots, \mathcal{L}_N^N\} \subset \mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})^*$ is a set of N linearly independent bounded linear functionals on $\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})$. Denote the representer of $\mathcal{L}_i^N$ according to the Riesz representation theorem by L_i^N, $i = 1,\ldots,N$, i.e.,*

$$\mathcal{L}_i^N F = (F, L_i^N)_{\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})} = F *_{\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})} L_i^N \text{ for all } F \in \mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}}).$$

Then any function of the form

$$S := \sum_{i=1}^{N} a_i^N L_i^N$$

is called an $\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})$*-spline function relative to* $\mathcal{L}_1^N,\ldots,\mathcal{L}_N^N$*. The scalars* $a_1^N,\ldots,a_N^N$ *are called the coefficients of the* $\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})$*-spline. The space of all* $\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})$*-splines relative to* $\mathcal{L}_1^N,\ldots,\mathcal{L}_N^N$ *is denoted by*

$$\mathcal{S}_{\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})}(\mathcal{L}_1^N,\ldots,\mathcal{L}_N^N) := \mathrm{span}_{i=1,\ldots,N}\{L_i^N\}.$$

Lemma 4.2 *Let* $\{A_n\}_{n\in\mathbb{N}_0}\subset\mathbb{R}_0^+$ *be a sequence of non-negative real numbers, let* $r\in\mathbb{R}^+$ *and* $h\in\{0,1,2\}$*. Assume that* $\{\mathcal{L}_1^N,\ldots,\mathcal{L}_N^N\}\subset\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})^*$ *is a set of* N *linearly independent bounded linear functionals on* $\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})$ *and denote their representers according to the Riesz representation theorem by* $L_1^N,\ldots,L_N^N\in\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})$*. Then, for every* $\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})$*-spline* $S\in\mathcal{S}_{\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})}(\mathcal{L}_1^N,\ldots,\mathcal{L}_N^N)$*,* $S=\sum\limits_{i=1}^N a_i^N L_i^N$*, relative to* $\mathcal{L}_1^N,\ldots,\mathcal{L}_N^N$ *and for every* $G\in\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})$

$$(G,S)_{\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})} = G *_{\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})} S = \sum_{i=1}^N a_i^N \mathcal{L}_i^N G.$$

Proof: The assertion follows immediately from the fact that $S=\sum\limits_{i=1}^N a_i^N L_i^N$ and that $L_i^N,\ 1\le i\le N$, are the representers of $\mathcal{L}_i^N,\ 1\le i\le N$ in $\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})$. □

The $\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})$-Spline Interpolation Problem Relative to a Set of Linearly Independent Bounded Linear Functionals $\mathcal{L}_1^N,\ldots,\mathcal{L}_N^N$

We now turn our attention to the mathematical properties of $\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})$-spline interpolation. Using the same notations and assumptions as in Definition 4.1, we let

$$\mathcal{I}^F_{\mathcal{L}_1^N,\ldots,\mathcal{L}_N^N} := \left\{G\in\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})\,|\,\mathcal{L}_i^N G=\mathcal{L}_i^N F \text{ for all } i=1,\ldots,N\right\}$$

denote the space of all interpolating functions in $\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})$ for $F\in\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})$ with respect to $\mathcal{L}_1^N,\ldots,\mathcal{L}_N^N$, and start with

Theorem 4.3 *Let* $\{A_n\}_{n\in\mathbb{N}_0}\subset\mathbb{R}_0^+$ *be a sequence of non-negative real numbers, let* $r\in\mathbb{R}^+$ *and* $h\in\{0,1,2\}$*. Assume that* $\{\mathcal{L}_1^N,\ldots,\mathcal{L}_N^N\}\subset\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})^*$ *is a set of* N *linearly independent bounded linear functionals on* $\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})$ *and denote their representers according to the Riesz representation theorem by* $L_1^N,\ldots,L_N^N\in\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})$*.*
Then, for each $F\in\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})$ *there exists one and only one interpolating* $\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})$*-spline* $S^F_{\mathcal{L}_1^N,\ldots,\mathcal{L}_N^N}\in\mathcal{S}_{\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})}(\mathcal{L}_1^N,\ldots,\mathcal{L}_N^N)\cap\mathcal{I}^F_{\mathcal{L}_1^N,\ldots,\mathcal{L}_N^N}$*.*
$S^F_{\mathcal{L}_1^N,\ldots,\mathcal{L}_N^N}$ *has the representation*

$$S^F_{\mathcal{L}_1^N,\ldots,\mathcal{L}_N^N} = \sum_{i=1}^N a_i^N L_i^N,$$

where the coefficients $a_1^N, \ldots, a_N^N$ *are uniquely determined by the linear system of equations*

$$\sum_{i=1}^{N} a_i^N (L_i^N, L_k^N)_{\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})} = \mathcal{L}_k^N F \text{ for } k = 1, \ldots, N. \tag{4.1}$$

Furthermore, the interpolating $\mathcal{H}(\{A_n\}; h; \overline{\Omega_r^{ext}})$*-spline* $S^F_{\mathcal{L}_1^N,\ldots,\mathcal{L}_N^N}$ *has the following properties:*

(i) $S^F_{\mathcal{L}_1^N,\ldots,\mathcal{L}_N^N}$ *is the* $\mathcal{H}(\{A_n\}; h; \overline{\Omega_r^{ext}})$*-orthogonal projection of* F *onto the space* $\mathcal{S}_{\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})}(\mathcal{L}_1^N, \ldots, \mathcal{L}_N^N)$.

(ii) $S^F_{\mathcal{L}_1^N,\ldots,\mathcal{L}_N^N}$ *is the interpolant in* $\mathcal{H}(\{A_n\}; h; \overline{\Omega_r^{ext}})$ *with minimum norm, i.e.*

$$\|G\|^2_{\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})} = \|S^F_{\mathcal{L}_1^N,\ldots,\mathcal{L}_N^N}\|^2_{\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})} + \|G - S^F_{\mathcal{L}_1^N,\ldots,\mathcal{L}_N^N}\|^2_{\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})}$$

for all $G \in \mathcal{I}^F_{\mathcal{L}_1^N,\ldots,\mathcal{L}_N^N}$ *(first minimum property).*

(iii) If $S \in \mathcal{S}_{\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})}(\mathcal{L}_1^N, \ldots, \mathcal{L}_N^N)$ *and* $G \in \mathcal{I}^F_{\mathcal{L}_1^N,\ldots,\mathcal{L}_N^N}$*, then*

$$\|S - G\|^2_{\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})} = \|S^F_{\mathcal{L}_1^N,\ldots,\mathcal{L}_N^N} - G\|^2_{\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})} + \|S - S^F_{\mathcal{L}_1^N,\ldots,\mathcal{L}_N^N}\|^2_{\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})}$$

(second minimum property).

Proof: Let $F \in \mathcal{H}(\{A_n\}; h; \overline{\Omega_r^{ext}})$. Its interpolating $\mathcal{H}(\{A_n\}; h; \overline{\Omega_r^{ext}})$-spline $S^F_{\mathcal{L}_1^N,\ldots,\mathcal{L}_N^N}$ has to satisfy the conditions

$$\mathcal{L}_k^N S^F_{\mathcal{L}_1^N,\ldots,\mathcal{L}_N^N} = \sum_{i=1}^{N} a_i^N \mathcal{L}_k^N L_i^N = \sum_{i=1}^{N} a_i^N (L_i^N, L_k^N)_{\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})} \stackrel{!}{=} \mathcal{L}_k^N F$$

for $k = 1, \ldots, N$, which leads to the linear system of equations (4.1). $(L_i, L_k)_{1 \le i,l \le N}$ is the Gram matrix of the basis $\{L_i^N\}$ of the spline space $\mathcal{S}_{\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})}(\mathcal{L}_1^N, \ldots, \mathcal{L}_N^N)$ and thus the linear equation system (4.1) is uniquely solvable. The $\mathcal{H}(\{A_n\}; h; \overline{\Omega_r^{ext}})$-orthogonal projector $P : \mathcal{H}(\{A_n\}; h; \overline{\Omega_r^{ext}}) \to \mathcal{S}_{\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})}(\mathcal{L}_1^N, \ldots, \mathcal{L}_N^N)$ is the projection operator onto $\mathcal{S}_{\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})}(\mathcal{L}_1^N, \ldots, \mathcal{L}_N^N)$ which additionally satisfies for every $F \in \mathcal{H}(\{A_n\}; h; \overline{\Omega_r^{ext}})$

$$(PF, S)_{\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})} = (F, S)_{\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})} \text{ for all } S \in \mathcal{S}_{\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})}(\mathcal{L}_1^N, \ldots, \mathcal{L}_N^N),$$

and in particular

$$(PF, L_i^N)_{\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})} = (F, L_i^N)_{\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})} \text{ for } i = 1, \ldots, N,$$

i.e., $\mathcal{L}_i^N(PF) = \mathcal{L}_i^N(F)$ for $i = 1, \ldots, N$. But these are just the interpolation equations stated above, which are uniquely solvable, and therefore $PF = S^F_{\mathcal{L}_1^N,\ldots,\mathcal{L}_N^N}$, which proves assertion (i).

(ii) is a special case of (iii), which is obtained if we let in (iii) $S = 0$. In order to prove (iii), let $S \in \mathcal{S}_{\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})}(\mathcal{L}_1^N, \ldots, \mathcal{L}_N^N)$ and $G \in \mathcal{I}^F_{\mathcal{L}_1^N,\ldots,\mathcal{L}_N^N}$. Then

$$\begin{aligned}\|S - G\|^2_{\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})} &= \|S - S^F_{\mathcal{L}_1^N,\ldots,\mathcal{L}_N^N} + S^F_{\mathcal{L}_1^N,\ldots,\mathcal{L}_N^N} - G\|^2_{\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})} \\ &= \|S^F_{\mathcal{L}_1^N,\ldots,\mathcal{L}_N^N} - G\|^2_{\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})} + \|S - S^F_{\mathcal{L}_1^N,\ldots,\mathcal{L}_N^N}\|^2_{\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})} \\ &+ 2(S^F_{\mathcal{L}_1^N,\ldots,\mathcal{L}_N^N} - G, S - S^F_{\mathcal{L}_1^N,\ldots,\mathcal{L}_N^N})_{\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})}.\end{aligned}$$

Since $G, S^F_{\mathcal{L}_1^N,\ldots,\mathcal{L}_N^N} \in \mathcal{I}^F_{\mathcal{L}_1^N,\ldots,\mathcal{L}_N^N}$, it holds that

$$\mathcal{L}_i^N(S^F_{\mathcal{L}_1^N,\ldots,\mathcal{L}_N^N} - G) = (S^F_{\mathcal{L}_1^N,\ldots,\mathcal{L}_N^N} - G, L_i^N)_{\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})} = 0$$

for $i = 1, \ldots, N$. But since $S - S^F_{\mathcal{L}_1^N,\ldots,\mathcal{L}_N^N} \in \mathcal{S}_{\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})}(\mathcal{L}_1^N, \ldots, \mathcal{L}_N^N)$, this implies that $(S^F_{\mathcal{L}_1^N,\ldots,\mathcal{L}_N^N} - G, S - S^F_{\mathcal{L}_1^N,\ldots,\mathcal{L}_N^N})_{\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})} = 0$. □

It should be noted that the demand of linear independence of the bounded linear functionals guarantees that the matrix in (4.1) is invertible. In case that $\mathcal{L}_1^N, \ldots, \mathcal{L}_N^N$ are not linearly independent there exists more than one set of spline coefficients which solve (4.1), but the interpolating spline is still uniquely determined because it is the orthogonal projection onto the spline space.

Theorem 4.4 *Let $\{A_n\}_{n\in\mathbb{N}_0} \subset \mathbb{R}_0^+$ be a sequence of non-negative real numbers, $h \in \{0, 1, 2\}$ and $F \in \mathcal{H}(\{A_n\}; h; \overline{\Omega_r^{ext}})$. Suppose that $\{\mathcal{L}_1^N, \ldots, \mathcal{L}_N^N\}_{N\in\mathbb{N}}$ is a hierarchical sequence of sets of linearly independent bounded linear functionals on $\mathcal{H}(\{A_n\}; h; \overline{\Omega_r^{ext}})$, i.e. $\mathcal{L}_i^N = \mathcal{L}_i^M$ for $i = 1, \ldots, N$ and for all $N \leq M$, such that* $\operatorname{span}\{\mathcal{L}_1^1, \mathcal{L}_1^2, \mathcal{L}_2^2, \mathcal{L}_1^3, \ldots\}$ *is dense in the dual space $\mathcal{H}(\{A_n\}; h; \overline{\Omega_r^{ext}})^*$ of $\mathcal{H}(\{A_n\}; h; \overline{\Omega_r^{ext}})$. Then*

$$\lim_{N\to\infty} \|F - S^F_{\mathcal{L}_1^N,\ldots,\mathcal{L}_N^N}\|_{\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})} = 0.$$

Proof: For $N \in \mathbb{N}$ and $i = 1, \ldots, N$, let L_i^N be the representer of $\mathcal{L}_i^N$ in $\mathcal{H}(\{A_n\}; h; \overline{\Omega_r^{ext}})$ according to the Riesz representation theorem. Let $L \in \mathcal{H}(\{A_n\}; h; \overline{\Omega_r^{ext}})$. Due to the Riesz representation theorem there exists one and only one $\mathcal{L} \in \mathcal{H}(\{A_n\}; h; \overline{\Omega_r^{ext}})^*$ such that $\mathcal{L}F = (F, L)_{\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})}$ for all $F \in \mathcal{H}(\{A_n\}; h; \overline{\Omega_r^{ext}})$. Since $\operatorname{span}\{\mathcal{L}_1^2, \mathcal{L}_1^2, \mathcal{L}_2^2, \mathcal{L}_1^3, \ldots\}$ is dense in $\mathcal{H}(\{A_n\}; h; \overline{\Omega_r^{ext}})^*$, given $\varepsilon > 0$, there exists an element $\mathcal{J} \in \operatorname{span}\{\mathcal{L}_1^1, \mathcal{L}_1^2, \mathcal{L}_2^2, \mathcal{L}_1^3, \ldots\}$ such that $\|\mathcal{J} - \mathcal{L}\|_{\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})^*} \leq \varepsilon$.

Let $J \in \mathcal{H}(\{A_n\}; h; \overline{\Omega_r^{ext}})$ be the representer of $\mathcal{J}$ in $\mathcal{H}(\{A_n\}; h; \overline{\Omega_r^{ext}})$. Then $\|J - L\|_{\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})} = \|\mathcal{J} - \mathcal{L}\|_{\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})^*} \leq \varepsilon$ and thus $\operatorname{span}\{L_1^1, L_1^2, L_2^2, L_1^3, \ldots\}$ is dense in $\mathcal{H}(\{A_n\}; h; \overline{\Omega_r^{ext}})$.

Using the Gram-Schmidt orthonormalization procedure, we may successively construct an orthonormal system $\{\tilde{L}_1, \tilde{L}_2, \ldots\}$ in $\operatorname{span}\{L_1^N, \ldots, L_N^N, \ldots\}$ such that $\operatorname{span}\{\tilde{L}_1, \ldots, \tilde{L}_N\} = \operatorname{span}\{\mathcal{L}_1^N, \ldots, \mathcal{L}_N^N\}$ for all $N \in \mathbb{N}$. Let $F \in \mathcal{H}(\{A_n\}; h; \overline{\Omega_r^{ext}})$ be such that $(F, \tilde{L}_i)_{\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})} = 0$ for all $i \in \mathbb{N}$. Then $F = 0$, because $\operatorname{span}\{\tilde{L}_1, \tilde{L}_2, \ldots\}$ is dense in $\mathcal{H}(\{A_n\}; h; \overline{\Omega_r^{ext}})$. But this implies that $\{\tilde{L}_1, \tilde{L}_2, \ldots\}$ is a complete orthonormal system for $\mathcal{H}(\{A_n\}; h; \overline{\Omega_r^{ext}})$. Since the $\mathcal{H}(\{A_n\}; h; \overline{\Omega_r^{ext}})$-spline interpolant $S^F_{\mathcal{L}_1^N,\ldots,\mathcal{L}_N^N}$ is the orthogonal projection onto $\operatorname{span}\{L_1^N, \ldots, L_N^N\} = \operatorname{span}\{\tilde{L}_1, \ldots, \tilde{L}_N\}$, it may be represented by the truncated Fourier series of F with respect to the complete orthonormal system $\{\tilde{L}_1, \tilde{L}_2, \ldots\}$, i.e.,

$$S^F_{\mathcal{L}_1^N,\ldots,\mathcal{L}_N^N} = \sum_{i=1}^{N} (F, \tilde{L}_i)_{\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})} \tilde{L}_i,$$

and hence $\lim_{N\to\infty} S^F_{\mathcal{L}_1^N,\ldots,\mathcal{L}_N^N} = F$ in $\mathcal{H}(\{A_n\}; h; \overline{\Omega_r^{ext}})$-sense. □

4.2 Spline Smoothing

In the presence of error-affected data it is advisable to perform an $\mathcal{H}(\{A_n\}; h; \overline{\Omega_r^{ext}})$-spline smoothing ($\mathcal{H}(\{A_n\}; h; \overline{\Omega_r^{ext}})$-spline approximation) for $F \in \mathcal{H}(\{A_n\}; h; \overline{\Omega_r^{ext}})$ instead of an interpolation. More precisely, let $\mathcal{L}_i^N : \mathcal{H}(\{A_n\}; h; \overline{\Omega_r^{ext}}) \to \mathbb{R}$, $i = 1, \ldots, N$, be N linearly independent bounded linear functionals, and assume that the data $\mathcal{L}_i^N F + \varepsilon_i$, $i = 1, \ldots, N$, of $F \in \mathcal{H}(\{A_n\}; h; \overline{\Omega_r^{ext}})$ is known, where $\varepsilon_1, \ldots, \varepsilon_N$ are measurement errors. Then we compute an $\mathcal{H}(\{A_n\}; h; \overline{\Omega_r^{ext}})$-spline $S \in \mathcal{S}_{\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})}(\mathcal{L}_1^N, \ldots, \mathcal{L}_N^N)$, which satisfies

$$\mathcal{L}_i^N S \approx \mathcal{L}_i^N F + \varepsilon_i \quad \text{for all} \quad i = 1, \ldots, N,$$

instead of $\mathcal{L}_i^N S = \mathcal{L}_i^N F + \varepsilon_i$, $i = 1, \ldots, N$, but is smoother than the interpolating spline. This situation is considered in the next theorem.

Theorem 4.5 *Let $\{A_n\}_{n\in\mathbb{N}_0} \subset \mathbb{R}_0^+$ be a sequence of non-negative real numbers, let $h \in \{0, 1, 2\}$, and assume that $\mathcal{L}_1^N, \ldots, \mathcal{L}_N^N \in \mathcal{H}(\{A_n\}; h; \overline{\Omega_r^{ext}})^*$ are N linearly independent bounded linear functionals. Denote the uniquely determined representer of $\mathcal{L}_i^N$ in $\mathcal{H}(\{A_n\}; h; \overline{\Omega_r^{ext}})$ by L_i^N. Let $F \in \mathcal{H}(\{A_n\}; h; \overline{\Omega_r^{ext}})$ and let $\tau_1^N, \ldots, \tau_N^N \in \mathbb{R}^+$ and $\lambda \in \mathbb{R}^+$ be positive real numbers. Then there exists one and only one $\mathcal{H}(\{A_n\}; h; \overline{\Omega_r^{ext}})$-spline*

$$S = \sum_{i=1}^{N} a_i^N L_i^N$$

relative to $\mathcal{L}_1^N, \ldots, \mathcal{L}_N^N$ with coefficient vector $a^N = (a_1^N, \ldots, a_N^N)^T$ which minimizes the functional

$$\mu(a^N) = \mu(a_1^N, \ldots, a_N^N) := \sum_{i=1}^{N} \left(\frac{\mathcal{L}_i^N S - \mathcal{L}_i^N F}{\tau_i^N} \right)^2 + \lambda \|S\|^2_{\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})}. \tag{4.2}$$

The coefficient vector a^N *of this minimizing* $\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})$*-spline is the uniquely determined solution of the linear equation system*

$$\sum_{i=1}^{N} a_i^N (L_i^N, L_k^N)_{\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})} + \lambda(\tau_k^N)^2 a_k^N = \mathcal{L}_k^N \quad \textit{for} \quad k = 1, \ldots, N. \tag{4.3}$$

Proof: The proof is technical and can for instance be found in [He2002]. □.

The parameters $\tau_1^N, \ldots, \tau_N^N$ are weight factors for the measurements $\mathcal{L}_i^N F$, $i = 1, \ldots, N$. If we want to treat all measurements in an equal way, we choose $\tau_1 = \ldots = \tau_N =: \tau$ with some $\tau > 0$, and multiplication of the functional (4.2) with τ^2 yields

$$\tau^2 \mu(a^N) = \sum_{i=1}^{N} (\mathcal{L}_i^N S - \mathcal{L}_i^N F)^2 + \tau^2 \lambda \|S\|^2_{\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})}. \tag{4.4}$$

As the minimization of (4.4) for $\tau := \tilde{\tau} \in \mathbb{R}^+$ and $\lambda := \tilde{\lambda} \in \mathbb{R}^+$ yields the same spline as the minimization of (4.4) for $\tau := 1$ and $\lambda := \tilde{\tau}^2 \tilde{\lambda}$, we may simply assume $\tau = 1$ in case $\tau_1^N = \ldots = \tau_N^N$.

The smoothing parameter λ in the functional (where we set $\tau_1 = \ldots = \tau_N = \tau := 1$)

$$\mu(a^N) = \mu(a_1^N, \ldots, a_N^N) := \sum_{i=1}^{N} \left(\mathcal{L}_i^N S - \mathcal{L}_i^N F\right)^2 + \lambda \|S\|^2_{\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})}. \tag{4.5}$$

(and more general in the functional (4.2)) determines the weighting between interpolation and smoothness of the $\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})$-spline which minimizes the functional. For $\lambda = 0$ this minimizing $\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})$-spline S is just the interpolating $\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})$-spline of F with respect to $\mathcal{L}_1^N, \ldots, \mathcal{L}_N^N$. The larger the smoothing parameter λ, the more weight is put on the norm $\|S\|_{\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})}$ of S, i.e., the smoother is the spline S.

If we drop the demand that the N bounded linear functionals are linearly independent, the linear equation system (4.3) is still uniquely solvable in case of $\lambda > 0$. Note that the condition of the matrix in the linear equation system (4.3) becomes the better the larger we choose $\lambda > 0$. Therefore, spline smoothing can be used to stabilize the linear equation system in the case that the matrix of the corresponding $\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})$-spline interpolation problem is extremely ill-conditioned.

We now turn to the question of the choice of the smoothing parameter λ (and the parameters $\tau_1, \ldots, \tau_N$) if $\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})$-spline smoothing is used in the presence of noisy data $\mathcal{L}_i^N F + \varepsilon_i$, $i = 1, \ldots, N$. Here we restrict ourselves to the case $\tau_1 = \ldots = \tau_N = \tau := 1$. We assume that we have some a priori information on the variance σ^2 of the noise. The next theorem gives a criterion of how to choose a suitable smoothing parameter $\lambda = \lambda(\tilde{\sigma})$ in dependence of the empirical standard deviation $\tilde{\sigma}$.

Theorem 4.6 *Let* $\{A_n\}_{n\in\mathbb{N}_0} \subset \mathbb{R}_0^+$ *be a sequence of non-negative real numbers, let* $h \in \{0.1.2\}$*, and assume that* $\mathcal{L}_1^N, \ldots, \mathcal{L}_N^N \subset \mathcal{H}(\{A_n\}; h; \overline{\Omega_r^{ext}})^*$ *are* N *linearly independent bounded linear functionals. Denote the uniquely determined representer of* $\mathcal{L}_i^N$ *by* L_i^N*,* $i = 1, \ldots, N$*. Suppose* $F \in \mathcal{H}(\{A_n\}; h; \overline{\Omega_r^{ext}})$*, and let*

$$S_0^F = \sum_{i=1}^{N} a_0^N L_i^N := S^F_{\mathcal{L}_1^N, \ldots, \mathcal{L}_N^N}$$

be the uniquely determined interpolating spline of F *relative to the bounded linear functionals* $\mathcal{L}_1^N, \ldots, \mathcal{L}_N^N$*. Let* $\mathcal{E}_1, \ldots, \mathcal{E}_N$ *be continuous random variables with expectation value* $\mathrm{E}[\mathcal{E}_i] = 0$ *for all* $i = 1, \ldots, N$ *and identical variance* $\mathrm{E}[\varepsilon_i^2] = \sigma^2$ *for all* $i = 1, \ldots, N$*. Suppose* $\varepsilon_i \in \mathbb{R}$ *is a value of the random variable* $\mathcal{E}_i$*,* $i = 1, \ldots, N$*. If* N *is large, the variance* σ^2 *is approximately equal to the empirical variance* $\tilde{\sigma}^2$*, defined by*

$$\tilde{\sigma}^2 := \frac{1}{N-1} \sum_{i=1}^{M} \varepsilon_i^2,$$

with the empirical standard deviation $\tilde{\sigma}$*, given by* $\tilde{\sigma} := \sqrt{\tilde{\sigma}^2}$*. Let*

$$S^F_{\lambda,\tilde{\sigma}} = \sum_{i=1}^{N} (a^N_{\lambda,\tilde{\sigma}})_i L_i^N$$

denote the spline in $\mathcal{S}_{\mathcal{H}(\{A_n\}; h; \overline{\Omega_r^{ext}})}(\mathcal{L}_1^N, \ldots, \mathcal{L}_N^N)$ *which minimizes the linear functional*

$$\mu_{\lambda,\tilde{\sigma}}(a^N) := \sum_{i=1}^{N} \left(\mathcal{L}_i^N S - (\mathcal{L}_i^N F + \varepsilon_i)\right)^2 + \lambda \|S\|^2_{\mathcal{H}(\{A_n\}; h; \overline{\Omega_r^{ext}})}, \quad \textit{where} \quad S = \sum_{i=1}^{N} a_i^N L_i^N.$$

If $(\mathcal{L}_1^N F + \varepsilon_1, \ldots, \mathcal{L}_N^N F + \varepsilon_N) \neq 0$ *and* $\sqrt{N-1}\,\tilde{\sigma} < |(\mathcal{L}_1^N F + \varepsilon_1, \ldots, \mathcal{L}_N^N F + \varepsilon_N)|$*, there exists one and only one* $\lambda = \lambda(\tilde{\sigma}) \in \mathbb{R}^+$ *such that*

$$\frac{1}{N-1} \sum_{i=1}^{N} \left(\mathcal{L}_i^N S^F_{\lambda(\tilde{\sigma}),\tilde{\sigma}} - (\mathcal{L}_i^N F + \varepsilon_i)\right)^2 = \tilde{\sigma}^2,$$

and the following estimate is valid:

$$\|S_0^F - S^F_{\lambda(\tilde{\sigma}),\tilde{\sigma}}\|^2_{\mathcal{H}(\{A_n\}; h; \overline{\Omega_r^{ext}})} \leq 4\,|a_0^N|\,\sqrt{N-1}\,\tilde{\sigma}. \tag{4.6}$$

Proof: This theorem is proved in [He2002]. □

The estimate (4.6) implies convergence in the sense that if the empirical variance $\tilde{\sigma}^2$ tends to zero, then $\lim_{\tilde{\sigma}\to 0} S^F_{\lambda(\tilde{\sigma}),\tilde{\sigma}} = S_0^F$.

In practical applications the empirical variance $\tilde{\sigma}^2$ will not be available, but if a reasonable estimate $\hat{\sigma}^2$ of the variance σ^2 is known, then we try to find a smoothing parameter $\lambda = \lambda(\tilde{\sigma})$ such that

$$\frac{1}{N-1}\sum_{i=1}^{N}\left(\mathcal{L}_i^N S^F_{\lambda(\tilde{\sigma}),\tilde{\sigma}} - (\mathcal{L}_i^N F + \varepsilon_i)\right)^2 = \hat{\sigma}^2.$$

The calculations in the proof of Theorem 4.6 in [He2002] show that the function $\lambda \mapsto \sum_{i=1}^N \left(\mathcal{L}_i^N S^F_{\lambda(\tilde{\sigma}),\tilde{\sigma}} - (\mathcal{L}_i^N F + \varepsilon_i)\right)^2$ is strict monotonically increasing in $\lambda \in \mathbb{R}^+$ if $(\mathcal{L}_1^N F + \varepsilon_1, \ldots, \mathcal{L}_N^N F + \varepsilon_N) \neq 0$. Therefore, the value of $\lambda = \lambda(\tilde{\sigma})$ can numerically be easily searched.

As another method for the choice of the smoothing parameter we mention cross validation (CV) and generalized cross validation (GCV). For more details on scattered data interpolation and approximation, and parameter choice by cross validation and generalized cross validation, the reader is referred to [Wa1990].

4.3 Approximation of Bounded Linear Functionals

Theorem 4.7 *Let $\{A_n\}_{n\in\mathbb{N}_0} \subset \mathbb{R}_0^+$ be a sequence of non-negative real numbers and let $h \in \{0,1,2\}$. Suppose that $\mathcal{L} \in \mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})^*$ is a bounded linear functional on $\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})$ and $\mathcal{L}_1^N,\ldots,\mathcal{L}_N^N \in \mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})^*$ are $N \in \mathbb{N}$ linearly independent bounded linear functionals on $\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})$. Denote the representers (according to the Riesz representation theorem) of $\mathcal{L}$ and $\mathcal{L}_1^N,\ldots,\mathcal{L}_N^N$ by L and $L_1^N,\ldots,L_N^N$, respectively. The $\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})^*$- best approximation $\mathcal{J}^{\mathcal{L}}_{\mathcal{L}_1^N,\ldots,\mathcal{L}_N^N}$ of $\mathcal{L}$ in the space* $\mathrm{span}\{\mathcal{L}_1^N,\ldots,\mathcal{L}_N^N\}$, *i.e., $\mathcal{J}^{\mathcal{L}}_{\mathcal{L}_1^N,\ldots\mathcal{L}_N^N}$ satisfies $\mathcal{J}^{\mathcal{L}}_{\mathcal{L}_1^N,\ldots,\mathcal{L}_N^N} \in \mathrm{span}\{\mathcal{L}_1^N,\ldots,\mathcal{L}_N^N\}$ and*

$$\|\mathcal{J}^{\mathcal{L}}_{\mathcal{L}_1^N,\ldots,\mathcal{L}_N^N} - \mathcal{L}\|_{\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})^*} \leq \|\mathcal{J} - \mathcal{L}\|_{\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})^*}$$

for all $\mathcal{J} \in \mathrm{span}\{\mathcal{L}_1^N,\ldots\mathcal{L}_N^N\}$, is uniquely determined. $\mathcal{J}^{\mathcal{L}}_{\mathcal{L}_1^N,\ldots\mathcal{L}_N^N}$ has the representation

$$\mathcal{J}^{\mathcal{L}}_{\mathcal{L}_1^N,\ldots\mathcal{L}_N^N} = \sum_{i=1}^N a_i^N \mathcal{L}_i^N,$$

where the coefficients are the uniquely determined solution of the linear system of equations

$$(L, L_k^N)_{\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})} = \sum_{i=1}^N a_i^N (L_i^N, L_k^N)_{\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})} \text{ for } k = 1,\ldots,N. \tag{4.7}$$

Proof: Compute $\|\mathcal{J} - \mathcal{L}\|_{\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})^*}$ for some $\mathcal{J} = \sum_{i=1}^N a_i^N \mathcal{L}_i^N$ in $\mathrm{span}\{\mathcal{L}_1^N,\ldots,\mathcal{L}_N^N\}$. Due to the Riesz representation theorem,

$$\|\mathcal{J} - \mathcal{L}\|_{\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})^*} = \left\|\sum_{i=1}^N a_i^N L_i^N - L\right\|_{\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})}.$$

Since $\text{span}\{L_1^N, \ldots, L_N^N\}$ is a finite-dimensional subspace of $\mathcal{H}(\{A_n\}; h; \overline{\Omega_r^{ext}})$, L can be uniquely decomposed into $L = PL + (Id - P)L$, where P is the orthogonal projection operator onto $\text{span}\{L_1^N, \ldots, L_N^N\}$. Therefore $\|\mathcal{J} - \mathcal{L}\|_{\mathcal{H}(\{A_n\}; h; \overline{\Omega_r^{ext}})^*}$ becomes minimal if and only if $\mathcal{J}$ is the unique element in $\text{span}\{\mathcal{L}_1^N, \ldots, \mathcal{L}_N^N\}$, which is represented by PL. As an orthogonal projection operator P satisfies the condition $(PL, S)_{\mathcal{H}(\{A_n\}; h; \overline{\Omega_r^{ext}})} = (L, S)_{\mathcal{H}(\{A_n\}; h; \overline{\Omega_r^{ext}})}$ for all functions $S \in \text{span}\{L_1^N, \ldots, L_N^N\}$, which is the case if and only if

$$(PL, L_i^N)_{\mathcal{H}(\{A_n\}; h; \overline{\Omega_r^{ext}})} = (L, L_i^N)_{\mathcal{H}(\{A_n\}; h; \overline{\Omega_r^{ext}})} \text{ for } i = 1, \ldots, N.$$

But this leads to the linear equation system (4.7). □

The linear equation system (4.7) coincides with the linear equation system (4.1) determining the coefficients of the interpolating $\mathcal{H}(\{A_n\}; h; \overline{\Omega_r^{ext}})$-spline of L with respect to the bounded linear functionals $\mathcal{L}_1^N, \ldots, \mathcal{L}_N^N$. Thus $\mathcal{H}(\{A_n\}; h; \overline{\Omega_r^{ext}})$-spline interpolation and $\mathcal{H}(\{A_n\}; h; \overline{\Omega_r^{ext}})^*$-best approximation are equivalent problems.

Discretization of Convolutions

A special application of $\mathcal{H}(\{A_n\}; h; \overline{\Omega_r^{ext}})$-spline interpolation (or, equivalently, best approximation in $\mathcal{H}(\{A_n\}; h; \overline{\Omega_r^{ext}})^*$,) is the discretization of convolutions

$$F *_{\mathcal{H}(\{A_n\}; h; \overline{\Omega_r^{ext}})} G = (F, G)_{\mathcal{H}(\{A_n\}; h; \overline{\Omega_r^{ext}})} \text{ for } F, G \in \mathcal{H}(\{A_n\}; h; \overline{\Omega_r^{ext}}).$$

Let, without loss of generality, F be explictly known and assume that G is known via the discrete values $\mathcal{L}_1^N G, \ldots, \mathcal{L}_N^N G$, where $\mathcal{L}_1^N, \ldots, \mathcal{L}_N^N$ are linearly independent bounded linear functionals on $\mathcal{H}(\{A_n\}; h; \overline{\Omega_r^{ext}})$. We regard G as the representer of the bounded linear functional $\mathcal{L} : \mathcal{H}(\{A_n\}; h; \overline{\Omega_r^{ext}}) \to \mathbb{R}$, $F \mapsto F *_{\mathcal{H}(\{A_n\}; h; \overline{\Omega_r^{ext}})} G$ (according to the Riesz representation theorem) and approximate G by its spline interpolant $S^G_{\mathcal{L}_1^N, \ldots, \mathcal{L}_N^N}$ relative to $\mathcal{L}_1^N, \ldots, \mathcal{L}_N^N$,

$$S^G_{\mathcal{L}_1^N, \ldots, \mathcal{L}_N^N} = \sum_{i=1}^{N} a_i^N L_i^N,$$

where the coefficients are uniquely determined by the linear equation system

$$(G, L_k^N)_{\mathcal{H}(\{A_n\}; h; \overline{\Omega_r^{ext}})} = \sum_{i=1}^{N} a_i^N (L_i^N, L_k^N)_{\mathcal{H}(\{A_n\}; h; \overline{\Omega_r^{ext}})}, \; k = 1, \ldots, N.$$

Substitution of G by $S^G_{\mathcal{L}_1^N, \ldots, \mathcal{L}_N^N}$ then yields the approximation formula

$$F *_{\mathcal{H}(\{A_n\}; h; \overline{\Omega_r^{ext}})} G \approx F *_{\mathcal{H}(\{A_n\}; h; \overline{\Omega_r^{ext}})} S^G_{\mathcal{L}_1^N, \ldots, \mathcal{L}_N^N} = \sum_{i=1}^{N} a_i^N \mathcal{L}_i^N F. \tag{4.8}$$

We now turn to the question of convergence of the numerical discretization rule (4.8).

Theorem 4.8 *Let* $\{A_n\}_{n\in\mathbb{N}_0} \subset \mathbb{R}_0^+$ *be a sequence of non-negative real numbers, let* $r \in \mathbb{R}^+$, $h \in \{0,1,2\}$ *and let* $F, G \in \mathcal{H}(\{A_n\}; h; \overline{\Omega_r^{ext}})$. *Assume that* $\{\mathcal{L}_1^N, \ldots, \mathcal{L}_N^N\}_{N\in\mathbb{N}} \subset \mathcal{H}(\{A_n\}; h; \overline{\Omega_r^{ext}})^*$ *is a sequence of sets of* N *linearly independent bounded linear functionals on* $\mathcal{H}(\{A_n\}; h; \overline{\Omega_r^{ext}})$*, which is hierarchical, i.e.,* $\mathcal{L}_i^N = \mathcal{L}_i^M$ *for* $i = 1, \ldots, N$ *and for all* $N \leq M$. *Furthermore, assume that* $\operatorname{span}\{\mathcal{L}_1^1, \mathcal{L}_1^2, \mathcal{L}_2^2, \mathcal{L}_1^3, \ldots\}$ *is dense in* $\mathcal{H}(\{A_n\}; h; \overline{\Omega_r^{ext}})^*$. *Denote the representer of* $\mathcal{L}_i^N$ *by* L_i^N, $i = 1, \ldots, N$, *and the* $\mathcal{H}(\{A_n\}; h; \overline{\Omega_r^{ext}})$*-spline interpolant of* G *relative to* $\mathcal{L}_1^N, \ldots, \mathcal{L}_N^N$ *by* $S^G_{\mathcal{L}_1^N,\ldots,\mathcal{L}_N^N}$,

$$S^G_{\mathcal{L}_1^N,\ldots,\mathcal{L}_N^N} = \sum_{i=1}^N a_i^N L_i^N.$$

Then the numerical rule (4.8) is exact in the limit $N \to \infty$*, i.e.*

$$\lim_{N\to\infty} |(F,G)_{\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})} - \sum_{i=1}^N a_i^N \mathcal{L}_i^N F| = 0.$$

Proof:

$$\begin{aligned} |(F,G)_{\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})} &- \sum_{i=1}^N a_i^N (F, L_i^N)_{\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})}| \\ &= |(F, G - S^G_{\mathcal{L}_1^N,\ldots,\mathcal{L}_N^N})_{\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})}| \\ &\leq \|F\|_{\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})} \|G - S^G_{\mathcal{L}_1^N,\ldots,\mathcal{L}_N^N}\|_{\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})}. \end{aligned}$$

Theorem 4.4 implies that

$$\lim_{N\to\infty} \|G - S^G_{\mathcal{L}_1^N,\ldots,\mathcal{L}_N^N}\|_{\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})} = 0,$$

which yields the desired result. □

Theorem 4.7 and Theorem 4.8 are rather abstract in so far, as they do not yield any information what kind of hierarchical systems $\{\mathcal{L}_1^N, \ldots, \mathcal{L}_N^N\}_{N\in\mathbb{N}}$ satisfy the property $\overline{\operatorname{span}\{\mathcal{L}_1^1, \mathcal{L}_1^2, \mathcal{L}_2^2, \ldots\}}^{\|\cdot\|_{\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})}} = \mathcal{H}(\{A_n\}; h; \overline{\Omega_r^{ext}})$. In Section 4.6 we will meet concrete sets of evaluation functionals which have exactly this property.

4.4 Reproducing Kernel Hilbert Spaces

We will now turn our attention to the special case of $\mathcal{H}(\{A_n\}; h; \overline{\Omega_r^{ext}})$-spline interpolation when $\mathcal{H}(\{A_n\}; h; \overline{\Omega_r^{ext}})$ is a reproducing kernel Hilbert space. In this case all evaluation functionals are bounded, and what is even more important, their representers (according to the Riesz representation theorem) have a simple representation in terms of the reproducing kernel. We give a summary of the well-known properties of reproducing kernel Hilbert spaces (for a general treatment of reproducing kernel Hilbert spaces, see e.g. the fundamental article by Aronszajn [Ar1950]):

Definition 4.9 *Let $r \in \mathbb{R}^+$. A separable Hilbert space $\mathcal{H}$ of real-valued functions on $\overline{\Omega_r^{ext}}$ with inner product $(\cdot,\cdot)_{\mathcal{H}}$ is called a reproducing kernel Hilbert space if there exists a kernel-function $K_{\mathcal{H}} : \overline{\Omega_r^{ext}} \times \overline{\Omega_r^{ext}} \to \mathbb{R}$ with the properties:*
(i) $K_{\mathcal{H}}(x,\cdot) \in \mathcal{H}$ for every fixed $x \in \overline{\Omega_r^{ext}}$ and all $F \in \mathcal{H}$,
(ii) $K_{\mathcal{H}}$ satisfies the reproducing property $(F, K_{\mathcal{H}}(x,\cdot))_{\mathcal{H}} = F(x)$ for all $x \in \overline{\Omega_r^{ext}}$.

The next lemma gives a necessary and sufficient condition under which $\mathcal{H}$ is a reproducing kernel Hilbert space.

Lemma 4.10 *Let $r \in \mathbb{R}^+$. A separable Hilbert space $\mathcal{H}$ of functions on $\overline{\Omega_r^{ext}}$ with inner product $(\cdot,\cdot)_{\mathcal{H}}$ is a reproducing kernel Hilbert space if and only if the evaluation functionals $\mathcal{L}_x : \mathcal{H} \to \mathbb{R},\ F \mapsto F(x)$, are bounded for all $x \in \overline{\Omega_r^{ext}}$.*

Proof: $\Rightarrow$: Assume that $(\mathcal{H}, (\cdot,\cdot)_{\mathcal{H}})$ is a reproducing kernel Hilbert space with reproducing kernel $K_{\mathcal{H}}$. Let $x \in \overline{\Omega_r^{ext}}$ and $\mathcal{L}_x : F \mapsto F(x)$. Then $K_{\mathcal{H}}(x,\cdot) \in \mathcal{H}$ and $(F, K_{\mathcal{H}}(x,\cdot))_{\mathcal{H}} = F(x)$. Consequently,

$$|\mathcal{L}_x F| = |(F, K_{\mathcal{H}}(x,\cdot))_{\mathcal{H}}| \leq \|F\|_{\mathcal{H}} \|K_{\mathcal{H}}(x,\cdot)\|_{\mathcal{H}},$$

and thus $\mathcal{L}_x$ is bounded.
$\Leftarrow$: Assume that for all $x \in \overline{\Omega_r^{ext}}$ the evaluation functional $\mathcal{L}_x$ is bounded. According to the Riesz representation theorem there exists one and only one $L_x \in \mathcal{H}$ such that $\mathcal{L}_x F = (F, L_x)_{\mathcal{H}}$ for all $F \in \mathcal{H}$. Define $K_{\mathcal{H}}(x,\cdot) := L_x$ for $x \in \overline{\Omega_r^{ext}}$. Then $K_{\mathcal{H}}$ fulfills the properties of a reproducing kernel, and thus $\mathcal{H}$ is a reproducing kernel Hilbert space. □

Lemma 4.11 *Let $(\mathcal{H}, (\cdot,\cdot)_{\mathcal{H}})$ be a reproducing kernel Hilbert space of real-valued functions defined on $\overline{\Omega_r^{ext}}$. Then the reproducing kernel is uniquely determined and has the representation*

$$K_{\mathcal{H}}(x,y) = \sum_{n\in\mathbb{N}_0} H_n(x) H_n(y) \tag{4.9}$$

for each complete orthonormal system $\{H_n\}_{n\in\mathbb{N}_0}$ in $\mathcal{H}$.

Proof: Uniqueness: Assume that $K_{\mathcal{H}}$ and $\tilde{K}_{\mathcal{H}}$ are two reproducing kernels for $\mathcal{H}$. By the reproducing property,

$$\begin{aligned} & (K_{\mathcal{H}}(x,\cdot), F)_{\mathcal{H}} = (\tilde{K}_{\mathcal{H}}(x,\cdot), F)_{\mathcal{H}} \text{ for all } F \in \mathcal{H} \\ \Leftrightarrow\ & (K_{\mathcal{H}}(x,\cdot) - \tilde{K}_{\mathcal{H}}(x,\cdot), F)_{\mathcal{H}} = 0 \text{ for all } F \in \mathcal{H}. \end{aligned}$$

Hence, $K_{\mathcal{H}}(x,\cdot) = \tilde{K}_{\mathcal{H}}(x,\cdot)$ in $\mathcal{H}$-sense for all $x \in \overline{\Omega_r^{ext}}$ and therefore $K_{\mathcal{H}} = \tilde{K}_{\mathcal{H}}$.
In order to prove the representation of $K_{\mathcal{H}}$, we let $x \in \overline{\Omega_r^{ext}}$ be fixed and calculate the coefficients of $K_{\mathcal{H}}(x,\cdot)$ with respect to the complete orthonormal system $\{H_n\}_{n\in\mathbb{N}_0}$. By definition of the reproducing kernel,

$$(K_{\mathcal{H}}(x,\cdot), H_n)_{\mathcal{H}} = H_n(x)$$

for $n \in \mathbb{N}_0$. Hence, the representation (4.9) is valid. □

Lemma 4.12 *Let $r \in \mathbb{R}^+$, let $h \in \{0,1,2\}$ and let $\{A_n\}_{n\in\mathbb{N}_0} \subset \mathbb{R}_0^+$ be a sequence of non-negative real numbers. $\mathcal{H}(\{A_n\}; h; \overline{\Omega_r^{ext}})$ is a reproducing kernel Hilbert space if and only if $\{A_n\}_{n\in\mathbb{N}_0}$ is summable. The corresponding reproducing kernel is given by*

$$K_{\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})}(x,y) = \sum_{n\in\mathcal{N}} \sum_{k=1}^{2n+1} \frac{1}{A_n} H_{n,k}(r;h;x) \frac{1}{A_n} H_{n,k}(r;h;y), \tag{4.10}$$

where $x,\ y \in \overline{\Omega_r^{ext}}$.

