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Some new aspects of Optimal Portfolios and Option Pricing

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Introduction

The main two problems of continuous-time financial mathematics are option pricing and portfolio optimization. The first of these problems is concerned with valuing derivative contracts on stocks (or other underlyings) which have a non-linear payoff structure such as all kind of options. The other important topic, portfolio optimization, consists of the search for the best investment strategy of an investor who is trading securities at a financial market.

In this thesis, various new aspects of the above major topics of financial mathematics will be discussed. In all our considerations we will assume the standard diffusion type setting for securitiy prices which is today well-know under the term "Black-Scholes model". This setting and the basic results of option pricing and portfolio optimization are surveyed in the first chapter.

The next three chapters deal with generalizations of the standard portfolio problem, also know as "Merton's problem". Here, we will always use the stochastic control approach as introduced in the seminal papers by Merton (1969, 1971, 1990). Although this approach is known for some time now, there are a lot of natural generalizations of the problem which are not treated in the literature.

One such problem is the very realistic setting of an investor who is faced with fixed monetary streams. More precisely, in addition to maximizing the utility from final wealth via choosing an investment strategy, the investor also has to fulfill certain consumption needs (such as paying a monthly rent) that can be deterministic or even stochastic.Also the opposite situation, an additional income stream (such as a payin scheme) can now be taken into account in our portfolio optimization problem. We consider various such examples and solve them on one hand via classical stochastic control methods (such as setting up a corresponding Hamilton- Jacobi-Bellman equation and proving a corresponding verification theorem (see Korn and Korn (2001)) and on the other hand show by means of a general separation theorem how the problem solution can be reduced to that of well-examined subproblems. This together with some numerical examples forms Chapter 2.

Chapter 3 is mainly concerned with the portfolio problem if the investor has different lending and borrowing rates. Even more, the borrowing rate depends on the percentage of his holdings which is already financed by a credit. Again, this is a very natural problem and is not yet treated in the literature in the form we consider. We give explicit solutions (where possible) and numerical methods to calculate the optimal strategy in the cases of log utility and HARA utility for three different modelling approaches of the dependence of the borrowing rate on the fraction of wealth financed by a credit.

A further generalization of the standard Merton problem consists in considering simultaneously the possibilities for continuous and discrete consumption (with respect to time). In our general approach there is a possibility for assigning the different consumption times different weights which is a generalization of the usual way of making them comparable via discounting. To solve this problem some new verification theorems have to be set up and have to be proved. Also, the martingale optimality principle of stochastic control (see Korn (2003)) proves to be very usefull in this chapter and is adapted to the special problems we are looking at. Again, all our findings are illustrated by some numerical examples.

The final two chapters of this thesis look at numerical methods for calculating option prices. Although, the option pricing problem in a complete market setting such as the one we are considering here is fully understood, there often remain numerical problems with the only remaining task, the computation of the expectation of the discounted final option payoff. Very often the payoff of so-called exotic options is highly complicated and can depend on the whole path of the underlying's price over the whole life time of the option. This makes it very difficult and sometimes impossible to have an explicit analytical formula for the option price. In such a situation, numerical methods are needed. Besides the classical candidates such as Monte Carlo simulation, tree methods or solving a corresponding partial differential equation, typically methods which are tailored to the exact specification of the option come into the game and prove to be efficient.

Chapter 5 deals with the special case of pricing basket options. Here, the main problem is not path-dependence but the multi-dimensionality which makes it impossible to give usuefull analytical representations of the option price. We review the literature and compare six different numerical methods in a systematic way. Thereby we also look at the influence of various parameters such as strike, correlation, forwards or volatilities on the performance of the different numerical methods.

The problem of pricing Asian options on average spot with average strike is the topic of Chapter 6. We here apply the bivariate normal distribution to obtain an approximate option price. This method proves to be very reliable and efficient for the valuation of different variants of Asian options on average spot with average strike.

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1 Preliminaries

1.1 The Economy and Some Basic Definitions

1.1.1 Introduction

In this section we introduce the underlying economy, modeled by a stock market and a money-market account. This economy will be used in this doctoral thesis for both option pricing and portfolio optimization. It is based on variants of the well-known lognormal model.

1.1.2 The Model

We consider a security market consisting of an interest-bearing cash account and n risky assets. The uncertainty is modeled by a probability space $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbf{P}\right)$. The time period is the finite interval $[0,T]$. The flow of information is given by the natural filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$, i.e. the $\mathbf{P}$-augmention of an (independent) n-dimensional Brownian filtration. Without loss of generality we set $\mathcal{F}_T = \mathcal{F}$, so that all observable events are known. All traders are assumed to be price takers, and there are no transaction costs. The cash account is modeled by the differential equation

$$dB(t) = B(t)r(t)dt, \tag{1.1}$$

where $r(t)$ is a bounded, positive and progressively measurable process. The price process of the i-th $(i = 1, \ldots, n)$ risky asset is given by

$$dS_i(t) = S_i(t)\left[(b_i(t) - d_i(t))dt + \sum_{j=1}^{n} \sigma_{ij}(t)dW_j(t)\right] \tag{1.2}$$

with

$$b(t) = (b_1(t), \ldots, b_n(t))', \quad d(t) = (d_1(t), \ldots, d_n(t))', \tag{1.3}$$

denoting the drift vector and dividend-yield vector, and

$$\sigma(\mathbf{t}) = \begin{pmatrix} \sigma_{11}(t) & \cdots & \sigma_{1n}(t) \\ \vdots & \ddots & \\ \sigma_{n1}(t) & \cdots & \sigma_{nn}(t) \end{pmatrix} \tag{1.4}$$

the volatility matrix. Let $W(t)$ be an n-dimensional Brownian motion, where the individual Brownian motions are independent. The coefficients $b_i(t)$,$d_i(t)$ and $\sigma_{ij}(t)$ are assumed to be bounded, progressively $\mathcal{F}_t$-measurable processes. The dividend yields are assumed to be nonnegative, that means $d_i(t) \geq 0$ for all $t \in [0,T]$ and $i = 1, \ldots, n$. In addition $\sigma\sigma'$ has to be a strictly positive definite $n \times n$-matrix, i.e. it exists some constant $K > 0$ with $x'\sigma(t)\sigma'(t)x \geq Kx'x$ for all $x \in I\!R^n$ and for all $t \in [0,T]$ $\mathbf{P}$–a.s..

In the following we will present the corresponding definitions of trading strategies and wealth processes, which are used in portfolio optimization and option pricing.

Definition 1.1

i) A **trading strategy** *φ is a $I\!R^{n+1}$-valued, $\{\mathcal{F}_s\}_{s\in[0,T]}$ -progressively measurable process*

$$\varphi(t) := (\varphi_0(t), \varphi_1(t), \ldots, \varphi_n(t))'$$

with

$$\int_0^T |\varphi_0(s)B(s)|ds < \infty \quad \mathbf{P}\text{–}a.s.,$$

$$\sum_{j=1}^n \int_0^T (\varphi_i(s)S_i(s))^2\, ds < \infty \quad \mathbf{P}\text{–}a.s., \; for\ i = 1, \ldots, n.$$

The value $x_0 := \varphi_0(0)B(0) + \sum_{i=1}^n \varphi_i(0)S_i(0)$ is called initial wealth.

ii) Let φ be a trading strategy with initial wealth $x_0 > 0$. The process

$$X(t) := \varphi_0(t)B(t) + \sum_{i=1}^n \varphi_i(t)S_i(t)$$

is called **wealth process** *corresponding to φ with initial wealth x_0.*

iii) A nonnegative, $\{\mathcal{F}_s\}_{s\in[0,T]}$ -progressively measurable, real-valued process $c(s)$, $s \in [0,T]$ with

$$\int_0^T c(s)ds < \infty \qquad \mathbf{P}\text{–}a.s.$$

is called **consumption process**.

Remark: The restrictions in Definition i) and iii) ensure, that the Itô-integral of the corresponding wealth process in Definition 1.2 below is well defined, i.e. in $H^2[0,T]$ (see Korn&Korn (2001) for a detailed definition).

Definition 1.2
A pair (φ, c) consisting of a trading strategy φ and a consumption process c is called **self-financing**, *if the associated wealth process $X(t)$ satisfies $\mathbf{P}$–a.s.:*

$$\begin{aligned} X(t) &= x_0 + \int_0^t \varphi_0(s)dB(s) + \sum_{i=1}^n \int_0^t \varphi_i(s)dS_i(s) \\ &\quad + \sum_{i=1}^n \int_0^t \varphi_i(s)d_i(s)S_i(s)ds - \int_0^t c(s)ds \end{aligned}$$

Remark: Self-financing means, that no fresh capital is added to the wealth process during the life-time, and that all consumption is financed via capital withdrawing from the rebalanced portfolio with inital capital x_0. The reason for the 4.th summand is that we assume that dividens are reinvested; this will be discussed in more detail later on.
We now introduce the process of fractions of wealth invested in the stocks, the portfolio process, as another way of describing an investor's trading activities.

Definition 1.3
Let $X(t) \equiv x$ be the wealth of an investor at time t:

i) *Consider a pair (π, c) where $c(t)$ is a consumption process and $\pi(t)$ a progressively measurable process with*

$$\pi(s) := (\pi_1(s), \ldots, \pi_n(s))'$$

satisfying the restriction $\int_t^T \pi_i^2(s)ds < \infty$ $\mathbf{P}$–a.s. for all $i \in \{1, \ldots, n\}$. Then $\pi(.)$ is called a **self-financing portfolio process**, *if $\pi_i(s)$ is the percentage of total wealth invested in the stock S_i at time s and if the corresponding wealth process $X^\pi(s)$ is generated by a self-financing trading strategy φ via*

$$X^\pi(s)\pi_i(s) = \varphi_i(s)S_i(s)$$

for all $i \in \{1, \ldots, n\}$. This set is denoted by $\mathcal{S}(\mathbf{t}, \mathbf{x})$.

ii) *The set*

$$\mathcal{A}^+(\mathbf{t}, \mathbf{x}) = \{(\pi, c) \in \mathcal{S}(t, x) | X(s) \geq 0 \;\; \mathbf{P}\text{–}a.s. \forall s \in [t, T]\}$$

is said to be the set of **admissible portfolio-consumption processes**.

Remark 1.4

i) The percentage of wealth invested in the saving accounts $\pi_0(t)$ is given by $\pi_0(t) = 1 - \pi'(t)\underline{1}$, where $\underline{1} = (1, \ldots, 1)'$. If $\sum_{i=1}^n \pi_i > 1$ the investor is actually borrowing money, which is explicitly admitted.

ii) The condition $\int_t^T \pi_i^2(s)ds < \infty$ $\mathbf{P}$–a.s. ensures (by Theorem 1.5), that the equation (1.6) defining the wealth process has a unique solution.

iii) The addional condition $X(s) \geq 0$ $\mathbf{P}$–a.s. $\forall s \in [t, T]$ in $\mathcal{A}^+(\mathbf{t}, \mathbf{x})$ guarantees, that the investor never ends up in debts. For these strategies the pairs (φ, c) and $(\pi, c) \in \mathcal{A}^+(t, x)$ are equivalent descriptions of the investor's actions. However using the portfolio process enables us deriving a nice differential equation for the evolution

of the wealth process.
Let (φ, c) be a self-financing strategy with $X(t) > 0 \, \forall t \in [0,T]$, then:

$$\begin{aligned} dX(t) &= \varphi_0(t)dB(t) + \sum_{i=1}^{n} \varphi_i(t)dS_i(t) + \sum_{i=1}^{n} \varphi_i(t)d_i(t)S_i(t)dt - c(t)dt \\ &= \varphi_0 B(t)r(t)dt + \sum_{i=1}^{n} \varphi_i(t)S_i(t)\left(b_i(t)dt + \sum_{j=1}^{n} \sigma_{ij}(t)dW_j(t)\right) \\ &\quad -c(t)dt \\ &= (1-\pi(t)'\underline{1})X(t)r(t)dt \\ &\quad + \sum_{i=1}^{n} X(t)\pi_i(t)\left(b_i(t)dt + \sum_{j=1}^{n} \sigma_{ij}(t)dW_j(t)\right) - c(t)dt \\ &= X(t)\left[r(t)dt + \pi'(t)(b(t) - \underline{1}r(t))dt + \pi'(t)\sigma(t)dW(t)\right] - c(t)dt \end{aligned}$$

iv) A wealth process $X^{\pi,c}$ written in terms of a fixed self-financing trading strategy $(\pi, c) \in \mathcal{A}^+(\mathbf{t}, \mathbf{x})$ with reinvested dividends has exactly the same form as a corresponding wealth process without dividends. If the dividends are consumed, the drift of the wealth process changes to $b(t) - d(t)$. That means, if we don't use the dividends for additional consumption, but reinvest them in the wealth process and keep the strategy π, they do not affect the wealth process. Nevertheless, dividends have of course to be taken into account, when the stock drifts are estimated.

The following theorem ensures the existence, uniqueness and the explicit form of the solution of the wealth equation (1.6) and will also prove to be very useful throughout this dissertation.

Theorem 1.5 (Variation of Constants)
Let $W(t)$ be a n-dimensional Brownian motion. Let $x \in I\!R$ and A, a, S_j, σ_j be progressively measurable, real-valued processes with

$$\int_0^t (|A(s) + |a(s)|) \, ds < \infty \text{ for all } t \geq 0 \;\; \mathbf{P}\text{–a.s.}$$

$$\int_0^t (|S_j(s) + |\sigma_j(s)|) \, ds < \infty \text{ for all } t \geq 0 \;\; \mathbf{P}\text{–a.s.}$$

Then the stochastic differential equation

$$\begin{aligned} dX(t) &= (A(t)X(t) + a(t))\,dt + \sum_{j=1}^{n} (S_j(t)X(t) + \sigma_j(t))\,dW_j(t) \\ X(0) &= x \end{aligned}$$

possesses the unique solution $\{(X(t), \mathcal{F}_t)_{t\geq 0}$ *given by*

$$X(t) = Z(t)\left(x + \int_0^t \frac{1}{Z(s)}\left(a(s) - \sum_{j=1}^n S_j(s)\sigma_j(s)\right)ds + \sum_{j=1}^n \int_0^t \frac{\sigma_j(s)}{Z(s)}dW_j(s)\right) \quad (1.5)$$

where

$$Z(t) = \exp\left(\int_0^t \left(A(s) - \frac{1}{2}||S(s)||^2\right)ds + \int_0^t S(s)dW(s)\right)$$

is the unique solution of the homogeneous equation

$$\begin{aligned} dZ(t) &= Z(t)(A(t)dt + S(t)'dW(t)) \\ Z(0) &= 1. \end{aligned}$$

Proof.See Korn&Korn (2001), Theorem 2.54 □

We sum up our considerations in the following theorem:

Corollary 1.6 (Wealth process)
The wealth process $X^{(\pi,c)}(t)$ *for* $(\pi, c) \in \mathcal{A}^+(\mathbf{t}, \mathbf{x})$ *is well-defined.*
If the dividends are (immediately) consumed its evolution is described by

$$dX(t) = \left[X(t)\left(r(t) + \pi'(t)(b(t) - d(t) - r(t)\underline{1})\right) - c(t)\right]dt + X(t)\pi'(t)\sigma(t)dW(t)$$

If the dividends are (immediately) reinvested the wealth equation has the following form:

$$dX(t) = \left[X(t)\left(r(t) + \pi'(t)(b(t) - r(t)\underline{1})\right) - c(t)\right]dt + X(t)\pi'(t)\sigma(t)dW(t) \quad (1.6)$$

The explicit solution of (1.6) is given by :

$$\begin{aligned} X(t) &= Z(t)\left(x_0 + \int_0^t \frac{-c(s)}{Z(s)}ds\right) \\ Z(t) &= \exp\left(\int_0^t \left(r(s) + \pi'(s)(b(s) - r(s)\underline{1}) - \frac{1}{2}\pi'(s)\sigma(s)\sigma'(s)\pi(s)\right)ds \right. \\ &\qquad \left. + \int_0^t \pi'(s)\sigma(s)dW(s)\right) \end{aligned}$$

Proof.

i) The processes $b(.)$,$d(.)$,$\sigma(.)$ and $r(.)$ are bounded by definition. In addition we have $\int_0^t \pi_i(s)ds < \infty$, $\int_0^t \pi_i(s)\pi_j(s)ds < \infty \forall i, j \in \{1, \ldots, n\}$ by $\int_0^T \pi_i^2(s)ds < \infty$ and $\int_0^t c(s)ds < \infty$ ($\mathbf{P}$–a.s.in each case). The process $X(t)$ is continuous, hence (pathwise) bounded on [0,T]. Hence

$$\int_0^t X(s)\left(r(s) + \pi'(s)(b(s) - r(s)\underline{1}) - \frac{1}{2}\pi'(s)\sigma(s)\sigma'(s)\pi(s)\right)ds < \infty \ \ \mathbf{P}\text{–a.s.}$$

for all $t \in [0,T]$, if we also take into accout that $\sigma(s)\sigma(s)'$ is uniformely positive definite. A similar reasoning yields $\int_0^t X^2(s)\pi'(s)\sigma(s)\sigma'(s)\pi(s)ds < \infty$. So the Itô-process $X(t)$ is well-defined.

ii) The explicit solution is an application of the Variation of Constants Theorem 1.5.

□

Remark: In this thesis we generally assume, that the dividends are reinvested in both cases of portfolio optimization as well as option pricing.

Remark 1.7 (Alternative representation of stock processes)
An alternative - seemingly easier interpretable - representation of the price processes could be

$$d\widetilde{S}_i(t) = \widetilde{S}_i(t)\left[(b_i(t) - d_i(t))dt + \widetilde{\sigma}_i(t)d\widetilde{W}_i(t)\right], \tag{1.7}$$

whereby $\widetilde{W}(t)$ is a n-dimensional Brownian motion with some strictly positive definite correlation matrix $\rho(t)$ (otherwise arbitrage opportunities may occur), i.e. the one-dimensional Brownian motions $\widetilde{W}_i$ and $\widetilde{W}_j$ have a correlation of $\rho_{ij}(t)$ at time t. The strictly positive definiteness ensures the existence of a particular upper triangular matrix $A(t)$, such that we have

$$A(t)A(t)^T = \rho(t).$$

This matrix can be easily determined by Cholesky-decomposition. Observe, that with

$$\widetilde{\sigma}(t) = \begin{pmatrix} \widetilde{\sigma}_1(t) & \cdots & 0 \\ \vdots & \ddots & \\ 0 & \cdots & \widetilde{\sigma}_n(t) \end{pmatrix}$$

and $(W_1, \ldots, W_n)$ independent Brownian motions we obtain

$$\begin{pmatrix} \int_0^t \tilde{\sigma}_1(s)d\widetilde{W}_1(s) \\ \vdots \\ \int_0^t \tilde{\sigma}_n(s)d\widetilde{W}_n(s) \end{pmatrix}$$

$$\stackrel{d}{=} N\left(\begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} \int_0^t \tilde{\sigma}_1^2(s)ds & \cdots & \int_0^t \tilde{\sigma}_1(s)\tilde{\sigma}_n(s)\rho_{1n}(s)ds \\ \vdots & \ddots & \vdots \\ \int_0^t \tilde{\sigma}_n(s)\tilde{\sigma}_1(s)\rho_{n1}(s)ds & \cdots & \int_0^t \tilde{\sigma}_n^2(s)ds \end{pmatrix}\right)$$

$$= N\left(\begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}, \int_0^t \widetilde{\sigma}(t)A(s)A^T(s)\widetilde{\sigma}(t)ds\right)$$

$$\stackrel{d}{=} \int_0^t \begin{pmatrix} \widetilde{A}_{11}(s) & \cdots & \widetilde{A}_{1n}(s) \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \widetilde{A}_{nn}(s) \end{pmatrix} \begin{pmatrix} dW_1(s) \\ \vdots \\ dW_n(s) \end{pmatrix}$$

with $\widetilde{A} = \widetilde{\sigma}A$, e.g. $\widetilde{A}_{ij} = \tilde{\sigma}_i A_{ij}$, for $j = 1, \ldots, n$ and $i = 1, \ldots, n$.
So the price processes (1.7) are equivalent to (1.2) for the choice of $\sigma = \widetilde{A}$. For a rigorous proof we refer to Björk[98] Proposition 3.19. For convenience in this thesis, the representation (1.2) is used for portfolio optimization and the representation (1.7) for option pricing.

1.2 Portfolio Optimization

1.2.1 Introduction

The portfolio and consumption problem is a well-studied problem in mathematical finance. It is concerned with optimal use of an inital capital of x for consumption ("living well") and investment ("getting rich") and is thus maybe the most natural task of an investor. The first mathematical approach to portfolio optimization was proposed by H. Markowitz (1952). He used a one-period setting in which he balanced the return and the risk of the portfolio. More exactly, at the initial time the parts of wealth invested in the particular (risky) assets are chosen, such that the mean and the variance of the return on the total wealth have the best possible fit to the investor's preferences. Due to its simplicity and plausibility it became popular in theory and practice. Even today it is widely used. However, this model has the drawback, that it is not able to react on the future market behavior, as it considers only buy-and-hold strategies.

In the late 1960's Merton (1969,1971) introduced a continuous-time model which incorporated this desired reaction feature. Research on this area continued in the 1990's, (see e.g. Karatzas and Shreve (1998), Korn (1997), Merton (1990)) accompanied by the growing popularity of option pricing. Today, the main challenges are the consideration of market imperfections like crashes, or generally not hedgeable risks, transactions costs, illiquidity or interest rate risk. The theory of portfolio optimization is mainly used by insurance companies and institutional investors to manage their portfolios.

1.2.2 The Model

In this thesis we consider continuous-time portfolio optimization based on the seminal papers by Merton (see Merton (1969,1971)). The market model and the corresponding trading strategies and portfolio processes are already described in Section 1. In Merton's framework portfolio optimization consists of maximising the expected utility from terminal wealth and/or consumption until the time horizon of an investor who is endowed with a fixed initial capital. More precisely, the portfolio optimization problem of an investor is about the determination of an optimal investment strategy in the market securities. "Optimal" means, that expected utility is maximised by choosing among admissible strategies, a task which will be made more precise shortly. The investor starts with an initial wealth of $x > 0$ at time $t = 0$. In the beginning this initial wealth is invested in different assets and he is allowed to adjust his holdings continuously up to a fixed planning horizon T. During the whole period $[0, T]$, parts of the wealth can be consumed to realise utility. We are considering self-financing portfolio processes. This leads us to the following definition:

Definition 1.8 (General Portfolio Problem)
The problem

$$V(t,x) := \sup_{(\pi,c)\in\mathcal{A}(t,x)} J(t,x;(\pi,c)) \tag{1.8}$$

with wealth process

$$dX^{\pi,c}(t) = X^{\pi,c}(t)\ \left[\big(r(t)+\pi'(t)(b(t)-r(t)\underline{1})\big)dt - \pi'(t)\sigma(t)dW(t)\right] - c(t)dt \tag{1.9}$$

$$X(0) = x_0$$

where $b(t), r(t), \sigma(t)$ are bounded, $\mathcal{F}_t$-progressively measurable processes defined in equations (1.1) - (1.4) and

$$J(t,x;(\pi,c)) = E^{t,x}\left[\int_t^T U_1(s,c(s))ds + U_2(X^{\pi,c}(T))\right] \tag{1.10}$$

$$\mathcal{A}(t,x) = \left\{(\pi,c)\in\mathcal{A}^+(t,x)\middle| E^{t,x}\left[\int_t^T U_1^-(s,c(s))ds + U_2^-(X^{\pi,c}(T))\right] < \infty\right\} \tag{1.11}$$

is called the continuous portfolio optimization problem.
$V(t,x)$ is called the value function, $J(t,x;(\pi,c))$ the utility functional, U_1, U_2 utility functions and $\mathcal{A}(t,x)$ is the set of admissible portfolio processes.

Definition 1.9 (Utility function)

i) A function $U : (0,\infty) \to I\!R$ which is strictly concave, continuously differentiable, and satisfies

$$U'(0) := \lim_{x\downarrow 0} U'(x) = +\infty \quad \text{and} \quad U'(\infty) := \lim_{x\to\infty} U'(x) = 0$$

is called utility function.

ii) Let $U : [0,T]\times(0,\infty) \to I\!R$ be continuous, such that for all $t\in[0,T]$ the function $U(t,\cdot)$ is a utility function in terms of i). Then U is also called utility function.

The following theorem enables us to prove some boundedness conditions needed later on.

Theorem 1.10 (Existence and uniqueness of solutions of SDEs)
Let

$$\begin{aligned} dX(t) &= b(t,X(t))dt + \sigma(t,X(t))dW(t) \\ X(0) &= x \end{aligned} \tag{1.12}$$

describe an n-dimensional stochastic process with

$$b : [0,\infty)\times I\!R^b \to I\!R^n, \quad \sigma : [0,\infty)\times I\!R^b \to I\!R^{n,n}$$

Let the coefficients $b(t,x)$, $\sigma(t,x)$ *of the stochastic differential equation (1.12) be continuous functions with*

$$||b(t,x) - b(t,y)|| + ||\sigma(t,x) - \sigma(t,y)|| \leq K||x-y|| \quad (1.13)$$

$$||b(t,x)||^2 + ||\sigma(t,x)||^2 \leq K^2(1+||x||^2) \quad (1.14)$$

for all $t \geq 0$, $\quad x, y \in I\!R^n$ *and a constant* $K > 0$ *(where* $||.||$ *denotes the Euclidean norm of suitable dimension). Then there exists a continuous strong solution* $\{(X(t), \mathcal{F}_t)\}_{t\in[0,T]}$ *of equation (1.12) satisfying*

$$E\left(||X(t)||^2\right) \leq C\left(1+||x||^2\right) \text{ for all } t \in [0,T]$$

for some constant $C = C(K,T)$ *and* $T > 0$*. Further, X(t) is unique up to indistinguishability.*

Proof. See Korn&Korn (2001), Theorem 3.22 □

Proposition 1.11
Under the conditions of Theorem 1.10 the solution $X(t)$ *of the stochastic differential equation satisfies for* $m \geq 1$

$$E\left(\max_{0\leq s\leq t} ||X(s)||^{2m}\right) \leq C\left(1+||x||^{2m}\right)e^{Ct}$$

for all $t \in [0,T]$ *and a suitable constant* $C = C(T,K,m,d)$*, where* $T \geq 0$ *is a fixed constant.*

Proof. See Korn&Korn (2001), Lemma 3.23 □

Remark: Only in very simple cases, e.g. log-utility, the optimisation problem can be solved by simple straight-forward algebra. One approach to solve the optimization problem is the martingale method (not to mix up with *Martinale Optimality principle* introduced in the next theorem). Thereby the optimization is separated into a static problem, namely the determination of the optimal payoff profile, and a representation problem, i.e. the computation of the portfolio process corresponding to the optimal payoff profile. The drawback of this method is that it cannot be applied to incomplete market and it contains some "contra-intuitive" and "inconvenient" transformations. The most widely used approach is therefore the *Hamilton-Jacobi-Bellman framework*, where the problem is reduced to the determination of the solution of a PDE. To apply it, we need further assumptions, which are in practice however no restrictions.

1.2.3 The Hamilton Jacobi Bellman - Theorem

Definition 1.12 (Standard HJB Portfolio Problem)
The problem

$$V(t,x) := \sup_{(\pi,c)\in\mathcal{A}^{\mathcal{H}}(t,x)} J(t,x;(\pi,c)) \tag{1.15}$$

$$J(t,x;(\pi,c)) = E^{t,x}\left[\int_t^T U_1(s,c(s))ds + U_2(X^{\pi,c}(T))\right] \tag{1.16}$$

with wealth process

$$dX^{\pi,c}(t) = X^{\pi,c}(t)\ \left[\left(r(t)+\pi'(t)(b(t)-r(t)\underline{1})\right)dt - \pi'(t)\sigma(t)dW(t)\right] - c(t)dt \tag{1.17}$$

$$X(0) = x_0$$

is called **Hamilton Jacobi Bellman optimization problem** *if the following conditions are fulfilled:*

i) The market coefficients $b(t), r(t), \sigma(t)$ are at most functions of time, wealth and the control:

$$b(s) := F^b(s, X(s), (\pi,c)(s)), \tag{1.18}$$

$$r(s) := F^r(s, X(s), (\pi,c)(s)), \tag{1.19}$$

$$\sigma(s) := F^\sigma(s, X(s), (\pi,c)(s)), \tag{1.20}$$

with

$$F^b(s,\mathbf{x},(\pi,c)) \in C^b([0,T]\times I\!R\times I\!R^{n+1}, I\!R^n), \tag{1.21}$$

$$F^r(s,\mathbf{x},(\pi,c))) \in C^b([0,T]\times I\!R\times I\!R^{n+1}, I\!R), \tag{1.22}$$

$$F^\sigma(s,\mathbf{x},(\pi,c))) \in C^b([0,T]\times I\!R\times I\!R^{n+1}, I\!R^{n\times n}), \tag{1.23}$$

whereby C^b denotes the appropriate set of continuous and bounded functions.

ii) The utility function $U_1(t,c)$ and $U_2(x)$ are restricted by

$$U_1(s,c) \leq C\left(1+|c|^k\right) \text{ for all } s\in[0,T], c\in I\!R, \tag{1.24}$$

$$U_2(x) \leq C\left(1+|x|^k\right) \text{ for all } x\in I\!R, \tag{1.25}$$

for some real constant C and integer k.

iii) The new set of admissible controls is given by:

$$\begin{aligned}\mathcal{A}^{\mathcal{H}}(t,x) &= \Bigg\{(\pi,c)\in\mathcal{A}^{+}(t,x)\Bigg| E\left(\int_t^T |(\pi(s),c(s))|^k ds\right)<\infty\,, \\ &\qquad E\left(\sup_{s\in[t,T]}|X^{\pi,c}(s)|^k ds\right)<\infty, \forall k\in\mathbb{N}, \\ &\qquad \exists k_1,k_2 : c(t,X^{\pi,c}(t))\le k_1+k_2X^{\pi,c}(t)\Bigg\}\end{aligned} \tag{1.26}$$

Remark: Note, that $\mathcal{A}^{\mathcal{H}}\subset\mathcal{A}$, since $U_1(s,c)$ and $U_2(x)$ are restricted by inequalities (1.24,1.25) and the conditions on $\pi(s), c(s)$ and $X^{\pi,c}(s)$ as defined in $\mathcal{A}^{\mathcal{H}}$. Thus, the coefficients of the wealth process are slightly restricted, in particular, they cannot be dependent from the paths of stocks anymore.
We now state a very useful principle for proving optimality of a control strategy.

Theorem 1.13 (The Martingale Optimality Principle)
Let (π^*,c^*) be an admissible control. We denote its corresponding utility functional by

$$G(t,x)=E^{t,x}\left[\int_t^T U_1(s,c^*(s))ds+U_2(X^{\pi^*,c^*}(T))\right]$$

Furthermore, consider

$$w^{\pi,c,t,x}(\theta) = \int_t^\theta U_1(s,c(s))ds+G(\theta,X^{\pi,c}(\theta))\,,\ X^{\pi,c}(t)=x$$

If $w^{\pi,c,t,x}(\theta)$ is a supermartingale for all $(\pi,c)\in\mathcal{A}(t,x)$, then (π^*,c^*) is indeed the optimal control, i.e. we have

$$G(t,x)=\sup_{(\pi,c)\in\mathcal{A}^{\mathcal{H}}(t,x)} E^{t,x}\left[\int_t^T U_1(s,c(s))+U_2(X^{\pi,c}(T))\right]=V(t,x)$$

Remark: Observe that we get $E^{t,x}\left[w^{\pi^*,c^*,t,x}(\theta)\right] = G(t,x)$ for all $\theta\in[t,T]$, thus $w^{\pi^*,c^*,t,x}(\theta)$ is a martingale.

PROOF. Let (π, c) be an arbitrary admissible control. Then our assumptions leads to:

$$\begin{aligned} E^{t,x}\left[\int_t^T U_1(s,c(s))ds + U_2(X^{\pi,c}(T))\right] &= E^{t,x}\left[\int_t^T U_1(s,c(s))ds + G(T,X^{\pi,c}(T))\right] \\ &= E^{t,x}[w^{\pi,c,t,x}(T)] \\ &\leq E^{t,x}[w^{\pi,c,t,x}(t)] \\ &= G(t,x) \\ &= E^{t,x}\left[\int_t^T U_1(s,c^*(s))ds + U_2(X^{\pi^*,c^*}(T))\right] \end{aligned}$$

□

Further Notations:

$$\begin{aligned} Q &= [t,T) \times I\!R && \text{(inner domain of value function)} \\ \partial Q &= T \times I\!R && \\ U^\pi &\subset I\!R \quad \text{compact} && \text{(domain of portfolio process)} \\ U &= U^\pi \times I\!R && \text{(domain of portfolio and consumption process)} \end{aligned}$$

For $G \in C^{1,2}(Q)$,$(t,x) \in Q$, $(\hat{\pi},\hat{c}) \in U$ let:

$$\mathcal{A}^{(\hat{\pi},\hat{c})} := \frac{\partial}{\partial t} + \frac{1}{2}x^2\hat{\pi}'\sigma\sigma'\hat{\pi}\frac{\partial^2}{\partial x^2} + \left[x(r + \hat{\pi}'(b-r)\underline{1}) - \hat{c}\right]\frac{\partial}{\partial x}$$

Remark: We can interpret the following HJB-theorem as a corrolary to the Martingale Optimality principle.

Theorem 1.14 (The Standard HJB Verification Theorem)
Let

$$G \in C^{1,2}(Q) \cap C(\bar{Q}) \text{ with } |G(t,x)| \leq K(1+|x|^k) \tag{1.27}$$

for some suitable constants $K > 0, k \in I\!N$, be a solution to the Hamilton-Jacobi-Bellman equation

$$\sup_{(\hat{\pi},\hat{c})\in U} \left\{A^{(\hat{\pi},\hat{c})}G(t,x) + U_1(t,\hat{c})\right\} = 0 \quad , \quad (t,x) \in Q, \tag{1.28}$$

$$G(T,x) = U_2(x) \quad , \quad x \in I\!R^n. \tag{1.29}$$

Then we achieve:

a) $G(t,x) \geq J(t,x;(\pi,c))$ *for all* $(t,x) \in Q$ *and* $(\hat{\pi},\hat{c})(.) \in \mathcal{A}^{\mathcal{H}}(t,x)$.

b) *If for all* $(t,x) \in Q$ *there exists a* $(\pi,c)^*(.) \in \mathcal{A}^{\mathcal{H}}(t,x)$ *with*

$$(\pi,c)^*(s) \in \arg\max_{(\hat{\pi},\hat{c})\in U} \left(A^{(\hat{\pi},\hat{c})} G(s, X^{(\pi,c)^*}(s)) + U_1(s,\hat{c}(s)) \right)$$

for all $s \in [t,T]$, *where* $X^{(\pi,c)^*}(s)$ *is the controlled process corresponding to* $(\pi,c)^*$ *via equation1.9), we obtain:*

$$G(t,x) = V(t,x) = J(t,x;(\pi,c)^*)$$

In particular, $(\pi,c)^*$ *is an optimal control and* $G(t,x)$ *coincides with the value function.*

Proof.
Let

$$w^{\pi,c,t,x}(\theta) \;=\; \int_t^\theta U_1(s,c(s))ds + G(\theta, X^{\pi,c}(\theta))\,,\; X^{\pi,c}(t) = x$$

we will show, that $w^{\pi,c,t,x}(\theta)$ is a supermartingale for all $(\pi,c) \in \mathcal{A}^{\mathcal{H}}(t,x)$. Then $(\pi,c)^*$ is indeed the optimal control, i.e. we have

$$G(t,x) = \sup_{(\pi,c)\in\mathcal{A}^{\mathcal{H}}(t,x)} E^{t,x}\left[\int_t^T U_1(s,c(s)) + U_2(X^{\pi,c}(T))\right].$$

We will do this in two steps:

i) Let us first consider an auxiliary problem where we are only considering a bounded domain for the wealth process $X^{\pi,c}(.)$. This is achieved by suitably stopping the process. Let therefore:

$$O_p \;=\; \{x \in I\!R \big| |x| < p\},$$

$$\tau_{\theta,p} \;=\; \inf\{s \geq t \big| (s,X(s)) \notin [t,\theta]\times O_p\}.$$

Hence the SDE for $w^{\pi,c,t,x}(.)$ reads as

$$w^{\pi,c,t,x}(\tau_{\theta,p}) \;=\; w^{\pi,c,t,x}(t) + \int_t^{\tau_{\theta,p}} \mathcal{A}^{(\pi,c)} G(s,X^{\pi,c}(s)) + U_1(s,c(s))ds$$

$$+ \int_t^{\tau_{\theta,p}} \left[X^{\pi,c}(s)(r + \pi'(b - r\underline{1})) - c(s)\right] G_x(t, X^{\pi,c}(s))dW(s)$$

For $(\pi, c) \in \mathcal{A}^{\mathcal{H}}(t,x)$ we obtain

$$\begin{aligned} E^{t,x}\left[w^{\pi,c,t,x}(\tau_{\theta,p})\right] &= w^{\pi,c,t,x}(t) + E^{t,x}\left[\int_t^{\tau_{\theta,p}} \underbrace{\mathcal{A}^{(\pi,c)}G(s,X^{\pi,c}(s)) + U_1(s,c(s))}_{\leq 0} ds\right] \\ &+ \underbrace{E^{t,x}\left[\int_t^{\tau_{\theta,p}} \left[X^{\pi,c}(s)(r+\pi'(b-r\underline{1})) - c(s)\right] G_x(s,X^{\pi,c}(s))dW(s)\right]}_{=0} \end{aligned}$$

The less-equal relation is valid by construction of $G(t,x)$ in equation (1.28). Since O_p is bounded and $G_x(.,.)$ is continuous, $G_x(.,.)$ is bounded on $[t,\theta] \times O_p$. As by the definition of $\mathcal{A}^{\mathcal{H}}(t,x)$ we also have

$$E^{t,x}\left[\int_t^{\theta} \left[X^{\pi,c}(s)(r+\pi'(b-r\underline{1})) - c(s)\right]^2 ds\right] < \infty$$

we realize that the whole integrand is in $L^2[0,\theta]$ and the expectation of the stochastic integral vanishes leading to

$$w^{\pi,c,t,x}(t) \geq E^{t,x}\left[w^{\pi,c,t,x}(\tau_{\theta,p})\right], \tag{1.30}$$

or equivalently in the usual notation

$$G(t,X^{\pi,c}(t)) \geq E^{t,x}\left[\int_t^{\tau_{\theta,p}} U_1(s,c(s))ds + G(t,X^{\pi,c}(\tau_{\theta,p}))\right]. \tag{1.31}$$

ii) Now we have to show, that the relation (1.31) is valid for an unrestricted domain, i.e. the situation we are originally considering. We are going to proof that via the following equation:

$$\begin{aligned} &\lim_{p\to\infty} E^{t,x}\left[\int_t^{\tau_{\theta,p}} U_1(s,c(s))ds + G(\tau_{\theta,p}), X^{\pi,c}(\tau_{\theta,p}))\right] \\ = \;& E^{t,x}\left[\int_t^{\theta} U_1(s,c(s))ds + G(\theta,X^{\pi,c}(\theta))\right]. \end{aligned}$$

Since $\tau_{\theta,p} \to \theta$ for $p \to \infty$ $\mathbf{P}$–a.s., the polynomial boundedness of $U(.,.)$, the definition of $\mathcal{A}^{\mathcal{H}}$, and the dominated convergence theorem we obtain

$$\lim_{p\to\infty} E^{t,x}\left[\int_t^{\tau_{\theta,p}} U_1(s,c(s))ds\right] = E^{t,x}\left[\int_t^{\theta} U_1(s,c(s))ds\right]$$

yielding convergence for the first summand. For the second, note that we have convergence in probability of $G(\tau_{\theta,p}, X^{\pi,c}(\tau_{\theta,p}))$ to $G(\theta, X^{\pi,c}(\theta))$ by:

$$\begin{aligned} &P(|G(\tau_{\theta,p},X^{\pi,c}(\tau_{\theta,p})) - G(\theta,X^{\pi,c}(\theta))| > \epsilon) \\ \leq\;& P(||X^{\pi,c}(.)|| > p)) \\ \leq\;& \frac{1}{p^2}\underbrace{E^{t,x}\left[||X^{\pi,c}(.))||^2\right]}_{<\infty} \to 0, \text{ for } p\to\infty. \end{aligned}$$

which in particular implies convergence in distribution (see Billingsley(1968), Theorem 4.3, p.26). Alternatively, we could prove the above convergences by $\lim_{p\to\infty} G(\tau_{\theta,p}\,, X^{\pi,c}(\tau_{\theta,p})) = G(\theta, X^{\pi,c}(\theta))$ $\mathbf{P}$–a.s. due to continuity of G and $X^{\pi,c}(.)$ as we have $\tau_{\theta,p} \to \theta$ $\mathbf{P}$–a.s. for $p \to \infty$.

Using polynomial boundedness of G and boundedness in expectation of $X^{\pi,c}$ we are going to show that $\{G(\tau_{\theta,p}, X^{\pi,c}(\tau_{\theta,p}))\}_p$ is uniformly integrable. With

$$|G(\tau_{\theta,p}, X^{\pi,c}(\tau_{\theta,p}))| \leq K\left(1 + |X^{\pi,c}(\tau_{\theta,p})|^k\right) \leq K\left(1 + ||X^{\pi,c}(.)||^k\right)$$

and

$$E\left[||X^{\pi,c}(.))||^j\right] < \infty \quad \text{(via equation (1.26))}$$

we obtain

$$\begin{aligned} & E\left[\left(G(\tau_{\theta,p}, X^{\pi,c}(\tau_{\theta,p}))\right)^2\right] \\ \leq \quad & E\left[K^2\left(1 + 2||X^{\pi,c}(.)||^k + ||X^{\pi,c}(.))||^{2k}\right)\right] < \infty \end{aligned}$$

Hence, the family $\{G(\tau_{\theta,p}, X^{\pi,c}(\tau_{\theta,p}))\}_p$ is indeed uniformly integrable. Together with the convergence in distribution we obtain (see Billingsley(1968), Theorem 5.3 and 5.4))

$$\lim_{p\to\infty} E^{t,x}\left[G(\tau_{\theta,p}, X^{\pi,c}(\tau_{\theta,p}))\right] = E^{t,x}\left[G(\theta, X^{\pi,c}(\theta))\right].$$

iii) Thus in total we have

$$G(t, X^{\pi,c}(t)) \quad \geq \quad E^{t,x}\left[\int_t^\theta U_1(s, c(s))ds + G(t, X^{\pi,c}(\theta, X^{\pi,c}(\theta))\right]$$

for all $(\pi, c) \in \mathcal{A}^{\mathcal{H}}$ and $\theta \in [0, T]$. Especially for $\theta = T$, together with $(\pi, c)^*(.) := \arg\max_{(\hat{\pi},\hat{c})\in U}\left(A^{(\hat{\pi},\hat{c})}G(., X^{(\pi,c)^*}(.)) + U_1(., \hat{c}(.))\right)$ and noting that $w^{\pi^*,c^*,t,x}(.)$ is a martingale, we obtain

$$E^{t,x}\left[\int_t^T U_1(s, c^*(s))ds + U_2(X^{\pi^*,c^*}(T))\right] \quad \geq \quad E^{t,x}\left[\int_t^T U_1(s, c(s))ds + U_2(X^{\pi,c}(T))\right].$$

So finally we proved assertion a) and b).

□

Remark 1.15 (Proof of HJB-Theorem)
We showed for all $x \in I\!R$ and $t \in [0, T]$, that $w^{\pi,c,t,x}$ (defined in 1.13) is a super-martingale for an arbritary control in $\mathcal{A}^{\mathcal{H}}(t, x)$, and a martingale for the control defined by (1.28). So we basically proved conditions ensuring the *Martingale optimality principle*. The advantage of this principle over the standard HJB-theorem is, that it is a more general principle, which can be applied to a wider class of portfolio problems.

Remark 1.16 (Algorithm to solve the HJB-Equation)
The main consequence of Theorem 1.14 is that it offers an algorithm to solve our portfolio problem via solving the HJB-equation which will be done in the following steps:

i) Under the assumption that V(t,x) is concave the candidates for the optimal consumption and portfolio process are obtained by a formal maximization:

$$\begin{aligned}\pi^*(t) &= -(\sigma\sigma')^{-1}(b - r\underline{1})\frac{V_x(t,x)}{xV_{xx}(t,x)}\\ c^*(t) &= \left(\frac{\partial}{\partial c}U_2(t,.)\right)^{-1}(V_x(t,x))\end{aligned}$$

Then, plugging this into equation (1.28) leads to:

$$\begin{aligned}0 &= -\frac{1}{2}(b - r\underline{1})'(\sigma\sigma')^{-1}(b - r\underline{1})\frac{V_x^2(t,x)}{V_{xx}(t,x)} + rxV_x(t,x) + V_t(t,x)\\ &\quad -\left(\frac{\partial}{\partial c}U_2(t,.)\right)^{-1}(V_x(t,x))\,V_x(t,x) + U_2(t,c^*(t))\\ U_1(x) &= V(T,x)\end{aligned}$$

ii) We are left with the task to solve this equation, which is the hardest part. One way which will be successful for some particular examples is the so-called separation ansatz, e.g. we try an ansatz like $V(t,x) = f(t)^{(1-\gamma)}\frac{1}{\gamma}x^{\gamma}, f(T) = 1$ (for HARA-utility). Another possibility is simply to guess the optimal control or the value function and then checking the conditions of Theorem 1.14.

iii) Derive $\pi^*(t)$ and $c^*(t)$ explicitly out of the formal representations given in i).

iv) Check all assumptions made and needed, i.e. :

   a) Is $V(t,x)$ strictly concave and satisfies the polynomial growth condition (1.27) ?
   b) Is the corresponding wealth process (1.9) well defined ?
   c) Is $(\pi, c)(t) \in \mathcal{A}^{\mathcal{H}}(t,x)$ and π bounded ?

Remark 1.17 (Alternative representation of the wealth process)
For exponential utility, that means $U(x) = 1 - \exp(-\lambda t)$ for some $\lambda > 0$, it is optimal to invest a fixed amount of money in stocks, which is independent of the wealth. If we have an income stream it can be optimal to invest in stocks even if the actual wealth is zero (see Chapter 2). So for these and other examples it is more convenient (resp. necessary for the second case) to define the amount of money invested in particular stocks as the control

process (see Pliska (1986) or Korn(1997)). Then we obtain the following representation of the wealth process:

$$dX^{u,c}(t) = \left[(X^{u,c}(t)r(t) + u'(t)(b(t) - r(t)\underline{1}) - c(t)\right] dt - u'(t)\sigma(t)dW(t) \tag{1.32}$$

$$X(0) = x_0$$

Observe the relation $u(t) = \pi(t)X(t)$. The HJB-equation reads then as:

$$\sup_{(\hat{u},\hat{c})\in I\!R^{n+1}} \left\{ \frac{\partial G(t,x)}{\partial t} + \frac{1}{2}\hat{u}'\hat{\sigma}\hat{\sigma}'\hat{u}\frac{\partial^2 G(t,x)}{\partial x^2} + \left[xr + \hat{u}'(b-r)\underline{1} - \hat{c}\right] \frac{\partial G(t,x)}{\partial x} + U_2(t,\hat{c}) \right\} = 0$$

$$G(T,x) = U_1(x)$$

The corresponding process $(u^*(t), c^*(t))$ of this solution leads to a unique wealth process, if the conditions of the Variation of Constants Theorem 1.5, the polynomial boundedness condition of (1.24,1.25) and the corresponding conditions of $\mathcal{A}^{\mathcal{H}}$ are fulfilled.