Proof: $\Leftarrow$: Due to the addition theorem and the summability of $\{A_n\}_{n\in\mathbb{N}_0}$, we have

$$\begin{aligned}\|K_{\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})}(x,\cdot)\|^2_{\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})} &= \sum_{n\in\mathcal{N}} \sum_{k=1}^{2n+1} A_n^2 \left(\frac{1}{A_n^2} H_{n,k}(r;h;x)\right)^2 \\ &= \frac{1}{4\pi r^2} \sum_{n\in\mathcal{N}} \frac{2n+1}{A_n^2} \left(\frac{r}{|x|}\right)^{2(n+h+1)} \\ &\le \frac{1}{4\pi r^2} \sum_{n\in\mathcal{N}} \frac{2n+1}{A_n^2} < \infty\end{aligned}$$

for all $x \in \overline{\Omega_r^{ext}}$, and thus, $K_{\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})}(x,\cdot) \in \mathcal{H}(\{A_n\}; h; \overline{\Omega_r^{ext}})$ for all $x \in \overline{\Omega_r^{ext}}$. Hence, $(F, K_{\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})}(x,\cdot))_{\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})}$ is well-defined for $F \in \mathcal{H}(\{A_n\}; h; \overline{\Omega_r^{ext}})$, and a straightforward calculation yields $F(x)$.
$\Rightarrow$: Let $\mathcal{H}(\{A_n\}; h; \overline{\Omega_r^{ext}})$ be a reproducing kernel Hilbert space with reproducing kernel $K_{\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})}$. Then $K_{\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})}(x,\cdot) \in \mathcal{H}(\{A_n\}; h; \overline{\Omega_r^{ext}})$ for all $x \in \overline{\Omega_r^{ext}}$, and according to Lemma 4.11,

$$K_{\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})}(x,y) = \sum_{n\in\mathcal{N}} \sum_{k=1}^{2n+1} \frac{1}{A_n} H_{n,k}(r;h;x) \frac{1}{A_n} H_{n,k}(r;h;y),$$

where $x,\ y \in \overline{\Omega_r^{ext}}$. Hence, for $x \in \Omega_r$

$$\begin{aligned}\|K_{\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})}(x,\cdot)\|^2_{\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})} &= \sum_{n\in\mathcal{N}} \sum_{k=1}^{2n+1} A_n^2 \left(\frac{1}{A_n^2} H_{n,k}(r;h;x)\right)^2 \\ &= \frac{1}{4\pi r^2} \sum_{n\in\mathcal{N}} \frac{2n+1}{A_n^2} < \infty,\end{aligned}$$

i.e., $\{A_n\}_{n\in\mathbb{N}_0}$ is summable. □

Lemma 4.13 *Let $(\mathcal{H}, (\cdot,\cdot)_{\mathcal{H}})$ be a reproducing kernel Hilbert space with reproducing kernel $K_{\mathcal{H}}$. Furthermore, let $\mathcal{L}$ be a bounded linear functional on $\mathcal{H}$. Assume that $\{H_n\}_{n\in\mathbb{N}_0}$ is a complete orthonormal system in $\mathcal{H}$. The unique representer of $\mathcal{L}$ in $\mathcal{H}$ according to the Riesz representation theorem is given by*

$$L = \mathcal{L}K_{\mathcal{H}} = \sum_{n\in\mathbb{N}_0} (\mathcal{L}H_n)H_n. \tag{4.11}$$

Proof: It suffices to compute the Fourier coefficients of L:

$$(L, H_n)_{\mathcal{H}} = \mathcal{L}H_n.$$

Hence, the representation (4.11) is valid. □

4.5 Representation of Bounded Evaluation Functionals

In our applications we are interested in a simple representation (that can numerically easily be calculated) of bounded evaluation functionals in the Sobolev-like Hilbert spaces $\mathcal{H}(\{A_n\}; h; \overline{\Omega_r^{ext}})$, where $h \in \{1,2\}$ and where $\{A_n\}_{n\in\mathbb{N}_0} \subset \mathbb{R}_0^+$ is a sequence of non-negative real numbers with $A_n \geq 1$ for almost all $n \in \mathcal{N}(A_n)$.
From Lemma 4.12 we know that whenever the sequence $\{A_n\}_{n\in\mathbb{N}_0}$ is summable, $\mathcal{H}(\{A_n\}; h; \overline{\Omega_r^{ext}})$ is a reproducing kernel Hilbert space and, by Lemma 4.10, the evaluation functional $\mathcal{L}_x : \mathcal{H}(\{A_n\}; h; \overline{\Omega_r^{ext}}) \to \mathbb{R}$, $\mathcal{L}_x F := F(x)$, is bounded for each $x \in \overline{\Omega_r^{ext}}$. By Lemma 4.13, its representer $L_x \in \mathcal{H}(\{A_n\}; h; \overline{\Omega_r^{ext}})$ is given by

$$L_x = \mathcal{L}_x K_{\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})}(\cdot,\cdot) = K_{\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})}(x,\cdot), \tag{4.12}$$

where $K_{\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})}(\cdot,\cdot) : \overline{\Omega_r^{ext}} \times \overline{\Omega_r^{ext}} \to \mathbb{R}$ denotes the reproducing kernel of $\mathcal{H}(\{A_n\}; h; \overline{\Omega_r^{ext}})$.
We will see, that under the first mentioned weaker assumptions on $\{A_n\}_{n\in\mathbb{N}_0}$ all evaluation functionals in points $x \in \Omega_r^{ext}$ are still bounded. Furthermore, there are special choices for $\{A_n\}_{n\in\mathbb{N}_0}$ such that the representers of bounded evaluation functionals are available as elementary functions and can be easily implemented.

Theorem 4.14 *Let $\{A_n\}_{n\in\mathbb{N}_0} \subset \mathbb{R}_0^+$ be a sequence of non-negative real numbers with $A_n \geq 1$ for almost all $n \in \mathcal{N} := \mathcal{N}(A_n)$. Then the evaluation functional $\mathcal{L}_x : \mathcal{H}(\{A_n\}; h; \overline{\Omega_r^{ext}}) \to \mathbb{R}$, $\mathcal{L}_x F := F(x)$, is bounded for each $x \in \Omega_r^{ext}$, and its representer $L_x \in \mathcal{H}(\{A_n\}; h; \overline{\Omega_r^{ext}})$ is given by*

$$L_x = \sum_{n\in\mathcal{N}} \sum_{k=0}^{2n+1} \frac{1}{A_n} H_{n,k}(r; h; x) \frac{1}{A_n} H_{n,k}(r; h; \cdot). \tag{4.13}$$

Proof: Let $x \in \Omega_r^{ext}$ and L_x be given according to (4.13).

$$\begin{aligned}
\|L_x\|^2_{\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})} &= \sum_{n\in\mathcal{N}} \sum_{k=0}^{2n+1} \frac{1}{A_n^2} H_{n,k}(r;h;x)^2 \\
&= \frac{1}{4\pi r^2} \sum_{n\in\mathcal{N}} \frac{2n+1}{A_n^2} \left(\frac{r}{|x|}\right)^{2(n+h+1)} \\
&< \infty
\end{aligned}$$

due to the quotient criterion, and thus $L_x \in \mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})$. Consequently, the convolution $(L_x, F)_{\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})}$ is well-defined for all $F \in \mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})$, and

$$(L_x, F)_{\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})} = \sum_{n\in\mathcal{N}} \sum_{k=0}^{2n+1} F^r_{n,k} H_{n,k}(r;h;x) = F(x) = \mathcal{L}_x F.$$

Due to the Riesz representation theorem

$$\|\mathcal{L}_x\|_{\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})^*} = \|L_x\|_{\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})} < \infty,$$

which completes the proof. □

Remark 4.15 *Under the assumptions of Theorem 4.14 we can furthermore show that the series (4.13) is uniformly convergent in $\overline{\Omega_r^{ext}}$, where we may use arguments analogous to those which lead to the conclusion that $\|L_x\|_{\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})} < \infty$ for $x \in \Omega_r^{ext}$. Hence L_x is a continuous function on $\overline{\Omega_r^{ext}}$.*

We now turn our attention to special choices of $\{A_n\}_{n\in\mathbb{N}_0} \subset \mathbb{R}_0^+$ with $A_n \geq 1$ for almost all $n \in \mathcal{N}(A_n)$ which are of major importance for the numerical treatment of $\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})$-spline interpolation and $\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})^*$-best approximation, respectively. As already mentioned, these are those cases in which the representers of bounded evaluation functionals on $\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})$ have a representation as an elementary function.

For $s \in (0,1)$ we introduce the families of univariate functions:

$$L_s : [-1,1] \to \mathbb{R}^+, \ t \mapsto 1 + s^2 - 2st$$

and

$$Q_s : [-1,1] \to \mathbb{R}^+, \ t \mapsto \frac{1}{4\pi} \frac{1-s^2}{(L_s(t))^{3/2}}.$$

Spaces $\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})$ with Representers of 'Abel-Poisson-Kernel Type'

Let $q \in (0,1]$ and $A_n = q^{-n/2}$ for $n \in \mathbb{N}_0$. Let $x \in \Omega_r^{ext}$ in case $q = 1$ and $x \in \overline{\Omega_r^{ext}}$ if $q \in (0,1)$. Then the evaluation functional $\mathcal{L}_x : \mathcal{H}(\{q^{-n/2}\};h;\overline{\Omega_r^{ext}}) \to \mathbb{R}$, $F \mapsto \mathcal{L}_x(F) := F(x)$, is bounded and its representer $L_x \in \mathcal{H}(\{q^{-n/2}\};h;\overline{\Omega_r^{ext}})$ is given by $L_x = K_{\mathcal{H}(\{q^{-n/2}\};h;\overline{\Omega_r^{ext}})}(x,\cdot)$, where

$$\begin{aligned} & K_{\mathcal{H}(\{q^{-n/2}\};h;\overline{\Omega_r^{ext}})}(x,y) \\ &= \sum_{n\in\mathbb{N}_0}\sum_{k=1}^{2n+1} q^n H_{n,k}(r;h;x)H_{n,k}(r;h;y) \\ &= \frac{1}{r^2}\left(\frac{r^2}{|x||y|}\right)^{h+1}\frac{1}{4\pi}\sum_{n\in\mathbb{N}_0}(2n+1)\left(q\frac{r^2}{|x||y|}\right)^n P_n\left(\frac{x}{|x|}\cdot\frac{y}{|y|}\right) \\ &= \frac{1}{r^2}\left(\frac{r^2}{|x||y|}\right)^{h+1} Q_{\left(q\frac{r^2}{|x||y|}\right)}\left(\frac{x}{|x|}\cdot\frac{y}{|y|}\right) \quad \text{for} \quad y \in \overline{\Omega_r^{ext}}. \end{aligned} \tag{4.14}$$

Spaces $\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})$ with Representers of 'Singularity-Kernel Type'

Let $q \in (0,1]$ and $A_n = q^{-n/2}\left(n+\frac{1}{2}\right)^{1/2}$ for $n \in \mathbb{N}_0$. Let $x \in \Omega_r^{ext}$ in case $q = 1$ and $x \in \overline{\Omega_r^{ext}}$ if $q \in (0,1)$. Then the evaluation functional $\mathcal{L}_x : \mathcal{H}(\{q^{-n/2}\left(n+\frac{1}{2}\right)^{1/2}\};h;\overline{\Omega_r^{ext}}) \to \mathbb{R}$, $F \mapsto \mathcal{L}_x(F) := F(x)$ is bounded and its representer $L_x \in \mathcal{H}(\{q^{-n/2}\left(n+\frac{1}{2}\right)^{1/2}\};h;\overline{\Omega_r^{ext}})$ is given by $L_x = K_{\mathcal{H}(\{q^{-n/2}\left(n+\frac{1}{2}\right)^{1/2}\};h;\overline{\Omega_r^{ext}})}(x,\cdot)$, where

$$\begin{aligned} & K_{\mathcal{H}(\{q^{-n/2}\left(n+\frac{1}{2}\right)^{1/2}\};h;\overline{\Omega_r^{ext}})}(x,y) \\ &= \sum_{n\in\mathbb{N}_0}\sum_{k=1}^{2n+1} \frac{q^n}{\left(n+\frac{1}{2}\right)} H_{n,k}(r;h;x)H_{n,k}(r;h;y) \\ &= \frac{1}{2\pi r^2}\left(\frac{r^2}{|x||y|}\right)^{h+1}\sum_{n\in\mathbb{N}_0}\left(q\frac{r^2}{|x||y|}\right)^n P_n\left(\frac{x}{|x|}\cdot\frac{y}{|y|}\right) \\ &= \frac{1}{2\pi r^2}\left(\frac{r^2}{|x||y|}\right)^{h+1}\frac{1}{\left(L_{\left(q\frac{r^2}{|x||y|}\right)}\left(\frac{x}{|x|}\cdot\frac{y}{|y|}\right)\right)^{1/2}} \quad \text{for} \quad y \in \overline{\Omega_r^{ext}}. \end{aligned}$$

Spaces $\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})$ with Representers of 'Logarithmic Kernel Type'

Let $q \in (0,1]$ and $A_n = q^{-n/2}\left((2n+1)(n+1)\right)^{1/2}$ for $n \in \mathbb{N}_0$. Let $x \in \Omega_r^{ext}$ in case $q = 1$ and $x \in \overline{\Omega_r^{ext}}$ if $q \in (0,1)$. Then the evaluation functional $\mathcal{L}_x : \mathcal{H}(\{q^{-n/2}\left((2n+1)(n+1)\right)^{1/2}\};h;\overline{\Omega_r^{ext}}) \to \mathbb{R}$, $F \mapsto \mathcal{L}_x(F) := F(x)$, is bounded and its representer $L_x \in \mathcal{H}(\{q^{-n/2}\left((2n+1)(n+1)\right)^{1/2}\};h;\overline{\Omega_r^{ext}})$ is given by

$L_x = K_{\mathcal{H}(\{q^{-n/2}((2n+1)(n+1))^{1/2}\};h;\overline{\Omega_r^{ext}})}(x,\cdot)$, where

$$\begin{aligned} & K_{\mathcal{H}(\{q^{-n/2}((2n+1)(n+1))^{1/2}\};h;\overline{\Omega_r^{ext}})}(x,y) \\ = & \sum_{n\in\mathbb{N}_0}\sum_{k=1}^{2n+1} \frac{q^n}{(2n+1)(n+1)} H_{n,k}(r;h;x)H_{n,k}(r;h;y) \\ = & \frac{1}{4\pi r^2}\left(\frac{r^2}{|x||y|}\right)^{h+1} \sum_{n\in\mathbb{N}_0} \frac{1}{n+1}\left(q\frac{r^2}{|x||y|}\right)^n P_n\left(\frac{x}{|x|}\cdot\frac{y}{|y|}\right) \\ = & \frac{1}{4\pi r^2}\frac{\left(\frac{r^2}{|x||y|}\right)^h}{q} \ln\left(1+\frac{2q\frac{r^2}{|x||y|}}{\left(L_{\left(q\frac{r^2}{|x||y|}\right)}\left(\frac{x}{|x|}\cdot\frac{y}{|y|}\right)\right)^{1/2}+1-q\frac{r^2}{|x||y|}}\right) \quad \text{for} \quad y\in\overline{\Omega_r^{ext}}. \end{aligned}$$

In all three cases $\{A_n\}_{n\in\mathbb{N}_0}$ is summable if $q\in(0,1)$, and $K_{\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})}(\cdot,\cdot)$ then coincides with the reproducing kernel of $\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})$. Moreover, $K_{\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})}(x,y)>0$ for all $y\in\overline{\Omega_r^{ext}}$ and all $x\in\Omega_r^{ext}$ in case $q=1$ and $x\in\overline{\Omega_r^{ext}}$ if $q\in(0,1)$, respectively.

The parameter $q\in(0,1]$ is a shape parameter and determines the decay behaviour and thus the spatial localization of the representer of the evaluation functional $\mathcal{L}_x = K_{\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})}(x,\cdot)$ in $x\in\Omega_r^{ext}$ (or $x\in\overline{\Omega_r^{ext}}$, respecitively, if $\{A_n\}_{n\in\mathbb{N}_0}$ is summable). Figure 4.1 shows the Abel-Posson-kernel, the Singularity kernel and the Logarithmic kernel on the unit sphere for $q=0.95$. These functions are nothing else but the representers of the evaluation functionals in $x=(1,0,0)$ in the spaces $\mathcal{H}(\{A_n\};0;\overline{\Omega_1^{ext}})$ for the above choices of the sequence $\{A_n\}_{n\in\mathbb{N}_0}$. It is clearly visible that their decay behaviour is quite different.

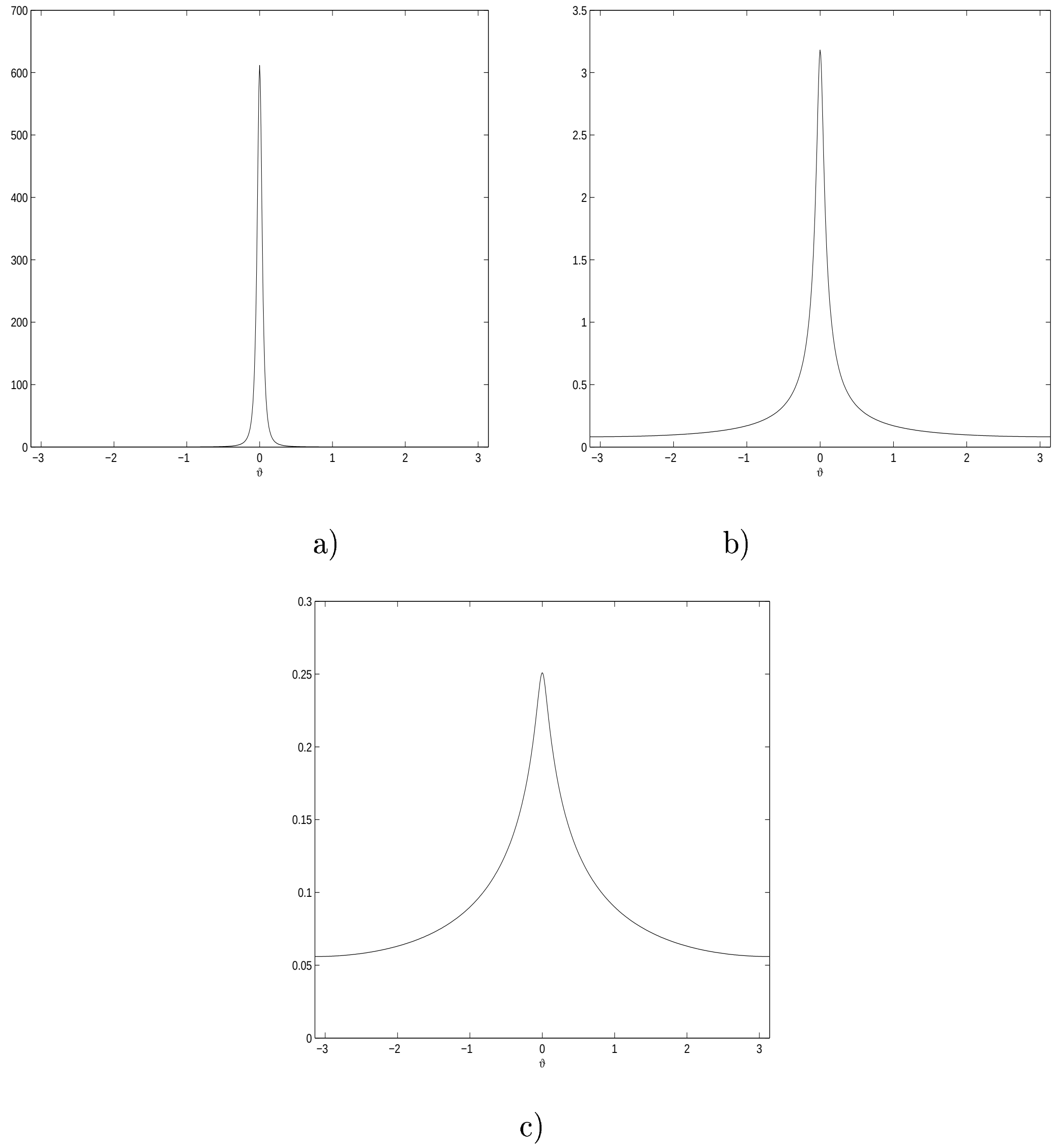


Figure 4.1: *a) Abel-Poisson kernel, b) Singularity kernel, and c) Logarithmic kernel on the unit sphere in case* $q = 0.95$.

Note that we may exploit the space-localizing properties of the representers of bounded evaluation functionals in order to treat the $\mathcal{H}(\{A_n\}; h; \overline{\Omega_r^{ext}})$-spline interpolation problem and the $\mathcal{H}(\{A_n\}; h; \overline{\Omega_r^{ext}})$-spline approximation problem 'locally'. This has to be understood in the following sense: Assume that we want to compute a spline reconstruction of a function of class $\mathcal{H}(\{A_n\}; h; \overline{\Omega_r^{ext}})$ only on a subdomain $\mathcal{M}$ of a $\mathcal{C}^{(2)}$-regular surface Σ in Ω_r^{ext} (e.g., the orbit of the satellite in the SST-problem and SGG-problem in Problem 3.1) and that the bounded evaluation functionals are evaluation functionals in points on

Σ. Then it suffices to solve the $\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})$-spline interpolation or spline smoothing problem only with respect to evaluation functionals in points in some subdomain $\widetilde{\mathcal{M}} \subset \Sigma$, which is a suitable neighbourhood for $\mathcal{M}$, for the following reasons: The representer of a point evaluation in x is a function with a strong space-localization around x. As a spline S is a linear combination of such representers, a representer of a point evaluation in a point far away from $\mathcal{M}$ yields only a negligible contribution to $S|_{\mathcal{M}}$. The subdomain $\widetilde{\mathcal{M}}$ has to be chosen somewhat larger than $\mathcal{M}$, because we have to take into account that errors due to Gibbs phenomena close to the boundaries of $\widetilde{\mathcal{M}}$ are likely to occur. (However, due to the spatial localization of the representers of the evaluation functionals these errors should be localized in space, too.)

4.6 Spline Interpolation with Data Given on a $C^{(2)}$-regular Surface

In order to solve the SST-problem and the SGG-problem in its discrete formulation (see Problem 3.8) we want to approximate the right-hand side $G \in \mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})$ of the operator equation $\Lambda F = G$, where $h = 1$ for $\Lambda = \Lambda^{SST}$ and $h = 2$ for $\Lambda = \Lambda^{SGG}$ with the help of the known data $\{(x_i^N, G(x_i^N))|\, i = 1,\ldots,N\}$ on the 'orbital surface' Σ_S. One approach to do this is to compute the interpolating $\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})$-spline of G from the measured data, and this is justified if we can show that the interpolating spline $S^G_{\mathcal{L}_1^N,\ldots,\mathcal{L}_N^N}$ (of G with respect to the evaluation functionals $\mathcal{L}_i : G \mapsto G(x_i^N)$, $i = 1,\ldots,N$) converges to G, when $X := \{x_1, x_2, \ldots\}$ is a dense subset of Σ_S and when $N \to \infty$. Therefore, we present convergence theorems for interpolating $\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})$-splines in this section. However, it should be noted that these convergence results yield no quantitative information about the quality of $S^G_{\mathcal{L}_1^N,\ldots,\mathcal{L}_N^N}$ as an approximation of G. But for a large number of measurements this spline can be expected to be also a suitable approximation.

Theorem 4.16 *Let $\{A_n\}_{n\in\mathbb{N}_0} \subset \mathbb{R}^+$ be a sequence of positive real numbers which satisfy $A_n \geq 1$ for almost all $n \in \mathbb{N}_0$ and let $h \in \{0,1,2\}$. Suppose furthermore, that $\Sigma \subset \Omega_r^{ext}$ is a $C^{(2)}$-regular surface. Let $X := \{x_1, x_2, \ldots\} \subset \Sigma$ be a dense pointset in Σ with corresponding evaluation functionals $\{\mathcal{L}_1, \mathcal{L}_2, \ldots\}$ on $\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})$ given by $\mathcal{L}_i : \mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}}) \to \mathbb{R}$, $F \mapsto F(x_i)$, $i = 1,2,\ldots$. The evaluation functionals $\mathcal{L}_i$, $i = 1,2,\ldots$, are bounded, and their representers $L_i \in \mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})$, $i = 1,2,\ldots$, are given by*

$$L_i = \sum_{n=0}^{\infty}\sum_{k=1}^{2n+1} \frac{1}{A_n^2} H_{n,k}(r;h;x_i) H_{n,k}(r;h;\cdot).$$

Then, $\operatorname{span}\{L_i | i = 1,2,\ldots\}$ *is dense in $\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})$, i.e.,*

$$\overline{\operatorname{span}\{L_i | i = 1,2,\ldots\}}^{\|\cdot\|_{\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})}} = \mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}}). \tag{4.15}$$

If the sequence $\{A_n\}_{n\in\mathbb{N}_0}$ *is additionally summable the statement even holds true for a* $C^{(2)}$*-regular surface* $\Sigma \subset \overline{\Omega_r^{ext}}$.

Proof: The boundedness of the evaluation functionals $\mathcal{L}_i$ and the representation of their representers $L_i \in \mathcal{H}(\{A_n\}; h; \overline{\Omega_r^{ext}})$ are a consequence of Theorem 4.14. In order to show (4.15), we have to verify that $\mathrm{span}\{L_i | i = 1, 2, \ldots\}$ contains a complete orthonormal system in $\mathcal{H}(\{A_n\}; h; \overline{\Omega_r^{ext}})$. Using the Gram-Schmidt orthonormalization procedure, we may construct successively an orthonormal system $\{\tilde{L}_j | \, j = 1, 2, \ldots, \tilde{N}\}$ in $\mathrm{span}\{L_i | \, i = 1, 2, \ldots, N\}$, where $\tilde{N} \leq N$, such that $\mathrm{span}\{\tilde{L}_j | \, j = 1, 2, \ldots\} = \mathrm{span}\{L_i | \, i = 1, 2, \ldots\}$. The system $\{\tilde{L}_j | \, j = 1, 2, \ldots\}$ is a complete orthonormal system in $\mathcal{H}(\{A_n\}; h; \overline{\Omega_r^{ext}})$ if we can show that whenever $F \in \mathcal{H}(\{A_n\}; h; \overline{\Omega_r^{ext}})$ is orthogonal to all $\tilde{L}_j$, $j = 1, 2, \ldots$, it follows that $F = 0$.
Let therefore $F \in \mathcal{H}(\{A_n\}; h; \overline{\Omega_r^{ext}})$ be given such that $(F, \tilde{L}_j)_{\mathcal{H}(\{A_n\}; h; \overline{\Omega_r^{ext}})} = 0$ for all $j = 1, 2, \ldots$. Since $\mathrm{span}\{\tilde{L}_j | \, j = 1, 2, \ldots\} = \mathrm{span}\{L_i | \, i = 1, 2, \ldots\}$, we have that

$$F(x_i) = \mathcal{L}_i F = (F, L_i)_{\mathcal{H}(\{A_n\}; h; \overline{\Omega_r^{ext}})} = 0 \text{ for } i = 1, 2, \ldots. \tag{4.16}$$

Since $F|_\Sigma$ is continuous and the pointset X is dense in Σ, (4.16) implies that $F|_\Sigma = 0$.
The function

$$F(x) \left(\frac{|x|}{r} \right)^h = \sum_{n=0}^{\infty} \sum_{k=1}^{2n+1} F_{n,k}^r H_{n,k}(r; x) \tag{4.17}$$

is harmonic in Ω_r^{ext} and vanishes on Σ. Due to the uniqueness of the solution of the exterior Dirichlet boundary value problem for the Laplace equation with boundary data given on Σ, $F(x)(|x|/r)^h = 0$ for all $x \in \overline{\Sigma^{ext}}$. By harmonic continuation, the function $x \mapsto F(x)(|x|/r)^h$ vanishes everywhere in Ω_r^{ext}. This means, however, that its series expansion (4.17) vanishes in Ω_r^{ext} and the Fourier coefficients $F_{n,k}^r$, $n \in \mathbb{N}_0$, $1 \leq k \leq 2n+1$ all have to be zero. Hence $F = 0$ and $\{\tilde{L}_j | j = 1, 2, \ldots\}$ is a complete orthonormal system in $\mathcal{H}(\{A_n\}; h; \overline{\Omega_r^{ext}})$.
If $\{A_n\}_{n\in\mathbb{N}_0}$ is additionally summable, all evaluation functionals are bounded and the proof is analogous for the case $\Sigma \subset \overline{\Omega_r^{ext}}$. □

Theorem 4.17 *Let the assumptions and notation be the same as in Theorem 4.16. Let* $F \in \mathcal{H}(\{A_n\}; h; \overline{\Omega_r^{ext}})$ *and denote its interpolating* $\mathcal{H}(\{A_n\}; h; \overline{\Omega_r^{ext}})$*-spline with respect to the evaluation functionals* $\{\mathcal{L}_1, \ldots, \mathcal{L}_N\}$, $N \in \mathbb{N}$, *by* $S^F_{\mathcal{L}_1,\ldots,\mathcal{L}_N}$. *Then the following convergence results hold true:*

(i) $\lim_{N\to\infty} \|F - S^F_{\mathcal{L}_1,\ldots,\mathcal{L}_N}\|_{\mathcal{H}(\{A_n\}; h; \overline{\Omega_r^{ext}})} = 0$,

(ii) $\lim_{N\to\infty} \|F - S^F_{\mathcal{L}_1,\ldots,\mathcal{L}_N}\|_{C(\overline{\Omega_{r+\delta}^{ext}})} = 0$ *for all* $\delta > 0$,

(iii) $\lim_{N\to\infty} \|F - S^F_{\mathcal{L}_1,\ldots,\mathcal{L}_N}\|_{C(\overline{\Sigma^{ext}})} = 0$.

If the sequence $\{A_n\}_{n\in\mathbb{N}_0}$ *is additionally summable, the statements (i), (ii) and (iii) are even valid for a* $C^{(2)}$*-regular surface* $\Sigma \subset \overline{\Omega_r^{ext}}$.

Proof: The spline interpolation operator which maps $F \in \mathcal{H}(\{A_n\}; h; \overline{\Omega_r^{ext}})$ onto its interpolating $\mathcal{H}(\{A_n\}; h; \overline{\Omega_r^{ext}})$-spline $S^F_{\mathcal{L}_1,\ldots,\mathcal{L}_N}$ is an orthogonal projection onto span$\{L_1, \ldots, L_N\}$. According to Theorem 4.16 we may use the Gram-Schmidt orthonormalization procedure to successively construct a complete orthonormal system $\{\tilde{L}_j\}_{j\in\mathbb{N}}$ of $\mathcal{H}(\{A_n\}; h; \overline{\Omega_r^{ext}})$ such that for all $N \in \mathbb{N}$, span$\{L_1, \ldots, L_N\}$ = span$\{\tilde{L}_1, \ldots, \tilde{L}_{\tilde{N}}\}$, where $\tilde{N} = \tilde{N}(N) \leq N$. Then

$$S^F_{\mathcal{L}_1,\ldots,\mathcal{L}_N} = \sum_{j=1}^{\tilde{N}} (F, \tilde{L}_j)_{\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})} \tilde{L}_j,$$

which is nothing else but the truncated Fourier series of F with respect to the complete orthonormal system $\{\tilde{L}_j\}_{j\in\mathbb{N}}$, and which converges to F in $\mathcal{H}(\{A_n\}; h; \overline{\Omega_r^{ext}})$-sense for $N \to \infty$. Thus, statement (i) follows.
Assertions (ii) and (iii) are a consequence of the estimate

$$\begin{aligned}
&|F(x) - S^F_{\mathcal{L}_1,\ldots,\mathcal{L}_N}(x)| \\
&= \left| \left(F - S^F_{\mathcal{L}_1,\ldots,\mathcal{L}_N}, \sum_{n=0}^{\infty} \sum_{k=1}^{2n+1} \frac{1}{A_n^2} H_{n,k}(r; h; x) H_{n,k}(r; h; \cdot) \right)_{\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})} \right| \\
&\leq \; \|F - S^F_{\mathcal{L}_1,\ldots,\mathcal{L}_N}\|_{\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})} \left(\sum_{n=0}^{\infty} \frac{2n+1}{4\pi r^2 A_n^2} \left(\frac{r}{|x|} \right)^{2(n+h+1)} \right)^{1/2}
\end{aligned}$$

for x with $|x| \geq r + \gamma$, where Γ is an arbitary $C^{(2)}$-regular surface in Ω_r^{ext} and $\gamma := \inf_{x\in\Gamma} |x| - r > 0$, and for $x \in \overline{\Omega_r^{ext}}$ in case $\{A_n\}_{n\in\mathbb{N}_0}$ is summable. Due to the quotient criterion the sum is in both cases finite and it is uniformly bounded in x. As a consequence of (i), there exists for each $\varepsilon > 0$ an $N_0 = N_0(\varepsilon)$ such that $\sup_{x\in\overline{\Gamma^{ext}}} |F(x) - S^F_{\mathcal{L}_1,\ldots,\mathcal{L}_N}(x)| \leq \varepsilon$ for all $N, M \geq N_0$. Hence $\{S^F_{\mathcal{L}_1,\ldots,\mathcal{L}_N}\}_{N\in\mathbb{N}}$ converges uniformly to F in $\overline{\Gamma^{ext}}$ (and $\overline{\Omega_r^{ext}}$ in case $\{A_n\}_{n\in\mathbb{N}_0}$ is summable), which implies (ii) and (iii). □

Chapter 5

Scaling Functions, Wavelets and Regularization by Multiresolution

Let $\{A_n\}_{n\in\mathbb{N}_0} \subset \mathbb{R}_0^+$ be a sequence of non-negative real numbers, let $r_1, r_2 \in \mathbb{R}^+$ and $h_1, h_2 \in \{0,1,2\}$. Suppose that $\Lambda : \mathcal{H}(\{A_n\}; h_1; \overline{\Omega_{r_1}^{ext}}) \to \mathcal{H}(\{A_n\}; h_2; \overline{\Omega_{r_2}^{ext}})$ is a pseudodifferential operator whose symbol $\{\Lambda^\wedge(n)\}_{n\in\mathbb{N}_0}$ satisfies $\lim_{n\to\infty} |\Lambda^\wedge(n)| = 0$ and which is therefore compact. The main objective of this chapter is to construct a regularization to the unbounded inverse $\Lambda^{-1} : \mathcal{H}(\{A_n(\Lambda^\wedge(n))^{-1}\}; h_2; \overline{\Omega_{r_2}^{ext}}) \to \mathcal{H}(\{A_n\}; h_1; \overline{\Omega_{r_1}^{ext}})$ which leads to a multiresolution analysis of the space $\mathcal{H}(\{A_n\}; h_1; \overline{\Omega_{r_1}^{ext}})$, i.e. to a nested sequence of approximation subspaces $\{\mathcal{V}_j^\Lambda(h; \overline{\Omega_{r_1}^{ext}})\}_{j\in\mathbb{N}_0} \subset \mathcal{H}(\{A_n\}; h_1; \overline{\Omega_{r_1}^{ext}})$ which admits a representation of the approximate solution to the ill-posed pseudodifferential operator equation $\Lambda F = G$, $F \in \mathcal{H}(\{A_n\}; h_1; \overline{\Omega_{r_1}^{ext}})$, $G \in \mathcal{H}(\{A_n\}; h_2; \overline{\Omega_{r_2}^{ext}})$ at different scales j of space-momentum resolution.

Section 5.1 introduces the (technical) notion of $\mathcal{H}(\{A_n\}; h_1; \overline{\Omega_{r_1}^{ext}})$-$\mathcal{H}(\{A_n\}; h_2; \overline{\Omega_{r_2}^{ext}})$ product kernels. These are functions on $\overline{\Omega_{r_1}^{ext}} \times \overline{\Omega_{r_2}^{ext}}$ which are characterized by a sequence of real numbers called the symbol of the product kernel.

First we explain the construction principles of a multiresolution analysis of $\mathcal{H}(\{A_n\}; h; \overline{\Omega_r^{ext}})$, where $r \in \mathbb{R}^+$ and $h \in \{0,1,2\}$, because after these preparations the construction of a regularization multiresolution analysis is straightforward. In Section 5.2 we therefore introduce scaling functions for $\mathcal{H}(\{A_n\}; h; \overline{\Omega_r^{ext}})$. These are families of $\mathcal{H}(\{A_n\}; h; \overline{\Omega_r^{ext}})$-$\mathcal{H}(\{A_n\}; h; \overline{\Omega_r^{ext}})$ product kernels $\{\Phi_j\}_{j\in\mathbb{N}_0}$ which depend on a so-called scale parameter $j \in \mathbb{N}_0$, and which may be generated as dilated versions of the mother scaling function Φ_0 at scale $j = 0$. The scale parameter j is a measure for increasing space localization of the product kernel Φ_j. The construction of an approximate identity in $\mathcal{H}(\{A_n\}; h; \overline{\Omega_r^{ext}})$, i.e., a family $\{P_j\}_{j\in\mathbb{N}_0}$ of bounded linear operators $P_j : \mathcal{H}(\{A_n\}; h; \overline{\Omega_r^{ext}}) \to \mathcal{H}(\{A_n\}; h; \overline{\Omega_r^{ext}})$ which approximates the identity operator $Id : \mathcal{H}(\{A_n\}; h; \overline{\Omega_r^{ext}}) \to \mathcal{H}(\{A_n\}; h; \overline{\Omega_r^{ext}})$, on the basis of a scaling function for $\mathcal{H}(\{A_n\}; h; \overline{\Omega_r^{ext}})$ then leads to a nested sequence of approximation subspaces (scale spaces) $\{\mathcal{V}_j(h; \overline{\Omega_r^{ext}})\}_{j\in\mathbb{N}_0} \subset \mathcal{H}(\{A_n\}; h; \overline{\Omega_r^{ext}})$.

The wavelet which corresponds to a scaling function for $\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})$ is defined in Section 5.3 as the family of $\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})$-$\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})$ product kernels $\{\Psi_j\}_{j\in\mathbb{N}_0}$ whose symbols are related to the symbols of the kernels Φ_j, $j\in\mathbb{N}_0$, via a so-called refinement equation. This equation reflects the difference between two members of the family $\{\Phi_j\}_{j\in\mathbb{N}_0}$ at consecutive scales. With the help of the wavelet corresponding to $\{\Phi_j\}_{j\in\mathbb{N}_0}$ we may then introduce a second family of bounded linear operators $\{S_j\}_{j\in\mathbb{N}_0}$, $S_j:\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})\to\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})$ with the property $P_j+S_j=P_{j+1}$, $j\in\mathbb{N}_0$. This construction leads to a further decomposition of the scale spaces according to $\mathcal{V}_j(h;\overline{\Omega_r^{ext}})+\mathcal{W}_j(h;\overline{\Omega_r^{ext}})=\mathcal{V}_{j+1}(h;\overline{\Omega_r^{ext}})$, where $\{\mathcal{W}_j(h;\overline{\Omega_r^{ext}})\}_{j\in\mathbb{N}_0}\subset\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})$ denotes a family of so-called detail spaces which contain the detail information which has to be added to the approximation of a function of class $\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})$ at scale j in order to obtain its approximation at scale $j+1$.

Section 5.4 finally transfers the results of Sections 5.2 and 5.3 to the case of a regularization multiresolution analysis corresponding to the ill-posed pseudodifferential operator equation $\Lambda F=G$, $F\in\mathcal{H}(\{A_n\};h_1;\overline{\Omega_{r_1}^{ext}})$, $G\in\mathcal{H}(\{A_n\};h_2;\overline{\Omega_{r_2}^{ext}})$.

The presentation of the concepts is oriented on [Fr1999].

5.1 Product Kernels

Definition 5.1 *Let* $\{A_n\}_{n\in\mathbb{N}_0}\subset\mathbb{R}_0^+$ *be a sequence of non-negative real numbers, let* $\mathcal{N}=\mathcal{N}(\{A_n\})$ *and let* $r_1,r_2\in\mathbb{R}^+$ *and* $h_1,h_2\in\{0,1,2\}$. *Suppose furthermore that* $\{K^\wedge(n)\}_{n\in\mathcal{N}}\subset\mathbb{R}$ *is a sequence of real numbers which satisfies*

$$\sum_{n\in\mathcal{N}}(2n+1)\left(K^\wedge(n)\right)^2\frac{1}{A_n^2}<\infty \tag{5.1}$$

and

$$\sum_{n\in\mathcal{N}}(2n+1)(K^\wedge(n))^2<\infty. \tag{5.2}$$

Then the function $K:\overline{\Omega_{r_1}^{ext}}\times\overline{\Omega_{r_2}^{ext}}\to\mathbb{R}$, *defined by*

$$K(x,y):=\sum_{n=0}^{\infty}\sum_{k=1}^{2n+1}K^\wedge(n)\frac{1}{A_n}H_{n,k}(r_1;h_1;x)\frac{1}{A_n}H_{n,k}(r_2;h_2;y), \tag{5.3}$$

where $x\in\overline{\Omega_{r_1}^{ext}}$ *and* $y\in\overline{\Omega_{r_2}^{ext}}$, *is called an* $\mathcal{H}(\{A_n\};h_1;\overline{\Omega_{r_1}^{ext}})$-$\mathcal{H}(\{A_n\};h_2;\overline{\Omega_{r_2}^{ext}})$ *product kernel with symbol* $\{K^\wedge(n)\}_{n\in\mathcal{N}}$. *In case that* $r_1=r_2$ *and* $h_1=h_2$, K *is simply called an* $\mathcal{H}(\{A_n\};h_1;\overline{\Omega_{r_1}^{ext}})$*-product kernel.*

Lemma 5.2 *Let $\{A_n\}_{n\in\mathbb{N}_0} \subset \mathbb{R}_0^+$ be a sequence of non-negative real numbers, let $r_1, r_2 \in \mathbb{R}^+$ and $h_1, h_2 \in \{0,1,2\}$. Suppose that $K : \overline{\Omega_{r_1}^{ext}} \times \overline{\Omega_{r_2}^{ext}} \to \mathbb{R}$ is an $\mathcal{H}(\{A_n\}; h_1; \overline{\Omega_{r_1}^{ext}})$-$\mathcal{H}(\{A_n\}; h_2; \overline{\Omega_{r_2}^{ext}})$ product kernel with symbol $\{K^\wedge(n)\}_{n\in\mathcal{N}}$. Then*
(i) $K(x,\cdot) \in \mathcal{H}(\{A_n\}; h_2; \overline{\Omega_{r_2}^{ext}})$ for all $x \in \overline{\Omega_{r_1}^{ext}}$,
(ii) $K(\cdot,y) \in \mathcal{H}(\{A_n\}; h_1; \overline{\Omega_{r_1}^{ext}})$ for all $y \in \overline{\Omega_{r_2}^{ext}}$ and
(iii) $K(\cdot,\cdot) *_{\mathcal{H}(\{A_n\};h_2;\overline{\Omega_{r_2}^{ext}})} F \in \mathcal{H}(\{A_n\}; h_1; \overline{\Omega_{r_1}^{ext}})$ for all $F \in \mathcal{H}(\{A_n\}; h_2; \overline{\Omega_{r_2}^{ext}})$ and
(iv)

$$K(x,\cdot) *_{\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})} F = \sum_{n\in\mathcal{N}} \sum_{k=1}^{2n+1} K^\wedge(n) F_{n,k}^r \, H_{n,k}(r_1; h_1; x). \tag{5.4}$$

Proof: Assertions (i) and (ii) are a consequence of condition (5.1), since

$$\begin{aligned} \|K(x,\cdot)\|^2_{\mathcal{H}(\{A_n\};h_2;\overline{\Omega_{r_2}^{ext}})} &= \sum_{n\in\mathcal{N}} \sum_{k=1}^{2n+1} (K^\wedge(n))^2 \frac{1}{A_n^4} A_n^2 \left(H_{n,k}(r_1; h_1; x)\right)^2 \\ &\leq \sum_{n\in\mathcal{N}} \frac{2n+1}{4\pi r_1^2} (K^\wedge(n))^2 \frac{1}{A_n^2} < \infty, \end{aligned}$$

and

$$\|K(\cdot,y)\|_{\mathcal{H}(\{A_n\};h_1;\overline{\Omega_{r_1}^{ext}})} < \infty$$

by analogous arguments.
Condition (5.2) implies that $\{(K^\wedge(n))^2\}_{n\in\mathbb{N}_0}$ is bounded. Thus

$$\begin{aligned} \|K(\cdot,\cdot) *_{\mathcal{H}(\{A_n\};h_2;\overline{\Omega_{r_2}^{ext}})} F\|^2_{\mathcal{H}(\{A_n\};h_1;\overline{\Omega_{r_1}^{ext}})} &= \sum_{n\in\mathcal{N}} \sum_{k=1}^{2n+1} A_n^2 (K^\wedge(n))^2 \left(F_{n,k}^{r_2}\right)^2 \\ &\leq \sup_{n\in\mathcal{N}} (K^\wedge(n))^2 \|F\|^2_{\mathcal{H}(\{A_n\};h_2;\overline{\Omega_{r_2}^{ext}})} < \infty, \end{aligned}$$

which proves (iii).
Property (iv) can be verified by elementary calculations. □

Equation (5.4) reveals that the convolution between a product kernel and a function is nothing else but a multiplication of the Fourier coefficients of the function with the the symbol of the kernel.

Remark 5.3 *Let the assumptions be the same as in Definition 5.1 and suppose that $\{A_n\}_{n\in\mathbb{N}_0}$ additionally satisfies $A_n \geq 1$ for almost all $n \in \mathbb{N}_0$. Then (5.2) implies (5.1).*

Definition 5.4 *Let the assumptions be the same as in Definition 5.1. An $\mathcal{H}(\{A_n\}; h_1; \overline{\Omega_{r_1}^{ext}})$-$\mathcal{H}(\{A_n\}; h_2; \overline{\Omega_{r_2}^{ext}})$ product kernel K with symbol $\{K^\wedge(n)\}_{n\in\mathcal{N}}$ is called bandlimited if there exists an $N \in \mathcal{N}$ such that $K^\wedge(N) \neq 0$ and $K^\wedge(n) = 0$ for all $n > N$, $n \in \mathcal{N}$. N is called the band limit of K.*

Finally, we show that the convolution of two product kernels (supposed it is well-defined) is again a product kernel.

Lemma 5.5 *Let* $\{A_n\}_{n\in\mathbb{N}_0} \subset \mathbb{R}_0^+$ *be a sequence of non-negative real numbers, and let* $r_1, r_2, r_3 \in \mathbb{R}^+$ *and* $h_1, h_2, h_3 \in \{0,1,2\}$. *Suppose that* $K : \overline{\Omega_{r_1}^{ext}} \times \overline{\Omega_{r_2}^{ext}} \to \mathbb{R}$ *is an* $\mathcal{H}(\{A_n\}; h_1; \overline{\Omega_{r_1}^{ext}})$-$\mathcal{H}(\{A_n\}; h_2; \overline{\Omega_{r_2}^{ext}})$ *product kernel with symbol* $\{K^\wedge(n)\}_{n\in\mathcal{N}}$ *and that* $L : \overline{\Omega_{r_2}^{ext}} \times \overline{\Omega_{r_3}^{ext}} \to \mathbb{R}$ *is an* $\mathcal{H}(\{A_n\}; h_2; \overline{\Omega_{r_2}^{ext}})$-$\mathcal{H}(\{A_n\}; h_3; \overline{\Omega_{r_3}^{ext}})$ *product kernel with symbol* $\{L^\wedge(n)\}_{n\in\mathcal{N}}$. *Then the convolution* $K(x,\cdot) *_{\mathcal{H}(\{A_n\};h_2;\overline{\Omega_{r_2}^{ext}})} L(\cdot,y)$, $(x,y) \in \overline{\Omega_{r_1}^{ext}} \times \overline{\Omega_{r_3}^{ext}}$ *is well-defined, yields*

$$K(x,\cdot) *_{\mathcal{H}(\{A_n\};h_2;\overline{\Omega_{r_2}^{ext}})} L(\cdot,y) = \sum_{n\in\mathcal{N}} \sum_{k=1}^{2n+1} K^\wedge(n) L^\wedge(n) \frac{1}{A_n^2} H_{n,k}(r_1; h_1; x) H_{n,k}(r_3; h_3; y)$$

and defines an $\mathcal{H}(\{A_n\}; h_1; \overline{\Omega_{r_1}^{ext}})$-$\mathcal{H}(\{A_n\}; h_3; \overline{\Omega_{r_3}^{ext}})$ *product kernel denoted by* $K *_{\mathcal{H}(\{A_n\};h_2;\overline{\Omega_{r_2}^{ext}})} L$.

5.2 Scaling Functions

Definition 5.6 *Let* $\{A_n\}_{n\in\mathbb{N}_0} \subset \mathbb{R}_0^+$ *be a sequence of non-negative real numbers,* $r \in \mathbb{R}^+$ *and* $h \in \{0,1,2\}$. *Let* $\{\Phi_j\}_{j\in\mathbb{N}_0}$ *be a family of* $\mathcal{H}(\{A_n\}; h; \overline{\Omega_r^{ext}})$*-product kernels* $\Phi_j : \overline{\Omega_r^{ext}} \times \overline{\Omega_r^{ext}} \to \mathbb{R}$ *with symbol* $\{(\Phi_j)^\wedge(n)\}_{n\in\mathcal{N}}$, *i.e., for* $j \in \mathbb{N}_0$, Φ_j *is defined by*

$$\Phi_j(x,y) := \sum_{n\in\mathcal{N}} \sum_{k=1}^{2n+1} (\Phi_j)^\wedge(n) \frac{1}{A_n^2} H_{n,k}(r; h; x) H_{n,k}(r; h; y),\ \ x, y, \in \overline{\Omega_r^{ext}},$$

and for all $j \in \mathbb{N}_0$ *the symbol* $\{(\Phi_j)^\wedge(n)\}_{n\in\mathcal{N}}$ *satisfies conditions (5.1) and (5.2) in Definition 5.1. Suppose that the family of sequences* $\{\{(\Phi_j)^\wedge(n)\}_{n\in\mathcal{N}}\}_{j\in\mathbb{N}_0}$ *has the following additional properties:*
(i) $(\Phi_j)^\wedge(n) \geq 0$ *for all* $n \in \mathcal{N}$ *and* $j \in \mathbb{N}_0$,
(ii) $(\Phi_{j'})^\wedge(n) \geq (\Phi_j)^\wedge(n)$ *for all* $n \in \mathcal{N}$ *and all* $j, j' \in \mathbb{N}_0$ *with* $j' \geq j$, *and*
(iii) $\lim_{j\to\infty} (\Phi_j)^\wedge(n) = 1$ *for all* $n \in \mathcal{N}$.
Then $\{\Phi_j\}_{j\in\mathbb{N}_0}$ *is called a (scale discrete) linear scaling function for the Hilbert space* $\mathcal{H}(\{A_n\}; h; \overline{\Omega_r^{ext}})$. *The family of sequences* $\{\{(\Phi_j)^\wedge(n)\}_{n\in\mathcal{N}}\}_{j\in\mathbb{N}_0}$ *is called the generating symbol of the linear scaling function.*

Definition 5.7 *Let* $\{A_n\}_{n\in\mathbb{N}_0} \subset \mathbb{R}_0^+$ *be a sequence of non-negative real numbers, let* $r \in \mathbb{R}^+$ *and* $h \in \{0,1,2\}$. *Suppose that* $\{\Phi_j\}_{j\in\mathbb{N}_0}$ *is a linear scaling function for* $\mathcal{H}(\{A_n\}; h; \overline{\Omega_r^{ext}})$ *with generating symbol* $\{\{(\Phi_j)^\wedge(n)\}_{n\in\mathcal{N}}\}_{j\in\mathbb{N}_0}$. *Then the iterated product kernel* $\{\Phi_j^{(2)}\}_{j\in\mathbb{N}_0}$, *defined by*

$$\Phi_j^{(2)}(x,y) := \Phi_j(x,\cdot) *_{\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})} \Phi_j(\cdot,y),\ \ x, y, \in \overline{\Omega_r^{ext}},$$

is called a (scale discrete) bilinear scaling function for $\mathcal{H}(\{A_n\}; h; \overline{\Omega_r^{ext}})$ *with generating symbol* $\{\{(\Phi_j)^\wedge(n)\}_{n\in\mathcal{N}}\}_{j\in\mathbb{N}_0}$.

Lemma 5.8 *Let the assumptions be the same as in Definition 5.6. Then the iterated kernel* $\{\Phi_j^{(2)}\}_{j\in\mathbb{N}_0}$ *introduced in Definition 5.7 is a linear scaling function for* $\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})$ *with generating symbol* $\{\{(\Phi_j^{(2)})^\wedge(n)\}_{n\in\mathcal{N}}\}_{j\in\mathbb{N}_0}$ *given by* $(\Phi_j^{(2)})^\wedge(n) = ((\Phi_j)^\wedge(n))^2$.

Proof: We have to verify that $\{\{((\Phi_j)^\wedge(n))^2\}_{n\in\mathbb{N}_0}\}_{j\in\mathbb{N}_0}$ satisfies properties (i) to (iii) in Definition 5.6.
(i) is clear and (ii) follows from the fact that $(\Phi_{j'})^\wedge(n) \geq (\Phi_j)^\wedge(n)$ for all $n \in \mathcal{N}$ and all $j, j' \in \mathbb{N}_0$ with $j' \geq j$. (iii) is a consequence of the fact that $\lim_{j\to\infty}(\Phi_j)^\wedge(n) = 1$ for all $n \in \mathcal{N}$. □

In the next definition we introduce a special type of a linear scaling function for $\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})$ which shows a certain 'reproducing property'. This property can be exploited to construct a recursive scheme for the fast evaluation of the multiscale approximation of the solution to an ill-posed pseudodifferential operator equation.

Definition 5.9 *Let* $\{A_n\}_{n\in\mathbb{N}_0} \subset \mathbb{R}_0^+$ *be a sequence of non-negative real numbers, let* $r \in \mathbb{R}^+$ *and* $h \in \{0,1,2\}$. *Assume that* $\{\{(\Phi_j^R)^\wedge(n)\}_{n\in\mathcal{N}}\}_{j\in\mathbb{N}_0} \subset \mathbb{R}^+$ *is a family of sequences of non-negative real numbers, which satisfies the conditions ((5.1) and (5.2) in Definition 5.1 and in addition*
(i) $(\Phi_j^R)^\wedge(n) > 0$ *for all* $n \in \mathcal{N}$ *and* $j \in \mathbb{N}_0$,
(ii) $(\Phi_{j'}^R)^\wedge(n) \geq (\Phi_j^R)^\wedge(n)$ *for all* $n \in \mathcal{N}$ *and all* $j, j' \in \mathbb{N}_0$ *with* $j' \geq j$,
(iii) $\lim\limits_{j\to\infty} (\Phi_j^R)^\wedge(n) = 1$ *for all* $n \in \mathcal{N}$, *and*
(iv) $(\Phi_j^R)^\wedge(n) = \left((\Phi_{j+1}^R)^\wedge(n)\right)^2$ *for all* $n \in \mathcal{N}$ *and* $j \in \mathbb{N}_0$.
Then the family of $\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})$*-product kernels* $\{\Phi_j^R\}_{j\in\mathbb{N}_0}$, *defined by*

$$\Phi_j^R(x,y) := \sum_{n\in\mathcal{N}} \sum_{k=1}^{2n+1} (\Phi_j^R)^\wedge(n) \frac{1}{A_n^2} H_{n,k}(r;h;x) H_{n,k}(r;h;y), \;\; x, y, \in \overline{\Omega_r^{ext}}$$

is called a reproducing (R-scale) scaling function for $\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})$ *with generating symbol* $\{\{(\Phi_j^R)^\wedge(n)\}_{n\in\mathcal{N}}\}_{j\in\mathbb{N}_0}$.

Remark 5.10 *Note that in Definition 5.9 we need the generating symbol of a reproducing scaling function to satisfy* $(\Phi_j^R)^\wedge(n) > 0$ *for all* $n \in \mathcal{N}$ *and* $j \in \mathbb{N}_0$ *in contrast to Definition 5.6 of a linear scaling function, where we allowed that* $(\Phi_j)^\wedge(n) \geq 0$ *for all* $n \in \mathcal{N}$ *and* $j \in \mathbb{N}_0$. *This is due to the fact that whenever there exists a* $j_0 \in \mathbb{N}_0$ *and an* $n_0 \in \mathcal{N}$ *such that* $(\Phi_{j_0})^\wedge(n_0) = 0$, *the reproducing property (iv) in Definition 5.9 implies that* $(\Phi_j)^\wedge(n_0) = 0$ *for all* $j \in \mathbb{N}_0$, *and thus condition (iii) cannot be satisfied.*

Lemma 5.11 *Let* $\{A_n\}_{n\in\mathbb{N}_0} \subset \mathbb{R}_0^+$ *be a sequence of non-negative real numbers,* $r \in \mathbb{R}^+$ *and* $h \in \{0,1,2\}$. *Let* $\{\Phi_j^R\}_{j\in\mathbb{N}_0}$ *be a reproducing scaling function for* $\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})$. *Then*

$$\Phi_j^R = \Phi_{j+1}^R *_{\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})} \Phi_{j+1}^R \text{ for all } j \in \mathbb{N}_0.$$

Proof: This follows from property (iv) in Definition 5.9. □

Definition 5.12 *Let $\{A_n\}_{n\in\mathbb{N}_0} \subset \mathbb{R}_0^+$ be a sequence of real numbers, let $r \in \mathbb{R}^+$ and $h \in \{0,1,2\}$, and assume that $\{\Phi_j\}_{j\in\mathbb{N}_0}$ is either a linear, bilinear or reproducing scaling function for $\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})$. Define the family $\{P_j\}_{j\in\mathbb{N}_0}$ of operators by*

$$P_j : \mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}}) \to \mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}}),\ F \mapsto P_j(F) := \Phi_j *_{\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})} F$$

and the family of scale spaces $\{\mathcal{V}_j(h;\overline{\Omega_r^{ext}})\}_{j\in\mathbb{N}_0}$ by

$$\mathcal{V}_j(h;\overline{\Omega_r^{ext}}) := \mathrm{im}(P_j) = P_j(\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})).$$

Note that due to Lemma 5.2

$$(P_jF)(x) = \Phi_j(x,\cdot) *_{\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})} F = \sum_{n\in\mathcal{N}} \sum_{k=1}^{2n+1} (\Phi_j)^\wedge(n) F_{n,k}^r\, H_{n,k}(r;h,x).$$

The fact that a scaling function $\{\Phi_j\}_{j\in\mathbb{N}_0}$ has the property that $\lim_{j\to\infty}(\Phi_j)^\wedge(n) = 1$ for all $n \in \mathcal{N}$ implies that P_jF should in some sense converge to F if $j \to \infty$. This is investigated in the next theorem.

Theorem 5.13 *Let $\{A_n\}_{n\in\mathbb{N}_0}$ be a sequence of non-negative real numbers, $r \in \mathbb{R}^+$ and $h \in \{0,1,2\}$, and assume that $\{\Phi_j\}_{j\in\mathbb{N}_0}$ is either a linear, bilinear or reproducing scaling function for $\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})$. The family of operators $\{P_j\}_{j\in\mathbb{N}_0}$ introduced in Definition 5.12 generates an approximate identity in $\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})$, i.e.,*

$$\lim_{j\to\infty} \|F - P_jF\|_{\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})} = 0$$

for all $F \in \mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})$.

Proof:

$$\begin{aligned}
\lim_{j\to\infty} \|F - P_jF\|_{\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})} &= \lim_{j\to\infty} \|F - \Phi_j(\cdot,\cdot) *_{\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})} F\|_{\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})} \\
&= \lim_{j\to\infty} \left(\sum_{n\in\mathcal{N}} \sum_{k=1}^{2n+1} A_n^2 (1 - (\Phi_j)^\wedge(n))^2 (F_{n,k}^r)^2 \right)^{1/2} \\
&= \left(\sum_{n\in\mathcal{N}} \sum_{k=1}^{2n+1} A_n^2 \left(\lim_{j\to\infty} (1 - (\Phi_j)^\wedge(n))^2 \right) (F_{n,k}^r)^2 \right)^{1/2} \\
&= 0,
\end{aligned}$$

where we may interchange sum and limit due to the fact that $F \in \mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})$ and $|1 - (\Phi_j)^\wedge(n)| \le 2$ for all $j \in \mathbb{N}_0$ and for all $n \in \mathcal{N}$. □

Theorem 5.14 (Muliresolution Analysis) *Let* $\{A_n\}_{n\in\mathbb{N}_0} \subset \mathbb{R}_0^+$ *be a sequence of non-negative real numbers,* $r \in \mathbb{R}^+$ *and* $h \in \{0,1,2\}$*, and let* $\{\Phi_j\}_{j\in\mathbb{N}_0}$ *either be a linear, bilinear or reproducing scaling function for* $\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})$*. Then the family of scale spaces* $\{\mathcal{V}_j(h;\overline{\Omega_r^{ext}})\}_{j\in\mathbb{N}_0}$ *corresponding to the family of operators* $\{P_j\}_{j\in\mathbb{N}_0}$ *introduced in Definition 5.12, is a multiresolution analysis of* $\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})$*, i.e.,*

(i) $\mathcal{V}_0(h;\overline{\Omega_r^{ext}}) \subset \ldots \subset \mathcal{V}_j(h;\overline{\Omega_r^{ext}}) \subset \mathcal{V}_{j'}(h;\overline{\Omega_r^{ext}}) \subset \ldots \subset \mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})$ *for* $j' \geq j$, $j \in \mathbb{N}_0$,

and

(ii) $\overline{\bigcup_{j\in\mathbb{N}_0} \mathcal{V}_j(h;\overline{\Omega_r^{ext}})}^{\|\cdot\|_{\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})}} = \mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})$.

Proof: As Φ_j is a product kernel we know by Lemma 5.2 that $\Phi_j *_{\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})} F$ is a function in $\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})$. Thus, $\mathcal{V}_j(h;\overline{\Omega_r^{ext}}) \subset \mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})$ for all $j \in \mathbb{N}_0$. In order to verify $\mathcal{V}_j(h;\overline{\Omega_r^{ext}}) \subset \mathcal{V}_{j'}(h;\overline{\Omega_r^{ext}})$ for $j' \geq j$, we have to show that for every $F \in \mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})$, $P_jF \in \mathcal{V}_{j'}(h;\overline{\Omega_r^{ext}})$. But this means that there exists an $G \in \mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})$ such that $P_{j'}G = P_jF$.

$$\begin{aligned} P_jF &= \sum_{n\in\mathcal{N}} \sum_{k=1}^{2n+1} (\Phi_j)^\wedge(n) F_{n,k}^r \, H_{n,k}(r;h;\cdot) \\ &\stackrel{!}{=} \sum_{n\in\mathcal{N}} \sum_{k=1}^{2n+1} (\Phi_{j'})^\wedge(n)\, G_{n,k}^r \, H_{n,k}(r;h;\cdot) = P_{j'}G \\ \Leftrightarrow \quad & (\Phi_j)^\wedge(n) F_{n,k}^r \stackrel{!}{=} (\Phi_{j'})^\wedge(n)\, G_{n,k}^r \quad \text{for all} \quad n \in \mathcal{N}. \end{aligned}$$

Define $G \in \mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})$ by

$$G_{n,k}^r := \begin{cases} \frac{(\Phi_j)^\wedge(n)}{(\Phi_{j'})^\wedge(n)} F_{n,k}^r & \text{if} \quad (\Phi_{j'})^\wedge(n) \neq 0 \\ 0 & \text{if} \quad (\Phi_{j'})^\wedge(n) = 0. \end{cases}$$

Because of $0 \leq (\Phi_j)^\wedge(n) \leq (\Phi_{j'})^\wedge(n)$ and $F \in \mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})$ the function G is obviously in $\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})$, and by construction $P_{j'}G = P_jF$. Statement (ii) is a direct consequence of Theorem 5.13. □

5.3 Wavelets

Definition 5.15 *Let* $\{A_n\}_{n\in\mathbb{N}_0} \subset \mathbb{R}_0^+$ *be a sequence of non-negative real numbers, let* $r \in \mathbb{R}^+$ *and* $h \in \{0,1,2\}$*. Let* $\{\Phi_j\}_{j\in\mathbb{N}_0}$ *be a linear scaling function for* $\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})$ *with generating symbol* $\{\{(\Phi_j)^\wedge(n)\}_{n\in\mathcal{N}}\}_{j\in\mathbb{N}_0}$*. Define the family of sequences of non-negative real numbers* $\{\{(\Psi_j)^\wedge(n)\}_{n\in\mathcal{N}}\}_{j\in\mathbb{N}_0} \subset \mathbb{R}_0^+$ *by*

$$(\Psi_j)^\wedge(n) := (\Phi_{j+1})^\wedge(n) - (\Phi_j)^\wedge(n) \quad \textit{for all} \quad n \in \mathcal{N} \quad \textit{and all} \quad j \in \mathbb{N}_0.$$

Then the family $\{\Psi_j\}_{j\in\mathbb{N}_0}$ *of* $\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})$*-product kernels* $\Psi_j : \overline{\Omega_r^{ext}} \times \overline{\Omega_r^{ext}} \to \mathbb{R}$*, defined by*

$$\Psi_j(x,y) := \sum_{n\in\mathcal{N}} \sum_{k=1}^{2n+1} (\Psi_j)^\wedge(n) \frac{1}{A_n^2} H_{n,k}(r;h;x) H_{n,k}(r;h;y),\ x,y \in \overline{\Omega_r^{ext}},\ j \in \mathbb{N}_0,$$

is called the (scale discrete) linear wavelet corresponding to the linear scaling function $\{\Phi_j\}_{j\in\mathbb{N}_0}$*. The family* $\{\{(\Psi_j)^\wedge(n)\}_{n\in\mathcal{N}}\}_{j\in\mathbb{N}_0}$ *is called the generating symbol of the linear wavelet corresponding to* $\{\Phi_j\}_{j\in\mathbb{N}_0}$*.*

Definition 5.16 *Let* $\{A_n\}_{n\in\mathbb{N}_0} \subset \mathbb{R}_0^+$ *be a sequence of non-negative real numbers, let* $r \in \mathbb{R}^+$ *and* $h \in \{0,1,2\}$*. Let* $\{\Phi_j^{(2)}\}_{j\in\mathbb{N}_0}$ *be a bilinear scaling function for* $\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})$ *with generating symbol* $\{\{(\Phi_j)^\wedge(n)\}_{n\in\mathcal{N}}\}_{j\in\mathbb{N}_0}$*. Let* $\{\{(\Psi_j)^\wedge(n)\}_{n\in\mathcal{N}}\}_{j\in\mathbb{N}_0}$ *and* $\{\{(\tilde{\Psi}_j)^\wedge(n)\}_{n\in\mathcal{N}}\}_{j\in\mathbb{N}_0}$ *be families of sequences of non-negative real numbers which satisfy the refinement equation*

$$(\tilde{\Psi}_j)^\wedge(n)(\Psi_j)^\wedge(n) = ((\Phi_{j+1})^\wedge(n))^2 - ((\Phi_j)^\wedge(n))^2 \tag{5.5}$$

for all $n \in \mathcal{N}$ *and* $j \in \mathbb{N}_0$*, where we set* $(\tilde{\Psi}_j)^\wedge(n) = (\Psi_j)^\wedge(n) = 0$ *whenever the right-hand-side of (5.5) is equal to zero. Then the family* $\{\Psi_j\}_{j\in\mathbb{N}_0}$ *of* $\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})$*-product kernels* $\Psi_j : \overline{\Omega_r^{ext}} \times \overline{\Omega_r^{ext}} \to \mathbb{R}$*, defined by*

$$\Psi_j(x,y) := \sum_{n\in\mathcal{N}} \sum_{k=1}^{2n+1} (\Psi_j)^\wedge(n) \frac{1}{A_n^2} H_{n,k}(r;h;x) H_{n,k}(r;h;y),\ x,y \in \overline{\Omega_r^{ext}},\ j \in \mathbb{N}_0,$$

is called a primal wavelet corresponding to the bilinear scaling function $\{\Phi_j^{(2)}\}_{j\in\mathbb{N}_0}$*. The family* $\{\tilde{\Psi}_j\}_{j\in\mathbb{N}_0}$ *of* $\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})$*-product kernels* $\tilde{\Psi}_j : \overline{\Omega_r^{ext}} \times \overline{\Omega_r^{ext}} \to \mathbb{R}$*, defined by*

$$\tilde{\Psi}_j(x,y) := \sum_{n\in\mathcal{N}} \sum_{k=1}^{2n+1} (\tilde{\Psi}_j)^\wedge(n) \frac{1}{A_n^2} H_{n,k}(r;h;x) H_{n,k}(r;h;y),\ x,y \in \overline{\Omega_r^{ext}},\ j \in \mathbb{N}_0,$$

is called the dual wavelet accompanying $\{\Psi_j\}_{j\in\mathbb{N}_0}$*.*
The families $\{\{(\Psi_j)^\wedge(n)\}_{n\in\mathcal{N}}\}_{j\in\mathbb{N}_0}$ *and* $\{\{(\tilde{\Psi}_j)^\wedge(n)\}_{n\in\mathcal{N}}\}_{j\in\mathbb{N}_0}$*, respectively, are called the generating symbol of the primal and the dual wavelet, respectively.*

The next definition gives two methods of how to construct a primal wavelet corresponding to a bilinear scaling function $\{\Phi_j^{(2)}\}_{j\in\mathbb{N}_0}$ and its accompanying dual wavelet.

Definition 5.17 *Let* $\{\Phi_j\}_{j\in\mathbb{N}_0}$ *be a linear scaling function for* $\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})$ *and let* $\{\Phi_j^{(2)}\}_{j\in\mathbb{N}_0}$ *be the bilinear scaling function for* $\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})$ *defined via* $\Phi_j^{(2)} := \Phi_j *_{\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})} \Phi_j$ *with generating symbol* $\{\{(\Phi_j)^\wedge(n)\}_{n\in\mathcal{N}}\}_{j\in\mathbb{N}_0}$*. Let the generating* *symbols* $\{\{(\Psi_j^P)^\wedge(n)\}_{n\in\mathcal{N}}\}_{j\in\mathbb{N}_0}$ *and* $\{\{(\tilde{\Psi}_j^P)^\wedge(n)\}_{n\in\mathcal{N}}\}_{j\in\mathbb{N}_0}$ *of a primal wavelet corresponding to* $\{\Phi_j^{(2)}\}_{j\in\mathbb{N}_0}$ *and its accompanying dual wavelet be defined by*

$$(\Psi_j^P)^\wedge(n) = (\tilde{\Psi}_j^P)^\wedge(n) := \left(((\Phi_{j+1})^\wedge(n))^2 - ((\Phi_j)^\wedge(n))^2\right)^{1/2} \quad \textit{for all} \quad n \in \mathcal{N},\ j \in \mathbb{N}_0.$$

Then $\{\Psi_j^P\}_{j\in\mathbb{N}_0}$, *is called the P-scale (packet-scale) wavelet corresponding to the bilinear scaling function* $\{\Phi_j^{(2)}\}_{j\in\mathbb{N}_0}$.
Let the generating symbol $\{\{(\Psi_j^M)^\wedge(n)\}_{n\in\mathcal{N}}\}_{j\in\mathbb{N}_0}$ *of a primal wavelet corresponding to* $\{\Phi_j^{(2)}\}_{j\in\mathbb{N}_0}$ *be defined by*

$$(\Psi_j^M)^\wedge(n) := (\Phi_{j+1})^\wedge(n) - (\Phi_j)^\wedge(n) \quad \textit{for all} \quad n \in \mathcal{N},\ j \in \mathbb{N}_0.$$

The generating symbol $\{\{(\tilde{\Psi}_j^M)^\wedge(n)\}_{n\in\mathcal{N}}\}_{j\in\mathbb{N}_0}$ *of its accompanying dual wavelet is given by*

$$(\tilde{\Psi}_j^M)^\wedge(n) := (\Phi_{j+1})^\wedge(n) + (\Phi_j)^\wedge(n) \quad \textit{for all} \quad n \in \mathbb{N}_0,\ j \in \mathbb{N}_0.$$

Then $\{\Psi_j^M\}_{j\in\mathbb{N}_0}$ *is called the M-scale (modified-packet-scale) primal and* $\{\tilde{\Psi}_j^M\}_{j\in\mathbb{N}_0}$ *the accompanying M-scale dual wavelet corresponding to the bilinear scaling function* $\{\Phi_j^{(2)}\}_{j\in\mathbb{N}_0}$.

Definition 5.18 *Let* $\{\Phi_j^R\}_{j\in\mathbb{N}_0}$ *be a reproducing scaling function for* $\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})$ *with generating symbol* $\{\{(\Phi_j^R)^\wedge(n)\}_{n\in\mathcal{N}}\}_{j\in\mathbb{N}_0}$. *Let* $\{\{(\Psi_j^{RM})^\wedge(n)\}_{n\in\mathcal{N}}\}_{j\in\mathbb{N}_0}$ *and* $\{\{(\tilde{\Psi}_j^{RM})^\wedge(n)\}_{n\in\mathcal{N}}\}_{j\in\mathbb{N}_0}$ *be the families of sequences of non-negative real numbers given by*

$$(\Psi_j^{RM})^\wedge(n) := (\Phi_{j+1}^R)^\wedge(n) - (\Phi_j^R)^\wedge(n) \quad \textit{for} \quad n \in \mathcal{N},\ j \in \mathbb{N}_0$$

and

$$(\tilde{\Psi}_j^{RM})^\wedge(n) := (\Phi_{j+1}^R)^\wedge(n) + (\Phi_j^R)^\wedge(n) \quad \textit{for} \quad n \in \mathcal{N},\ j \in \mathbb{N}_0,$$

respectively.
Then the familes $\{\Psi_j^{RM}\}_{j\in\mathbb{N}_0}$ *and* $\{\tilde{\Psi}_j^{RM}\}_{j\in\mathbb{N}_0}$ *of* $\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})$*-product kernels, defined by*

$$\Psi_j^{RM}(x,y) := \sum_{n\in\mathcal{N}}\sum_{k=1}^{2n+1}(\Psi_j^{RM})^\wedge(n)\frac{1}{A_n^2}H_{n,k}(r;h;x)H_{n,k}(r;h;y),\ x,y\in\overline{\Omega_r^{ext}},\ j\in\mathbb{N}_0,$$

and

$$\tilde{\Psi}_j^{RM}(x,y) := \sum_{n\in\mathcal{N}}\sum_{k=1}^{2n+1}(\tilde{\Psi}_j^{RM})^\wedge(n)\frac{1}{A_n^2}H_{n,k}(r;h;x)H_{n,k}(r;h;y),\ x,y\in\overline{\Omega_r^{ext}},\ j\in\mathbb{N}_0,$$

are called the primal M-scale wavelet corresponding to $\{\Phi_j^R\}_{j\in\mathbb{N}_0}$ *and its accompanying dual M-scale wavelet.*

Lemma 5.19 *Let $\{\Phi_j^R\}_{j\in\mathbb{N}_0}$ be a reproducing scaling function for $\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})$. Let $\{\Psi_j^{RM}\}_{j\in\mathbb{N}_0}$ and $\{\tilde{\Psi}_j^{RM}\}_{j\in\mathbb{N}_0}$ denote the primal M-scale wavelet corresponding to $\{\Phi_j^R\}_{j\in\mathbb{N}_0}$ and its accompanying dual M-scale wavelet. Then*

$$\Psi_j^{RM} = \tilde{\Psi}_{j+1}^{RM} *_{\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})} \Psi_{j+1}^{RM} \text{ for all } j \in \mathbb{N}_0.$$

Proof: By construction, $\{\{(\Psi_j^{RM})^\wedge(n)\}_{n\in\mathcal{N}}\}_{j\in\mathbb{N}_0}$ and $\{\{(\tilde{\Psi}_j^{RM})^\wedge(n)\}_{n\in\mathcal{N}}\}_{j\in\mathbb{N}_0}$ satisfy the refinement equation

$$\begin{aligned} (\tilde{\Psi}_{j+1}^{RM})^\wedge(n)(\Psi_{j+1}^{RM})^\wedge(n) &= ((\Phi_{j+2}^R)^\wedge(n))^2 - ((\Phi_{j+1}^R)^\wedge(n))^2 \\ &= (\Phi_{j+1}^R)^\wedge(n) - (\Phi_j^R)^\wedge(n) \\ &= (\Psi_j^{RM})^\wedge(n), \end{aligned}$$

which implies the statement. □

Definition 5.20 *Let $\{A_n\}_{n\in\mathbb{N}_0} \subset \mathbb{R}_0^+$ be a sequence of non-negative real numbers, let $r \in \mathbb{R}^+$ and $h \in \{0,1,2\}$. Suppose that $\{\Phi_j\}_{j\in\mathbb{N}_0}$ is a linear scaling function for $\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})$ and let $\{\Psi_j\}_{j\in\mathbb{N}_0}$ be the corresponding linear wavelet. Define the family of operators $\{S_j\}_{j\in\mathbb{N}_0}$ by*

$$S_j : \mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}}) \to \mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}}),\ F \mapsto S_j(F) := \Psi_j *_{\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})} F.$$

If $\{\Phi_j^{(2)}\}_{j\in\mathbb{N}_0}$ is a bilinear scaling function for $\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})$ and $\{\Psi_j\}_{j\in\mathbb{N}_0}$ and $\{\tilde{\Psi}_j\}_{j\in\mathbb{N}_0}$ are a primal wavelet corresponding to $\{\Phi_j^{(2)}\}_{j\in\mathbb{N}_0}$ and its accompanying dual wavelet, define the family of operators$\{S_j\}_{j\in\mathbb{N}_0}$ by

$$\begin{aligned} &S_j : \mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}}) \to \mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}}), \\ &F \mapsto S_j(F) := \tilde{\Psi}_j *_{\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})} \Psi_j *_{\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})} F. \end{aligned}$$

In case that $\{\Phi_j^R\}_{j\in\mathbb{N}_0}$ is a reproducing scaling function for $\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})$ and $\{\Psi_j^{RM}\}_{j\in\mathbb{N}_0}$ and $\{\tilde{\Psi}_j^{RM}\}_{j\in\mathbb{N}_0}$ are the primal M-scale wavelet corresponding to $\{\Phi_j^R\}_{j\in\mathbb{N}_0}$ and its accompanying dual M-scale wavelet, the family of operators $\{S_j\}_{j\in\mathbb{N}_0}$ is defined by

$$\begin{aligned} &S_j : \mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}}) \to \mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}}), \\ &F \mapsto S_j(F) := \Psi_j^{RM} *_{\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})} F = \tilde{\Psi}_{j+1}^{RM} *_{\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})} \Psi_{j+1}^{RM} *_{\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})} F. \end{aligned}$$

The family of detail spaces $\{\mathcal{W}_j(h;\overline{\Omega_r^{ext}})\}_{j\in\mathbb{N}_0}$ is in all three cases defined by

$$\mathcal{W}_j(h;\overline{\Omega_r^{ext}}) := \mathrm{im}(S_j) = S_j(\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})).$$

Theorem 5.21 *Let $\{A_n\}_{n\in\mathbb{N}_0} \subset \mathbb{R}_0^+$ be a sequence of non-negative real numbers, let $r \in \mathbb{R}^+$ and $h \in \{0,1,2\}$. Suppose that $\{\Phi_j\}_{j\in\mathbb{N}_0}$ is a linear scaling function for $\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})$ (and $\{\Phi_j^{(2)}\}_{j\in\mathbb{N}_0}$ and $\{\Phi_j^R\}_{j\in\mathbb{N}_0}$ a bilinear and a reproducing scaling function for $\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})$, respectively). Let the linear wavelet $\{\Psi_j\}_{j\in\mathbb{N}_0}$ corresponding to $\{\Phi_j\}_{j\in\mathbb{N}_0}$*

(and a primal wavelet and its accompanying dual wavelet $\{\Psi_j\}_{j\in\mathbb{N}_0}$ *and* $\{\tilde{\Psi}_j\}_{j\in\mathbb{N}_0}$ *corresponding to* $\{\Phi_j^{(2)}\}_{j\in\mathbb{N}_0}$ *and the M-scale primal wavelet and its accompanying dual M-scale wavelet* $\{\Psi_j^{RM}\}_{j\in\mathbb{N}_0}$ *and* $\{\tilde{\Psi}_j^{RM}\}_{j\in\mathbb{N}_0}$ *corresponding to* $\{\Phi_j^R\}_{j\in\mathbb{N}_0}$*, respectively). Then the families of operators* $\{P_j\}_{j\in\mathbb{N}_0}$ *and* $\{S_j\}_{j\in\mathbb{N}_0}$ *defined according to Definitions 5.12 and 5.20, respectively, satisfy the relations*

$$P_{j+1} = P_j + S_j \text{ for all } j \in \mathbb{N}_0 \tag{5.6}$$

and

$$P_{J+1} = P_{J_0} + \sum_{j=J_0}^{J} S_j = P_0 + \sum_{j=0}^{J} S_j, \ J, J_0 \in \mathbb{N}_0, \ J > J_0, \tag{5.7}$$

and each function $F \in \mathcal{H}(\{A_n\}; h; \overline{\Omega_r^{ext}})$ *can be reconstructed in* $\mathcal{H}(\{A_n\}; h; \overline{\Omega_r^{ext}})$*-sense by*

$$F = P_{J_0}F + \lim_{J\to\infty} \sum_{j=J_0}^{J} S_j F = P_0 F + \lim_{J\to\infty} \sum_{j=0}^{J} S_j F. \tag{5.8}$$

Proof: Relations (5.6) and (5.7) can be easily verified by just inserting the definitions of the respective scaling functions and their corresponding wavelets into the definition of $\{P_j\}_{j\in\mathbb{N}_0}$ and $\{S_j\}_{j\in\mathbb{N}_0}$, respectively. The limit relation (5.8) then follows by Theorem 5.13. □

An immediate consequence of Theorem 5.21 is a further decomposition of the scale spaces $\{\mathcal{V}_j(h; \overline{\Omega_r^{ext}})\}_{j\in\mathbb{N}_0}$:

Corollary 5.22 *Let the assumptions be the same as in Theorem 5.21. Let the families of scale and detail spaces* $\{\mathcal{V}_j(h; \overline{\Omega_r^{ext}})\}_{j\in\mathbb{N}_0}$ *and* $\{\mathcal{W}_j(h; \overline{\Omega_r^{ext}})\}_{j\in\mathbb{N}_0}$ *in* $\mathcal{H}(\{A_n\}; h; \overline{\Omega_r^{ext}})$ *be given according to Definition 5.12 and 5.20, respectively. Then*

$$\mathcal{V}_{j+1}(h; \overline{\Omega_r^{ext}}) = \mathcal{V}_j(h; \overline{\Omega_r^{ext}}) + \mathcal{W}_j(h; \overline{\Omega_r^{ext}}) \text{ for all } j \in \mathbb{N}_0$$

and

$$\mathcal{V}_{J+1}(h; \overline{\Omega_r^{ext}}) = \mathcal{V}_{J_0}(h; \overline{\Omega_r^{ext}}) + \sum_{j=J_0}^{J} \mathcal{W}_j(h; \overline{\Omega_r^{ext}}) = \mathcal{V}_0(h; \overline{\Omega_r^{ext}}) + \sum_{j=0}^{J} \mathcal{W}_j(h; \overline{\Omega_r^{ext}})$$

for all $J_0, J \in \mathbb{N}_0, \ J_0 < J$.

Proof: The statement can be verified by calculations analogous to those in the proof of Theorem 5.14. □

Relation (5.8) in Theorem 5.21 can be interpreted in the following way: $P_{J_0}F$ is a 'basic approximation' or a low-pass-filtered version of the function F, whereas S_jF, $j \in \{J_0, J_0+1, \ldots, J\}$, are band-pass-filtered versions or details of F. Thus the operators P_j can be interpreted as low-pass filters and the operators S_j as band-pass filters, and S_j describes the difference between the two low-pass filters P_j and P_{j+1}.

5.4 Regularization by Multiresolution

In this section we will use the concepts of multiresolution analysis to construct a regularizing filter to an ill-posed pseudodifferential operator equation $\Lambda F = G$, where we assume that $\{A_n\}_{n\in\mathbb{N}_0} \subset \mathbb{R}_0^+$ is a sequence of non-negative real numbers, $r_1, r_2 \in \mathbb{R}^+$, $h_1, h_2 \in \{0,1,2\}$ and $\Lambda : \mathcal{H}(\{A_n\}; h_1; \overline{\Omega_{r_1}^{ext}}) \to \mathcal{H}(\{A_n\}; h_2; \overline{\Omega_{r_2}^{ext}})$ is an injective (and compact) pseudodifferential operator, whose symbol $\{\Lambda^\wedge(n)\}_{n\in\mathbb{N}_0}$ satisfies the limit relation $\lim_{n\to\infty} |\Lambda^\wedge(n)| = 0$, and $F \in \mathcal{H}(\{A_n\}; h_1; \overline{\Omega_{r_1}^{ext}})$, $G \in \mathcal{H}(\{A_n\}; h_2; \overline{\Omega_{r_2}^{ext}})$. For the simplicity of the notation of the singular system we will always assume that $\Lambda^\wedge(n) > 0$ for all $n \in \mathcal{N}$. This means no loss of generality for the discussion of the SST-/SGG-problem. The notion of a regularization multiresolution analysis itself traces back to [Schn1997] and [FrSchn1998].

Definition 5.23 *Let $\{A_n\}_{n\in\mathbb{N}_0} \subset \mathbb{R}_0^+$ be a sequence of non-negative real numbers, let $r_1, r_2 \in \mathbb{R}^+$ and $h_1, h_2 \in \{0,1,2\}$. Suppose that $\Lambda : \mathcal{H}(\{A_n\}; h_1; \overline{\Omega_{r_1}^{ext}}) \to \mathcal{H}(\{A_n\}; h_2; \overline{\Omega_{r_2}^{ext}})$ is an injective pseudodifferential operator whose symbol $\{\Lambda^\wedge(n)\}_{n\in\mathbb{N}_0}$ satisfies $\Lambda^\wedge(n) > 0$ for all $n \in \mathcal{N}$ and $\lim_{n\in\mathcal{N},\, n\to\infty} \Lambda^\wedge(n) = 0$, with singular system $(\Lambda^\wedge(n), A_n^{-1} H_{n,k}(r_1; h_1; \cdot), A_n^{-1} H_{n,k}(r_2; h_2; \cdot))$, $n \in \mathcal{N}$, $1 \le k \le 2n+1$. Let $\{\Phi_j^\Lambda\}_{j\in\mathbb{N}_0}$ be a family of $\mathcal{H}(\{A_n\}; h_1; \overline{\Omega_{r_1}^{ext}})$-$\mathcal{H}(\{A_n\}; h_2; \overline{\Omega_{r_2}^{ext}})$ product kernels $\Phi_j^\Lambda : \overline{\Omega_{r_1}^{ext}} \times \overline{\Omega_{r_1}^{ext}} \to \mathbb{R}$ with symbols $\{(\Phi_j^\Lambda)^\wedge(n)\}_{n\in\mathbb{N}_0}$, i.e., for $j \in \mathbb{N}_0$, Φ_j^Λ is defined by*

$$\Phi_j^\Lambda(x,y) := \sum_{n\in\mathcal{N}} \sum_{k=1}^{2n+1} (\Phi_j^\Lambda)^\wedge(n) \frac{1}{A_n^2} H_{n,k}(r_1; h_1; x) H_{n,k}(r_2; h_2; y),\ x \in \overline{\Omega_{r_1}^{ext}},\ y \in \overline{\Omega_{r_2}^{ext}}$$

and for all $j \in \mathbb{N}_0$ and $\{(\Phi_j^\Lambda)^\wedge(n)\}_{n\in\mathbb{N}_0}$ satisfies conditions (5.1) and (5.2) in Definition 5.1. Suppose that the family of sequences $\{\{(\Phi_j^\Lambda)^\wedge(n)\}_{n\in\mathcal{N}}\}_{j\in\mathbb{N}_0}$ has the additional properties

(i) $(\Phi_j^\Lambda)^\wedge(n) \ge 0$ for all $n \in \mathcal{N}$ and $j \in \mathbb{N}_0$,

(ii) $(\Phi_{j'}^\Lambda)^\wedge(n) \ge (\Phi_j^\Lambda)^\wedge(n)$ for all $n \in \mathcal{N}$ and all $j, j' \in \mathbb{N}_0$ with $j' \ge j$, and

(iii) $\lim_{j\to\infty} (\Phi_j^\Lambda)^\wedge(n) = (\Lambda^\wedge(n))^{-1}$ for all $n \in \mathcal{N}$.

Then $\{\Phi_j^\Lambda\}_{j\in\mathbb{N}_0}$ is called a (scale discrete) linear regularization scaling function for $\Lambda^{-1} : \mathcal{H}(\{A_n(\Lambda^\wedge(n))^{-1}\}; h_2; \overline{\Omega_{r_2}^{ext}}) \to \mathcal{H}(\{A_n\}; h_1; \overline{\Omega_{r_1}^{ext}})$. The family of sequences $\{\{(\Phi_j^\Lambda)^\wedge(n)\}_{n\in\mathcal{N}}\}_{j\in\mathbb{N}_0}$ is called the generating symbol of the linear regularization scaling function.

In order to construct a linear regularization scaling function $\{\Phi_j^\Lambda\}_{j\in\mathbb{N}_0}$ for $\Lambda^{-1} : \mathcal{H}(\{A_n(\Lambda^\wedge(n))^{-1}\}; h_2; \overline{\Omega_{r_2}^{ext}}) \to \mathcal{H}(\{A_n\}; h_1; \overline{\Omega_{r_1}^{ext}})$ we start from a linear scaling function $\{\Phi_j\}_{j\in\mathbb{N}_0}$ for $\mathcal{H}(\{A_n\}; h_2; \overline{\Omega_{r_2}^{ext}})$. Under the assumption that its generating symbol $\{\{(\Phi_j)^\wedge(n)\}_{n\in\mathcal{N}}\}_{j\in\mathbb{N}_0}$ satisfies certain summability conditions with respect to $\{\Lambda^\wedge(n)\}_{n\in\mathbb{N}_0}$ and $\{A_n\}_{n\in\mathbb{N}_0}$, we may define $\Phi_j^\Lambda := \Lambda^{-1}\Phi_j$, $j \in \mathbb{N}_0$. Alternatively, we may choose a regularization $\{\tilde{T}_j\}_{j\in\mathbb{N}_0}$ for $\Lambda^{-1} : \mathcal{H}(\{A_n(\Lambda^\wedge(n))^{-1}\}; h_2; \overline{\Omega_{r_2}^{ext}}) \to \mathcal{H}(\{A_n\}; h_1; \overline{\Omega_{r_1}^{ext}})$ which is based on a regularizing filter, and apply it to $\{\Phi_j\}_{j\in\mathbb{N}_0}$. In this case we define $\Phi_j^\Lambda := \tilde{T}_j \Phi_j$, $j \in \mathbb{N}_0$.

Theorem 5.24 *Let $\{A_n\}_{n\in\mathbb{N}_0} \subset \mathbb{R}_0^+$ be a sequence of non-negative real numbers, let $r_1, r_2 \in \mathbb{R}^+$ and $h_1, h_2 \in \{0,1,2\}$. Suppose that $\Lambda : \mathcal{H}(\{A_n\}; h_1; \overline{\Omega_{r_1}^{ext}}) \to \mathcal{H}(\{A_n\}; h_2; \overline{\Omega_{r_2}^{ext}})$ is an injective pseudodifferential operator whose symbol $\{\Lambda^\wedge(n)\}_{n\in\mathbb{N}_0}$ satisfies $\Lambda^\wedge(n) > 0$ for all $n \in \mathcal{N}$ and $\lim_{n\in\mathcal{N},\, n\to\infty} \Lambda^\wedge(n) = 0$. Suppose that $\{\Phi_j\}_{j\in\mathbb{N}_0}$ is a linear scaling function for $\mathcal{H}(\{A_n\}; h_2; \overline{\Omega_{r_2}^{ext}})$ with generating symbol $\{\{(\Phi_j)^\wedge(n)\}_{n\in\mathcal{N}}\}_{j\in\mathbb{N}_0}$.*
(i) Let $\{\{(\Phi_j)^\wedge(n)\}_{n\in\mathcal{N}}\}_{j\in\mathbb{N}_0}$ satisfy the summability conditions

$$\sum_{n\in\mathcal{N}} (2n+1) \left(\frac{(\Phi_j)^\wedge(n)}{\Lambda^\wedge(n)}\right)^2 \frac{1}{A_n^2} < \infty$$

and

$$\sum_{n\in\mathcal{N}} (2n+1) \left(\frac{(\Phi_j)^\wedge(n)}{\Lambda^\wedge(n)}\right)^2 < \infty.$$

Then $\{\{(\Phi_j^\Lambda)^\wedge(n)\}_{n\in\mathcal{N}}\}_{j\in\mathbb{N}_0}$, defined by

$$(\Phi_j^\Lambda)^\wedge(n) := (\Lambda^\wedge(n))^{-1}(\Phi_j)^\wedge(n), \quad \textit{for} \quad n \in \mathcal{N},\ j \in \mathbb{N}_0,$$

is the generating symbol of a linear regularization scaling function for $\Lambda^{-1} : \mathcal{H}(\{A_n(\Lambda^\wedge(n))^{-1}\}; h_2; \overline{\Omega_{r_2}^{ext}}) \to \mathcal{H}(\{A_n\}; h_1; \overline{\Omega_{r_1}^{ext}})$.
(ii) Let $q : \mathbb{N}_0 \times (0, \|\Lambda\|] \to \mathbb{R}$, $(j, \Lambda^\wedge(n)) \mapsto q(j, \Lambda^\wedge(n))$, $n \in \mathcal{N}$, be a regularizing filter for $\Lambda F = G$, $F \in \mathcal{H}(\{A_n\}; h_1; \overline{\Omega_{r_1}^{ext}})$, $G \in \mathcal{H}(\{A_n\}; h_2; \overline{\Omega_{r_2}^{ext}})$ with the following additional properties:

$$0 \le q(j,\sigma) \le 1 \quad \textit{for all} \quad j \in \mathbb{N}_0,\ \sigma \in (0, \|\Lambda\|]$$

and

$$q(j,\sigma) \le q(j',\sigma) \quad \textit{for all} \quad j, j' \in \mathbb{N}_0 \quad \textit{with} \quad j \le j' \quad \textit{and all} \quad \sigma \in (0, \|\Lambda\|].$$

Then $\{\{(\Phi_j^\Lambda)^\wedge(n)\}_{n\in\mathcal{N}}\}_{j\in\mathbb{N}_0}$, defined by

$$(\Phi_j^\Lambda)^\wedge(n) := (\Lambda^\wedge(n))^{-1} q(j, \Lambda^\wedge(n))(\Phi_j)^\wedge(n),\ n, j \in \mathbb{N}_0,$$

is the generating symbol of a linear regularization scaling function for $\Lambda^{-1} : \mathcal{H}(\{A_n(\Lambda^\wedge(n))^{-1}\}; h_2; \overline{\Omega_{r_2}^{ext}}) \to \mathcal{H}(\{A_n\}; h_1; \overline{\Omega_{r_1}^{ext}})$.

Proof: In both cases we have to verify that $\{\{(\Phi_j^\Lambda)^\wedge(n)\}_{n\in\mathcal{N}}\}_{j\in\mathbb{N}_0}$ satisfies conditions (5.1) and (5.2). In (i) we simply demand that those two conditions are satisfied. For (ii), note that due to the fact that q is a regularizing filter, there exists a constant $C(j) > 0$ such that for all $j \in \mathbb{N}_0$, $q(j, \Lambda^\wedge(n)) \leq C(j)\Lambda^\wedge(n)$ for $n \in \mathcal{N}$. This estimate and the fact that $\{\Phi_j\}_{j\in\mathbb{N}_0}$ is a linear scaling function for $\mathcal{H}(\{A_n\}; h_2; \overline{\Omega_{r_2}^{ext}})$ imply that for all $j \in \mathbb{N}_0$,

$$\begin{aligned}\sum_{n\in\mathcal{N}}(2n+1)((\Phi_j^\Lambda)^\wedge(n))^2\frac{1}{A_n^2} &= \sum_{n\in\mathcal{N}}(2n+1)\left(\frac{(\Phi_j)^\wedge(n)q(j,\Lambda^\wedge(n))}{\Lambda^\wedge(n)}\right)^2\frac{1}{A_n^2}\\ &\leq \sum_{n\in\mathcal{N}}(2n+1)\left(\frac{(\Phi_j)^\wedge(n)C(j)\Lambda^\wedge(n)}{\Lambda^\wedge(n)}\right)^2\frac{1}{A_n^2}\\ &= (C(j))^2\sum_{n\in\mathcal{N}}(2n+1)((\Phi_j)^\wedge(n))^2\frac{1}{A_n^2} < \infty.\end{aligned}$$

Condition (5.2) follows by analogous arguments. Properties (i) to (iii) in Definition 5.23 are satisfied in case (i) as well as in case (ii), since on the one hand the generating symbol $\{\{(\Phi_j)^\wedge(n)\}_{n\in\mathcal{N}}\}_{j\in\mathbb{N}_0}$ of $\{\Phi_j\}_{j\in\mathbb{N}_0}$ fulfills conditions (i) to (iii) in Definition 5.6 and on the other hand $q(j, \Lambda^\wedge(n)) \geq 0$ for all $j \in \mathbb{N}_0$ and all $n \in \mathcal{N}$ and $\lim_{j\to\infty} q(j, \Lambda^\wedge(n)) = 1$ for all $n \in \mathcal{N}$. □

The construction of a bilinear regularization scaling function for $\Lambda^{-1} : \mathcal{H}(\{A_n(\Lambda^\wedge(n))^{-1}\}; h_2; \overline{\Omega_{r_2}^{ext}}) \to \mathcal{H}(\{A_n\}; h_1; \overline{\Omega_{r_1}^{ext}})$ is a little bit more complicated as in the case of a bilinear scaling function, since we now have to use the convolution of two different product kernels. Here different combinations are imaginable. One choice is to convolute an $\mathcal{H}(\{A_n\}; h_1; \overline{\Omega_{r_1}^{ext}})$ product kernel $\tilde{\Phi}_j^\Lambda$ against an $\mathcal{H}(\{A_n\}; h_1; \overline{\Omega_{r_1}^{ext}})$-$\mathcal{H}(\{A_n\}; h_2; \overline{\Omega_{r_2}^{ext}})$ product kernel Φ_j^Λ. We omit the discussion of bilinear regularization scaling functions here, since they are not used in our numerical computations.