Remark 1.18 (HJB-equation and log-utility)
Often the HJB-theorem is used, to find the optimal control for log-utility. Technically this is wrong, because the log and its value function violates the polynomial boundedness condition (1.27) required by this theorem. Practically this is no problem, since - roughly spoken - the value function is something like $V(t,x) = \beta_1(t)\log(\beta_2(t)x) + f(\pi, r, c, b, \sigma, t)$, and thus, the x-variable in its derivatives always cancels out against the x-variable of the corresponding coefficients in the Itô-Integral. So the Itô-Integral is always sufficiently well behaving, such that the argumentation of the HJB-proof is still valid.

1.2.4 Examples

Example 1.19 (HARA-Utility with deterministic process parameters)
Suppose there is no consumption and the final-utility is given by the HARA-function:

$$U_1(t,x) := 0, \quad U_2(x) = \frac{1}{\gamma}x^\gamma, \quad c \equiv 0$$

where $0 < \gamma < 1$. Let

$$b(t) \in C^b([0,T] \to I\!R^n), \quad r(t) \in C^b([0,T] \to I\!R), \quad \sigma(t)C^b([0,T] \to I\!R^{n\times n})$$

bounded and continuous functions. Our guess for the value function, generalized from the case for constant coefficients, is:

$$G(t,x) = \frac{1}{\gamma}x(t)^\gamma \exp\left(\gamma \int_t^T \kappa(s)ds\right)$$

with

$$\kappa(s) = r(s) + \frac{1}{2}(b(s) - r(s)\underline{1})'(\sigma(s)\sigma(s)')^{-1}(b(s) - r(s)\underline{1})\frac{1}{1-\gamma} \quad .$$

Now we check, that this indeed is a solution of the HJB-equation (where we omit the time variable s for lucidity):

$$\begin{aligned} & 0 = -\frac{1}{2}(b - r\underline{1})'(\sigma\sigma')^{-1}(b - r\underline{1})\frac{G_x^2(t,x)}{G_{xx}(t,x)} + rxG_x(t,x) + G_t(t,x) \\ \iff \quad & 0 = -\frac{1}{2}(b - r\underline{1})'(\sigma\sigma')^{-1}(b - r\underline{1})\frac{1}{\gamma - 1}x(t)^\gamma \exp\left(\int_t^T \gamma\kappa(s)ds\right) \\ & + rx(t)^\gamma \exp\left(\int_t^T \gamma\kappa(s)ds\right) - \frac{1}{\gamma}x(t)^\gamma \exp\left(\int_t^T \gamma\kappa(s)ds\right)\gamma\kappa(s) \\ \iff \quad & 0 = \frac{1}{2}(b - r\underline{1})'(\sigma\sigma')^{-1}(b - r\underline{1})\frac{1}{1-\gamma} + r - \kappa(s) \end{aligned}$$

Note, that the value function is concave, thus we can derive $\pi^*(t)$ by formal maximisation of the HJB-equation:

$$\pi^*(t) = \frac{1}{1-\gamma}(\sigma(t)\sigma(t)')^{-1}(b(t) - r(t)\underline{1}) = -(\sigma\sigma')^{-1}(b - r\underline{1})\frac{G_x(t,x)}{xG_{xx}(t,x)}$$

Finally we check the assumptions:

a)

$$\left|\frac{1}{\gamma}x^\gamma \exp\left(\gamma \int_t^T \kappa(s)ds\right)\right| \le x\left|\frac{1}{\gamma}\exp\left(\gamma\int_0^T |\kappa(s)|ds\right)\right|$$

Thus the polynomial growth condition (1.27) is valid and $V(t,x)$ is strictly concave.

b) We prove the assumptions of (1.10), since these are sufficient conditions for the existence of a solution of the corresponding wealth process (1.9):

$$\begin{aligned} b(t,x) &= \left[\left(r(t) + \frac{1}{1-\gamma}(b(t) - r(t)\underline{1})'(\sigma(t)\sigma(t)')^{-1}(b(t) - r(t)\underline{1}\right)\right] x \\ \sigma(t,x) &= \left[\frac{1}{1-\gamma}(b(t) - r(t)\underline{1})'\sigma(t)')^{-1}\right] x \end{aligned}$$

Since the market parameters $b(t), r(t), \sigma(t)$ are bounded and $b(t,x)$ and $\sigma(t,x)$ are linear in x, the assumptions (1.13,1.14) are valid. Thus the equation (1.17) has a unique solution.

c) The process $\pi^*(t)$ is bounded, since the market parameters are bounded and $\sigma\sigma'$ satisfies the uniformly positive definiteness conditions. By Proposition (1.11) we have $\pi^*(t) \in \mathcal{A}^{\mathcal{H}}(t,x)$.

Remark: The reason why we did not model market coefficients that depend on the wealth and the control, is that we cannot give an explicit solution for the value function.

Example 1.20 (Log-Utility with stochastic process parameters)
Suppose there is no consumption and the final-utility is given by the natural logarithm:

$$U_1(t,x) := 0, \quad U_2(x) = \log(x), \quad c \equiv 0$$

Let:

$$\begin{aligned}
b(.) &\equiv \{b(s), \mathcal{F}_s : s \in [0,T]\}, \\
r(.) &\equiv \{r(s), \mathcal{F}_s : s \in [0,T]\}, \\
\sigma(.) &\equiv \{\sigma(s), \mathcal{F}_s : s \in [0,T]\} > C \qquad \mathbf{P}\text{–a.s.}
\end{aligned}$$

be progressivly measurable and bounded processes with $x'\sigma(s)\sigma'(s)x > Cx'x$ for some $C > 0$ and all $s \in [0,T]$.

$$\begin{aligned}
V(t,x) &:= \sup_{\pi \in \mathcal{A}(t,x)} E^{t,x}[U_2(X^\pi(T))] \\
&= \log(x) + \sup_{\pi \in \mathcal{A}(t,x)} E^{t,x}\Bigg[\int_t^T r(s) + \pi'(s)(b(s) - r(s)\underline{1}) \\
&\qquad -\frac{1}{2}\pi'(s)\sigma(s)\sigma'(s)\pi(s)ds + \int_t^T \pi'(s)\sigma(s)dWs\Bigg] \\
&\le \log(x) + E^{t,x}\left[\int_t^T \sup_{\{\hat{\pi}_s : \mathcal{F}_s - meas.\}} \left\{r(s) + \hat{\pi}'_s(b(s) - r(s)\underline{1}) - \frac{1}{2}\hat{\pi}'_s\sigma(s)\sigma'(s)\hat{\pi}_s\right\} ds\right]
\end{aligned}$$

So the pointwise optimal control is

$$\hat{\pi}^*_s = (\sigma(s)\sigma(s)')^{-1}(b(s) - r(s)\underline{1}),$$

and we define it pathwise as:

$$\pi^*(s) := \hat{\pi}^*_s$$

Since $\pi^*(s)$ is progressivly measurable and bounded too, we found the optimal control.

1.3 Pricing Derivatives with Martingale Methods

1.3.1 Introduction

The valuation of derivative securities has been the object of a long quest. A model of describing the random behavior of speculative asset prices was initially proposed by Bachelier (1900). The development of a rigorous theory of option pricing, however, only dates back to the 1970's. Black and Scholes (1973) proposed a valuation formula for European options which is consistent with the absence of arbitrage opportunities in the financial market. This model and the underlying methodology are refined and extended by Merton (1973). An equivalent approach based on an appropriately chosen "risk neutral" valuation operator was pioneered by Cox and Ross (1976). The foundations and principles underlying these valuation methods are identified and characterized in the seminal paper of Harrison and Kreps (1979).
In the 1990's derivatives became more and more popular. They were used for hedging market risks and simply for speculation. Thus, many new types of derivatives were invented with complicated contract specifications. As a consequence of that an enormous number of mathematical papers were published in this research area. However, the pricing is still based on the framework pioneered by Black, Merton and Scholes. So in 1997 it was well-deserved that Merton and Scholes were awarded the Nobel prize in Economic Science for their work to determine the value of derivatives. Black already died in 1995.

1.3.2 The Model

A derivative security is a financial contract whose payoff depends on the price(s) of some underlying asset(s). In their most general form, derivative securities generate a flow of payments over periods of time as well as cash payments at specific dates. In addition, the cash flow needs to be paid at fixed points in time or during fixed periods of time. Some derivative securities involve cash flows paid at prespecified random times or even at (random) times which are chosen by the holder of the contract.

Definition 1.21
Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space. A probability measure $\mathbf{Q}$ on $(\Omega, \mathcal{F})$ is **absolutely continuous** *relative to $\mathbf{P}$ if*

$$\forall A \in \mathcal{F} : \mathbf{P}(A) = 0 \Longrightarrow \mathbf{Q}(A) = 0 \ .$$

Theorem 1.22
A probability measure $\mathbf{Q}$ is **absolutely continuous** *relative to $\mathbf{P}$if and only if there exists a nonnegative random variable Z on $(\Omega, \mathcal{F})$ such that we have*

$$\forall A \in \mathcal{A} : \mathbf{Q}(A) = \int_A Z(\omega) \mathbf{dP}(\omega) \ .$$

$Z = \mathbf{dQ}/\mathbf{dP}$ is said to be the density of $\mathbf{Q}$ relative to $\mathbf{P}$.

PROOF. The suffiency of the Theorem 1.22 is obvious, the converse is a version of the Radon-Nikodym theorem (Williams (1991), Chapter 5.14). □

Definition 1.23
The probability measures $\mathbf{P}$ *and* $\mathbf{Q}$ *are* **equivalent** *, if each one is absolutely continuous relative to the other.*

Remark: Note that if $\mathbf{Q}$ is absolutely continuous relative to $\mathbf{P}$, with density Z, then $\mathbf{P}$ and $\mathbf{Q}$ are equivalent if and only if $\mathbf{P}(Z > 0) = 1$.

For presenting the modern approach to option pricing we need the following result relating Brownian motions under a change to an equivalent probability measure.

Theorem 1.24 (Girsanov's Theorem)
Let $\theta = \{\theta(t), \mathcal{F}_t : t \in [0,T]\}$ *be an* n*-dimensional adapted process with*

$$\int_0^T \theta_i^2(s)ds < \infty \quad \mathbf{P}\text{–}a.s.\ \forall i \in \{0,\dots,n\}$$

and $(\eta(t;\theta))_{0\le t\le T}$ *defined by*

$$\eta(t;\theta) := \exp\Big(-\sum_{i=1}^{n}\int_0^t \theta_i(s)dW_i(s) - \frac{1}{2}\int_0^t ||\theta(s)||^2 ds\Big),$$

be a martingale, where $W(t)$ *is an* n*-dimensional Brownian motion. Then*

$$W_t^* = W(t) + \int_0^t \theta_s ds$$

is an n*-dimensional standard Brownian motion with respect to the probability measure* $\mathbf{P}^{(\eta)}$ *with density* $\eta(T;\theta)$ *relative to* $\mathbf{P}$*.*

PROOF. See Korn and Korn (2001), Theorem 3.11 □

Proposition 1.25 (Novikov condition)
A sufficient condition that $\eta(t;\theta)$ *is a martingale is the so-called Novikov condition:*

$$E\left[\exp\left(\tfrac{1}{2}\int_0^t ||\theta(s)||^2 ds\right)\right] < \infty$$

PROOF. See Karatzas&Shreve (1991). □

Definition 1.26

i) Let $\mathbf{Q}$ *denote the probability measure, defined by* $\mathbf{dQ} = \eta(T, \sigma^{-1}(b-r))\mathbf{dP}$*. The measure* $\mathbf{Q}$ *is per definition absolutely continuous relative to* $\mathbf{P}$*.*

ii) *With* E^* *we denote the expected value w.r.t to* $\mathbf{Q}$. *Let* Y $\mathcal{F}_T$*-measurable, then*

$$\begin{aligned}
& E^*\left[\int_0^T B^{-1}(s)c(s)ds + B^{-1}(T)Y\right] \\
= \; & E\left[\eta(T;\sigma^{-1}(b-r))\left(\int_0^T B^{-1}(s)c(s)ds + B^{-1}(T)Y\right)\right]
\end{aligned}$$

if the expected values exist.

Remark 1.27

i) Since μ, r, σ are bounded and as $\sigma\sigma'$ is uniformely positive definite, the Novikov condition (1.25) is fulfilled, and $W(t)+\int_0^t \sigma^{-1}(s)(b(s)-r(s))ds$ is a $\mathbf{Q}$-standard Brownian motion.

ii) In our setting $\mathbf{Q}$ is the unique equivalent martingale measure.

iii) The definition of the Itô integral remains valid when changing to an equivalent probability measure (Lamberton and Lapeyre (1996), p. 79).

Remark 1.28 (The stock price process under the measure Q)
The probability measure $\mathbf{Q}$ is often called equivalent martingale measure, since the wealth process (1.6) and stocks without dividend yields discounted by $B(t)$ are martingales with respect to this measure. More precisely, let $\widetilde{S}_i(t) := S_i(t)/B(t)$ be the discounted stock price. Then by Itô's Lemma

$$\begin{aligned}
d\widetilde{S}_i(t) &= -r(t)\exp\left(\int_0^t -r(s)ds\right) S_i(t)dt + \exp\left(\int_0^t -r(s)ds\right) dS_i(t) \\
&= \widetilde{S}_i(t)\left((b_i(t)-d_i(t)-r(t))dt + \sum_{j=1}^n \sigma_{ij}(t)dW_j(t)\right).
\end{aligned}$$

Written in one equation for all stocks this reads as

$$\begin{aligned}
d\widetilde{S}(t) &= \widetilde{S}(t)\left((b(t)-d(t)-r(t)\underline{1})dt + \sigma(t)dW(t)\right) \\
&= \widetilde{S}(t)\left(-d(t) + \sigma(t)\left(\sigma^{-1}(t)(\mu(s)-r(s))dt + dW(t)\right)\right) \\
&= \widetilde{S}_t\left(-d(t) + \sigma(t)dW_t^*\right) \;,
\end{aligned}$$

where $W_t^* := \int_0^t \sigma^{-1}(s)(b(s)-r(s))ds + W(t)$.
Now we consider $\widetilde{S}(t)$ under $\mathbf{Q}$. According to the theorem of Variation of constants $\widetilde{S}_i(t)$ is given by:

$$\widetilde{S}_i(t) = \widetilde{S}_i(0)\;\exp\left(-\int_0^t d_i(s) - \frac{1}{2}||\sigma_{i\cdot}(s)||^2 ds + \int_0^t \sigma_{i\cdot}(s)dW_s^*\right)$$

where $\sigma_{i\cdot} = (\sigma_{i1}, \ldots, \sigma_{in})$. So if $d_i \equiv 0$, $\widetilde{S}(t)$ is a **Q**-martingale (since the Novikov Condition is fulfilled with $\theta = -\sigma(s)$). Hence $S(t)$ is given under **Q** by:

$$S_i(t) = S_i(0)\ \exp\left(\int_0^t \left(r(s) - d_i(s) - \tfrac{1}{2}||\sigma_{i\cdot}(s)||^2\right) ds + \int_0^t \sigma_{i\cdot}(s) dW_s^*\right)$$

So for the case that $r(.), b(.), d(.), \sigma(.)$ are deterministic, we just have to replace the stock drift $b(.)$ by the interest rate $r(.)$ to obtain the stock price distributions under the martingale measure, roughly spoken.

Definition 1.29 (Contingent Claim)
A (European) contingent claim Y ist a nonnegative payoff at time T, more precisely Y is a nonnegative $\mathcal{F}_T$-measurable random variable with $E^*[B^{-1}(T)Y] < \infty$ for the equivalent martingale measures **Q**. *The set of contingent claims is denoted by I.*

Examples:

i) $Y = (S_1(T) - K)^+ \, , K \in I\!R^+$; vanilla call ($x^+ \equiv \max(x, 0)$)

ii) $Y = (S_1(T) - K)^+ 1_{\{\forall t \in [0,T]:\, S_1(t) > H\}} \, , H \in I\!R^+$; "down-and-out call"-barrier option

iii) $Y = (\sum_{i=1}^m S_1(t_i) - K)^+; 0 < t_1 < \ldots < t_i < \ldots < t_m = T$; Asian option

iv) $Y = (\sum_{i=1}^n a_i S_i(T) - K)^+$; basket option

Definition 1.30
The contigent claim Y is said to be **attainable** *under a consumption processs c if there exists an admissible trading strategy $(\pi, c) \in \mathcal{A}^+(t, x)$ with corresponding $X^{(\pi,c)}(t)$ (defined in Corollar 1.5) and*

$$Y = X^{\pi,c}(T) \ \ \mathbf{P}\text{–}a.s.$$

such that $\hat{X^\pi}(t) = X^\pi / B(t)$ is a martingale with respect to the equivalent martingale measure **Q**.

The foregoing definition is the basis of the so-called replication approach to option pricing (see also Definition (1.32)). Its applicability relies heavily on the fact that trading strategies attaining the final payment of a contingent claim exist. In our setting this is ensured by the following theorem:

Theorem 1.31 (Completeness of the market)
Let $Y \in I$ be a contigent claim and c a consumption process. Assume further that our standard requirements for the market coefficients are satisfied.

Assertion:

*i) If there exists a trading strategy $(\pi, c) \in \mathcal{A}^+(0, x)$ with $X^{\pi,c}(T) \geq Y$ **P**–a.s., then*

$$E^*\left[\int_0^T c(s)/B(s) ds + Y/B(T)\right] \leq x. \tag{1.33}$$

ii) Let Y be a contingent claim and $x = E^*[\int_0^T c(s)/B(s)ds + Y/B(T)]$. Then there exists a trading strategy $(\pi, c) \in \mathcal{A}^+(0, x)$, and the corresponding wealth process satisfies $Y = X^{\pi,c}(T)$ $\mathbf{P}$–a.s. .

PROOF.

i) Let $(\pi, c) \in \mathcal{A}^+(0, x)$ with $X_T \geq Y$ $\mathbf{P}$–a.s. and $\widetilde{X}^{\pi,c}(t) = X^{\pi,c}(t)/B(t)$, then:

$$\begin{aligned} d\widetilde{X}^{\pi,c}(s) &= B^{-1}(s)X^{\pi,c}(s)[\big(r(s) + \pi'(t)(b(s) - r(s))\big)ds + \pi'(s)\sigma(s)dW(s)] \\ &\quad -B^{-1}(s)c(s)ds - B^{-1}(s)r(s)X^{\pi,c}(s)ds \\ &= \widetilde{X}^{\pi,c}(s)[\big(\pi'(t)(b(s) - r(s))ds + \pi'(s)\sigma(s)dW(s)] - c(s)/B(s)ds \end{aligned}$$

Hence:

$$X^{\pi,c}(t)/B(t) + \int_0^t c(s)/B(s)ds \tag{1.34}$$

$$= x + \int_0^t \widetilde{X}^{\pi,c}(s)\big(\pi'(t)(b(s) - r(s))ds + \int_0^t \widetilde{X}^{\pi,c}(s)\pi'(s)\sigma(s)dW(s)$$

$$= x + \int_0^t \widetilde{X}^{\pi,c}(s)\pi'(s)\sigma(s)dW^*(s) \tag{1.35}$$

The Brownian motion $W^*(.)$ is defined by $W^*(t) = \int_0^t \sigma^{-1}(s)(b(s) - r(s))ds + W(t)$. Due to $(\pi, c) \in \mathcal{A}^+(t, x)$ (1.34) is nonnegative. The term (1.35) is a continuous local $\mathbf{Q}$-martingale. Hence, (1.35) is a nonnegative $\mathbf{Q}$-super-martingale (see Karatzas&Shreve (1991), Chapter 1, Problem 5.19). Taking the expectation for $t = T$ yields

$$\begin{aligned} & E^*\left[X^{\pi,c}(T)/B(T) + \textstyle\int_0^T c(s)/B(s)ds\right] \\ \leq\ & E^*\left[x + \textstyle\int_0^T \widetilde{X}^{\pi,c}(s)\pi'(s)\sigma(s)dW_s^*\right] = x\ . \end{aligned}$$

Since Y is attainable by the strategy (π, c), we get

$$\begin{aligned} & E^*\left[Y/B(T) + \textstyle\int_0^T c(s)/B(s)ds\right] \\ \leq\ & E^*\left[X^{\pi,c}(T)/B(T) + \textstyle\int_0^T c(s)/B(s)ds\right] \leq x \end{aligned}$$

ii) The term

$$M_t = E\left[\textstyle\int_0^T \eta(T;\theta)B^{-1}(s)c(s) + \eta(T;\theta)B^{-1}(T)Y \middle| \mathcal{F}_t\right]\ ,$$

with $\eta(T;\theta)$ defined in Girsanov's theorem with $\theta = (\mu - r)/\sigma$, is a $\mathbf{P}$-martingale. According to the martingale-representation theorem (Karatzas & Shreve (1988), Chapter 3, Theorem 4.15 and Problem 4.16) M_t can be written as Itô integral

$$M_t = M_0 + \int_0^t \Phi_s dW(s) \ , \tag{1.36}$$

with $\{\Phi_t\}_{t\in[0,T]}$ some n-dimensional, progressively measurable process with $\int_0^T \Phi_t^2 dt < \infty$, $\mathbf{P}$–a.s.. Via the Bayes' Rule (Karatzas&Shreve (1988), Chapter 3, Lemma 5.3) we obtain

$$\eta(t;\theta)^{-1} M_t = M_t^* \equiv E^* \left[\int_0^T B^{-1}(s)c(s) + B^{-1}(T)Y \middle| \mathcal{F}_t \right] \ ,$$

Observe that $\eta(t;\theta) = 1 + \int_0^t \eta(s;\theta)(-\theta(s))dWs$. Hence by the multidimensional Itô-formula we get

$$M_t^* = M_0^* + \int_0^t \Phi^*(s)dW_s^* \ , \Phi^*(s) \equiv \frac{\Phi(s) + M(s)\theta(s)}{P(s;\theta)}.$$

Now let:

$$\pi(s) = \begin{cases} \frac{(\sigma^{-1})'(s)\Phi^*(s)}{\widetilde{X}^{\pi,c}(s)} & : \ \widetilde{X}^{\pi,c}(s) > 0 \\ 0 & : \ \widetilde{X}^{\pi,c}(s) = 0 \end{cases} \tag{1.37}$$

Plugging this into (1.35) yields for $\forall t \in [0,T]$:

$$\begin{aligned} \widetilde{X}^{\pi,c}(t) &= x + \int_0^t \Phi_s^* dW_s^* \\ &= x - M_0^* + M_t^* \\ &= x - E^* \left[\int_0^T B^{-1}(s)c(s) + B^{-1}(T)Y \right] \\ &\quad + E^*[B^{-1}(T)Y \mid \mathcal{F}_t] \ . \end{aligned} \tag{1.38}$$

Now let $t = T$:

$$D(0,T)X_T = \underbrace{x - E^* \left[\int_0^T B^{-1}(s)c(s) + B^{-1}(T)Y \right]}_{=0 \text{ by assumption}} + B^{-1}(T)Y \tag{1.39}$$

This yields $X_T = Y$. Further, by our assumption on the market coefficients, $\pi(s)$ satisfies all the requirements on a portfolio process. □

Remark: By (1.38) we can conclude $X_t = E^*[e^{-\int_t^T r(s)ds} Y \mid \mathcal{F}_t]$, if interest rates are deterministic.

Definition 1.32 (Arbitrage Opportunity)
A trading strategy $(\pi, c) \in \mathcal{A}(0,0)$ *is called an arbitrage opportunity , if it satisfies* $\mathbf{P}(\int_0^T c(s)ds + X_T \geq 0) = 1$ *and* $\mathbf{P}(\int_0^T c(s)ds + X_T > 0) > 0$ *.*

Remark: It follows from the Completeness of the Market Theorem (1.31) Part i), that the market contains no arbitrage opportunities.

Definition 1.33 (Rational Price)
The rational price of a contingent claim (Y, c) *at time* t *is the infimum over the prices over all trading strategies* $(\pi, c) \in \mathcal{A}(t, x)$ *with* $X_T = Y$*.*

Corollary 1.34 (Formula for the rational price)
The rational price of a contigent claim (Y, c) *is given by*

$$V_{Y,c}(t) = E^* \left[\int_t^T D(t,s)c(s)ds + D(t,T)Y \middle| \mathcal{F}_t \right] \ , \forall t \in [0,T]$$

and there exists a trading strategy (π, c) *with which we are able to replicate the payoffs.*

Proof. Due to Theorem (1.31) Part ii) there exists a trading strategy $(\pi, c) \in \mathcal{A}(0, \hat{x})$, with $\hat{x} = E^* \left[\int_0^T D(0,s)c(s)ds + D(0,T)Y \right]$ such that $Y = X^{\pi,c}(T)$ $\mathbf{P}$–a.s.. Then with part ii):

$$E^* \left[\int_0^T D(0,s)c(s)ds + D(0,T)Y \right] \leq V_Y(0) \leq \hat{x} = E^* \left[\int_0^T D(0,s)c(s)ds + D(0,T)Y \right]$$

In the case of $t \in (0,T]$ it can be argued in a similar way. □

Completeness of our model is mainly due to the fact that we have exactly as many risky assets as the dimension of the Brownian motion which is the source of randomness in our model. This is also underlined by:

Remark 1.35 (Björk's Meta-Theorem)
Let **M** denote the number of underlying **traded** assets in the model **excluding** the risk free assets, and let **R** denote the number of random sources. Generically we have the following relations:

i) The model is arbitrage free if and only if $M \leq R$.

ii) The model is complete if and only if $M \geq R$.

iii) The model is complete and arbitrage free if and only if $M = R$.

Remark 1.36

i) Björk gave neither an exact formulation nor a proof of his theorem, but this "theorem", or rule of thumb, is nevertheless extremely useful for intuition, when dealing with market models.

In chapter (1.1.2) we required, that the product of the volatility matrix with its transposed counterpart is strictly positive definite. This implies that the volatility matrix itself has full rank. Hence the effective number of random sources is equal to the number of trades assets. Therefore we ensured that the market is complete and arbitrage free.

ii) Now we present an example of an arbitrage opportunity for $2{=}M > R{=}1$. Imagine two assets driven by the same one-dim. Brownian motion:

$$\begin{aligned} S_1(t) &= S_1(t)(\mu_1 dt + \sigma_1 dW(t)) \quad , \quad \mu_1 > r, \sigma_1 \in I\!R^+ \\ S_2(t) &= S_2(t)(\mu_2 dt + \sigma_2 dW(t)) \quad , \quad \mu_2 > r, \sigma_2 \in I\!R^+ \end{aligned}$$

Let $\pi_1(\pi_2)$ be the part of wealth, invested in $S_1(S_2)$ and $1-\pi_1-\pi_2$ the part invested in the money market account. Then the wealth process reads as:

$$dX(t) = X(t)\left[\pi_1\sigma_1\left(\frac{\mu_1 - r}{\sigma_1}dt + dW(t)\right) + \pi_2\sigma_2\left(\frac{\mu_2 - r}{\sigma_2}dt + dW(t)\right) + rdt\right]$$

If the so-called "market prices of risk" satisfy $\frac{\mu_1 - r}{\sigma_1} > \frac{\mu_2 - r}{\sigma_2}$ then choose π_1 and π_2, such that $\pi_1\sigma_1 + \pi_2\sigma_2 = 0$ and $\pi_1 > 0$ (or $\pi_2 > 0$, if the market price of risk from S_2 is bigger). As a consequence, we created a trading strategy, which is completely riskless, but has a greater return than the money market account. Hence we have an arbitrage strategy, by lending money and investing it in this portfolio.

iii) The following is an example of an incomplete market:
Let us assume that we have a market consisting of two assets, driven by two correlated Brownian motions, and an exchange option on this assets, namely $\max[S_2(T) - S_1(T), K]$. By theorem (1.31) there exists exactly one trading strategy, consisting of the two assets and the money market account. Now suppose we are not allowed to trade in the second asset (which means that our market has two random sources, but only one tradable asset), then of course we are not able to hedge this option.

1.3.3 Examples

Corollary 1.37 (Black and Scholes 1973)
Let the interest rate $r(.)$, the volatility $\sigma(.)$, and the dividend yield $d(.)$ be deterministic functions of time with $\int_t^T \sigma(s)^2 ds > 0 \quad \forall t \in [0,T]$. Let $S(t) \in I\!R^+$ be the stock price at time $t \in [0,T]$. The rational price of a European call ($Y = (S(T)-K)^+$) resp. put ($Y = (K - S(T))^+$) with "strike price" K at time t is given by:

$$\begin{aligned} V_C(S,K,t) &= Se^{-\int_t^T d(s)ds}N(d_1) - Ke^{-\int_t^T r(s)ds}N(d_2), \\ V_P(S,K,t) &= Ke^{-\int_t^T r(s)ds}N(-d_2) - Se^{-\int_t^T d(s)ds}N(-d_1) \end{aligned}$$

where

$$d_1 = \frac{\log(S(t)/K) + \int_t^T \left(r(s) - d(s) + 0.5\sigma^2(s)\right) ds}{\sqrt{\int_t^T \sigma^2(s) ds}}$$

$$d_2 = d_1 - \sqrt{\int_t^T \sigma^2(s) ds}$$

and where $N(x)$ denotes the standard normal distribution function.

PROOF. Due to Corollary (1.34) we obtain

$$\begin{aligned} V_C(S,K,t) &= E^*[e^{-r(T-t)}(S(T)-K)^+ \mid \mathcal{F}_t] \\ &= E^*[e^{-r(T-t)}(Se^{\int_t^T r(s)-d(s)-0.5\sigma^2(s))ds+\int_t^T \sigma(s)dW(s)} - K)^+] \end{aligned}$$

whereby $W(s)$ denotes the standard Brownian motion with respect to $\mathbf{Q}$. As $\sigma(.)$ is deterministic, $\int_t^T \sigma(s)dW(s)$ is distributed as $\sqrt{\int_t^T \sigma^2(s)ds}X$, with X being standard normally distributed. Therefore:

$$\begin{aligned} & S(T) && \geq K \\ \Longleftrightarrow\quad & Se^{\int_t^T r(s)-d(s)-0.5\sigma^2(s)ds+\sqrt{\int_t^T \sigma^2(s)ds}X} && \geq K \\ \Longleftrightarrow\quad & X && \geq -d_2 \end{aligned}$$

Plugging this into the above equation yields

$$\begin{aligned} V_C(S,K,t) &= \int_{-d_2}^{\infty} e^{-r(T-t)} \left(S(t)e^{\int_t^T r(s)-d(s)-0.5\sigma^2(s)ds+\sqrt{\int_t^T \sigma^2(s)ds}\, x} - K\right) \frac{e^{-x^2/2}}{\sqrt{2\pi}}\, dx \\ &= \int_{-\infty}^{d_2} \left(S(t)e^{-\int_t^T d(s)ds} e^{\int_t^T -0.5\sigma^2(s)ds-\sqrt{\int_t^T \sigma^2(s)ds}\, x} - e^{-r(T-t)} K\right) \frac{e^{-x^2/2}}{\sqrt{2\pi}}\, dx \\ &= \int_{-\infty}^{d_2} \left(Se^{-\int_t^T d(s)ds} \frac{e^{-(x+\sqrt{\int_t^T \sigma^2(s)ds})^2/2}}{\sqrt{2\pi}} - e^{-r(T-t)} K \frac{e^{-x^2/2}}{\sqrt{2\pi}}\right) dx \\ &= \int_{-\infty}^{d_1} S(t)e^{-\int_t^T d(s)ds} \frac{e^{-x^2/2}}{\sqrt{2\pi}} dx - \int_{-\infty}^{d_2} e^{-r(T-t)} K \frac{e^{-x^2/2}}{\sqrt{2\pi}}\, dx \\ &= Se^{-\int_t^T d(s)ds} N(d_1) - Ke^{-r(T-t)} N(d_2) \end{aligned}$$

The formula for the put option can be determined either by a similar calculation or by the put-call parity $V_C(S,K,t) - V_P(S,K,t) = Se^{-\int_t^T d(s)ds} - Ke^{-r(T-t)}$. □

2 Optimal Portfolios with Fixed Monetary Streams

2.1 Introduction

In contrast to the standard setting of the portfolio problem as presented in Chapter 1 a small investor often has to take into account additional constraints like:

- continuous consumption requirements ("daily living expenses")
- expenses occurring regularly at fixed time instants (such as rents, insurance fees,..)
- income occurring at fixed times (such as the investor's salary).

This has the particular consequence that only some of the investor's money can be used for investment purposes during a subset of the whole investment period $[0, T]$. We will show how an investor can still make use of it by considering a generalized portfolio problem with given consumption and investment streams.

The contributions of this section will consist of

- presenting a generalized setting for the standard continuous-time portfolio problem allowing for the consideration of additional consumption and investment requirements
- an explicit solution of the generalized problem via an explicit solution of a Hamilton-Jacobi-Bellman equation with additional boundary constraints (thereby adding a new example of an explicit solution to a stochastic control problem to the literature)
- a second solution method based on a general separation theorem between constrained and unconstrained investment that allows for dealing with general requirements
- some explicitly solved realistic examples of constrained portfolio problems.

This problem and also our findings are similar to the results of El Karoui and Jeanblanc-Picqué (1998), but differ in both the methods used and in some aspects of the model. In particular, we will rely on our Seperation Theorem and on the stochastic control approach via solving the HJB equation explicitly.

This chapter is organized as follows: We will state the problem together with some notations in the next subsection. The solution of this problems by classical HJB-methods forms Section 2.3 while Section 2.4 will contain the solution method based on our Separation Theorem. Some more examples and final remarks rounds off this chapter.

2.2 The Model and Some Basic Definitions

We consider a standard n-dimensional Black-Scholes type securities market as introduced in Section 1.1, but we restrict to the case of constant market coefficients. This market consists of a riskless bond and n risky assets with prices given by

$$\begin{aligned} dB(t) &= B(t)r(t)dt, \\ dS_i(t) &= S_i(t)\left[b_i dt + \sum_{j=1}^{n} \sigma_{ij} dW_j(t)\right], \ i = 1, \ldots, m, \end{aligned}$$

where $W(t)$ is an n-dimensional Brownian motion. To model the consumption requirements and/or the income streams of our investor we consider continuous monetary streams c(t) with the additional feature of

$$\int_0^T |c(s)|ds < \infty \qquad \mathbf{P}\text{–a.s.}$$

and discrete monetary streams given by a finite sequence of square integrable random variables B(i) which are $\mathcal{F}_{t_i}$-measurable. The discrete monetary stream takes place at the times $t_1, \ldots, t_m$ with $0 < t_1 < \ldots < t_m \leq T$, where $\mathcal{F}_t$ is assumed to be the Brownian filtration. In the presence of those monetary streams the wealth process corresponding to a portfolio process $\pi(t)$ (i.e. the process of the fractions of wealth invested in the different securities at time t) satisfies equations

$$dX^\pi(t) = \left[X^\pi(t)\left(r(1-\pi'(t)\underline{1}) + \pi'(t)\mu\right) - c(t)\right]dt + X^\pi(t)\pi'(t)\sigma dW(t) \tag{2.1}$$

on $[t_i, t_{i+1})$ for $i = 0, \ldots, m$, with $t_0 = 0, t_{n+1} = T$. At time instants $t_1, \ldots, t_m$ we have the following jump condition

$$X^\pi(t_i) = X^\pi(t_i-) - B_i\,. \tag{2.2}$$

In this setting, our goal is to maximise the utililty of the final wealth, i.e.

$$V(t,x) = \sup_{\pi \in \mathcal{A}^{\mathcal{H}}(t,x)} E^{t,x}\left[U(X^\pi(T))\right]\,. \tag{2.3}$$

where $\mathcal{A}^{\mathcal{H}}(t,x)$ is the set of admissible portfolio processes defined in equation (1.26).

Typical examples will be:

i) Continuous consumption requirements, i.e. $c(t) \equiv c > 0$

ii) A constant pay in scheme, i.e. $c(t) \equiv -d < 0$

iii) Monthly payments, $B_i = B > 0$

2.3 Problems with Fixed Consumption/Income: the HJB-Solution

In this section we demonstrate that for a variety of cases the above described control problem can be explicitly solved via setting up a corresponding Hamilton-Jacobi-Bellman equation and then solving it.

2.3.1 Constant continuous consumption requirements

We start with the case of a constant consumption process and power utility, i.e. we assume

$$c(t) \equiv -c \tag{2.4}$$

for all $t \in [0, T]$ and some $c > 0$ and the HARA-function as final utility, i.e. $U(x) = \frac{1}{\gamma} x^{\gamma}$, $\gamma \in (0, 1)$. Then, the HJB-equation corresponding to our problem (2.3) has the form

$$\max_{\pi \in U^{\pi}} \left\{ {}^1\!/\!{}_2 \pi^2 \sigma^2 x^2 V_{xx}(t, x) + (rx + \pi (b - r) x - c) V_x(t, x) + V_t(t, x) \right\} = 0 \tag{2.5}$$

with the obvious final condition

$$V(T, x) = {}^1\!/\!{}_\gamma x^{\gamma} \tag{2.6}$$

and the boundary condition

$$V\left(t, \tfrac{c}{r}(1 - \exp(-r(T - t)))\right) = 0 \tag{2.7}$$

which results from the fact that as soon as the minimum amount of money to satisfy the future consumption requirements is reached by the wealth process all risky investments are stopped. The form of this boundary is a consequence of the equation governing the future consumption requirements process $X_c(t)$ given by the ordinary differential equation

$$X_c^{'}(t) = rX_c(t) - c\,,\; X_c(T) = 0$$

which is uniquely solved by

$$X_c(t) = \tfrac{c}{r}(1 - \exp(-r(T - t)))\,.$$

Note in particular that therefore the investor's initial capital x has to be bigger than

$$x_c = X_c(0) = \tfrac{c}{r}(1 - \exp(-rT))\ .$$

Otherwise, the consumption requirements cannot be satisfied.

Hence, given our Verification Theorem 1.14 (or e.g. Fleming and Soner (1993) or Korn and Korn (2001)) one only has to solve the HJB-equation (2.5)) together with the boundary conditions (2.6, 2.7). This will be done in the next theorem:

Theorem 2.1 Optimal control with continuous consumption
Let our initial capital x satisfy

$$x > c\frac{1-\exp(-rT)}{r}\,. \tag{2.8}$$

Then, the value function V(t, x) of our optimisation problem (2.1-2.3) with a given consumption rate of $c \geq 0$ *is given by*

$$\begin{aligned} V(t,x) &= E^{t,x}\left(\tfrac{1}{\gamma}X(T)^{\gamma}\right) \qquad (2.9)\\ &= \tfrac{1}{\gamma}\left(x-\tfrac{c}{r}\left(1-\exp\left(-r\left(T-t\right)\right)\right)\right)^{\gamma}\exp\left(\gamma\left(r+1/2\left(\tfrac{b-r}{\sigma}\right)^{2}\tfrac{1}{1-\gamma}\right)(T-t)\right) \end{aligned}$$

for all pairs (t,x) *with* $t \in [0,T]$ *and* $x \in \left[\frac{c}{r}\left(1-\exp\left(-r\left(T-t\right)\right)\right),\infty\right)$.
The corresponding optimal portfolio process has the form

$$\pi(t) = \frac{b-r}{\sigma^2(1-\gamma)}\left[1-\frac{c}{rx}\left(1-\exp\left(-r\left(T-t\right)\right)\right)\right]. \tag{2.10}$$

Remark
Note the following limiting behaviour of the portfolio process

$$\pi(t) \to \begin{cases} 0, \text{if } x \downarrow \frac{c}{r}\left(1-\exp\left(-r\left(T-t\right)\right)\right) \\ \frac{b-r}{\sigma^2(1-\gamma)}, \text{if } x \to \infty \end{cases}$$

i.e. the influence of the consumption vanishes if the wealth process approaches infinity while the consumption requirements do not permit stock investment if all the capital is needed for consumption. In particular, the boundary condition (2.7) is met.

PROOF.
Standard verification theorems yield that a smooth and polynomially bounded solution $V(t,x)$ of the HJB-equation (2.5-2.6) is indeed the value function of our optimisation problem. In doing the first step to arrive at this solution, we perform the optimisation in (2.5-2.6) which results in the candidate

$$\pi(t) = -\frac{b-r}{\sigma^2}\frac{V_x}{xV_{xx}}$$

for the optimal portfolio process and hence leads to the equation

$$V_t + (rx-c)V_x - 1/2\left(\frac{b-r}{\sigma}\right)^2\frac{V_x^2}{V_{xx}} = 0\,,$$

which has to hold for all pairs (t,x) with $t \in [0,T]$ and $x \in \left[\frac{c}{r}(1-\exp(-r(T-t))),\infty\right)$ as points outside this set cannot guarantee to satisfy the consumption requirements for sure. We now verify that $V(t,x)$ as given in (2.9) solves this equation. To simplify this we introduce

$$A = \left(x - c\frac{1-\exp(-r(T-t))}{r}\right), \quad B = \exp\left(\gamma\left(r+1/2\left(\tfrac{b-r}{\sigma}\right)^2\tfrac{1}{1-\gamma}\right)(T-t)\right).$$

Via $V = \frac{1}{\gamma} A^\gamma B$ this leads to

$$\begin{aligned} V_t &= A^{\gamma-1} c \exp\left(-r\left(T-t\right)\right) B - A^\gamma B \left(r + 1/2 \tfrac{1}{1-\gamma} \left(\tfrac{b-r}{\sigma}\right)^2\right) \\ V_x &= A^{\gamma-1} B \\ V_{xx} &= \left(\gamma - 1\right) A^{\gamma-2} B \end{aligned}$$

and to

$$\begin{aligned} & V_t + \left(rx - c\right) V_x - 1/2 \left(\frac{b-r}{\sigma}\right)^2 \frac{V_x^2}{V_{xx}} \\ = \; & \left(rx-c\right) A^{\gamma-1} B + 1/2 \left(\tfrac{b-r}{\sigma}\right)^2 \tfrac{1}{1-\gamma} A^\gamma B + A^{\gamma-1} c \exp\left(-r\left(T-t\right)\right) B \\ & - A^\gamma B \left(r + 1/2 \left(\tfrac{b-r}{\sigma}\right)^2 \tfrac{1}{1-\gamma}\right) \\ = \; & A^{\gamma-1} B \left[\left(rx-c\right) + c\exp\left(-r\left(T-t\right)\right) - Ar\right] = 0. \end{aligned}$$

As a further result we obtain the optimal portfolio process as

$$\pi\left(t\right) = -\frac{b-r}{\sigma^2} \frac{V_x}{x V_{xx}} = \frac{b-r}{\sigma^2\left(1-\gamma\right)} \left(1 - \tfrac{c}{rx}\left(1 - \exp\left(-r\left(T-t\right)\right)\right)\right).$$

□

Remark 2.2 (Decomposition of wealth process)
The form of equation (2.9) implies that the value

$$A := \left(x - c\frac{1-\exp\left(-r\left(T-t\right)\right)}{r}\right) \tag{2.11}$$

would be the capital that an investor starting at time t with a capital of x can use for investment. Indeed, the optimal utility as described in equation (2.9) can be reached by using this amount of money and investing it according to the optimal portfolio process in the pure optimal terminal wealth problem (see e.g. Korn (1997)),

$$\tilde{\pi}\left(t\right) = \frac{b-r}{\sigma^2\left(1-\gamma\right)}, \tag{2.12}$$

leading to exactly this expected utility. Further, the remaining amount of money equals $X_c(t)$, the process describing the evolution of the money needed for future consumption requirements. Thus, the optimal wealth process starting with initial value of x in t=0 must have the form

$$\begin{aligned} X\left(t\right) = \; & \left(x - \tfrac{c}{r}\left(1-\exp\left(-rT\right)\right)\right) \exp\left[\left(r + \left(\tfrac{b-r}{\sigma}\right)^2 \tfrac{1}{1-\gamma} - 1/2\left(\tfrac{b-r}{\sigma(1-\gamma)}\right)^2\right) t \right. \\ & \left. + \frac{b-r}{\sigma(1-\gamma)} W(t)\right] + \tfrac{c}{r}\left(1 - \exp\left(-r\left(T-t\right)\right)\right) \end{aligned} \tag{2.13}$$

Note that the form of the optimal portfolio process as given in (2.10) corresponds exactly to the strategy of dividing the initial capital into

$$x = x_1 + x_2 = c\frac{1 - \exp(-rT)}{r} + (x - x_1)$$

and then leaving x_1 in the bond to pay out all the consumption requirements, taking the remaining part x_2 and investing it so as to solve a portfolio problem without any consumption at all. The second term in the brackets of the relation defining $\pi(t)$ is thus a consequence of the consumption requirements. It can easily be verified that $\pi(t)$ satisfies all the integrability requirements of a portfolio process.

Remark 2.3 (General given continuous consumption rate requirements)
Having seen both the relevant idea and the solution of the HJB-equation in the above constant case, it is easy to figure out the necessary ingredients to solve the problem in the non-constant continuous case. In fact the only difference is that now the required initial capital that has to be put aside at the beginning is obtained from the solution of the differential equation

$$X_c^{'}(t) = rX_c(t) - c(t), \quad X_c(T) = 0$$

i.e from

$$X_c(t) = \int_t^T \exp(-r(s-t))c(s)ds$$

as

$$x_c = X_c(0) = \int_0^T \exp(-rs)c(s)ds$$

The solution of the corresponding HJB-equation is then totally similar.

2.3.2 Lump Sum Consumption

In contrast to the previous section we now assume that consumption takes place at fixed time instants $t_1, \ldots, t_m$ with $0 < t_1 < \ldots < t_m \leq T$ and is required to equal (non-stochastic) amounts $C_i > 0$ at times t_i. This is now a consumption stream with all mass concentrated at isolated time points. However, the idea to put aside at $t = 0$ the required money to satisfy the needs for consumption and to invest the remaining capital as if there were no consumption at all, will stay valid here, too. Note that for paying in a consumption of $B_i \equiv C_i$ at time t_i (i.e. to pay out $C_i > 0$)one needs an amount of money of

$$C_i e^{-r(t_i - t)}$$

at time $t \leq t_i$ to attain C_i via riskless investment on $[t, t_i]$. We therefore get the following condition for the wealth process to satisfy

$$X(t) \geq \sum_{i:t_i>t} C_i e^{-r(t_i-t)} \quad , \quad t \in [0,T]. \tag{2.14}$$

Note that by the form of this requirement we also indicate that $X(t)$ is the wealth at time t *after* the possible consumption at time t has been made. More precisely, we have

$$X(t_i) = X(t_i-) - C_i. \tag{2.15}$$

As $X(t)$ is discontinuous at the times of consumption t_i, we cannot expect the value function

$$V(t,x) = \sup_{\pi \in \mathcal{A}^{\mathcal{H}}(t,x)} E^{t,x}\left[\frac{1}{\gamma} X^{\pi}(T)^{\gamma}\right] \tag{2.16}$$

to be continuous at t_i. Instead, we must have

$$V(t_i, x - C_i) = V(t_i-, x) \tag{2.17}$$

for all x satisfying (2.15) in place of $X(t)$. However, on intervalls (t_i, t_{i+1}) $V(t,x)$ should satisfy the usual HJB-Equation as we will prove in the verification theorem below. We summarize our consideration in

Theorem 2.4 Optimal control with lump sum consumption

For a given set of consumption requirements $C_i > 0$ at times t_i, $i = 0, \ldots, m$ with $0 \le t_1 < \ldots < t_m \le T$, let our initial capital satisfy

$$x > \sum_{i:t_i>t} C_i e^{-r(t_i - t)}\,. \tag{2.18}$$

Then the value function of problem (2.16) is given by

$$V(t,x) = \frac{1}{\gamma}\left(x - \sum_{i:t_i>t} C_i e^{-r(t_i-t)}\right)^{\gamma} e^{\gamma\left(r + \frac{1}{2}\left(\frac{b-r}{\sigma}\right)^2 \frac{1}{1-\gamma}\right)(T-t)} \tag{2.19}$$

for all pairs (t,x) with $t \in [0,T]$ and $x \in \left[\sum_{i:t_i>t} C_i e^{-r(t_i-t)}, \infty\right)$. The corresponding optimal portfolio process has the form

$$\pi(t) = \frac{b-r}{\sigma^2(1-\gamma)}\left[1 - \frac{\sum_{i:t_i>t} C_i e^{-r(t_i-t)}}{x}\right]. \tag{2.20}$$

PROOF.
The Verification Theorem (2.5) below indicates that $V(t,x)$ is the unique (piecewise) smooth solution of the corresponding HJB equation that also satisfies the jump condition (2.17). Similar as in the proof of Theorem (2.1), we can verify that $V(t,x)$ as given in (2.19) above has these properties and hence coincides with the value function. One can also obtain the optimal portfolio process then directly as

$$\pi(t) = -\frac{b-r}{\sigma^2}\frac{V_x(t,x)}{xV_{xx}(t,x)} = \frac{b-r}{\sigma^2(1-\gamma)}\left[1 - \frac{\sum_{i:t_i>t} C_i e^{-r(t_i-t)}}{x}\right]$$

where at times t_i we have taken the right-continuous limit of the derivatives. □

It thus only remains to prove the verification theorem:

Theorem 2.5 Verification theorem for lump sum consumption
Let $G(t,x)$ be a polynomially bounded solution of

$$\sup_{\pi\in[-\alpha,\alpha]}\left\{\frac{1}{2}\sigma^2\pi^2x^2G_{xx}(t,x)+x[r+\pi(b-r)]G_x(t,x)+G_t(t,x)\right\}=0 \tag{2.21}$$

for all $t\in[0,T]\setminus\{t_1,\dots,t_m\}, x>\sum_{i:t_i>t}c_ie^{-r(t_i-t)}$ and some fixed $\alpha>0$.

$$G(t_i,x-C_i) = G(t_i-,x) \tag{2.22}$$

$$G\left(t,\sum_{i:t_i>t}C_ie^{-r(t_i-t)}\right) = 0 \tag{2.23}$$

$$G(T,x) = \frac{1}{\gamma}x^\gamma \tag{2.24}$$

which is in $C^{1,2}$ on $(t_i,t_{i+1}), i=0,\dots,m$ with $t_0=0, t_{m+1}=T$. Let further be

$$\pi^*(t,x)=-\frac{b-r}{\sigma^2}\frac{G_x(t,x)}{xG_{xx}(t,x)}\in(-\alpha,\alpha)$$

for suitable $\alpha>0$ (where in points t_i we take the right hand limits of the derivatives). Then, $g(t,x)$ coincides with the value function $V(t,x)$, and $\pi^*(t,X^{\pi^*}(t))$ is an optimal portfolio process.

Proof.
Let $G(t,x)$ be the asserted solution of (2.21)-(2.24). Let $\pi(.)$ be a portfolio process with corresponding wealth process $X^\pi(t)$ satisfying the initial condition (2.14) and $\pi(t)\in[-\alpha,\alpha]$. We then have:

$$\begin{aligned}G(t,X^\pi(t)) &= G(t_{i^e},X^\pi(t_{i^e}))+\int_{t_{i^e}}^t G_xX^\pi(s)\sigma\pi(s)dW(s)\\ &+ \int_{t_{i^e}}^t\left[G_t+G_xX^\pi(s)(r+\pi(s)(b-r))+\frac{1}{2}\sigma^2\pi(s)^2X^\pi(s)^2G_{xx}\right]ds\end{aligned}$$

for $i^e=\max\{i|t_i\le t\}$. From equations (2.15) and (2.22) we conclude

$$G(t_{i^e},X^\pi(t_{i^e}))=G(t_{i^e}-,X^\pi(t_{i^e})+C_i)=G(t_{i^e}-,X^\pi(t_{i^e}-)) \quad .$$

Thus, starting at (t_s, x) we can apply the Itô-formula to obtain inductively

$$\begin{aligned} G(t, X^\pi(t)) &= G(t_s, x) \\ &+ \int_{t_s}^{t_{i^b}} \left[G_t + G_x X^\pi(s)(r + \pi(s)(b-r)) + \frac{1}{2}\sigma^2 \pi(s)^2 X^\pi(s)^2 G_{xx} \right] ds \\ &+ \int_{t_s}^{t_{i^b}} G_x X^\pi(s) \sigma \pi(s) dW(s) \\ &+ \sum_{i=i^b}^{i^e-1} \int_{t_i}^{t_{i+1}} \left[G_t + G_x X^\pi(s)(r + \pi(s)(b-r)) + \frac{1}{2}\sigma^2 \pi(s)^2 X^\pi(s)^2 G_{xx} \right] ds \\ &+ \sum_{i=i^b}^{i^e-1} \int_{t_i}^{t_{i+1}} G_x X^\pi(s) \sigma \pi(s) dW(s) \\ &+ \int_{t_{i^e}}^{t} \left[G_t + G_x X^\pi(s)(r + \pi(s)(b-r)) + \frac{1}{2}\sigma^2 \pi(s)^2 X^\pi(s)^2 G_{xx} \right] ds \\ &+ \int_{t_{i^e}}^{t} G_x X^\pi(s) \sigma \pi(s) dW(s) \end{aligned}$$

where $i^b = \min\{i : t_i > t_s\}$. Due to the definition of $G(t,x)$ in (2.21), the facts that $\pi^*(t)$ attains the supremum in (2.21), and lies in $[-\alpha, \alpha]$ we have

$$E^{t_s,x}\left(G(t, X^\pi(t))\right) \leq E^{t_s,x}\left(G(t, X^{\pi^*}(t))\right)$$

for all $t \in [0,T]$ and $\pi \in \mathcal{A}^{\mathcal{H}}(t_s, x)$ (note the polynomiality of $G(t,x)$ and the boundedness of $\pi(.)$, and $\pi^*(t)$) and in particular

$$\begin{aligned} E^{t_s,x}\left(\frac{1}{\gamma}\left(X^\pi(T)\right)^\gamma\right) &= E^{t_s,x}\left(G(T, X^\pi(T))\right) \\ &\leq E^{t_s,x}\left(G(T, X^{\pi^*}(T))\right) = E^{t_s,x}\left(\frac{1}{\gamma}\left(X^{\pi^*}(T)\right)^\gamma\right) \end{aligned}$$

As $\pi^*(t) \in (-\alpha, \alpha)$, $\pi^*(t)$ is an (interior) optimal control, which is still optimal if we make α arbitrarily large. Hence $V(t,x) = E^{t_s,x}\left(G(T, X^{\pi^*}(T))\right)$, and using (2.21) we get

$$V(t,x) = E^{t_s,x}\left(G(T, X^{\pi^*}(T))\right) = G(t,x)$$

□

2.3.3 Generalized Consumption and Income

In the following we investigate the portolio problem with both consumption and income simultaneously. In both cases we deal with continuous and discrete monetary streams. More precisely, we assume that discrete consumption and income takes place at fixed time instants $t_1, \ldots, t_m$ with $0 < t_1 < \ldots < t_m \leq T$ and is required to equal values D_i at times t_i, where $D_i > 0$ means consumption and $D_i < 0$ means income. We denote the continuous monetary stream by $c(t)$, where again $c(t) > 0$ stands for consumption and $c(t) < 0$ for income. Having seen both the relevant idea and the solution of the HJB-equation in Section 2.2, it is easy to figure out the necessary ingredients to solve the problem in the generalized case. Of course, if the value of future obligations is positive, we then do not have to set aside capital at the beginning. Just the opposite, as we are certain to get more capital in the future we can already take advantage of it. More precisely, we raise a credit to invest future income today to get a higher overall-return.
Observe, that the sign of the present value of future consumption and income can be changing over time. The main idea now is to add this present value - independent of its sign - to our wealth and to invest this then obtained capital as if there were no consumption or income at all.
The value of discrete streams D_i with $t_i > t$ equals

$$\sum_{i:t_i>t} D_i e^{-r(t_i-t)}.$$

The value of the continuous monetary stream $c(s)$ at time t equals

$$X_c(t) = \int_t^T \exp(-r(s-t))c(s)ds.$$

In total, we get the following condition on the wealth process

$$X(t) \geq X_c(t) + \sum_{i:t_i>t} D_i e^{-r(t_i-t)} \quad , \quad t \in [0,T]. \tag{2.25}$$

We solve this optimisation problem by using $u(t)$, the amounts of money invested in the stocks as control process, instead of $\pi(t)$. The wealth process then has the representation

$$dX(t) = [X(t)r + (b-r)u(t) + c(t)]dt + u(t)\sigma dW_t \tag{2.26}$$

on (t_i, t_{i+1}) and the jump condition equals

$$X(t_i) = X(t_i-) - D_i. \tag{2.27}$$

Note that by the form of this requirement we also indicate that $X(t)$ is the wealth at time t after the discrete payment at time t has been made. We get the following value function

$$V(t,x) = \sup_{u \in \mathcal{A}^*(t,x)} E^{t,x}\left[\frac{1}{\gamma}X^u(T)^\gamma\right] \tag{2.28}$$

with the obvious jump condition

$$V(t_i, x - D_i) = V(t_i-, x) \tag{2.29}$$

for all x satisfying equation (2.25) in place of $X(t)$ and with $\mathcal{A}^*(t,x)$ being the corresponding admissible set of controls for $u(t)$. However, on intervalls (t_i, t_{i+1}) $V(t,x)$ should satisfy the usual HJB-Equation as we will prove in the verification theorem below.

Remark
The corresponding boundary condition of the value function is

$$V\left(t, X_c(t) + \sum_{i:t_i>t} D_i e^{-r(t_i-t)}\right) = 0\,.$$

So if the value of future streams is positive at a particular time instant $\tilde{t} \in [0,T]$, the domain of $V(t,x)$ and the corresponding control includes points (t,x) with x=0 (in particular $(\tilde{t},0)$). Looking at the optimal control (2.20) we see that just copying the methods of Sections 2.3.1 or 2.3.2 cannot work, since π would not be defined for x=0. We therefore overcome this problem by choosing as control $u(t)$, the process of money invested in the stock instead of the portfolio process $\pi(t)$. However, the main ideas will stay valid here.

Theorem 2.6 Optimisation with consumption and income
For a given set of monetary streams D_i at discrete times t_i, $i = 1,\ldots,m$,with $0 \le t_1 < \ldots < t_m \le T$ and a continuous stream $c(s)$ with present value $X_c(t) = \int_t^T \exp(-r(t-s))c(s)ds$, let our initial capital satisfy

$$x > X_c(t) + \sum_{i:t_i>t} D_i e^{-r(t_i-t)}. \tag{2.30}$$

Then the value function of problem (2.28) is given by

$$V(t,x) = \frac{1}{\gamma}\left(x - X_c(t) - \sum_{i:t_i>t} D_i e^{-r(t_i-t)}\right)^{\gamma} e^{\gamma\left(r+\frac{1}{2}\left(\frac{b-r}{\sigma}\right)^2\frac{1}{1-\gamma}\right)(T-t)} \tag{2.31}$$

for all pairs (t,x) with $t \in [0,T]$ and $x \in \left[X_c(t) + \sum_{i:t_i>t} D_i e^{-r(t_i-t)}, \infty\right)$. The corresponding process of amounts of money invested in the stock has the form

$$u^*(t) = \frac{b-r}{\sigma^2(1-\gamma)}\left[X(t) - X_c(t) - \sum_{i:t_i>t} D_i e^{-r(t_i-t)}\right]\,.$$

Proof.
The HJB-equation corresponding to our problem has the form

$$\sup_{u\in\mathbb{R}}\left\{{}^1\!/\!_2 u^2\sigma^2 V_{xx}(t,x) + (rx + (b-r)u + c(t))\,V_x(t,x) + V_t(t,x)\right\} = 0 \tag{2.32}$$

for all pairs (t,x) satisfying the constrains $t \in [0,T] \setminus \{t_1,\ldots,t_m\}$ and $x \in \left(X_c(t) + \sum_{i:t_i>t} D_i e^{-r(t_i-t)}, \infty\right)$ and boundary conditions

$$\begin{aligned} V(T,x) &= {}^1\!/\!_\gamma x^\gamma \\ V\left(t, X_c(t) + \sum_{i:t_i>t} D_i e^{-r(t_i-t)}\right) &= 0 \\ V(t_i, x - D_i) &= V(t_i-, x) \end{aligned}$$

The Verification Theorem 2.7 below indicates that $V(t,x)$ is the unique (piecewise) smooth solution of the corresponding HJB equation that also satisfies the jump condition (2.29). We will verify that $V(t,x)$ as given in (2.31) above has these properties and hence coincides with the value function. In doing the first step to arrive at this solution we perform the optimisation in (2.32) which results in the candidate for the optimal portfolio process

$$u(t) = -\frac{b-r}{\sigma^2}\frac{V_x(t,x)}{V_{xx}(t,x)}\,,$$

where at times t_i we have taken the right-continuous limit of the derivatives. As a consequence this leads to the equation

$$V_t + (rx + c(t))\,V_x - {}^1\!/_2\left(\frac{b-r}{\sigma}\right)^2\frac{V_x^2}{V_{xx}} = 0\,,$$

which has the same domain as the HJB-equation (2.32). We now verify that $V(t,x)$ as given in (2.31) solves this equation. To make this easier we introduce

$$A = x - X_c(t) - \sum_{i:t_i>t} D_i e^{-r(t_i-t)}, \quad B = \exp\left(\gamma\left(r + {}^1\!/_2\left(\tfrac{b-r}{\sigma}\right)^2\tfrac{1}{1-\gamma}\right)(T-t)\right).$$

This leads to

$$\begin{aligned} V_t &= A^{\gamma-1}\left(-X_c'(t) - r\sum_{i:t_i>t} D_i e^{-r(t_i-t)}\right)B - A^\gamma B\left(r + {}^1\!/_2\tfrac{1}{1-\gamma}\left(\tfrac{b-r}{\sigma}\right)^2\right) \\ V_x &= A^{\gamma-1}B \\ V_{xx} &= (\gamma-1)\,A^{\gamma-2}B \end{aligned}$$

and to

$$\begin{aligned} & V_t + (rx + c(t))\,V_x - {}^1\!/_2\left(\frac{b-r}{\sigma}\right)^2\frac{V_x^2}{V_{xx}} \\ =\; & (rx + c(t))\,A^{\gamma-1}B + {}^1\!/_2\left(\tfrac{b-r}{\sigma}\right)^2\tfrac{1}{1-\gamma}A^\gamma B \\ & + A^{\gamma-1}\left(-X_c'(t) - r\sum_{i:t_i>t} D_i e^{-r(t_i-t)}\right)B - A^\gamma B\left(r + {}^1\!/_2\left(\tfrac{b-r}{\sigma}\right)^2\tfrac{1}{1-\gamma}\right) \\ =\; & A^{\gamma-1}B\left[(rx + c(t)) + \left(-X_c'(t) - r\sum_{i:t_i>t} D_i e^{-r(t_i-t)}\right) - Ar\right] \\ =\; & A^{\gamma-1}B\left[c(t) - X_c'(t) + X_c(t)\right] = 0. \end{aligned}$$

As a further result we obtain the optimal portfolio process

$$u^*(t) = -\frac{b-r}{\sigma^2}\frac{V_x}{V_{xx}} = \frac{b-r}{\sigma^2(1-\gamma)}\left(X(t) + Y(t) + \sum_{i:t_i>t} D_i e^{-r(t_i-t)}\right),$$

where at times t_i we have taken the right-continuous limit of the derivatives.

□

It thus only remains to prove the verification theorem.

Theorem 2.7 Verification theorem for income and consumption
Let G(t,x) be a polynomially bounded solution of

$$\sup_{\pi\in\mathbb{R}}\left\{\frac{1}{2}\sigma^2u^2G_{xx}(t,x)+[xr+(b-r)u]G_x(t,x)+G_t(t,x)\right\}=0 \tag{2.33}$$

for all $t\in[0,T]\setminus\{t_1,\ldots,t_m\}, x>-Y(t)-\sum_{i:t_i>t}D_ie^{-r(t_i-t)}$ *and*

$$G(t_i-,x) = G(t_i,x+D_i) \tag{2.34}$$

$$G\left(t,-Y(t)-\sum_{i:t_i>t}D_ie^{-r(t_i-t)}\right) = 0 \tag{2.35}$$

$$G(T,x) = \frac{1}{\gamma}x^\gamma \tag{2.36}$$

which is in $C^{1,2}$ *on* $(t_i,t_{i+1}), i=0,\ldots,m$ *with* $t_0=0, t_{n+1}=T$*. Le further be*

$$u^*(t,x)=-\frac{b-r}{\sigma^2}\frac{G_x(t,x)}{G_{xx}(t,x)}$$

(where in points t_i *we take the right hand limits of the derivatives). Then,* $G(t,x)$ *coincides with the value function* $V(t,x)$*, and* $u^*(t,X^{u^*}(t))$ *is an optimal control process for problem (2.28).*

Proof.
Let $G(t,x)$ be the asserted solution of (2.33-2.36). Let $u(.)$ be the process of the amounts of money invested in the stock with corresponding wealth process $X^u(t)$ satisfying (2.26). Again, we apply the Itô formula to obtain inductively for each admissible control process $u(.)$

$$\begin{aligned} & G(t,X^\pi(t)) \\ = \; & G(t_s,x) \\ + \; & \int_{t_s}^{t_{ib}}\left[G_t+G_x\Big(X^u(s)r+u(s)(b-r)+c(s)\Big)+\frac{1}{2}\sigma^2u(s)^2G_{xx}\right]ds \\ + \; & \int_{t_s}^{t_{ib}}G_x\sigma u(s)dW(s)\end{aligned}$$

$$
\begin{aligned}
&+ \sum_{i=i^b}^{i^e} \int_{t_i}^{t_{i+1}} \left[G_t + G_x \Big(X^u(s)r + u(s)(b-r) + c(s) \Big) + \frac{1}{2}\sigma^2 u(s)^2 G_{xx} \right] ds \\
&+ \sum_{i=i^b}^{i^e} \int_{t_i}^{t_{i+1}} G_x \sigma u(s) dW(s) \\
&+ \int_{t_{i^e}}^{t} \left[G_t + G_x \Big(X^u(s)r + u(s)(b-r) + c(s) \Big) + \frac{1}{2}\sigma^2 u(s)^2 G_{xx} \right] ds \\
&+ \int_{t_{i^e}}^{t} G_x \sigma u(s) dW(s)
\end{aligned}
$$

where $i^b = \min\{i : t_i > t_s\}$ and $i^e = \max\{i : t_i \le t\}$. Due to the form of $u^*(t,x)$ (an affine linear function of $X^*(t)$), $X^*(t)$ is the unique solution of the corresponding wealth equation. Further as $G(t,x)$ solves (2.33) and the fact that $u^*(t)$ attains the supremum in (2.33) we have:

$$
\begin{aligned}
E^{t_s,x}\left(\frac{1}{\gamma} \left(X^u(T) \right)^\gamma \right) &= E^{t_s,x}\left(G(T, X^u(T)) \right) \\
&\le E^{t_s,x}\Big(G(T, X^{u^*}(T)) \Big) = E^{t_s,x}\left(\frac{1}{\gamma} \Big(X^{u^*}(T) \Big)^\gamma \right)
\end{aligned}
$$

where the expectations are finite due to the polynomiality of $G(t,x)$. Thus finally we obtain:

$$
V(t,x) = E^{t_s,x}\Big(g(T, X^{\pi^*}(T)) \Big) = G(t,x).
$$

□

2.4 A Separation Theorem for Requirements

The solutions obtained above by the HJB-technology all show a very natural formal separation of the initial wealth in an amount necessary to cover the consumption requirements and a remaining part which the investor can use to speculate in an optimal way. In this section we will show that there is a general separation principle that can also be used to cover cases such as regular pay-in schemes. The basic idea behind this approach is the fact that due to our assumption of a complete market the additional monetary streams of the investor can be hedged by suitable investment strategies. To state the separation theorem, we have to introduce some notation :

Since we assumed a complete market, each discrete monetary stream B_i maturing at t_i $(i = 1, \dots, m)$ can be hedged with an appropriate self-financing portfolio process $\pi_i(.)$ and a corresponding wealth process with initial wealth given by:

$$\begin{aligned} dX_i(t) &= X_i(t)\left[(r + \pi_i'(t)(b - r\underline{1}))dt + \pi_i'(t)\sigma dW_t\right] \quad (2.37) \\ X_i(0) &= x_i := E^*[e^{-\int_0^{t_i} r(s)ds} B_i] \\ X_i(t_i) &= B_i \ \ \mathbf{P}\text{–a.s.} \end{aligned}$$

Observe that $\pi_i(.)$ is not the optimal control for the i-th stock, but for the i-th "option" B_i. For the continuous streams we have:

$$\begin{aligned} dX_c(t) &= X_c(t)\left[(r + \pi_c'(t)(b - r\underline{1}))dt + \pi_c'(t)\sigma dW_t\right] - c(t)dt \quad (2.38) \\ X_c(0) &= x_c := E^*\left[\int_0^T e^{-\int_0^t r(s)ds} c(t)dt\right] \\ X_c(T) &= 0 \ \ \mathbf{P}\text{–a.s.} \end{aligned}$$

Remember that in both cases the wealth process is almost surely nonnegative (in both the objective- and martingale probability measure) at any time. So the strategies contain no bankruptcy risk.

Instead of trying to maximise the utility of a single wealth process which contains all payments streams as defined by equations (2.1) and (2.2), we will try to maximise the utility of a decomposed wealth process:

$$\overline{X}^{\tilde{\pi}}(t) = \widetilde{X}^{\tilde{\pi}}(t) + \sum_{j=s(t)}^{n} X_j(t) + X_c(t) \quad , s(t) = \min\{i : t_i > t\} \quad (2.39)$$

where on $[t_i, t_{i+1})$ we have the evolution described by

$$d\overline{X}^{\tilde{\pi}}(t) = d\widetilde{X}^{\tilde{\pi}}(t) + \sum_{j=i+1}^{m} dX_j(t) + dX_c(t) \quad , \quad (2.40)$$

and at time instants t_i we have the "jump conditions"

$$\begin{aligned} \overline{X}^{\tilde{\pi}}(t_i) &= \overline{X}^{\tilde{\pi}}(t_i-) - B_i \quad (2.41) \\ &= \widetilde{X}^{\tilde{\pi}}(t_i-) + \sum_{j=i+1}^{m} X_j(t_i-) + X_c(t_i-) + \underbrace{X_i(t_i-) - B_i}_{=0} \quad (2.42) \end{aligned}$$

The separated wealth processes satisfies:

$$\begin{aligned} d\widetilde{X}^{\tilde{\pi}}(t) &= \widetilde{X}^{\tilde{\pi}}(t)\left[(r + \tilde{\pi}'(t)(b - r\underline{1}))dt + \tilde{\pi}'(t)\sigma dW_t\right] \quad (2.43) \\ \widetilde{X}^{\tilde{\pi}}(0) &= x - \sum_{j=1}^{n} x_j - x_c \end{aligned}$$

Observe that $\overline{X}^{\tilde{\pi}}(T) = \widetilde{X}^{\tilde{\pi}}(T)$ and $\widetilde{X}^{\tilde{\pi}}(t)$ contains no consumption requirements. We will show that the processes $\overline{X}^{\tilde{\pi}}(t)$ and $X^{\pi}(t)$ as defined in (2.1) and (2.2) are equivalent under the constraint that both have to be nonnegative and the controls have to be admissible. So if our objective is to maximise the final utility, we would of course prefer to maximise $\overline{X}^{\tilde{\pi}}(T)$, because that means we have to deal with the simple wealth process $\widetilde{X}^{\tilde{\pi}}(t)$. Let us summarize our considerations in the following theorem:

Theorem 2.8 *Let $X^{\pi}(t)$ be the wealth process given by (2.1,2.2), $\widetilde{X}^{\tilde{\pi}}(t)$ the process given by (2.43) and $\tilde{\mathcal{A}}^{\mathcal{H}}(t,x)$ the set of its admissible portfolio processes. Then:*

i) For every $\tilde{\pi} \in \tilde{\mathcal{A}}^{\mathcal{H}}(t,x)$, there exists a $\pi \in \mathcal{A}^{\mathcal{H}}(t,x)$ with

$$X^{\pi}(t) = \overline{X}^{\tilde{\pi}}(t) \forall t \in [0,T]$$

ii) For every $\pi \in \mathcal{A}^{\mathcal{H}}(t,x)$, there exists a $\tilde{\pi} \in \tilde{\mathcal{A}}^{\mathcal{H}}(t,x)$ with

$$X^{\pi}(t) = \overline{X}^{\tilde{\pi}}(t) \forall t \in [0,T]$$

iii) By the above assertions it follows

$$\sup_{\pi \in \mathcal{A}^{\mathcal{H}}(t,x)} E\left[U(X^{\pi}(T))\right] = \sup_{\tilde{\pi} \in \tilde{\mathcal{A}}^{\mathcal{H}}(t,x)} E\left[U(\widetilde{X}^{\tilde{\pi}}(T))\right]$$

PROOF.
1. Let $t \in [t_i, t_{i+1})$. Assume that $X(t_i) = \widetilde{X}(t_i)$. Define $\pi(t)$ as:

$$\begin{aligned} \pi(t) &= \frac{\left(X^{\pi}(t) - \sum_{j=i+1}^{m} X_j(t) - X_c(t)\right)\tilde{\pi}(t)}{X^{\pi}(t)} \\ &+ \frac{\sum_{j=i+1}^{m} X_j(t)\pi_j(t) + X_c(t)\pi_c(t)}{X^{\pi}(t)} \end{aligned} \tag{2.44}$$

Note that the value $X^{\pi}(t)$ $(\tilde{X}^{\tilde{\pi}}(t))$ does not depend on $\pi(t)$ $(\tilde{\pi}(t))$ but of course on $\pi(s)(\tilde{\pi}(s)), s \in [0,t)$. Therefore, the definitions of $\pi(t)$ $(\tilde{\pi})$ are explicit ones and not implicit.

This yields:

$$\begin{aligned}
X^\pi(t) &= X^\pi(t_i) + \int_{t_i}^t X^\pi(s)[r + \pi'(s)(b - r\underline{1})] - c(s)\,ds + \int_{t_i}^t X^\pi(s)\pi'(s)\sigma dW_s \\
&= X^\pi(t_i) + \int_{t_i}^t X^\pi(s)r + \left(X^\pi(s) - \sum_{j=i+1}^m X_j(s) - X_c(s)\right)[\tilde{\pi}'(s)(b - r\underline{1})]\,ds \\
&\quad + \int_{t_i}^t \left(X^\pi(s) - \sum_{j=i+1}^m X_j(s) - X_c(s)\right)\tilde{\pi}'(s)\sigma dW_s \\
&\quad + \sum_{j=i+1}^m \left\{\int_{t_i}^t X_j(s)[\pi_j'(s)(b - r\underline{1})]\,ds + \int_{t_i}^t X_j(s)\pi_j'(s)\sigma dW_s\right\} \\
&\quad + \int_{t_i}^t X_c(s)[\pi_c'(s)(b - r\underline{1})] - c(s)\,ds + \int_{t_i}^t X_c(s)\pi_c'(s)\sigma dW_s \\
&= \left(X^\pi(t_i) - \sum_{j=i+1}^m X_j(t_i) - X_c(t_i)\right) \\
&\quad + \int_{t_i}^t \left(X^\pi(s) - \sum_{j=i+1}^m X_j(s) - X_c(s)\right)[r + \tilde{\pi}'(s)(b - r\underline{1})]\,ds \\
&\quad + \int_{t_i}^t \left(X^\pi(s) - \sum_{j=i+1}^m X_j(s) - X_c(s)\right)\tilde{\pi}'(s)\sigma dW_s \\
&\quad + \sum_{j=i+1}^m \underbrace{\left\{X_j(t_i) + \int_{t_i}^t X_j(s)[r + \pi_j'(s)(b - r\underline{1})]\,ds + \int_{t_i}^t X_j(s)\pi_j'(s)\sigma dW_s\right\}}_{X_j(t)} \\
&\quad + \underbrace{X_c(t_i) + \int_{t_i}^t X_c(s)[r + \pi_c'(s)(b - r\underline{1})] - c(s)\,ds + \int_{t_i}^t X_c(s)\pi_c'(s)\sigma dW_s}_{X_c(t)}
\end{aligned}$$

$\Longrightarrow$

$$\begin{aligned}
&\left(X^{\pi}(t)-\sum_{j=i+1}^{m}X_j(t)-X_c(t)\right)\\
=&\left(X^{\pi}(t_i)-\sum_{j=i+1}^{m}X_j(t_i)-X_c(t_i)\right)\\
+&\int_{t_i}^{t}\left(X^{\pi}(s)-\sum_{j=i+1}^{m}X_j(s)-X_c(s)\right)[r+\tilde{\pi}'(s)(b-r\underline{1})]\,ds\\
+&\int_{t_i}^{t}\left(X^{\pi}(s)-\sum_{j=i+1}^{m}X_j(s)-X_c(s)\right)\tilde{\pi}'(s)\sigma dW_s
\end{aligned}$$

So $X^{\pi}(t)-\sum_{j=i+1}^{m}X_j(t)-X_c(t)$ follows the same dynamics as $\widetilde{X}^{\tilde{\pi}}(t)$, hence

$$X^{\pi}(t)=\widetilde{X}^{\tilde{\pi}}(t)+\sum_{j=i+1}^{m}X_j(t)-X_c(t)$$

for all $t\in[t_i,t_{i+1})$. The assertion for all $t\in[0,T]$ follows by induction and due to the fact that $X_i(t_i)=B_i$ almost surely.

2. Let $t\in[t_i,t_{i+1})$. Assume that $\widetilde{X}(t_i)=X(t_i)$. Define $\tilde{\pi}(t)$ as:

$$\tilde{\pi}(t)=\frac{\pi(t)\left(\widetilde{X}^{\tilde{\pi}}(t)+\sum_{j=i+1}^{m}X_j(t)+X_c(t)\right)-\sum_{j=i+1}^{m}X_j(t)\pi_j(t)+X_c(t)\pi_c(t)}{\widetilde{X}^{\tilde{\pi}}(t)}$$

Plugging this into (2.43) yields:

$$\begin{aligned}
\widetilde{X}^{\tilde{\pi}}(t)\;=\;&\widetilde{X}^{\tilde{\pi}}(t_i)+\int_{t_i}^{t}r\widetilde{X}^{\tilde{\pi}}(s)+\left(\widetilde{X}^{\tilde{\pi}}(s)+\sum_{j=i+1}^{m}X_j(s)+X_c(s)\right)\pi'(s)(b-r\underline{1})ds\\
&-\int_{t_i}^{t}\sum_{j=i+1}^{m}X_j(s)\pi_j'(s)(b-r\underline{1})+X_c(s)\pi_c'(s)(b-r\underline{1})ds\\
&+\int_{t_i}^{t}\left(\widetilde{X}(s)+\sum_{j=i+1}^{m}X_j(s)+X_c(s)\right)\pi'(s)\sigma dW_s\\
&-\int_{t_i}^{t}\left(\sum_{j=i+1}^{m}X_j(s)\pi_j'(s)+X_c(s)\pi_c'(s)\right)\sigma dW_s
\end{aligned}$$

$$
\begin{aligned}
= \;& \left(\widetilde{X}(t_i) + \sum_{j=i+1}^{m} X_j(t_i) + X_c(t_i)\right) \\
&+ \int_{t_i}^{t} \left(\widetilde{X}(s) + \sum_{j=i+1}^{m} X_j(s) + X_c(s)\right)(r + \pi'(s)(b - r\underline{1}))ds \\
&+ \int_{t_i}^{t} \left(\widetilde{X}(s) + \sum_{j=i+1}^{m} X_j(s) + X_c(s)\right)\pi'(s)\sigma dW_s \\
&- \sum_{j=i+1}^{m} \underbrace{\left\{X_j(t_i) + \int_{t_i}^{t} X_j(s)[r + \pi_j'(s)(b - r\underline{1})] + \int_{t_i}^{t} X_j(s)\pi_j'(s)\sigma dW_s\right\}}_{X_j(t)} \\
&- \underbrace{\left\{X_c(t_i) + \int_{t_i}^{t} X_c(s)[r + \pi_c'(s)(b - r\underline{1})]ds + \int_{t_i}^{t} X_c(s)\pi_c'(s)\sigma dW_s\right\}}_{X_c(t)}
\end{aligned}
$$

$\Longrightarrow$

$$
\begin{aligned}
&\left(\widetilde{X}^{\tilde{\pi}}(t) + \sum_{j=i+1}^{m} X_j(t) + X_c(t)\right) \\
= \;& \left(\widetilde{X}^{\tilde{\pi}}(t_i) + \sum_{j=i+1}^{m} X_j(t_i) + X_c(t_i)\right) \\
+ \;& \int_{t_i}^{t} \left(\widetilde{X}^{\tilde{\pi}}(s) + \sum_{j=i+1}^{m} X_j(s) + X_c(s)\right)\pi'(s)(b - r\underline{1})ds \\
+ \;& \int_{t_i}^{t} \left(\widetilde{X}^{\tilde{\pi}}(s) + \sum_{j=i+1}^{m} X_j(s) + X_c(s)\right)\pi'(s)\sigma dW_s
\end{aligned}
$$

Thus $\left(\widetilde{X}^{\tilde{\pi}}(t) + \sum_{j=i+1}^{m} X_j(t) + X_c(t)\right)$ follows the same dynamics as $X^{\pi}(t)$.

3. From 1. we conclude:

$$\sup_{\pi \in \mathcal{A}^{\mathcal{H}}(t,x)} E\left[U(X^{\pi}(T))\right] \geq \sup_{\tilde{\pi} \in \tilde{\mathcal{A}}^{\mathcal{H}}(t,x)} E\left[U(\widetilde{X}^{\tilde{\pi}}(T))\right]$$

And from 2.:

$$\sup_{\pi \in \mathcal{A}^{\mathcal{H}}(t,x)} E\left[U(X^{\pi}(T))\right] \leq \sup_{\tilde{\pi} \in \tilde{\mathcal{A}}^{\mathcal{H}}(t,x)} E\left[U(\widetilde{X}^{\tilde{\pi}}(T))\right]$$

So the assertion is proved.

□

Remark 2.9 (The Use of the Separation Theorem)
Our Separation Theorem results in the following algorithm to solve the portfolio problem:

i) Calculate the Black-Scholes-prices x_j of the outstanding contingent claims B_j ($j = 1, \dots, m$) and the BS-price x_c of the continuous monetary stream $c(t)$.

ii) Substract these prices from the initial wealth:

$$\widetilde{X}^{\tilde{\pi}}(0) = \overline{X}^{\tilde{\pi}}(0) - \sum_{j=1}^{n} x_j - x_c \quad .$$

iii) Solve the simplified optimization problem (without consumption)

$$\tilde{\pi}^* = \arg \max_{\tilde{\pi} \in \tilde{\mathcal{A}}(t,x)} E\left[U(\widetilde{X}^{\tilde{\pi}}(T))\right]$$

by the usual HJB-method, whereby $\widetilde{X}^{\tilde{\pi}}(t)$ is given by (2.43).

iv) Calculate the optimal control for the total wealth by formula (2.44).

To illustrate our Separation Theorem we solve again the continuous consumption problem:

Example 2.10 (Continuous Consumption with HARA–Utility)
Let $c(t) \equiv c$ be a constant consumption stream, x our total initial wealth and the market parameters r, b, σ constant. Then the initial capital needed to cover this consumption reads as:

$$x_c \equiv X_c(0) = E^*\left[\int_0^T e^{-rt} c dt\right] = \tfrac{c}{r}\left(1 - \exp\left(-rT\right)\right) .$$

Thus the wealth process of our decomposed optimisation problem is given by

$$\begin{aligned} d\widetilde{X}^{\tilde{\pi}}(t) &= \widetilde{X}^{\tilde{\pi}}(t)\left[(r + \tilde{\pi}'(t)(b - r\underline{1}))dt + \tilde{\pi}'(t)\sigma dW_t\right] , \\ \widetilde{X}^{\tilde{\pi}}(0) &= x - \tfrac{c}{r}\left(1 - \exp\left(-rT\right)\right) , \end{aligned}$$

The solution of the new optimisation problem

$$\widetilde{\pi}^*(t) = \arg \max_{\tilde{\pi} \in \tilde{\mathcal{A}}(t,x)} E^{t,x}\left[\frac{1}{\gamma}(\widetilde{X}^{\tilde{\pi}}(T))^{\gamma}\right]$$

is well-known and reads as

$$\widetilde{\pi} = \frac{(\sigma\sigma')^{-1}}{1-\gamma}(b - r\underline{1}) .$$

The composed wealth process reads as:

$$\begin{aligned}
\bar{X}^{\tilde{\pi}}(t) &= \widetilde{X}^{\tilde{\pi}}(t) + X_c(t) \\
&= \left(x - \frac{c}{r}\left(1 - \exp(-rT)\right)\right) \exp\left[\left(r + \left((b - r\underline{1})' \frac{(\sigma\sigma')^{-1}}{1-\gamma}(b - r\underline{1})\right)\right.\right. \\
&\quad \left.\left. -\frac{1}{2}(b - r\underline{1})' \frac{(\sigma\sigma')^{-1}}{(1-\gamma)^2}(b - r\underline{1})\right) t + \frac{(b - r\underline{1})'}{1-\gamma}\sigma^{-1'} W(t)\right] \\
&\quad + \tfrac{c}{r}\left(1 - \exp\left(-r\left(T - t\right)\right)\right)
\end{aligned}$$

Since the consumption stream is deterministic, its replicating portfolio π_c is equal to zero. With formula (2.44) we can determine the optimal portfolio process of our original problem:

$$\begin{aligned}
\pi(t) &= \frac{\left(X^{\pi}(t) - X_c(t)\right)\tilde{\pi}(t)}{X^{\pi}(t)} \\
&= \frac{(\sigma\sigma')^{-1}}{1-\gamma}(b - r\underline{1}) \left[1 - \frac{c}{rX^{\pi}(t)}\left(1 - \exp\left(-r\left(T - t\right)\right)\right)\right].
\end{aligned}$$

Comparing the wealth process with (2.13) and the optimal portfolio process with (2.10), we can conclude that these results are confirmed by the solutions from the HJB-approach.

Example 2.11 (HARA-Utility with an option)
Let B be a vanilla call with strike price K, x our total initial wealth and the market parameters r, b, σ are constant. Then the initial capital needed to cover this consumption reads as:

$$x_1 \equiv X_1(0) = E^*\left[e^{-rT}(S(T) - K)^+\right] = V(S(0), K, 0)$$

with

$$V(s,k,t) = sN\left(\frac{\log(s/k) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\right) - ke^{-r(T-t)}N\left(\frac{\log(s/k) + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\right)$$

The initial wealth $\widetilde{X}^{\tilde{\pi}}(0)$ of our isolated wealth process $\widetilde{X}^{\tilde{\pi}}(.)$ is given by $x - x_1$. The solution of the optimisation problem

$$\widetilde{\pi}^*(t) = \arg \max_{\tilde{\pi} \in \tilde{\mathcal{A}}(t,x)} E^{t,x}\left[\frac{1}{\gamma}(\widetilde{X}^{\tilde{\pi}}(T))^{\gamma}\right]$$

remains the same

$$\widetilde{\pi} = \frac{(\sigma\sigma')^{-1}}{1-\gamma}(b - r\underline{1}).$$

The composed wealth process reads as:

$$\begin{aligned}\bar{X}^{\tilde{\pi}}(t) &= \widetilde{X}^{\tilde{\pi}}(t) + X_1(t)\\ &= \Big(x - \frac{c}{r}\left(1-\exp(-rT)\right)\Big)\exp\Bigg[\Bigg(r + \Big((b-r\underline{1})'\frac{(\sigma\sigma')^{-1}}{1-\gamma}(b-r\underline{1})\Big)\\ &\quad -\frac{1}{2}(b-r\underline{1})'\frac{(\sigma\sigma')^{-1}}{(1-\gamma)^2}(b-r\underline{1})\Bigg)t + \frac{(b-r\underline{1})'}{1-\gamma}\sigma^{-1'}W(t)\Bigg] + V(S(t),K,t)\end{aligned}$$

The control needed to replicate the option is given by:

$$\pi_1(t) = \frac{S(t)}{X_1(t)}N\left(\frac{\log(S(t)/K) + (r+\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\right)$$

Since the consumption stream is deterministic, its replicating portfolio π_c is equal to zero. With formula (2.44) we can determine the optimal portfolio process of our original problem:

$$\begin{aligned}\pi(t) &= \frac{\left(X^{\pi}(t) - X_1(t)\right)\tilde{\pi}(t) + X_1(t)\pi_1(t)}{X^{\pi}(t)}\\ &= \frac{(\sigma\sigma')^{-1}}{1-\gamma}(b-r\underline{1})\left[1-\frac{X_1(t)}{X^{\pi}(t)}\right] + \frac{S(t)}{X^{\pi}(t)}N\left(\frac{\log(S(t)/K) + (r-d+0.5\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\right)\end{aligned}$$

Observe that we need to know not only the wealth, but also the stock price $S(t)$ to determine the optimal control, which indicates that we solved a problem with a corresponding two dimensional HJB-equation, which - if the solution cannot be guessed - may become quite involved.

2.5 Numerical Illustration and Conclusions

To illustrate the behaviour of the portfolio process in the different situations presented so far we give some numerical examples:

Figure 1 corresponds to the continuous consumption case in Section 2.3.1 with $T = 1$, $b = 12\%$, $r = 5\%$, $\sigma = 20\%$, $\gamma = 0.5$ and consumption rate $c(t) \equiv +500$. The optimal control without consumption would be $\pi' = \frac{b-r}{(1-\gamma)\sigma^2} = 3.5$. We see that for $t \to 1$ and x constant the optimal control $\pi(t,x)$ converges to 3.5, since the amount of consumption, which has to be financed from the wealth is decreasing with time, and so we have more and more money left over to invest in stocks. For increasing wealth x and constant t the optimal control converges again to 3.5, since the role of consumption compared with total wealth can then be neglected. On the other hand, for $x \to c\frac{1-\exp(-r(T-t))}{r}$, $\pi(t,x)$ converges to zero, since if $X(t) = c\frac{1-\exp(-r(T-t))}{r}$, all wealth is needed to finance future consumption.

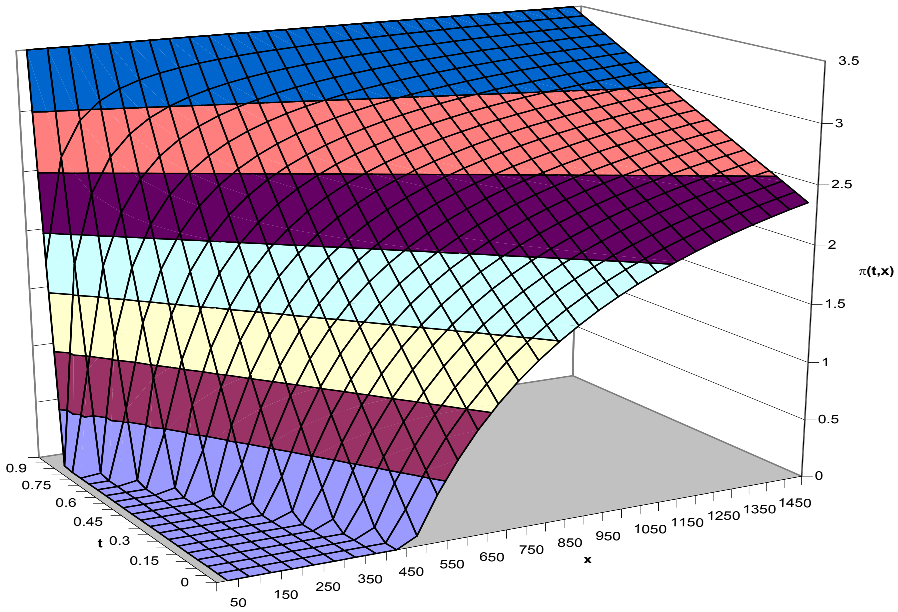


Figure 1: Optimal control π with continuous consumption

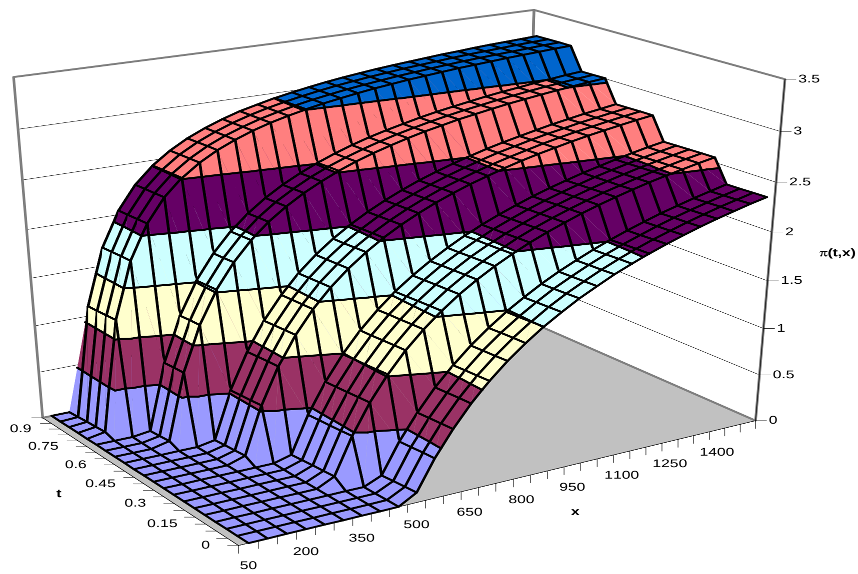


Figure 2: Optimal control π with lump sum consumption

Figure 2 shows the optimal control for the discrete consumption case in Section 2.3.2 with same stock parameters as above but lump sum consumption with $\triangle t = 0.2$ and $D_i \equiv -100$. It is not surprising, that we get jumps at consumption time instants. Besides this effect, the behaviour coincides with that of Figure 1.