In analogy to Section 5.2, we are, however, interested in regularization scaling functions which show a certain reproducing property. The construction of a reproducing regularization scaling function for $\Lambda^{-1} : \mathcal{H}(\{A_n(\Lambda^\wedge(n))^{-1}\}; h_2; \overline{\Omega_{r_2}^{ext}}) \to \mathcal{H}(\{A_n\}; h_1; \overline{\Omega_{r_1}^{ext}})$ is based on a reproducing scaling function for $\mathcal{H}(\{A_n\}; h_1; \overline{\Omega_{r_1}^{ext}})$ and is analogous to the construction in Theorem 5.24:

Definition 5.25 *Let $\{A_n\}_{n\in\mathbb{N}_0} \subset \mathbb{R}_0^+$ be a sequence of non-negative real numbers, let $r_1, r_2 \in \mathbb{R}^+$ and $h_1, h_2 \in \{0,1,2\}$. Suppose that $\Lambda : \mathcal{H}(\{A_n\}; h_1; \overline{\Omega_{r_1}^{ext}}) \to \mathcal{H}(\{A_n\}; h_2; \overline{\Omega_{r_2}^{ext}})$ is an injective pseudodifferential operator whose symbol $\{\Lambda^\wedge(n)\}_{n\in\mathbb{N}_0}$ satisfies $\Lambda^\wedge(n) > 0$ for all $n \in \mathcal{N}$ and $\lim_{n\in\mathcal{N},\, n\to\infty} \Lambda^\wedge(n) = 0$. Suppose that $\{\Phi_j^R\}_{j\in\mathbb{N}_0}$ is a reproducing scaling function for $\mathcal{H}(\{A_n\}; h_1; \overline{\Omega_{r_1}^{ext}})$ with generating symbol $\{\{(\Phi_j^R)^\wedge(n)\}_{n\in\mathcal{N}}\}_{j\in\mathbb{N}_0}$ which satisfies*

$$\sum_{n\in\mathcal{N}}(2n+1)\left(\frac{(\Phi_j^R)^\wedge(n)}{\Lambda^\wedge(n)}\right)^2\frac{1}{A_n^2} < \infty \tag{5.9}$$

and

$$\sum_{n\in\mathcal{N}}(2n+1)\left(\frac{(\Phi_j^R)^\wedge(n)}{\Lambda^\wedge(n)}\right)^2 < \infty. \tag{5.10}$$

Define the sequence of non-negative real numbers $\{\{((\Phi_j^\Lambda)^R)^\wedge(n)\}_{n\in\mathcal{N}}\}_{j\in\mathbb{N}_0} \subset \mathbb{R}^+$ *by*

$$((\Phi_j^\Lambda)^R)^\wedge(n) := (\Lambda^\wedge(n))^{-1} q(j,\Lambda^\wedge(n))(\Phi_j^R)^\wedge(n) \quad \text{for all} \quad n\in\mathcal{N},\ j\in\mathbb{N}_0.$$

The family of $\mathcal{H}(\{A_n\};h_1;\overline{\Omega_{r_1}^{ext}})$-$\mathcal{H}(\{A_n\};h_2;\overline{\Omega_{r_2}^{ext}})$ *product kernels* $\{(\Phi_j^\Lambda)^R\}_{j\in\mathbb{N}_0}$, *defined by*

$$(\Phi_j^\Lambda)^R(x,y) := \sum_{n\in\mathcal{N}} \sum_{k=1}^{2n+1} ((\Phi_j^\Lambda)^R)^\wedge(n) \frac{1}{A_n^2} H_{n,k}(r_1;h_1;x) H_{n,k}(r_2;h_2;y),$$

where $x\in\overline{\Omega_{r_1}^{ext}}$ *and* $y\in\overline{\Omega_{r_2}^{ext}}$, *is called the reproducing regularization scaling function for* $\Lambda^{-1}:\mathcal{H}(\{A_n(\Lambda^\wedge(n))^{-1}\};h_2;\overline{\Omega_{r_2}^{ext}})\to\mathcal{H}(\{A_n\};h_1;\overline{\Omega_{r_1}^{ext}})$ *corresponding to* $\{\Phi_j^R\}_{j\in\mathbb{N}_0}$ *with generating symbol* $\{\{((\Phi_j^\Lambda)^R)^\wedge(n)\}_{n\in\mathcal{N}}\}_{j\in\mathbb{N}_0}$.

The reproducing property of a reproducing regularization scaling function is specified in the next lemma.

Lemma 5.26 *Let* $\{A_n\}_{n\in\mathbb{N}_0}\subset\mathbb{R}_0^+$ *be a sequence of non-negative real numbers, let* $r_1,r_2\in\mathbb{R}^+$, *and let* $h_1,h_2\in\{0,1,2\}$. *Suppose* $\Lambda:\mathcal{H}(\{A_n\};h_1;\overline{\Omega_{r_1}^{ext}})\to\mathcal{H}(\{A_n\};h_2;\overline{\Omega_{r_2}^{ext}})$ *is an injective pseudodifferential operator whose symbol* $\{\Lambda^\wedge(n)\}_{n\in\mathbb{N}_0}$ *satisfies* $\Lambda^\wedge(n)>0$ *for all* $n\in\mathcal{N}$ *and* $\lim_{n\in\mathcal{N},\ n\to\infty}\Lambda^\wedge(n)=0$. *Let* $\{(\Phi_j^\Lambda)^R\}_{j\in\mathbb{N}_0}$ *be a reproducing regularization scaling function for* $\Lambda^{-1}:\mathcal{H}(\{A_n(\Lambda^\wedge(n))^{-1}\};h_2;\overline{\Omega_{r_2}^{ext}})\to\mathcal{H}(\{A_n\};h_1;\overline{\Omega_{r_1}^{ext}})$ *corresponding to a reproducing scaling function* $\{\Phi_j^R\}_{j\in\mathbb{N}_0}$ *for* $\mathcal{H}(\{A_n\};h_1;\overline{\Omega_{r_1}^{ext}})$. *Then*

$$(\Phi_j^\Lambda)^R = \Phi_{j+1}^R *_{\mathcal{H}(\{A_n\};h_1;\overline{\Omega_{r_1}^{ext}})} (\Phi_{j+1}^\Lambda)^R \ \text{for}\ j\in\mathbb{N}_0.$$

Proof: According to Definition 5.25, $\{\{((\Phi_j^\Lambda)^R)^\wedge(n)\}_{n\in\mathcal{N}}\}_{j\in\mathbb{N}_0}$ satisfies the relation

$$((\Phi_j^\Lambda)^R)^\wedge(n) = (\Lambda^\wedge(n))^{-1}(\Phi_j^R)^\wedge(n) = (\Lambda^\wedge(n))^{-1}((\Phi_{j+1}^R)^\wedge(n))^2$$

for all $j\in\mathbb{N}_0$, $n\in\mathcal{N}$, and the statement then follows by simple computations. □

After these preparations, we define a family of linear operators as follows:

Definition 5.27 *Let* $\{A_n\}_{n\in\mathbb{N}_0}\subset\mathbb{R}_0^+$ *be a sequence of non-negative real numbers, let* $r_1,r_2\in\mathbb{R}^+$ *and* $h_1,h_2\in\{0,1,2\}$. *Suppose that* $\Lambda:\mathcal{H}(\{A_n\};h_1;\overline{\Omega_{r_1}^{ext}})\to\mathcal{H}(\{A_n\};h_2;\overline{\Omega_{r_2}^{ext}})$ *is an injective pseudodifferential operator whose symbol* $\{\Lambda^\wedge(n)\}_{n\in\mathbb{N}_0}$ *satisfies* $\Lambda^\wedge(n)>0$ *for all* $n\in\mathcal{N}$ *and* $\lim_{n\in\mathcal{N},\ n\to\infty}\Lambda^\wedge(n)=0$. *Suppose that* $\{\Phi_j^\Lambda\}_{j\in\mathbb{N}_0}$ *is either a linear or a reproducing regularization scaling function for* $\Lambda^{-1}:\mathcal{H}(\{A_n(\Lambda^\wedge(n))^{-1}\};h_2;\overline{\Omega_{r_2}^{ext}})\to\mathcal{H}(\{A_n\};h_1;\overline{\Omega_{r_1}^{ext}})$. *Define the family of linear operators* $\{T_j\}_{j\in\mathbb{N}_0}$ *by*

$$T_j:\mathcal{H}(\{A_n\};h_2;\overline{\Omega_{r_2}^{ext}})\to\mathcal{H}(\{A_n\};h_1;\overline{\Omega_{r_1}^{ext}}),\ G\mapsto T_jG:=\Phi_j^\Lambda *_{\mathcal{H}(\{A_n\};h_2;\overline{\Omega_{r_2}^{ext}})} G$$

and the family of scale spaces $\{\mathcal{V}_j^\Lambda(h_1;\overline{\Omega_{r_1}^{ext}})\}_{j\in\mathbb{N}_0}$ *by*

$$\mathcal{V}_j^\Lambda(h_1;\overline{\Omega_{r_1}^{ext}}) := \operatorname{im}(T_j) = T_j(\mathcal{H}(\{A_n\};h_2;\overline{\Omega_{r_2}^{ext}})),\ j\in\mathbb{N}_0.$$

Theorem 5.28 *Let $\{A_n\}_{n\in\mathbb{N}_0} \subset \mathbb{R}_0^+$ be a sequence of non-negative real numbers, let $r_1, r_2 \in \mathbb{R}^+$ and $h_1, h_2 \in \{0,1,2\}$. Suppose that $\Lambda : \mathcal{H}(\{A_n\}; h_1; \overline{\Omega_{r_1}^{ext}}) \to \mathcal{H}(\{A_n\}; h_2; \overline{\Omega_{r_2}^{ext}})$ is an injective pseudodifferential operator whose symbol $\{\Lambda^\wedge(n)\}_{n\in\mathbb{N}_0}$ satisfies $\Lambda^\wedge(n) > 0$ for all $n \in \mathcal{N}$ and $\lim_{n\in\mathcal{N},\, n\to\infty} \Lambda^\wedge(n) = 0$. Suppose that $\{\Phi_j^\Lambda\}_{j\in\mathbb{N}_0}$ is either a linear or reproducing regularization scaling function for $\Lambda^{-1} : \mathcal{H}(\{A_n(\Lambda^\wedge(n))^{-1}\}; h_2; \overline{\Omega_{r_2}^{ext}}) \to \mathcal{H}(\{A_n\}; h_1; \overline{\Omega_{r_1}^{ext}})$ with generating symbol $\{\{(\Phi_j^\Lambda)^\wedge(n)\}_{n\in\mathcal{N}}\}_{j\in\mathbb{N}_0}$. The family of linear operators $\{T_j\}_{j\in\mathbb{N}_0}$ introduced in Definition 5.27 is a regularization of the inverse operator $\Lambda^{-1} : \mathcal{H}(\{A_n(\Lambda^\wedge(n))^{-1}\}; h_2; \overline{\Omega_{r_2}^{ext}}) \to \mathcal{H}(\{A_n\}; h_1; \overline{\Omega_{r_1}^{ext}})$ with discrete regularization parameter $j \in \mathbb{N}_0$.*

Proof: We show that a suitably defined function $f : \mathbb{N}_0 \times (0, \|\Lambda\|] \to \mathbb{R}$, which assumes the values

$$f(j, \Lambda^\wedge(n)) := \Lambda^\wedge(n)(\Phi_j^\Lambda)^\wedge(n) \text{ for } j \in \mathbb{N}_0,\ n \in \mathcal{N},$$

is a regularizing filter to the ill-posed pseudodifferential operator equation $\Lambda F = G$, $F \in \mathcal{H}(\{A_n\}; h_1; \overline{\Omega_{r_1}^{ext}})$, $G \in \mathcal{H}(\{A_n\}; h_2; \overline{\Omega_{r_2}^{ext}})$. Thus we have to verify conditions (i) to (iii) in Theorem 3.7. In order to verify (i), note that conditions (i) to (iii) in Definition 5.23 imply that $0 \le (\Phi_j^\Lambda)^\wedge(n) \le (\Lambda^\wedge(n))^{-1}$ for all $j \in \mathbb{N}_0$, $n \in \mathcal{N}$. This estimate in combination with $\Lambda^\wedge(n) > 0$ for all $n \in \mathcal{N}$ implies that $f(j, \Lambda^\wedge(n)) \le 1$ for all $n \in \mathcal{N}$ and $j \in \mathbb{N}_0$. Condition (ii) in Theorem 3.7 is satisfied, because $\{\Phi_j\}_{j\in\mathbb{N}_0}$ is a sequence of product kernels and satisfies for every fixed $j \in \mathbb{N}_0$ the condition $\sum_{n\in\mathcal{N}}((\Phi_j^\Lambda)^\wedge(n))^2 < \infty$. This implies that $\lim_{n\in\mathcal{N},\, n\to\infty}(\Phi_j^\Lambda)^\wedge(n) = 0$ and there exists a constant $C(j) > 0$ such that $|(\Phi_j^\Lambda)^\wedge(n)| \le C(j)$ for all $n \in \mathcal{N}$. Thus, $|f(j, \Lambda^\wedge(n))| \le C(j)\Lambda^\wedge(n)$ for all $n \in \mathcal{N}$. Condition (iii) in Definition 5.23 finally ensures that $\lim_{j\to\infty} f(j, \Lambda^\wedge(n)) = 1$ for all $n \in \mathcal{N}$, which completes the proof. □

The next theorem shows that due to our construction we obtain a hierarchical sequence of approximation spaces in the sense of a multiresolution analysis of $\mathcal{H}(\{A_n\}; h_1; \overline{\Omega_{r_1}^{ext}})$.

Theorem 5.29 (Regularization Multiresolution Analysis) *Let $\{A_n\}_{n\in\mathbb{N}_0} \subset \mathbb{R}_0^+$ be a sequence of non-negative real numbers, let $r_1, r_2 \in \mathbb{R}^+$ and $h_1, h_2 \in \{0,1,2\}$. Suppose that $\Lambda : \mathcal{H}(\{A_n\}; h_1; \overline{\Omega_{r_1}^{ext}}) \to \mathcal{H}(\{A_n\}; h_2; \overline{\Omega_{r_2}^{ext}})$ is an injective pseudodifferential operator whose symbol $\{\Lambda^\wedge(n)\}_{n\in\mathbb{N}_0}$ satisfies $\Lambda^\wedge(n) > 0$ for all $n \in \mathcal{N}$ and $\lim_{n\in\mathcal{N},\, n\to\infty} \Lambda^\wedge(n) = 0$. Suppose that $\{\Phi_j^\Lambda\}_{j\in\mathbb{N}_0}$ is either a linear or a reproducing regularization scaling function for $\Lambda^{-1} : \mathcal{H}(\{A_n(\Lambda^\wedge(n))^{-1}\}; h_2; \overline{\Omega_{r_2}^{ext}}) \to \mathcal{H}(\{A_n\}; h_1; \overline{\Omega_{r_1}^{ext}})$ with generating symbol $\{\{(\Phi_j^\Lambda)^\wedge(n)\}_{n\in\mathcal{N}}\}_{j\in\mathbb{N}_0}$. Then the family of scale spaces $\{\mathcal{V}_j^\Lambda(h_1; \overline{\Omega_{r_1}^{ext}})\}_{j\in\mathbb{N}_0}$ corresponding to the family of linear operators $\{T_j\}_{j\in\mathbb{N}_0}$ introduced in Definition 5.27, is a multiresolution analalsysis of $\mathcal{H}(\{A_n\}; h_1; \overline{\Omega_{r_1}^{ext}})$, i.e.,*

(i) $\mathcal{V}_0^\Lambda(h_1; \overline{\Omega_{r_1}^{ext}}) \subset \ldots \subset \mathcal{V}_j^\Lambda(h_1; \overline{\Omega_{r_1}^{ext}}) \subset \mathcal{V}_{j'}^\Lambda(h_1; \overline{\Omega_{r_1}^{ext}}) \subset \ldots \subset \mathcal{H}(\{A_n\}; h_1; \overline{\Omega_{r_1}^{ext}})$ for $j' \ge j, j \in \mathbb{N}_0$

and

(ii) $\overline{\bigcup_{j\in\mathbb{N}_0} \mathcal{V}_j^{\Lambda}(h_1;\overline{\Omega_{r_1}^{ext}})}^{\|\cdot\|_{\mathcal{H}(\{A_n\};h_1;\overline{\Omega_{r_1}^{ext}})}} = \mathcal{H}(\{A_n\};h_1;\overline{\Omega_{r_1}^{ext}})$.

Proof: The proof can be carried out in analogy to the proof of Theorem 5.14. □

In analogy to Section 5.3, we wish to decompose the approximation spaces $\mathcal{V}_j^{\Lambda}(h_1;\overline{\Omega_{r_1}^{ext}})$, $j \in \mathbb{N}_0$ into detail spaces, such that we can analyse the difference between two subsequent j-level regularizations $T_{j+1}G$ and T_jG of the problem $\Lambda F = G$, $F \in \mathcal{H}(\{A_n\};h_1;\overline{\Omega_{r_1}^{ext}})$, $G \in \mathcal{H}(\{A_n\};h_2;\overline{\Omega_{r_2}^{ext}})$. We thus introduce the notion of regularization wavelets corresponding to a regularization scaling function for $\Lambda^{-1} : \mathcal{H}(\{A_n(\Lambda^\wedge(n))^{-1}\};h_2;\overline{\Omega_{r_2}^{ext}}) \to \mathcal{H}(\{A_n\};h_1;\overline{\Omega_{r_1}^{ext}})$.

Definition 5.30 *Let $\{A_n\}_{n\in\mathbb{N}_0} \subset \mathbb{R}_0^+$ be a sequence of non-negative real numbers, let $r_1, r_2 \in \mathbb{R}^+$ and $h_1, h_2 \in \{0,1,2\}$. Suppose that $\Lambda : \mathcal{H}(\{A_n\};h_1;\overline{\Omega_{r_1}^{ext}}) \to \mathcal{H}(\{A_n\};h_2;\overline{\Omega_{r_2}^{ext}})$ is an injective pseudodifferential operator whose symbol $\{\Lambda^\wedge(n)\}_{n\in\mathbb{N}_0}$ satisfies $\Lambda^\wedge(n) > 0$ for all $n \in \mathcal{N}$ and $\lim_{n\in\mathcal{N},\, n\to\infty} \Lambda^\wedge(n) = 0$. Suppose that $\{\Phi_j^{\Lambda}\}_{j\in\mathbb{N}_0}$ is a linear regularization scaling function for $\Lambda^{-1} : \mathcal{H}(\{A_n(\Lambda^\wedge(n))^{-1}\};h_2;\overline{\Omega_{r_2}^{ext}}) \to \mathcal{H}(\{A_n\};h_1;\overline{\Omega_{r_1}^{ext}})$ with generating symbol $\{\{(\Phi_j^{\Lambda})^\wedge(n)\}_{n\in\mathcal{N}}\}_{j\in\mathbb{N}_0}$. Define the family of sequences of non-negative real numbers $\{\{(\Psi_j^{\Lambda})^\wedge(n)\}_{n\in\mathcal{N}}\}_{j\in\mathbb{N}_0} \subset \mathbb{R}_0^+$ by*

$$(\Psi_j^{\Lambda})^\wedge(n) := (\Phi_{j+1}^{\Lambda})^\wedge(n) - (\Phi_j^{\Lambda})^\wedge(n) \quad \text{for all} \quad n \in \mathcal{N} \quad \text{and all} \quad j \in \mathbb{N}_0.$$

Then the family $\{\Psi_j^{\Lambda}\}_{j\in\mathbb{N}_0}$ of $\mathcal{H}(\{A_n\};h_1;\overline{\Omega_{r_1}^{ext}})$-$\mathcal{H}(\{A_n\};h_2;\overline{\Omega_{r_2}^{ext}})$ product kernels $\Psi_j^{\Lambda} : \overline{\Omega_{r_1}^{ext}} \times \overline{\Omega_{r_2}^{ext}} \to \mathbb{R}$, defined by

$$\Psi_j^{\Lambda}(x,y) := \sum_{n\in\mathcal{N}} \sum_{k=1}^{2n+1} (\Psi_j^{\Lambda})^\wedge(n) \frac{1}{A_n^2} H_{n,k}(r_1;h_1;x) H_{n,k}(r_2;h_2;y),\ x \in \overline{\Omega_{r_1}^{ext}},\ y \in \overline{\Omega_{r_2}^{ext}},$$

is called the (scale discrete) linear regularization wavelet corresponding to the linear regularization scaling function $\{\Phi_j^{\Lambda}\}_{j\in\mathbb{N}_0}$. The family $\{\{(\Psi_j^{\Lambda})^\wedge(n)\}_{n\in\mathcal{N}}\}_{j\in\mathbb{N}_0}$ is called the generating symbol of $\{\Psi_j^{\Lambda}\}_{j\in\mathbb{N}_0}$.

Definition 5.31 *Let $\{A_n\}_{n\in\mathbb{N}_0} \subset \mathbb{R}_0^+$ be a sequence of non-negative real numbers, let $r_1, r_2 \in \mathbb{R}^+$ and $h_1, h_2 \in \{0,1,2\}$. Suppose that $\Lambda : \mathcal{H}(\{A_n\};h_1;\overline{\Omega_{r_1}^{ext}}) \to \mathcal{H}(\{A_n\};h_2;\overline{\Omega_{r_2}^{ext}})$ is an injective pseudodifferential operator whose symbol $\{\Lambda^\wedge(n)\}_{n\in\mathbb{N}_0}$ satisfies $\Lambda^\wedge(n) > 0$ for all $n \in \mathcal{N}$ and $\lim_{n\in\mathcal{N},n\to\infty} \Lambda^\wedge(n) = 0$. Let $\{\Phi_j^R\}_{j\in\mathbb{N}_0}$ be a reproducing scaling function for $\mathcal{H}(\{A_n\};h_1;\overline{\Omega_{r_1}^{ext}})$ whose generating symbol $\{\{(\Phi_j^R)^\wedge(n)\}_{n\in\mathcal{N}}\}_{j\in\mathbb{N}_0}$, satisfies conditions (5.9) and (5.10) in Definition 5.25, and let $\{(\Phi_j^{\Lambda})^R\}_{j\in\mathbb{N}_0}$ denote the reproducing regularization scaling function for $\Lambda^{-1} : \mathcal{H}(\{A_n(\Lambda^\wedge(n))^{-1}\};h_2;\overline{\Omega_{r_2}^{ext}}) \to \mathcal{H}(\{A_n\};h_1;\overline{\Omega_{r_1}^{ext}})$ corresponding to $\{\Phi_j^R\}_{j\in\mathbb{N}_0}$ with generating symbol $\{\{((\Phi_j^{\Lambda})^R)^\wedge(n)\}_{n\in\mathcal{N}}\}_{j\in\mathbb{N}_0}$. Let $\{\{((\Psi_j^{\Lambda})^{RM})^\wedge(n)\}_{n\in\mathcal{N}}\}_{j\in\mathbb{N}_0}$ and $\{\{(\tilde{\Psi}_j^{RM})^\wedge(n)\}_{n\in\mathcal{N}}\}_{j\in\mathbb{N}_0}$ be the families of sequences of non-negative real numbers given by*

$$((\Psi_j^{\Lambda})^{RM})^\wedge(n) := ((\Phi_{j+1}^{\Lambda})^R)^\wedge(n) - ((\Phi_j^{\Lambda})^R)^\wedge(n) \quad \text{for} \quad n \in \mathcal{N},\ j \in \mathbb{N}_0$$

and

$$(\tilde{\Psi}_j^{RM})^\wedge(n) := (\Phi_{j+1}^R)^\wedge(n) + (\Phi_j^R)^\wedge(n) \quad for \quad n \in \mathcal{N},\ j \in \mathbb{N}_0,$$

respectively.
Define the family $\{(\Psi_j^\Lambda)^{RM}\}_{j\in\mathbb{N}_0}$ *of* $\mathcal{H}(\{A_n\};h_1;\overline{\Omega_{r_1}^{ext}})$-$\mathcal{H}(\{A_n\};h_2;\overline{\Omega_{r_2}^{ext}})$ *product kernels by*

$$\Psi_j^{RM}(x,y) := \sum_{n\in\mathcal{N}}\sum_{k=1}^{2n+1} (\Psi_j^{RM})^\wedge(n)\frac{1}{A_n^2} H_{n,k}(r_1;h_1;x)H_{n,k}(r_2;h_2;y),\ x\in\overline{\Omega_{r_1}^{ext}},\ y\in\overline{\Omega_{r_2}^{ext}},$$

and the family $\{\tilde{\Psi}_j^{RM}\}_{j\in\mathbb{N}_0}$ *of* $\mathcal{H}(\{A_n\};h_1;\overline{\Omega_{r_1}^{ext}})$ *product kernels by*

$$\tilde{\Psi}_j^{RM}(x,y) := \sum_{n\in\mathcal{N}}\sum_{k=1}^{2n+1} (\tilde{\Psi}_j^{RM})^\wedge(n)\frac{1}{A_n^2} H_{n,k}(r_1;h_1;x)H_{n,k}(r_1;h_1;y),\ x,y\in\overline{\Omega_{r_1}^{ext}}.$$

$\{(\Psi_j^\Lambda)^{RM}\}_{j\in\mathbb{N}_0}$ *is called the primal M-scale regularization wavelet for* $\Lambda^{-1} : \mathcal{H}(\{A_n(\Lambda^\wedge(n))^{-1}\};h_2;\overline{\Omega_{r_2}^{ext}}) \to \mathcal{H}(\{A_n\};h_1;\overline{\Omega_{r_1}^{ext}})$ *corresponding to* $\{\Phi_j^R\}_{j\in\mathbb{N}_0}$*, and* $\{\tilde{\Psi}_j^{RM}\}_{j\in\mathbb{N}_0}$ *is called its accompanying dual M-scale wavelet.*

Lemma 5.32 *Let* $\{A_n\}_{n\in\mathbb{N}_0}\subset\mathbb{R}_0^+$ *be a sequence of non-negative real numbers, let* $r_1,r_2\in\mathbb{R}^+$ *and* $h_1,h_2\in\{0,1,2\}$*. Suppose that* $\Lambda : \mathcal{H}(\{A_n\};h_1;\overline{\Omega_{r_1}^{ext}}) \to \mathcal{H}(\{A_n\};h_2;\overline{\Omega_{r_2}^{ext}})$ *is an injective pseudodifferential operator whose symbol* $\{\Lambda^\wedge(n)\}_{n\in\mathbb{N}_0}$ *satisfies* $\Lambda^\wedge(n)>0$ *for all* $n\in\mathcal{N}$ *and* $\lim_{n\in\mathcal{N},\ n\to\infty}\Lambda^\wedge(n)=0$*. Let* $\{\Phi_j^R\}_{j\in\mathbb{N}_0}$ *be a reproducing scaling function for* $\mathcal{H}(\{A_n\};h_1;\overline{\Omega_{r_1}^{ext}})$ *whose symbol* $\{\{(\Phi_j^R)^\wedge(n)\}_{n\in\mathcal{N}}\}_{j\in\mathbb{N}_0}$ *satisfies conditions (5.9) and (5.10) in Definition 5.25. Let* $\{(\Psi_j^\Lambda)^{RM}\}_{j\in\mathbb{N}_0}$ *and* $\{\tilde{\Psi}_j^{RM}\}_{j\in\mathbb{N}_0}$ *denote the primal M-scale regularization wavelet for* $\Lambda^{-1} : \mathcal{H}(\{A_n(\Lambda^\wedge(n))^{-1}\};h_2;\overline{\Omega_{r_2}^{ext}}) \to \mathcal{H}(\{A_n\};h_1;\overline{\Omega_{r_1}^{ext}})$ *corresponding to* $\{\Phi_j^R\}_{j\in\mathbb{N}_0}$ *and its accompanying dual M-scale wavelet. Then*

$$(\Psi_j^\Lambda)^{RM} = \tilde{\Psi}_{j+1}^{RM} *_{\mathcal{H}(\{A_n\};h_1;\overline{\Omega_{r_1}^{ext}})} (\Psi_{j+1}^\Lambda)^{RM} \ for\ all\ j\in\mathbb{N}_0.$$

Proof: By construction, $\{\{((\Psi_j^\Lambda)^{RM})^\wedge(n)\}_{n\in\mathcal{N}}\}_{j\in\mathbb{N}_0}$ and $\{\{(\tilde{\Psi}_j^{RM})^\wedge(n)\}_{n\in\mathcal{N}}\}_{j\in\mathbb{N}_0}$ satisfy the refinement equation

$$\begin{aligned}(\tilde{\Psi}_{j+1}^{RM})^\wedge(n)((\Psi_{j+1}^\Lambda)^{RM})^\wedge(n) &= (\Lambda^\wedge(n))^{-1}\left(((\Phi_{j+2}^R)^\wedge(n))^2 - ((\Phi_{j+1}^R)^\wedge(n))^2\right)\\ &= (\Lambda^\wedge(n))^{-1}\left((\Phi_{j+1}^R)^\wedge(n) - (\Phi_j^R)^\wedge(n)\right)\\ &= ((\Psi_j^\Lambda)^{RM})^\wedge(n),\end{aligned}$$

which implies the statement. □

Definition 5.33 *Let* $\{A_n\}_{n\in\mathbb{N}_0}\subset\mathbb{R}_0^+$ *be a sequence of non-negative real numbers, let* $r_1,r_2\in\mathbb{R}^+$ *and* $h_1,h_2\in\{0,1,2\}$*. Suppose that* $\Lambda : \mathcal{H}(\{A_n\};h_1;\overline{\Omega_{r_1}^{ext}}) \to \mathcal{H}(\{A_n\};h_2;\overline{\Omega_{r_2}^{ext}})$ *is an injective pseudodifferential operator whose symbol* $\{\Lambda^\wedge(n)\}_{n\in\mathbb{N}_0}$ *satisfies* $\Lambda^\wedge(n)>0$ *for all* $n\in\mathcal{N}$ *and* $\lim_{n\in\mathcal{N},\ n\to\infty}|\Lambda^\wedge(n)|=0$*. Suppose that* $\{\Phi_j^\Lambda\}_{j\in\mathbb{N}_0}$ *is a linear regularization scaling function for* $\Lambda^{-1} : \mathcal{H}(\{A_n(\Lambda^\wedge(n))^{-1}\};h_2;\overline{\Omega_{r_2}^{ext}}) \to \mathcal{H}(\{A_n\};h_1;\overline{\Omega_{r_1}^{ext}})$ *and let*

$\{\Psi_j^\Lambda\}_{j\in\mathbb{N}_0}$ *be the corresponding linear regularization wavelet. Define the family of operators* $\{R_j\}_{j\in\mathbb{N}_0}$ *by*

$$R_j : \mathcal{H}(\{A_n\}; h_2; \overline{\Omega_{r_2}^{ext}}) \to \mathcal{H}(\{A_n\}; h_1; \overline{\Omega_{r_1}^{ext}}),\ G \mapsto R_j(G) := \Psi_j^\Lambda *_{\mathcal{H}(\{A_n\};h_2;\overline{\Omega_{r_2}^{ext}})} G.$$

In case that $\{\Phi_j^R\}_{j\in\mathbb{N}_0}$ *is a reproducing scaling function for* $\mathcal{H}(\{A_n\}; h_1; \overline{\Omega_{r_1}^{ext}})$*, whose generating symbol satisfies conditions (5.9) and (5.10) in Definition 5.25, and* $\{(\Psi_j^\Lambda)^{RM}\}_{j\in\mathbb{N}_0}$ *and* $\{\tilde{\Psi}_j^{RM}\}_{j\in\mathbb{N}_0}$ *are the primal M-scale regularization wavelet corresponding to* $\{\Phi_j^R\}_{j\in\mathbb{N}_0}$ *and its accompanying dual M-scale wavelet, the family of operators* $\{R_j\}_{j\in\mathbb{N}_0}$ *is defined by*

$$R_j : \mathcal{H}(\{A_n\}; h_2; \overline{\Omega_{r_2}^{ext}}) \to \mathcal{H}(\{A_n\}; h_1; \overline{\Omega_{r_1}^{ext}}),$$

$$\begin{aligned} G \mapsto R_j(G) &:= (\Psi^\Lambda)_j^{RM} *_{\mathcal{H}(\{A_n\};h_2;\overline{\Omega_{r_2}^{ext}})} G \\ &= \tilde{\Psi}_{j+1}^{RM} *_{\mathcal{H}(\{A_n\};h_1;\overline{\Omega_{r_1}^{ext}})} (\Psi_{j+1}^\Lambda)^{RM} *_{\mathcal{H}(\{A_n\};h_2;\overline{\Omega_{r_2}^{ext}})} G. \end{aligned}$$

The detail spaces $\{\mathcal{W}_j^\Lambda(h_1; \overline{\Omega_{r_1}^{ext}})\}_{j\in\mathbb{N}_0}$ *are in both cases defined by*

$$\mathcal{W}_j^\Lambda(h_1; \overline{\Omega_{r_1}^{ext}}) := \mathrm{im}(R_j) = R_j(\mathcal{H}(\{A_n\}; h_2; \overline{\Omega_{r_2}^{ext}})).$$

Theorem 5.34 *Let the assumptions be the same as in Definition 5.27 and Definition 5.33. Then the families of operators* $\{T_j\}_{j\in\mathbb{N}_0}$ *and* $\{R_j\}_{j\in\mathbb{N}_0}$ *defined according to Definitions 5.27 and 5.33, respectively, satisfy the relations*

$$T_{j+1} = T_j + R_j \text{ for all } j \in \mathbb{N}_0, \tag{5.11}$$

$$T_{J+1} = T_{J_0} + \sum_{j=J_0}^{J} R_j = T_0 + \sum_{j=0}^{J} R_j,\ J, J_0 \in \mathbb{N}_0,\ J > J_0, \tag{5.12}$$

and for $G \in \mathrm{im}(\Lambda)$ *the solution* F *to the operator equation* $\Lambda F = G$ *can be reconstructed in* $\mathcal{H}(\{A_n\}; h_1; \overline{\Omega_{r_1}^{ext}})$*-sense according to*

$$F = T_{J_0}G + \lim_{J\to\infty} \sum_{j=J_0}^{J} R_j G = T_0 G + \lim_{J\to\infty} \sum_{j=0}^{J} R_j G. \tag{5.13}$$

Proof: Relations (5.11) and (5.12) can be easily verified by just inserting the definitions of the respective regularization scaling functions and their corresponding regularization wavelets into the definition of $\{T_j\}_{j\in\mathbb{N}_0}$ and $\{R_j\}_{j\in\mathbb{N}_0}$, respectively.
The limit relation (5.13) then follows by Theorem 5.28. □

The desired decomposition of the scale spaces $\{\mathcal{V}_j^\Lambda(h_1; \overline{\Omega_{r_1}^{ext}})\}_{j\in\mathbb{N}_0}$ is an immediate consequence of Theorem 5.34:

Corollary 5.35 *Let the assumptions be the same as in Theorem 5.34 and let the families of scale and detail spaces* $\{\mathcal{V}_j^\Lambda(h_1;\overline{\Omega_{r_1}^{ext}})\}_{j\in\mathbb{N}_0}$ *and* $\{\mathcal{W}_j^\Lambda(h_1;\overline{\Omega_{r_1}^{ext}})\}_{j\in\mathbb{N}_0}$ *be given according to Definition 5.27 and 5.33, respectively. Then*

$$\mathcal{V}_{j+1}^\Lambda(h_1;\overline{\Omega_{r_1}^{ext}}) = \mathcal{V}_j^\Lambda(h_1;\overline{\Omega_{r_1}^{ext}}) + \mathcal{W}_j^\Lambda(h_1;\overline{\Omega_{r_1}^{ext}}) \text{ for all } j \in \mathbb{N}_0$$

and

$$\mathcal{V}_{J+1}^\Lambda(h_1;\overline{\Omega_{r_1}^{ext}}) = \mathcal{V}_{J_0}^\Lambda(h_1;\overline{\Omega_{r_1}^{ext}}) + \sum_{j=J_0}^{J} \mathcal{W}_j^\Lambda(h_1;\overline{\Omega_{r_1}^{ext}}) = \mathcal{V}_0^\Lambda(h_1;\overline{\Omega_{r_1}^{ext}}) + \sum_{j=0}^{J} \mathcal{W}_j^\Lambda(h_1;\overline{\Omega_{r_1}^{ext}})$$

for all $J_0, J \in \mathbb{N}_0,\ J_0 < J$.

The sequence of regularized solutions $\{T_jG\}_{j\in\mathbb{N}_0}$ can be interpreted as a sequence of low-pass filtered versions of the 'true' solution $F = \Lambda^{-1}G$, and the sequence of 'details' $\{R_jG\}_{j\in\mathbb{N}_0}$ can be thought of as a sequence of band-pass filtered versions of $F = \Lambda^{-1}G$. According to Definition 3.4 the regularized solution is also called the j-level regularization of the problem $\Lambda F = G$ corresponding to the regularization $\{T_j\}_{j\in\mathbb{N}_0}$.

5.5 Examples

In this section we go back to the 'continuous' formulation of the SST-problem and the SGG-problem in Problem 3.1 and present some examples of regularization scaling functions and wavelets which can be used to construct a j-level regularization $F_j = T_jG$ to Problem 3.1. It should be noted that most regularization schemes in the mathematical literature are applicable for a huge class of operators, but not every regularization scheme leads to a regularization scaling function for an inverse Λ^{-1} to a given injective compact operator Λ, and whether it leads to such a regularization scaling function for Λ^{-1} or not is highly dependent on the operator Λ (or more precisely on its singular values). Therefore we restrict the discussion for the remainder of this chapter to the SST-operator and the SGG-operator as defined in Chapter 2. For this chapter we thus assume that $\{A_n\}_{n\in\mathbb{N}_0} \subset \mathbb{R}^+$ is a sequence of positive real numbers which satisfies $A_n \geq 1$ for almost all $n \in \mathbb{N}_0$. The mathematical formulation of the SST-problem and the SGG-problem in Problem 3.1 yields an operator equation $\Lambda F = G$, where the pseudodifferential operator $\Lambda : \mathcal{H}(\{A_n\};\overline{\Omega_R^{ext}}) \to \mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})$ is either the SST-operator Λ^{SST} with symbol $\{(\Lambda^{SST})^\wedge(n)\}_{n\in\mathbb{N}_0}$ given by

$$(\Lambda^{SST})^\wedge(n) = \frac{n+1}{r}\left(\frac{R}{r}\right)^n$$

or the SGG-operator Λ^{SGG} with symbol $\{(\Lambda^{SGG})^\wedge(n)\}_{n\in\mathbb{N}_0}$ given by

$$(\Lambda^{SGG})^\wedge(n) = \frac{(n+1)(n+2)}{r^2}\left(\frac{R}{r}\right)^n .$$

For the remainder of this section we assume that Λ is always either Λ^{SST} or Λ^{SGG}.

Two important examples of linear regularization scaling functions for $\Lambda^{-1} : \mathcal{H}(\{A_n(\Lambda^\wedge(n))^{-1}\}; h; \overline{\Omega_r^{ext}}) \to \mathcal{H}(\{A_n\}; \overline{\Omega_R^{ext}})$ are the Tikhonov and the Tikhonov-Philips regularization scaling function:

Example 5.36 (Tikhonov Regularization Scaling Function) *Let* $\{\gamma_j\}_{j\in\mathbb{N}_0} \subset \mathbb{R}^+$ *be a monotonically decreasing sequence of positive real numbers with* $\lim_{j\to\infty}\gamma_j = 0$. *The sequence of non-negative real numbers* $\{\{(\Phi_j^\Lambda)^\wedge(n)\}_{n\in\mathbb{N}_0}\}_{j\in\mathbb{N}_0} \subset \mathbb{R}_0^+$, *defined by*

$$(\Phi_j^\Lambda)^\wedge(n) := \frac{\Lambda^\wedge(n)}{(\Lambda^\wedge(n))^2 + \gamma_j} \quad \textit{for} \quad n \in \mathbb{N}_0,\ j \in \mathbb{N}_0,$$

is the generating symbol of the linear Tikhonov regularization scaling function for $\Lambda^{-1} : \mathcal{H}(\{A_n(\Lambda^\wedge(n))^{-1}\}; h; \overline{\Omega_r^{ext}}) \to \mathcal{H}(\{A_n\}; \overline{\Omega_R^{ext}})$.

Example 5.37 (Tikhonov-Philips Regularization Scaling Function) *Let* $\{\gamma_j\}_{j\in\mathbb{N}_0} \subset \mathbb{R}^+$ *be a monotonically decreasing sequence of positive real numbers with* $\lim_{j\to\infty}\gamma_j = 0$. *The sequence of non-negative real numbers* $\{\{(\Phi_j^\Lambda)^\wedge(n)\}_{n\in\mathbb{N}_0}\}_{j\in\mathbb{N}_0} \subset \mathbb{R}_0^+$, *defined by*

$$(\Phi_j^\Lambda)^\wedge(n) := \frac{\Lambda^\wedge(n)}{(\Lambda^\wedge(n))^2 + \frac{\gamma_j}{R^2}(n(n+1)) + \frac{1}{4}} \quad \textit{for} \quad n \in \mathbb{N}_0,\ j \in \mathbb{N}_0,$$

is the generating symbol of the linear Tikhonov-Philips regularization scaling function for $\Lambda^{-1} : \mathcal{H}(\{A_n(\Lambda^\wedge(n))^{-1}\}; h; \overline{\Omega_r^{ext}}) \to \mathcal{H}(\{A_n\}; \overline{\Omega_R^{ext}})$.

In order to see that the Tikhonov and the Tikhonov-Philips regularization scaling function are well-defined we have to make sure that the kernels Φ_j^Λ, $j \in \mathbb{N}_0$, are well-defined product kernels. The regularization quality of $\{\Phi_j^\Lambda\}_{j\in\mathbb{N}_0}$is clear from the well-known properties of the Tikhonov and Tikhonov-Philips regularization, see for example [EnHaNe1996]. First, note that

$$0 \le \frac{\Lambda^\wedge(n)}{(\Lambda^\wedge(n))^2 + \gamma_j \frac{1}{R^2}\left(n(n+1) + \frac{1}{4}\right)} \le \frac{\Lambda^\wedge(n)}{(\Lambda^\wedge(n))^2 + \gamma_j C}$$

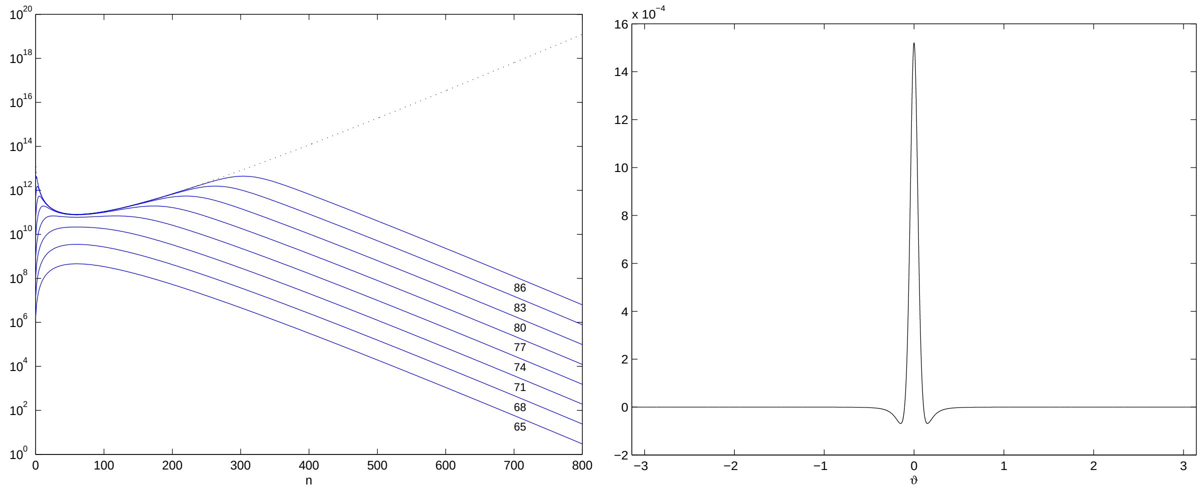
with the constant $C := 1/(4R^2)$. Therefore it suffices to show that $\{\Phi_J^\Lambda\}_{j\in\mathbb{N}_0}$ is well-defined in the Tikhonov case. We have to show that

$$\sum_{n\in\mathbb{N}_0} (2n+1)((\Phi_j^\Lambda)^\wedge(n))^2 < \infty \quad \text{for all} \quad j \in \mathbb{N}_0.$$

But

$$\sum_{n\in\mathbb{N}_0} (2n+1)\frac{\Lambda^\wedge(n)}{(\Lambda^\wedge(n))^2 + \gamma_j} \le \sum_{n\in\mathbb{N}_0} (2n+1)\frac{\Lambda^\wedge(n)}{\gamma_j} < \infty$$

because for $\Lambda \in \{\Lambda^{SST}, \Lambda^{SGG}\}$ the symbol of Λ is dominated by the factor $(\frac{R}{r})^n$ and $(\frac{R}{r})^n < 1$.



a) b)

Figure 5.1: *a) Symbol of the Tikhonov regularization scaling function for* $\gamma_j = 2^{-(j+1)}$ *with* $j \in \{65, 68, 71, 74, 77, 80, 83, 86\}$. *b) Tikhonov regularization scaling function for* $j = 68$.

Finally we will discuss the space-localizing properties of the Tikhonov regularization scaling function. To do this, one variable of the kernel is kept fixed and we regard the function $y \mapsto \Phi_j^\Lambda(x, y)$. Figure 5.1 b) clearly shows that the Tikhonov regularization scaling function is highly space-localizing in a neighbourhood of the point x. Therefore, the Tikhonov regularization scaling function is well suited for the a local solution of a regularization problem from locally given data. More precisely, suppose the discretization of a convolution $\Phi_j^\Lambda(n) *_{\mathcal{H}} (\{A_n\}; h; \overline{\Omega_r^{ext}})G$ between a right-hand side G of our operator equation $\Lambda F = G$ and the Tikhonov scaling function leads to a formula

$$(\Phi_j^\Lambda(n) *_{\mathcal{H}} (\{A_n\}; h; \overline{\Omega_r^{ext}})G)(y) \approx \sum_{i=1}^{N} a_i^N \, \Phi_j^\Lambda(y, x_i^N) \tag{5.14}$$

with coefficients a_i^N, $i = 1, \ldots, N$, that depend on G, and a global pointgrid $x_1^N, \ldots, x_N^N$ on the sphere in Ω_r^{ext} close to Ω_r. If we are only interested in a local model for points y in some subdomain $\mathcal{M}$ of Ω_R, then only those terms $a_i^N \, \Phi_j^\Lambda(y, x_i^N)$ in the sum (5.14) with the point x_i^N in some neighbourhood of $\mathcal{M}$ yield an essential contribution to the sum. Thus all terms $a_i^N \, \Phi_j^\Lambda(y, x_i^N)$ with x_i^N outside this neighbourhood may be omitted.

According to Theorem 5.24, we may construct further examples of a regularization scaling function for $\Lambda^{-1} : \mathcal{H}(\{A_n(\Lambda^\wedge(n))^{-1}\}; h; \overline{\Omega_r^{ext}}) \to \mathcal{H}(\{A_n\}; \overline{\Omega_R^{ext}})$ with the help of a suitable linear scaling function for $\mathcal{H}(\{A_n\}; h; \overline{\Omega_r^{ext}})$. The most simple construction starts from a band-limited scaling function $\{\Phi_j\}_{j\in\mathbb{N}_0}$ for $\mathcal{H}(\{A_n\}; h; \overline{\Omega_r^{ext}})$ and applies the inverse operator Λ^{-1} to Φ_j, $j \in \mathbb{N}_0$. Due to the bandlimited nature of the scaling function the two sums in Theorem 5.24 are always finite in this case.