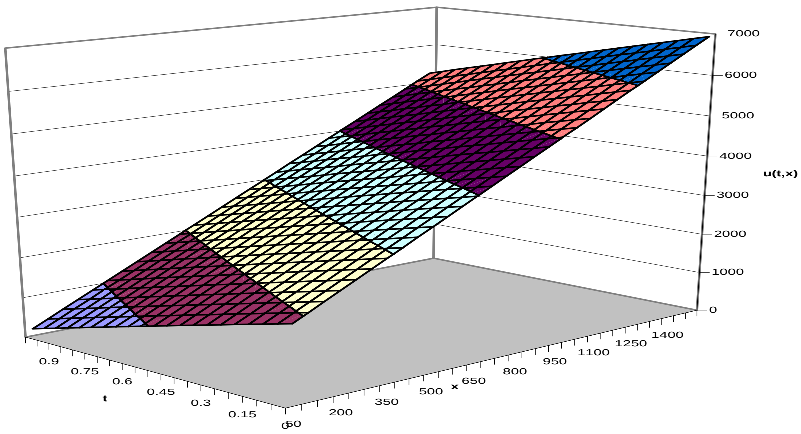


Figure 3: Optimal control u with continuous income

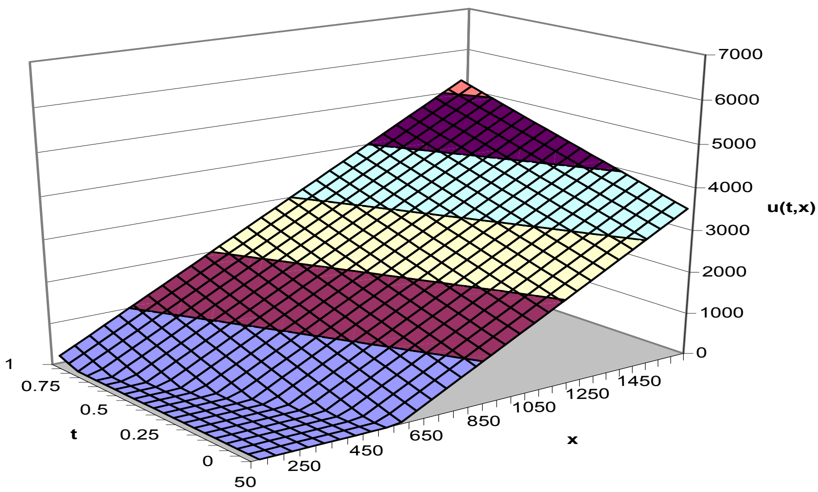


Figure 4: Optimal control u with continuous consumption

Figure 3 and Figure 4 illustrate the difference between continuous income and consumption, where we used the same parameters as before, except $c(t) = +500$ for the income rate. Note, that we changed the control process to be the amount of money invested in the stock instead of the portfolio process. In the case of income the optimal control decreases over time, because the amount of future income decreases. In the case for consumption it is just the other way around, i.e. the optimal control increases, since the money needed to finance future consumption decreases.

Conclusions

As private equity plans on one hand are getting more and more into fashion we believe that the results of this chapter have a practical relevance. Further, the case of an a priori fixed consumption plan seems to be much more realistic than that of a random consumption as treated in the standard formulation of the portfolio problem. With regard to this argument and our results one can thus always concentrate on the pure terminal wealth problem.
Even more general problems can be treated with our approach and are subjects of future research. Two possible candidates are: Optimal portfolios with fixed consumption/income and a loan dependent interest rate (see the next chapter and also Krekel (2001) for a related problem) and optimal portfolios with crash possibilities and fixed consumption/income (see Korn (2001)).

3 Optimal Portfolios with loan-dependent Interest Rates

3.1 Introduction

In the classical Merton framework the optimal trading strategy is to invest a multiple of the total wealth in the stocks for both cases, logarithmic as well as HARA-utility. With common market parameters this factor is often bigger than one (see Chapter 1 for the typical forms of the optimal portfolios). In other words, the investor is advised to borrow a multiple of his own wealth to speculate in risky assets. Of course in the presence of possible crashes no rational investor would do so, because this can result in immediate bankruptcy (see Korn and Wilmott (2001) who investigate optimal portfolios under the threat of a crash). On the other hand, since the default probability of this particular credit is much higher, the counterpart lending the money will definitely claim higher yields than that for government bonds. To take this effect into account, we introduce a loan-dependent interest rate, which we call in addition credit margin or interest rate spread. If the investor's bond position gets more and more negative, the risk of the lender will be almost the same, as if he invests in the stocks themselves. Therefore in a single stock setting, the loan-dependent interest rate should be modelled to converge (w.r.t. the control) to the return of the stock.

3.2 Model

We consider a security market consisting of an interest-bearing cash account and n risky assets as introduced in Section 1.1 by equations (1.1), (1.2) with the following modifications: We assume the volatility and the stock drift to be constant, but more importantly, distinguish between the interest rates for borrowing and lending. This feature will be modeled via a control dependent interest rate $r(t) := r(\pi_t)$, where $r(.) : I\!R^n \to I\!R$ is a left-continuous and bounded function, which will be defined later on. In financial terms, the bank or the financial market calls for a compensation in form of higher interest rates if it seems to be more risky to lend money to the investor. So we end up with the following processes. The cash account is modelled by the differential equation

$$dB(t) = B(t)r(\pi(t))dt,$$

where $r(\pi(t))$ is still a bounded, strictly positive and progressively measurable process, as defined in Section 1.1. Note in particular that with this kind of modeling each investor may be faced with a different evolution of his account. The price process of the i-th risky asset $S_i(t)$ defined by

$$dS_i(t) = S_i(t)[b_i dt + \sum_{j=1}^{n} \sigma_{ij} dW_j(t)]$$

remains the same with $\sigma\sigma'$ a strictly positive definite $N \times N$-matrix. As usual the investor starts with an initial wealth $x_0 > 0$ at time $t = 0$. In the beginning this initial wealth is invested in different assets and the investor is allowed to adjust his holdings continuously up to a fixed planning horizon T. His investment behavior is modeled by a portfolio process

$\pi(t) = (\pi_1(t), \ldots, \pi_n(t))$ as introduced in Definition (1.3) . Observe that $1 - \sum_{i=1}^n \pi_i(t)$ is the percentage of wealth invested in the savings account. So if $\sum_{i=1}^n \pi_i(t) > 1$ the investor is actually borrowing money and the interest rate spread comes into the game. With the new feature of control dependent interest rate the wealth process reads explicitely as

$$dX(t) = X(t)\left[\,\big(r(\pi(t))(1 - \pi'(t)\underline{1}) + \pi'(t)b\big)dt + \pi'(t)\sigma dW(t)\,\right], \tag{3.1}$$

with $X(0) = x_0$. Note that the presence of $r(\pi(t))$ introduces a non-linear dependence of the wealth process on $\pi(t)$. The properties of $r(.)$ ensure the existence of a solution of the SDE (3.1) The investor is only allowed to choose a portfolio process which is admissible and thus leads to a positive wealth process X^π. The final wealth is given by

$$X^\pi(T) = x_0 e^{\int_0^T \left(r(\pi(t))(1-\pi'(t)\underline{1})+\pi'(t)b-\frac{1}{2}\pi'(t)\sigma\sigma'\pi(t)\right)dt+\int_0^T \pi'(t)\sigma dW(t)}. \tag{3.2}$$

We will solve the optimization problem without consumption, i.e.

$$\max_{\pi(.)\in\mathcal{A}^{\mathcal{H}}(0,x_0)} E(U(X^\pi(T))), \tag{3.3}$$

with Log and HARA-utility. Note that in expression (3.3) we wrote *max* instead of *sup*. Via a new verification theorem we will show that such a maximum exists.

We suggest three ways of modeling $r(.)$ which covers all practical needs, and also prove to be quite usefull for numerical calculations. Let $\bar{r}$ be the interest rate for a positive cash account and $\hat{\pi}$ a real-valued vector, which denotes the percentage of wealth invested in the particular stocks. Keep in mind that $\hat{\pi}'\underline{1} = \sum_{i=1}^n \hat{\pi}_i$ is the total percentage of wealth invested in stocks. Our considered functions read as follows:

i) Step function

$$r(\hat{\pi}) = \bar{r} \; + \sum_{j=0}^{m-1} \lambda_i \mathbf{1}_{(\alpha_j,\alpha_{j+1}]}(\hat{\pi}'\underline{1}) \tag{3.4}$$

where $-\infty = \alpha_0 < 1 \leq \alpha_1 < \ldots < \alpha_j < \alpha_{j+1} < \ldots < \alpha_m = \infty$ and $0 = \lambda_0 < \lambda_1 < \ldots < \lambda_j < \lambda_{j+1} < \ldots < \lambda_{m-1} < \infty$.

ii) Frequency polygon

$$r(\hat{\pi}) \;=\; \bar{r} \; + \sum_{j=0}^{m-1}(r_j + \mu_j(\hat{\pi}'\underline{1} - \alpha_j))\mathbf{1}_{[\alpha_j,\alpha_{j+1})}(\hat{\pi}'\underline{1}) \tag{3.5}$$

$$r_j \;=\; \sum_{l=1}^{j} \mu_{l-1}(\alpha_l - \alpha_{l-1}), \quad j = 1, \ldots, m-1$$

where $-\infty = \alpha_0 < 1 \leq \alpha_1 < \ldots < \alpha_i < \alpha_{j+1} < \ldots < \alpha_m = \infty$, $\mu_j \geq 0$ for all $j = 1, \ldots, m-2$ and $\mu_0 = 0 = \mu_{m-1}, r_0 = 0$.

iii) Logistic function

$$r(\hat{\pi}) = \bar{r} + \lambda \frac{e^{\alpha\hat{\pi}'\underline{1}+\beta}}{e^{\alpha\hat{\pi}'\underline{1}+\beta}+1} \tag{3.6}$$

with $\lambda > 0$, $\alpha > 0$ and $\beta \in I\!R$.

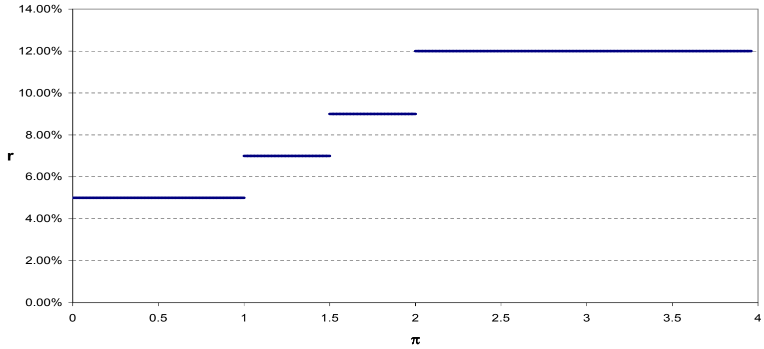


Figure 5: Step function

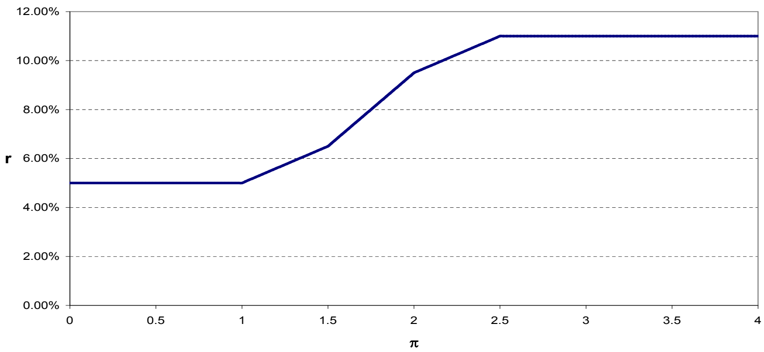


Figure 6: Frequency polygon

Note that simple dependencies, like $r(\hat{\pi}) = \bar{r}$ for $\hat{\pi}'\underline{1} \leq 1$ and $r(\hat{\pi}) = \bar{r} + \lambda$ for $\hat{\pi}'\underline{1} > 1$ can be modeled with the help of the step function. See Korn (1995) for the treatment of an option pricing problem in the presence of such a setting. With the frequency polygon we are able to model smoothly increasing credit spreads. In these cases, the optimisation problem (3.3) can be solved analytically, although we have to deal with some subcases separately.
The logistic function can be unterstood as a differentiable approximation of a frequency polygon with just one triangle. The main reason for its introduction is for numerical computations, because it is twice continuously differentiable and can be handled without considering subcases separately. An analytical solution is not available, but this does not matter with regard to the use in a numerical context.

In Section 3.3 we solve the optimization problem for the logarithmic utility and in Section 3.4 for the HARA utility. Section 3.5 gives the conclusion.

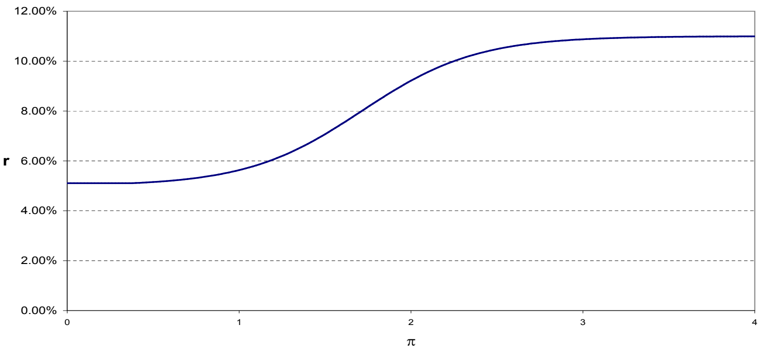


Figure 7: Logistic function

3.3 Logarithmic Utility

Let $U(x) = \log(x)$, then we have the following optimization problem

$$\begin{aligned} V(t,x) &:= \sup_{\pi(.)\in\mathcal{A}^{\mathcal{H}}(t,x)} E^{t,x}(\log(X^\pi(T))) \\ &= \sup_{\pi(.)\in\mathcal{A}^{\mathcal{H}}(t,x)} \Bigg\{ \log(x) + E\Bigg[\int_t^T \big(r(\pi(t))(1-\pi'(t)\underline{1}) + \pi'(t)b \\ &\qquad -\frac{1}{2}\pi'(t)\sigma\sigma'\pi(t)\big)dt\Bigg] + E\left[\int_t^T \pi'(t)\sigma dW(t)\right]\Bigg\}, \end{aligned} \tag{3.7}$$

where $r(x)$ is given by (3.4),(3.5) or (3.6). Using Fubini's Theorem for $\pi(t) \in L^2[0,T]$ and observing that in this case the whole process in the expectation-brackets is in $L^2[0,T]$, yields

$$\begin{aligned} & V(t,x) \\ = \ & \log(x) + \sup_{\pi(.)\in\mathcal{A}^{\mathcal{H}}(t,x)} \int_t^T E\left[\big(r(\pi(t))(1-\pi'(t)\underline{1}) + \pi'(t)b - \frac{1}{2}\pi'(t)\sigma\sigma'\pi(t)\big)\right] dt \\ \le \ & \log(x) + \int_t^T \sup_{\substack{\hat\pi(t) \\ \{\mathcal{F}_t-\text{meas.}\}}} E\Big[\big(r(\hat\pi(t))(1-\hat\pi'(t)\underline{1}) + \hat\pi'(t)b - \frac{1}{2}\pi'(t)\sigma\sigma'\pi(t)\big)\Big] dt. \end{aligned}$$

Notice, that we have changed from functional to pointwise optimization, leading to the inequality sign. Since there is nothing stochastic or time-dependent within the brackets of the expected value (besides the control process $\hat\pi(t)$ which however is at our disposal), we obtain that

$$V(t,x) \le \log(x) + \sup_{\hat\pi\in\mathbb{R}^n} \big\{ r(\hat\pi)(1-\hat\pi'\underline{1}) + \hat\pi'b - \frac{1}{2}\hat\pi'\sigma\sigma'\hat\pi\big\}(T-t). \tag{3.8}$$

We introduce the following notations to study the question of existence of a maximum:

$$D_j := \{(x_1, \ldots, x_n)' : \alpha_j < \sum_{i=1}^{n} x_i \leq \alpha_{j+1}\} \tag{3.9}$$

$$H_j := \{x \in I\!R^n : \sum_{i=1}^{n} x_i = \alpha_j\} \tag{3.10}$$

$$\overline{D_j} := D_j \cup H_j \tag{3.11}$$

$$M_j^{S\theta}(x) := (\bar{r} + \lambda_j)(1 - x'\underline{1}) + x'b - \frac{1}{2}\Theta x'\sigma\sigma' x \tag{3.12}$$

where $j = 0, \ldots, m-1$.

Remark 3.1

i) Observe that $\{D_j\}_{j=0,\ldots,m-1}$ is a partition of $I\!R^n$, i.e. $I\!R^n = \dot{\bigcup}_{j=0}^{m-1} D_j$, and $\overline{D_j} \cap U = (H_j \cup D_j) \cap U$ for any compact set $U \subset I\!R^n$.

ii) In the step-function case of the following Proposition 3.2 the term $M_j^{S\theta}(x)$ is needed to determine the local maxima.

iii) We include a real number $\Theta \in (0, \infty)$ in front of the quadratic term, because we are going to use this notation and the upcoming calculations again in the section on HARA-utility.

Proposition 3.2 : Existence of the maximum
Let:

$$M^{\theta}(x) := r(x)(1 - x'\underline{1}) + x'b - \frac{1}{2}\Theta x'\sigma\sigma' x \tag{3.13}$$

with r(x) being either a step function, frequency polygon or logistic function as given by (3.4)-(3.6) and $\Theta \in (0, \infty)$*. Then there is an*

$$x^* \in U := \bar{D}_c(\underline{0}) = \{x \in I\!R^n : \|x - (0, \ldots, 0)\| \leq c\},$$

for a suitable c, such that we have

$$M^{\theta}(x^*) = \sup_{x \in I\!R^n} M^{\theta}(x) = \sup_{x \in U} M^{\theta}(x),$$

or in other words

$$x^* = \arg\max_{x \in U} M^{\theta}(x).$$

Proof.
Boundedness:
Since $\sigma\sigma'$ is strictly positive definite and $r(x)$ is bounded, $M^\theta(x)$ is bounded from above and $M^\theta(x) \to -\infty$ for $\|x\| \to \infty$. Hence the supremum is finite and located in a compact domain U, i.e. we have

$$\sup_{x\in U} M^\theta(x) = \sup_{x\in \mathbb{R}^n} M^\theta(x).$$

The same holds of course for the local suprema

$$\sup_{x\in D_j\cap U} M^\theta(x) = \sup_{x\in D_j} M^\theta(x)$$

for all $j = 0, \dots, m-1$.

Existence:
If $r(x)$ is a *frequency polygon* or a *logistic function*, the existence of the maximum follows by continuity of $M^\theta(x)$ and compactness of U.

Let $r(x)$ be a *step function* as given in (3.4). Because $M_j^{S\theta}$ is continuous we get:

$$\sup_{x\in U} M^{S\theta}(x) = \max_j \sup_{x\in D_j\cap U} M_i^{S\theta}(x) = \max_j \max_{x\in \overline{D_j\cap U}} M_j^{S\theta}(x).$$

Observe, that for all $x \in H_j$, $j = 1, \dots, m-1$, we have $M_{j-1}^{S\theta}(x) \geq M_j^{S\theta}(x)$, since $\lambda_{j-1} < \lambda_j$ and $x'\underline{1} \geq 1$ in H_j. Thus $\arg\max_{x\in\overline{D_j\cap U}} M_j^{S\theta}(x)$ is always in $D_j \cap U$ and not in $H_j \cap U$. Hence there exists an index l and $x_l \in D_l$, such that $\sup_{x\in U} M^{S\theta}(x) = M_l^{S\theta}(x_l)$.

Consequently, for all three functions there exists x^* with

$$x^* = \arg\max_{x\in U} M^\theta(x).$$

□

Now we are able to proof our verification theorem:

Theorem 3.3 : Optimal control for Log-utility
Let

$$\pi^*(.) \equiv \hat{\pi} = \arg\max_{x\in\mathbb{R}^n} \left\{ r(x)(1 - x'\underline{1}) + x'b - \frac{1}{2}x'\sigma\sigma' x \right\}. \tag{3.14}$$

The constant process π^* defined by (3.14) is the optimal control and we get

$$V(t,x) = \log(x) + \left(r(\hat{\pi})(1 - \hat{\pi}'\underline{1}) + \hat{\pi}'b - \frac{1}{2}\hat{\pi}'\sigma\sigma'\hat{\pi} \right)(T-t).$$

Proof. From Proposition 3.2 and equation (3.8) we obtain:

$$E^{t,x}(\log(X^{\pi^*}(T))) \leq V(t,x) \leq \underbrace{\log(x) + \left(r(\hat{\pi})(1 - \hat{\pi}'\underline{1}) + \hat{\pi}'b - \frac{1}{2}\hat{\pi}'\sigma\sigma'\hat{\pi} \right)(T-t)}_{=E^{t,x}(\log(X^{\pi^*}(T)))}$$

Since $\pi^*(.)$ is constant, it is an element of $\mathcal{A}^{\mathcal{H}}(0, x_0)$, and thus admissible.

□

Remark 3.4
The remaining question is, how to determine the optimal control. If $r(\hat{\pi})$ is a step function or a frequency polygon as given in equations (3.4) and (3.5), we can determine the maximum explicitly, using the partition $\{D_j\}_{j=0,\ldots,m-1}$ of $I\!R^n$. We investigate $M_j^\theta(x)$ separately on the sets $\overline{D_j}$. Since $M_j^\theta(x)$ are downwards opened parabolas (in both cases), we can determine the local maxima. Then we compare these maxima to obtain the absolute maximum and the corresponding optimal control. If $r(\hat{\pi})$ is a logistic function, we have to calculate the maximum via numerical methods.

In the following we consider all these cases explicitly.

3.3.1 Step function

Theorem 3.5 : Optimal Portfolios for step functions and Log-utility

Let $V^S(t,x)$ be the value function for logarithmic utility with $r(x)$ being a step function defined by equation (3.4). In addition, let $M^{S\Theta}$ be the function to be maximized in Proposition 3.2 and which corresponds to this step function $r(x)$, i.e.

$$M^{S\Theta}(x) \quad = \quad \left[\bar{r} \ + \sum_{j=0}^{m-1} \lambda_j \mathbf{1}_{(\alpha_j,\alpha_{j+1}]}(x'\underline{1})\right](1 - x'\underline{1}) + x'b - \frac{1}{2}\Theta x'\sigma\sigma' x, \tag{3.15}$$

where λ_j and α_j as given in equation (3.4). Then there exists an optimal (constant) control

$$\pi^*(.) = \hat{\pi} = \arg\max_{x \in I\!R^n} M^{S1}(x)$$

such that

$$V^S(t,x) \equiv \sup_{\pi(.)\in\mathcal{A}^{\mathcal{H}}(t,x)} E^{t,x}(\log(X^\pi(T))) = E^{t,x}(\log(X^{\pi^*}(T))).$$

The value $\hat{\pi}$ is explicitly given below (with $\Theta = 1$):

i) In the one-dimensional case holds

$$\hat{\pi} \quad = \quad \arg\max_{\{\hat{\pi}_j : j=0,\ldots,m-1\}} M_j^{S\theta}(\hat{\pi}_j) \qquad \text{where} \tag{3.16}$$

$$\hat{\pi}_j \quad = \quad \max\left(\alpha_j, \min\left(\alpha_{j+1}, \frac{b - r - \lambda_j}{\Theta\sigma^2}\right)\right). \tag{3.17}$$

ii) The multidimensional case leads to

$$\hat{\pi} = \underset{\{\hat{\pi}_j : j=0,\ldots,m-1\}}{\arg\max} M_j^{S\theta}(\hat{\pi}_j) \qquad \text{where} \tag{3.18}$$

$$M_j^{S\Theta}(x) = [\bar{r} + \lambda_j]\,(1 - x'\underline{1}) + x'b - \frac{1}{2}\Theta x'\sigma\sigma' x, \tag{3.19}$$

and

$$\hat{\pi}_j = \begin{cases} \frac{1}{\theta}(\sigma^*\sigma^{*\prime})^{-1} b^{*u} & : \quad v_j \notin \overline{D_j} \quad \wedge \quad \operatorname{dist}(H_j, v_j) > \operatorname{dist}(H_{j+1}, v_j) \\ v_j & : \quad v_j \in \overline{D_j} \\ \frac{1}{\theta}(\sigma^*\sigma^{*\prime})^{-1} b^{*d} & : \quad v_j \notin \overline{D_j} \quad \wedge \quad \operatorname{dist}(H_j, v_j) < \operatorname{dist}(H_{j+1}, v_j) \end{cases} \tag{3.20}$$

with

$$v_j = \frac{1}{\theta}(\sigma\sigma')^{-1}\left(b - (\bar{r} + \lambda_j)\underline{1}\right)$$

and $\sigma^* \in I\!R^{(n-1)\times(n-1)}$ *with* $\sigma^*_{ki} = \sigma_{ki} - \sigma_{ni}$ *and* $b_k^{*u} = b_k - b_n - \theta\alpha_{j+1}\sum_{i=1}^n \sigma_{ni}\sigma^*_{ki}$ *resp.* $b_k^{*d} = b_k - b_n - \theta\alpha_j \sum_{i=1}^n \sigma_{ni}\sigma^*_{ki}$.

The function dist(.,.) *is defined as*

$$\operatorname{dist}(X, Y) := \inf_{\{x\in X, y\in Y\}} ||x - y||,$$

where $X, Y \subset I\!R^n$ *and* $||.||$ *is the Euclidean norm.*

Proof. As proved in Theorem 3.3, the optimal control exists and is given by

$$\pi^*(.) \equiv \hat{\pi} = \arg\max_{x\in I\!R^n} M^{S\Theta}(x)$$

with $\Theta = 1$. As stated in the proof of Proposition 3.2 we know that

$$\max_{x\in I\!R^n} M^{S\theta}(x) = \max_{\{j:0,\ldots,m-1\}} \max_{x\in\overline{D_j}} M_j^{S\theta}(x).$$

Hence, it holds:

$$\arg\max_{x\in U} M^{\theta}(x) = \underset{\{\hat{\pi}_j : j=0,\ldots,m-1\}}{\arg\max} \left\{ M_j^{S\theta}(\hat{\pi}_j) \right\} = \arg\max_{x\in\overline{D_j}} M_j^{S\theta}(x).$$

As mentioned before, we achieve the local maxima and corresponding arguments on the sets $\overline{D_j}$ and then compare them in order to obtain the absolute maximum. Thus we are only left with verification of $\hat{\pi}_j$.

One-dimensional case
Each $M_j^{S\theta}(x)$ is a downwards opening parabola with an obviously unique unconstrained maximum (with unconstrained maximum, we mean the maximum w.r.t. the domain $\mathbb{R}^n$). So we can determine the local maximum (i.e. w.r.t. D_j) by calculating the zero point of its derivative (w.r.t $\mathbb{R}^n$) and check its position relative to $\overline{D_j}$. If the unconstrained maximum is in $\overline{D_j} = [\alpha_j, \alpha_{j+1}]$ we have already found the local maximum. If it lies on the right(left) side of the interval, the local maximum is achieved in α_{j+1} (α_j) .

Multidimensional case
Again, the first step is to determine the unconstrained maxima for the different λ_j by defining

$$\begin{aligned} v_j &:= \arg\max_{x\in\mathbb{R}^n} \left\{(\bar{r} + \lambda_j)(1 - x'\underline{1}) + x'b - \frac{1}{2}\Theta x'\sigma\sigma' x\right\} \qquad (3.21) \\ &= \frac{1}{\Theta}(\sigma\sigma')^{-1}\left(b - (\bar{r} + \lambda_j)\underline{1}\right). \end{aligned}$$

Observe, that $\sigma\sigma'$ is regular, as stated in Proposition (3.2) . If $v_j \in \overline{D_j}$, then we have found the local maximum of case λ_j and so $\hat{\pi}_j = v_j$.
If $v_j \notin \overline{D_j}$, then the local maximum must lie in one of the hyperplanes H_j respectively H_{j+1}, since $-\sigma\sigma'$ is strictly negative definite and $M_j^{S\Theta}(x)$ therefore strictly concave. If $\text{dist}(H_j, v_j) > (<) \text{ dist}(H_{i+1}, v_j)$ then $\hat{\pi}_j$ lies in H_{j+1} (H_j). Thus we have to calculate the maximum under the constraint $\hat{\pi}'\underline{1} = \alpha$, with $\alpha = \alpha_j$ resp. $\alpha = \alpha_{j+1}$. We will realize it by $(\hat{\pi}_j)_n = \alpha - \sum_{i=1}^{n-1}(\hat{\pi}_j)_i$.
In the following we have to use the components of the vector $\hat{\pi}$ and b explicitly to continue our calculations. So for ease of notation we neglect the index j of λ_j and v_j:

$$\begin{aligned} v &= \arg\max_{x\in H} \left\{(\bar{r} + \lambda)(1 - x'\underline{1}) + x'b - \frac{1}{2}\Theta\hat{\pi}'\sigma\sigma' x\right\} \\ &= \arg\max_{(x_1,\dots,x_{n-1})\in\mathbb{R}^{n-1}} \Bigg\{\bar{r} + \lambda + \sum_{k=1}^{n-1} x_k(b_k - \bar{r} - \lambda) + (\alpha - \sum_{k=1}^{n-1} x_k)(b_n - \bar{r} - \lambda) \\ &\qquad -\frac{1}{2}\Theta\sum_{i=1}^{n}\left(\sum_{k=1}^{n-1} x_k\sigma_{ki} + (\alpha - \sum_{k=1}^{n-1} x_k)\sigma_{ni}\right)^2\Bigg\} \\ &= \arg\max_{(x_1,\dots,x_{n-1})\in\mathbb{R}^{n-1}} \Bigg\{\bar{r} + \lambda + \sum_{k=1}^{n-1} x_k(b_k - b_n) + \alpha(b_n - \bar{r} - \lambda) \\ &\qquad -\frac{1}{2}\Theta\sum_{i=1}^{n}\left(\sum_{k=1}^{n-1} x_k(\sigma_{ki} - \sigma_{ni}) + \alpha\sigma_{ni}\right)^2\Bigg\} \end{aligned}$$

Now let $b^* \in \mathbb{R}^{n-1}$ with $b_k^* = b_k - b_n (k = 1,\dots,n-1)$ and $\sigma^* \in \mathbb{R}^{(n-1)\times(n-1)}$ with $\sigma_{ki}^* = \sigma_{ki} - \sigma_{ni}$ $(k = 1,\dots,n-1)$. Observe, that $\text{rank}(\sigma^*) = n - 1$, otherwise this would lead to the contradiction rank(σ) $< n$. Thus σ^* is regular. So we achieve

$$v = \underset{(x_1,\dots,x_{n-1})\in\mathbb{R}^{n-1}}{\arg\max} \Bigg\{ (1-\alpha)(\bar{r} \ + \lambda) + \alpha b_n + \sum_{k=0}^{n-1} x_k b_k^* + \\ -\frac{1}{2}\Theta \sum_{i=1}^{n} \left(\left(\sum_{k=1}^{n-1} x_k \sigma_{ki}^* \right)^2 + 2\alpha\sigma_{ni} \sum_{k=1}^{n-1} x_k \sigma_{ki}^* + \alpha^2 \sigma_{ni}^2 \right) \Bigg\}$$

$$= \underset{(x_1,\dots,x_{n-1})\in\mathbb{R}^{n-1}}{\arg\max} \Bigg\{ (1-\alpha)(\bar{r} \ + \lambda) + \alpha b_n - \frac{1}{2}\Theta \sum_{i=1}^{n} \alpha^2 \sigma_{ni}^2 \\ + \sum_{k=1}^{n-1} x_k \left(b_k^* - \Theta\alpha \sum_{i=1}^{n} \sigma_{ni}\sigma_{ki}^* \right) - \frac{1}{2}\Theta \sum_{i=1}^{n} \left(\left(\sum_{k=1}^{n-1} x_k \sigma_{ki}^* \right)^2 \right) \Bigg\}.$$

Thus, with $b_k^{**} = b_k^* - \Theta\alpha \sum_{i=1}^{n} \sigma_{ni}\sigma_{ki}^*$, we obtain the usual representation

$$v = \underset{(x_1,\dots,x_{n-1})\in\mathbb{R}^{n-1}}{\arg\max} \left\{ (1-\alpha)(\bar{r} \ + \lambda) + \alpha b_n - \frac{1}{2}\Theta \sum_{i=1}^{n} \alpha^2 \sigma_{ni}^2 + x' b^{**} - \frac{1}{2}\Theta x' \sigma^* \sigma^{*\prime} x \right\},$$

which yields the solution

$$v = \frac{1}{\Theta} (\sigma^* \sigma^{*\prime})^{-1} b^{**}.$$

□

Remark 3.6

i) Note, that v does not depend on λ or $\bar{r}$, because these quantities are fixed on the hyperplanes H_j. If the unconstrained maximum of the $M_j^{S\theta}(x)$ lies in the sets $\{\overline{D_l} : l \le j\}$, then the absolute maximum cannot lie in one of the sets $\{\overline{D_l} : l > j\}$, because $M_j^{S\theta}(x) > M^{S\theta}(x)$ for all $x \in \bigcup_{l \ge j} \overline{D_l}$ (via $\lambda_j < \lambda_{j+1}$). So, if we are stepwise increasing j (beginning at 0) we can stop the maximum-search, if the above condition is fulfilled.

ii) In the one-dimensional case we see from the above equations that this method can be used to bound $\pi(t)$ by an arbitrary boundary α_m choosing $\lambda_{m-1} = b - r$, since this case implies $v_{m-1} = 0$.

Example 3.7
Let $r(x)$ be modeled as in Figure 5, i.e.

$$r(x) = \begin{cases} 5\% & : \quad x\underline{1} \le 1 \\ 7\% & : \quad 1 < x\underline{1} \le 1.5 \\ 9\% & : \quad 1.5 < x\underline{1} \le 2 \\ 12\% & : \quad 2.5 < x\underline{1} \end{cases}.$$

Let $b = 12\%$ and $\sigma = 20\%$. Then $\pi^*(.) = \frac{12\%-7\%}{20\%^2} = 1.25$. For comparison: If we would have $r(x) \equiv 5\%$ then the optimal control would yield $\frac{12\%-5\%}{20\%^2} = 1.75$.

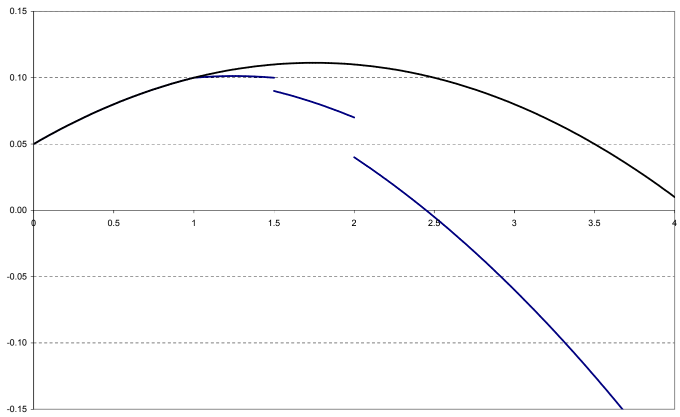


Figure 8: Parabolas M^{S1} with r(.) step function and r flat

In Figure 8 we plotted the corresponding function M^{S1} (which we have to maximise) with r modeled as step function and with r flat. Note, that generally there are jumps at α_j, except for the case when $\alpha_j = 1.0$. Since the coefficient of $r(x)$ is $(1-x)$, the parabola is continuous in $x = 1$, although $r(x)$ jumps at that point.

3.3.2 Frequency polygon

The procedure is similar to the one for step functions, i.e. we determine the maxima piecewise on $\overline{D_j}$ and then we compare them to obtain the absolute maximum. In preparation for the next section, we again include a parameter $\theta \in (0, \infty)$ in front of the square term.

Theorem 3.8 : Optimal Portfolios for polygons and Log-utility

Let $V^P(t,x)$ be the value function with logarithmic utility and with $r(\hat{\pi})$ being a frequency polygon as defined in equation (3.5). In addition, let $M^{P\Theta}$ be the corresponding function to be maximized in Proposition 3.2, i.e.

$$M^{P\theta}(x) \quad = \quad \left[\bar{r} \; + \sum_{j=0}^{m-1} (r_j + \mu_j(x'\underline{1} - \alpha_j))\mathbf{1}_{[\alpha_j,\alpha_{j+1})}(x'\underline{1})\right](1 - x'\underline{1}) + x'b - \frac{1}{2}\Theta x'\sigma\sigma' x,$$

with α_i, μ_i, r_i given in (3.5). Then there exists a constant control $\pi^*(.) = \hat{\pi} = \arg\max_{x \in I\!R^n} M^{P\theta}(x)$ such that we have

$$V^P(t,x) \equiv \sup_{\pi(.)\in\mathcal{A}^{\mathcal{H}}(t,x)} E^{t,x}(\log(X^{\pi}(T))) = E^{t,x}(\log(X^{\pi^*}(T))).$$

The value $\hat{\pi}$ is explicitly given below (with $\Theta = 1$):

i) In the one-dimensional case we have

$$\hat{\pi} = \underset{\{\hat{\pi}_j : j=0,\dots,m-1\}}{\arg\max} \left\{ M_j^{P\theta}(\hat{\pi}_j) \right\} \quad \text{where} \tag{3.22}$$

$$\hat{\pi}_j = \max\left(\alpha_j, \min\left(\alpha_{j+1}, \frac{b-\bar{r}-r_j+\mu_j(1+\alpha_j)}{\Theta\sigma^2+2\mu_j}\right)\right) \tag{3.23}$$

ii) For the multidimensional case we get

$$\hat{\pi} = \underset{\{\hat{\pi}_j : j=0,\dots,m-1\}}{\arg\max} \left\{ M_j^{P\theta}(\hat{\pi}_j) \right\} \quad \text{where} \tag{3.24}$$

$$M_j^{P\Theta}(x) = (\bar{r} + r_j + \mu_j(x'\underline{1}-\alpha_j))(1-x'\underline{1}) + x'b - \frac{1}{2}\Theta x'\sigma\sigma' x \tag{3.25}$$

for

$$\hat{\pi}_j = \begin{cases} \frac{1}{\theta}(\sigma^*\sigma^{*\prime})^{-1}b^{*u} & : \; v_j \notin \overline{D_j} \quad \wedge \quad \text{dist}(H_j, v_j) > \text{dist}(H_{j+1}, v_j) \\ v_j & : \; v_j \in \overline{D_j} \\ \frac{1}{\theta}(\sigma^*\sigma^{*\prime})^{-1}b^{*d} & : \; v_j \notin \overline{D_j} \quad \wedge \quad \text{dist}(H_j, v_j) < \text{dist}(H_{j+1}, v_j) \end{cases} \tag{3.26}$$

with

$$v_j = (\Theta\sigma\sigma' + 2\mu_j\underline{1}\,\underline{1}')^{-1}\,(b - \underline{1}(\bar{r} - r_j + \mu_j(1+\alpha_j)))$$

and $\sigma^* \in I\!R^{(n-1)\times(n-1)}$ *with* $\sigma^*_{ki} = \sigma_{ki} - \sigma_{ni}$ *and* $b_k^{*u} = b_k - b_n - \theta\alpha_{j+1}\sum_{i=1}^n \sigma_{ni}\sigma^*_{ki}$ *resp.* $b_k^{*d} = b_k - b_n - \theta\alpha_j \sum_{i=1}^n \sigma_{ni}\sigma^*_{ki}$.

PROOF. Again, due to Theorem 3.3 and Proposition 3.2, the optimal control exists and is given by

$$\pi^*(.) \equiv \hat{\pi} = \underset{x\in I\!R^n}{\arg\max}\, M^{P\Theta}(x)$$

$$= \underset{\{\hat{\pi}_j : j=0,\dots,m-1\}}{\arg\max}\, M_j^{P\theta}(\hat{\pi}_j) \text{ with } \hat{\pi}_j = \underset{x\in\overline{D_j}}{\arg\max}\, M_j^{P\theta}(x),$$

with $\Theta = 1$. Again, only the form of $\hat{\pi}_j$ has to be verified, using the fact that equation (3.25) can be rewritten as

$$M_j^{P\Theta}(x) = (\bar{r} + r_j - \mu_j\alpha_j) + x'\big(b - \underline{1}(\bar{r} + r_j - \mu_j(1+\alpha_j))\big) - \frac{1}{2}x'(\Theta\sigma\sigma' + 2\mu_j\underline{1}\,\underline{1}')x.$$

Because $M^{P\theta}(x)$ is continuous, the above procedure is valid. More precisely, due to continuity, we get that $\sup_{x\in U} M^{P\theta}(x) = \max_j \sup_{x\in D_j} M_j^{P\theta}(x) = \max_j \max_{x\in\overline{D_j}} M_j^{P\theta}(x)$, and thus the above equation follows. Observe, that $\mu_j\underline{1}\,\underline{1}'$ is positive semidefinite, since $\mu_j > 0$ and $\hat{\pi}'\underline{1}\,\underline{1}'u = (\sum_{j=1}^n \hat{\pi}_j)^2 \geq 0$. Thus $\Theta\sigma\sigma' + 2\mu_j\underline{1}\,\underline{1}'$ is still strictly positive definite. So as before, we are concerned with downwards opening parabolas.

One-dimensional case
The argumentation is exactly the same as in the proof for step functions. But in contrast to the step function, we have to check all intervals in order to get the absolute maximum. More precisely, due to "strong" increasing slopes, it can happen that the unconstrained maximum lies in the interior of an interval, but the absolute maximum lies in an interval right from it on the real line. For demonstration, we are going to give an example in Remark 3.10.

Multidimensional case
Let $\Phi_j = \bar{r} + r_j - \mu_j \alpha_j$, $\Psi_j = \bar{r} + r_j - \mu_j(1+\alpha_j))$ and $M_j^{P\Theta}$ be the parabola on $\overline{D_j}$, i.e.:

$$M_j^{P\Theta}(x) = \left\{ \Phi_j + x'\big(b - \underline{1}\Psi_j\big) - \frac{1}{2}x'(\Theta\sigma\sigma' + 2\mu_j \underline{1}\,\underline{1}')x \right\} \tag{3.27}$$

The first step is to determine the unconstrained maximum.

$$\begin{aligned} v_j &:= \arg \max_{x \in \mathbb{R}^n} \left\{ \Phi_j + x'\big(b - \underline{1}\Psi_j\big) - \frac{1}{2}x'(\Theta\sigma\sigma' + 2\mu_j \underline{1}\,\underline{1}')x \right\} \\ &= (\Theta\sigma\sigma' + 2\mu_j \underline{1}\,\underline{1}')^{-1}\,(b - \underline{1}(\bar{r} - r_j + \mu_j(1+\alpha_j))) \end{aligned}$$

If $v_j \in \overline{D_j}$, then we have already found the local maximum, thus

$$\hat{\pi}_j = v_j.$$

If $v_j \notin \overline{D_j}$, then the local maximum must lie in one of the hyperplanes H_j respectively H_{j+1}, since $-\sigma\sigma'$ is strictly negative definite and $M_j^{P\Theta}$ therefore strictly concave. Again we calculate the maximum under the constraint $v_j{}'\underline{1} = \alpha$, with $\alpha = \alpha_j$ resp. $\alpha = \alpha_{j+1}$. As before we use the components of the vector $\hat{\pi}$ and b explicitly to continue our calculations, and neglect the index j:

$$\begin{aligned} v &= \arg\max_{x \in H} \left\{ \Phi + x'\big(b - \underline{1}\Psi\big) - \frac{1}{2}x'(\Theta\sigma\sigma' + 2\mu \underline{1}\,\underline{1}')x \right\} \\ &= \arg\max_{(x_1,\dots,x_{n-1}) \in \mathbb{R}^{n-1}} \left\{ \Phi + \sum_{k=1}^{n-1} x_k(b_k - \Psi) + (\alpha - \sum_{k=1}^{n-1} x_k)(b_n - \Psi) \right. \\ &\quad \left. -\frac{1}{2}\Theta \sum_{i=1}^{n} \left(\sum_{k=1}^{n-1} x_k \sigma_{ki} + (\alpha - \sum_{k=1}^{n-1} x_k)\sigma_{ni} \right)^2 - \mu \underbrace{\left(\sum_{k=1}^{n-1} x_k + (\alpha - \sum_{k=1}^{n-1} x_k) \right)^2}_{=\alpha^2} \right\} \end{aligned}$$

$$= \arg\max_{(x_1,\dots,x_{n-1})\in \mathbb{R}^{n-1}} \Bigg\{ \Phi + \alpha(b_n - \Psi) - \mu\alpha^2 + \sum_{k=1}^{n-1} x_k(b_k - b_n)$$
$$- \frac{1}{2}\Theta \sum_{i=1}^{n} \left(\sum_{k=1}^{n-1} x_k(\sigma_{ki} - \sigma_{ni}) + \alpha\sigma_{ni} \right)^2 \Bigg\}$$

Analogously to the proof of Theorem 3.5 let $b^* \in \mathbb{R}^{n-1}$ with $b_k^* = b_k - b_n$ and $\sigma^* \in \mathbb{R}^{(n-1)\times(n-1)}$ with $\sigma_{ki}^* = \sigma_{ki} - \sigma_{ni}$. Again we have $\operatorname{rank}(\sigma^*) = n-1$. Thus σ^* is regular and

$$v = \arg\max_{(x_1,\dots,x_{n-1})\in \mathbb{R}^{n-1}} \Bigg\{ \Phi + \alpha(b_n - \Psi) - \mu\alpha^2 + \sum_{k=1}^{n-1} x_k b_k^*$$
$$- \frac{1}{2}\Theta \sum_{i=1}^{n} \left(\left(\sum_{k=1}^{n-1} x_k\sigma_{ki}^* \right)^2 + 2\alpha\sigma_{ni} \sum_{k=1}^{n-1} x_k\sigma_{ki}^* + \alpha^2\sigma_{ni}^2 \right) \Bigg\}$$
$$= \arg\max_{(x_1,\dots,x_{n-1})\in \mathbb{R}^{n-1}} \Bigg\{ \Phi + \alpha(b_n - \Psi) - \mu\alpha^2 - \frac{1}{2}\Theta \sum_{i=1}^{n} \alpha^2\sigma_{ni}^2$$
$$+ \sum_{k=1}^{n-1} x_k \left(b_k^* - \Theta\alpha \sum_{i=1}^{n} \sigma_{ni}\sigma_{ki}^* \right) - \frac{1}{2}\Theta \sum_{i=1}^{n} \left(\left(\sum_{k=1}^{n-1} x_k\sigma_{ki}^* \right)^2 \right) \Bigg\}.$$

Hence, with $b_k^{**} = b_k^* - \Theta\alpha \sum_{i=1}^{n} \sigma_{ni}\sigma_{ki}^*$, we obtain the usual representation

$$v = \arg\max_{(x_1,\dots,x_{n-1})\in \mathbb{R}^{n-1}} \left\{ \Phi + \alpha(b_n - \Psi) - \mu\alpha^2 - \frac{1}{2}\Theta \sum_{i=1}^{n} \alpha^2\sigma_{ni}^2 + x'b^{**} - \frac{1}{2}\Theta x'\sigma^*\sigma^{*\prime}x \right\},$$

which yields the solution

$$v = \frac{1}{\Theta}(\sigma^*\sigma^{*\prime})^{-1} b^{**}.$$

It is worthwhile to note that the maximum does neither depend on the interest rate $\bar{r} + r_j$, nor on μ_j, and the calculation of the maximum is exactly the same as for step functions. This is not suprising, since r is fixed on these hyperplanes.

□

Example 3.9
Let $r(x)$ be modeled as in Figure 6, i.e.

$$r(x) = \begin{cases} 5\% & : \quad x \le 1 \\ 5\% + (x-1) * 3\% & : \quad 1 < x \le 1.5 \\ 6.5\% + (x-1.5) * 6\% & : \quad 1.5 < x \le 2 \\ 9.5\% + (x-2) * 3\% & : \quad 2 < x \le 2.5 \\ 11\% & : \quad 2.5 < x \end{cases},$$

Let $b = 12\%$ and $\sigma = 20\%$, then the optimal control equals $\pi^*() = \frac{12\%-5\%+3\%*2}{20\%^2+2*3\%} = 1.3$. For comparison: If $r(x) \equiv 5\%$ then again the optimal control equals 1.75.

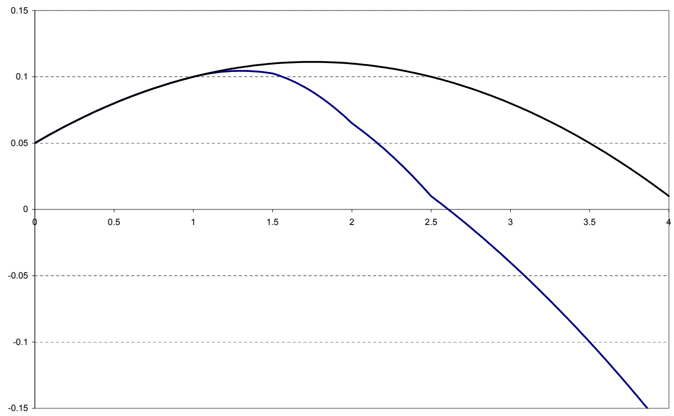


Figure 9: Parabolas M^{P1} with r(.) frequency polygon and r flat

Note that the parabola is generally not differentiable in the α_i.