Example 5.38 (Bandlimited Regularization Scaling Functions) *Let $\varphi : [0,\infty) \to \mathbb{R}$ denote a function with the following properties:*
(i) $\varphi(0) = 1$,
(ii) $\varphi(x) = 0$ for $x \in [1,\infty)$ and
(iii) φ is monotonically decreasing.
Let $\{m_j\}_{j\in\mathbb{N}_0} \subset \mathbb{N}_0$ be a monotonically increasing sequence of non-negative integers. Define the generating symbol $\{\{(\Phi_j)^\wedge(n)\}_{n\in\mathbb{N}_0}\}_{j\in\mathbb{N}_0}$ of a scaling function $\{\Phi_j\}_{j\in\mathbb{N}_0}$ for $\mathcal{H}(\{A_n\}; h; \overline{\Omega_r^{ext}})$ by

$$(\Phi_j)^\wedge(n) := \phi(n/(m_j + 1)),\ n \in \mathbb{N}_0,\ j \in \mathbb{N}_0.$$

For all $j \in \mathbb{N}_0$ Φ_j is a bandlimited $\mathcal{H}(\{A_n\}; h; \overline{\Omega_r^{ext}})$ product kernel with band limit m_j. Then $\{\{(\Phi_j^\Lambda)^\wedge(n)\}_{n\in\mathbb{N}_0}\}_{j\in\mathbb{N}_0}$, defined by

$$(\Phi_j^\Lambda)^\wedge(n) := (\Lambda^\wedge(n))^{-1}(\Phi_j)^\wedge(n) \quad \text{for} \quad n \in \mathbb{N}_0,\ j \in \mathbb{N}_0,$$

is the generating symbol of a bandlimited regularization scaling function for $\Lambda^{-1} : \mathcal{H}(\{A_n(\Lambda^\wedge(n))^{-1}\}; h; \overline{\Omega_r^{ext}}) \to \mathcal{H}(\{A_n\}; \overline{\Omega_R^{ext}})$. Two popular choices of φ are

$$\varphi(x) = \begin{cases} 1 & \text{for} \quad x \in [0,1) \\ 0 & \text{for} \quad x \in [1,\infty), \end{cases}$$

which leads to the so-called Shannon scaling function for $\mathcal{H}(\{A_n\}; h; \overline{\Omega_r^{ext}})$, and

$$\varphi(x) = \begin{cases} (1-x)^2(1+2x) & \text{for} \quad x \in [0,1) \\ 0 & \text{for} \quad x \in [1,\infty), \end{cases}$$

which leads to the so-called Cubic Polynomial (CP) scaling function for $\mathcal{H}(\{A_n\}; h; \overline{\Omega_r^{ext}})$.

Finally, we also give an example of a reproducing regularization scaling function for $\Lambda^{-1} : \mathcal{H}(\{A_n(\Lambda^\wedge(n))^{-1}\}; h; \overline{\Omega_r^{ext}}) \to \mathcal{H}(\{A_n\}; \overline{\Omega_R^{ext}})$. The construction uses the so-called exponential Gauss-Weierstrass scaling function for $\mathcal{H}(\{A_n\}; \overline{\Omega_R^{ext}})$ in combination with the inverse operator Λ^{-1}.

Example 5.39 (Exponential Gauss-Weierstrass Regularization Scaling Function) *Let the sequence $\{\{(\Phi_j^R)^\wedge(n)\}_{n\in\mathbb{N}_0}\}_{j\in\mathbb{N}_0} \subset \mathbb{R}_0^+$ be given by*

$$(\Phi_j^R)^\wedge(n) := e^{-2^{-j}\alpha n(n+1)},\ n \in \mathbb{N}_0,\ j \in \mathbb{N}_0.$$

The exponential Gauss-Weierstrass scaling function $\{\Phi_j^R\}_{j\in\mathbb{N}_0}$ *for* $\mathcal{H}(\{A_n\};\overline{\Omega_R^{ext}})$ *is given by*

$$\Phi_j^R(x,y) := \sum_{n\in\mathcal{N}}\sum_{k=1}^{2n+1}(\Phi_j^R)^\wedge(n)\frac{1}{A_n^2}H_{n,k}(R;\cdot)H_{n,k}(R;\cdot),\ x,y\in\overline{\Omega_R^{ext}},\ j\in\mathbb{N}_0.$$

The Gauss-Weierstrass scaling function is a reproducing scaling function for $\mathcal{H}(\{A_n\};\overline{\Omega_R^{ext}})$. *The exponential Gauss-Weierstrass regularization scaling function* $\{(\Phi_j^\Lambda)^R\}_{j\in\mathbb{N}_0}$ *for the* $\Lambda^{-1}:\mathcal{H}(\{A_n(\Lambda^\wedge(n))^{-1}\};h;\overline{\Omega_r^{ext}})\to\mathcal{H}(\{A_n\};\overline{\Omega_R^{ext}})$ *then is the reproducing regularization scaling function for* $\Lambda^{-1}:\mathcal{H}(\{A_n(\Lambda^\wedge(n))^{-1}\};h;\overline{\Omega_r^{ext}})\to\mathcal{H}(\{A_n\};\overline{\Omega_R^{ext}})$ *corresponding to* $\{\Phi_j^R\}_{j\in\mathbb{N}_0}$. *Its generating symbol* $\{\{((\Phi_j^\Lambda)^R)^\wedge(n)\}_{n\in\mathbb{N}_0}\}_{j\in\mathbb{N}_0}$ *is given by*

$$((\Phi_j^\Lambda)^R)^\wedge(n) := (\Lambda^\wedge(n))^{-1}(\Phi_j^R)^\wedge(n)\quad \textit{for}\quad n\in\mathbb{N}_0,\ j\in\mathbb{N}_0.$$

In order to see that the exponential Gauss-Weierstrass regularization scaling function is well-defined, we have to verify that

$$\sum_{n\in\mathbb{N}_0}(2n+1)(\Lambda^\wedge(n))^{-2}((\Phi_j^R)^\wedge(n))^2<\infty\quad\text{for all}\quad j\in\mathbb{N}_0.$$

Since we regard $\Lambda\in\{\Lambda^{SST},\Lambda^{SGG}\}$, the operator symbol $\{\Lambda^\wedge(n)\}_{n\in\mathbb{N}_0}$ is given by $\Lambda^\wedge(n)=p(n)(\frac{R}{r})^n$, where $p(x)=(x+1)/r$ in case of SST and $p(x)=(x+1)(x+2)/r^2$ in case of SGG.

$$\begin{aligned}
&\sum_{n\in\mathbb{N}_0}(2n+1)(\Lambda^\wedge(n))^{-2}((\Phi_j^R)^\wedge(n))^2\\
&= \sum_{n\in\mathbb{N}_0}(2n+1)(p(n))^2\left(\frac{r}{R}\right)^{2n}e^{-2^{-j+1}\alpha n(n+1)}\\
&= \sum_{n\in\mathbb{N}_0}\frac{2n+1}{(p(n))^2}e^{2n\ln(\frac{r}{R})-2^{-j+1}\alpha n(n+1)}\\
&= \sum_{n\in\mathbb{N}_0}\frac{2n+1}{(p(n))^2}\left(e^{2(\ln(\frac{r}{R})-2^{-j}\alpha(n+1))}\right)^n.
\end{aligned}$$

Due to the quotient criterion the sum is finite, if the term in the brackets satisfies

$$e^{2(\ln(\frac{r}{R})-2^{-j}\alpha(n+1))}<C<1\quad\text{for all}\quad n\geq N_0 \tag{5.15}$$

for some consant C, $0<C<1$, and some $N_0\in\mathbb{N}$. As $\ln(\frac{r}{R})$ is a constant and $\lim_{n\to\infty}2^{-j}\alpha(n+1)=\infty$ condition (5.15) is satisfied.

Part II

Numerical Realization

Chapter 6

The Schwarz Alternating Algorithm

After the theoretical preparations in Part I we now turn our attention to the numerical computation of the multiscale approximation of the regularized solution to the SST-/SGG-Problem 3.8 in its discrete formulation. Thus the general assumptions throughout Part II are as follows:

Let $\{A_n\}_{n\in\mathbb{N}_0} \subset \mathbb{R}^+$ be a sequence of positive real numbers which satisfy $A_n \geq 1$ for almost all $n \in \mathbb{N}_0$. According to our geometrical concept, we assume that $\Sigma_E, \Sigma_S \subset \mathbb{R}^3$ are $\mathcal{C}^{(2)}$-regular surfaces with $\sup_{y\in\Sigma_E}|y| < \inf_{x\in\Sigma_S}|x|$ and $R, r \in \mathbb{R}^+$ are real numbers with $R < \inf_{y\in\Sigma_E}|y|$, $\sup_{y\in\Sigma_E}|y| < r < \inf_{x\in\Sigma_S}|x|$. Furthermore, the operator $\Lambda : \mathcal{H}(\{A_n\};\overline{\Omega_R^{ext}}) \to \mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})$ with singular system $(\Lambda^\wedge(n), H_{n,k}(R;\cdot), H_{n,k}(r;h;\cdot))$, $n \in \mathbb{N}_0$, $1 \leq k \leq 2n+1$, shall be either the SST-operator Λ^{SST}, where $h = 1$, or the SGG-operator Λ^{SGG}, where $h = 2$. Note that for the brevity of the notation we will in the sequel always refer to the above assumptions as the general assumptions of the numerical part without explicitly repeating them.

In order to obtain an approximate solution to the non-discrete SST/SGG-problem $\Lambda F = G$, $F \in \mathcal{H}(\{A_n\};\overline{\Omega_R^{ext}})$, $G \in \mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})$, we first construct an $\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})$-spline S^G, which is a sensible approximation of the right-hand side G. This spline has to be computed with the help of the given data $\{(x_i^N, G(x_i^N)|\, i = 1,\ldots,N\}$, where $X^N := \{x_1^N,\ldots,x_N^N\} \subset \Sigma_S$ is a set of mutually distinct points on the 'orbital surface'. In the case of exact data (no measurement noise) this can be done by solving the $\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})$-spline interpolation problem with respect to the bounded linear measurement functionals $\mathcal{L}_1^N,\ldots,\mathcal{L}_N^N$ given by $\mathcal{L}_i : \mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}}) \to \mathbb{R}$, $H \mapsto \mathcal{L}_i^N(H) := H(x_i^N)$, for the right-hand side G. In case of noisy data we have to solve an $\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})$-spline approximation problem (with respect to the same bounded linear functionals) for the given error-affected data of G as explained in Section 4.2. The $\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})$-spline S^G obtained as the solution to the $\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})$ spline interplation problem or the $\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})$ spline smoothing problem (dependent on the data situation) is an approximation of the right-hand-side G of $\Lambda F = G$. An approximation to the solution $F = \Lambda^{-1}G$ can then be calculated by convolving S^G with a suitable regularization scaling function $\{\Phi_j^\Lambda\}_{j\in\mathbb{N}_0}$ for Λ^{-1}. This aspect is discussed in Chapter 7.

In this chapter we will be only concerned with the calculation of an interpolating or approximating $\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})$-spline for the right-hand side $G \in \mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})$. In order to solve the linear equation system of the $\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})$-spline interpolation or approximation problem in a fast, efficient and easily implementable way with modest storage requirement, a domain decomposition method, namely a variant of the multiplicative Schwarz algorithm (or Schwarz alternating procedure (SAP)), is applied. The Schwarz alternating procedure is first formulated for the $\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})$-spline interplation problem in Section 6.1 and its convergence is investigated. At the end of this section we explain how the Schwarz alternating procedure can also be applied to the solution of the $\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})$-spline smoothing problem and even more to the solution of any linear equation system with a symmetric positive definite matrix. In Section 6.2 the numerical implementation of the SAP algorithm is presented. As the idea behind the multiplicative Schwarz algorithm is more simple to understand for the $\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})$-spline interpolation problem, the motivation below is given for the $\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})$-spline interpolation scenario.

We want to solve the $\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})$-spline interplation problem with respect to N samples of $G \in \mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})$ taken in the points $\{x_1^N,\ldots,x_N^N\}$ on the satellite orbit Σ_S. Under our assumptions the evaluation functionals $\mathcal{L}_i^N : \mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}}) \to \mathbb{R}$, $G \mapsto \mathcal{L}_i^N(G) := G(x_i^N)$, $1 \le i \le N$ are bounded and we demand in addition that they are linearly independent. Let $L_i^N \in \mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})$ denote the representers of $\mathcal{L}_i^N$, $1 \le i \le N$. According to Chapter 4 we have to solve the linear equation system

$$\sum_{l=1}^{N} a_l^N (L_i^N, L_l^N)_{\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})} = (G, L_i^N)_{\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})} = G(x_i^N),\ 1 \le i \le N, \tag{6.1}$$

in order to obtain the coefficients $a_1^N,\ldots,a_N^N$ of the $\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})$-spline interpolant

$$S^G_{\mathcal{L}_1^N,\ldots,\mathcal{L}_N^N} = \sum_{i=1}^{N} a_i^N L_i^N.$$

of G relative to $\{\mathcal{L}_1^N,\ldots,\mathcal{L}_N^N\}$. The numerical solution of the linear equation system (6.1) for large data sets comprising more than $N = 10000$ samples of G on the satellite orbit is the crucial point and the big challenge in our approach to the SST-/SGG problem with respect to stability, storage requirement and computation time.

In our numerical computations we always choose the sequence $\{A_n\}_{n\in\mathbb{N}_0}$, and consequently the space $\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})$, in such a way that the Riesz representers L_i^N, $1 \le i \le N$, can be represented according to $L_i^N = K_{\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})}(x_i^N,\cdot)$, where $K_{\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})} : \overline{\Omega_r^{ext}} \times \overline{\Omega_r^{ext}} \to \mathbb{R}$ is a kernel function which has a representation as an elementary function. Examples of spaces with this property are given in Section 4.5. Apart from the demand that the representers L_i^N, $i = 1,\ldots,N$, are available as elementary functions, it is necessary to choose

$\{A_n\}_{n\in\mathbb{N}_0}$ in such a way that the space localization of $L_1^N, \ldots, L_N^N$ is adapted to the density of the point grid $X^N \subset \Sigma_S$ in which the samples of G are taken. This will be investigated in Chapter 8.

One strategy to solve large linear systems of type (6.1) is to start from an iterative solution method like the Conjugate Gradient (CG) method or the Generalized Minimal Residual (GMRES) method. In this type of equation solver a matrix-vector multiplication has to be performed in each iterative step, i.e., N summations of the form $\sum_{i=1}^{N} \tilde{a}_i^N K_{\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})}(x_i^N, x_l^N)$, where $\tilde{a}^N \in \mathbb{R}^N$, have to be carried out. The idea then is to accelerate the above summations with the help of a Fast Multipole Method (FMM), which exploits the spatial localization of the kernel $K_{\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})}$ in order to compute an approximation of the sum $\sum_{i=1}^{N} \tilde{a}_i^N K_{\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})}(x_i^N, x_l^N)$. This method has recently been successfully applied in the Geomathematics Group in Kaiserslautern for the case that $K_{\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})}$ is the singularity kernel (see [Gl2001], [Mi2001] and the references therein).

A second approach to solve the $\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})$-spline interpolation problem for a large number of samples is to apply a domain decomposition method, which 'splits' the large linear equation system into a number of smaller ones which are alternatingly solved in an iterative algorithm. The Schwarz alternating procedure is such an iterative solution method in which the original set of bounded linear functionals $\Xi^N := \{\mathcal{L}_1^N, \ldots, \mathcal{L}_N^N\} \subset \mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})^*$ is partitioned into $M \in \mathbb{N}$ possibly overlapping subsets $\Xi_1^N, \ldots, \Xi_M^N$. The smaller $\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})$-spline interpolation problems relative to the sets Ξ_i^N, $i \in \{1, \ldots, M\}$, are then alternatingly solved in an iterative procedure, and the sequence of iterates converges to the solution of the large $\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})$-spline interpolation problem relative to Ξ^N.

6.1 The Schwarz Alternating Algorithm for the Spline Interpolation Problem

The so-called Schwarz alternating procedure (SAP) (or multiplicative Schwarz algorithm) was introduced by H. Schwarz in [Schw1890], and is, together with its additive variant, one of the best known and most widely used domain decomposition principles which is applied in many methods for the fast solution of partial differential equations. The idea that the Schwarz alternating procedure could be applied to the $\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})$-spline interpolation problem was inspired by the publication [BeLiBi2000] by R. K. Beatson, W. A. Light and S. Billings, in which the radial basis function interpolation problem was solved in a fast and efficient way with the help of the Schwarz algorithm. The analogy between the radial basis function interpolation problem and the $\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})$-spline interpolation problem is based on the fact that both problems can be formulated in terms of orthogonal projections in exactly the same way. Since in [BeLiBi2000] the applicability of the multiplicative Schwarz algorithm is based on its formulation as a sequence of orthogonal

projections onto overlapping subspaces, the method can be completely transferred to the $\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})$-spline interpolation problem. Moreover, the Schwarz alternating procedure can be applied to the solution of the spline smoothing problem and even more to the solution of any linear equation system with a symmetric positive definite matrix. This will be briefly scetched at the and of this section.
Due to Theorem 4.3 the $\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})$-spline interpolation problem can be formulated in the following two equivalent ways:

$\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})$-Spline Interpolation Problem: Formulation 1

Let $\Xi^N := \{\mathcal{L}_1^N,\ldots,\mathcal{L}_N^N\} \subset \mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})^*$ be a set of $N \in \mathbb{N}$ linearly independent bounded linear functionals on $\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})$ and let $L_i^N \in \mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})$ denote the representers of $\mathcal{L}_i^N$, $i \in \{1,\ldots,N\}$ according to the Riesz representation theorem. Find $S^G_{\mathcal{L}_1^N,\ldots,\mathcal{L}_N^N} = \sum_{i=1}^N a_i^N L_i^N \in \mathcal{S}_{\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})}(\mathcal{L}_1^N,\ldots,\mathcal{L}_N^N) = \mathrm{span}(L_1^N,\ldots,L_N^N)$ such that

$$\mathcal{L}_k^N S^G_{\mathcal{L}_1^N,\ldots,\mathcal{L}_N^N} = \sum_{i=1}^N a_i^N (L_k^N, L_i^N)_{\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})} \stackrel{!}{=} \mathcal{L}_k^N G \text{ for } k = 1,\ldots,N.$$

$\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})$-Spline Interpolation Problem: Formulation 2

Let $P_{\Xi^N} : \mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}}) \to \mathcal{S}_{\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})}(\mathcal{L}_1^N,\ldots,\mathcal{L}_N^N)$ denote the orthogonal projector onto the N-dimensional spline space $\mathcal{S}_{\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})}(\mathcal{L}_1^N,\ldots,\mathcal{L}_N^N)$. Find the orthogonal projection $P_{\Xi^N}G$ of a given function $G \in \mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})$ which is only known indirectly in form of the values $\mathcal{L}_k^N G$, $k \in \{1,\ldots,N\}$.

The orthogonal projector P_{Ξ^N} is just the interpolation operator which maps the function $G \in \mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})$ onto its interpolating spline $S^G_{\mathcal{L}_1^N,\ldots,\mathcal{L}_N^N}$.

The multiplicative Schwarz method allows it to partition the set of bounded linear functionals Ξ^N into $M \in \mathbb{N}$ possibly overlapping subsets $\Xi_j^N \subset \Xi^N$, $j \in \{1,\ldots,M\}$ and successively solve the smaller $\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})$-spline interpolation problems with respect to the subsets Ξ_j^N, $j \in \{1,\ldots,M\}$ in an iterative algorithm. This procedure yields a sequence of iterates which converge to the solution $P_{\Xi^N}G$ of the large interpolation problem. To be more specific, we consider a partition $\{\Xi_1^N,\ldots,\Xi_M^N\} \subset \Xi^N$, $\Xi_j^N := \{\mathcal{L}_1^{N_j},\ldots,\mathcal{L}_{N_j}^{N_j}\}$, such that

$$\bigcup_{j=1}^M \Xi_j^N = \Xi^N.$$

In general, the subsets $\Xi_1^N,\ldots,\Xi_M^N$ need not be pairwise disjoint, i.e., there exist indices $i,j \in \{1,\ldots,M\}$, $i \neq j$, such that

$$\Xi_i^N \cap \Xi_j^N \neq \emptyset.$$

In this case we will speak of two overlapping subsets Ξ_i^N and Ξ_j^N.

For each subset Ξ_j^N, $j \in \{1,\ldots,M\}$ we may formulate the $\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})$-spline interpolation problem as follows: Let $L_i^{N_j} \in \mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})$ denote the representer of $\mathcal{L}_i^{N_j}$, $i \in 1,\ldots,N_j$, according to the Riesz representation theorem. Given $G \in \mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})$, find $S^G_{\mathcal{L}_1^{N_j},\ldots,\mathcal{L}_{N_j}^{N_j}}$ in $\mathcal{S}_{\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})}(\mathcal{L}_1^{N_j},\ldots,\mathcal{L}_{N_j}^{N_j}) := \operatorname{span}(L_1^{N_j},\ldots,L_{N_j}^{N_j})$ such that

$$\mathcal{L}_i^{N_j} S^G_{\mathcal{L}_1^{N_j},\ldots,\mathcal{L}_{N_j}^{N_j}} = \mathcal{L}_i^{N_j} G \text{ for } i = 1,\ldots,N_j.$$

The corresponding interpolation operator is the orthogonal projector onto the space $\mathcal{S}_{\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})}(\mathcal{L}_1^{N_j},\ldots,\mathcal{L}_{N_j}^{N_j})$, i.e.,

$$P_{\Xi_j^N} : \mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}}) \to \mathcal{S}_{\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})}(\mathcal{L}_1^{N_j},\ldots,\mathcal{L}_{N_j}^{N_j}),\ G \mapsto P_{\Xi_j^N} G := S^G_{\mathcal{L}_1^{N_j},\ldots,\mathcal{L}_{N_j}^{N_j}}.$$

With the help of the orthogonal projectors $P_{\Xi_j^N}$ the multiplicative Schwarz algorithm reads as follows:

Algorithm 6.1 (Multiplicative Schwarz Algorithm)

Given $\varepsilon > 0$, $G \in \mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})$.
$R_0 := G$
$S_0^G := 0$
$n := 0$
while $\frac{|(\mathcal{L}_1^N R_{nM},\ldots,\mathcal{L}_N^N R_{nM})^T|}{|(\mathcal{L}_1^N G,\ldots,\mathcal{L}_N^N G)^T|} > \varepsilon$
  for $j = 1 : M$
    $S^G_{nM+j} := S^G_{nM+(j-1)} + P_{\Xi_j^N} R_{nM+(j-1)}$
    $R_{nM+j} := R_{nM+(j-1)} - P_{\Xi_j^N} R_{nM+(j-1)}$
  end
  n:= n+1
end

The functions S^G_{nM+j} are called the iterates (of G), and the functions R_{nM+j} are called the residuals. The sequence $\{S^G_{nM}\}_{n\in\mathbb{N}_0}$ converges to the spline $P_{\Xi^N} G = S^G_{\mathcal{L}_1^N,\ldots,\mathcal{L}_N^N}$. As we will see below, the formulation of the SAP algorithm in terms of orthogonal projections is essential for the analysis of its convergence. For the practical implementation we have to rewrite Algorithm 6.1 in matrix formulation. This is done in Section 6.2.

The next lemma is meant to give some further insight into the construction of Algorithm 6.1.

Lemma 6.2 *Let the notation and assumptions be the same as in Algorithm 6.1 and denote the orthogonal projector onto* $\left(S_{\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})}(\mathcal{L}_1^{N_j},\ldots,\mathcal{L}_{N_j}^{N_j})\right)^\perp$, $j \in \{1,\ldots,M\}$, *by*

$Q_j : \mathcal{H}(\{A_n\}; h; \overline{\Omega_r^{ext}}) \to \left(S_{\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})}(\mathcal{L}_1^{N_j}, \ldots, \mathcal{L}_{N_j}^{N_j})\right)^\perp$, *i.e.* $Q_j := Id - P_{\Xi_j^N}$. *Then the following identities hold true:*

(i) $S^G_{nM+j} = \sum_{i=1}^{j} P_{\Xi_i^N}(R_{nM+(i-1)}) + \sum_{l=0}^{n-1}\sum_{i=1}^{M} P_{\Xi_i^N}(R_{lM+(i-1)})$ *for* $n \in \mathbb{N}_0,\ j \in \{1, \ldots, M\}$.

(ii) $R_{nM+j} = G - S^G_{nM+j}$ *for* $n \in \mathbb{N}_0$, $j \in \{1, \ldots, M\}$.

(iii) $S^G_{nM+j} = S^G_{nM+(j-1)} + P_{\Xi_j^N}\left(G - S^G_{nM+(j-1)}\right)$ *for* $n \in \mathbb{N}_0$, $j \in \{1, \ldots, M\}$.

(iv) $R_{nM+j} = (Q_j \ldots Q_1)(Q_M \ldots Q_1)^n G$ *for* $n \in \mathbb{N}_0$, $j \in \{1, \ldots, M\}$.

(v) $S^G_{nM+j} = G - (Q_j \ldots Q_1)(Q_M \ldots Q_1)^n G$ *for* $n \in \mathbb{N}_0$, $j \in \{1, \ldots, M\}$.

(vi) If Algorithm 6.1 is executed for $P_{\Xi^N} G$ *instead of* G*, then the calculated iterates are identical, i.e.*

$$S^G_{nM+j} = S^{P_{\Xi^N} G}_{nM+j} \text{ for all } n \in \mathbb{N}_0 \text{ and all } j \in \{1, \ldots, M\}.$$

Proof: All identities can be verified by induction over $nM + j$. For the details we refer the reader to [He2002]. □

Identity (iii) in Lemma 6.2 shows that each step in the multiplicative Schwarz algorithm is of the form

$$S^G_{nM+j} = S^G_{nM+(j-1)} + P_{\Xi_j^N}\left(G - S^G_{nM+(j-1)}\right),$$

which means that within each iteration step n a new iterate S^G_{nM+j} with respect to j is calculated from the old iterate $S^G_{nM+(j-1)}$ by solving the $\mathcal{H}(\{A_n\}; h; \overline{\Omega_r^{ext}})$-spline interpolation problem relative to the subset Ξ_j^N for the old residual $R_{nM+(j-1)} = G - S^G_{nM+(j-1)}$ and adding this 'correction term' to the old iterate. In the multiplicative Schwarz algorithm the approximate solution of the large linear system for the residual $R_{nM} = G - S^G_{nM}$ is computed alternatingly with the help of the projectors $P_{\Xi_N^1}, \ldots, P_{\Xi_M^N}$. It should be noted that there also exists a so-called additive Schwarz algorithm in which the approximate solution of the large linear system for the residual $R_{nM} = G - S^G_{nM}$ is computed simultaneously with the help of the projectors $P_{\Xi_1^N}, \ldots, P_{\Xi_M^N}$ in each iteration. Obviously, that variant is very well suited for parallelization.

While it is quite obvious that identities (i) to (iii) illustrate the construction principle of Algorithm 6.1, identities (iv) to (vi) are needed to prove its convergence.

Due to identities (ii) and (iv) in Lemma 6.2, the approximation error with respect to the iterate S^g_{nM} in Algorithm 6.1 can be represented according to

$$G - S^G_{nM} = R_{nM} = (Q_M \ldots Q_1)^n G.$$

The convergence of $\{S_{nM}^G\}_{n\in\mathbb{N}_0}$ to G in $\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})$-sense then follows from Theorem 6.3, which is a special case of the results published by I. Halperin in [Ha1962]. In our special case the Schwarz alternating procedure can be viewed as an algorithm operating on a finite dimensional space according to Lemma 6.2 (vi). Therefore it is also possible to give an estimate of the convergence rate of Algorithm 6.1. This result is stated in Theorem 6.6 below.

Theorem 6.3 *Let $\mathcal{H}$ be a Hilbert space with inner product $(\cdot,\cdot)_{\mathcal{H}}$, let $\mathcal{H}_1,\dots,\mathcal{H}_M$, $M\in\mathbb{N}$, be closed subspaces of $\mathcal{H}$, and let $P_i:\mathcal{H}\to\mathcal{H}_i$, $i\in\{1,\dots,M\}$ be the orthogonal projector onto $\mathcal{H}_i$. Denote by $Q:\mathcal{H}\to\bigcap_{i=1}^M\mathcal{H}_i$ the orthogonal projector onto $\bigcap_{i=1}^M\mathcal{H}_i$, and define $P:\mathcal{H}\to\mathcal{H}$ by $P:=P_M\dots P_1$. Then $\{P^n\}_{n\in\mathbb{N}_0}$ converges pointwise to Q, i.e.,*

$$\lim_{n\to\infty}\|P^nF-QF\|_{\mathcal{H}}=0 \textit{ for all } F\in\mathcal{H}.$$

Proof: The proof can be found for a more general case in [Ha1962]. □

In order to prove the convergence of $\{S_{nM}^G\}_{n\in\mathbb{N}_0}$ to $S^G_{\mathcal{L}_1^N,\dots,\mathcal{L}_N^N}=P_{\Xi^N}G$ in $\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})$-sense, we make use of the following technical lemma:

Lemma 6.4 *Let $\mathcal{H}$ be a Hilbert space with inner product $(\cdot,\cdot)_{\mathcal{H}}$, and let $\mathcal{H}_1,\dots,\mathcal{H}_M$, $M\in\mathbb{N}$, be closed subspaces of $\mathcal{H}$. Then*

$$(\mathcal{H}_1+\dots+\mathcal{H}_M)^\perp=\bigcap_{i=1}^M\mathcal{H}_i^\perp.$$

Proof: The statement follows by elementary calculations. □

Theorem 6.5 *Let the notation and assumptions be the same as in Algorithm 6.1. Then the sequence of iterates $\{S_{nM}^G\}_{n\in\mathbb{N}_0}$ of $G\in\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})$ converges to $S^G_{\mathcal{L}_1^N,\dots,\mathcal{L}_N^N}=P_{\Xi^N}G$.*

Proof: Due to Lemma 6.2 (v) it holds that

$$P_{\Xi^N}G-S_{nM}^G=(P_{\Xi^N}-Id)G+(G-S_{nM}^G)=(P_\Xi-Id)G+(Q_M\dots Q_1)^nG.$$

Theorem 6.3 tells us that the sequence of operators $\{(Q_M\dots Q_1)^n\}_{n\in\mathbb{N}}$ converges pointwise to the orthogonal projector $Q:\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})\to\bigcap_{j=1}^M\mathrm{im}(Q_j)$ onto $\mathrm{im}(Q):=\bigcap_{j=1}^M\mathrm{im}(Q_j)$. But

$$\mathrm{im}(Q_j)=\mathrm{im}(Id-P_{\Xi_j^N})=\left(\mathcal{S}_{\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})}(\mathcal{L}_1^{N_j},\dots,\mathcal{L}_{N_j}^{N_j})\right)^\perp.$$

According to Lemma 6.4,

$$\mathrm{im}(Q)\;=\;\bigcap_{j=1}^M\left(\mathcal{S}_{\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})}(\mathcal{L}_1^{N_j},\dots,\mathcal{L}_{N_j}^{N_j})\right)^\perp$$

$$\begin{aligned} &= \left(\sum_{j=1}^{M} \mathcal{S}_{\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})}(\mathcal{L}_1^{N_j}, \dots, \mathcal{L}_{N_j}^{N_j}) \right)^{\perp} \\ &= \left(\mathcal{S}_{\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})}(\mathcal{L}_1^{N}, \dots, \mathcal{L}_{N}^{N}) \right)^{\perp} \\ &= \operatorname{im}(Id - P_{\Xi^N}). \end{aligned}$$

Consequently, $Q = Id - P_{\Xi^N}$, since the orthogonal projector onto $\operatorname{im}(Id - \mathcal{P}_{\Xi^N})$ is uniquely determined. Hence,

$$\lim_{n\to\infty} P_{\Xi^N}G - S_{nM}^G = (P_{\Xi^N} - Id)G + \lim_{n\to\infty} (Q_M \dots Q_1)^n G = 0.$$

□

Since due to Lemma 6.2 (vi), Algorithm 6.1 can be viewed as an algorithm operating on the finite dimensional spline space $\mathcal{S}_{\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})}(\mathcal{L}_1^N, \dots, \mathcal{L}_N^N)$, we can get the following estimate of its convergence rate:

Theorem 6.6 *Let the notation and the assumptions be the same as in Algorithm 6.1. Then the sequence $\{S_{nM}^G\}_{n\in\mathbb{N}_0}$ of iterates of $G \in \mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})$ converges to $S_{\mathcal{L}_1^N,\dots,\mathcal{L}_N^N}^G = P_{\Xi^N}G$ in $\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})$-sense, and the error estimate*

$$\|P_{\Xi^N}G - S_{nM}^G\|_{\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})} \le C^n \|P_{\Xi^N}G\|_{\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})}$$

holds true with some positive constant $C < 1$, which is independent of G.

Proof: For a detailed proof of the statement the reader is referred to [He2002]. It uses the fact that due to Lemma 6.2 (vi) $S_{nM}^G = S_{nM}^{P_{\Xi^N}G}$, and thus it is sufficient to prove the convergence of Algorithm 6.1 for input functions from $\operatorname{im}(P_{\Xi^N}) = \mathcal{S}_{\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})}(\mathcal{L}_1^N, \dots, \mathcal{L}_N^N)$. Due to Lemma 6.2 (v) the approximation error for $G \in \mathcal{S}_{\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})}(\mathcal{L}_1^N, \dots, \mathcal{L}_N^N)$ can then be written as

$$\|G - S_{nM}^G\|_{\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})} = \|\tilde{Q}^n G\|_{\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})},$$

where $\tilde{Q} := \tilde{Q}_M \dots \tilde{Q}_1$, $\tilde{Q}_j := Q_j|_{\mathcal{S}_{\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})}(\mathcal{L}_1^N,\dots,\mathcal{L}_N^N)}$ and $Q_j := Id - P_{\Xi_j^N}$. Using the fact that $\mathcal{S}_{\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})}(\mathcal{L}_1^N, \dots, \mathcal{L}_N^N)$ is a finite dimensional space with a compact unit sphere it is now rather easy to show that $\|\tilde{Q}\| < 1$. □

Note that Theorem 6.3 gives convergence of the sequence of operators $\{(Q_M \dots Q_1)^n\}_{n\in\mathbb{N}}$ in pointwise sense, whereas Theorem 6.6 shows that in case that the multiplicative Schwarz algorithm can be viewed as an algorithm which operates on a finite-dimensional space, we even obtain convergence in the operator norm ($\|P_{\Xi^N}G\|_{\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})} \le \|G\|_{\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})}$, as P_{Ξ^N} is an orthogonal projector).

It is also possible to solve the linear equation system of an $\mathcal{H}(\{A_n\}; h; \overline{\Omega_r^{ext}})$-spline smoothing problem with the multiplicative Schwarz alternating procedure. We sketch the basic idea behind the application of the Schwarz alternating procedure to the smoothing spline problem only briefly and refer for a detailed explanation and analysis to [He2002]. But the implementation of the algorithm and its formulation in MATLAB pseudocode in Section 6.2 includes the situation of spline interpolation as well as spline smoothing, and the Schwarz alternating procedure is used in our numerical studies for the solution of spline problems.

The crucial point in the application of the Schwarz alternating procedure to the spline interpolation problem is that the spline interpolation problem can be formulated in terms of orthogonal projectors, and the convergence analysis of the algorithm is based on this formulation. This leads to the question whether the spline smoothing problem or more precisely, the linear equation system that has to be solved for the calculation of the smoothing spline, can be interpreted as some kind of orthogonal projection problem.

The matrix of the spline smoothing problem is a symmetric positive definite $N \times N$-matrix A and has a Cholesky factorization $A = L\,L^T$, where L is a uniquely determined lower triangular matrix with positive diagonal entries. Denoting the row vectors of the matrix L by $v_1, \ldots, v_N$, we see that the matrix is actually the Gram matrix of the basis $v_1, \ldots, v_N$. The determination of the representation $f = \sum_{i=1}^{N} x_i\, v_i$ of a vector $f \in \mathbb{R}^N$ with respect to the basis $v_1, \ldots, v_N$ from the knowledge of the inner products (f, v_i), $i = 1, \ldots, N$, leads to the linear equation system

$$\sum_{i=1}^{N} x_i\,(v_i, v_k) = (f, v_k)\,, \quad k = 1, \ldots, N \qquad \Longleftrightarrow \qquad A\,x = (f, v_k)_{k=1,\ldots,N},\,. \tag{6.2}$$

This equation system has the same matrix as the linear equation system of the spline smoothing problem, and a suitable choice of the vector f yields exactly the linear equation system of the spline smoothing problem. The idea for the solution of the spline smoothing problem with the Schwarz alternating procedure is to regard the linear equation system (6.2) as the linear equation system that belongs to the representation problem of $f \in \mathbb{R}^N$ mentioned above. This representation problem can be interpreted as the trivial orthogonal projection problem find $w \in \mathbb{R}^N$ such that $Id\,w = f$, where Id is the identity operator. This trivial orthogonal projection problem can be solved with the Schwarz alternating procedure and its numerical realization as the representation problem (6.2) leads to a Schwarz alternating procedure that solves the linear equatin system of the spline smoothing problem. As this approach to the spline smoothing problem uses only the fact that it has a symmetric positive definite matrix, the same idea allows the application of the Schwarz alternating procedure for the solution of any linear equation system with a symmetric positive definite matrix.

6.2 Implementation of an SAP Equation Solver

The Schwarz algorithm has not been applied in the Geomathematics Group before and is investigated within the Geomathematics Group for the solution of the $\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})$-spline interpolation problem and the $\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})$-spline smoothing problem in [Gu2002], [He2002], and this thesis for the first time. In this section the details of the implementation are discussed in the next three subsections. The code itself is written in C, and a formulation of the algorithm in MATLAB-pseudo-code can be found at the end of this section on page 98. The subdivision of the original pointset into overlapping subsets which corresponds to the partitioning of the evaluation functionals into subsets is based on a geometrical subdivision of the sphere into subdomains of identical surface areas. The subproblems are solved with a direct solver, and we explain how the update can be calculated without storage of the complete matrix of the initial large spline interpolation or spline smoothing problem.

Subdivision of the Pointset into Overlapping Subsets

Our implementation of the SAP algorithm makes use of the fact that the only information on the subdivision we need to store in order to perform the SAP iterations are the indices of the points which belong to a certain subset. Apart from that we are free to apply any criterion we consider to be appropriate to perform the subdivision. Since in our numerical tests of the SAP algorithm we solve the $\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})$-spline interpolation and $\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})$-spline smoothing problem with test data given on a sphere or an ellipsoid of revolution with an eccentricity close to zero, we have implemented a subdivision scheme which assumes that the points are given in geographical spherical polar coordinates $(\rho,\varphi_g,\vartheta_g)$, $\rho > r$, $\varphi_g \in [-\pi,\pi)$ and $\vartheta_g \in [-\pi/2,\pi/2]$ and that the φ_g- and ϑ_g-coordinates lie in a certain subdomain $[\varphi^0_{g_{min}},\varphi^0_{g_{max}}] \times [\vartheta^0_{g_{min}},\vartheta^0_{g_{max}}]$, $\varphi^0_{g_{min}} < \varphi^0_{g_{max}}$, $\vartheta^0_{g_{min}} < \vartheta^0_{g_{max}}$, of the φ_g-ϑ_g-plane $[-\pi,\pi) \times [-\pi/2,\pi/2]$. Note that this assumption means no loss of generality, since it can always be fulfilled by simply shifting the φ_g-coordinates of the points in φ_g-direction, and we are only interested in determining the point indices. We will simply call this domain the φ_g-ϑ_g-box in which the data points are given.

In the sequel we briefly explain the geometrical idea behind the angular test which is performed to obtain the indices of the points in the overlapping subdomains:
The φ_g-ϑ_g-box defines a domain on the unit sphere, and we first consider a subdivision of this domain into disjoint subdomains of equal surface areas, i.e., we choose a subdivision of the φ_g-ϑ_g-box into $M \in \mathbb{N}$ subdomains such that

$$\left(\varphi^i_{g_{max}} - \varphi^i_{g_{min}}\right)\left(t^i_{g_{max}} - t^i_{g_{min}}\right) = \left(\varphi^j_{g_{max}} - \varphi^j_{g_{min}}\right)\left(t^j_{g_{max}} - t^j_{g_{min}}\right) \tag{6.3}$$

for $i,j \in \{1,\ldots,M\}$, where $\varphi^i_{g_{max}} > \varphi^i_{g_{min}}$, $\vartheta^i_{g_{max}} > \vartheta^i_{g_{min}}$ and $t^i_{g_{max}} := \sin(\vartheta^i_{g_{max}})$, $t^i_{g_{min}} := \sin(\vartheta^i_{g_{min}})$ for $i \in \{1,\ldots,M\}$. Special care has to be taken if one of the poles $(\pi,\pi/2)$ or $(-\pi,-\pi/2)$ is contained in the φ_g-ϑ_g-box, which leads to the following case analysis:

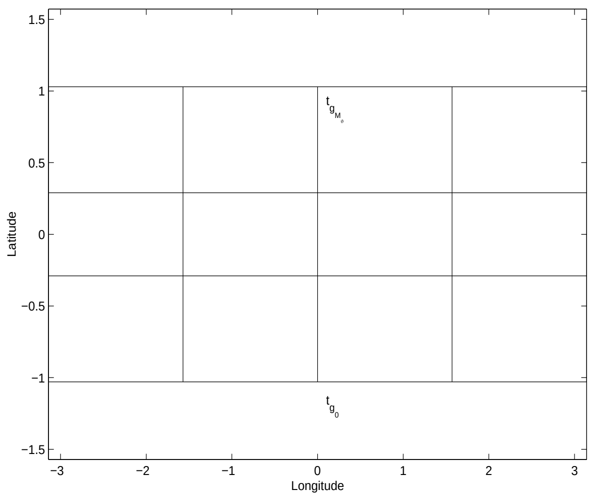


Figure 6.1: *Illustration of the subdivision scheme.*

Case 1: The φ_g-ϑ_g-box contains both poles.
As illustrated in Figure 6.1, we subdivide the φ_g-ϑ_g-box into two polar segments and a ring segment consisting of $M_\varphi M_\vartheta$ subdomains. Thus the total number of subdomains is given by $M = M_\varphi M_\vartheta + 2$. In the ring segment the subdivision is equidistant with respect to φ_g and $t_g := \sin(\vartheta_g)$. Denote $\Delta t := t_{g_{i+1}} - t_{g_i},\ i = 0, \ldots, M_\vartheta - 1$. We have to compute $t_{g_{M_\vartheta}}$ and Δt. The surface area of a subdomain in the ring segment is equal to $\Delta\varphi\Delta t$, where

$$\Delta\varphi := \frac{\varphi^0_{g_{max}} - \varphi^0_{g_{min}}}{M_\varphi}.$$

Condition (6.3) then reads

$$\begin{aligned} &(M_\varphi \Delta\varphi)(1 - t_{g_{M_\vartheta}}) = (M_\varphi \Delta\varphi)(t_{g_0+1}) = \Delta\varphi\Delta t \\ &\Leftrightarrow M_\varphi(1 - t_{g_{M_\vartheta}}) = M_\varphi(t_{g_0} + 1) = \Delta t. \end{aligned} \tag{6.4}$$

The first equality in (6.4) yields $t_{g_{M_\vartheta}} = -t_{g_0}$. As the φ_g-ϑ_g-box cointains both poles, $t^0_{g_{max}} = 1$ and $t^0_{g_{min}} = -1$. Thus

$$\sum_{i=0}^{M_\vartheta - 1} (t_{g_{i+1}} - t_{g_i}) + (1 - t_{g_{M_\vartheta}}) + (t_{g_0} + 1) = 2,$$

and a simple calculation yields

$$t_{g_0} = -\frac{M_\varphi M_\vartheta}{2 + M_\varphi M_\vartheta} \quad \text{and} \quad \Delta t = \frac{2M_\varphi}{2 + M_\varphi M_\vartheta}.$$

Case 2: The φ_g-ϑ_g-box contains exactly one pole.
Assume that the φ_g-ϑ_g-box contains only the North pole. In analogy to case 1 we subdivide the φ_g-ϑ_g-plane into one polar segment and a ring segment consisting of $M_\varphi M_\vartheta$ subdomains such that the total number of subdomains is given by $M = M_\varphi M_\vartheta + 1$. Again, the subdivision is equidistant with respect to φ_g and $t_g := \sin(\vartheta_g)$ in the ring segment. $t_{g_0} = \sin(\vartheta^0_{g_{min}})$, and we have to compute $t_{g_{M_\vartheta}}$ and Δt. Observing that

$$\Delta\varphi = \frac{\varphi^0_{g_{max}} - \varphi^0_{g_{min}}}{M_\varphi} \text{ and } \Delta t = \frac{t_{g_{M_\vartheta}} - t_{g_0}}{M_\vartheta},$$

condition (6.3) now reads

$$(M_\varphi \Delta\varphi)(1 - t_{g_{M_\vartheta}}) = \Delta\varphi\Delta t \Leftrightarrow M_\varphi(1 - t_{g_{M_\vartheta}}) = \Delta t,$$

and an easy computation gives

$$\Delta t = M_\varphi \frac{1 - t_{g_0}}{1 + M_\varphi M_\vartheta}$$

and consequently,

$$t_{g_{M_\vartheta}} = t_{g_0} + M_\vartheta \Delta t.$$

In case that only the South pole is contained in the φ_g-ϑ_g-box, analogous considerations yield

$$\Delta t = M_\varphi \frac{1 + t_{g_{M_\vartheta}}}{1 + M_\varphi M_\vartheta} \quad \text{and} \quad t_{g_0} = t_{g_{M_\vartheta}} - M_\vartheta \Delta t.$$

Case 3: The φ_g-ϑ_g-box contains no poles.
In this case the φ_g-ϑ_g-box is subdivided into $M = M_\varphi M_\vartheta$ subdomains, where $t_{g_0} = \sin(\vartheta^0_{g_{min}})$, $t_{g_{M_\vartheta}} = \sin(\vartheta^0_{g_{max}})$, and

$$\Delta\varphi = \frac{\varphi^0_{g_{max}} - \varphi^0_{g_{min}}}{M_\varphi} \quad \text{and} \quad \Delta t := \frac{t_{g_{M_\vartheta}} - t_{g_0}}{M_\vartheta}.$$

Based on the initial subdivision of the φ_g-ϑ_g-box into M subdomains $[\varphi^i_{g_{min}}, \varphi^i_{g_{max}}) \times [\vartheta^i_{g_{min}}, \vartheta^i_{g_{max}})$, $i \in \{1, \ldots, M\}$, the limiting angles of the overlapping subdomains are obtained as follows: We specify numbers $p_\varphi,\ p_\vartheta \in (0,1)$. For each segment i which has a neighbouring segment in $(+\varphi_g)$-direction, we assign

$$\varphi^i_{g_{max}} \leftarrow \varphi^i_{g_{max}} + p_\varphi \Delta\varphi$$

and for each segment i which has an adjacent segment in $(-\varphi_g)$-direction, we assign

$$\varphi^i_{g_{min}} \leftarrow \varphi^i_{g_{min}} - p_\varphi \Delta\varphi.$$

Here we observe the overlap due to the topology of the sphere in case that $\varphi^0_{g_{min}} = -\pi$ and $\varphi^0_{g_{max}} = \pi$.
For each segment i which has an adjacent segment j in $(+\varphi_g)$-direction, we assign

$$\vartheta^i_{g_{max}} \leftarrow \begin{cases} \arcsin(\sin(\vartheta^i_{g_{max}}) + p_t(1 - t_{g_{M_\vartheta}})), & \text{if segment j is the North Pole} \\ \arcsin(\sin(\vartheta^i_{g_{max}}) + p_t\Delta t) & \text{else,} \end{cases}$$

and for each segment i which has a neighbouring segment j in $(-\varphi_g)$-direction, we assign

$$\vartheta^i_{g_{min}} \leftarrow \begin{cases} \arcsin(\sin(\vartheta^i_{g_{min}}) - p_t(1 + t_{g_0})), & \text{if segment j is the South Pole} \\ \arcsin(\sin(\vartheta^i_{g_{min}}) - p_t\Delta t) & \text{else.} \end{cases}$$

Numerical Solution of the Spline Problem for the Subsets

In order to solve the spline interpolation or spline smoothing equations related to the subsets, we use two solution algorithms which are designed for the direct solution of linear systems with symmetric coefficient matrices, namely the well-known (and stable) Cholesky decomposition for positive definite symmetric systems and the Parlett-Reid algorithm for symmetric indefinite systems with Bunch-Kaufman pivoting. Both algorithms are implemented in the FORTRAN software package LAPACK, which is electronically available via Netlib (see `http://www.netlib.org/index.html` for an overview of the available routines and the LAPACK user manual [AnBaBi1995]) and expect the coefficient matrix of the linear system which has to be solved to be given in packed store-by-column format. This means a storage requirement for a matrix $K_j \in \mathbb{R}^{N_j \times N_j}$ of order $\mathcal{O}(N_j^2/2)$, $j = 1, \dots, M$. In our implementation of the SAP method we either use the LAPACK double precision routine `dpptrf` to compute the Cholesky factorizations of the coefficient matrices corresponding to the subproblems in the setup step or the LAPACK routine `dpttrf` to compute their LDL^T-factorizations by means of the Parlett-Reid algorithm with Bunch-Kaufman pivoting. Here L denotes a lower triangular and D a diagonal matrix. The factorized coefficient matrices K_j are kept in the store, which leads to a total memory requirement of $\mathcal{O}(N^2/(2M))$ for the storage of the M submatrices, which means a reduction of the storage requirement by a factor M in comparison to the one of the large matrix, provided that the overlap of the subsystems is not too large. Within the SAP iterations the solution of the linear systems is computed with the routines `dpptrs` and `dpttrs`, respectively, which perform the back (or forward) substitution.