Remark 3.10
In this remark we show that for the case of frequency polygons it is necessary to calculate the maximums for all domains D_j. Let $r(x)$ be modeled as follows

$$r(x) = \begin{cases} 5\% & : \quad x \le 1 \\ 5\% + (x-1)*3\% & : \quad 1 < x \le 1.5 \\ 6.5\% + (x-1.5)*6\% & : \quad 1.5 < x \le 2 \\ 9.5\% + (x-2)*10\% & : \quad 2 < x \le 2.5 \\ 14.5\% & : \quad 2.5 < x \end{cases},$$

Assuming $b = 30\%$ and $\sigma = 20\%$, the unconstrained maximum of $M_3^{P1}(x)$ is in D_3, but the absolute maximum is achieved in D_4, which is proved by the following calculations:

$$\begin{aligned} \hat{\pi}_3 &= \max\left(\alpha_3, \min\left(\alpha_4, \frac{b - \bar{r} - r_3 + \mu_3(1+\alpha_3)}{\sigma^2 + 2\mu_3}\right)\right) \\ &= \max\left(2, \min\left(3, \frac{0.3 - 0.05 - 0.065 + 0.1(1+2)}{0.2^2 + 2*0.1}\right)\right) = 2.104166 \end{aligned}$$

implies

$$M_3^{P\Theta}(\hat{\pi}_3) = (0.05 + 0.065 + 0.1(\hat{\pi}_3 - 2))(1 - \hat{\pi}_3) + \hat{\pi}_3 0.3 - \frac{1}{2}\hat{\pi}_3^2 0.2^2 = 0.4042$$

but also

$$\hat{\pi}_4 = \max\left(\alpha_4 \frac{b - \bar{r} - r_4}{\sigma^2}\right)) = \max\left(2.5, \frac{0.3 - 0.05 - 0.095}{0.2^2}\right) = 3.875$$

and

$$M_4^{P\Theta}(\hat{\pi}_4) = 0.145(1 - \hat{\pi}_4) + \hat{\pi}_4 0.3 - \frac{1}{2}\hat{\pi}_4^2 0.2^2 = 0.4453.$$

3.3.3 Logistic function

The optimal control is given by

$$\hat{\pi} \;=\; \arg\max_{x\in I\!R^n} \left\{ \left[\bar{r} \;+ \lambda \frac{e^{\alpha x'\underline{1}+\beta}}{e^{\alpha x'\underline{1}+\beta}+1} \right] (1 - x'\underline{1}) + x'b - \frac{1}{2} x'\sigma\sigma' x \right\} .$$

Since $r(x)$ is bounded, we achieve very loosely speaking, a kind of downwards opened parabola (at least in the asymptotic sense for $||x|| \to \infty$). Thus, there exists an absolute maximum. In the one-dimensional case we have to solve the following equation

$$\underbrace{(1-x)\lambda \frac{\alpha e^{\alpha x+\beta}}{(e^{\alpha x+\beta}+1)^2}}_{A(x)} - \underbrace{\lambda \frac{\alpha e^{\alpha x+\beta}}{e^{\alpha x+\beta}+1}}_{B(x)} \overset{!}{=} x\sigma^2 - b + \bar{r} \; .$$

Since $\lim_{x\to\infty} A(x) = 0$,$\lim_{x\to-\infty} A(x) = 0$, $A(x)$ is bounded. In conjunction with the boundedness of $B(x)$ and continuity we can infer that the above equation has at least one solution $\tilde{\pi}$. Furthermore we can determine the solutions of the above equations by some numerical procedure, and then we can conclude the absolute maximum as before, by comparing these results. But with suitable parameters the above equation will only have one solution. Because $\pi^*(t) \equiv \hat{\pi}$ is constant, it belongs to $\mathcal{A}^{\mathcal{H}}(0, x_0)$. Thereby the original optimisation problem is solved too.

Example 3.11
Let $r(x)$ be modeled as in Figure 7, i.e. $\bar{r} = 5\%$, $\lambda = 6\%$, $\alpha = 3$ and $\beta = 4$. Then the optimal control equals 1.26.

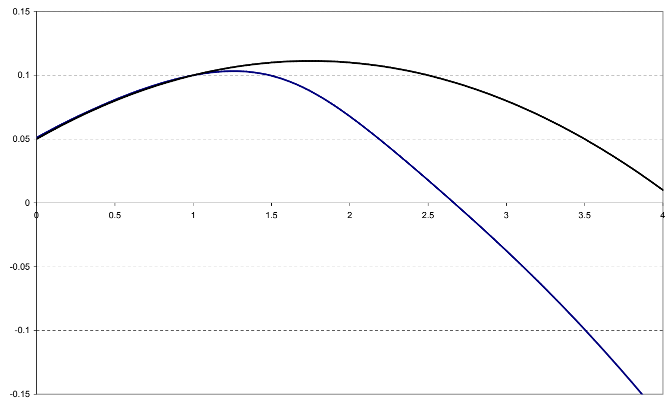


Figure 10: Parabolas with r(.) logistic function and r flat

3.4 HARA Utility

Let $U(x) = \frac{1}{\gamma}x^\gamma$ with $\gamma \in (0,1)$. In this case we have the following optimization problem

$$\begin{aligned} V(t,x) &:= \frac{1}{\gamma} \sup_{\pi(.)\in\mathcal{A}^{\mathcal{H}}(t,x)} E^{t,x}((X^\pi(T))^\gamma) \qquad (3.28) \\ &= \frac{1}{\gamma} \sup_{\pi(.)\in\mathcal{A}^{\mathcal{H}}(t,x)} \left\{ x^\gamma E e^{\gamma\left[\int_t^T \left(r(\pi(t))(1-\pi'(t)\underline{1})+\pi'(t)b - \frac{1}{2}\pi'(t)\sigma\sigma'\pi(t)\right)dt + \int_t^T \pi(t)\sigma dW(t)\right]} \right\}. \end{aligned}$$

Again, this optimization problem is solved by a pointwise maximization. But, due to the non-linear structure of the above term, the correctness cannot be shown by some simple inequaltities as in the logarithmic case. Instead of this, we will show the correnctness via our Verification Theorem 1.14.

Theorem 3.12 : Verification with HARA utility
The value function with HARA utility is given by

$$V(t,x) = \frac{1}{\gamma}x^\gamma e^{\gamma\left[(r(\hat\pi)(1-\hat\pi'\underline{1})+\hat\pi'b - \frac{1}{2}(1-\gamma)\hat\pi'\sigma\sigma'\hat\pi\right](T-t)}$$

and the optimal control exists and equals

$$\pi^*(.) \equiv \hat\pi = \arg\max_x \left[r(x)(1-x'\underline{1}) + x'b - \frac{1}{2}(1-\gamma)x'\sigma\sigma'x \right].$$

PROOF. The existence of the maximum was already shown in Proposition 3.2. So it is left, to check the conditions of the Verification Theorem 1.14. Let

$$G(t,x) = \frac{1}{\gamma}x^\gamma e^{\gamma\left[(r(\hat\pi)(1-\hat\pi'\underline{1})+\hat\pi'b - \frac{1}{2}(1-\gamma)\hat\pi'\sigma\sigma'\hat\pi\right](T-t)}.$$

The function $G(t,x)$ is sufficiently smooth and polynomially bounded. In addition, we have to show that for all $t \in [0,T]$ and $x \in I\!R^+$

$$\sup_{\pi\in I\!R^+} (\mathcal{A}^\pi G(t,x)) = 0\,,$$

holds where

$$\mathcal{A}^\pi = \frac{\partial}{\partial t} + \left[r(\pi)(1-\pi'\underline{1}) + \pi'b\right] x\frac{\partial}{\partial x} + \frac{1}{2}x^2\pi'\sigma\sigma'\pi\frac{\partial^2}{\partial^2 x}\,.$$

Let for all $\pi \in I\!R^n$ and $(t,x) \in [0,T] \times I\!R^+$

$$\mathcal{A}^\pi G(t,x) \le 0.$$

This is equivalent to

$$\left(\frac{\partial}{\partial t} + \left[r(\pi)(1-\pi'\underline{1}) + \pi' b\right] x \frac{\partial}{\partial x} + \frac{1}{2}\pi'\sigma\sigma'\pi x^2 \frac{\partial^2}{\partial^2 x}\right) G(t,x) \leq 0$$

$$\Leftrightarrow \quad \frac{1}{\gamma} e^{\gamma\left[r(\hat\pi)(1-\hat\pi'\underline{1})+\hat\pi' b-\frac{1}{2}(1-\gamma)\hat\pi'\sigma\sigma'\hat\pi\right](T-t)}\Big\{ - x^\gamma \gamma \Big[r(\hat\pi)(1-\hat\pi'\underline{1}) + \hat\pi'$$

$$-\frac{1}{2}(1-\gamma)\hat\pi'\sigma\sigma'\hat\pi\Big] + \left[r(\pi)(1-\pi'\underline{1}) + \pi' b\right] x\gamma x^{\gamma-1} + \frac{1}{2}\pi'\sigma\sigma'\pi x^2\gamma(\gamma-1)x^{\gamma-2}\Big\} \leq 0$$

$$\Leftrightarrow \quad r(\pi)(1-\pi'\underline{1}) + \pi' b - \frac{1}{2}(1-\gamma)\pi'\sigma\sigma'\pi \leq r(\hat\pi)(1-\hat\pi'\underline{1}) + \hat\pi' b - \frac{1}{2}(1-\gamma)\hat\pi'\sigma\sigma'\hat\pi$$

which follows by the construction of $\hat\pi$, and finally the assertion follows. Verification is now completed by also noting $G(T,x) = \frac{1}{\gamma}x^\gamma$. □

As in the case with logarithmic utility the optimization problem is reduced to the maximization of downwards opening parabolas. Hence, the further steps will be very similar.

3.4.1 Step function

Theorem 3.13 : Optimal Portfolios for step functions and HARA utility

Let $V^S(t,x)$ be the solution of the HARA-portfolio problem (3.28) with $r(x)$ step function defined by (3.4). In addition, let

$$M^{S(1-\gamma)}(x) = \left[\bar r \; + \sum_{j=0}^{m-1} \lambda_j \mathbf{1}_{(\alpha_j,\alpha_{j+1}]}(x'\underline{1})\right](1-x'\underline{1}) + x'b - \frac{1}{2}(1-\gamma)x'\sigma\sigma' x$$

with α_j, λ_j given in (3.4). Then there exists an optimal (constant) control

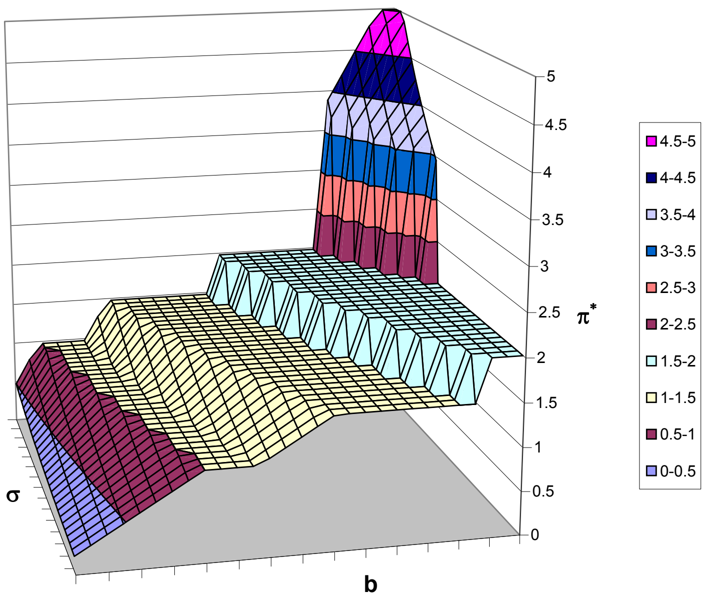
$$\pi^*(.) = \hat\pi = \arg\max_{x\in \mathbb{R}^n} M^{S(1-\gamma)}(x)$$

such that

$$V^S(t,x) \equiv \sup_{\pi(.)\in\mathcal{A}^{\mathcal{H}}(t,x)} E^{t,x}\left(\frac{1}{\gamma}(X^\pi(T))^\gamma\right) = E^{t,x}\left(\frac{1}{\gamma}(X^{\pi^*}(T))^\gamma\right).$$

The solution $\hat\pi$ is explicitly given in Theorem 3.5 with $\Theta = (1-\gamma)$.

Proof. The existence of the maximum of $M^{S(1-\gamma)}(x)$ was shown in Proposition 3.2. The correctness of the value function was proved in Theorem 3.12. The determination of $\hat\pi$ was already shown in Theorem 3.5. □



Example 3.14

Figure 11: Optimal control with $r(.)$ step function and HARA utility ($\gamma = 0.5$)

Remark 3.15 (Optimal control π^* for the case of step functions)
In Figure 11 we plotted the optimal control π^* for the case of step function HARA-utility against the volatility σ and the stock drift b. We observe the well known and natural result, that the optimal control π^* increases when the asset drift increases resp. the volatility decreases. But there is a new feature: Note that if $\alpha_1 = 1$ there is no jump, because the parabola is continuous at this point, as explained before. Otherwise there are plateaus on levels which equal the points of discontinuity of $r(\pi)$, given by the α_i. This behaviour can be explained due to the formulas (3.16) and (3.17) in Theorem 3.5. Observe that a change of σ and b does not necessarily affect the optimal control, if it lies actually on one of the points of discontinuity of the step function. This can be ovserved in Figure 8 showing a typical problem, to be maximised in order to find the optimal control. These function are discontinuous, in particular the maximum can lie on a point of discontinuity, thus a change of the parameters σ and b does not have to imply a change of the maximum.
There is also an economic interpretation for this behavior: On these regions it is not benefitial to increase π when the stock drift (slightly) increases, because the loss due to the more expensive payments of interest (via the upwards-jump of $r(\pi)$) is higher than the benefit due to the higher position in the stock. Conversely, it is not benefitial, to reduce the stock positions when b (slightly) decreases, because r would not fall, and thus the gain from decreasing interest payments would not be higher than the loss via the shortage of the stock-position. If the drift is strongly changing the above effects beat their counterparts and the optimal control is jumping to the next plateau.

3.4.2 Frequency Polygon

Theorem 3.16 : Optimal Portfolios with polygons and HARA utility

Let $V^P(t,x)$ be the value function given in equation (3.28) with $r(x)$ given by a frequency polygon as defined by equation (3.5). In addition, let $M^{P(1-\gamma)}(x)$ be the corresponding function to be maximized to find the optimal control with $r(x)$ frequency polygon, i.e.

$$\begin{aligned} M^{P(1-\gamma)}(x) &= \left[\bar{r} + \sum_{j=0}^{m-1} \left[(r_j + \mu_j(x'\underline{1} - \alpha_j)) \right] \mathbf{1}_{[\alpha_j, \alpha_{j+1})}(x'\underline{1}) \right] (1 - x'\underline{1}) \\ & \quad + x'b - \frac{1}{2}(1-\gamma)x'\sigma\sigma'x \end{aligned}$$

with α_j, μ_j, r_j given in (3.5). Then there exists a constant control

$$\pi^*(.) = \hat{\pi} = \arg\max_{x \in I\!R^n} M^{P\theta}(x)$$

such that

$$V^P(t,x) \equiv \sup_{\pi(.) \in \mathcal{A}^{\mathcal{H}}(t,x)} E^{t,x}\left(\frac{1}{\gamma}(X^\pi(T))^\gamma \right) = E^{t,x}\left(\frac{1}{\gamma}(X^{\pi^*}(T))^\gamma \right).$$

The number $\hat{\pi}$ is explicitly given in Theorem 3.8 with $\Theta = (1-\gamma)$.

Proof. The existence of the maximum was shown in Proposition 3.2. The correctness of the value function was proved in Theorem 3.12. The determination of $\hat{\pi}$ is exactly the same as in Theorem 3.8. □

Remark 3.17 (Optimal control π^* for the case of frequency polygons)
In Figure 12 we plotted the optimal control π^* in case of the frequency polygon and the HARA-utility against the volatility σ and the stock drift b. Again, we observe the obvious behavior, that the optimal control π^* increases when the asset drift increases resp. the volatility decreases. But on the points of discontinuity of the first derivative, i.e. α_j, we observe different properties: In $\alpha_1 = 1.0$ there is a sharp bend on the surface, instead of a plateau. In $\alpha_2 = 1.5$ there is again a small plateau. Then π is slightly increasing between 1.5 and 2.5 and then it jumps to a value at 3.5. So there are still plateaus, but they are smaller as in the step function case, since the dependency of the interest rate and the control is continuous.
The plateau at 1.5 can be explained by the the slopes of the frequency polygon: The slope between 1 and 1.5 is smaller than the one between 1.5 and 2, so - from an economic point of view - it is not benefitial to increase the (relative) investment in the stocks, since the (relative) loan payments would increase "over-proportional". Or from an analytical point of view: Due to the stronger slope the maximum does not change with an increasing b and decreasing σ.

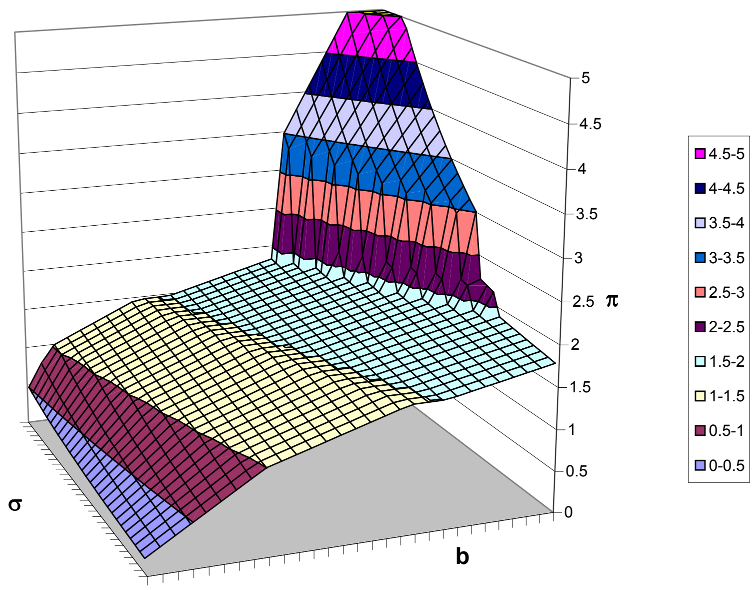


Figure 12: Optimal control with $r(.)$ frequency polygon and HARA utility ($\gamma = 0.5$)

The sharp bend can be explained by formula (3.23): For $x \leq 1$ the derivative of the optimal control w.r.t b is $1/(0.5{*}\sigma^2)$, but for $1 < x < 1.5$ it is $1/(0.5{*}\sigma^2 + 2{*}\mu_1) = 1/(0.5{*}\sigma^2 + 0.06)$, thus clearly smaller.

3.5 Conclusions

Optimal control for other dependencies

Note that in the case of frequency polygons, the value function is a continuous function from the space of frequency polygons to the real numbers, because the apex and the maximum function is continuous. Let $\tilde{r}(\hat{\pi}) : I\!R^n \to I\!R$ be a bounded and continuously differentiable function. Since $\tilde{r}(\hat{\pi})$ is bounded and continuous, the maximum of the corresponding function $M^\theta(x)$ and thus an optimal control u^* exists. We can restrict the domain of $\tilde{r}(\hat{\pi})$ to a compact set, which is sufficiently large such that $\hat{\pi}$ lies in it. On this compact set $\tilde{r}(\hat{\pi})$ can be uniformely approximated by a sequence of frequency polygons $P_n(\hat{\pi})$, i.e. $\|P_n(.) - \tilde{r}(.)\| \to 0$. Hence via the above noted continuity and the same techniques as in Theorem 1.14 in order to show uniform integrability we can conclude

$$E^{t,x}\left[U\left(X^{\pi^{*,P_n(.)}}\right)\right] \quad \to \quad E^{t,x}\left[U\left(X^{\pi^{*,\tilde{r}(.)}}\right)\right],$$

where U equals Log or HARA utility and $\pi^{*,f(.)}$ denotes the optimal control with control depending interest rate $r(t) = f(\pi(t))$. Unfortunately, the optimal control does not necessarily converge, because 'arg max' is not continuous. But if π^* is unique (which means the difference between the absolute maximum and nearest local maximum is greater than zero), we obtain convergence of the control too, as $\pi^{*,P_n(.)}$ cannot alternate between two local maxima, if n is sufficiently high.

For the ease of notation we have not included time-dependent market parameters $b(t), \sigma(t)$ and a time-dependent funtion $r(t, x)$, but note that this would not cause any technical difficulties. In addition it would be interesting to solve the problem with a jump-diffusion stock process.

Closing remarks

We have shown that a control-depending interest rate can be easily included into portfolio optimization. We provided explicit solutions for step functions and frequency polygons in both cases of logarithmic and HARA utility. In addition we have shown convergence of the optimal control; a feature, which is generally hard to obtain in portfolio optimization.
Independent from interest rate risk, this method can also be used to avoid high controls, in a sense of an implicit risk controlling.

4 Optimal Continuous and Discrete Consumption

4.1 Introduction

The standard literature (e.g. Merton(1990) or Korn&Korn(2001)) of portolio optimisation typically deals with the case of a continuous consumption stream and/or one final payment. Of course, in the real market consumption takes place at discrete, maybe periodical time instants. So a continuous consumption stream is more or less an approximation of the discrete consumption in the real market. The longer the time spaces between the consumption payments are, the more unsatisfactory this approximation becomes. In this chapter we will present the solution of the lump sum consumption problem for HARA-utility and LOG-utility, i.e. we explicitly consider consumption at a discrete time set. Furthermore we will add time-dependent weigths to the particular consumptions. These weights give the investor more possibilities to take his preferences into account and can also be used to discount the payments. As a side-effect, we will derive more general solutions for continuous consumptions.

4.2 Model

Since consumption takes place at discrete time instants $0 < t_1 < t_2 < \ldots < t_n = T$, we get a new evolution for the wealth process. The process $X^{\pi,\mathbf{B}}(t)$, where $\mathbf{B} = (B_1, \ldots, B_n)$ denotes the consumption payments, reads as

$$dX^{\pi,\mathbf{B}}(t) = X^{\pi,\mathbf{B}}(t)\Big[\big(r(t) + \pi'(t)(b(t) - r(t)\underline{1})\big)\,dt + \pi'(t)\sigma dW_t\Big] \tag{4.1}$$

for all $t \in (t_i, t_{i+1}), i = 0, \ldots, n-1$ and $t = 0$. At t_i $(\forall i = 1, \ldots, n-1)$ we clearly must have

$$X^{\pi,\mathbf{B}}(t_i) = X^{\pi,\mathbf{B}}(t_i-) - B_i, \tag{4.2}$$

where $X^{\pi,\mathbf{B}}(t_i-)$ denote the lefthand limit. In this chapter we assume that the market coefficients $r(t), b(t)$ and $\sigma(t)$ have the properties of Definition 1.12, i.e. they are continuous, bounded and are function of time, wealth and of π.

Definition 4.1 (Lump Sum Optimization Problem)
The problem of deriving a solution to

$$V^L(t,x) := \sup_{(\pi,\mathbf{B})\in\mathcal{A}^L(t,x)} J^L(t,x;(\pi,\mathbf{B})) \tag{4.3}$$

where

$$J^L(t,x;(\pi,\mathbf{B})) = E^{t,x}\left[\sum_{i:t<t_i<T} \alpha_i U_i(B_i) + \alpha_n U_n(B_n)\right] \tag{4.4}$$

and

$$\begin{aligned}\mathcal{A}^L(t,x) &= \Bigg\{(\pi,\,\mathbf{B})\Bigg| X^{\pi,\mathbf{B}}(t) \geq 0 \;\; \mathbf{P}\text{–}a.s.\forall s \in [t,T],\; B_i \,\text{w.r.t Def. (1.29)} \\ &\qquad E\left(\int_t^T |(\pi(s)|^k ds\right) < \infty\,,\; E^{t,x}\left[\sum_{i:t<t_i<T} \alpha_i U_i^-(B_i)\right] < \infty\Bigg\} \end{aligned} \tag{4.5}$$

is called a lump sum optimization problem, where the U_i *are the utility functions and the* α_i *are the real-valued weights. The process* π *is a portfolio process as defined in Definition 1.3 and the* $B_j \geq 0$ *are* $\mathcal{F}_{t_j}$*-measurable random variables (in other words "contingent claims").*

Theorem 4.2 (The martingale optimality principle for lump sum consumption) *Let* (π^*, B^*) *be an admissible control. We define the corresponding value function by*

$$G(t,x) = E^{t,x}\left[\sum_{i:t<t_i<T} \alpha_i U_i\left(B_i^*\right) + \alpha_n U_n\left(B_n^*\right)\right].$$

Furthermore, let

$$w^{\pi,\mathbf{B},t,x}(\theta) = \sum_{i:t<t_i\leq\theta, i\neq n} \alpha_i U_i\left(B_i\right) + G(\theta, X^{\pi,\mathbf{B}}(\theta)) \;\; with \;\; X(t) = x.$$

If $w^{\pi,\mathbf{B},t,x}(\theta)$ *is a supermartingale for all* $(\pi,\mathbf{B}) \in \mathcal{A}^L(t,x)$*, then* $(\pi^*,\mathbf{B}^*)$ *is indeed the optimal control, i.e. we have*

$$G(t,x) = \sup_{(\pi,\mathbf{B})\in\mathcal{A}^L(t,x)} E^{t,x}\left[J^L(t,x;(\pi,\mathbf{B}))\right] \equiv V^L(t,x).$$

PROOF.
Let $(\pi,\mathbf{B})$ be an admissible control and for notational convenience assume $t < T$:

$$\begin{aligned} E^{t,x}\left[\sum_{i:t<t_i\leq T} \alpha_i U_i\left(B_i\right)\right] &= E^{t,x}\left[\sum_{i:t<t_i<T} \alpha_i U_i\left(B_i\right)\right] + G(T, X^{\pi,c}(T)) \\ &= E^{t,x}[w^{\pi,\mathbf{B},t,x}(T)] \\ &\leq E^{t,x}[w^{\pi,\mathbf{B},t,x}(t)] \\ &= G(t,x) \\ &= E^{t,x}\left[\sum_{i:t<t_i\leq T} \alpha_i U_i\left(B_i^*\right)\right] \end{aligned}$$

□

Remark 4.3

i) Observe, that $E^{t,x}\left[w^{\pi^*,\mathbf{B}^*,t,x}(\theta)\right] = G(t,x)$ for all $\theta \in [t,T]$ implies that $w^{\pi^*,\mathbf{B}^*,t,x}(\theta)$ is a martingale.

ii) Note that even though π does not occur explicitly in the definition of $G(t,x)$ and $w^{\pi,\mathbf{B},t,x}(\theta)$ in Theorem 4.2 and in its proof, the optimal consumption $\mathbf{B}$ and the connected functions of course depends on π.

iii) The optimal consumption B_n at the last time instant T is obviously equal to $X(T)$. For the ease of notation we wrote B_n instead of $X(T)$.

Let us first recall some notations, which will be useful for proving a verification theorem corresponding to Theorem (4.2):

$$O = I\!R^n$$

$$U \subset I\!R \quad \text{compact}$$

For $G \in C^{1,2}(Q)$,$(t,x) \in Q$, $\hat{\pi} \in U^\pi$ let

$$\mathcal{A}^{\hat{\pi}}(t,x) := \frac{\partial}{\partial t} + \frac{1}{2}x^2\hat{\pi}'\sigma\sigma'\hat{\pi}\frac{\partial^2}{\partial x^2} + x(r + \hat{\pi}'(b-r)\underline{1})\frac{\partial}{\partial x}.$$

Theorem 4.4 Lump Sum Verification theorem
Let

$$G \in C^{1,2}([t_i, t_{i+1}) \times O) \cap C(T \times O) \text{ with } |G(t,x)| \leq K(1 + |x|^k) \tag{4.6}$$

for some suitable constants $K > 0, k \in I\!N$, be a solution to the piecewise Hamilton-Jacobi-Bellman equation for all $t \in [0,T] \setminus \{t_1, \ldots, t_n\}$ and $x > 0$:

$$\sup_{\pi \in U} \{\mathcal{A}^\pi(t,x)G(t,x)\} = 0 \tag{4.7}$$

with the jump conditions

$$G(t_i-, x) = \alpha_i U_i(b_i^*(x)) + G(t_i, x - b_i^*(x)) \tag{4.8}$$

$$b_i^*(x) = \arg\max_b \left[\alpha_i U_i(b) + G(t_i, x-b)\right] \tag{4.9}$$

$$G(T,x) = \alpha_n U_n(x) \tag{4.10}$$

Furthermore define

$$\pi^*(t,x) = -\frac{b-r}{\sigma^2}\frac{G_x(t,x)}{xG_{xx}(t,x)} \tag{4.11}$$

(where in the points t_i we take the right hand limits of the derivatives). Then, $G(t,x)$ coincides with the value function $V^L(t,x)$, and $(\pi^*(t), \mathbf{B})$ is an optimal control process for problem (4.3). Note, that we wrote B_i as $b_i(X(t_i))$, whereby $b_i(.)$ is a real-valued function.

Proof.
The proof is similar to the continuous case in Chapter 1. That means, we will prove the desired assertion first on a bounded domain, and then we will generalize it to an unbounded one. Let $G(t,x)$ be the asserted solution of equations (4.7)-(4.10) and

$$\begin{aligned} O_p &= \{x \in I\!R \big| |x| < p\}, \\ \tau_{\theta,p} &= \inf\{s \geq t \big| (s, X(s)) \notin [t,\theta] \times O_p\}. \end{aligned}$$

Now we apply Itô's formula to the utility process and take the expectation in one step:

$$\begin{aligned}
& E\left[w^{\pi,\mathbf{B},t,x}(\tau_{\theta,p}) \Big| \mathcal{F}_t \right] \\
= \ & w^{\pi,\mathbf{B},t,x}(\tau_{\theta,p}) \\
+ \ & E\left[\int_t^{\min(\min\{t_i : t_i > t\}, \tau_{\theta,p})} \underbrace{\mathcal{A}^\pi(t, X^{\pi,\mathbf{B}}(s)) G(s, X^{\pi,\mathbf{B}}(s))}_{\leq 0} ds \Big| \mathcal{F}_t \right] \\
+ \ & \underbrace{E\left[\int_t^{\min(\min\{t_i : t_i > t\}, \tau_{\theta,p})} X^{\pi,\mathbf{B}}(s) \pi'(s) \sigma(s) G_x(s, X^{\pi,\mathbf{B}}(s)) dW(s) \Big| \mathcal{F}_t \right]}_{=0} \\
+ \ & \sum_{i : t < t_i < \tau_{\theta,p}} E\left[\underbrace{G(t_i, X^{\pi,\mathbf{B}}(t_i)) - G(t_i -, X^{\pi,\mathbf{B}}(t_i -)) + \alpha_i U_i\left(b_i^*(X^{\pi,\mathbf{B}}(t_i)) \right)}_{\leq 0} \Big| \mathcal{F}_t \right] \\
+ \ & \sum_{i : t < t_i, t_{i+1} < \tau_{\theta,p}} E\left[\int_{t_i}^{t_{i+1}} \underbrace{\mathcal{A}^\pi(t, X^{\pi,\mathbf{B}}(s)) G(s, X^{\pi,\mathbf{B}}(s))}_{\leq 0} ds \Big| \mathcal{F}_t \right] \\
+ \ & \sum_{i : t < t_i, t_{i+1} < \tau_{\theta,p}} \underbrace{E\left[\int_{t_i}^{t_{i+1}} X^{\pi,\mathbf{B}}(s) \pi'(s) \sigma(s) G_x(s, X^{\pi,\mathbf{B}}(s)) dW(s) \Big| \mathcal{F}_t \right]}_{=0} \\
+ \ & E\left[\int_{\max\{t_i : t < t_i \leq \tau_{\theta,p}\}}^{\tau_{\theta,p}} \underbrace{\mathcal{A}^\pi(t, X^{\pi,\mathbf{B}}(s)) G(s, X^{\pi,\mathbf{B}}(s))}_{\leq 0} ds \Big| \mathcal{F}_t \right] \\
+ \ & \underbrace{E\left[\int_{\max\{t_i : t < t_i \leq \tau_{\theta,p}\}}^{\tau_{\theta,p}} X^{\pi,\mathbf{B}}(s) \pi'(s) \sigma(s) G_x(s, X^{\pi,\mathbf{B}}(s)) dW(s) \Big| \mathcal{F}_t \right]}_{=0}
\end{aligned}$$

The inequality-relation for the term $\mathcal{A}^\pi(t, X(s))G(s, X(s)))$ is valid by equation (4.7). For the second inequality-relation in the jumps at times t_i we observe from (4.8) and (4.9), that

that for all $b \in [0, x]$

$$G(t_i-, x) \quad \geq \quad \alpha_i U_i(b) + G(t_i, x - b).$$

Since O_p is bounded and $G_x(.,.)$ is continuous, $G_x(.,.)$ is bounded on $[t, \theta] \times O_p$. In addition, it holds

$$E^{t,x}\left[\int_t^\theta \left[X^{\pi,\mathbf{B}}(s)(r + \pi'(b - r\underline{1}))\right]^2 ds\right] < \infty$$

by the definition of $\mathcal{A}^L(t, x)$. So the whole integrand is in $L^2[0, \theta]$ and the expectation of the stochastic integral equals zero. Thus we shown so far that

$$w^{\pi,\mathbf{B},t,x}(t) \quad \geq \quad E^{t,x}\left[w^{\pi,\mathbf{B},t,x}(\tau_{\theta,p})\right]. \tag{4.12}$$

Now we have to show the same for the unrestricted problem, i.e.

$$w^{\pi,\mathbf{B},t,x}(t) \quad \geq \quad E^{t,x}\left[w^{\pi,\mathbf{B},t,x}(\theta)\right] \tag{4.13}$$

due to

$$\lim_{p \to \infty} E^{t,x}\left[w^{\pi,\mathbf{B},t,x}(\tau_{\theta,p})\right] \quad = \quad E^{t,x}\left[w^{\pi,\mathbf{B},t,x}(\theta)\right]. \tag{4.14}$$

Since the argumentation of the proof is the same as in the Verification Theorem 1.14 in Chapter 1, we will not repeat it here. Hence $w^{\pi,\mathbf{B},t,x}(\theta)$ is a martingale for $(\pi^*, \mathbf{B}^*)$, and otherwise a supermartingale. The assertion now follows by the martingale optimality principle.

□

4.3 HARA Utility

Now we present the explicit solution for HARA-Utility, which is defined by

$$V^L(t, x) := \sup_{(\pi,,\mathbf{B}) \in \mathcal{A}^L(t,x)} E^{t,x}\left[\sum_{i:t_i > t \cup n} \alpha_i \frac{1}{\gamma} B_i^\gamma\right].$$

We start with a heuristic derivation which is followed by a rigorous proof.
First consider the jump condition, that means at time instants t_i the next equation has to hold

$$V(t_i-, x) \quad = \quad \sup_{b_i}\left[\alpha_i \frac{1}{\gamma} b_i{}^\gamma + V(t_i, x - b_i)\right]. \tag{4.15}$$

This equation can be interpreted as follows: The supremum over b_i means, that the consumption has to be chosen, such that the sum of the utility of the actual consumption and expected utility of future consumption is maximised. The value function is right-continuous w.r.t t in t_i. The jump in the value function at t_i is exactly the realized utility.

We guess the solution to be of the form

$$V^L(t,x) = \frac{1}{\gamma} x^\gamma f(t)^{1-\gamma} \tag{4.16}$$

leading to

$$V^L(t_i-, x) \;=\; \sup_{b_i} \Big[\alpha_i \frac{1}{\gamma} {b_i}^\gamma + \frac{1}{\gamma}(x - b_i)^\gamma f(t_i)^{1-\gamma} \Big]. \tag{4.17}$$

To determine the maximum, we set the derivative with resp. to b_i equal to zero:

$$\begin{aligned}
& \alpha_i {b_i}^{\gamma-1} - (x-b_i)^{\gamma-1} f(t_i)^{1-\gamma} = 0 \\
\Leftrightarrow \quad & \alpha_i {b_i}^{\gamma-1} = (x-b_i)^{\gamma-1} \left(\frac{1}{f(t_i)} \right)^{\gamma-1} \\
\Leftrightarrow \quad & \alpha_i^{\frac{1}{\gamma-1}} b_i f(t_i) = x - b_i \\
\Leftrightarrow \quad & b_i = \frac{x}{\alpha_i^{\frac{1}{\gamma-1}} f(t_i) + 1} = \frac{x}{\left(\frac{1}{\alpha_i}\right)^{\frac{1}{1-\gamma}} f(t_i) + 1}
\end{aligned} \tag{4.18}$$

Plugging this back into (4.15) yields:

$$\begin{aligned}
V(t_i-, x) &= \frac{1}{\gamma} \left[\alpha_i \left(\frac{x}{\alpha_i^{\frac{1}{\gamma-1}} f(t_i) + 1} \right)^\gamma + \left(x - \left(\frac{x}{\alpha_i^{\frac{1}{\gamma-1}} f(t_i) + 1} \right) \right)^\gamma f(t_i)^{1-\gamma} \right] \\
&= \frac{1}{\gamma} \left[\alpha_i \left(\frac{x}{\alpha_i^{\frac{1}{\gamma-1}} f(t_i) + 1} \right)^\gamma + \left(x \left(\frac{\alpha_i^{\frac{1}{\gamma-1}} f(t_i)}{\alpha_i^{\frac{1}{\gamma-1}} f(t_i) + 1} \right) \right)^\gamma f(t_i)^{1-\gamma} \right] \\
&= \frac{1}{\gamma} \left[\alpha_i \left(\frac{x}{\alpha_i^{\frac{1}{\gamma-1}} f(t_i) + 1} \right)^\gamma + \left(\frac{\alpha_i^{\frac{1}{\gamma-1}} x}{\alpha_i^{\frac{1}{\gamma-1}} f(t_i) + 1} \right)^\gamma f(t_i) \right] \\
&= \frac{1}{\gamma} \left[\left(\alpha_i^{\frac{\gamma}{\gamma-1}} f(t_i) + \alpha_i \right) \left(\frac{x}{\alpha_i^{\frac{1}{\gamma-1}} f(t_i) + 1} \right)^\gamma \right] \\
&= \frac{1}{\gamma} \alpha_i \left[\left(\alpha_i^{\frac{1}{\gamma-1}} f(t_i) + 1 \right) \left(\frac{x}{\alpha_i^{\frac{1}{\gamma-1}} f(t_i) + 1} \right)^\gamma \right] \\
&= \frac{1}{\gamma} x^\gamma \alpha_i \left(\alpha_i^{\frac{1}{\gamma-1}} f(t_i) + 1 \right)^{1-\gamma}
\end{aligned} \tag{4.19}$$

Using again the guess of V^L given by equation (4.16) leads us to

$$\frac{1}{\gamma}x^\gamma f(t_i-)^{1-\gamma} = \frac{1}{\gamma}x^\gamma \alpha_i \left(\alpha_i^{\frac{1}{\gamma-1}} f(t_i) + 1\right)^{1-\gamma} \tag{4.20}$$

$$\Longrightarrow \quad f(t_i-) = \alpha_i^{\frac{1}{1-\gamma}} \left(\alpha_i^{\frac{1}{\gamma-1}} f(t_i) + 1\right) \tag{4.21}$$

$$\Longrightarrow \quad f(t_i) = f(t_i-) - \alpha_i^{\frac{1}{1-\gamma}}. \tag{4.22}$$

The case when there is only the final utility left leads us to the following deduction

$$f(t) = \sum_{\{i:t_i>t\}\cup n} \alpha_i^{\frac{1}{1-\gamma}} g_i(t),$$

$$g_i(t) = \exp\left(\frac{\gamma}{1-\gamma}\int_t^{t_i} \kappa(s)ds\right),$$

where $\kappa(s)$ denotes the drift coefficient of the value function of the standard HARA-problem. We summarize the above considerations in the following theorem:

Theorem 4.5 (Optimal portfolio for Lump Sum HARA-Utility)
The value function of the portfolio problem

$$V^L(t,x) := \sup_{(\pi,\mathbf{B})\in\mathcal{A}^L(t,x)} E^{t,x}\left[\sum_{\{i:t_i>t\}\cup n} \alpha_i \frac{1}{\gamma} B_i^\gamma\right] \tag{4.23}$$

is given by

$$V^L(t,x) = \frac{1}{\gamma}x^\gamma \left(\sum_{\{i:t_i>t\}\cup n} \alpha_i^{\frac{1}{1-\gamma}} \exp\left(\frac{\gamma}{1-\gamma}\int_t^{t_i} \kappa(s)ds\right)\right)^{1-\gamma} \tag{4.24}$$

where

$$\kappa(s) = r(s) + \frac{1}{2}(b(s) - r(s)\underline{1})'(\sigma(s)\sigma(s)')^{-1}(b(s) - r(s)\underline{1})\frac{1}{1-\gamma}.$$

The optimal control equals

$$\pi^*(t) = \frac{1}{1-\gamma}(\sigma(t)\sigma(t)')^{-1}(b(t) - r(t)\underline{1}) \tag{4.25}$$

and the optimal consumption reads as

$$b_i^*(x) = \frac{x\alpha_i^{\frac{1}{1-\gamma}}}{\sum_{j=i}^n \alpha_j^{\frac{1}{1-\gamma}} \exp\left(\frac{\gamma}{1-\gamma}\int_{t_i}^{t_j} \kappa(s)ds\right)}. \tag{4.26}$$

Remark 4.6

i) The reason for the "$\cup n$" in the sums of equations (4.23) and (4.24) is the following: We defined the value function as sum of all future utility, but did not define a jump in wealth at the final time instant $t_n = T$, such that $G(T, x) > 0$ holds. So we need the "$\cup n$" to cover the case of $t = T$.

ii) The portfolio process is the same as in the Example 1.19 with only a final utility.

iii) The optimal consumption B_i at times t_i has the natural behavior of being a linear function in wealth x. It is growing with the corresponding weight α_i. Note in particular that it contains some non-linearity: The optimal consumption (for $x = 1$) is not just α_i devided by the sum of the expected HARA-utility of the remaining time instants. Instead all summands are taken to the power of $1/(1-\gamma)$.

PROOF.
Our candidate for the value function $V(t, x)$ is concave, so we can formally derive the optimal process π^* and check if it fulfills the HJB-equation. First we calculate the derivatives

$$
\begin{aligned}
V_x(t,x) &= x^{\gamma-1}\left(\sum_{j=i}^{n}\alpha_j^{\frac{1}{1-\gamma}}\exp\left(\frac{\gamma}{1-\gamma}\int_t^{t_j}\kappa(s)ds\right)\right)^{1-\gamma}, \\
V_{xx}(t,x) &= (\gamma-1)x^{\gamma-2}\left(\sum_{j=i}^{n}\alpha_j^{\frac{1}{1-\gamma}}\exp\left(\frac{\gamma}{1-\gamma}\int_t^{t_j}\kappa(s)ds\right)\right)^{1-\gamma}, \\
V_t(t,x) &= \frac{1}{\gamma}x^{\gamma}(1-\gamma)\left(\sum_{j=i}^{n}\alpha_j^{\frac{1}{1-\gamma}}\exp\left(\frac{\gamma}{1-\gamma}\int_t^{t_j}\kappa(s)ds\right)\right)^{-\gamma} \\
&\quad * \left(-\kappa(s)\sum_{j=i}^{n}\alpha_j^{\frac{1}{1-\gamma}}\exp\left(\frac{\gamma}{1-\gamma}\int_t^{t_j}\kappa(s)ds\right)\right) \\
&= \frac{1}{\gamma}x^{\gamma}(1-\gamma)\left(\sum_{j=i}^{n}\alpha_j^{\frac{1}{1-\gamma}}\exp\left(\frac{\gamma}{1-\gamma}\int_t^{t_j}\kappa(s)ds\right)\right)^{1-\gamma}(-\kappa), \\
A(s) &= (b(s)-r(s)\underline{1})'(\sigma(s)\sigma(s)')^{-1}(b(s)-r(s)\underline{1}).
\end{aligned}
$$

By the HJB-equation and by the above derivatives we check that the following equations are true

$$
\begin{aligned}
& V_t(t,x) + r(t)xV_x(t,x) - \frac{1}{2}A(t)\frac{V_x^2(t,x)}{V_{xx}(t,x)} = 0 \\
\Leftrightarrow \quad & \frac{1-\gamma}{\gamma}x^\gamma(-\kappa(t)) + r(t)x^\gamma - \frac{1}{2}A(s)\frac{x^\gamma}{\gamma-1} = 0 \\
\Leftrightarrow \quad & \frac{\gamma-1}{\gamma}\kappa(t) + r(t) + \frac{1}{2}A(t)\frac{1}{1-\gamma} = 0 \\
\Leftrightarrow \quad & \kappa(t) = \frac{\gamma}{1-\gamma}\left(r(t) + \frac{1}{2}A(t)\frac{1}{1-\gamma}\right)
\end{aligned}
$$

From the derivatives we can easily conclude, that the trading strategy (4.25) is the optimal one for this value function. By equation (4.9) and due to the construction of V^L in (4.17) and (4.18) holds

$$
b_i^* = \arg\max_b \left[\alpha_i \frac{1}{\gamma} b^\gamma + V(t_i, x-b)\right].
$$

The jump condition (4.8) actually also is true by construction, but we check it once again:

$$
\begin{aligned}
& V(t_i-, x) = V(t_i, x - b_i^*) + \alpha_i \frac{1}{\gamma}x^\gamma \\
\Leftrightarrow \quad & \frac{1}{\gamma}x^\gamma\left(\sum_{j=i}^n \alpha_j^{\frac{1}{1-\gamma}} e^{\kappa(t_j-t)}\right)^{1-\gamma} = \frac{1}{\gamma}\left(x\frac{\alpha_i^{\frac{1}{\gamma-1}}f(t_i)}{\alpha_i^{\frac{1}{\gamma-1}}f(t_i)+1}\right)^\gamma \left(\sum_{j=i+1}^n \alpha_j^{\frac{1}{1-\gamma}} e^{\kappa(t_j-t)}\right)^{1-\gamma} \\
& \qquad + \alpha_i\frac{1}{\gamma}\left(\frac{x}{\alpha_i^{\frac{1}{\gamma-1}}f(t_i)+1}\right)^\gamma \\
\Leftrightarrow \quad & \frac{1}{\gamma}x^\gamma\left(f(t_i) + \alpha_i^{\frac{1}{1-\gamma}}\right)^{1-\gamma} = \frac{1}{\gamma}\left(x\frac{\alpha_i^{\frac{1}{\gamma-1}}f(t_i)}{\alpha_i^{\frac{1}{\gamma-1}}f(t_i)+1}\right)^\gamma f(t_i)^{1-\gamma} + \alpha_i\frac{1}{\gamma}\left(\frac{x}{\alpha_i^{\frac{1}{\gamma-1}}f(t_i)+1}\right)^\gamma \\
\Leftrightarrow \quad & \frac{1}{\gamma}x^\gamma\left(f(t_i) + \alpha_i^{\frac{1}{1-\gamma}}\right)^{1-\gamma} = \frac{1}{\gamma}x^\gamma\left[\left(\frac{1}{f(t_i)+\alpha_i^{\frac{1}{1-\gamma}}}\right)^\gamma f(t_i) + \alpha_i\alpha_i^{\frac{\gamma}{1-\gamma}}\left(\frac{1}{f(t_i)+\alpha_i^{\frac{1}{1-\gamma}}}\right)^\gamma\right] \\
\Leftrightarrow \quad & \frac{1}{\gamma}x^\gamma\left(f(t_j) + \alpha_i^{\frac{1}{1-\gamma}}\right)^{1-\gamma} = \frac{1}{\gamma}x^\gamma\left[\left(\frac{1}{f(t_j)+\alpha_j^{\frac{1}{1-\gamma}}}\right)^\gamma \left(f(t_j) + \alpha_j^{\frac{1}{1-\gamma}}\right)\right]
\end{aligned}
$$

□

From the solution of lump sum utility we can guess the solution for the continuous case:

Theorem 4.7 (Optimal portfolio for continuous HARA-Utility)
The value function of the portfolio problem

$$V(t,x) := \sup_{(\pi,c)\in\mathcal{A}^{\mathcal{H}}(t,x)} E^{t,x}\left[\int_t^T \alpha(s)\frac{1}{\gamma}c(s)^\gamma ds + \beta\frac{1}{\gamma}X^{\pi,c}(T)^\gamma\right],$$

i.e., a problem as in Definition 1.12 with $U_2(t,c) = \alpha(s)\frac{1}{\gamma}c(s)^\gamma$ *and* $U_1(x) = \beta\frac{1}{\gamma}x^\gamma$ *and with a wealth process fulfilling equation (1.9), that means*

$$dX^{\pi,c}(t) = X^{\pi,c}(t)\left[\big(r(t)+\pi'(t)(b(t)-r(t)\underline{1})\big)dt - \pi'(t)\sigma(t)dW(t)\right] - c(t)dt,$$

is given by

$$V(t,x) = \frac{1}{\gamma}x^\gamma\left(\int_t^T \alpha(s)^{\frac{1}{1-\gamma}}\exp\left(\frac{\gamma}{1-\gamma}\int_t^s \kappa(z)dz\right)ds + \beta^{\frac{1}{1-\gamma}}\exp\left(\frac{\gamma}{1-\gamma}\int_t^T \kappa(z)dz\right)\right)^{1-\gamma},$$

where

$$\kappa(s) = r(s) + \frac{1}{2}(b(s)-r(s)\underline{1})'(\sigma(s)\sigma(s)')^{-1}(b(s)-r(s)\underline{1})\frac{1}{1-\gamma}.$$

The optimal control equals

$$\pi^*(t) = \frac{1}{1-\gamma}(\sigma(t)\sigma(t)')^{-1}(b(t)-r(t)\underline{1})$$

and the optimal consumption reads as

$$c^*(t) = \frac{x\alpha(s)^{\frac{1}{1-\gamma}}}{\int_t^T \alpha(s)^{\frac{1}{1-\gamma}}\exp\left(\frac{\gamma}{1-\gamma}\int_t^s \kappa(z)dz\right)ds + \beta^{\frac{1}{1-\gamma}}\exp\left(\frac{\gamma}{1-\gamma}\int_t^T \kappa(z)dz\right)}.$$

PROOF.
First we check, whether the HJB-equation is fulfilled. For this purpose we introduce some notations:

$$\begin{aligned}
\Phi(t) &= \int_t^T \alpha(s)^{\frac{1}{1-\gamma}}\exp\left(\frac{\gamma}{1-\gamma}\int_t^s \kappa(z)dz\right)ds + \beta^{\frac{1}{1-\gamma}}\exp\left(\frac{\gamma}{1-\gamma}\int_t^T \kappa(z)dz\right),\\
V(t,x) &= \frac{1}{\gamma}x^\gamma\Phi(t)^{1-\gamma},\\
V_x(t,x) &= x^{\gamma-1}\Phi(t)^{1-\gamma},\\
V_{xx}(t,x) &= (\gamma-1)x^{\gamma-2}\Phi(t)^{1-\gamma},\\
V_t(t,x) &= \frac{1}{\gamma}x^\gamma(1-\gamma)\Phi(t)^{-\gamma}\left[-\alpha(t)^{\frac{1}{1-\gamma}} - \frac{\gamma}{1-\gamma}\kappa(t)\Phi(t)\right],\\
&= -\frac{1}{\gamma}x^\gamma(1-\gamma)\Phi(t)^{-\gamma}\alpha(t)^{\frac{1}{1-\gamma}} - x^\gamma\Phi(t)^{1-\gamma}\kappa(t)
\end{aligned}$$

$$\left(\frac{\partial U_2(t,c)}{\partial c}\right)^{-1}(y) = \left(\frac{y}{\alpha(t)}\right)^{\frac{1}{\gamma-1}},$$

$$c^*(t) = \frac{\alpha(t)^{\frac{1}{1-\gamma}}x}{\Phi(t)},$$

$$A(t) = (b(t) - r(t)\underline{1})'(\sigma(t)\sigma(t)')^{-1}(b(t) - r(t)\underline{1}).$$

Now we prove that the HJB-equation holds:

$$-\frac{1}{2}A(t)\frac{V_x^2(t,x)}{V_{xx}(t,x)} + r(t)xV_x(t,x) + V_t(t,x)$$

$$-\left(\frac{\partial U_2(t,c)}{\partial c}\right)^{-1}(V_x(t,x))V_x(t,x) + U_2(t,c^*(t)) = 0$$

$$\Leftrightarrow \quad -\frac{1}{2}A(s)\frac{x^\gamma}{\gamma-1}\Phi(t)^{1-\gamma} + r(t)x^\gamma\Phi(t)^{1-\gamma} - x^\gamma\Phi(t)^{1-\gamma}\kappa(t)$$

$$-\frac{1}{\gamma}x^\gamma(1-\gamma)\Phi(t)^{-\gamma}\alpha(t)^{\frac{1}{1-\gamma}} - \left(\frac{x^{\gamma-1}\Phi(t)^{1-\gamma}}{\alpha(t)}\right)^{\frac{1}{\gamma-1}} x^{\gamma-1}\Phi(t)^{1-\gamma}$$

$$+\alpha(t)\frac{1}{\gamma}\left(\frac{\alpha(t)^{\frac{1}{1-\gamma}}x}{\Phi(t)}\right)^\gamma = 0$$

$$\Leftrightarrow \quad \left(r(t) + \frac{1}{2}A(s)\frac{1}{1-\gamma} - \kappa(t)\right)x^\gamma\Phi(t)^{1-\gamma}$$

$$-\frac{1}{\gamma}(1-\gamma)\alpha(t)^{\frac{1}{1-\gamma}}x^\gamma\Phi(t)^{-\gamma} - \alpha(t)^{\frac{1}{1-\gamma}}x^\gamma\Phi(t)^{-\gamma}$$

$$+\frac{1}{\gamma}\alpha(t)^{\frac{1}{1-\gamma}}x^\gamma\Phi(t)^{-\gamma} = 0$$

The first line of the last equation is zero and the remainder collapses also to zero. So the HJB-equation is fulfilled. In addition $\pi^*(t)$ is bounded and equals $-(\sigma\sigma')^{-1}(b - r\underline{1})\frac{V_x(t,x)}{xV_{xx}(t,x)}$ and the value function $V(t,x)$ is concave.

□

Example 4.8 (Optimal continuous consumption and final payment for HARA-Utility)

Let

$$\alpha(t) = e^{-\delta t}; \quad \beta = 1$$

and the market parameters r, b, σ to be constant.
The optimal control equals:

$$\pi^*(t) = \frac{1}{1-\gamma}(\sigma\sigma')^{-1}(b - r\underline{1}).$$

The optimal consumption is given by:

$$\begin{aligned} c^*(t) &= \frac{x(e^{-\delta t})^{\frac{1}{1-\gamma}}}{\int_t^T (e^{-\delta s})^{\frac{1}{1-\gamma}} \exp\left(\frac{\gamma}{1-\gamma}\kappa(s-t)\right) ds + \exp\left(\frac{\gamma}{1-\gamma}\kappa(T-t)\right)} \\ &= \frac{x(e^{-\delta t})^{\frac{1}{1-\gamma}}}{\exp\left(-\frac{\gamma}{1-\gamma}\kappa t\right) \int_t^T \exp\left(\frac{1}{1-\gamma}(\gamma\kappa-\delta)s\right) ds + \exp\left(\frac{\gamma}{1-\gamma}\kappa(T-t)\right)} \\ &= \frac{x(e^{-\delta t})^{\frac{1}{1-\gamma}}}{\exp\left(-\frac{\gamma}{1-\gamma}\kappa t\right) \frac{1-\gamma}{\gamma\kappa-\delta}\left[e^{\left(\frac{1}{1-\gamma}(\gamma\kappa-\delta)T\right)} - e^{\left(\frac{1}{1-\gamma}(\gamma\kappa-\delta)t\right)}\right] + \exp\left(\frac{\gamma}{1-\gamma}\kappa(T-t)\right)}. \end{aligned}$$

Remark 4.9 (Convergence of the HARA-Utility solutions)
In this remark we show that the discrete solution converges to the continuous solutions. Let without loss of generality $t = 0$ and $\alpha(s) : [0,T] \to \mathbb{R}^+$ be some real-valued continuous and bounded function and let $\beta = 0$. We define the corresponding discrete problem as follows

$$\triangle^n = \frac{T}{n}, \quad t_i^n = i * \triangle^n, \quad \alpha_i^n = \alpha(t_i^n)\triangle^n.$$

Let the consumption rate of the discrete solution be defined as

$$c^{*n}_{\theta(n)}(x) = \frac{b^{*n}_{\theta(n)}(x)}{\triangle^n},$$

where $\theta(n) = [\theta/\triangle^n]$ for some $\theta \in [0,T]$ and "$[y]$" is the biggest integer smaller than y. Observe that $(\alpha_i^n)^{\frac{1}{1-\gamma}} = (\triangle^n)^{\frac{\gamma}{1-\gamma}}\alpha(t_i^n)^{\frac{1}{1-\gamma}}\triangle^n$ such that

$$\begin{aligned} b^{*n}_\theta(x) &= \frac{x\,(\alpha^n_{\theta(n)})^{\frac{1}{1-\gamma}}}{\sum_{j=\theta(n)}^n (\alpha_j^n)^{\frac{1}{1-\gamma}} \exp\left(\frac{\gamma}{1-\gamma}\int_{t^n_{\theta(n)}}^{t_j^n} \kappa(s)ds\right)} \\ &= \frac{x\alpha(t^n_{\theta(n)})^{\frac{1}{1-\gamma}}\triangle^n}{\sum_{j=\theta(n)}^n \alpha(t_j^n)^{\frac{1}{1-\gamma}} \exp\left(\frac{\gamma}{1-\gamma}\int_{t^n_{\theta(n)}}^{t_j^n} \kappa(s)ds\right)\triangle^n}. \end{aligned}$$

By the definition of the Riemann-Integral we obtain

$$\begin{aligned}
\lim_{n\to\infty} c_\theta^{*n}(x) &= \frac{x\alpha(t_i^n)^{\frac{1}{1-\gamma}}}{\lim_{n\to\infty}\sum_{j=\theta(n)}^n \alpha(t_j^n)^{\frac{1}{1-\gamma}} \exp\left(\frac{\gamma}{1-\gamma}\int_{t_{\theta(n)}^n}^{t_j^n} \kappa(s)ds\right) \triangle^n} \\
&= \frac{x\alpha(t_i^n)^{\frac{1}{1-\gamma}}}{\int_\theta^T \alpha(s)^{\frac{1}{1-\gamma}} \exp\left(\frac{\gamma}{1-\gamma}\int_\theta^s \kappa(z)dz\right) ds} \\
&= c^*(\theta, x).
\end{aligned}$$

So we get the same consumption process and the same portfolio process. With our choice of α_n^i the disrete value function does not converge to the continuous value function. But this is just a matter of rescaling the weights α_i. With $\alpha_i^n = \alpha(t_i^n) * (\triangle^n)^{1-\gamma}$ we can conclude the convergence of consumption, control and value function.

4.4 Logarithmic Utility

Now we derive the corresponding theorems of the previous section for the LOG-Utility.

Theorem 4.10 (Optimal portfolio with Lump Sum LOG-Utility)
The value function of the portfolio problem

$$V(t,x) := \sup_{(\pi,\mathbf{B})\in\mathcal{A}^L(t,x)} E^{t,x}\left[\sum_{\{i:t_i>t\}\cup n} \alpha_j \log(B_j)\right]$$

equals

$$G(t,x) = \sum_{i:t_i>t\cup n} \alpha_j \left[\log\left(\frac{\alpha_i x}{\sum_{\{i:t_i>t\}\cup n}\alpha_j}\right) + \kappa(t_j - t)\right] \tag{4.27}$$

where

$$\kappa(s) = r(s) + \frac{1}{2}(b(s) - r(s)\underline{1})'(\sigma(s)\sigma(s)')^{-1}(b(s) - r(s)\underline{1}).$$

The optimal control is given by

$$\pi^*(t) = (\sigma(t)\sigma(t)')^{-1}(b(t) - r(t)\underline{1})$$

and the optimal consumption at t_i reads as

$$b_i^* = \frac{\alpha_i x}{\sum_{j=i}^n \alpha_j}.$$

Remark: The term in the brackets in equation (4.27) is the log-utility in t_j of $\left(\frac{\alpha_i x}{\sum_{j=i}^n \alpha_j}\right)$ invested in t under the optimal control in the standard log utilitiy problem. Thus we have a linear structure of the solution, since the value function is the sum of the discounted expected utlities to be achieved on the time instants.

Proof.
In order to proof this theorem we check the conditions of Theorem 4.4:

i.) Continuity condition in t_i as in equation (4.8):

$$\begin{aligned} G(t_i-,x) &= \alpha_i \log(b_i^*) + G(t_i, x-b_i^*) \\ &= \alpha_i \log\left(\frac{\alpha_i x}{\sum_{j=i}^n \alpha_j}\right) + G\left(t_i, x\frac{\sum_{j=i+1}^n \alpha_j}{\sum_{j=i}^n \alpha_j}\right) \\ &= \alpha_i \log\left(\frac{\alpha_i x}{\sum_{j=i}^n \alpha_j}\right) + \sum_{j=i+1}^n \alpha_j \left[\log\left(\frac{\alpha_i x}{\sum_{j=i}^n \alpha_j}\right) + \kappa(t_j - t)\right] \end{aligned}$$

ii.) Correctness of b_i^* as in equation (4.9):

$$b_i^* = \arg\max_{b_i} \left[\alpha_i \log(b_i) + G(t_i, x-b_i)\right]$$

The derivative with resp. to b_i has to be equal to zero, i.e.

$$\alpha_i \frac{1}{b_i} = \frac{1}{x-b_i} \sum_{j=i+1}^n \alpha_j$$

$$\Leftrightarrow \qquad x - b_i = \frac{b_i}{\alpha_i} \sum_{j=i+1}^n \alpha_j$$

$$\Leftrightarrow \qquad x = b_i \left(\frac{1}{\alpha_i} \sum_{j=i+1}^n \alpha_j + 1\right)$$

$$\Leftrightarrow \qquad b_i = \frac{\alpha_i x}{\sum_{j=i}^n \alpha_j}$$

iii.) Hamilton-Jacobi-Bellmann equation (4.7):

$$\begin{aligned} G(t,x) &= \sum_{j=i}^n \alpha_j \log\left(\frac{\alpha_i x}{\sum_{j=i}^n \alpha_j}\right) + \left(r + \frac{1}{2}A(s)\right) \sum_{j=i}^n \alpha_j (t_j - t) \\ G_x(t,x) &= \frac{1}{x} \sum_{j=i}^n \alpha_j \\ G_{xx}(t,x) &= -\frac{1}{x^2} \sum_{j=i}^n \alpha_j \\ G_t(t,x) &= -\left(r + \frac{1}{2}A(s)\right) \sum_{j=i}^n \alpha_j \end{aligned}$$

Hence, the following equation holds

$$G_t(t,x) + rxG_x(t,x) - \frac{1}{2}A(s)\frac{G_x^2(t,x)}{G_{xx}(t,x)} = 0$$

$$\Leftrightarrow \quad -\left(r + \frac{1}{2}A(s)\right)\sum_{j=i}^{n}\alpha_j + r\sum_{j=i}^{n}\alpha_j + A(s)\sum_{j=i}^{n}\alpha_j = 0.$$

iv.) Final Utility Condition (4.10):

$$\alpha_n \log(x) = G(T,x)$$

v.) Optimal control (4.11):

$$\pi^*(t,x) = -(\sigma(t)\sigma(t)')^{-1}(b(t) - r(t)\underline{1})\frac{G_x(t,x)}{xG_{xx}(t,x)} = (\sigma(t)\sigma(t)')^{-1}(b(t) - r(t)\underline{1})$$

Since the assumptions of Theorem 4.4 are fulfilled, the above statement follows directly.

□

As in the HARA-case we can guess the solution of the continuous case from the solution of the discrete case:

Theorem 4.11 (Optimal portfolio for continuous LOG-Utility)
The value function of the portfolio problem

$$V(t,x) := \sup_{(\pi,c)\in\mathcal{A}^{\mathcal{H}}(t,x)} E^{t,x}\left[\int_t^T \alpha(s)\log(s)ds + \beta\log(X^{\pi,c}(T))\right]$$

i.e. a problem as in Definition 1.12 with $U_2(t,c) = \alpha(s)\log(c)$ *and* $U_1(x) = \beta\log(x)$ *and wealth process (1.9)*

$$dX^{\pi,c}(t) \quad = \quad X^{\pi,c}(t)\ \left[\left(r(t) + \pi'(t)(b(t) - r(t)\underline{1})\right)dt - \pi'(t)\sigma(t)dW(t)\right] - c(t)dt$$

is given by

$$G(t,x) = \int_t^T \alpha(s)\left[\log\left(\frac{\alpha(s)x}{\int_t^T \alpha(z)dz + \beta}\right) + \int_t^s \kappa(z)dz\right]ds \tag{4.28}$$

$$+\beta\log\left(\frac{\beta x}{\int_t^T \alpha(z)dz + \beta}\right) + \int_t^T \kappa(z)dz \tag{4.29}$$

where

$$\kappa(s) = r(s) + \frac{1}{2}(b(s) - r(s)\underline{1})'(\sigma(s)\sigma(s)')^{-1}(b(s) - r(s)\underline{1}).$$

The optimal control reads as

$$\pi^*(t) = (\sigma(t)\sigma(t)')^{-1}(b(t) - r(t)\underline{1})$$

and the optimal consumption equals

$$c^*(t) = \frac{\alpha(s)x}{\int_t^T \alpha(z)dz + \beta}.$$

PROOF.
First we check, whether the HJB-equation is fulfilled. For this purpose we introduce some notations

$$
\begin{aligned}
\Phi^{\alpha}(s) &= \left[\log\left(\frac{\alpha(s)}{\int_t^T \alpha(z)dz+\beta}\right)+\int_t^s \kappa(z)dz\right], \\
\Phi_t^{\alpha}(s) &= \frac{\alpha(t)}{\int_t^T \alpha(z)dz+\beta}-\kappa(t), \\
\Phi^{\beta}(s) &= \left[\log\left(\frac{\beta}{\int_t^T \alpha(z)dz+\beta}\right)+\int_t^T \kappa(z)dz\right], \\
\Phi_t^{\beta}(s) &= \frac{\alpha(t)}{\int_t^T \alpha(z)dz+\beta}-\kappa(t), \\
G(t,x) &= \int_t^T \alpha(s)\left[\log(x)+\Phi(s)\right]ds+\beta\left[\log(x)+\Phi^{\beta}(s)\right], \\
G_x(t,x) &= \frac{1}{x}\left(\int_t^T \alpha(s)ds+\beta\right), \\
G_{xx}(t,x) &= -\frac{1}{x^2}\left(\int_t^T \alpha(s)ds+\beta\right), \\
G_t(t,x) &= -\alpha(t)\left[\log(x)+\Phi^{\alpha}(t)\right]+\int_t^T \alpha(s)\Phi_t^{\alpha}(s)ds+\beta\Phi_t^{\beta}(s),
\end{aligned}
$$

and

$$
\begin{aligned}
\left(\frac{\partial U_2(t,c)}{\partial c}\right)^{-1}(y) &= \frac{\alpha(t)}{y} \\
A(t) &= (b(t)-r(t)\underline{1})'(\sigma(t)\sigma(t)')^{-1}(b(t)-r(t)\underline{1}).
\end{aligned}
$$

Observe that, the optimal controls and final utility coincide with the above assertions:

$$
\begin{aligned}
c^*(t,x) &= \frac{\alpha(s)x}{\int_t^T \alpha(z)dz+\beta}=\left(\frac{\partial U_2(t,c)}{\partial c}\right)^{-1}(G(t,x)) \\
\pi^*(t) &= (\sigma(t)\sigma(t)')^{-1}(b(t)-r(t)\underline{1})=-(\sigma(t)\sigma(t)')^{-1}(b(t)-r(t)\underline{1})\frac{xG_x(t,x)}{G_{xx}(t,x)} \\
\beta\log(x) &= G(T,x)
\end{aligned}
$$

Now we check that the HJB-equation holds

$$
\begin{aligned}
&-\frac{1}{2}A(t)\frac{V_x^2(t,x)}{V_{xx}(t,x)}+r(t)xV_x(t,x)+V_t(t,x)\\
&-\left(\frac{\partial U_2(t,c)}{\partial c}\right)^{-1}(V_x(t,x))V_x(t,x)+U_2(t,c^*(t))=0\\
\Leftrightarrow\quad &+\frac{1}{2}A(t)\left(\int_t^T\alpha(s)+\beta\right)+r(t)\left(\int_t^T\alpha(s)+\beta\right)-\alpha(t)\left[\log(x)+\Phi(t)\right]\\
&+\int_t^T\alpha(s)\left(\frac{\alpha(t)}{\int_t^T\alpha(z)dz+\beta}-\kappa(t)\right)ds+\beta\left(\frac{\alpha(t)}{\int_t^T\alpha(z)dz+\beta}-\kappa(t)\right)\\
&-\alpha(s)+\alpha(t)\log\left(\frac{\alpha(t)x}{\int_t^T\alpha(z)dz}\right)=0
\end{aligned}
$$

$$
\begin{aligned}
\Leftrightarrow\quad &\left\{\left(r(t)+\frac{1}{t}A(s)-\kappa(t)\right)\left(\int_t^T\alpha(s)ds+\beta\right)\right\}\\
&+\left\{-\alpha(t)+\int_t^T\alpha(s)\frac{\alpha(t)}{\int_t^T\alpha(z)dz+\beta}+\beta\frac{\alpha(t)}{\int_t^T\alpha(z)dz+\beta}\right\}\\
&+\left\{-\alpha(t)\left[\log(x)+\Phi^\alpha(t)\right]+\alpha(t)\log\left(\frac{\alpha(t)x}{\int_t^T\alpha(z)dz+\beta}\right)\right\}=0
\end{aligned}
$$

The first line of the last equation is zero and the remainder collapse also to zero. So the HJB-equation is fulfilled. In addition $\pi^*(t)$ is bounded and equals $-(\sigma\sigma')^{-1}(b-r\underline{1})\frac{V_x(t,x)}{xV_{xx}(t,x)}$ and finally the value function $V(t,x)$ is concave.

□

Example 4.12 (Equally weighted continuous log-consumption)
The solution of the optimisation problem

$$V(t,x)=\sup_{(\pi,c)\in\mathcal{A}^{\mathcal{H}}(t,x)}E^{t,x}\left[\int_t^T\log(c(s))ds+\log(x)\right]$$

for fixed coefficients b,r,σ equals

$$
\begin{aligned}
\pi^*(t) &= \frac{b-r}{\sigma^2},\\
c^*(t) &= \frac{X^{\pi,c}(t)}{T-t+1}.
\end{aligned}
$$

Remark 4.13 (Convergence of discrete to continuous consumption)
At time t the consumption rate of the continuous case with $\beta = 1$ and $\alpha(t) = 1$ is given by

$$c(t) = \frac{x}{T - t + 1}$$

Now we calculate the corresponding consumption rate in the discrete setting with equally-spaced time instants $t_{i+1} - t_i := \triangle t := T/N$, $\alpha_i = \triangle t$ and $\alpha_n = 1$. With $i = [t/N]$ we obtain that

$$\frac{b_i}{\triangle t} = \frac{x}{\sum_{j=i}^{N-1} \triangle t + 1}.$$

Taking the limit we get:

$$\lim_{N \to \infty} \frac{b_i}{\triangle t} = \frac{x}{T - t + 1}.$$

So again we conclude convergence of the discrete consumption case to the continuous one.

4.5 Numerical Results and Conclusion

4.5.1 Optimal consumption for equal weights

In this subsection we are neither discounting the utilities nor do we have any other preferences w.r.t the different time instants, i.e. $\alpha_i = 1$ for all $i = 1, \dots, n$. Let us first investigate for a time horizon of twenty years the effect of different γ's on the optimal controls in case of HARA-Utility, where consumption takes place every two years. In the following table the ratios of optimal consumption $b_i(x)$ w.r.t x, i.e. $b_i(x)/x$ are shown:

Table 4.1: Optimal HARA Consumption w.r.t γ

$N = 10,\quad \triangle t_i = 2$ years, $\quad \alpha_i = 1,\quad b = 5\%,\quad \sigma = 30\%,\quad r = 0\%$

$\gamma \setminus t_i$	2	4	6	8	10	12	14	16	18	20
0,01	0,100	0,111	0,125	0,143	0,167	0,200	0,250	0,333	0,500	1,000
0,1	0,098	0,110	0,124	0,141	0,165	0,199	0,249	0,332	0,499	1,000
0,2	0,096	0,107	0,121	0,139	0,163	0,197	0,247	0,330	0,498	1,000
0,3	0,093	0,104	0,118	0,136	0,160	0,193	0,244	0,328	0,496	1,000
0,4	0,087	0,098	0,112	0,130	0,154	0,188	0,239	0,323	0,492	1,000
0,5	0,077	0,088	0,102	0,120	0,144	0,178	0,230	0,315	0,486	1,000
0,6	0,060	0,071	0,084	0,102	0,126	0,161	0,212	0,299	0,474	1,000
0,7	0,031	0,040	0,052	0,068	0,091	0,124	0,176	0,264	0,446	1,000
0,8	0,003	0,005	0,009	0,016	0,027	0,049	0,090	0,173	0,365	1,000
0,9	0,000	0,000	0,000	0,000	0,000	0,000	0,001	0,006	0,076	1,000
0,99	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	1,000

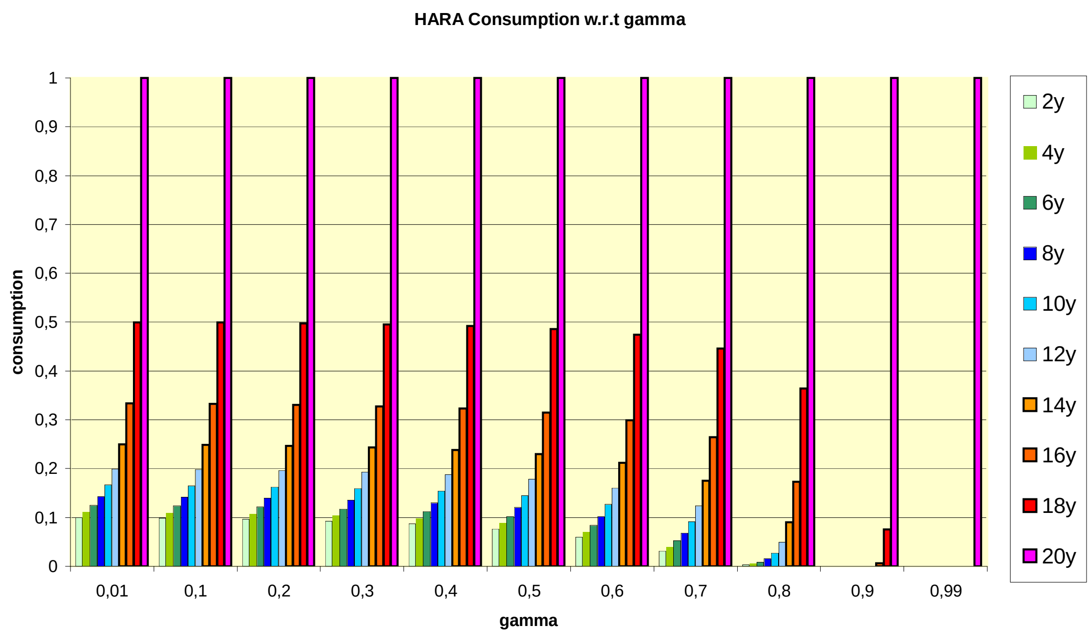


Figure 13: Absolute Optimal HARA Consumption w.r.t γ

For the ease of interpretation we rescale the consumption ratio, by the number of outstanding consumptions by

$$R_i = \frac{b_i(x)}{x}(n - i + 1).$$

So $R_i = 1$ means, that we have no preference about the time of consumption, $R_i > 1$, that we prefer to consume in the present (instead of the future) and $R_i < 1$ vice versa.

Table 4.2: Relative Optimal HARA Fraction R_i w.r.t γ

$N = 10$, $\triangle t_i = 2$ years, $\alpha_i = 1$, $b = 5\%$, $\sigma = 30\%$, $r = 0\%$

$\gamma \setminus t_i$	2	4	6	8	10	12	14	16	18	20
0,01	0,999	0,999	0,999	0,999	0,999	0,999	1,000	1,000	1,000	1,000
0,1	0,985	0,986	0,988	0,990	0,991	0,993	0,995	0,997	0,998	1,000
0,2	0,961	0,966	0,970	0,974	0,978	0,983	0,987	0,991	0,996	1,000
0,3	0,925	0,933	0,941	0,950	0,958	0,966	0,975	0,983	0,991	1,000
0,4	0,867	0,881	0,895	0,910	0,924	0,939	0,954	0,969	0,985	1,000
0,5	0,769	0,793	0,817	0,841	0,866	0,892	0,918	0,945	0,972	1,000
0,6	0,599	0,636	0,675	0,716	0,759	0,803	0,850	0,898	0,948	1,000
0,7	0,314	0,362	0,417	0,477	0,545	0,620	0,703	0,793	0,892	1,000
0,8	0,029	0,045	0,071	0,109	0,165	0,246	0,361	0,519	0,729	1,000
0,9	0,000	0,000	0,000	0,000	0,000	0,000	0,002	0,019	0,152	1,000
0,99	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	1,000

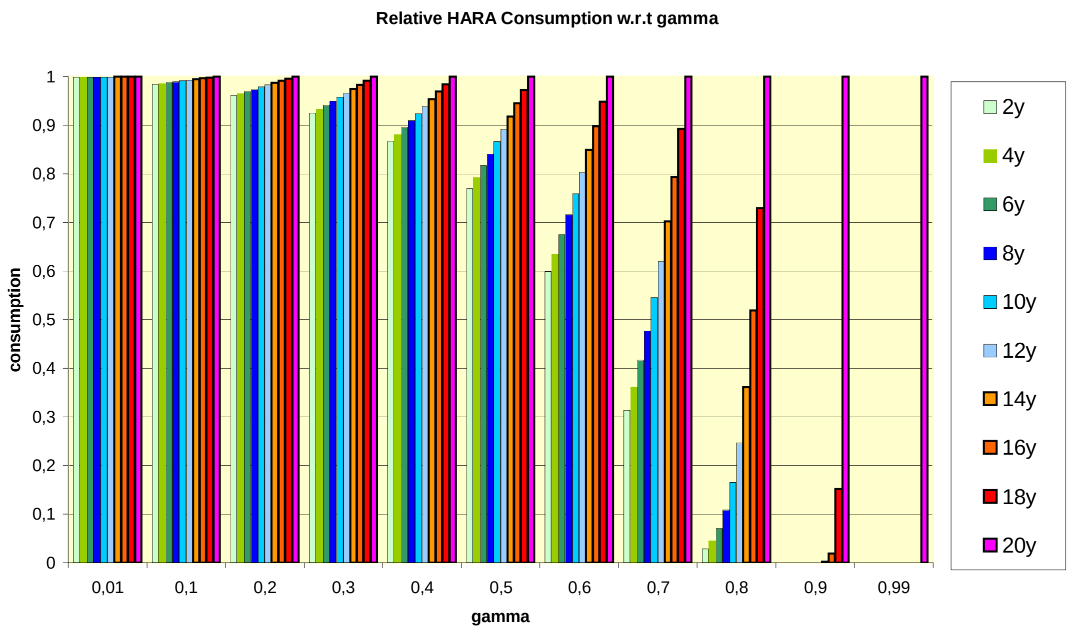


Figure 14: Relative Optimal HARA Consumption w.r.t γ

Table 4.2 has to be read as follows: For $\gamma = 0.4$ and $T_i = 4$ it is optimal to consume 88.1% of the amount, which would be optimal to consume if all time instants would be equally weighted.
From Table 4.2 and Figure 14 we can observe, that consumption is equally weighted for $\gamma \to 0$ and that all consumption takes place at the final time instant for $\gamma \to 1$. Let us examine this analytically :

Remark 4.14 (HARA-Utility at extreme cases)

i) For $\gamma \to 0$ we get:

$$\lim_{\gamma\to 0} \pi^*(t) = (\sigma(t)\sigma(t)')^{-1}(b(t) - r(t)\underline{1})$$

$$\lim_{\gamma\to 0} b_i^*(x) = \frac{\alpha_i x}{\sum_{j=i}^n \alpha_j}$$

So for $\gamma \to 0$ we get exactly the same optimal portfolio and consumptions as for logarithmic utility. Besides some possible weighting in the α_i, there are no preferences with regard to the times of consumption.

ii) For the case of $\gamma \to 1$
let us assume, that $r(s) > 0$ or $b(s) \neq r(s)$ for all $s \in [0, T]$. Then $\kappa(s)$ is positive, since $(\sigma(t)\sigma'(t))^-1$ is positive definite for all $s \in [0, T]$. So we conclude with (4.26) :

$$\lim_{\gamma\to 1} b_i^*(x) = \begin{cases} 0 & : \quad i \neq n \\ x & : \quad i = n \end{cases}$$

That means, for γ close to but still smaller than 1, we are extreme risk takers, and wait for consumption until the final utility, to take the chance to increase our wealth over time until the final time instant.

We here do not plot any results for the log-utility case, since the results would be almost the same as in the case of HARA-utility for $\gamma = 0.01$,

4.5.2 Optimal consumption for discounted utility

In this subsection we investigate the HARA-utility case with fixed $\gamma = 0.5$ as well as the LOG-utility case under discounted utility. Let δ be the discount factor, then we define $\alpha_i = e^{-\delta t_i}$. We now examine the solutions under different δ's.

Table 4.3: Relative Optimal LOG Consumption R_i w.r.t δ

$N = 10$, $\triangle t_i = 2$ years, $\alpha_i = e^{-\delta t_i}$, $b = 5\%$, $\sigma = 30\%$, $r = 0\%$

$\delta \setminus T_i$	2	4	6	8	10	12	14	16	18	20
0%	1,000	1,000	1,000	1,000	1,000	1,000	1,000	1,000	1,000	1,000
1%	1,092	1,082	1,071	1,061	1,051	1,040	1,030	1,020	1,010	1,000
2%	1,189	1,167	1,145	1,124	1,103	1,082	1,061	1,040	1,020	1,000
3%	1,291	1,256	1,222	1,189	1,156	1,123	1,092	1,061	1,030	1,000
4%	1,396	1,348	1,301	1,255	1,210	1,166	1,123	1,081	1,040	1,000
5%	1,505	1,443	1,382	1,323	1,265	1,209	1,155	1,101	1,050	1,000
6%	1,618	1,541	1,466	1,393	1,322	1,253	1,187	1,122	1,060	1,000
7%	1,734	1,641	1,551	1,464	1,379	1,298	1,219	1,143	1,070	1,000
8%	1,853	1,744	1,638	1,536	1,438	1,343	1,251	1,164	1,080	1,000
9%	1,974	1,848	1,727	1,610	1,497	1,388	1,284	1,184	1,090	1,000
10%	2,096	1,954	1,817	1,684	1,556	1,434	1,317	1,205	1,100	1,000

The results coincide with economic intuition, that means, the greater the discount factor δ, the bigger is the part of wealth consumed in the beginning of the time horizon.

Table 4.4: Relative Optimal HARA Consumption R_i w.r.t δ

$N = 10$, $\triangle t_i = 2$ years, $\alpha_i = e^{-\delta t_i}$, $b = 5\%$, $\sigma = 30\%$, $r = 0\%$, $\gamma = 0,5$

$\delta \setminus t_i$	2	4	6	8	10	12	14	16	18	20
0%	0,563	0,603	0,645	0,689	0,735	0,783	0,834	0,887	0,942	1,000
1%	0,695	0,725	0,756	0,788	0,821	0,855	0,890	0,925	0,962	1,000
2%	0,848	0,864	0,880	0,897	0,913	0,930	0,947	0,965	0,982	1,000
3%	1,020	1,018	1,016	1,013	1,011	1,009	1,007	1,004	1,002	1,000
4%	1,212	1,187	1,162	1,138	1,114	1,091	1,068	1,045	1,022	1,000
5%	1,420	1,369	1,319	1,270	1,222	1,176	1,130	1,086	1,042	1,000
6%	1,644	1,563	1,485	1,409	1,335	1,263	1,194	1,127	1,062	1,000
7%	1,879	1,767	1,658	1,552	1,451	1,353	1,258	1,168	1,082	1,000
8%	2,124	1,978	1,837	1,701	1,570	1,444	1,324	1,210	1,102	1,000
9%	2,375	2,195	2,021	1,853	1,691	1,537	1,390	1,252	1,122	1,000
10%	2,629	2,415	2,207	2,007	1,815	1,631	1,457	1,294	1,141	1,000

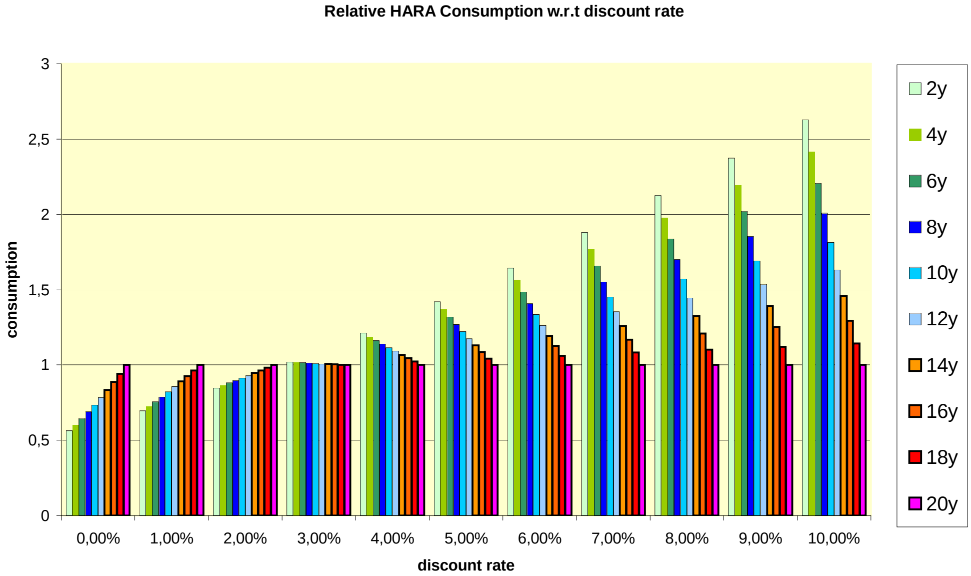


Figure 15: Relative Optimal HARA Consumption w.r.t ρ

In Figure 15 again we observe, that the more wealth is consumed in the beginning, the bigger the discount factor δ. But here, there seems to exist a position of equilibrium somewhere around $\delta = 3\%$, such that there is no preference with respect to the time of consumption. We will investigate this closer:

Remark 4.15 (Equilibrium of consumption)
For $\alpha_i = e^{-\delta t_i}$ the optimal consumption reads as follows

$$\begin{aligned} b_i^*(x) &= \frac{x e^{-\delta t_i \frac{1}{1-\gamma}}}{\sum_{j=i}^n e^{-\delta t_j \frac{1}{1-\gamma}} \exp\left(\frac{\gamma}{1-\gamma}\int_{t_i}^{t_j} \kappa(s) ds\right)} \\ &= \frac{x}{\sum_{j=i}^n \exp\left(\frac{1}{1-\gamma}\int_{t_i}^{t_j} \left(\gamma\kappa(s) - \delta\right) ds\right)}. \end{aligned}$$

So for b, r, σ constant and $\delta = \gamma\kappa$ we get $b_i^*(x) = \frac{x}{n-i+1}$, thus $R_i = 1$. This equality could in particular be used to determine the inverstor's γ, if he plans to consume in each year the same relative part of his wealth and ρ is known (e.g. equal to the interest rate).

It is clear that all consumption takes place at the first time instant for $\delta \to \infty$.
At last we give in Figures 17 and 18 the results for Log-Utility:

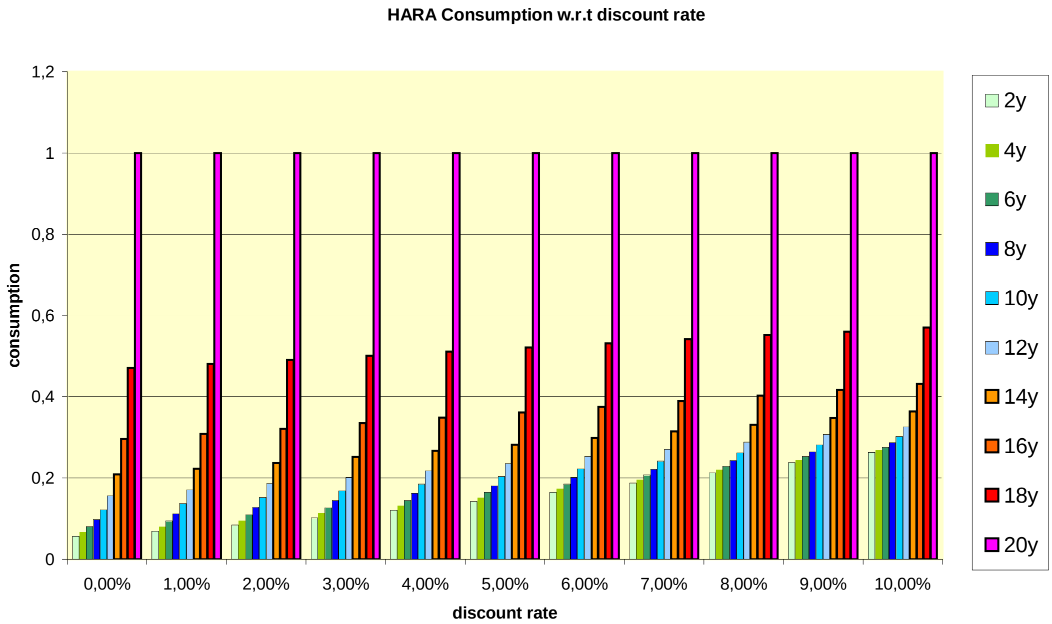


Figure 16: Relative Optimal HARA Consumption w.r.t ρ

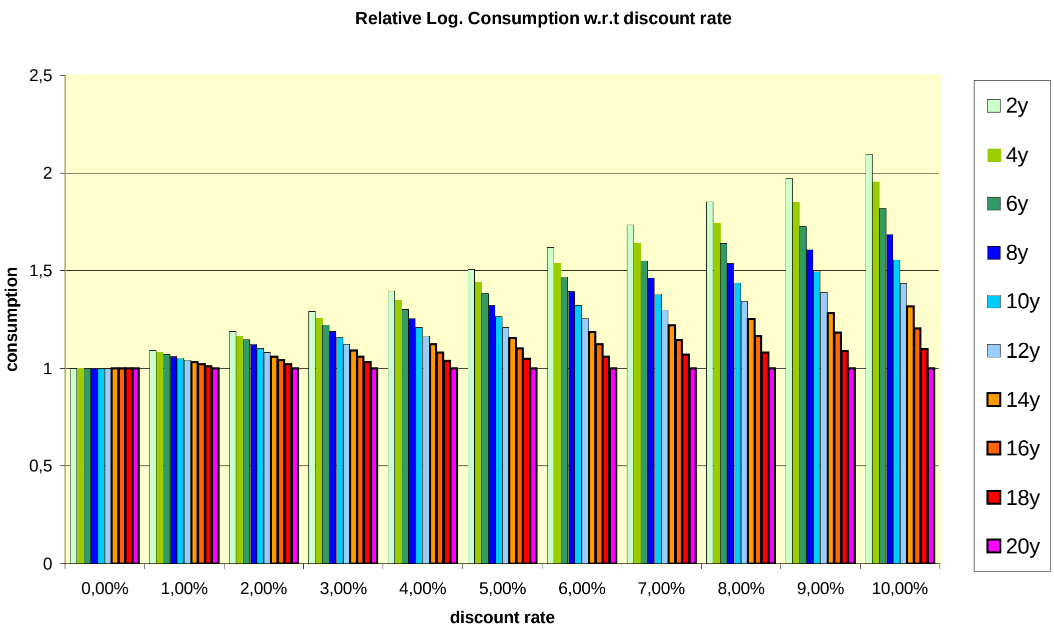


Figure 17: Relative Optimal LOG Consumption w.r.t ρ

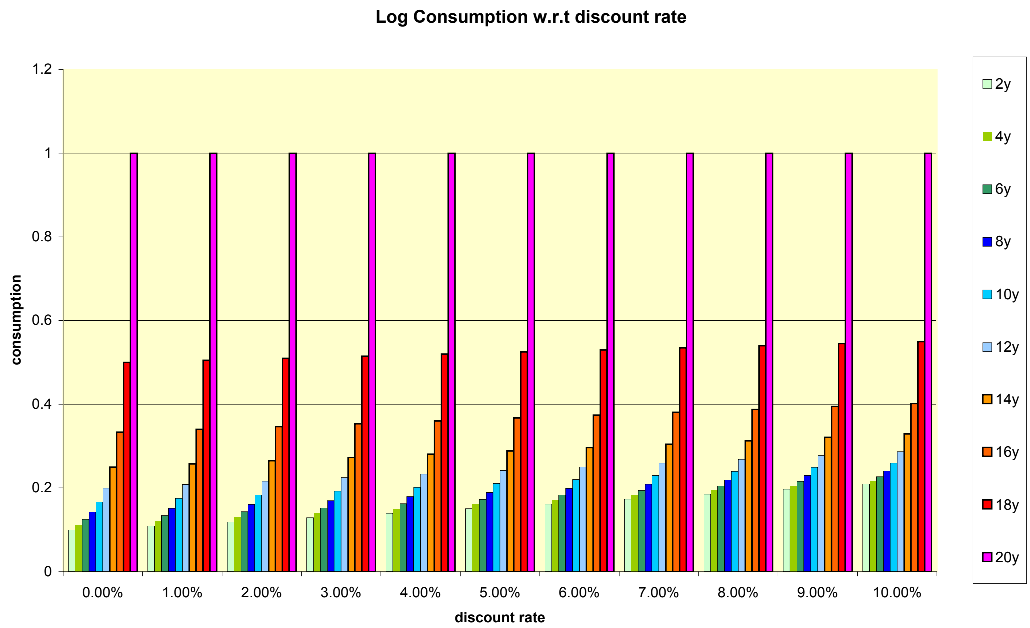


Figure 18: Absolute Optimal LOG Consumption w.r.t ρ

4.6 Conclusions

We defined a new family of lump sum optimization problems and derived an appropriate verification theorem for these. Due to this result we solved the optimization problem for HARA-Utility and LOG-Utility with time-dependent weights. The solutions of the lump sum problems enabled us to determine the solutions of the corresponding problems with continuous consumption. These continuous problems are also generalizations of the cases treated in the literature and we derived their solutions by using the standard verification theorem (for continuous consumption). To complete our considerations we showed the convergence of the discrete solutions to the continuous ones in special cases and provided some numerical results.