For general literature on the solution of symmetric systems and on the error analysis of the selected algorithms we refer the reader to [GoVLo1996] and the references therein.

The computation of the Cholesky factorization of a matrix $K_j \in \mathbb{R}^{N_j \times N_j}$ requires $N_j^3/3$ floating point operations (flops). The Parlett-Reid algorithm computes the LDL^T-factorization of a symmetric indefinite matrix $K_j \in \mathbb{R}^{N_j \times N_j}$ using Gauss transforms and requires $2N_j^3/3$ flops. The diagonal pivoting method of Bunch and Kaufman additionally involves $N_j^3/3$ flops and $\mathcal{O}(N_j^2)$ comparisons.

Residual Update

In each iterative step n an update of the residual has to be performed after the solution of each of the M subproblems. Having computed the solution $a^{N_j} = (a_1^{N_j}, \ldots, a_{N_j}^{N_j})^T$ to the linear equation system

$$\sum_{l=1}^{N_j} a_l^{N_j} (K_{\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})}(x_i^{N_j}, x_l^{N_j}) + \lambda \delta_{i,l}) = R_{nM+(j-1)}(x_i^{N_j}),\ 1 \leq i \leq N_j,$$

where $\lambda = 0$ incase of spline interpolation and $\lambda > 0$ in case of spline smoothing, we have to evaluate the sum

$$\sum_{i=1}^{N_j} a_i^{N_j} (K_{\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})}(x_i^{N_j}, x_l^{N}) + \lambda \delta_{l,i(j)})$$

for $l = 1, \ldots, N$, where we use the notation $x_i^{N_j} := x_{i(j)}^N$, $1 \leq j \leq M$, $1 \leq i \leq N_j$. This means that we have to multiply the submatrix $((K_{\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})}(x_i^{N_j}, x_l^N) + \lambda \delta_{l,i(j)})_{1 \leq i \leq N_j,\ 1 \leq l \leq N}$ of the coefficient matrix of the initial large system with the coefficient vector $a_i^{N_j}$. Here the submatrix entries are generated dynamically by calls of the kernel function $K_{\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})}$, while the point coordinates of the large system are read once in the setup step and kept in the memory. Obviously, the time requirement for the residual updates makes our algorithm less and less efficient the finer we subdivide.

This drawback can be overcome with the help of the Fast Multipole Method mentioned in the introduction to this chapter. Moreover, in an additive variant of the Schwarz algorithm the subproblems can be solved simultaneously on parallel computers and the residual update must only be computed once per iterative step. The design of an efficient solver which combines the (additive) Schwarz algorithm and fast mutipole techniques is a challenging task for future investigations.

SAP-Algorithm Including a Subdivision of the Initial Pointset in the φ_g-ϑ_g-Plane (and the Solution of the Linear Subsystems via Cholesky Decomposition)

INPUT
Limiting angles $\varphi^0_{g_{min}}$, $\varphi^0_{g_{max}}$, $\vartheta^0_{g_{min}}$, $\vartheta^0_{g_{max}}$ of φ_g-ϑ_g-box,
overlap p_t, p_φ,
number of subdomains in φ_g-direction M_φ and
number of subdomains in ϑ_g-direction M_ϑ, excluding polar caps,
variable $POLES$ indicating the presence of poles,
flag $PHI\_PERIODIC$ signalizing if overlap of subdomains
in φ_g-direction at $\varphi_g = -\pi$
has to be taken into account, smoothing parameter λ,
accurracy tolerance TOL,
maximal number of iterations $MAXIT$,
spherical polar coordinates of interpolation points $\{x^N_1, \ldots, x^N_N\}$
and data $b^N = (G(x^N_1), \ldots, G(x^N_N))^T$.

SETUP
(1) $l := 0$, $r^N := b^N$, $a^N := 0$.
(2) Compute the limiting angles of the overlapping subdomains
$X^N_1, \ldots, X^N_M$ in the φ_g-ϑ_g-plane,
determine the numbers of points $N_1, \ldots, N_M$ in $X^N_1, \ldots, X^N_M$
and store their indices in an index array *ind*, such that
$x^{N_j}_i = x^N_{ind_{ji}}$ for $j = 1, \ldots, M$ and $i = 1, \ldots, N_j$.
(3) Set up a 2D array K containing the matrices
$((K_{\mathcal{H}(\{A_n\};h;\overline{\Omega^{ext}_r})}(x^{N_j}_i, x^{N_j}_k) + \lambda\delta_{i,k})_{1\le i,k\le N_j}$ in symmetric
store-by-column format as row vectors.
(4) **for** $j = 1 : M$
    Compute the Cholesky factorization of $K_{j,.}$ using the
    LAPACK routine `dpptrf`.
end

ITERATIVE SOLUTION
(1) **while** $|r|/|b| > TOL$ and $l < MAXIT$
(2)   **for** $j = 1 : M$
(3)   **for** $i = 1 : N_j$
                $r^{N_j}_i := r^N_{ind_{ji}}$
            end
(4)   Solve $K_{j,.}a^{N_j} = r^{N_j}$ using the LAPACK routine
            `dpptrs`.
(5)   **for** $i = 1 : N_j$
$$r^N_i := r^N_i - \sum_{l=1}^{N_j} a^{N_j}_l (K_{\mathcal{H}(\{A_n\};h;\overline{\Omega^{ext}_r})}(x^N_i, x^{N_j}_l) + \lambda\delta_{i,ind_{jl}})$$
            end
(6)   **for** $i = 1 : N_j$
                $a^N_{ind_{ji}} := a^N_{ind_{ji}} + a^{N_j}_i$
            end
        end
(7) $l := l + 1$
end

Chapter 7

Multiscale Reconstruction of the Gravitational Potential

In this chapter we explain how the regularization schemes which were introduced in Section 5.4 can be discretized with the help of $\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})$-splines. Throughout the chapter we assume that the general assumptions of the numerical part, which are given on page 84, are satisfied. In particular, $\Lambda \in \{\Lambda^{SST}, \Lambda^{SGG}\}$ either denotes the SST- or the SGG-operator.

Let $\{\Phi_j^\Lambda\}_{j\in\mathbb{N}_0}$ be a linear regularization scaling function for the inverse operator Λ^{-1} with generating symbol $\{\{(\Phi_j^\Lambda)^\wedge(n)\}_{n\in\mathbb{N}_0}\}_{j\in\mathbb{N}_0}$, and let $\{\Psi_j^\Lambda\}_{j\in\mathbb{N}_0}$ denote the regularization wavelet corresponding to $\{\Phi_j^\Lambda\}_{j\in\mathbb{N}_0}$ with generating symbol $\{\{(\Psi_j^\Lambda)^\wedge(n)\}_{n\in\mathbb{N}_0}\}_{j\in\mathbb{N}_0}$. Let $\{T_j\}_{j\in\mathbb{N}_0}$ and $\{R_j\}_{j\in\mathbb{N}_0}$ be the bounded linear operators introduced in Definition 5.27 and Definition 5.33. In order to compute the j-level regularization T_jG, $j\in\mathbb{N}_0$ of the problem $\Lambda F = G$, $F\in\mathcal{H}(\{A_n\};\overline{\Omega_R^{ext}})$, $G\in\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})$, and the detail $R_{j-1}G$, $j\in\mathbb{N}_0$, we have to discretize the convolutions

$$T_jG = \Phi_j^\Lambda *_{\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})} G,\ \ j\in\mathbb{N}_0, \tag{7.1}$$

and

$$R_{j-1}G = \Psi_{j-1}^\Lambda *_{\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})} G,\ \ j\in\mathbb{N}, \tag{7.2}$$

with the help of the measurements $\{(x_i^N, G(x_i^N))\}_{1\le i\le N}$, taken in a set of N mutually distinct points $X^N := \{x_1^N,\dots,x_N^N\}$ on the satellite orbit. Assume that the bounded evaluation functionals $\mathcal{L}_i^N : \mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}}) \to \mathbb{R}$, $H\mapsto H(x_i^N)$, $i=1,\dots,N$ are linearly independent and denote their representers in $\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})$ by L_i^N, $i=1,\dots,N$. As explained in the last chapter we calculate an interpolating or smoothing $\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})$-spline

$$S^G = \sum_{i=1}^{N} a_i^N L_i^N$$

relative to $\mathcal{L}_1^N, \ldots, \mathcal{L}_N^N$ of the right-hand side G and replace G in the convolutions (7.1) and (7.2) by this spline. This leads to the discretization rules

$$T_j G \approx \sum_{i=1}^{N} a_i^N \Phi_j^\Lambda(\cdot, x_i^N), \;\; j \in \mathbb{N}_0, \tag{7.3}$$

and

$$R_{j-1} G \approx \sum_{i=1}^{N} a_i^N \Psi_{j-1}^\Lambda(\cdot, x_i^N), \;\; j \in \mathbb{N}. \tag{7.4}$$

Note that the $\mathcal{H}(\{A_n\}; \overline{\Omega_R^{ext}})$-$\mathcal{H}(\{A_n\}; h; \overline{\Omega_r^{ext}})$-product kernels Φ_j^Λ, $j \in \mathbb{N}_0$, and Ψ_{j-1}^Λ, $j \in \mathbb{N}$, are (at least for the examples we consider in this thesis and to our knowledge) not available as elementary functions. Thus, in practical computations we have to evaluate their truncated series expansions

$$\begin{aligned}(\Phi_j^\Lambda)^m(x,y) &= \sum_{n=0}^{m} \sum_{k=1}^{2n+1} \frac{1}{A_n^2} (\Phi_j^\Lambda)^\wedge(n) H_{n,k}(R;y) H_{n,k}(r;h;x) \\ &= \frac{1}{4\pi Rr} \left(\frac{r}{|x|}\right)^h \sum_{n=0}^{m} \frac{1}{A_n^2} (\Phi_j^\Lambda)^\wedge(n) \left(\frac{Rr}{|y||x|}\right)^{n+1} P_n\left(\frac{y}{|y|} \cdot \frac{x}{|x|}\right)\end{aligned}$$

and

$$\begin{aligned}(\Psi_{j-1}^\Lambda)^m(x,y) &= \sum_{n=0}^{m} \sum_{k=1}^{2n+1} \frac{1}{A_n^2} (\Psi_{j-1}^\Lambda)^\wedge(n) H_{n,k}(R;y) H_{n,k}(r;h;x) \\ &= \frac{1}{4\pi Rr} \left(\frac{r}{|x|}\right)^h \sum_{n=0}^{m} \frac{1}{A_n^2} (\Psi_{j-1}^\Lambda)^\wedge(n) \left(\frac{Rr}{|y||x|}\right)^{n+1} P_n\left(\frac{y}{|y|} \cdot \frac{x}{|x|}\right),\end{aligned}$$

where $m \in \mathbb{N}_0$, in $y \in \overline{\Omega_R^{ext}}$ and $x \in \Omega_r^{ext}$. This leads to an additional numerical error.

It is also possible to discretize the convolutions in a multiscale reconstruction

$$T_J G = T_{J_0} G + \sum_{j=J_0}^{J-1} R_j G$$

in a more efficient way than it is done in (7.1) and (7.2). This can be done with a so-called 'pyramid scheme'. In our situation the use of a pyramid scheme would mean that only the detail $R_{J-1}G$ at the highest scale J is discretized according to formula (7.4) and that the coefficients $a_1^N, \ldots, a_N^N$ in (7.4) are used for the recursive calculation of coefficients for the discretization of $R_{J-2}, \ldots, R_{J_0}G, T_{J_0}G$. For more information about pyramid schemes the reader is referred to [Fr1999] and [FrSchn1997].

Part III

Computational Results

Chapter 8

Numerical Test of the Schwarz Alternating Algorithm

In this chapter we apply the Schwarz alternating procedure to the $\mathcal{H}(\{A_n\}; h; \overline{\Omega_r^{ext}})$-spline interpolation problem. In Section 8.1 we test the implemented algorithm with respect to convergence, runtime and memory requirement, whereas an accuracy analysis of the computed interpolating splines is presented in Section 8.2.

All test computations are performed for an SGG-scenario (h=2) with simulated satellite data. We start with the most simple case of a spherical orbit with radius r_S and use a set of interpolation points which forms an approximate equidistribution on the orbital sphere. The notion of an equidistribution on the sphere is introduced in Section 8.3, where we present the point distributions which we use in our numerical tests. We then leave the sphere and consider an orbit in the shape of an ellipsoid of revolution with eccentricity e and ellipse parameter p, which approximates the orbit of the GOCE satellite. Pointsets on this ellipsoid are generated by projection of the spherical pointsets along the radial direction. We use the NASA model EGM96 (see [LeKeFa1998] and `http://cddisa.gsfc.nasa.gov/926/egm96/egm96.html`) to generate the SGG data. The EGM96 model provides a set of (real, fully normalized) spherical harmonic coefficients $\{V_{n,k}^R\}_{0\leq n\leq 360, -n\leq k\leq n}$ related to a spherical earth model with radius $R = 6378136.3$ m. The coefficients $\{V_{n,k}^R\}_{0\leq n\leq 360, -n\leq k\leq n}$ were obtained from a combination of satellite and terrestrial data material. The system of fully normalized spherical harmonics in terms of Legendre functions is an $\mathcal{L}^2(\Omega)$-orthonormal system commonly used in geosciences, and we refer the reader to [HeMo1967] for more details. In our test example we use the EGM96 model to compute the second order radial derivative G of the gravitational potential $V \in \mathrm{Pot}^{(0)}(\overline{\Omega_R^{ext}})$, in a set of N points $X^N := \{x_i^N, \ldots, x_N^N\}$ on the 'orbital surface' Σ_S, including contributions of outer harmonic degrees 3 up to 255, i.e.,

$$G(x_i^N) = \Gamma M \sum_{n=3}^{255} \frac{(n+1)(n+2)}{|x_i^N|^2} \sum_{k=-n}^{n} V_{n,k}^R \, H_{n,k}(R; x_i^N), \; i = 1, \ldots, N,$$

where $\Gamma M = 3986004.415 \cdot 10^8$ m^{3}s^{-2} is the product of the gravitational constant and the mass of the earth. Note that $\{H_{n,k}(R;\cdot)\}_{n\in\mathbb{N}_0,\ -n\leq k\leq n}$ here denotes the system of outer harmonics related to the system of fully normalized spherical harmonics.

The parameters of the orbital test geometries are as follows:

$$\begin{aligned} r_S &= 6628059 \text{ m}, \\ e &= 0.0045 \\ p &= 6628002.78 \text{ m}, \end{aligned}$$

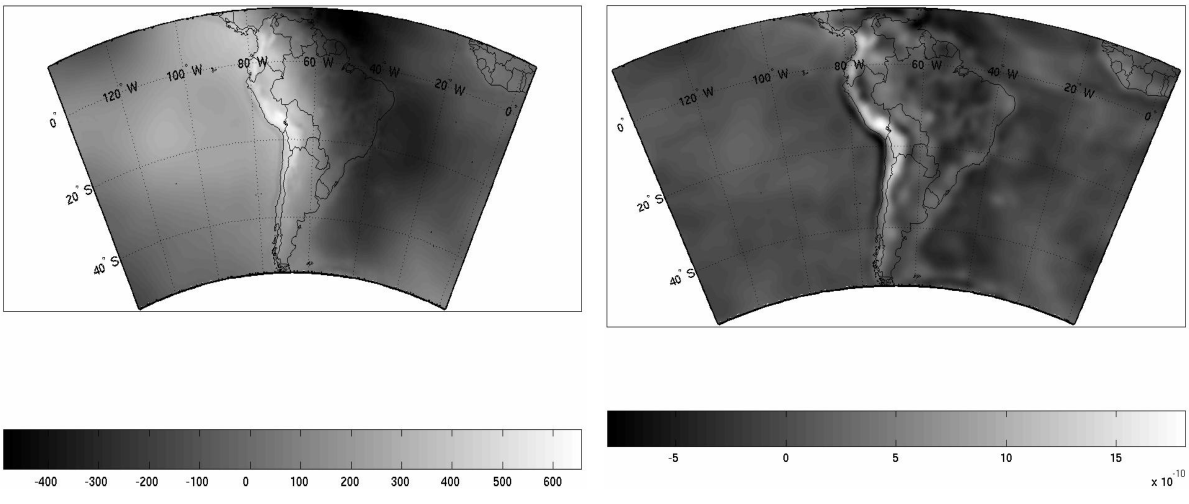
and the radius of the Bjerhammar sphere for the satellit orbit is given by

$$r = 6588310 \text{ m}.$$

We compute a local spline reconstruction of the simulated SGG data, where the longitudes φ_g and latitudes ϑ_g of the interpolation points $x_1^N, \ldots, x_N^N$ are taken within the φ_g-ϑ_g-box defined by

$$(\varphi_g, \vartheta_g) \in [-2.394, 0.052] \times [-0.944, 0.224].$$

Figure 8.1 shows the gravitational potential on the spherical earth's surface and the SGG signal at satellite height in the selected φ_g-ϑ_g-box.



a) b)

Figure 8.1: *a) Gravitational potential in φ_g-ϑ_g-box on the earth's surface and b) SGG-signal on the spherical test orbit, computed with the EGM96 model.*

As the simulated SGG-data which is regarded in this chapter is exact (i.e. without noise) the reconstruction of the second radial derivative on the 'orbital surface' Σ_S can be performed with an interpolating $\mathcal{H}(\{A_n\}; h; \overline{\Omega_r^{ext}})$-spline.

We restrict ourselves to the case that the sequence $\{A_n\}_{n\in\mathbb{N}_0}$ which defines the space $\mathcal{H}(\{A_n\}; 2; \overline{\Omega_r^{ext}})$ is given by $A_n := q^{-n/2}$ for $n \in \mathbb{N}_0$, where $q \in (0, 1]$. Let $x \in \Omega_r^{ext}$ if $q = 1$ and $x \in \overline{\Omega_r^{ext}}$ if $q \in (0, 1)$. According to the results in Chapter 4, the evaluation functional $\mathcal{L}_x : \mathcal{H}(\{A_n\}; h; \overline{\Omega_r^{ext}}) \to \mathbb{R}$, $G \mapsto \mathcal{L}_x(G) := G(x)$ is bounded and its representer $L_x \in \mathcal{H}(\{A_n\}; h; \overline{\Omega_r^{ext}})$ is of 'Abel-Poisson-kernel type' (4.14). As already mentioned there, the parameter $q \in (0, 1]$ can be interpreted as a 'shape parameter', which determines the decay behaviour and the space-localization of the Abel-Poisson kernel. Our numerical studies show that the accuracy of the interpolating $\mathcal{H}(\{A_n\}; h; \overline{\Omega_r^{ext}})$-spline is very sensitive to the variation of q and to the density of the interpolation points. The analysis of the dependence of the accuracy of the interpolating spline on these two parameters is investigated in Section 8.2.

8.1 Performance of the Multiplicative Schwarz Algorithm

We investigate the performance of the Schwarz alternating algorithm with respect to

- subdomain overlap,
- subdivision depth,
- total number of interpolation points,
- point distribution,
- orbit geometry and
- choice of the parameter $q \in (0, 1]$.

The numerical test is carried out with our first implementation of the Schwarz alternating algorithm which was designed for solving the $\mathcal{H}(\{A_n\}; h; \overline{\Omega_r^{ext}})$-spline interpolation problem on a φ_g-ϑ_g-box containing no poles. This solver uses a simpler subdivision scheme than the one described in Section 6.2, i.e., we perform a subdivision of the φ_g-ϑ_g-box into $M = M_\varphi M_\vartheta$ subdomains which have equal surface areas in the φ_g-ϑ_g-plane and then enlarge these subdomains by a certain overlap $d\varphi$ and $d\vartheta$ in φ_g- and ϑ_g-direction. This subdivision procedure leads to some variation in the surface areas covered by the subdomains on the sphere and thus to considerable variation in the numbers of points contained in the subsets X_j^N, $j \in \{1, \dots, M\}$ of X^N. Although one would intuitively expect that subdomains of varying surface area might slow down the algorithm we get rather good results with this simple subdivision scheme. Since this subdivision procedure cannot be applied if a pole is contained in the φ_g-ϑ_g-box and as we wished to avoid large variations of the point numbers in the subsets we later implemented the subdivision scheme described in Section 6.2 in the more general version of the SAP equation solver.

In the presentation of the results we always give the range of the point numbers N_j in the subsets X_j^N, $j \in \{1, \ldots, M\}$, and the total number of points in the overlap N_o (counted without multiplicities).
The study is carried out with the Cholesky variant of the SAP solver. We always work with an accuracy tolerance for the relative residual given by $TOL = 10^{-16}$. All time measurements are performed on a Pentium III (Coppermine) with cpu-speed 868.664 MHz, 1.4 GByte RAM, and operating system Red Hat Linux 7.1 with kernel 2.4.7. The C-code is compiled with the gcc 2.96 compiler using the optimization option -O3, and the installed C-library is glibc 2.2.2.

For comparison of the efficiency of the SAP solver, note that if we apply the LAPACK Cholesky routine `dppsv` to solve the $\mathcal{H}(\{A_n\}; h; \overline{\Omega_r^{ext}})$-spline interpolation equations for 10295 unknowns directly (without SAP), the total computation time amounts to 2.5 h, and the storage requirement is 405 MByte.

Performance of the SAP Algorithm for Varying Subdomain Overlap and Subdivision Depth

All numerical tests here are performed with data given on the spherical test orbit Ω_{r_S} in the points of a Reuter grid with grid parameter $\gamma = 200$ and a total number of $N = 10295$ points in the φ_g-ϑ_g-box. The parameter q which determines the sequence $\{A_n\}_{n \in \mathbb{N}_0}$ is chosen as $q = 0.95$.

We start with $M_\varphi = M_\vartheta = 4$ (i.e. M=16 subdomains) and determine the number of SAP iterations, the CPU time per iteration, the total computing time and the memory requirement for varying values of the overlap $d\varphi$ and $d\vartheta$. The results are listed in Table 8.1. Figure 8.2 shows these quantities in dependence of the oberlap $d\varphi$ in φ_g-direction for varying values of $d\vartheta$. In Figure 8.2 a) we see that for our test case the convergence of the method does not further improve if we choose an overlap $d\varphi > 0.2$ and $d\vartheta > 0.1$. Figure 8.2 c) shows that the total computation time seems to become minimal for a longitudinal overlap of $0.05 < d\varphi < 0.2$.

Table 8.1: Test of the SAP method for a local φ_g-ϑ_g-box with N=10295 points. The φ_g-ϑ_g-box was subdivided in $M_\varphi = M_\vartheta = 4$ subdomains in φ_g- and ϑ_g-direction, with varying overlap $d\varphi$ and $d\vartheta$. N_j gives the smallest and the largest number of points in a subdomain, N_o is the number of points (counted without multiplicity) which belong to more than one subdomain and It. is the number of iterations.

$d\vartheta$	$d\varphi$	N_j	N_o	$It.$	$t/It.$ in s	t_{tot} in min	Mem. in MByte
0.05	0.05	653-1096	3610	22	76.05	30.95	53
	0.1	706-1250	4555	13	86.14	21.25	77
	0.2	802-1560	6426	13	103.33	26.8	94
	0.3	900-1867	8288	13	120.49	33.13	127
0.1	0.05	759-1398	5985	22	91.92	38.0	77
	0.1	820-1594	6592	8	99.64	18.0	95
	0.2	932-1994	7799	6	124.3	24.0	137
	0.3	1045-2385	9001	6	143.29	27.0	187
0.15	0.05	903-1653	8419	22	110.56	46.37	107
	0.1	976-1885	8683	8	122.78	23.73	132
	0.2	1110-2359	9206	6	147.53	27.55	190
	0.3	1245-2820	9732	6	172.0	35.82	259

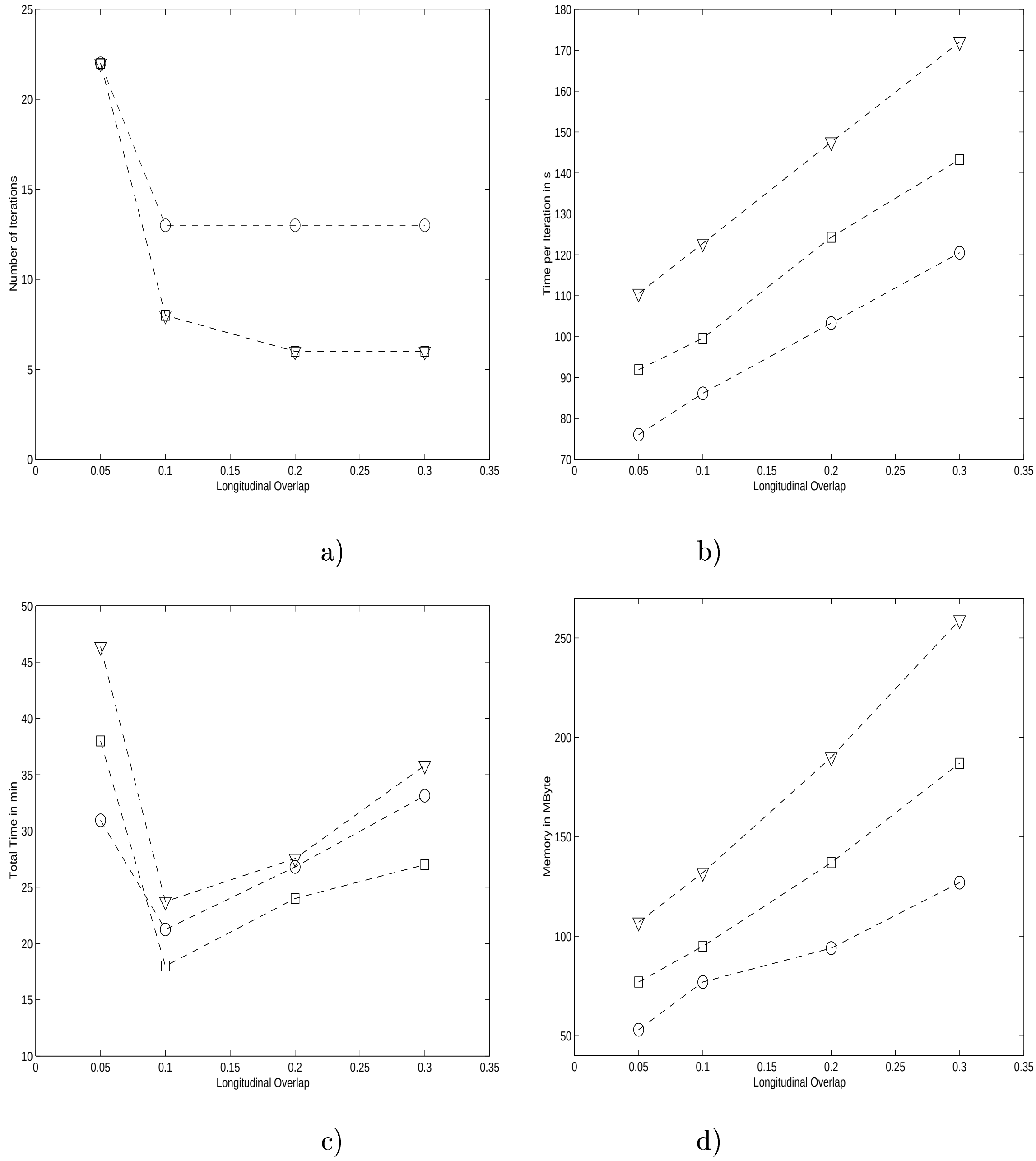


Figure 8.2: *SAP test performed on a Reuter grid with* $N = 10295$ *points in the selected parameter region on a spherical orbit with radius* $r_S = 6628059$ *m, parameter* $q = 0.95$ *and* $M_\varphi = M_\vartheta = 4$ *subdomains in* φ_g*- and* ϑ_g*-direction, respectively, for* $d\varphi \in \{0.05, 0.1, 0.2, 0.3\}$ *and* $d\vartheta = 0.05$ *(*$\circ$*),* $d\vartheta = 0.1$ *(*$\square$*) and* $d\vartheta = 0.15$ *(*$\bigtriangledown$*). a) Number of iterations, b) time per iteration in s, c) total runtime in min and d) memory requirement in MByte in dependence of the longitudinal and latitudinal subdomain overlap.*

In a second test we additionally vary the subdivision depth $M = M_\varphi M_\vartheta$, where $M_\varphi = M_\vartheta = n$ and $n \in \{5, \ldots, 10\}$. Here we relate the overlap of the subdomains to the overlap specified for the study with $M_\varphi = M_\vartheta = 4$ with overlap $d\vartheta = 0.1$ and $d\varphi \in \{0.1, 0.2, 0.3\}$ in such a way that in φ_g- and in ϑ_g-direction the overlap is a fixed multiple of the subdomain length. Thus we take $d\vartheta = 0.1 \cdot 4/M_\varphi$ and vary $d\varphi \in \{0.1 \cdot 4/M_\varphi, 0.2 \cdot 4/M_\varphi, 0.3 \cdot 4/M_\varphi\}$. The results of this study are listed in Table 8.2.

Table 8.2: Test of the SAP method for a local φ_g-ϑ_g-box with N=10295 points. The φ_g-ϑ_g-box was subdivided in $M = n^2$, $n \in \{5, \ldots, 10\}$, subdomains with varying overlap $d\varphi_g$ and $d\vartheta_g$. N_j gives the smallest and the largest number of points in a subdomain, N_o is the number of points (counted without multiplicity) which belong to more than one subdomain and It. is the number of iterations.

M	$d\vartheta_g$	$d\varphi_g$	N_j	N_o	$It.$	$t/It.$ in s	t_{tot} in min	Mem. in MByte
25	0.08	0.08	502-1024	6838	10	119.5	22.97	65
		0.16	577-1273	7960	7	144.16	21.75	95
		0.24	644-1525	9295	7	169.91	27.58	131
36	0.2/3	0.2/3	349-718	7099	13	121.42	27.8	48
		0.4/3	398-894	8297	9	147.44	24.45	70
		0.2	448-1076	9496	9	173.69	30.25	98
49	0.4/7	0.4/7	259-535	7415	18	124.92	38.43	37
		0.8/7	296-663	8530	11	152.40	28.27	55
		1.2/7	331-790	9657	11	179.74	35.18	77
64	0.05	0.05	193-415	7339	21	124.19	46.73	28
		0.1	221-516	8534	14	152.1	36.72	43
		0.15	248-621	9709	14	179.92	43.19	59
81	0.4/9	0.4/9	144-322	7446	27	125.65	57.88	23
		0.8/9	163-404	8614	17	154.12	44.87	35
		1.2/9	184-479	9777	17	182.32	53.12	48
100	0.04	0.04	116-270	7543	40	126.71	87.65	19
		0.08	132-336	8686	19	156.24	50.98	29
		0.12	149-403	9833	19	185.2	60.33	40

Figure 8.3 shows that for our test example the memory requirement is dramatically lowered with increasing subdivision depth. For fixed values of $d\varphi$ and $d\vartheta$ the number of necessary iterations increases with increasing number of subdomains. Considering the total runtime it therefore seems not advisable to increase the subdivision depth further than $M = 49$.

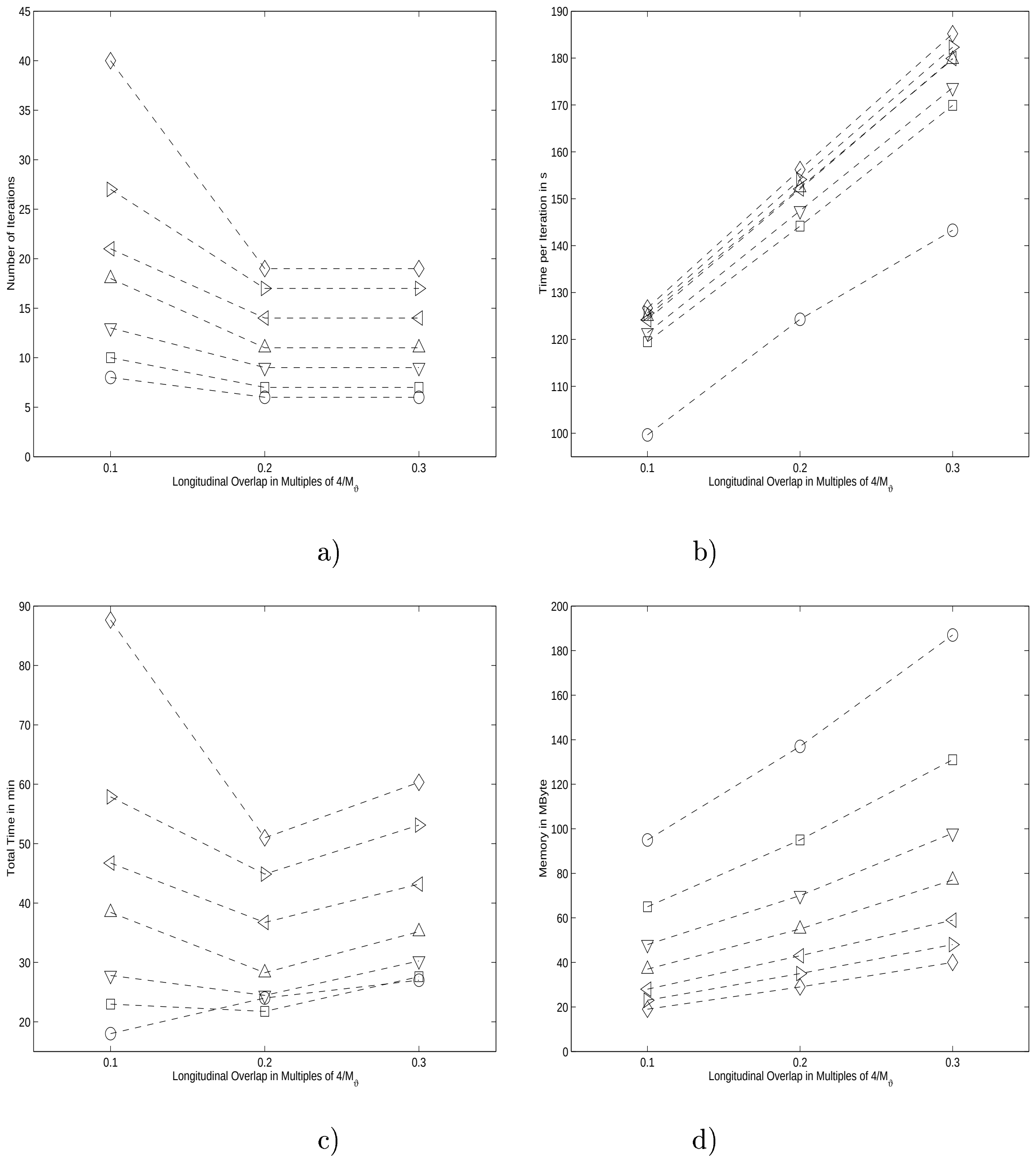


Figure 8.3: *SAP test performed on a Reuter grid with* $N = 10295$ *points in the selected* φ_g*-*ϑ_g*-box on the spherical test orbit* Ω_{r_S} *and parameter* $q = 0.95$ *for varying subdivision depth* $M = M_\vartheta^2$, $M_\vartheta \in \{4, \ldots, 10\}$ *and* $d\vartheta = 0.1 \cdot 4/M_\vartheta$, $d\varphi \in \{0.1 \cdot 4/M_\vartheta, 0.2 \cdot 4/M_\vartheta, 0.3 \cdot 4/M_\vartheta\}$. *a) Number of iterations, b) time per iteration in s, c) total runtime in min and d) memory requirement in MByte in dependence of the longitudinal overlap in multiples of* $4/M_\vartheta$ *where* $M_\vartheta = 4$ *(○),* $M_\vartheta = 5$ *(□),* $M_\vartheta = 6$ *(▽),* $M_\vartheta = 7$*, (△),* $M_\vartheta = 8$ *(◁),* $M_\vartheta = 9$ *(▷) and* $M_\vartheta = 10$ *(◇).*

Performance of the SAP Algorithm for Increasing Total Number of Interpolation Points

The increase of the total runtime is the reason why the convergence study for an increasing total number of interpolation points is only carried out for $M = M_\vartheta^2$, (i.e., $M_\varphi = M_\vartheta$), where $M_\vartheta \in \{4, \ldots, 7\}$ for a latitudinal and longitudinal overlap $d\vartheta = 0.1 \cdot 4/M_\vartheta$ and $d\varphi = 0.2 \cdot 4/M_\vartheta$, respectively. The results of this study are listed in Table 8.3 and plotted in Figure 8.4.

Table 8.3: Test of the SAP method on a φ_g-ϑ_g-box on the spherical test orbit Ω_{r_S} on a Reuter grid with grid parameter γ and N points in the φ_g-ϑ_g-box. The parameter q is chosen as $q = 0.95$ and the overlap is given by $d\vartheta = 0.1 \cdot 4/M_\varphi$ and $d\varphi = 0.2 \cdot 4/M_\vartheta$ for varying subdivision depth $M = M_\vartheta^2$. N_j gives the smallest and the largest number of points in a subdomain, N_o is the number of points (counted without multiplicity) which belong to more than one subdomain and It. is the number of iterations.

M	γ	N	N_j	N_o	$It.$	$t/It.$ in s	t_{tot} in min	Mem. in MByte
16	250	15958	1502-3032	12149	6	338.68	59.87	331
	300	23070	2142-4389	17592	6	720.54	149.33	693
25	250	15958	901-1972	12636	7	347.79	53.43	232
	300	23070	1300-2818	17195	8	735.24	134.16	478
	350	31235	1767-3906	24647	8	1356.32	286.1	878
36	250	15958	614-1387	12835	10	353.74	65.88	167
	300	23070	878-2052	18620	10	742.18	145.35	351
	350	31235	1188-2686	25286	10	1368.22	278.21	646
49	250	15958	441-1055	13083	12	361.23	76.48	129
	300	23070	655-1430	18808	14	750.24	189.03	262
	350	31235	861-2016	25588	14	1392.21	381.02	489
	400	40949	1148-2645	33758	15	2400.02	668.1	850

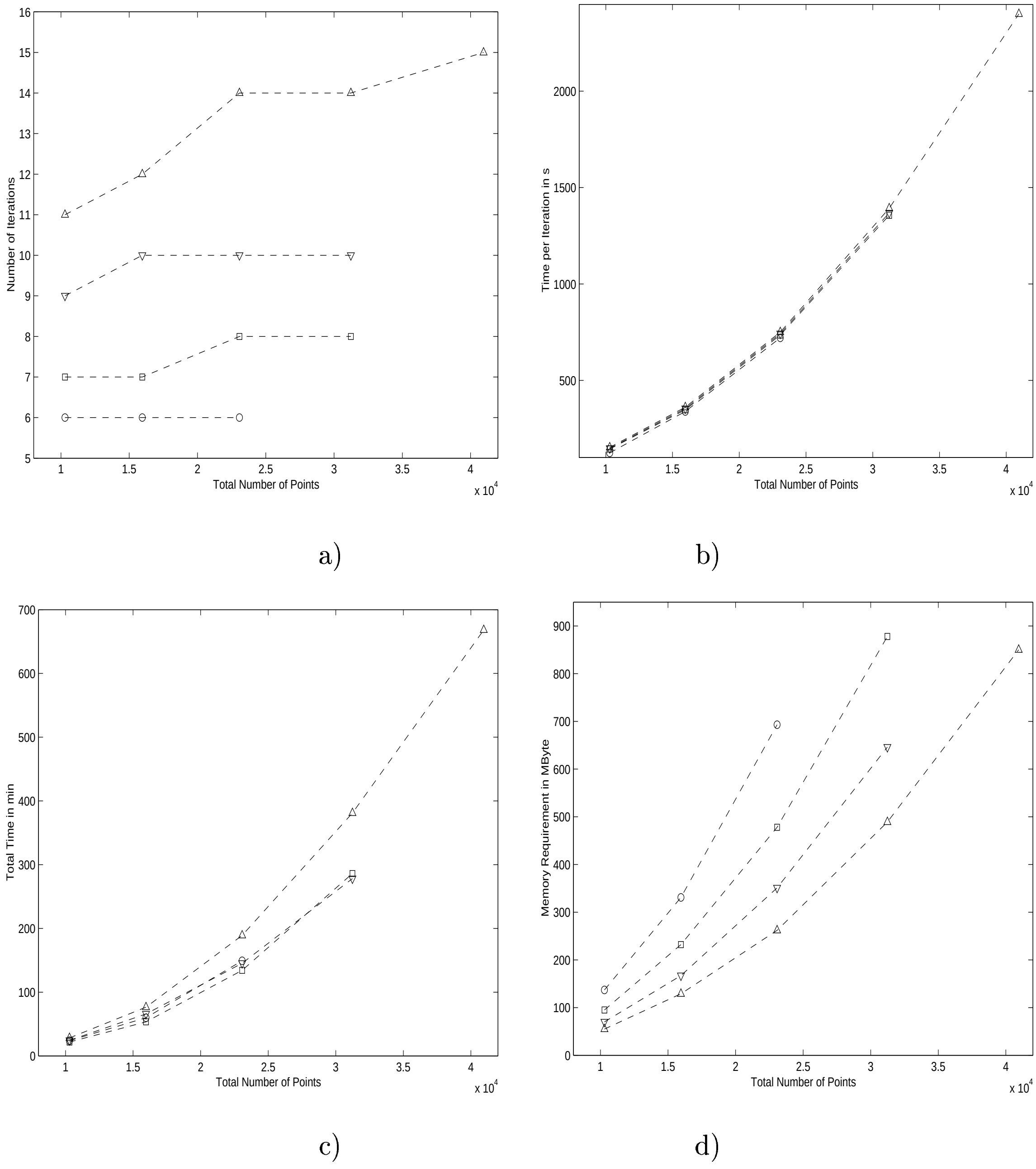


Figure 8.4: *Test of the SAP method performed on a Reuter grid with varying total number of points* $N \in \{10295, 15958, 23070, 31235\}$ *in the selected* φ_g-ϑ_g*-box. The parameter* q *is given by* $q = 0.95$ *and the latitudinal and longitudinal overlap by* $d\vartheta = 0.1 \cdot 4/M_\vartheta$ *and* $d\varphi = 0.2 \cdot 4/M_\vartheta$*, respectively, where* $M_\varphi = M_\vartheta = 4$ *(*$\circ$*),* $M_\varphi = M_\vartheta = 5$ *(*$\Box$*),* $M_\varphi = M_\vartheta = 6$ *(*$\bigtriangledown$*) and* $M_\varphi = M_\vartheta = 7$ *(*$\bigtriangleup$*). a) Number of iterations, b) time per iteration in s, c) total runtime in min and d) memory requirement in MByte.*

Performance of the SAP Algorithm for Varying Point Distribution

The dependence of the convergence of the SAP method on the point distribution is tested in a comparative study for four grid types (on the spherical test orbit Ω_{r_S}) which are listed in Section 8.3 and a subdivision into $M = M_\vartheta^2$ subdomains, (i.e., $M_\varphi = M_\vartheta$), where $M_\vartheta \in \{4, 5, 6\}$. The total number of points is equal to $N = 10224$ in case of the Brand grid and $N = 10295$ otherwise, and we choose $q = 0.95$, $d\vartheta = 0.1 \cdot 4/M_\vartheta$ and vary the longitudinal overlap $d\varphi \in \{0.1 \cdot 4/M_\vartheta, 0.2 \cdot 4/M_\vartheta, 0.3 \cdot 4/M_\vartheta\}$. The results obtained for the Brand grid and the grids based on the Hammersley and Corput-Halton sequence, respectively, are listed in Tables 8.4 to 8.6.

Table 8.4: Test of the SAP method for a Brand grid on the spherical test orbit Ω_{r_S} and $N = 10224$ points in the φ_g-ϑ_g-box.

M	$d\vartheta_g$	$d\varphi_g$	N_j	N_o	$It.$	$t/It.$ in s	t_{tot} in min	Mem. in MByte
16	0.1	0.1	696-1659	6546	7	113.87	17.63	97
		0.2	794-2067	7744	6	136.65	21.25	139
		0.3	892-2473	8931	6	159.31	27.8	189
25	0.08	0.08	434-1108	6900	10	117.92	22.03	67
		0.16	494-1387	8093	7	142.83	20.45	98
		0.24	555-1653	9254	7	167.15	25.93	134
36	0.2/3	0.2/3	282-810	7202	11	121.49	23.65	50
		0.4/3	322-1010	8336	9	147.57	25.08	73
		0.2	362-1209	9464	9	173.81	34.35	102

Table 8.5: Test of the SAP method for a grid based on the Hammersley sequence on the spherical test orbit Ω_{r_S} and $N = 10295$ points in the φ_g-ϑ_g-box.

M	$d\vartheta_g$	$d\varphi_g$	N_i	N_o	It.	$t/It.$ in s	t_{tot} in min	Mem. in MByte
16	0.1	0.1	828-1574	6622	7	115.71	17.62	96
		0.2	944-1962	7822	6	142.56	21.35	138
		0.3	1061-2352	9011	6	162.49	27.53	188
25	0.08	0.08	511-1027	6951	8	119.49	17.95	66
		0.16	583-1281	8137	7	144.36	20.02	96
		0.24	655-1534	9326	6	169.58	22.42	133
36	0.2/3	0.2/3	346-721	7154	11	122.06	23.63	48
		0.4/3	395-898	8332	7	148.02	19.27	71
		0.2	443-1074	9510	7	174.54	23.4	98

Table 8.6: Test of the SAP method for a grid based on the Corput-Halton sequence on the spherical test orbit Ω_{r_S} and $N = 10295$ points in the φ_g-ϑ_g-box.

M	$d\vartheta_g$	$d\varphi_g$	N_i	N_o	$It.$	$t/It.$ in s	t_{tot} in min	Mem. in MByte
16	0.1	0.1	828-1578	6629	7	116.84	17.75	96
		0.2	946-1965	7827	6	139.51	20.97	138
		0.3	1062-2355	9013	6	163.71	27.67	188
25	0.08	0.08	512-1029	6955	8	119.49	18.0	66
		0.16	584-1282	8142	7	144.6	20.32	96
		0.24	654-1536	9312	6	169.82	22.47	133
36	0.2/3	0.2/3	348-726	7167	11	122.09	23.67	48
		0.4/3	396-897	8342	7	148.27	19.25	71
		0.2	445-1074	9515	7	174.39	23.3	98

Figure 8.5 shows the number of iterations in dependence of the longitudinal overlap and the subdivision depth for the four different grid types. We see that the dependence of the convergence of the method on the subdivision depth is the same for each pair of grid types with similar law of generation, and we obtain a faster convergence of the SAP method for the Hammersley and the Corput-Halton grid for a subdivision into 36 subdomains than in the case of the Reuter and the Brand grid.