Our lump sum optimization problem is a very natural extension of the existing literature of portfolio optimization. It seems to be very useful in practice, since in the real world consumption is more likely to take place on a monthly or yearly base than on a continuous one. If the investor is sure about his consumption he can use our formulas to determine his personal risk aversion, namely the γ in the HARA-case. Future challenges of our problem could be time-dependent γ's, in order to take different risk preferences in the different life-spans of the investor into account. It seems to be very hard to solve this problem analytically, so this could be an application for numerical methods in portfolio optimization.

5 An Analysis of the Pricing Methods for Baskets Options

5.1 Introduction

A *basket option* is an option whose payoff depends on the average of some particular assets (for this reason it is called "basket" option). We are going to treat Calls and Puts on baskets. More precisely we define a basket of stocks by

$$B(T) = \sum_{i=1}^{n} w_i S_i(T),$$

where $B(T)$ is the weighted arithmetic average of n underlying stocks. Then the payoff of a Call($\theta = 1$) resp. Put($\theta = -1$) reads as

$$P_{Basket}(B(T), K, \theta) \quad = [\theta(B(T) - K)]^+.$$

We price these options with the Black-Scholes Model described in Chapter 1. Note that by the form of the payoff it is not necessary to distinguish between the trading date and the valuation date to calculate the values of these options, since they are not path-dependent. Hence without loss of generality we can set $t = 0$ and denote the remaining time to maturity with T. In order to ease the calculations we use the so-called *forward notation*. The T-forward price of stock i is given by

$$F_i^T = S_i(0) \exp\left(\int_0^T \Big(r(s) - d_i(s) \Big) \, ds \right).$$

On account of the fact that we are not explicitly performing any Itô-calculus in this chapter, we use the more expressive notation introduced in Remark (1.7). Thus we can rewrite the stock price as

$$S_i(T) = F_i^T \exp\left(- \int_0^T \frac{1}{2}\sigma_i^2 ds + \int_0^T \sigma_i dW_i(s) \right)$$

where the $W_i(.)$ are one-dimensional Brownian motions correlated to each other with ρ_{ij}. We define the discount factor as

$$Df(T) = \exp\left(- \int_0^T r(s) ds \right).$$

The forward-oriented notation has two advantages: Firstly, in the opposite to short rates and dividend yields, forward prices and discount factors are market-quotes. Secondly, from a computational point of view, it is less costly to work with single numbers, i.e. the forward prices and the discount factor, instead of several term-structures, namely the short rates and the dividend yields.

The problem of pricing these basket options in the Black-Scholes Model is the following: The stock prices are modelled by a geometric Brownian motion and are therefore log-normally distributed. Since the sum of log-normally distributed random variables is not log-normal, it is not possible to derive an (exact) closed-form solution. Due to the fact that we are dealing with a multidimensional process, only Monte Carlo or Quasi-Monte Carlo are suitable numerical methods to determine the values of these options. However, the aim of this chapter is to analyse existing analytical approximations, and find out which one provides the overall best results, i.e. the closest prices to Monte Carlo.

We test these methods:

- Beisser(1999) performs some conditional expectation technique,
- Gentle(1993) approximates the arithmethic average by a geometric one,
- Levy(1992) uses a log-normal distribution with matching moments, and
- Milevsky&Posner(1998) applies the reciprocal gamma distribution.

Not that all methods mentioned here use an approximation of the distribution $B(T)$ to calculate the basket option price in closed-form. Wid did not test the method of Hyun(1993), because it is an application of the method of Turnbull&Wakeman(1991) for Asian Options (Edgeworth expansion up to the 4th moment) and it is a well-known problem that this approximation gives really bad results for long maturities and high volatilites. The observation that the method of Levy almost always overprices, while the method of Beisser's approach provides a lower bound of the option values, leads us to test also the arithmetic mean of both (referred to as "Levy+Beisser").

The different approximations are discussed in the next section, while in the third section we calculate the reference prices by Monte Carlo simulation in order to be able to backtest the values of the different approaches. The last section of this chapter contains our conclusions.

5.2 The Valuation Methods

5.2.1 Beisser

The main idea behind the method of Beisser is the conditional expectation technique introduced by Rogers&Shi(1995) for the pricing of Asian Options. More precisely, to condition the expectation of the payoff with a normal distributed random variable Z using the tower law, then estimate the result by applying Jensen's inequality and derive a closed-form solution for this estimate. In a nutshell this reads as

$$\begin{aligned}
\mathrm{E}\left([B(T)-K]^{+}\right) &= \mathrm{E}\left(\mathrm{E}\left([B(T)-K]^{+}\Big|Z\right)\right) \\
&\geq \mathrm{E}\left(\mathrm{E}\left(\left[B(T)-K\Big|Z\right]^{+}\right)\right) \\
&= \sum_{i=1}^{n} w_i \left[\widetilde{F}_i^T N(d_{1i}) - \widetilde{K}_i N(d_{2i})\right]
\end{aligned}$$

where $\widetilde{F}_i^T, \widetilde{K}_i$ some adjusted parameters and d_{1i}, d_{2i} are the usual terms in this new notation.

5.2.2 Gentle

We can rewrite the payoff of the basket option as

$$P_{Basket}(B(T),K,\theta) = \left[\theta\left(\sum_{i=1}^{n} w_i S_i(T) - K\right)\right]^{+} = \left[\theta\left(\left(\sum_{i=1}^{n} w_i F_i^T\right)\sum_{i=1}^{n} a_i S_i^*(T) - K\right)\right]^{+},$$

where

$$\begin{aligned}
a_i &= \frac{w_i F_i^T}{\sum_{i=1}^{n} w_i F_i^T}, \\
S_i^*(T) &= \frac{S_i(T)}{F_i^T} = \exp\left(-\frac{1}{2}\int_0^T \sigma_i^2 ds + \int_0^T \sigma_i dW_i(s)\right).
\end{aligned}$$

Note that $\sum_{i=1}^{n} a_i S_i^*(T)$ is an arithmetic average with expectation equal to one as the weights sum up to 1. We approximate this sum by the geometric average

$$\widetilde{B}(T) = \left(\sum_{i=1}^{n} w_i F_i^T\right)\prod_{i=1}^{n} \left(S_i^*(T)\right)^{a_i}.$$

In contrast to the arithmetic average, the geometric average of log-normally distributed random variables is also log-normally distributed. Taking this into account Gentle performs a Black-Scholes calculation to obtain the price of the basket option

$$V_{Basket}(T) = Df(T)\theta\left(e^{M+\frac{1}{2}V^2} N(\theta d_1) - K^* N(\theta d_2)\right), \tag{5.1}$$

where $Df(T)$ is the discount factor, $N(\cdot)$ the distribution function of a standard normal random variable and

$$d_1 = \frac{M - \log K + V^2}{V}\,,$$

$$d_2 = d_2 - V\,,$$

$$M = \mathrm{E}(\log \tilde{B}(T)) = \log\left(\sum_{i=1}^{n} w_i F_i^T\right) - \frac{1}{2}\sum_{i=1}^{n} a_i \sigma_i^2 T \quad \text{and}$$

$$V^2 = \mathrm{Var}(\log \tilde{B}(T)) = \sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j \sigma_i \sigma_j \rho_{ij} T.$$

By the obvious fact that

$$\mathrm{E}\left[\tilde{B}(T) + \mathrm{E}[B(T)] - \mathrm{E}[\tilde{B}(T)]\right] = \mathrm{E}[B(T)],$$

Gentle corrects the mean of the geometric average by using the adjusted strike

$$K^* \quad = K - \left(\mathrm{E}(B(T)) - \mathrm{E}(\tilde{B}(T))\right)$$

where

$$\mathrm{E}(B(T)) \quad = \textstyle\sum_{i=1}^{n} w_i F_i^T, \quad \mathrm{E}(\tilde{B}(T)) = e^{M + \frac{1}{2}V^2}.$$

5.2.3 Levy

Levy's basic idea is to approximate the distribution of the basket by a log-normal distribution with mean $\widetilde{M}$ and variance $\widetilde{V}^2$. These two parameters are determined in such a way that they match the true moments of the arithmetic average, i.e.

$$\widetilde{M} = 2\log \mathrm{E}(B(T)) - 0.5 \log \mathrm{E}(B^2(T)) \quad \text{and}$$

$$\widetilde{V}^2 = \log \mathrm{E}(B^2(T)) - 2\log \mathrm{E}(B(T)),$$

$$\mathrm{E}(B^2(T)) = \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j F_i^T F_j^T \exp(\sigma_i \sigma_j \rho_{ij} T),$$

such that

$$\mathrm{E}(B(T)) = \mathrm{E}\left(e^X\right) \quad \text{and} \quad \mathrm{Var}(B(T)) = \mathrm{Var}\left(e^X\right),$$

where X is a normally distributed random variable with mean $\widetilde{M}$ and variance $\widetilde{V}^2$. The basket option price can now be evaluated analogously to (5.1).

5.2.4 Levy + Beisser

The prices in this method are the equally weighted arithmetic means of Beisser's and Levy's prices.

5.2.5 Milevsky and Posner

Milevsky&Posner(1998) use the reciprocal gamma distribution as an approximation for the distribution of the basket. The motivation is the fact that the distribution of correlated log-normally distributed random variables converges to the reciprocal gamma distribution as $n \to \infty$. Consequently, the first two moments of both distributions are matched to yield a closed-form solution. We used the two following ways of calculating the parameters of the gamma distribution. In the figures and tables *Milevsky* denotes the genuine choice of the parameters of the gamma distribution, and *Milevsky-Excel* a slightly modification found in Staunton(2002). We just state the Milevsky-Excel results in the tables as Milevsky-Excel clearly outperforms Milevsky.

5.2.6 Monte Carlo

The algorithm we applied consists of a Monte Carlo simulation using antithetic method and geometric mean as controle variate for variance reduction. The number of simulations was always chosen large enough to keep the standard deviation below 0.05.

5.3 Test Results

This section analyses the performance of the particular pricing methods. Normally, this is done changing the volatility and the strike. We will do this more systematically, i.e. we will change *all* relevant parameters individually, while the remaining ones are kept fixed to our standard scenario, and examine the results. That means in detail that four sets of tests are performed. They involve changing the correlations ρ_{ij}, strike K, forward prices F_i^T and the volatilities σ_i repectively. It is not necessary to change the discount factor, because this only a multiplicative factor in the formula. In addition, it makes no sense to change the weights w_i, because this is effectively the same as to vary the forwards, neither to change the the maturity, since this equivalent to changes of the volatility.

Our standard scenario is a call option on a basket with four stocks and a maturity of five years. The discount factor is fixed equal to one. Let i and j denote the indices of the stocks. The default parameters are

$$
\begin{aligned}
T &= 5.0, \\
Df(T) &= 1.0, \\
\rho_{ij} &= 0.5 \qquad (\text{for } i \neq j),
\end{aligned}
$$

$$K = 100,$$

$$F_i^T = 100,$$

$$\sigma_i = 40\% \quad \text{and}$$

$$w_i = \frac{1}{4}.$$

In the next section we compare the option prices calculated by the different methods. The results are presented in tables and plots. If the absolute prices are too close to allow for a judgement of the methods, the relative differences between the Monte Carlo prices and the prices calculated by Beisser, Levy and Levy+Beisser are also compared.

In addition we plot the *implicit distribution* of the particular approximations and compare them to the real ones calculated by Monte Carlo simulation. With *implicit distribution* we mean, that we derive the underlying distribution of the particular methods by an appropriate portfolio of calls. Consider the payoff of the following portfolio consisting only of Calls:

$$\begin{aligned} \Pi(B(T)) \quad = \quad & \alpha * \left[P_{Basket}\left(B(T), L - \frac{1}{\alpha}, 1\right) - P_{Basket}(B(T), L, 1) \right. \\ & \left. - \left(P_{Basket}(B(T), L + \triangle L, 1) - P_{Basket}\left(B(T), L + \triangle L + \frac{1}{\alpha}, 1\right)\right)\right] \end{aligned}$$

We notice that the payoff $\Pi(B(T))$ is explicitly given by

$$\Pi(B(T)) = \begin{cases} 0 & : \quad B(T) < L - \frac{1}{\alpha} \\ \alpha\left[B(T) - (L - \frac{1}{\alpha})\right] & : \quad L - \frac{1}{\alpha} \leq B(T) \leq L \\ 1 & : \quad L \leq B(T) \leq L + \triangle L \\ 1 - \alpha\left[B(T) - (L + \triangle L)\right] & : \quad L + \triangle L \leq B(T) \leq L + \triangle L + \frac{1}{\alpha} \\ 0 & : \quad B(T) > L + \triangle L + \frac{1}{\alpha} \end{cases} \tag{5.2}$$

For $\alpha \to \infty$ it is equal to:

$$\Pi(B(T)) = \begin{cases} 0 & : \quad B(T) < L \\ 1 & : \quad L \leq B(T) \leq L + \triangle L \\ 0 & : \quad B(T) > L + \triangle L \end{cases}$$

So for a sufficiently high α the *value* of our portfolio is approximately the probability that the price of the basket is at maturity in $[L, L + \triangle L]$. To calculate the whole implicit distribution, we shift the boundaries stepwise by $\triangle L$. Instead of applying the underlying distributions, we used this procedure, because we can not directly determine the distribution for Beisser's approximation. Besides, this procedure seems to be more objective and consistent to compare the approximations.

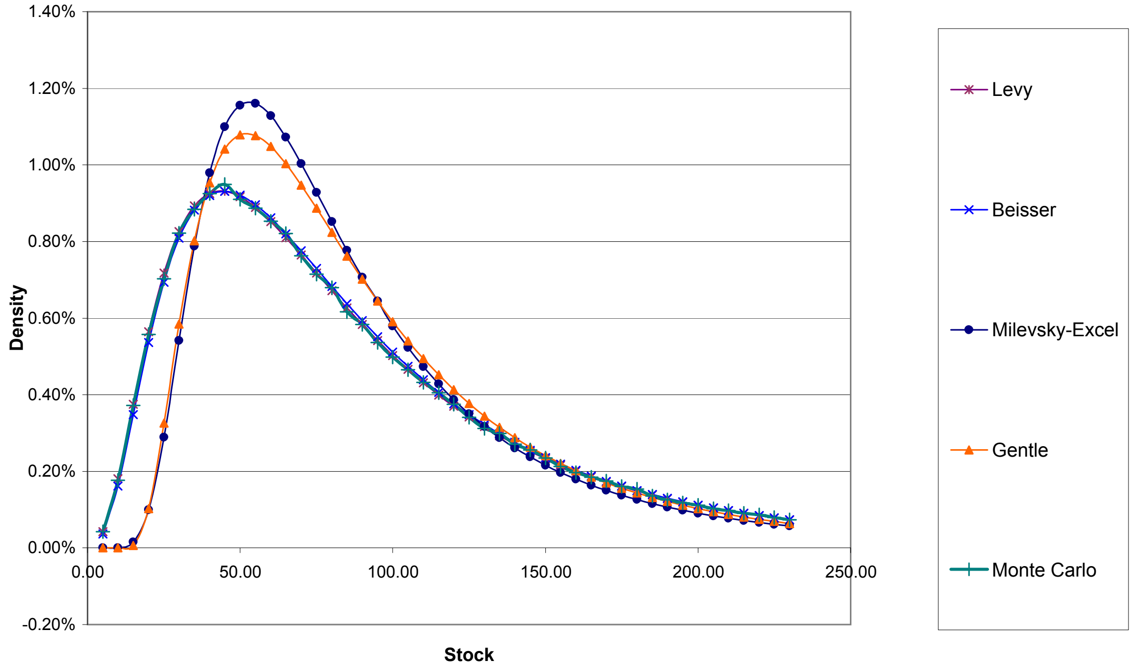


Figure 19: Densities for the standard scenario

In Figure 19 we plot the densities of the basket distributions for our standard scenario. Levy's and Beisser's distribution coincides with Monte Carlo, whereby Gentle's and Milevsky's one show evident deviations. We will have this result again for many of the forthcoming cases.

5.3.1 Effect of varying the Correlations

Two tests are performed to observe the effect of varying the correlation ρ_{ij} between stocks i and j. First at each step the correlation ρ_{ij} between all stocks i and j is set to the same value ρ, varied from 0.1 to 0.95. The results (Table 1) are graphed in Figures 20 and 21. Note that, except for Milevsky and Gentle, all methods perform reasonably well. Especially for $\rho \geq 0.8$, Beisser, Levy, Levy+Beisser and Monte Carlo give virtually the same price.

ρ	Milevsky -Excel	Gentle	Beisser	Levy	Levy+ Beisser	Monte Carlo	Standard Deviation
0,10	20,25	15,36	20,12	22,06	21,09	21,62	0,0319
0,30	22,54	19,62	24,21	25,17	24,69	24,97	0,0249
0,50	24,50	23,78	27,63	28,05	27,84	27,97	0,0187
0,70	26,18	27,98	30,62	30,75	30,69	30,72	0,0123
0,80	26,93	30,13	31,99	32,04	32,02	32,03	0,0087
0,95	27,97	33,41	33,92	33,92	33,92	33,92	0,0024
Dev.[1]	3,17	5,15	1,13	0,33	0,39	0	

Table 1: Varying the correlations simultaneously (Figure 20)

The deviation to Monte Carlo, denoted by "Dev.", is calculated as

$$Dev = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(MC_i - V_i)^2},$$

and is used as a performance index for the different methods.

The good performance of Levy, Gentle and Beisser for high correlations can be explained as follows: All three methods provide exactly the Black-Scholes prices for the special case that the number of stocks is one. For high correlations the distribution of the basket is approximatly the sum of the *same* (for $\rho = 1$ exactly the same) log-normal distribution, which is indeed again log-normal. As Levy uses a log-normal distribution with the correct moments, it has to be a good approximation for these cases. The same argumentation applies for Gentle. If we have effectively one stock the geometric and the arithmetic average is the same. The bad performance of Milevsky for high correlations can be explained by the fact, that with effectively one stock we are far away from "infinitly" many stocks, which was the motivation for this method.

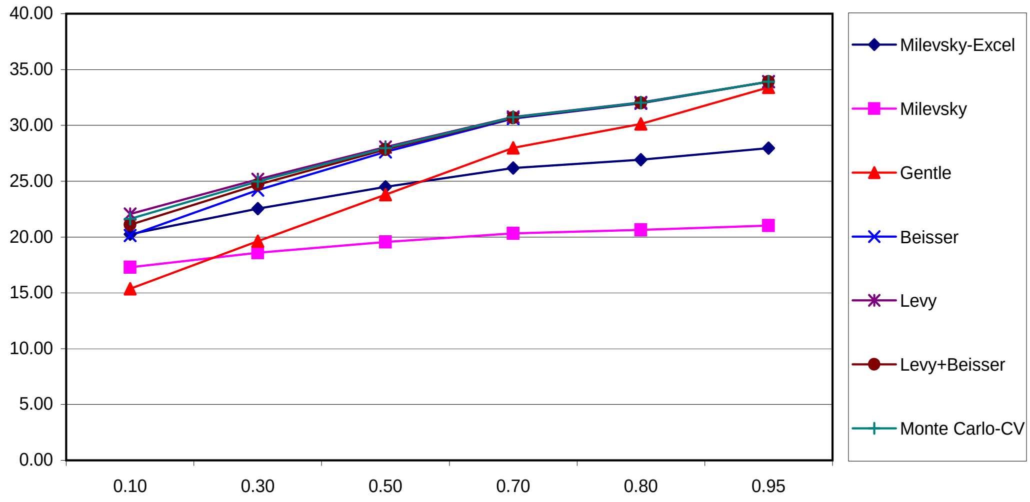


Figure 20: Varying the correlations simultaneously (Table 1)

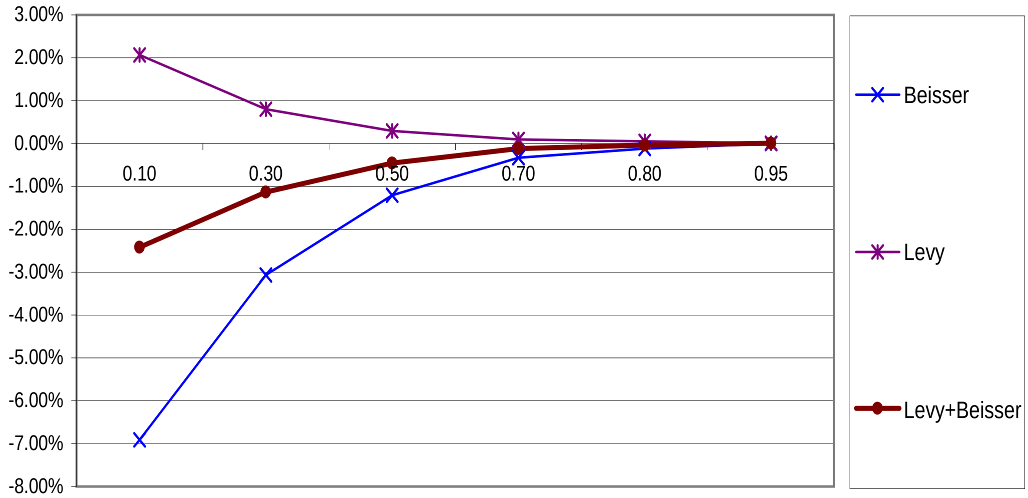


Figure 21: Varying the correlations simultaneously (Rel. Diff.)

The second test is a repetition of the first, but this time the correlation ρ_{12} between stocks 1 and 2 is kept constant at 95%. Figures 22 and 23 show the results. The plots corresponds to the results of the first test in Figures 20 and 21. Milevsky and Gentle performs worst (by a considerable margin) in both tests.

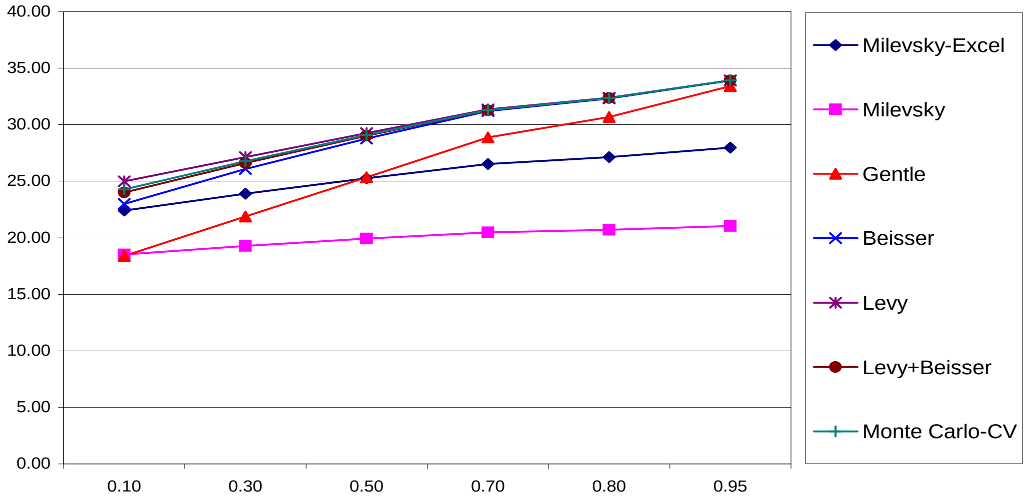


Figure 22: Varying the correlations sym. with fixed $\rho_{12} = 95\%$

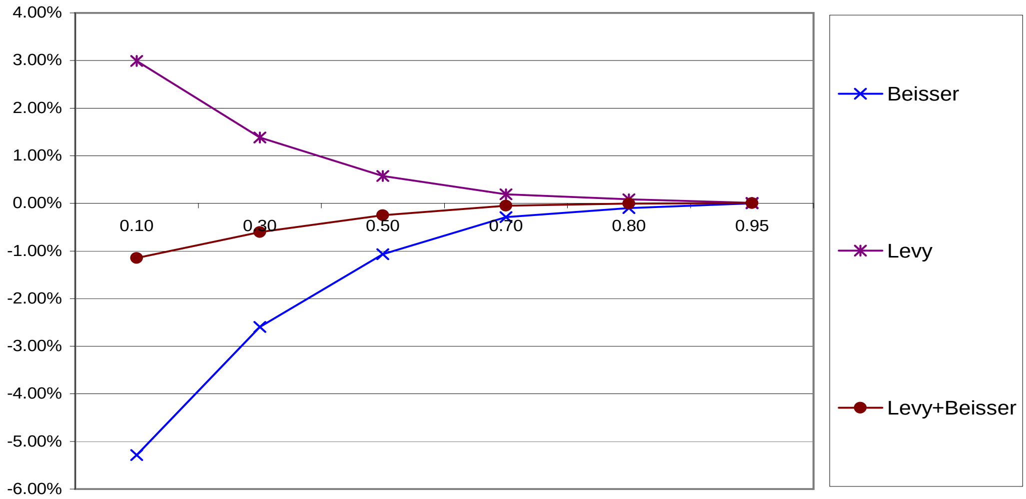


Figure 23: Varying the correlations sym. with fixed $\rho_{12} = 95\%$ (Rel. Diff.)

To analyse this more precisely we plot the corresponding densities for Figure 22 with $\rho_{12} = 95\%$, else $\rho_{ij} = 0.5$ in Figure 24:

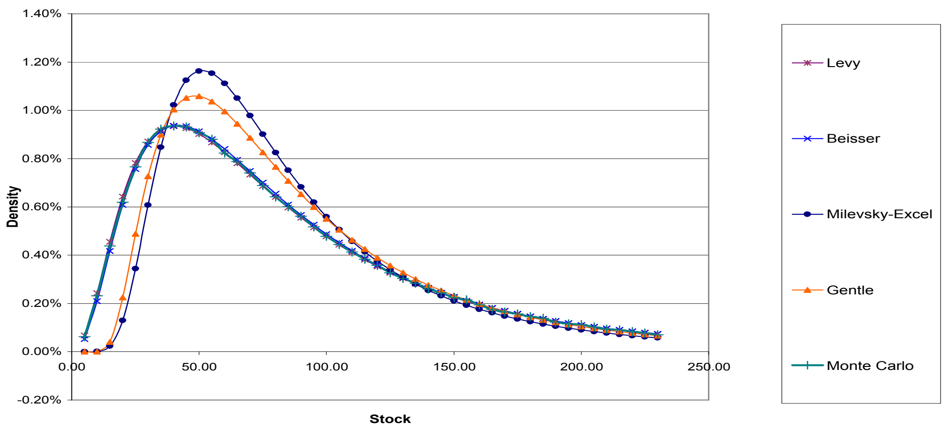


Figure 24: Densities for the standard scenario with $\rho_{12} = 95\%$

At last we use in Figure 25 a very inhomogeneous correlation matrix.:

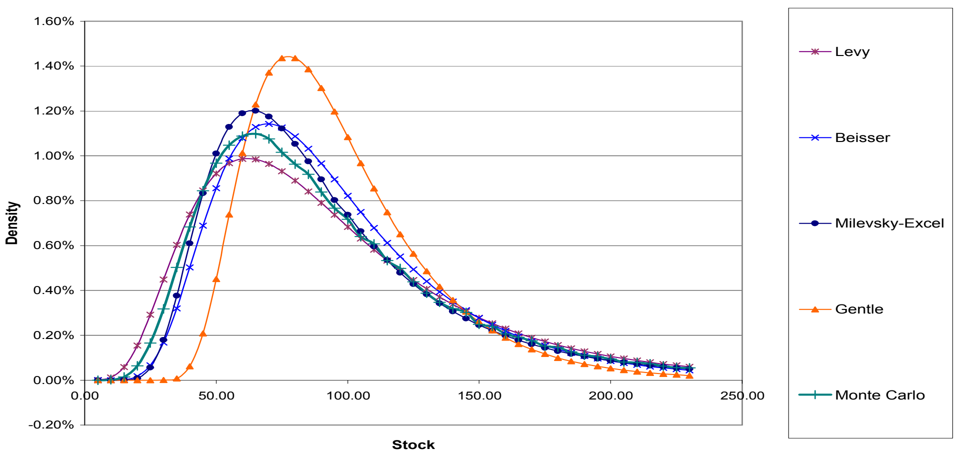


Figure 25: Densities for the standard scenario with inhomogenous correlation

The density plots are almost the same as for the case of our standard scenario in Figure 19. Thus the correlation does not have a really strong effect on the densities and consequently on the difference to the Monte Carlo prices. Beisser's and Levy's approaches are most able to deal with all cases of correlation matrices.

Summary
In a nutshell we come to the conclusion that the prices calculated by Levy's approach (whose method slightly overprices) are overall the closest to the Monte Carlo prices. Levy's approach is followed by Beisser's approximation (whose approach slightly underprices). The other two methods are not recommendable.

5.3.2 Effect of varying the Strikes

The strike K is varied from 50 to 150. Table 2 and Figures 26 and 27 contain the results.

K	Milevsky -Excel	Gentle	Beisser	Levy	Levy+ Beisser	Monte Carlo	Standard Deviation
50,00	51,93	51,99	54,16	54,34	54,25	54,28	0,0383
60,00	44,41	44,43	47,27	47,52	47,40	47,45	0,0375
70,00	38,03	37,93	41,26	41,57	41,41	41,50	0,0369
80,00	32,68	32,40	36,04	36,40	36,22	36,32	0,0363
90,00	28,22	27,73	31,53	31,92	31,73	31,85	0,0356
100,00	24,50	23,78	27,63	28,05	27,84	27,98	0,0350
110,00	21,39	20,46	24,27	24,70	24,48	24,63	0,0344
120,00	18,77	17,65	21,36	21,80	21,58	21,74	0,0338
130,00	16,57	15,27	18,84	19,28	19,06	19,22	0,0332
140,00	14,70	13,25	16,65	17,10	16,87	17,05	0,0326
150,00	13,10	11,53	14,75	15,19	14,97	15,15	0,0320
Dev.	3,03	3,74	0,32	0,06	0,13	0	

Table 2: Varying the strike (Figure 26)

The differences between the prices calculated by Monte Carlo and the approaches of Levy, Beisser and Levy+Beisser are relatively small. The price curves of the methods of Gentle and Milevsky run almost parallel to the Monte Carlo curve and represent an under-evaluation. The relative and absolute differences of all methods are generally increasing when K is growing, since the approximation of the real distributions in the tails is getting worse and the absolute prices are decreasing.

Summary
Again, overall Levy's approximation performs best and slightly overprices, while Beisser's approximation is the second best and always slightly underprices.

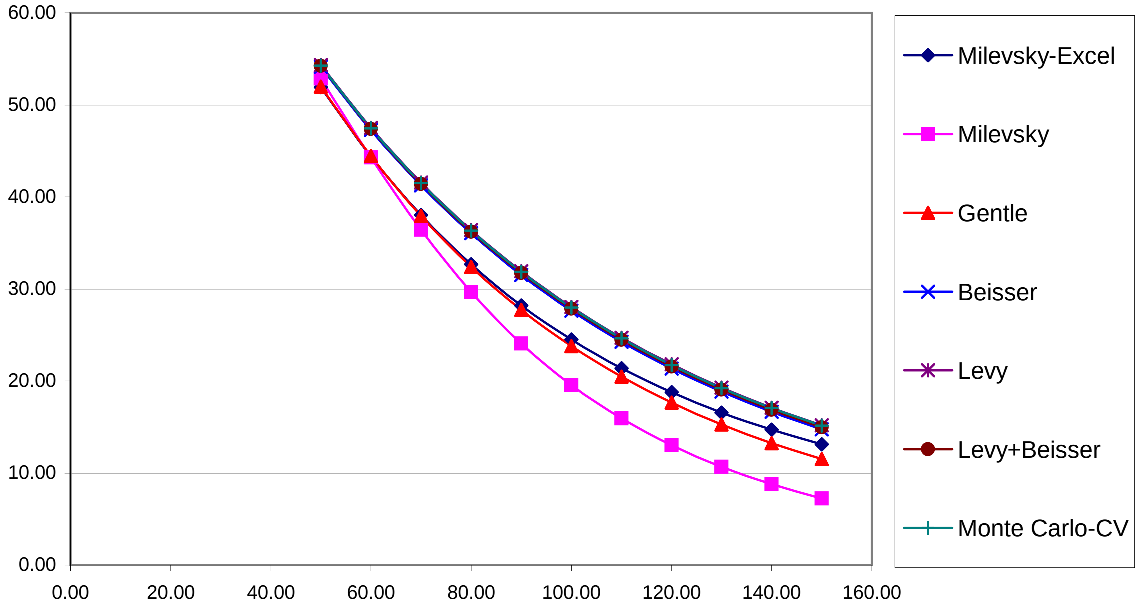


Figure 26: Varying the strike (Table 2)

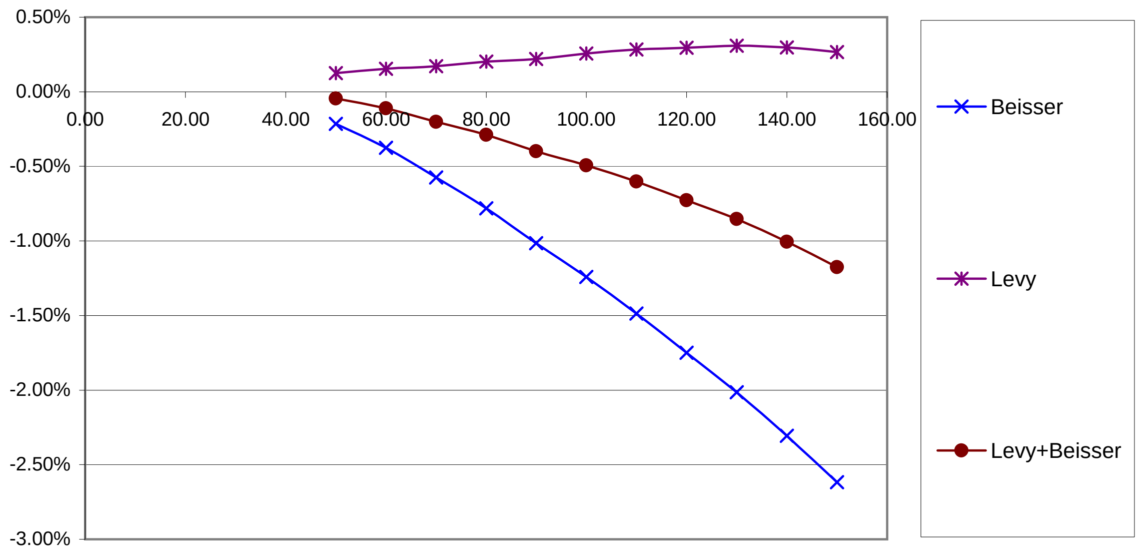


Figure 27: Varying the strike (Rel. Diff.)

5.3.3 Effect of varying the Forwards and Strikes

The forwards on all stocks are now set to the same value F which is varied between 50 and 150 in this set of tests. The results are presented in Table 3 and Figures 28 and 29. The price curves of Monte Carlo, Beisser, Levy and Levy+Beisser join to form one curve and curves of Milevsky and Gentle are clearly below these curve.

F	Milevsky -Excel	Gentle	Beisser	Levy	Levy+ Beisser	Monte Carlo	Standard Deviation
50,00	3,93	3,00	4,16	4,34	4,25	4,34	0,0141
60,00	6,56	5,53	7,27	7,52	7,40	7,50	0,0185
70,00	9,95	8,91	11,26	11,57	11,41	11,53	0,0227
80,00	14,10	13,13	16,04	16,40	16,22	16,35	0,0268
90,00	18,97	18,11	21,53	21,92	21,73	21,86	0,0309
100,00	24,50	23,78	27,63	28,05	27,84	27,98	0,0350
110,00	30,63	30,08	34,27	34,70	34,48	34,63	0,0391
120,00	37,32	36,91	41,36	41,80	41,58	41,71	0,0433
130,00	44,49	44,21	48,84	49,28	49,06	49,19	0,0474
140,00	52,08	51,92	56,65	57,10	56,87	57,00	0,0516
150,00	60,05	59,98	64,75	65,19	64,97	65,08	0,0556
Dev.	3,51	3,98	0,31	0,07	0,12	0	

Table 3: Varying the forwards sym. with $K = 100$ (Figure 28)

Then we repeat the test but keep the first forward price F_1^T fixed at 10. The results can be found in Figures 30 and 31 and are similar to the previous results.

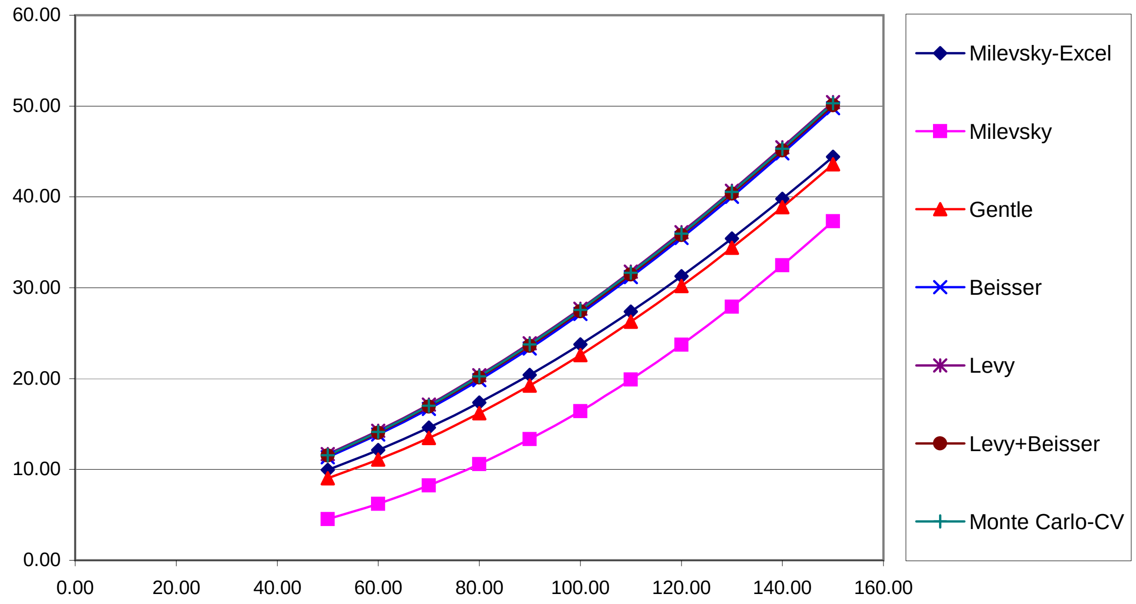


Figure 28: Varying the forwards sym. with $K = 100$ (Table 3)

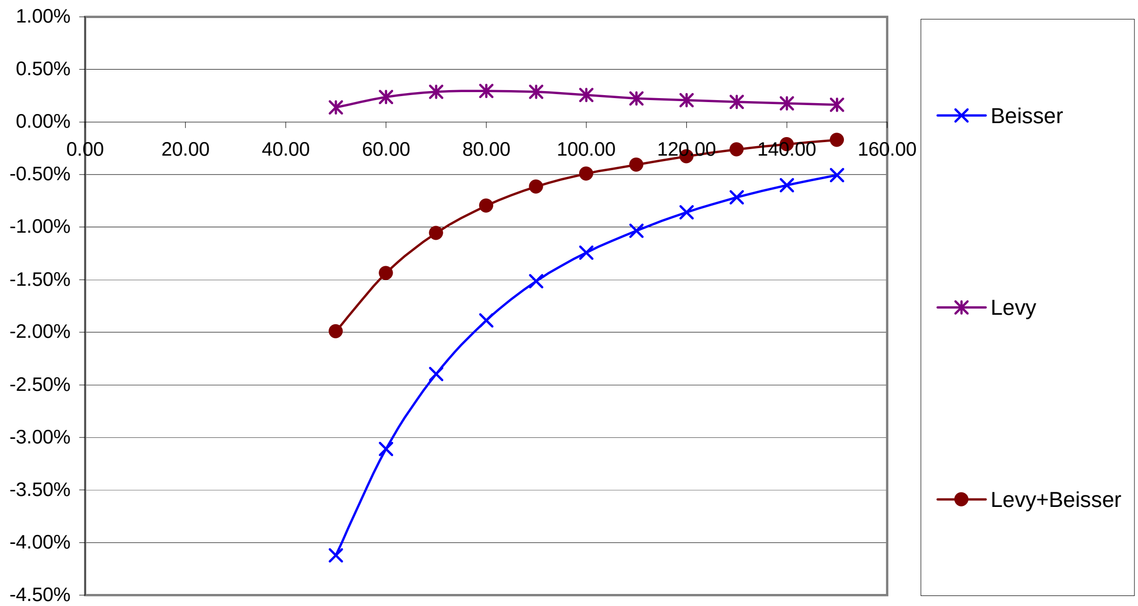


Figure 29: Varying the forwards sym. with $K = 100$ (Rel. Diff.)

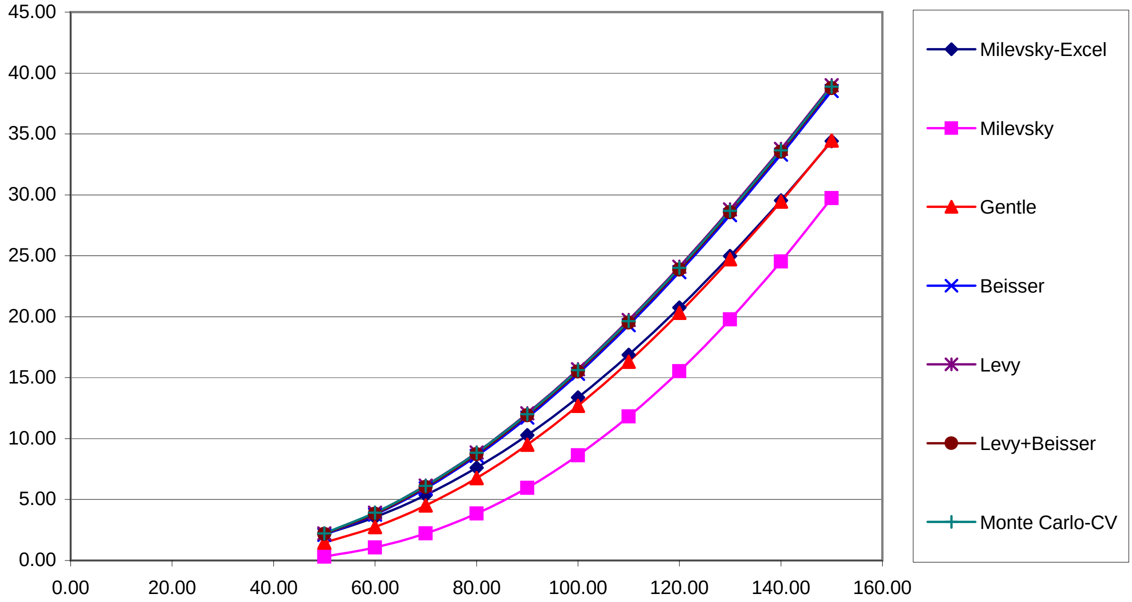


Figure 30: Varying the forwards sym. with $F_1^T = 10, K = 100$

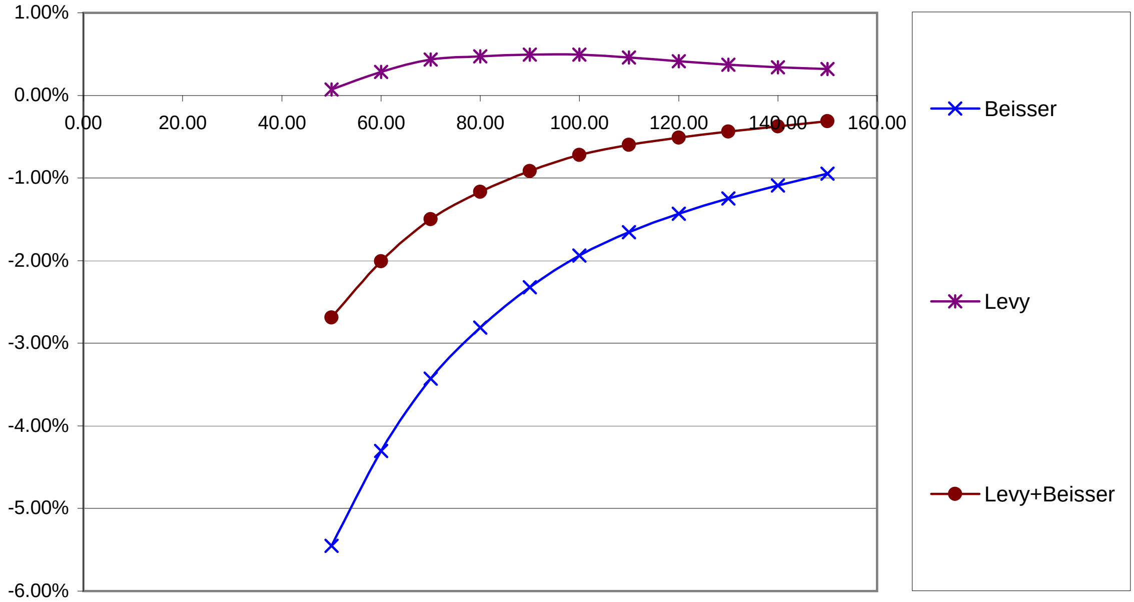


Figure 31: Varying the forwards sym. with $F_1^T = 10, K = 100$ (Rel. Diff.)

We again repeat the test but this time use $F_1^T = 20$ and the strike is set to

$$K = a \sum_{i=1}^{4} w_i F_i^T \tag{5.3}$$

with $a = 0.5$. The corresponding curves are in Figures 32 and 33. The results are again similar to the previous two tests, except that the differences between all methods are smaller, which comes from the fact that we have an *in-the-money* option and all methods are able to reproduce the forward of the basket. Consequently the resulting curves (mentioned in the first test of the forwards) almost merge into one (on the scale of Figure 32).