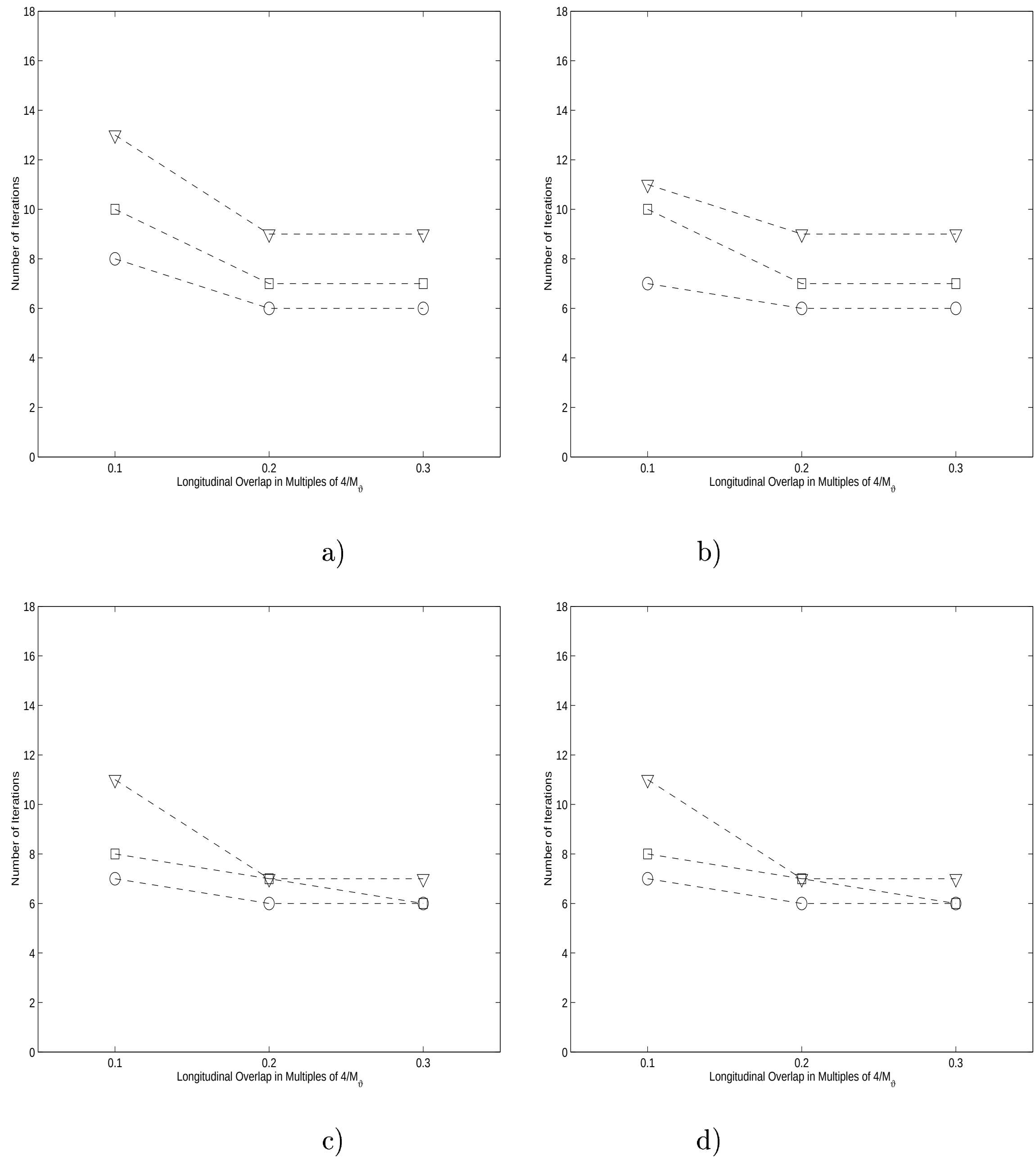


Figure 8.5: *Number of iterations in the SAP method performed a) on a Reuter grid with $N = 10295$, b) on a Brand grid with $N = 10224$, c) on a grid based on the Hammersley sequence with $N = 10295$ and d) on a grid based on the Corput-Halton sequence with $N = 10295$ grid points on the spherical test orbit Ω_{r_S} for varying subdivision depth $M = M_\vartheta^2$, where $M_\vartheta = 4$ ($\circ$), $M_\vartheta = 5$ ($\Box$) and $M_\vartheta = 6$ ($\bigtriangledown$), and latitudinal and longitudinal overlap given by $d\vartheta = 0.1 \cdot 4/M_\vartheta$ and $d\varphi \in \{0.1 \cdot 4/M_\vartheta, 0.2 \cdot 4/M_\vartheta, 0.3 \cdot 4/M_\vartheta\}$, respectively, and parameter $q = 0.95$.*

Performance of the SAP Method for Varying Orbit Geometry

The results in Table 8.7 show that for the choice of $q = 0.95$ the convergence behaviour of the Schwarz alternating algorithm does not significantly change when we go over from an orbital sphere with radius $r_S = 6628059$ m to an ellipsoid of revolution with eccentricity $e = 0.0045$ and ellipse parameter $p = 6628002.78$ m.

Table 8.7: Test of the SAP method for $q = 0.95$ and data given on a Reuter grid with $N = 10295$ points projected on an orbital ellipsoid of revolution with eccentricity $e = 0.0045$ and ellipse parameter $p = 6628002.78$ m for varying overlap and subdivision depth.

M	$d\vartheta_g$	$d\varphi_g$	$It.$	$t/It.$ in s	t_{tot} in min
16	0.1	0.1	7	115.49	18.02
		0.2	6	138.31	21.3
		0.3	6	161.27	27.92
25	0.08	0.08	9	118.53	20.43
		0.16	7	143.19	20.35
		0.24	7	168.03	25.32
36	0.2/3	0.2/3	11	121.58	23.9
		0.4/3	8	147.6	21.77
		0.2	8	173.82	26.48

Performance of the SAP Algorithm for Varying Parameter $q \in (0, 1]$

Finally, we study the dependence of the performance of the SAP method on the parameter $q \in (0, 1]$, which determines the decay behaviour of the Abel-Poisson-kernel. Table 8.1 contains the convergence results of a comparative test of the SAP method for $q = 0.92$ and $q = 0.95$, performed for the spherical test orbit Ω_{r_S} for a Reuter grid with $N = 10295$ points in the φ_g-ϑ_g-box and a subdivision into $M_\varphi = M_\vartheta = 4$ subdomains in φ_g- and ϑ_g-direction.

Table 8.8: Test of the SAP method for $q = 0.92$ and $q = 0.95$ and data given on the spherical test orbit Ω_{r_S} in $N = 10295$ points of a Reuter grid in the φ_g-ϑ_g-box, where $M_\varphi = M_\vartheta = 4$.

q	$d\vartheta$	$d\varphi$	$It.$	q	$d\vartheta$	$d\varphi$	$It.$
0.92	0.1	0.1	22	0.95	0.1	0.1	8
		0.2	11			0.2	6
		0.3	11			0.3	6
0.92	0.15	0.1	22	0.95	0.15	0.1	8
		0.2	7			0.2	6
		0.3	7			0.3	6

Note that in case that $d\vartheta = 0.05$ the method did not converge within 30 iterations for $d\varphi \in \{0.1, 0.2, 0.3\}$. In Section 8.2, where we investigate the accuracy of the interpolating $\mathcal{H}(\{A_n\}; h; \overline{\Omega_r^{ext}})$-spline in dependence of the parameter q for a larger number of interpolation points, we give supplementary results concerning the convergence of the SAP solver for varying values of q.

Convergence Rate

We conclude the numerical investigation of the SAP method with a closer examination of its convergence rate. Theorem 6.3 in Section 6.1 predicts an exponential decay of the residual of the n-th iterate S_{nM}^G (measured in the $\mathcal{H}(\{A_n\}; h; \overline{\Omega_r^{ext}})$-norm) with increasing number of iterations n according to

$$\frac{\|S^G_{\mathcal{L}_1^N,\ldots,\mathcal{L}_N^N} - S^G_{nM}\|_{\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})}}{\|S^G_{\mathcal{L}_1^N,\ldots,\mathcal{L}_N^N}\|_{\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})}} \leq C^n, \quad \text{where} \quad C < 1. \tag{8.1}$$

Since in our implementation of the SAP algorithm the coefficients of the iterates S_{nM}^G are not stored, we cannot evaluate the quantity on the left-hand side of (8.1). Our numerical experiments, however, confirm an exponential decay of the relative residual

$$\frac{\left|\left(S^G_{\mathcal{L}_1^N,\ldots,\mathcal{L}_N^N}(x_i^N) - S^G_{nM}(x_i^N)\right)^T_{1\leq i\leq N}\right|}{\left|(G(x_i^N))^T_{1\leq i\leq N}\right|},$$

which is computed after each iterative step. This is shown in in Figure 8.6, where the logarithm of the relative residual is plotted versus the number of SAP iterations. There we use the results we obtained in the study of the performance of the SAP algorithm for varying total number of interpolation points and a subdivision into $M = 49$ subdomains (see Table 8.3).

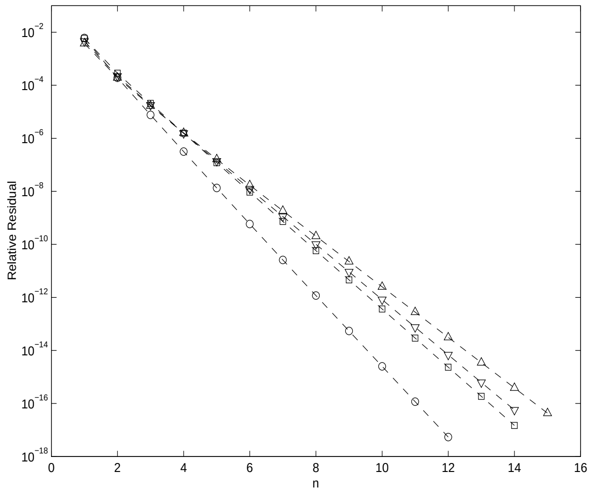


Figure 8.6: *Relative residual of the SAP iterates in dependence of the number of SAP iterations n for the numerical test with $N = 15958$ (o), $N = 23070$ (□), $N = 31235$ (▽) and $N = 40949$ (△) points of a Reuter grid in the φ_g-ϑ_g-box.*

8.2 Accuracy of the Approximation of the Signal

In this section we analyse the accuracy of the interpolating spline as an approximation of the SGG signal with respect to the parameter $q \in (0, 1]$ and the number of interpolation points in the φ_g-ϑ_g-box. As already mentioned in Section 4.5, we have to expect that Gibbs phenomena occur close to the boundaries of the φ_g-ϑ_g-box. Figure 8.7 shows the influence of this truncation effects for the interpolating spline which was computed for $q = 0.95$ and $N = 40949$ interpolation points on the spherical test orbit Ω_{r_S}. Here absolute errors larger than $8 \cdot 10^{-17}\mathrm{s}^{-2}$ and smaller than $-9 \cdot 10^{-17}\mathrm{s}^{-2}$ are set to $8 \cdot 10^{-17}\mathrm{s}^{-2}$ and $-9 \cdot 10^{-17}\mathrm{s}^{-2}$, respectively. Note that the signal itself varies between $-8 \cdot 10^{-10}\ s^{-2}$ and $18 \cdot 10^{-10}\ s^{-2}$ (see Figure 8.1 b)).

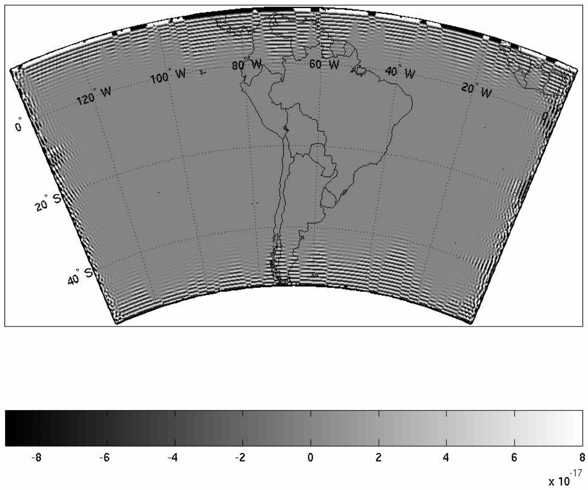


Figure 8.7: *Error* $S^G_{\mathcal{L}_1^N,\ldots,\mathcal{L}_N^N} - G$ *in* s^{-2} *for* $N = 40949$ *and* $q = 0.95$ *in the* φ_g*-*ϑ_g*-box. Absolute errors larger than* $8\cdot 10^{-17}\mathrm{s}^{-2}$ *and smaller than* $-9\cdot 10^{-17}\mathrm{s}^{-2}$ *are set to* $8\cdot 10^{-17}\mathrm{s}^{-2}$ *and* $-9\cdot 10^{-17}\mathrm{s}^{-2}$*, respectively. Note that the signal itself varies between* $-8\cdot 10^{-10}$ s^{-2} *and* $18\cdot 10^{-10}$ s^{-2} *(see Figure 8.1 b)).*

In our discussion of the quality of the interpolating spline as an approximation to the signal we evaluate the interpolating spline in 60830 points of an equiangular φ-ϑ-grid in a reconstruction window on the orbital surface, which is defined by

$$(\varphi_g, \vartheta_g) \in [-2.203, -0.14] \times [-0.754, 0.047].$$

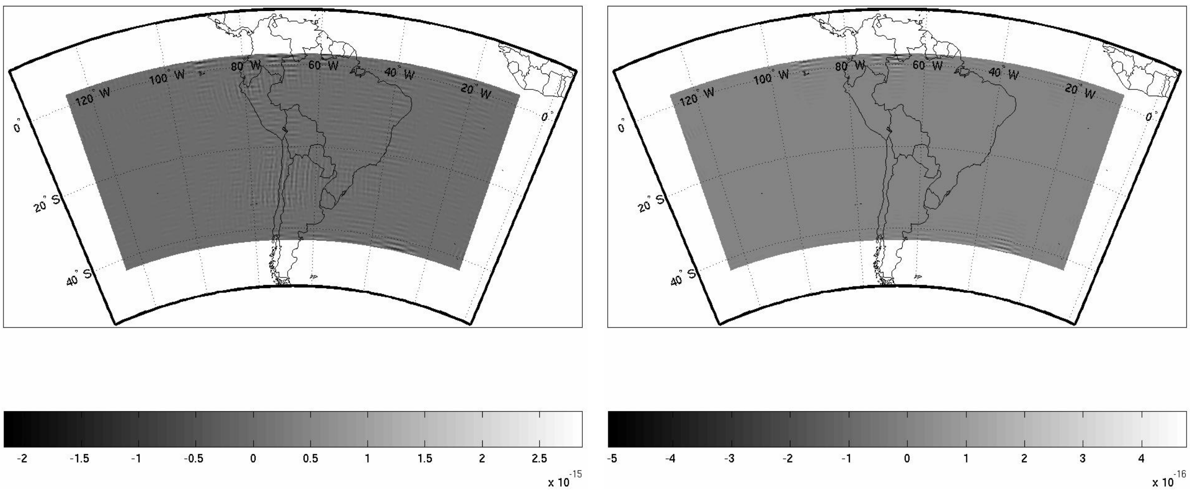
We always compute the maximal absolute error, the mean absolute error and the absolute rooted mean square error of the interpolating spline in the reconstruction window.

In our first study we analyse the influence of the number of interpolation points on the accuracy of the $\mathcal{H}(\{A_n\}; h; \overline{\Omega_r^{ext}})$-spline interpolant. Here we refer to the study we performed in Section 8.1 for a Reuter grid on the spherical test orbit Ω_{r_S} for $q = 0.95$ and a subdivision of the φ_g-ϑ_G-box into $M_\varphi = M_\vartheta = 7$ subdomains in φ_g- and ϑ_g-direction with overlap $d\varphi = 0.2 \cdot 4/7$ and $d\vartheta = 0.1 \cdot 4/7$. The results are given in Table 8.9.

Table 8.9: Test of the accuracy of the $\mathcal{H}(\{A_n\}; h; \overline{\Omega_r^{ext}})$-spline interpolant computed with the Cholesky-SAP solver for $q = 0.95$ and data given on a Reuter grid with varying number of grid points on the spherical test orbit Ω_{r_S} which is subdivided into $M = 7 \cdot 7$ subdomains with overlap $d\varphi = 0.2 \cdot 4/7$ and $d\vartheta = 0.1 \cdot 4/7$ (for further information see also Table 8.3).

N	max error in s^{-2}	mean error in s^{-2}	rms error in s^{-2}
10295	$2.118 \cdot 10^{-13}$	$1.550 \cdot 10^{-14}$	$3.755 \cdot 10^{-14}$
15958	$2.21 \cdot 10^{-13}$	$1.498 \cdot 10^{-14}$	$2.501 \cdot 10^{-14}$
23070	$2.865 \cdot 10^{-13}$	$6.401 \cdot 10^{-17}$	$2.101 \cdot 10^{-16}$
31235	$5.112 \cdot 10^{-16}$	$4.585 \cdot 10^{-18}$	$3.088 \cdot 10^{-17}$
40949	$9.37 \cdot 10^{-17}$	$7.041 \cdot 10^{-19}$	$5.174 \cdot 10^{-18}$

Figure 8.8 and Figure 8.9 show the error $S^G_{\mathcal{L}_1^N,\ldots,\mathcal{L}_N^N} - G$ of the interpolating splines, computed for $N = 23070$, $N = 31235$ and $N = 40949$, in the reconstruction window.



a) b)

Figure 8.8: *Error* $S^G_{\mathcal{L}_1^N,\ldots,\mathcal{L}_N^N} - G$ *in* s^{-2} *for a)* $N = 23070$*, and b)* $N = 31235$.

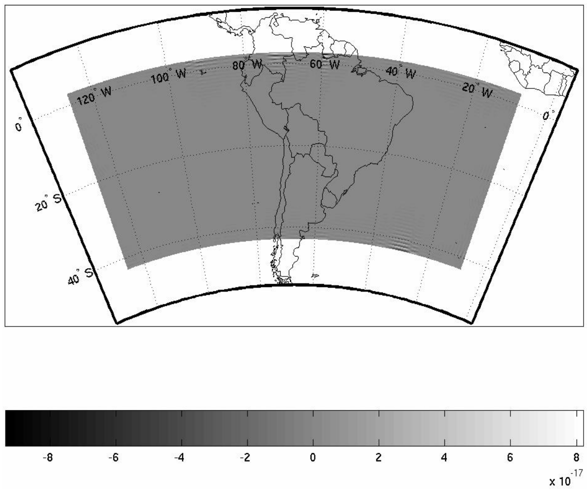


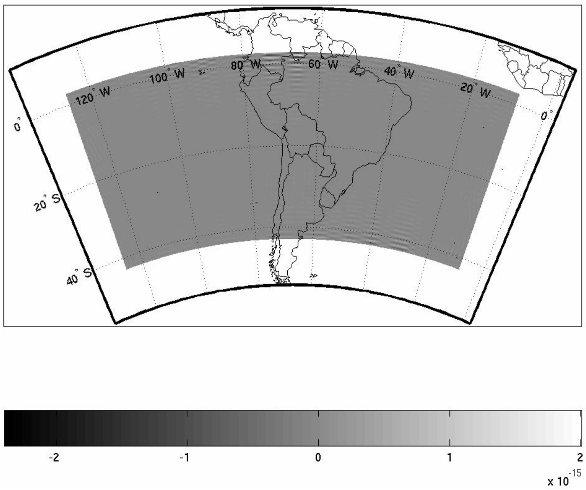
Figure 8.9: *Error* $S^G_{\mathcal{L}_1^N,\ldots,\mathcal{L}_N^N} - G$ *in* s^{-2} *for* $N = 40949$.

In order to study the influence of the parameter q on the accuracy of the $\mathcal{H}(\{A_n\}; h; \overline{\Omega_r^{ext}})$-spline we make two test series for different numbers of interpolation points. In order to keep the total computation time low, we select a Reuter grid with grid parameter $\gamma = 250$ and $\gamma = 300$, respectively, which corresponds to $N = 15958$ and $N = 23070$, respectively, interpolation points in the φ_g-ϑ_g-box and choose a subdivision into $M_\varphi = M_\vartheta = 5$ subdomains in φ_g- and ϑ_g-direction, with overlap $d\varphi = 0.16$ and $d\vartheta = 0.08$. Again, the test is performed for the spherical orbit Ω_{r_S}. The results are listed in Table 8.10, and we additionally give the number of SAP iterations to illustrate the influence of the parameter q on the convergence of the Schwarz alternating algorithm.

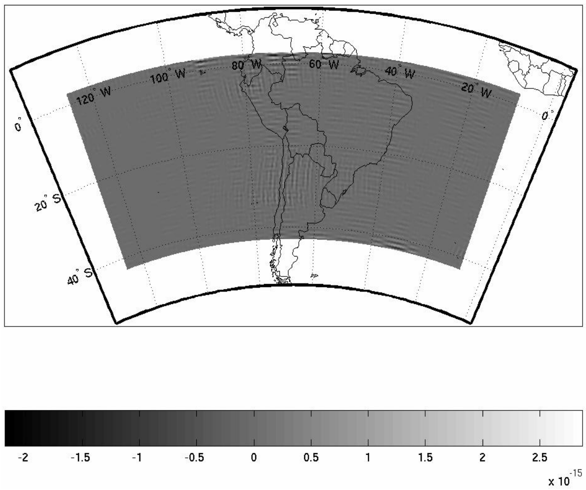
Table 8.10: Test of the accuracy of the $\mathcal{H}(\{A_n\}; h; \overline{\Omega_r^{ext}})$-spline interpolant computed with the SAP method for varying value of q and data given on a Reuter grid with $N = 15958$ and $N = 23070$ grid points in the φ_g-ϑ_g-box on the spherical test orbit which is subdivided into $M = 5 \cdot 5$ subdomains with overlap $d\varphi = 0.16$ and $d\vartheta = 0.08$.

N	q	It.	max error	mean error	rms error
15958	0.92	19	$1.150 \cdot 10^{-13}$	$1.269 \cdot 10^{-14}$	$1.951 \cdot 10^{-14}$
23070		21	$2.398 \cdot 10^{-15}$	$2.028 \cdot 10^{-17}$	$1.266 \cdot 10^{-16}$
15958	0.94	9	$1.566 \cdot 10^{-13}$	$1.422 \cdot 10^{-14}$	$2.215 \cdot 10^{-14}$
23070		10	$2.887 \cdot 10^{-15}$	$3.405 \cdot 10^{-17}$	$1.617 \cdot 10^{-16}$
15958	0.95	7	$2.118 \cdot 10^{-13}$	$1.559 \cdot 10^{-14}$	$2.478 \cdot 10^{-14}$
23070		8	$2.873 \cdot 10^{-15}$	$6.347 \cdot 10^{-17}$	$2.077 \cdot 10^{-16}$
15958	0.96	7	$2.901 \cdot 10^{-13}$	$1.775 \cdot 10^{-14}$	$2.924 \cdot 10^{-14}$
23070		7	$3.0972 \cdot 10^{-15}$	$1.581 \cdot 10^{-16}$	$4.247 \cdot 10^{-16}$
15958	0.97	7	$4.216 \cdot 10^{-13}$	$2.147 \cdot 10^{-14}$	$3.74 \cdot 10^{-14}$
23070		7	$4.749 \cdot 10^{-14}$	$7.491 \cdot 10^{-16}$	$2.314 \cdot 10^{-15}$

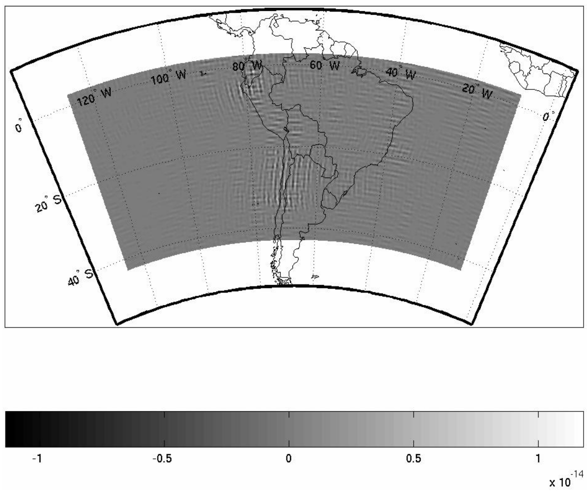
The results clearly show the limitations of the method. The convergence of the SAP algorithm is very good and independent of q for $q \geq 0.95$ in case $N = 15958$ and for $q \geq 0.96$ in case $N = 23070$. This behaviour is not astonishing, since the coefficient matrices corresponding to the subproblems are better conditioned if the Abel-Poisson kernel is strongly space-localizing. For smaller values of q we observe a rapid deterioration of the convergence if we choose q smaller than 0.94. Note that for $q = 0.91$ the Cholesky routine `dpptrf` already detects a singular matrix for one of the subproblems in case $N = 23070$ and the SAP algorithm does not converge within 30 iterations if $N = 15958$. Test computations using the Parlett-Reid algorithm to solve the interpolation equations for the subsets of X^N instead of the Cholesky algorithm were performed for $q = 0.92$ and $N = 23070$ and gave the same rate of convergence as the Cholesky variant of the SAP solver. An interpolating spline which is a good approximation of our signal is achieved if q is chosen small. Figure 8.8 shows the error $S^G_{\mathcal{L}_1^N,\ldots,L_N^N} - G$ for $N = 20370$ and $q = 0.92$, $q = 0.95$ and $q = 0.97$. For $q = 0.97$ we clearly see the effect of a too strong space localization of the Abel-Poisson kernel on the error.



a)



b)



c)

Figure 8.10: *Error* $S^G_{\mathcal{L}_1^N,\ldots,L_N^N} - G$ *in* s^{-2} *for* $N = 20370$ *and a)* $q = 0.92$*, b)* $q = 0.95$ *and c)* $q = 0.97$.

8.3 Choice of Pointsets

In this section we introduce the notion of an equidistribution on the sphere in the sense of [FrGeSchr1998] and give some examples of equidistributions which can be easily computed. It should be noted that even in the case of an orbital sphere such an equidistribution need not lead to a well-conditioned coefficient matrix in the linear equation system of the spline interpolation problem (6.1). Research on equidistributions on the sphere which lead to well-conditioned coefficient matrices in connection with the spline interpolation problem in $\mathcal{L}^2(\Omega)$ is, for example, currently done by I. H. Sloan and R. S. Womersley. The equidistributions computed by Sloan and Womersley require the numerical solution of large-scale optimization problems, which are far beyond the scope of this thesis. For more details, the reader is referred to [SlWo1999] and [SlWo2001] and the references therein.

Definition 8.1 *Let $\{A_n\}_{n\in\mathbb{N}_0} \subset \mathbb{R}^+$ be a summable sequence of non-negative real numbers, let $r \in \mathbb{R}^+$ and $h \in \{0,1,2\}$. A (hierarchical) sequence of point systems $\{X_N\}_{N\in\mathbb{N}} := \{x_1^N, \ldots, x_N^N\}_{N\in\mathbb{N}}$ on the sphere Ω_r is called a (hierarchical) equidistribution on Ω_r, if*

$$\int_{\Omega_r} F(x)\, d\omega(x) = \lim_{N\to\infty} \frac{4\pi}{N} \sum_{i=1}^{N} F(x_i^N)$$

for all $F \in \mathcal{H}(\{A_n\}; h; \overline{\Omega_r^{ext}})$.

The subsequent list of examples of equidistributions on the unit sphere is taken from [FrGeSchr1998], and the corresponding point distributions in the φ-ϑ-plane are illustrated in Figure 8.11.

Example 8.2 (Reuter Grid) *For $\gamma \in \mathbb{N}$ and $N = N(\gamma) \in \mathbb{N}$, define $X_N := \{(\varphi_{i,j}, \vartheta_i)\} \subset \Omega$ as follows:*

(i) $\vartheta_0 := 0$, $\varphi_{01} := 0$ *(North Pole)*

(ii) $\Delta\vartheta := \pi/\gamma$

(iii) $\vartheta_i := i\Delta\vartheta$, $1 \le i \le \gamma - 1$

(iv) $\gamma_i := \lfloor 2\pi/\arccos\left((\cos\Delta\vartheta - \cos^2\vartheta_i)/\sin^2\vartheta_i\right)\rfloor$

(v) $\varphi_{ij} := (j - 1/2)(2\pi/\gamma_i)$, $1 \le j \le \gamma_i$

(vi) $\vartheta_\gamma := \pi$, $\varphi_{i,j} = 0$ *(South Pole)*

The number of points $N(\gamma)$ for a given $\gamma \in \mathbb{N}$ can be estimated by

$$N(\gamma) \le 2 + \frac{4}{\pi}\gamma^2.$$

Example 8.3 (Brand Grid) *For $\gamma \in \mathbb{N}$ and $N = N(\gamma) \in \mathbb{N}$, define $X_N := \{(\varphi_{i,j}, \vartheta_i)\} \subset \Omega$ as follows:*

(i) $\vartheta_0 := 0$, $\varphi_{01} := 0$ *(North Pole)*

(ii) $\Delta\vartheta := \pi/\gamma$

(iii) $\vartheta_i := i\Delta\vartheta$, $1 \le i \le \gamma - 1$

(iv) $\begin{array}{lcl} i \le \gamma/2 & : & \gamma_i := 4i \\ i > \gamma/2 & : & \gamma_i := 4(\gamma - i) \end{array}$

(v) $\varphi_{ij} := j(2\pi/\gamma_i)$, $1 \le j \le \gamma_i$

(vi) $\vartheta_\gamma := \pi$, $\varphi_{i,j} = 0$ *(South Pole)*

The number of points $N(\gamma)$ for a given $\gamma \in \mathbb{N}$ is given by

$$N(\gamma) = 2 + 4\lfloor\frac{\gamma+1}{2}\rfloor\lfloor\frac{\gamma}{2}\rfloor.$$

A hierarchical equidistribution is obtained by successively doubling the grid parameter γ.

For the last two examples we introduce the so-called Van-der-Corput sequence $\{\Phi_p(n)\}_{n\in\mathbb{N}}$, where $p \in \{2, 3, \ldots\}$. For a given p consider the unique expansion of $n-1$, $n \in \mathbb{N}$, of the form

$$n - 1 = \sum_{j=0}^{s} a_j p^j,\ a_j \in \{0, \ldots, p-1\},\ s \in \mathbb{N}_0$$

and define

$$\Phi_p(n) := \sum_{j=0}^{s} a_j p^{-j-1}.$$

Example 8.4 (Equidistribution Based on the Hammersley Sequence) *Given $N \in \mathbb{N}$, define the system $X_N := \{(\varphi_n, \vartheta_n)\}_{1\le n\le N} \subset \Omega$ by*

$$(\varphi_n, \vartheta_n) := \left(2\pi\,\frac{n-1}{N},\ \arccos(2\Phi_2(n) - 1)\right),\ 1 \le n \le N.$$

The equidistribution obtained by this so-called Hammersley sequence is not hierarchical.

Example 8.5 (Equidistribution Based on the Corput-Halton Sequence) *Given $N \in \mathbb{N}$, define the system $X_N := \{(\varphi_n, \vartheta_n)\}_{1\le n\le N} \subset \Omega$ by*

$$(\varphi_n, \vartheta_n) := (2\pi\Phi_2(n),\ \arccos(2\Phi_3(n) - 1))\,,\ 1 \le n \le N.$$

The equidistribution based on this so-called Corput-Halton sequence is hierarchical.

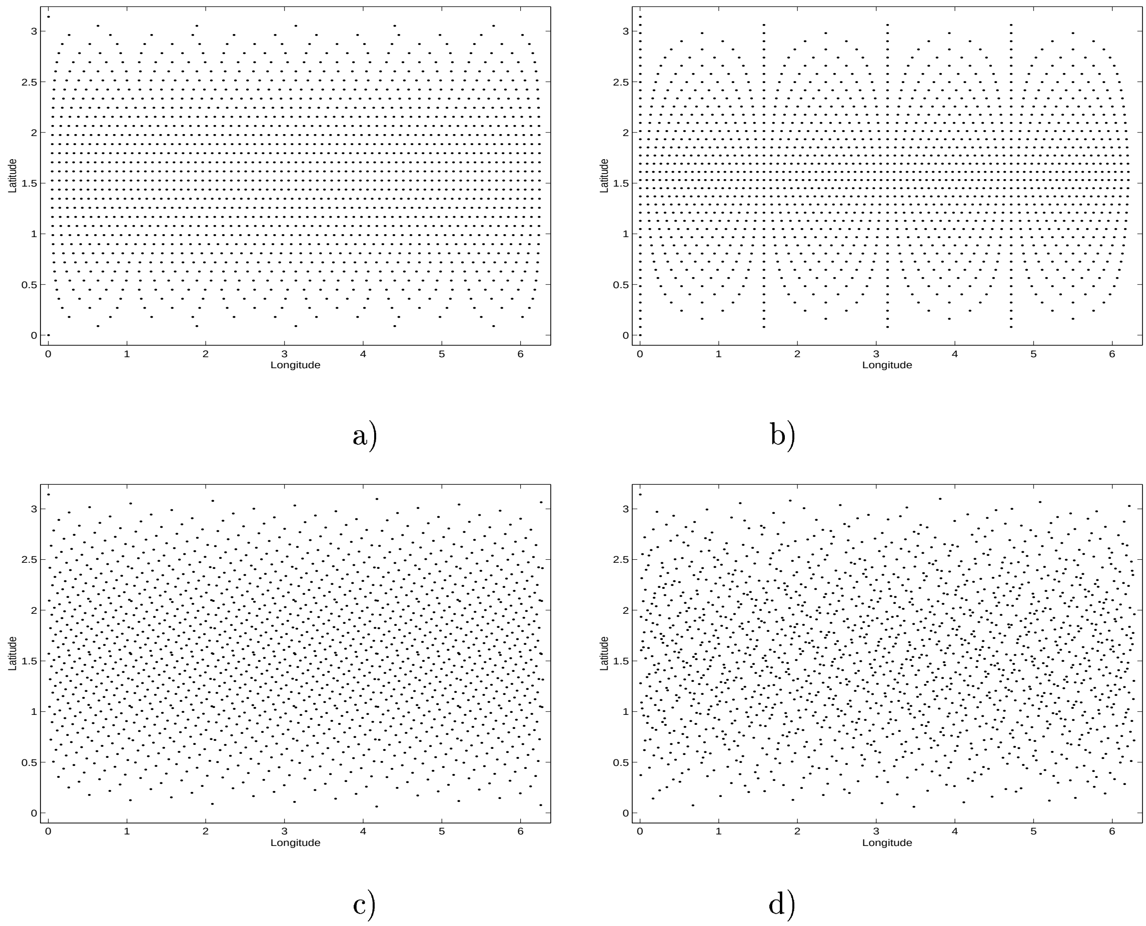


Figure 8.11: *a) Pointset of Example 8.2 (Reuter Grid) with grid parameter* $\gamma = 35$ *and* $N = 1542$ *points. b)Pointset of Example 8.3 (Brand Grid) with grid parameter* $\gamma = 39$ *and* $N = 1522$ *points. c) Pointset of Example 8.4 with* $N = 1542$ *points. d) Pointset of Example 8.5 with* $N = 1542$ *points.*

Chapter 9

Multiscale Reconstruction of the Gravitational Potential from Simulated SGG Data

In this chapter we present a numerical study of the discretization schemes proposed in Chapter 7 for the numerical computation of a regularized solution to the SGG-problem.

In our test example we use the EGM96 model to compute a local model of the second order radial derivative G of the gravitational potential $V \in \mathrm{Pot}^{(0)}(\overline{\Sigma_E^{ext}})$ in a set of N points $X^N := \{x_1^N, \ldots, x_N^N\}$ of a Reuter grid on a spherical satellite orbit with radius r_S, which approximates the GOCE orbit. Here we include contributions of outer harmonic degrees 36 up to 200, i.e., using the same notation as on page 102, our test data are given by

$$G(x_i^N) = \Gamma M \sum_{n=36}^{200} \frac{(n+1)(n+2)}{r_S^2} \sum_{k=-n}^{n} V_{n,k}^R \, H_{n,k}(R; x_i^N), \ i = 1, \ldots, N, \tag{9.1}$$

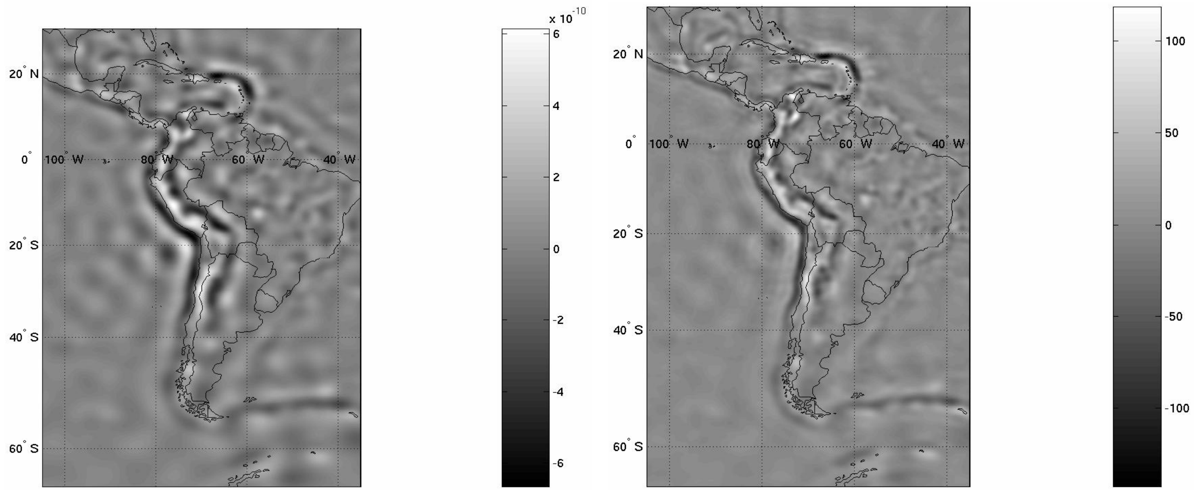
where R denotes the radius of the Bjerhammar sphere for the earth and r the radius of the Bjerhammar sphere for the satellite orbit. The model parameters are given by

$$\begin{aligned} r_S &= 6628059 \text{ m}, \\ r &= 6588310 \text{ m}, \\ R &= 6378136.4 \text{ m}, \\ \Gamma M &= 3986004.415 \cdot 10^8 \text{ m}^3\text{s}^{-2}, \end{aligned}$$

and the test data are generated within the longitude-latitude- (φ_g, ϑ_g)- box

$$(\varphi_g, \vartheta_g) \in [-1.833, -0.61] \times [-1.15, 0.524],$$

which comprises the complete South American continent. Figure 9.1 shows the simulated SGG-signal at satellite height and the corresponding band of the gravitational potential on Ω_R which we want to reconstruct.



a) b)

Figure 9.1: *a) Second order radial derivative of the earth's gravitational potential on* Ω_{r_S} *in* s^{-2}*, computed with the EGM96 model including all outer harmonic contributions from degrees 36 up to 200, and b) corresponding gravitational potential on the earth's surface* Ω_R *in* $m^2 s^{-2}$.

As in Chapter 8, we let $q \in (0,1]$ and define the sequence A_n by $A_n := q^{-n/2}$. Since $r_S > r$, the evaluation functionals $\mathcal{L}_i^N : \mathcal{H}(\{A_n\};2;\overline{\Omega_r^{ext}}) \to \mathbb{R}$, $H \mapsto \mathcal{L}_i^N(H) := H(x_i^N)$ are bounded for all choices of $q \in (0,1]$ and their representers $L_i^N \in \mathcal{H}(\{A_n\};2;\overline{\Omega_r^{ext}})$, $1 \leq i \leq N$ are given by $L_i^N = K_{\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})}(\cdot, x_i^N)$, where $K_{\mathcal{H}(\{A_n\};2;\overline{\Omega_r^{ext}})}$ is the kernel of 'Abel-Poisson type' given in (4.14).

In Section 9.1 we compute the 'exact' data $\{(x_i^N, G(x_i^N)\}_{1\leq i\leq N}$, according to (9.1), and calculate the interpolating $\mathcal{H}(\{A_n\};2;\overline{\Omega_r^{ext}})$-spline $S^G_{\mathcal{L}_i^N,\ldots,\mathcal{L}_N^N}$ which is a sensible approximation to G, supposed that the parameter q was appropriately chosen. $S^G_{\mathcal{L}_i^N,\ldots,\mathcal{L}_N^N}$ is then convolved with the Tikhonov regularization scaling function for the inverse operator $(\Lambda^{SGG})^{-1}$ in order to obtain a regularized solution to the SGG-problem.

After that the case of error-affected data is simulated by adding white noise to the function values $G(x_i^N)$, $1 \leq i \leq N$. In this case we first compute an approximating $\mathcal{H}(\{A_n\};2;\overline{\Omega_r^{ext}})$-spline for different values of the smoothing parameter λ. Then these $\mathcal{H}(\{A_n\};2;\overline{\Omega_r^{ext}})$-splines are convolved with the Tikhonov-regularization scaling function for $(\Lambda^{SGG})^{-1}$ in order to obtain a reconstruction of the potential. The comparison of the reconstruction of the potential for different smoothing parameters λ allows it to determine which parameter λ yielded the best reconstruction. This 'best' smoothing parameter is quite close to the one which is predicted in Theorem 4.6 in Section 4.2. The results of this study are presented in Section 9.2.

Let $\{\Phi_j^\Lambda\}_{j\in\mathbb{N}_0}$ denote the Tikhonov regularization scaling function for $(\Lambda^{SGG})^{-1}$. In order to obtain the j-level regularization T_jG corresponding to the family of operators $\{T_j\}_{j\in\mathbb{N}_0}$, $T_j : \mathcal{H}(\{A_n(\Lambda^\wedge(n))^{-1}\};2;\overline{\Omega_r^{ext}}) \to \mathcal{H}(\{A_n\};\overline{\Omega_R^{ext}})$, $H \mapsto T_jH := \Phi_j^\Lambda *_{\mathcal{H}(\{A_n\};2;\overline{\Omega_r^{ext}})} H$, replace the function G in T_jG by its interpolating ('exact data') or smoothing $\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})$-spline (noisy data) $S=\sum_{i=1}^N a_i^N K_{\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})}(\cdot,x_i^N)$. This yields the dicretization

$$T_jG = \Phi_j^\Lambda *_{\mathcal{H}(\{A_n\};h;\overline{\Omega_r^{ext}})} G \approx \sum_{i=1}^N a_i^N \Phi_j^\Lambda(\cdot,x_i^N),$$

which we evaluate in M points $\{y_1^M,\ldots,y_M^M\}$ of an equiangular $\varphi-\vartheta$-grid on Ω_R. The kernel functions Φ_j^Λ, $j\in\mathbb{N}_0$, are (to our knowledge) not available as elementary functions. Thus we have to evaluate a truncated series expansions

$$\begin{aligned}
(\Phi_j^\Lambda)^m(y,x) &= \sum_{n=0}^m \sum_{k=1}^{2n+1} \frac{\Lambda^\wedge(n)}{(\Lambda^\wedge(n))^2+\gamma_j}\frac{1}{A_n^2} H_{n,k}(R;y)H_{n,k}(r;2;x)\\
&= \sum_{n=0}^m \sum_{k=1}^{2n+1} \frac{\Lambda^\wedge(n)}{(\Lambda^\wedge(n))^2+\gamma_j} q^n H_{n,k}(R;y)H_{n,k}(r;2;x)\\
&= \frac{1}{4\pi Rr}\left(\frac{r}{r_S}\right)^3 \sum_{n=0}^m \frac{\Lambda^\wedge(n)}{(\Lambda^\wedge(n))^2+\gamma_j}(2n+1)q^n\left(\frac{r}{r_S}\right)^n P_n\left(\frac{y}{R}\cdot\frac{x}{r_S}\right) \qquad (9.2)
\end{aligned}$$

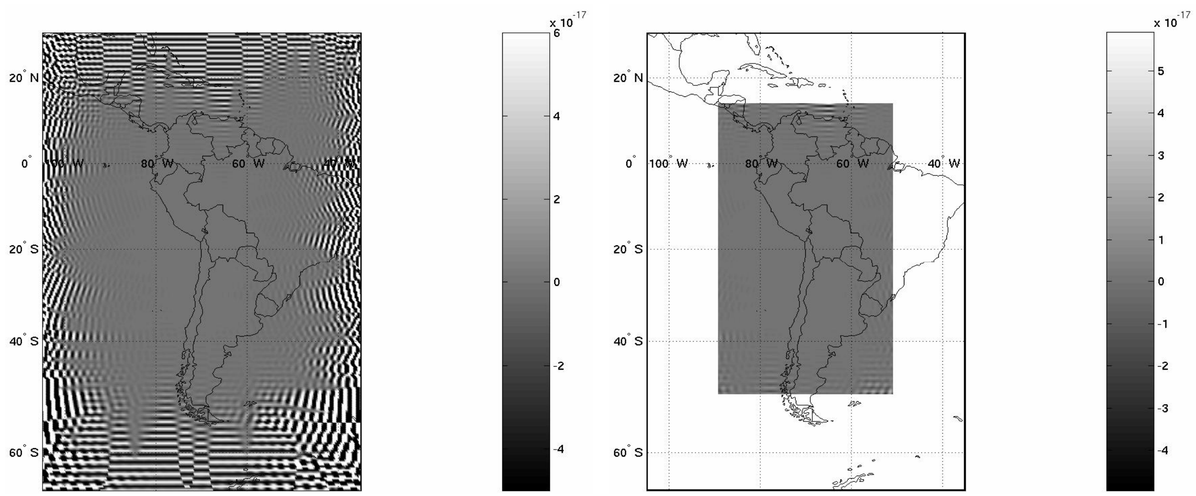
for all $y\in\{y_1^M,\ldots,y_M^M\}$ and all $x\in\{x_1^N,\ldots,x_N^N\}$, where a_i^N, $i=1,\ldots,N$. This can be done in a numerically stable way via adjoint summation of the Legendre polynomials, using the reconstruction formula (1.4). The algorithm can for example be found in [DeHo1993]. Note that in case of a spherical geometry Φ_j^Λ is a univariate function depending only on the spherical distance $t=(y/R)\cdot(x/r_s)$ of the points y/R and x/r_S. In spherical computations we can reduce the computational costs if we evaluate the series (9.2) in a suitably high number of nodes $t\in[-1,1]$ in a preprocessing step and obtain the values of $(\Phi_j^\Lambda)^m$ in the subintervals by linear interpolation of the a priori calculated nodal values. Our numerical computations for a spherical geometry are performed with a truncated Tikhonov regularization scaling function for $(\Lambda^{SGG})^{-1}$ with bandlimit $m=800$, which has been calculated in advance in 500000 nodes in the interval $[-1,1]$.

9.1 Reconstruction of the Potential from 'Exact' Data

All numerical tests are made for $q=0.92$. The $\mathcal{H}(\{A_n\};2;\overline{\Omega_r^{ext}})$-spline interpolation is performed with respect to $N=15768$ samples of G, taken in the points of a Reuter grid with grid-parameter 300 in the φ_g-ϑ_g-box on Ω_{r_S}. The linear equation system of the $\mathcal{H}(\{A_n\}; h; \overline{\Omega_r^{ext}})$-spline interpolation problem is solved with the SAP algorithm based on Cholesky decomposition of the coefficient matrices for the subproblems. Note that our numerical studies in Section 8.2 showed that for a Reuter grid on Ω_{r_S} with grid parameter 300 the Cholesky routine failed if the parameter q was chosen smaller than $q = 0.92$, thus we work here with the smallest possible value of $q = 0.92$ for a Reuter grid with grid parameter 300. Furthermore, the φ_g-ϑ_g-box is subdivided into 20 subdomains ($M_\varphi = 5$ and $M_{\vartheta_g} = 4$ subdivisions in φ_g and ϑ_g-direction, respectively) with an overlap $p_\varphi = p_\vartheta = 0.6$ (here we use the subdivision scheme described in Section 6.2). The accuracy tolerance for the relative residual is 10^{-16}. Figure 9.2 a) shows the error of the interpolating spline, evaluated in 74880 points of an equiangular φ-ϑ-grid in the φ_g-ϑ_g-box on Ω_{r_S}. In order to illustrate the influence of truncation errors close to the boundary, errors larger than $6 \cdot 10^{-17}$ s^{-2} and smaller than $-5 \cdot 10^{-17}$ s^{-2} are set to $6 \cdot 10^{-17}$ s^{-2} and $-5 \cdot 10^{-17}$ s^{-2}, respectively. Taking 10^{-17} s^{-2} as an acceptable error-level for the interpolating $\mathcal{H}(\{A_n\}; 2; \overline{\Omega_r^{ext}})$-spline, we select a parameter region, defined by

$$(\varphi_g, \vartheta_g) \in [-1.559, -0.883] \times [-0.874, 0.251],$$

as the window on Ω_R on which the gravitational potential shall be reconstructed.



a) b)

Figure 9.2: *a) Error* $S^G_{\mathcal{L}_i^N,\ldots,\mathcal{L}_N^N} - G$ *in the data window in* s^{-2}*. Values larger than* $6 \cdot 10^{-17}$ s^{-2} *and smaller than* $-5 \cdot 10^{-17}$ s^{-2} *are set to* $6 \cdot 10^{-17}$ s^{-2} *and* $-5 \cdot 10^{-17}$ s^{-2}*, respectively. b) Error* $S^G_{\mathcal{L}_i^N,\ldots,\mathcal{L}_N^N} - G$ *in the reconstruction window.*

The j-level regularization T_jG is evaluated in $M = 28080$ points of an equiangular φ-ϑ-grid in the reconstruction window on Ω_R for different values of the regularization parameter j.

In Table 9.1 we list the value of the parameter γ_j in the generating symbol of the Tikhonov regularization scaling function, the maximal and the mean absolute error and the rooted mean square (rms) error of the reconstructed gravitational potential.

Table 9.1: Reconstruction error in dependence of the parameter γ in the Tikhonov regularization.

j	γ	max error in m^2s^{-2}	mean error in m^2s^{-2}	rms error in m^2s^{-2}
1	$1.6941 \cdot 10^{-21}$	114.74	13.02	19.52
2	$8.4703 \cdot 10^{-22}$	111.06	12.21	18.39
3	$4.2352 \cdot 10^{-22}$	104.729	10.8969	16.56
4	$2.1176 \cdot 10^{-22}$	94.86	9.07	13.99
5	$1.0588 \cdot 10^{-22}$	81.58	7.00	11.01
6	$5.294 \cdot 10^{-23}$	66.16	5.11	8.18
7	$2.647 \cdot 10^{-23}$	50.60	3.65	5.89
8	$1.3235 \cdot 10^{-23}$	36.5	2.63	5.89
9	$6.6174 \cdot 10^{-24}$	24.93	1.9	2.92
10	$3.3087 \cdot 10^{-24}$	16.57	1.5	2.13
11	$1.6544 \cdot 10^{-24}$	11.57	1.66	2.1
12	$8.2718 \cdot 10^{-25}$	9.19	2.48	2.86
13	$4.1359 \cdot 10^{-25}$	8.04	3.79	3.94
14	$2.068 \cdot 10^{-25}$	6.93	4.8	4.93
15	$1.034 \cdot 10^{-25}$	7.56	4.86	5.22
16	$5.1699 \cdot 10^{-26}$	12.79	3.94	4.66
17	$2.5849 \cdot 10^{-26}$	20.5	4.76	6.11
18	$1.2925 \cdot 10^{-26}$	30.0	10.35	12.21

Figure 9.3 shows that the mean absolute error of the reconstructed gravitational potential becomes minimal for the choice $j = 10$, i.e., $\gamma = 1.6544 \cdot 10^{-24}$, but we also observe a second local minimum for $j = 16$, i.e., $\gamma = 5.1699 \cdot 10^{-26}$.

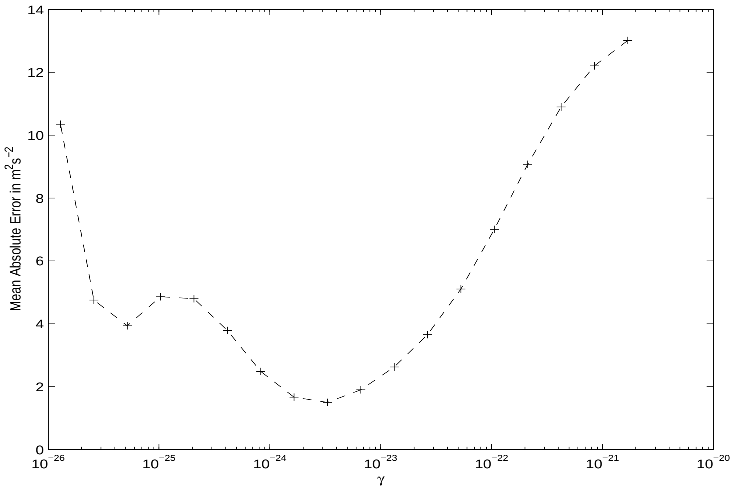


Figure 9.3: *Mean absolute error of the reconstructed gravitational potential in dependence of the parameter γ in the Tikhonov regularization.*

We do not plot all the partial reconstructions T_jG and details R_jG of the gravitational potential which were computed but only show those which are interesting. In order to be consistent with the notation in Chapter 5 we select a subsequence $\{\gamma_j\}_{1\leq j'(j)\leq 6}$ of $\{\gamma_j\}_{1\leq j\leq 18}$ by setting $j'(3) := 1$, $j'(7) := 2$, $j'(9) := 3$, $j'(10) := 4$, $j'(11) := 5$ and $j'(13) := 6$ for which we plot $T_{j'}G$ and $R_{j'}G$. Figure 9.4 shows the j'-level regularization at scales $j' = 1$, $j' = 2$ and $j' = 3$, together with the details $R_{j'}G$, i.e., we show the reconstructions $T_{j'}G$ of the gravitational potential at level j' and the detail $R_{j'}G$ which has to be added to obtain $T_{j'+1}G$ until we reach the parameter $j' = 3$ for which an 'optimal' approximation of the gravitational potential was obtained. After that, we show the partial reconstructions $T_{j'}G$ together with the error $T_{j'}G - F$. Thus Figure 9.5 shows the j'-level regularization at scales $j' = 4$, $j' = 5$ and $j' = 6$, together with the error $T_{j'}G - F$, $4 \leq j' \leq 6$.

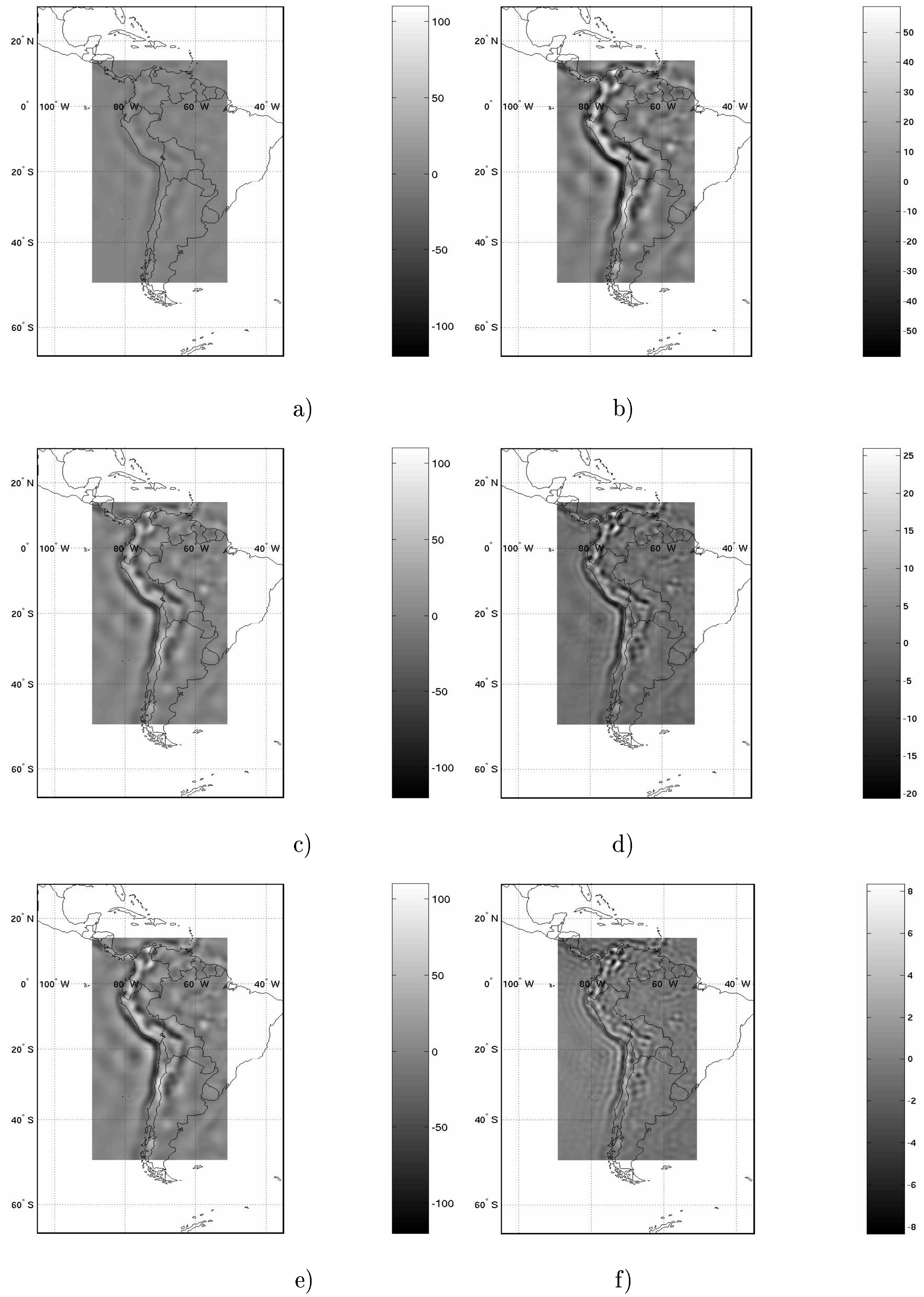


Figure 9.4: *a)* T_1G, *b)* R_1G, *c)* T_2G, *d)* R_2G, *e)* T_3G, *and f)* R_3G *in* m^2s^{-2}.

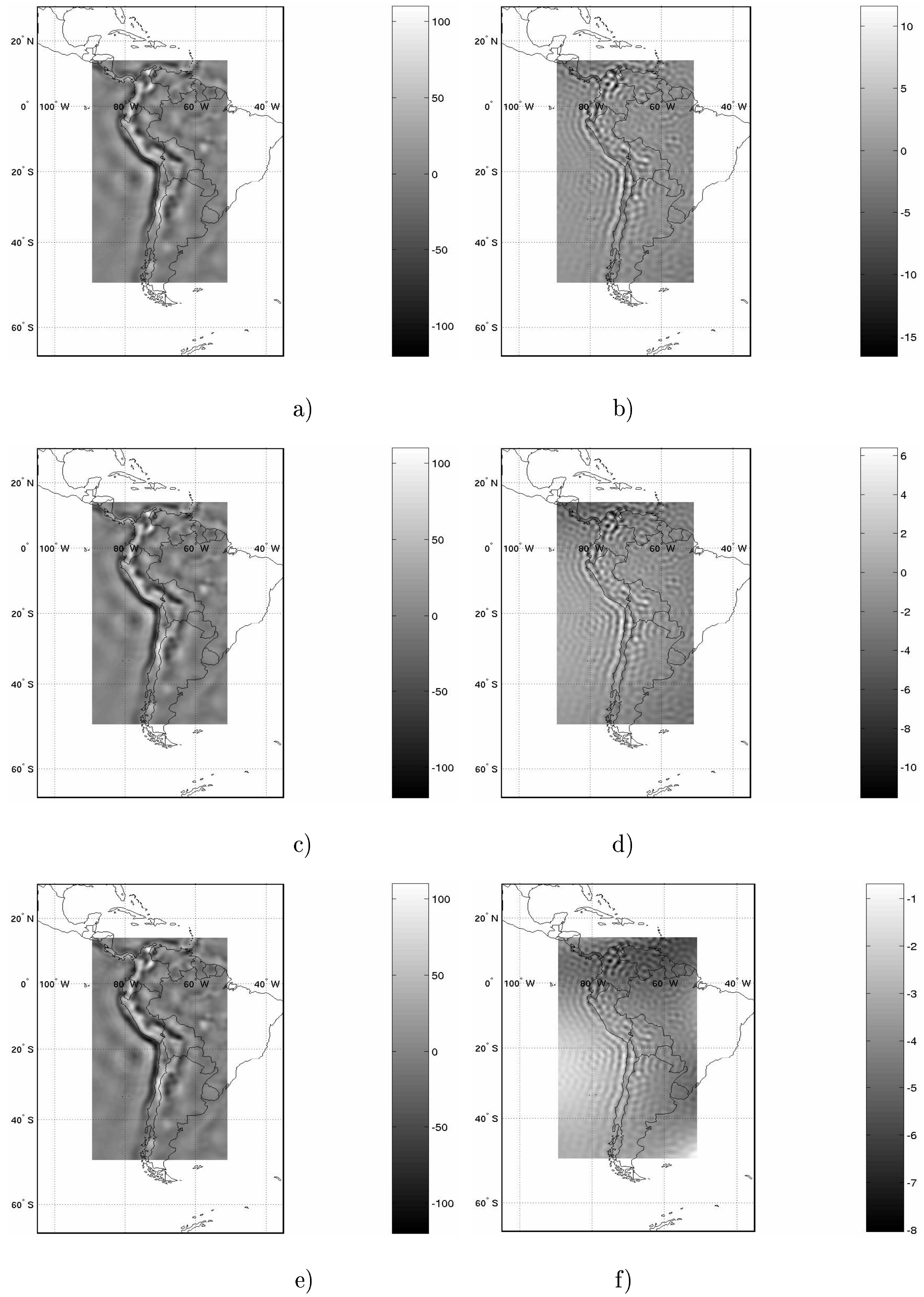


Figure 9.5: *a)* T_4G, *b)* $T_4G - F$, *c)* T_5G, *d)* $T_5G - F$, *e)* T_6G, *and f)* $T_6G - F$ *in* m^2s^{-2}.

9.2 Reconstruction of the Potential from Noisy Data

In order to investigate the multiscale reconstruction of the gravitational potential on the earth's surface (again using the Tikhonov regularization scaling function for $(\Lambda^{SGG})^{-1}$) in the presence of noisy data, we add to the simulated 'exact' data $\{(x_i^N, G(x_i^N)) | \, 1 \leq i \leq N\}$ of Section 9.1 white noise with mean value $\overline{\varepsilon} = 6.9787 \cdot 10^{-13}$ and standard deviation $\tilde{\sigma} = 8.0572 \cdot 10^{-13}$, which yields the error-affected data $\{(x_i^N, \tilde{G}(x_i^N)) | \, 1 \leq i \leq N\}$, where $\tilde{G}(x_i^N) := G(x_i^N) + \varepsilon_i$, $1 \leq i \leq N$. First the $\mathcal{H}(\{A_n\}; 2; \overline{\Omega_r^{ext}})$-spline smoothing problem with respect to the noisy data $\{(x_i^N, \tilde{G}(x_i^N)) | \, 1 \leq i \leq N\}$ for a set of different values of the smoothing parameter λ is solved.

Like in Section 9.1, we perform the numerical experiments for $q = 0.92$ and a Reuter grid with grid parameter $\gamma = 300$ and $N = 15768$ points in the φ_g-ϑ_g-box on the satellite orbit Ω_{r_S}. The j-level regularization $T_j G$ is again evaluated in $M = 28080$ points of an equiangular φ-ϑ-grid in the reconstruction window on Ω_R.

Due to the spherical orbit, all entries on the diagonal of $(K_{\mathcal{H}(\{A_n\};2;\overline{\Omega_r^{ext}})}(x_i^N, x_l^N))_{1 \leq i.l \leq N}$ are equal. Furthermore, the maximal entry in $(K_{\mathcal{H}(\{A_n\};2;\overline{\Omega_r^{ext}})}(x_i^N, x_l^N))_{1 \leq i.l \leq N}$ is on the diagonal, and assumes the value

$$K_{\mathcal{H}(\{A_n\};2;\overline{\Omega_r^{ext}})}(x_i^N, x_i^N) \approx 4.07630 \cdot 10^{-14} \ (\mathrm{m}^{-2}) \quad \text{for} \quad i = 1, \ldots, N.$$

which can be calculated easily by inserting the values of R, r, r_S and q into equation (4.14). We solve the $\mathcal{H}(\{A_n\}; 2; \overline{\Omega_r^{ext}})$-spline smoothing problem for the smoothing parameters $\lambda_k := 10^{-k}$, $k = 15, \ldots, 19$, and evaluate $T_j G \approx T_j S_{\lambda_k}$. The results of this study are listed in Table 9.2 and Table 9.3. Like in Section 9.1, we always list the value of the parameter γ in the Tikhonov regularization, and the maximal absolute error, the mean absolute error and the absolute rooted mean square (rms) error in the reconstruction window.

Table 9.2: Reconstruction error in dependence of the parameter γ in the Tikhonov regularization and the smoothing parameter λ_k, $k = 15, 16, 17$, in the $\mathcal{H}(\{A_n\}; 2; \overline{\Omega_r^{ext}})$-spline smoothing problem.

λ	j	γ	max error in m^2s^{-2}	mean error in m^2s^{-2}	rms error in m^2s^{-2}
10^{-15}	1	$2.647 \cdot 10^{-23}$	68.29	4.79	7.9
	2	$6.6174 \cdot 10^{-24}$	56.81	4.05	6.6
	3	$1.6544 \cdot 10^{-24}$	54.42	4.07	6.37
	4	$4.1359 \cdot 10^{-25}$	55.71	4.72	6.76
	5	$1.034 \cdot 10^{-25}$	55.72	4.91	6.9
	6	$2.5849 \cdot 10^{-26}$	54.03	5.67	7.69
	7	$6.4623 \cdot 10^{-27}$	66.06	15.49	16.95
10^{-16}	1	$2.647 \cdot 10^{-23}$	58.51	4.12	6.7
	2	$6.6174 \cdot 10^{-24}$	41.53	3.09	4.87
	3	$1.6544 \cdot 10^{-24}$	34.11	3.25	4.65
	4	$4.1359 \cdot 10^{-25}$	35.48	4.51	5.63
	5	$1.034 \cdot 10^{-25}$	36.13	4.45	5.72
	6	$2.5849 \cdot 10^{-26}$	35.39	6.11	7.93
	7	$6.4623 \cdot 10^{-27}$	50.12	20.37	21.85
10^{-17}	1	$4.2352 \cdot 10^{-22}$	104.93	10.90	16.57
	2	$1.0588 \cdot 10^{-22}$	82.36	7.03	11.06
	3	$2.647 \cdot 10^{-23}$	53.19	3.81	6.15
	4	$6.6174 \cdot 10^{-24}$	31.11	2.42	3.73
	5	$1.6544 \cdot 10^{-24}$	21.15	2.16	3.0
	6	$4.1359 \cdot 10^{-25}$	17.43	2.78	3.4
	7	$2.068 \cdot 10^{-25}$	18.45	3.01	3.67
	8	$1.034 \cdot 10^{-25}$	19.36	2.76	3.52
	9	$5.1699 \cdot 10^{-26}$	19.34	2.46	3.26
	10	$2.5849 \cdot 10^{-26}$	17.95	3.77	4.64
	11	$1.2925 \cdot 10^{-26}$	19.73	7.32	8.24
	12	$6.4623 \cdot 10^{-27}$	25.41	12.21	13.06

Table 9.3: Reconstruction error in dependence of the parameter γ in the Tikhonov regularization and the smoothing parameter λ_k, $k = 18, 19$, in the $\mathcal{H}(\{A_n\}; 2; \overline{\Omega_r^{ext}})$-spline smoothing problem.

λ	j	γ	max error in m^2s^{-2}	mean error in m^2s^{-2}	rms error in m^2s^{-2}
10^{-18}	1	$4.2352 \cdot 10^{-22}$	104.76	10.9	16.56
	2	$1.0588 \cdot 10^{-22}$	81.71	7.01	11.02
	3	$2.647 \cdot 10^{-23}$	51.11	3.69	5.94
	4	$1.3235 \cdot 10^{-23}$	27.99	2.13	3.28
	5	$6.6174 \cdot 10^{-24}$	26.18	2.03	3.10
	6	$3.3087 \cdot 10^{-24}$	17.91	1.63	2.33
	7	$1.6544 \cdot 10^{-24}$	12.45	1.53	1.98
	8	$8.2718 \cdot 10^{-25}$	9.5	1.57	1.92
	9	$4.1359 \cdot 10^{-25}$	7.98	1.53	1.87
	10	$2.068 \cdot 10^{-25}$	6.87	1.60	1.88
	11	$1.034 \cdot 10^{-25}$	7.53	1.92	2.37
	12	$5.1699 \cdot 10^{-26}$	11.98	2.67	3.44
	13	$2.5849 \cdot 10^{-26}$	16.58	4.03	5.00
	14	$1.2925 \cdot 10^{-26}$	21.04	5.7	7.0
	15	$6.4623 \cdot 10^{-27}$	25.17	7.66	9.24
10^{-19}	1	$4.2352 \cdot 10^{-22}$	104.72	10.9	16.56
	2	$1.0588 \cdot 10^{-22}$	81.59	7.00	11.01
	4	$2.647 \cdot 10^{-23}$	50.7	3.66	5.9
	5	$6.6174 \cdot 10^{-24}$	25.21	2.16	3.09
	6	$1.6544 \cdot 10^{-24}$	12.12	3.32	4.0
	7	$4.1359 \cdot 10^{-25}$	14.26	5.08	6.08
	8	$1.034 \cdot 10^{-25}$	21.84	6.5	7.69
	9	$2.5849 \cdot 10^{-26}$	50.07	14.63	17.75
	10	$6.4623 \cdot 10^{-27}$	84.81	30.07	36.23

Figure 9.6 shows the mean absolute error in dependence of the parameter γ of the Tikhonov regularization. The best results are obtained with the smoothing spline corresponding to the smoothing parameter $\lambda = 10^{-18}$. The minimal mean error obtained in this case is comparable to the one obtained in Section 9.1 for 'exact' data and the convolution of the Tikhonov regularization scaling function with an interpolating spline.

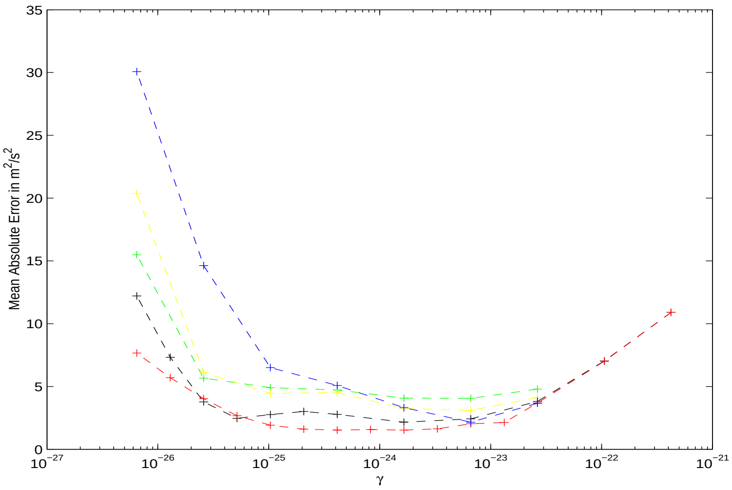
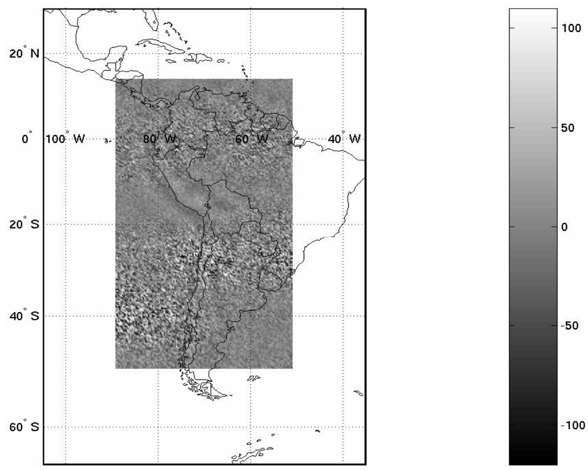


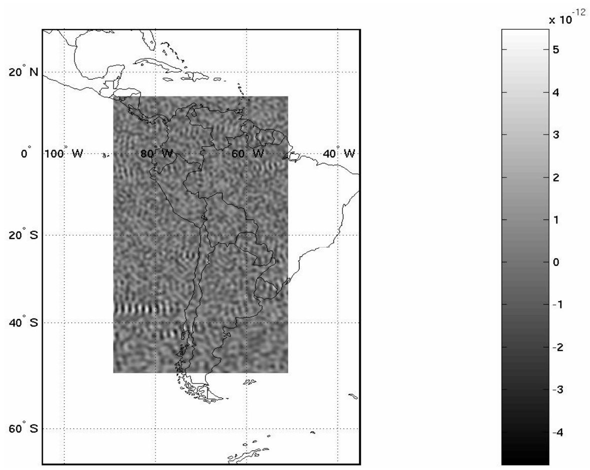
Figure 9.6: *Mean absolute error of the reconstructed gravitational potential in dependence of the parameter γ in the Tikhonov regularization, discretized with smoothing splines for smoothing parameters $\lambda = 10^{-15}$ (green), $\lambda = 10^{-16}$ (yellow), $\lambda = 10^{-17}$ (black), $\lambda = 10^{-18}$ (red) and $\lambda = 10^{-19}$ (blue).*

Before we present the numerical results obtained with the smoothing spline for $\lambda = 10^{-18}$ in more detail, we present the result we obtain at level $j = 1$ if we convolve the Tikhonov regularization scaling function with the interpolating $\mathcal{H}(\{A_n\}; 2; \overline{\Omega_r^{ext}})$-spline: Figure 9.7 a) shows the j-level regularization $T_j G$ we obtain with the interpolating $\mathcal{H}(\{A_n\}; 2; \overline{\Omega_r^{ext}})$-spline which has been calculated from the noisy data $\{(x_i^N, \tilde{G}(x_i^N)) | \, 1 \leq i \leq N\}$. Clearly, no reasonable result can be expected if we further decrease the parameter γ in the Tikhonov regularization. In Figure 9.7 b) we plot the deviation of the interpolating $\mathcal{H}(\{A_n\}; h; \overline{\Omega_r^{ext}})$-spline from the 'exact' SGG-data, and in Figure 9.7 c) we present the deviation of the approximating $\mathcal{H}(\{A_n\}; 2; \overline{\Omega_r^{ext}})$-spline S_λ corresponding to $\lambda = 10^{-18}$ from the 'exact' SGG data.

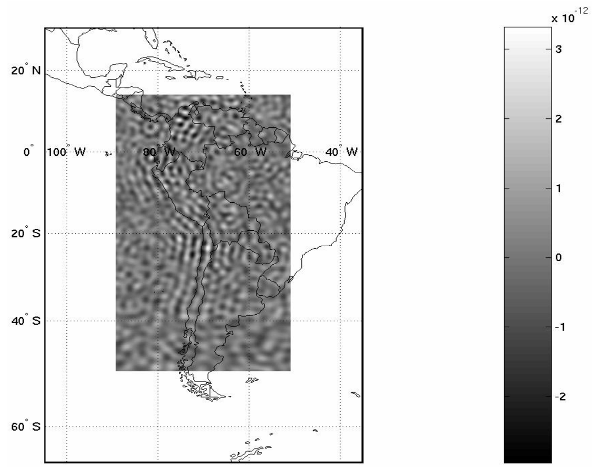
Finally, Figure 9.8 and Figure 9.9 illustrate the multiscale reconstruction of the gravitational potential based on the smoothing $\mathcal{H}(\{A_n\}; h; \overline{\Omega_r^{ext}})$-spline with smoothing parameter $\lambda = 10^{-18}$. We present the results in analogy to Section 9.1, i.e., we choose a subset $\{\gamma_{j'}\}_{1 \leq j' \leq 6}$ of $\{\gamma_j\}_{1 \leq j \leq 15}$, by setting $j'(1) := 1$, $j'(3) := 2$, $j'(7) := 3$, $j'(9) := 4$, $j'(11) := 5$, and $j'(13) := 6$, and plot $T_{j'} G$, $R_{j'} G$ and $T_{j'} G - F$.



a)



b)



c)

Figure 9.7: *a)* $T_1 G$ *in* $m^2 s^{-2}$*, computed with the interpolating* $\mathcal{H}(\{A_n\};2;\overline{\Omega_r^{ext}})$*-spline* $S^{\tilde{G}}_{\mathcal{L}_1^N,\dots,\mathcal{L}_N^N}$ *in* s^{-2}*, b)* $S^{\tilde{G}}_{\mathcal{L}_1^N,\dots,\mathcal{L}_N^N} - G$*, and in c)* $S_\lambda - G$ *in* s^{-2}*.*

Finally, we want to see whether the rule for the choice of the smoothing parameter λ which is given in Theorem 4.6 in Section 4.2 can be verified by our numerical experiments. Given the empirical variance $\tilde{\sigma}^2$ of the data, we suggest to choose choose λ according to the 'discrepancy principle'

$$\alpha^2 := \frac{1}{N-1}\sum_{i=1}^{N}\left(\sum_{l=1}^{N}(a^N_{\lambda(\sigma),\sigma})_l K_{\mathcal{H}(\{A_n\};2;\overline{\Omega_r^{ext}})}(x_l^N, x_i^N) - \big(G(x_i^N)+\varepsilon_i\big)\right)^2 \stackrel{!}{=} \tilde{\sigma}^2. \quad (9.3)$$

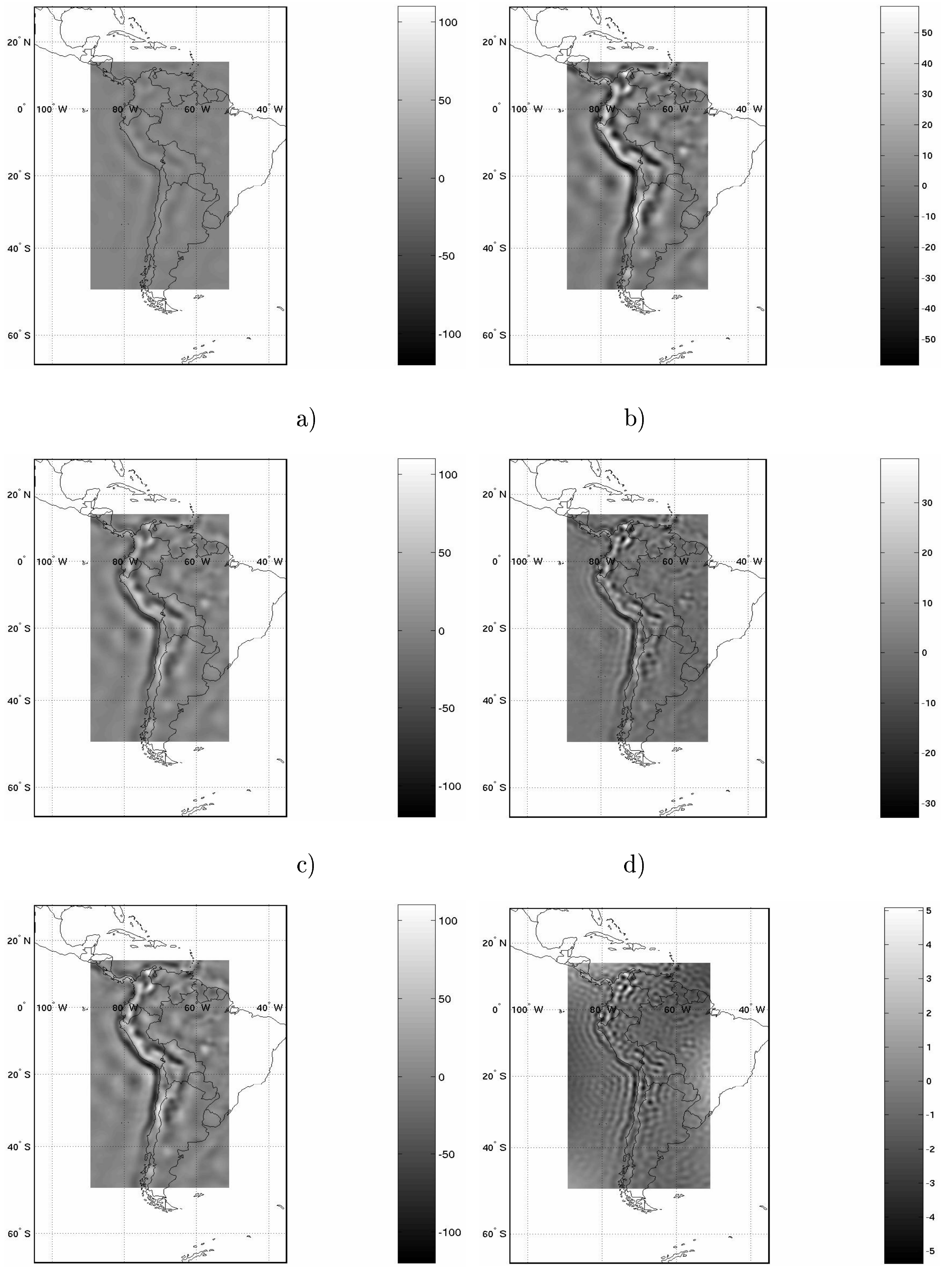


Figure 9.8: *a)* T_1G *b)* R_1G, *c)* T_2G, *d)* R_2G, *e)* T_3G, *and f)* R_3G *in* m^2s^{-2}.

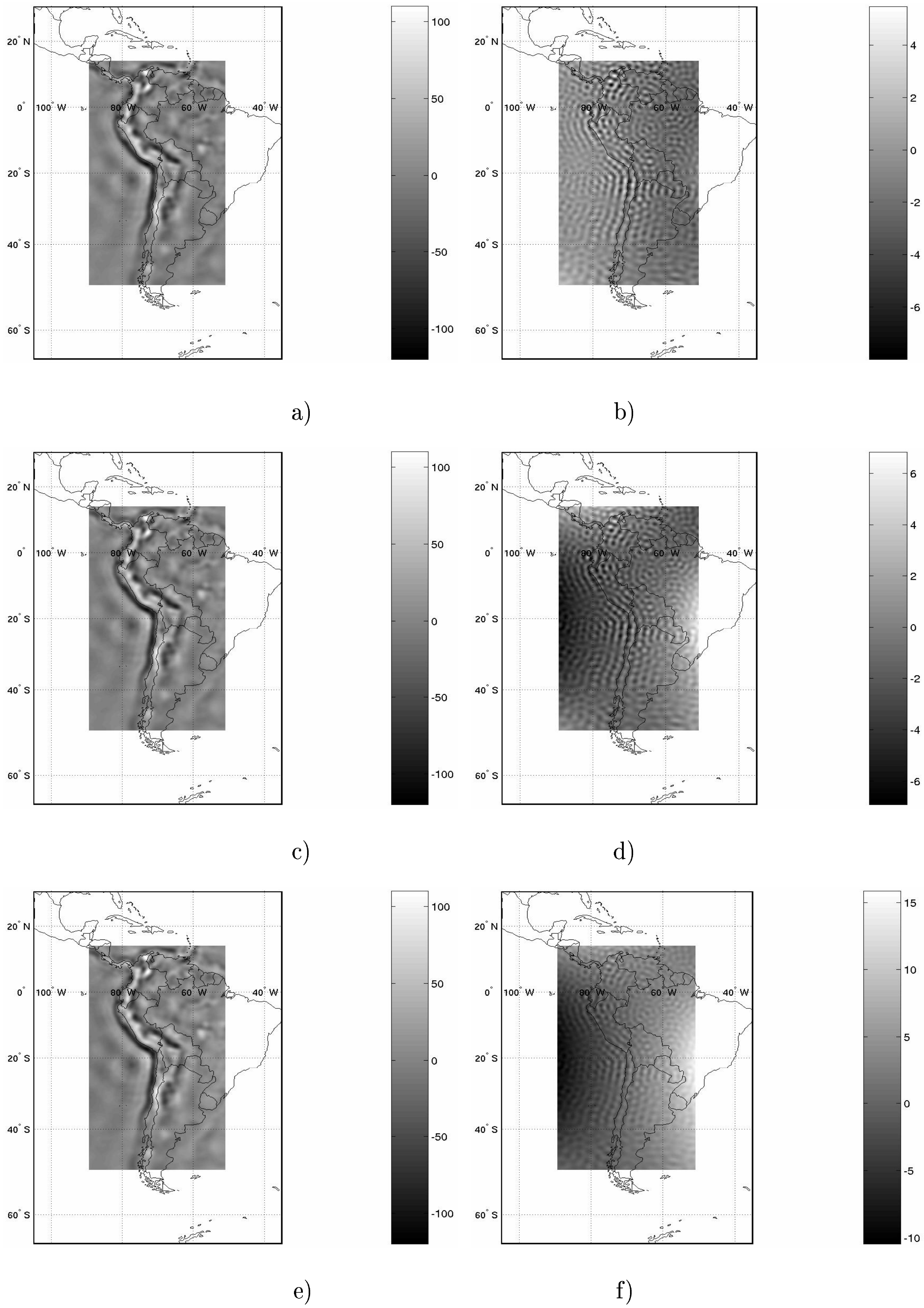


Figure 9.9: *a)* T_4G, *b)* $T_4G - F$, *c)* T_5G, *d)* $T_5G - F$, *e)* T_6G, *and e)* $T_6G - F$ *in* m^2s^{-2}.

Thus we compute the quantity α with the help of the spline coefficients of the smoothing splines for $\lambda_k := 10^{-k}$, $k = 15, \ldots, 19$. The results of this study are listed in Table 9.2 and are plotted in Figure 9.10. Figure 9.10 shows that for $\lambda = 10^{-18}$ the value of α is very close to the 'optimal' value $\alpha_{opt} = \tilde{\sigma} = 8.0572 \cdot 10^{-13}$.

Table 9.4: Discrepancy α in dependence of the smoothing parameter λ.

λ	α
10^{-15}	$9.09 \cdot 10^{-12}$
10^{-16}	$3.78 \cdot 10^{-12}$
10^{-17}	$1.53 \cdot 10^{-12}$
10^{-18}	$7.15 \cdot 10^{-13}$
10^{-19}	$5.69 \cdot 10^{-13}$

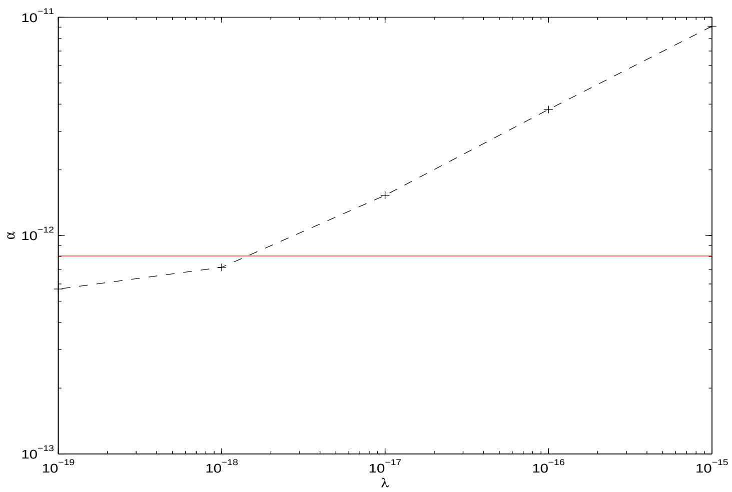


Figure 9.10: *Discrepancy α in dependence of the smoothing parameter λ. The horizontal line marks the value of the noise standard deviation $\sigma = 8.0572 \cdot 10^{-13}$.*

Conclusion and Recommendations for Future Investigations

The aim of this thesis is the study of certain methods for the modelling of the gravitational potential. Therefore, the methods were investigated with simulated SGG-data on a simplified spherical geometry on an (approximately) equidistributed point grid. However, the methods of spline interpolation and spline smoothing for the approximation of the signal G (second order radial derivative of the potential at the orbit) are not restricted to data on a sphere or any other regular surface but can cope with scattered data (i.e., data on a realistic orbit). The regularization with a Tikhonov scaling function is discretized by replacement of the right-hand side G (of the SGG-operator equation $\Lambda F = G$) by the spline, which immediately leads to a discretzation, which can be numerically calculated. The evaluation of this model of the potential demands more time when the points on the orbit or the points on the evaluation grid on the reference surface of the earth do not lie on a sphere, but this is also no real restriction of the advocated method to spherical geometries. Our numerical studies for spherical geometries yielded good numerical results, and the methods seem to be suitable for satellite-data on a realistic orbit.

The application of the methods in this thesis to the processing of satellite data on a realistic orbit seems especially interesting and promising for the following reasons: As already mentioned above the approximation of the satellite data, i.e., the first or second radial derivative of the potential at orbit height, was perfomed with the help of a spline, which can be calculated from scattered data. So there is no restriction to any special geometrical ordering of the data. It should be noted that the matrices that have to be solved for the calculation of the spline coefficients will probably be rather ill-conditioned if the data distribution is rather irregular. But even in this case there is always the possibility to stabilize the equation system by calculating a smoothing spline with a smoothing parameter chosen slightly greater than it is necessary due to the measurement noise. The other great advantage of the approximation of the signal by a spline is that the spline coefficients can be calculated with the Schwarz alternating procedure. The Schwarz alternating procedure splits the large equation system (for the calculation of the spline coefficients) into a number of smaller ones which are solved sucsessively in an iterative algorithm. Thus, a reduction of the runtime and of the memory requirement is achieved. This was confirmed by the numerical studies in this thesis. The implementation of the Schwarz alternating procedure for this thesis show how well it performs, but also how the implementation can be improved. A more sophisticated and much faster realization of the Schwarz alternating procedure will require a fast summation method for accelerated matrix vector multiplications. More precisely, the update is rather time consuming and this can be prevented if a so-called fast multipole method is used. Such methods are available for the singularity kernel (see [Gl2001]) and the spline interpolation can also be performed in the same way (with the Schwarz alternating procedure) using the singularity kernel. A sophisticated combination of the Schwarz alternating procedure with fast summation techniques can be expected to enable spline approximation and interpolation for huge data sets.

Finally, it should be noted that the discretization of the Tikhonov regularization as performed in this thesis, is only one possible way. Another variant of discretiztion is as follows: A Tikhonov regularization scaling function for $\mathcal{H}(\{1\}; h; \overline{\Omega_r^{ext}})$ is used for the regularization, where Ω_r is a sphere at mean orbit altitude. The convolutions are in this case simply $\mathcal{L}^2(\Omega_r)$-integrals and can be discretized with the help of a spherical integration formulas. The data of the signal G in this discretization is replaced by data of the spline which approximates G and which can easily be evaluated at the integration grid on the sphere Ω_r.

Bibliography

[AnBaBi1995] E. Anderson, Z. Bai, C. Bischof, J. Demmel, J. Dongarra, J. DuCroz, A. Greenbaum, S. Hammarling, A. McKenney, S. Ostrouchov, and D. Sorensen, *LAPACK User's Guide, Release 2.0, 2nd ed.*, SIAM Publications, Philadelphia, 1995.

[Ar1950] N. Aronszajn, *Theory of Reproducing Kernels*, Trans. Am. Math. Soc., 68 (1950), pp. 337-404.

[BeLiBi2000] R. K. Beatson, W. A. Light, S. Billings, *Fast Solution of the Radial Basis Function Interpolation Equations: Domain Decomposition Methods*, SIAM J. Sci. Comput., 22(5) (2000), pp. 1717-1740.

[DeHo1993] P. Deuflhard, A. Hohmann, *Numerische Mathematik I*, de Gruyter, Berlin, New York, 1993.

[EnHaNe1996] H. W. Engl, M. Hanke, and A. Neubauer, *Regularization of Inverse Problems*, Kluwer, Dordrecht, 1996.

[Fr1980] W. Freeden, *On the Approximation of the External Gravitational Potential with Closed Systems of (Trial) Functions*, Bull. Geod., 54 (1980), pp. 1-20.

[Fr1999] W. Freeden, *Multiscale Modelling of Spaceborne Geodata*, Teubner, Stuttgart, Leipzig, 1999.

[FrGeSchr1998] W. Freeden, T. Gervens, M. Schreiner, *Constructive Approximation on the Sphere (with Applications to Geomathematics)*, Clarendon Press, Oxford, 1998.

[FrSchn1997] W. Freeden, F. Schneider, *Runge-Walsh Wavelet Approximation for the Helmholtz Equation*, AGTM-Report No. 176, Geomathematics Group, University of Kaiserslautern, 1997.

[FrSchn1998] W. Freeden, F. Schneider, *Regularization Wavelets and Multiresolution*, Inverse Problems, 14 (1998), pp. 225-243.

[Gl2001] O. Glockner, *On Numerical Aspects of Gravitational Field Modelling from SST and SGG by Harmonic Splines and Wavelets (With Application to CHAMP Data)*, Ph.D. Thesis, University of Kaiserslautern, Geomathematics Group, 2001 (submitted).

[GoVLo1996] G. H. Golub, C. Van Loan, *Matrix Computations*, The Johns Hopkins University Press, Baltimore, London, 1996.

[Gu2002] M. Gutting, *Multiscale Gravitational Field Modelling from Oblique Derivatives*, Diploma Thesis, University of Kaiserslautern, Geomathematics Group, 2002 (in preparation).

[Ha1962] I. Halperin, *The Product of Projection Operators*, Acta Sci. Math. 23 (1962), pp. 96-99.

[HeMo1967] W. A. Heiskanen, H. Moritz, *Physical Geodesy*, W. H. Freeman and Company, 1967.

[He2002] K. Hesse, *Multiscale Gravitational Field Modelling Using Regularization, Projection and Domain Decomposition Methods (preliminary title)*, Ph.D. Thesis, University of Kaiserslautern, Geomathematics Group, 2002 (in preparation).

[Ki1996] A. Kirsch, *An Introduction into the Mathematical Theory of Inverse Problems*, App. Math. Sciences 120, Springer, New York, Berlin, Heidelberg, 1996.

[Kr1989] R. Kress, *Linear Integral Equations*, Springer, Berlin, Heidelberg, New York, 1989.

[LeKeFa1998] F. G. Lemoine, S.C. Kenyon, J. K. Factor, R. G. Trimmer, N. K. Pavlis, D. S. Chinn, C. M. Cox, S. M. Klosko, S. B. Luthcke, M. H. Torrence, Y. M. Wang, R. G. Williamson, E. C. Pavlis, R. H. Rapp, and T. R. Olson, *The Development of the Joint NASA GSFC and NIMA Geopotential Model EGM96*, NASA/TP-1998-206861, NASA Goddard Space Flight Center, Greenbelt, Maryland, USA, 1998.

[Mi2001] D. Michel, *On the Combination of Harmonic Splines and Fast Multipole Methods for CHAMP Data Modelling*, Diploma Thesis, University of Kaiserslautern, Geomathematics Group, 2001.

[Mik1970] S. G. Mikhlin, *An Advanced Course of Mathematical Physics*, North-Holland, Amsterdam, London, 1970.

[Mir1970] C. Miranda, *Partial Differential Equations of Elliptic Type*, Springer, New York, Heidelberg, Berlin, 1970.

[Mu1966] C. Müller, *Spherical Harmonics*, Lecture Notes in Mathematics 17, Springer, 1966.

[Ru1885] C. Runge, *Zur Theorie der eindeutigen analytischen Funktionen*, Acta Mathematica, 6 (1885), pp. 229-244.

[Schn1997] F. Schneider, *Inverse Problems in Satellite Geodesy and Their Approximate Solution by Splines and Wavelets*, Ph.D. Thesis, University of Kaiserslautern, Geomathematics Group, 1997.

[Schw1890] H. Schwarz, *Gesammelte mathematische Abhandlungen*, Vol. 2, Springer-Verlag Berlin, 1890 (pp. 133-134).

[SlWo1999] I. H. Sloan, R. S. Womersley, *How Good can Polynomial Interpolation on the Sphere be?*, Preprint, School of Mathematics, University of New South Wales, Sydney, Australia, 1999.

[SlWo2001] I. H. Sloan, R. S. Womersley, *Extremal Systems of Points and Numerical Integration on the Sphere*, Preprint, School of Mathematics, University of New South Wales, Sydney, Australia, 2001.

[Wa1990] G. Wahba, *Spline Models for Observational Data*, CBMS-NSF Regional Conference Series in Applied Mathematics, Vol. 59, SIAM, Philadelphia, 1990.

[Ya1950] H. Yamabe, *On an Extension of the Helly's Theorem*, Osaka Mathematical Journal, Vol. 2, No. 1 (1950), pp. 229-244.