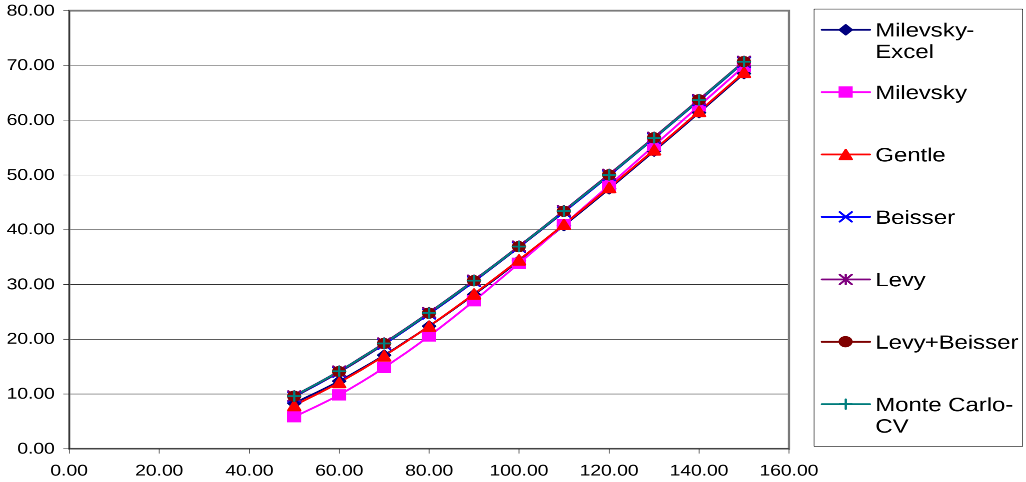


Figure 32: Varying the forwards sym. with $F_1^T = 20$ and $K = 0.5 \sum_{i=1}^{4} w_i F_i^T$

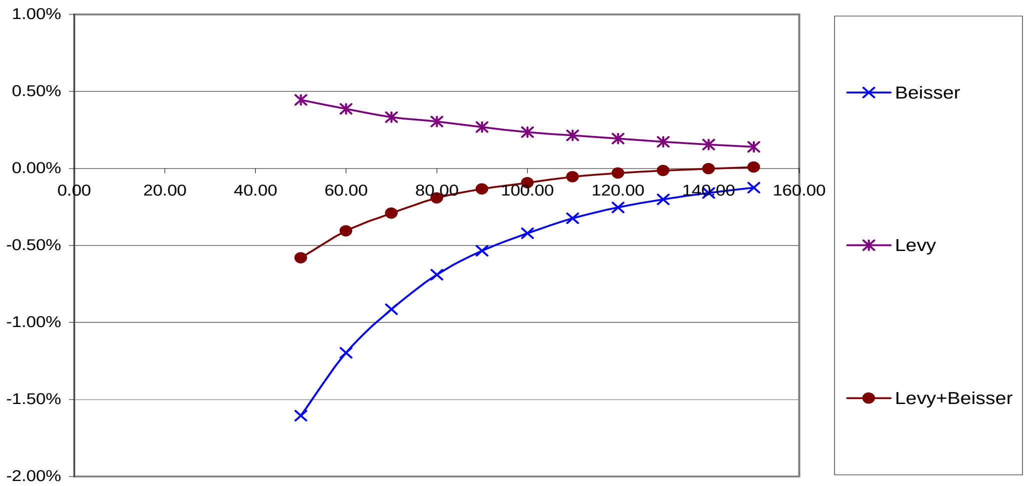


Figure 33: Varying the forwards sym. with $F_1^T = 20$ and $K = 0.5 \sum_{i=1}^{4} w_i F_i^T$ (Rel. Diff.)

The previous test is now repeated, but we set $a = 1.5$ in (5.3). The results are given in Figures 34 and 35.

The prices of Monte Carlo, Beisser, Levy and Levy+Beisser once again perform similarly (forming the upper curve). The difference between the Milevsky and Gentle from the Monte Carlo prices is more substantial than in the case with $a = 0.5$. Because we have an *out of the money* option, the tails of distributions are determining. Consequently, due to the fact that the approximating distributions differ in the tails more than at the mean (resp. the mass in the tails has more weight), the (relative) price differences become larger.

Note by Figure 35 that for the first time we have the effect that Levy's approximation also (slightly) underprices .

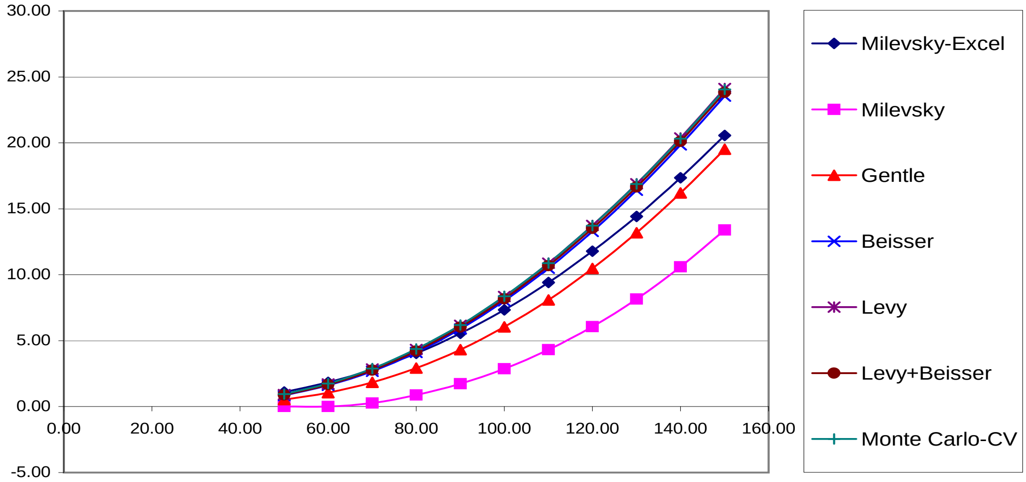


Figure 34: Varying the forwards sym. with $F_1^T = 20$ and $K = 1.5 \sum_{i=1}^4 w_i F_i^T$

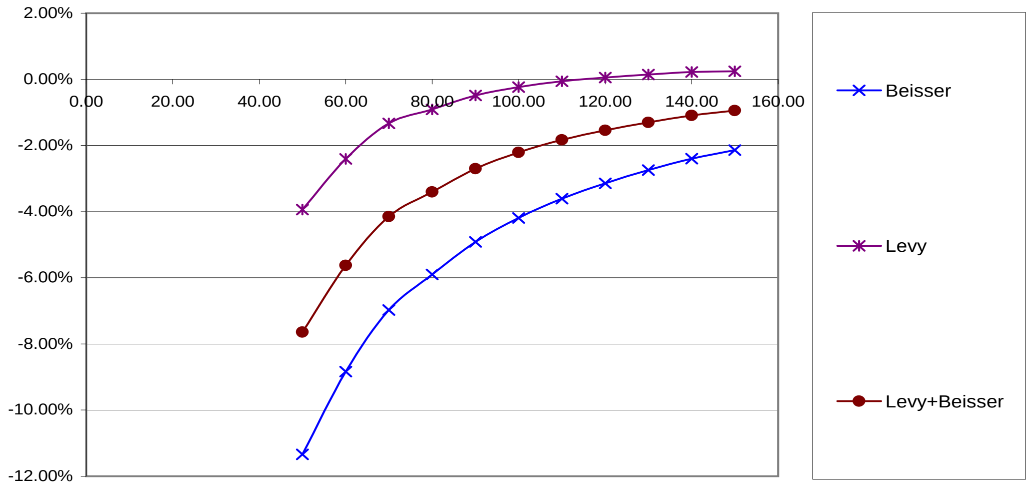


Figure 35: Varying the forwards sym. with $F_1^T = 20$ and $K = 1.5 \sum_{i=1}^4 w_i F_i^T$ (Rel. Diff.)

Finally, we repeat the last three tests but set $F_1^T = 200$. The resulting price curves are displayed in Figures 36, 38 and 40 (or compare Figures 37, 39 and 41 showing the relative differences) and form the same pattern as the those in Figures 30, 32 and 34 (or compare Figures 31, 33 and 35). The results are in general similar to those of the previous tests. Levy's approach (which again overprices) performs overall best, followed by the method of Beisser (which again underprices).

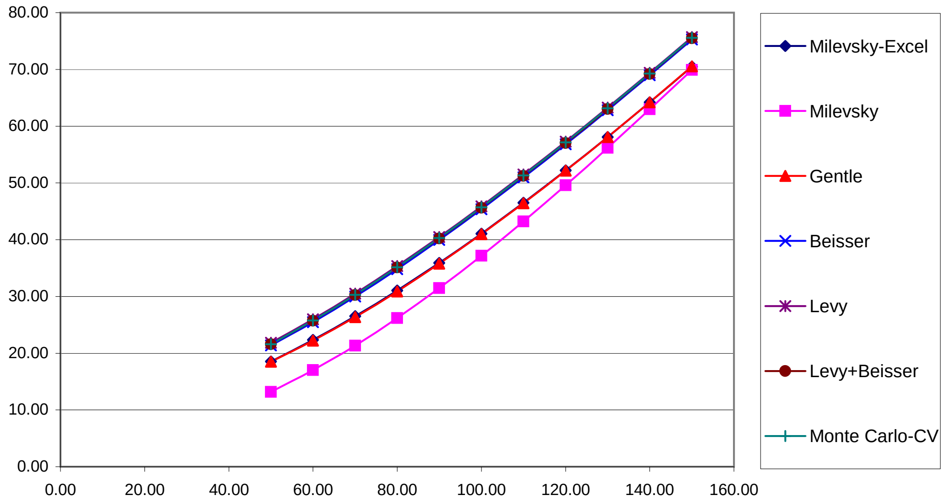


Figure 36: Varying the forwards sym. with $F_1^T = 200, K = 100$

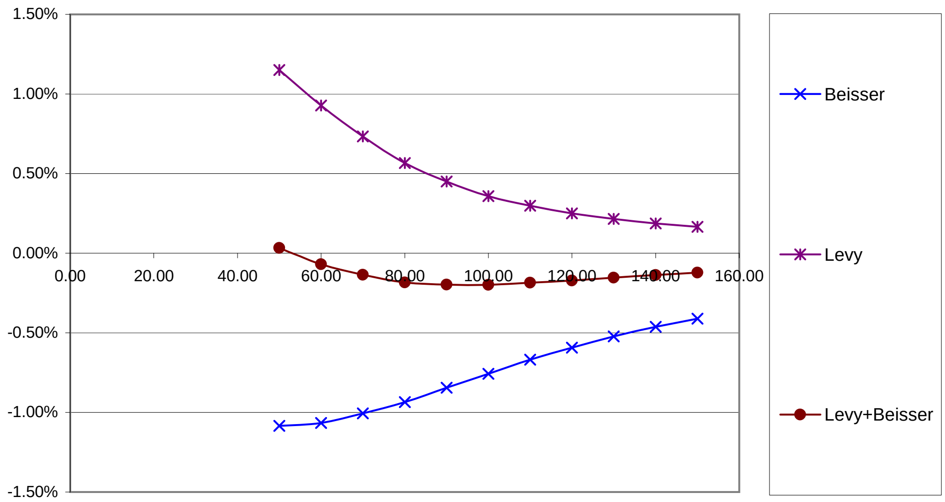


Figure 37: Varying the forwards sym. with $F_1^T = 200, K = 100$ (Rel. Diff.)

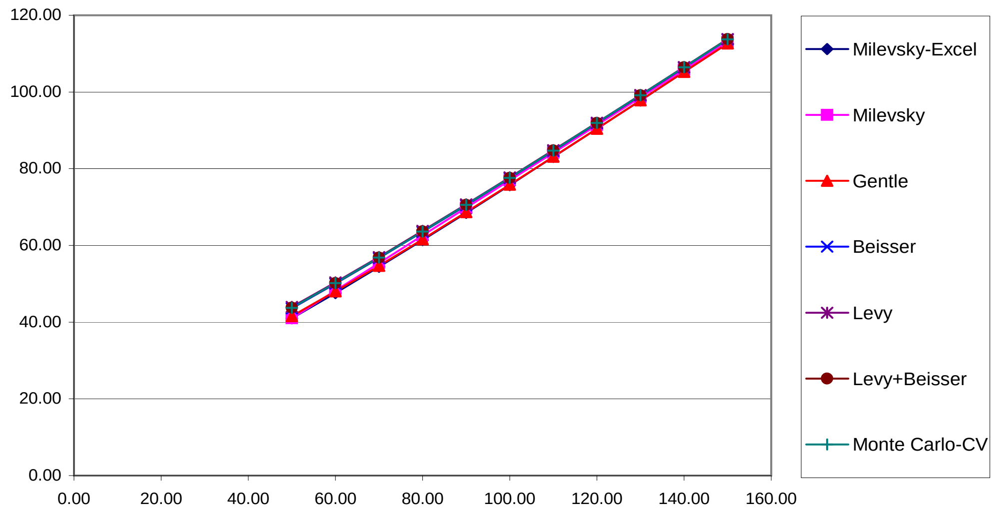


Figure 38: Varying the forwards sym. with $F_1^T = 200$ and $K = 0.5 \sum_{i=1}^{4} w_i F_i^T$

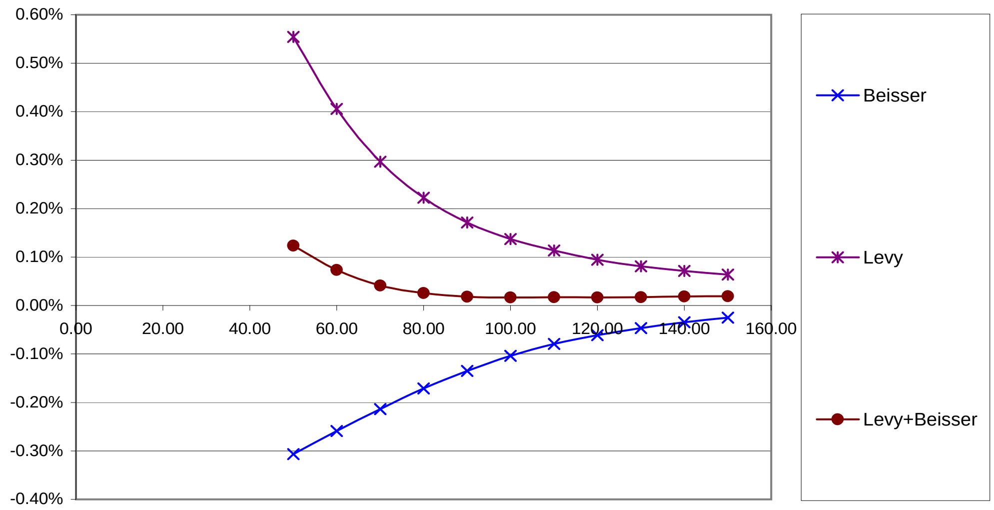


Figure 39: Varying the forwards sym. with $F_1^T = 200$ and $K = 0.5 \sum_{i=1}^{4} w_i F_i^T$ (Rel. Diff.)

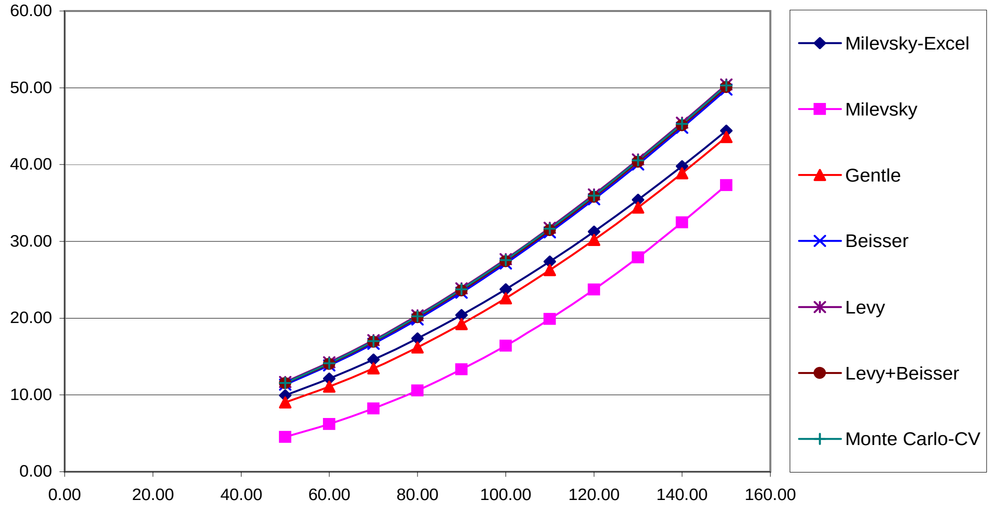


Figure 40: Varying the forwards sym. with $F_1^T = 200$ and $K = 1.5\sum_{i=1}^4 w_i F_i^T$

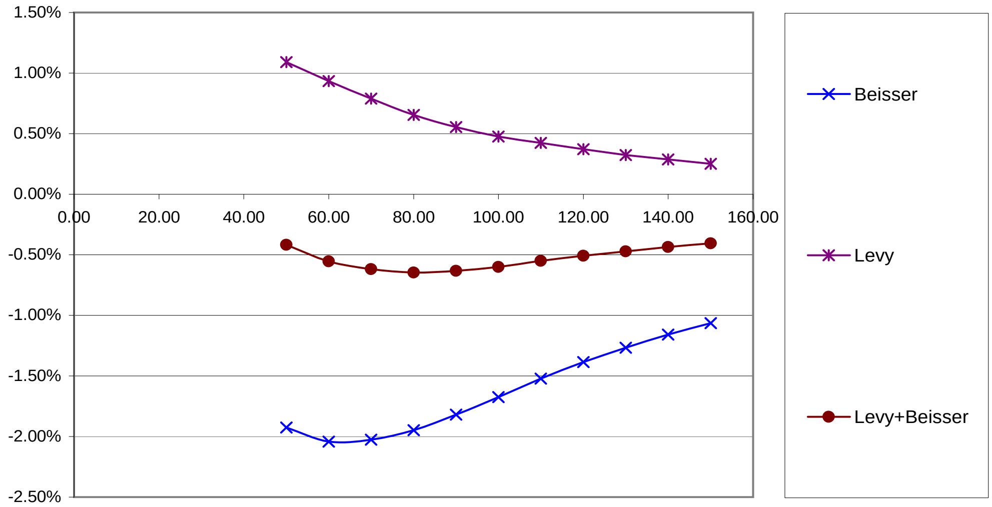


Figure 41: Varying the forwards sym. with $F_1^T = 200$ and $K = 1.5\sum_{i=1}^4 w_i F_i^T$ (Rel. Diff.)

At last we give in Figure 42 a density plot with $F_1^T = 10$ and $F_2^T = F_3^T = F_4^T = 100$. Due to the comparison with Figure 42 we observe, that inhomogenous forwards does not have an effect on the approximations of the densities w.r.t our standard scenario with homogenous parameters.

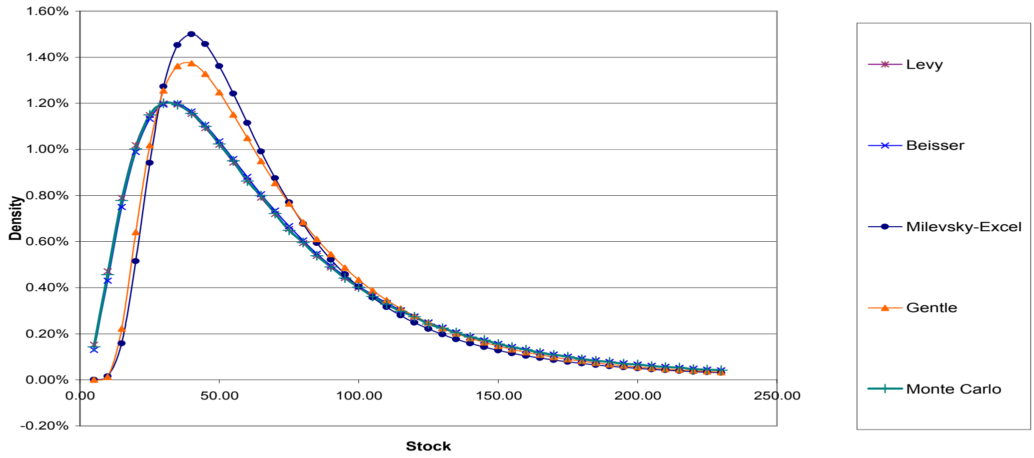


Figure 42: Densities for the standard scenario with $F_1^T = 10$

Summary

Levy's and Beisser's approach are most able to deal with varying forwards and strikes. Except for the case of a out-of-the-money Call with inhomogenous forwards, Levy's approximation always slightly overprices. As before Beisser's approximation slightly underprices. The other methods perform worse by a considerable margin.

5.3.4 Effect of varying the Volatilities

The next set of tests involves varying the volatilities σ_i. At each step, σ_i is set to the same value σ. The volatility σ is varied between 5% and 100%. The results (see table 4) are displayed in Figures 43 and 44.

σ	Milevsky -Excel	Gentle	Beisser	Levy	Levy+ Beisser	Monte Carlo	Standard Deviation
5 %	3,52	3,52	3,53	3,53	3,53	3,53	0,0014
10 %	6,99	6,98	7,04	7,05	7,05	7,05	0,0042
15 %	10,36	10,33	10,55	10,57	10,56	10,57	0,0073
20 %	13,59	13,52	14,03	14,08	14,06	14,08	0,0115
30 %	19,49	19,22	20,91	21,09	21,00	21,07	0,0237
40 %	24,50	23,78	27,63	28,05	27,84	27,98	0,0350
50 %	28,51	27,01	34,15	34,96	34,55	34,80	0,0448
60 %	31,56	28,84	40,41	41,78	41,10	41,44	0,0327
70 %	33,72	29,30	46,39	48,50	47,45	47,86	0,0490
80 %	35,15	28,57	52,05	55,05	53,55	54,01	0,0685
100 %	36,45	24,41	62,32	67,24	64,78	65,31	0,0996
Dev.	11,83	16,25	1,22	0,69	0,28	0	

Table 4: Varying the volatilities sym. with $K = 100$ (Figure 43)

The prices calculated by the different methods are more or less equal for rather "small" values of the volatility. They start to diverge at $\sigma \approx 20\%$. The Monte Carlo, Levy, Beisser and Levy+Beisser prices remain close, whereas the prices calculated by the other methods are too low.

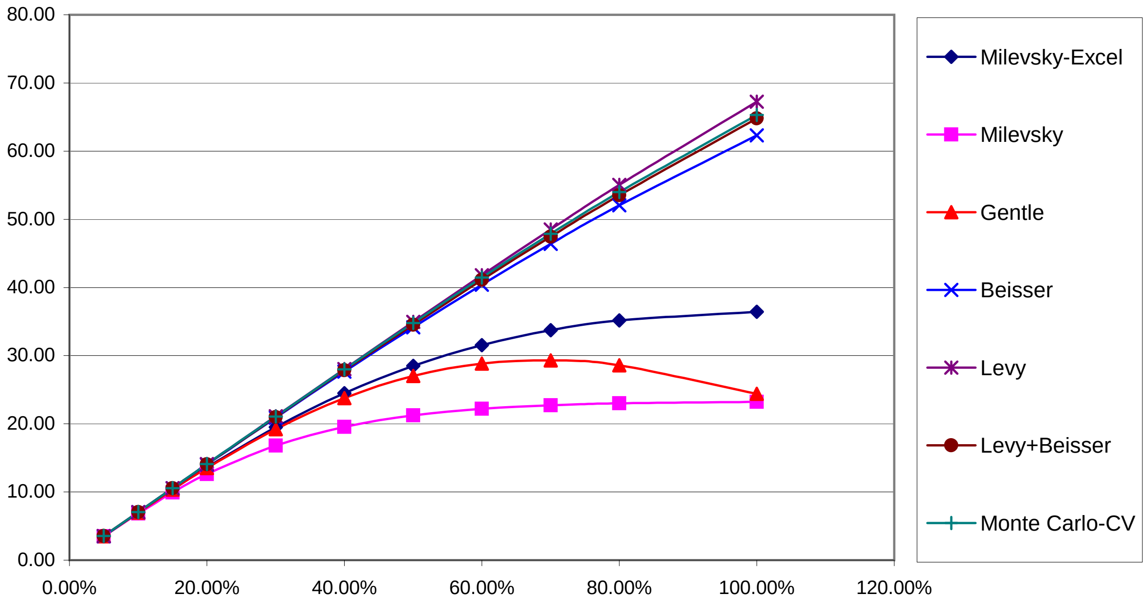


Figure 43: Varying the volatilities sym. with $K = 100$ (Table 4)

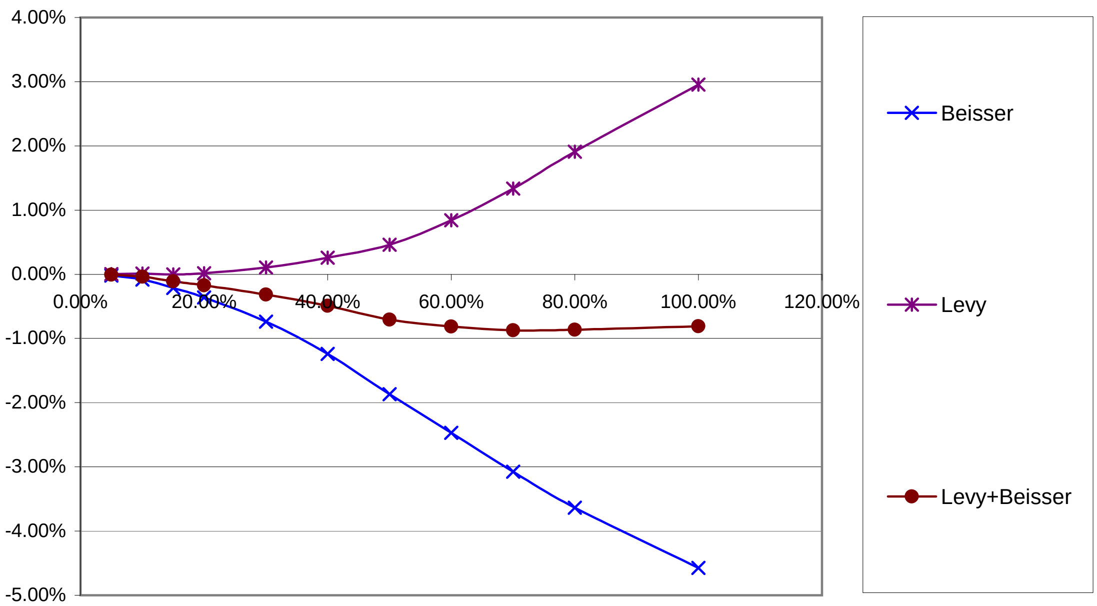


Figure 44: Varying the Volatilities sym. with $K = 100$ (Rel. Diff.)

The second test in Figure 45 is a repetition of the first, except that the volatility σ_1 of the first stock is kept fixed at 5%. This time the prices diverge much more. The methods Levy and Levy+Beisser are overpricing with all other methods underpricing. We note that Beisser's method performs best. Figure 45 portrays the results.

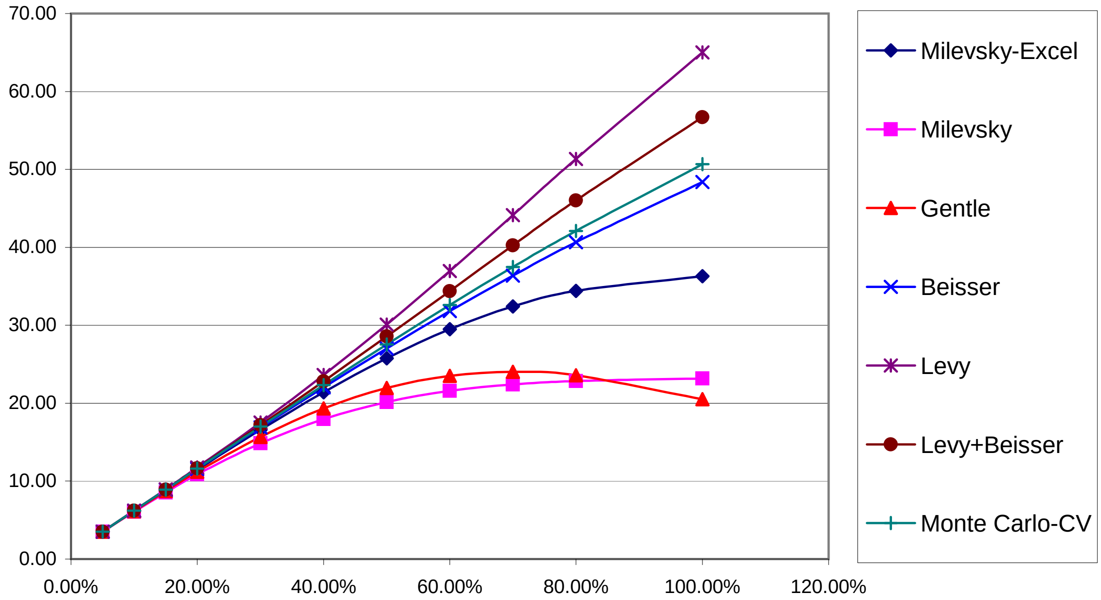


Figure 45: Varying the volatilities sym. with $\sigma_1 = 5\%, K = 100$

We now repeat the previous test but set the strike to $K = 50$. The results (see Figure 46) are similar to those of the previous test. However, the price curves only start to diverge for $\sigma \approx 30\%$. Gentle's approximation performs worst of all throughout the test. Worse results are obtained when repeating the previous test with the strike K set to 150. The price curves are graphed in Figure 47. The prices diverge more, since we have now an out-of-the-money option.

At last these three plots are graphed again but with $\sigma_1 = 100\%$. The general pattern (Figures 48, 49 and 50) looks completely different, i.e. one no longer observes a single curve "fanning out" with increasing σ (see Figures 45 to 47). However, changing the strike from 50 to 100 and then to 150 has the same effect as before (i.e. the price differences increase). This time the prices do not start out similarly and then diverge, instead they differ considerably from the beginnig. Gentle's approximation performs well for "small" values of σ, while the Levy and Levy+Beisser method also perform well for "large" σ. Beisser's approach again performs well for all values of σ.

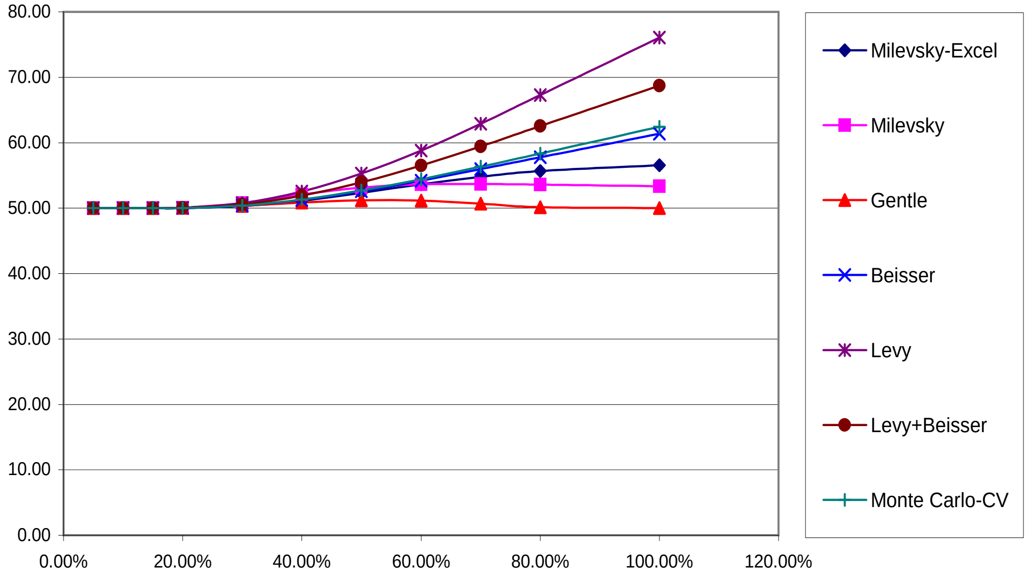


Figure 46: Varying the volatilities sym. with $\sigma_1 = 5\%, K = 50$

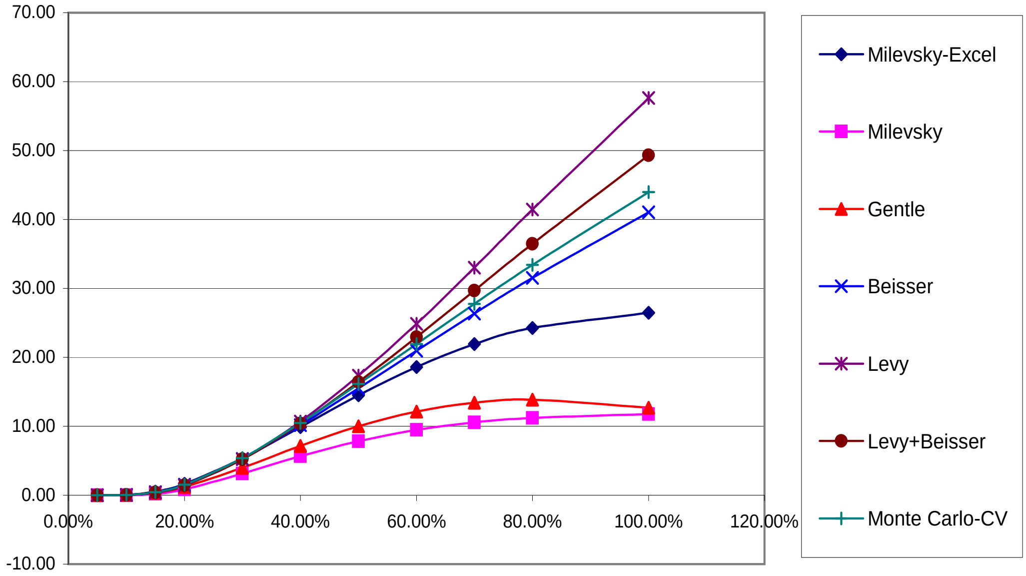


Figure 47: Varying the volatilities sym. with $\sigma_1 = 5\%, K = 150$

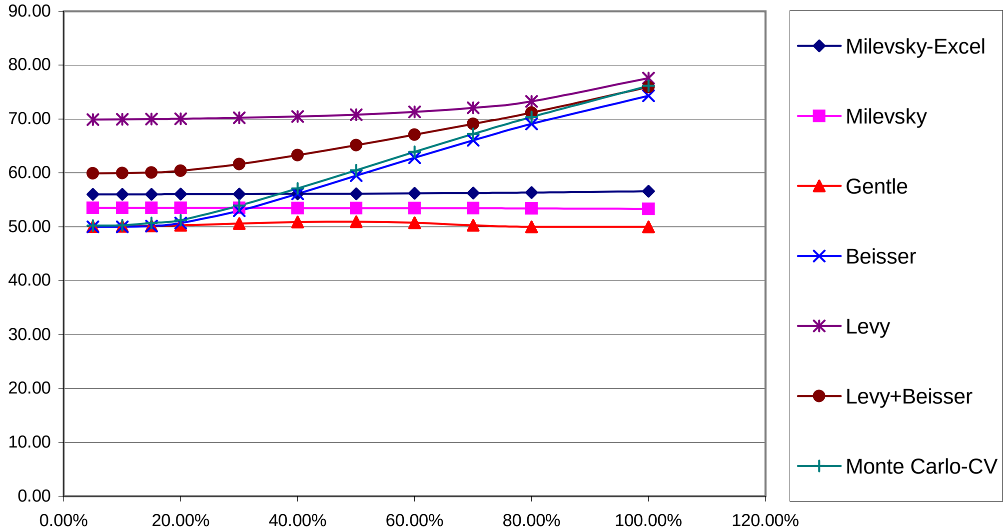


Figure 48: Varying the volatilities sym. with $\sigma_1 = 100\%, K = 50$ (Table 5)

σ	Milevsky -Excel	Gentle	Beisser	Levy	Levy+ Beisser	Monte Carlo-CV	StdDev
5 %	56,03	50,03	50,00	69,89	59,94	50,22	0,0812
10 %	56,04	50,08	50,00	69,93	59,97	50,27	0,0790
15 %	56,05	50,17	50,14	69,99	60,06	50,69	0,0772
20 %	56,05	50,29	50,70	70,05	60,38	51,25	0,0535
30 %	56,07	50,60	53,00	70,22	61,61	53,92	0,0428
40 %	56,10	50,88	56,11	70,46	63,28	57,12	0,0380
50 %	56,14	50,95	59,47	70,81	65,14	60,51	0,0416
60 %	56,19	50,74	62,84	71,32	67,08	63,91	0,0490
70 %	56,26	50,30	66,06	72,08	69,07	67,23	0,0606
80 %	56,36	50,00	69,08	73,27	71,17	70,40	0,0762
100 %	56,61	50,00	74,33	77,57	75,95	76,15	0,1018
Dev.	9,05	12,44	1,01	13,93	6,69	0,00	

Table 5: Varying the volatilities sym. with $\sigma_1 = 100\%, K = 50$ (Figure 48)

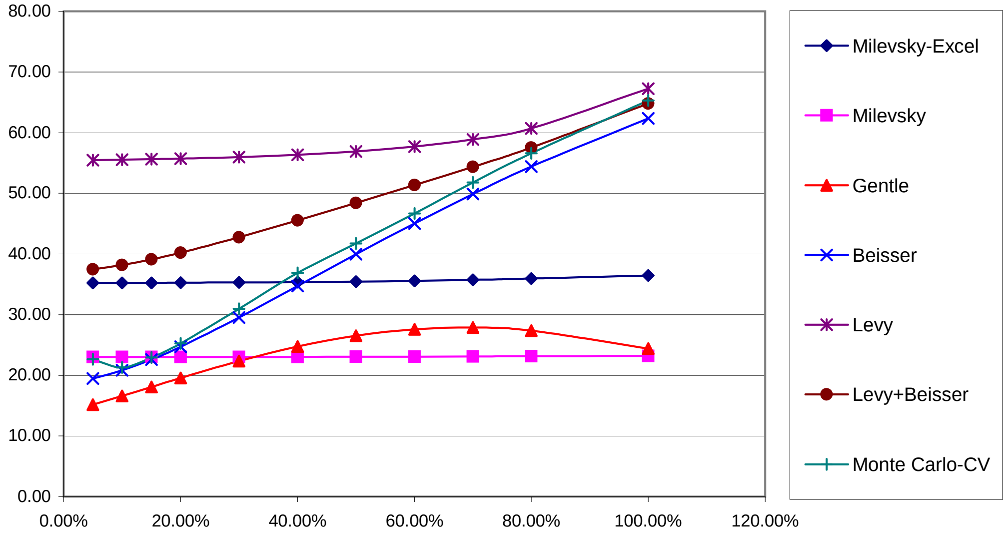


Figure 49: Varying the volatilities sym. with $\sigma_1 = 100\%, K = 100$ (Table 6)

σ	Milevsky -Excel	Gentle	Beisser	Levy	Levy+ Beisser	Monte Carlo-CV	StdDev
5 %	35,22	15,15	19,45	55,46	37,45	22,65	0,5594
10 %	35,23	16,60	20,84	55,52	38,18	21,30	0,3858
15 %	35,24	18,08	22,60	55,61	39,10	22,94	0,2660
20 %	35,26	19,56	24,69	55,71	40,20	25,24	0,2124
30 %	35,30	22,35	29,52	55,98	42,75	30,95	0,1603
40 %	35,36	24,73	34,72	56,35	45,54	36,89	0,1156
50 %	35,44	26,52	39,96	56,89	48,43	41,72	0,0894
60 %	35,56	27,59	45,05	57,68	51,36	46,68	0,0472
70 %	35,72	27,87	49,88	58,87	54,38	51,78	0,0587
80 %	35,93	27,38	54,39	60,70	57,54	56,61	0,0742
100 %	36,45	24,41	62,32	67,24	64,78	65,31	0,0996
Dev.	14,4816	19,1883	1,9254	22,7055	10,7802	0,00	

Table 6: Varying the volatilities sym. with $\sigma_1 = 100\%, K = 100$ (Figure 49)

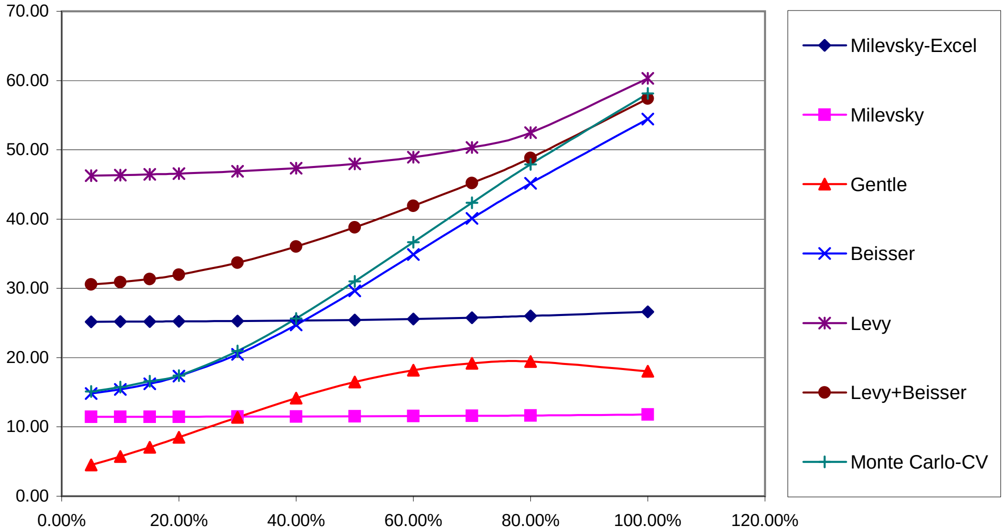


Figure 50: Varying the volatilities sym. with $\sigma_1 = 100\%, K = 150$ (Table 7)

σ	Milevsky -Excel	Gentle	Beisser	Levy	Levy+ Beisser	Monte Carlo-CV	StdDev
5 %	25,18	4,52	14,82	46,28	30,55	15,10	0,0765
10 %	25,19	5,73	15,42	46,36	30,89	15,73	0,0749
15 %	25,21	7,07	16,24	46,46	31,35	16,58	0,0734
20 %	25,23	8,49	17,34	46,58	31,96	17,42	0,0510
30 %	25,28	11,40	20,49	46,90	33,69	20,94	0,0407
40 %	25,35	14,15	24,74	47,35	36,04	25,64	0,0360
50 %	25,44	16,47	29,66	47,98	38,82	31,01	0,0394
60 %	25,58	18,18	34,87	48,92	41,90	36,66	0,0465
70 %	25,76	19,18	40,11	50,33	45,22	42,34	0,0579
80 %	26,01	19,46	45,18	52,50	48,84	47,88	0,0733
100 %	26,62	18,03	54,44	60,32	57,38	58,14	0,0988
Dev.	14,28	19,33	1,71	22,00	10,73	0,00	

Table 7: Varying the volatilities sym. with $\sigma_1 = 100\%, K = 150$ (Figure 50)

We plot the densities of the implicit basket distributions for inhomogenous volatilities in Figures 51 and 52.

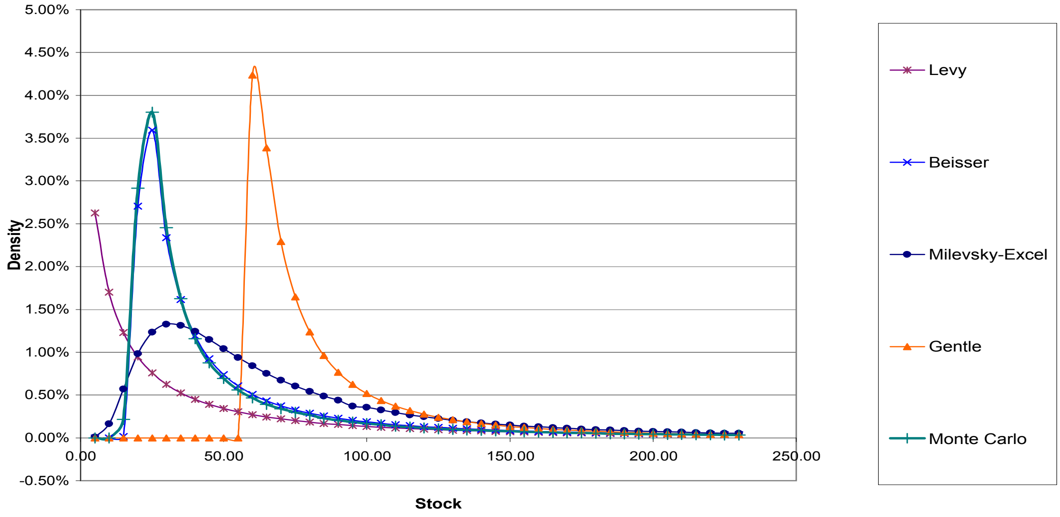


Figure 51: Densities for std. scenario with $\sigma_1 = 5\%, \sigma_2 = \sigma_3 = \sigma_4 = 100\%$ (Figure 45)

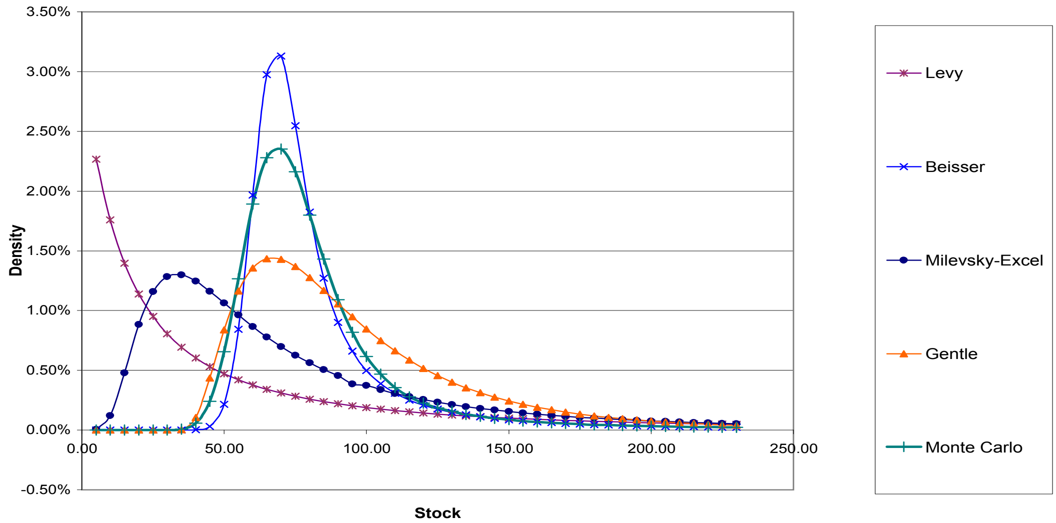


Figure 52: Densities for std. scenario with $\sigma_1 = 100\%, \sigma_2 = \sigma_3 = \sigma_4 = 10\%$ (Table 6)

In both cases Beisser's approach is the only method which reproduces the distribution of the basket. To our surprise, Levy's method shows the most evidents deviations to the real distribution. In these scenarios even Gentle's and Milevsky's approach is better.

At last we try an example with $\sigma_1 = 90\%, \sigma_2 = \sigma_3 = 50\%$ and $\sigma_4 = 10\%$ in Figure 53 to test if there is some "balancing" effect, i.e. observe that $(\sigma_1 + \sigma_4)/2 = \sigma_2$. We see there is one except for Levy's approach.

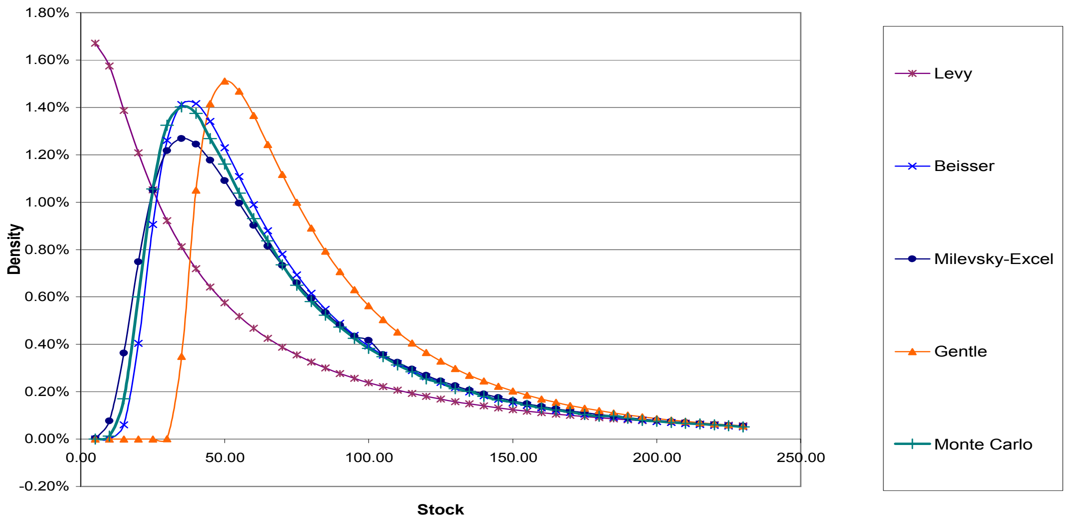


Figure 53: Densities for the standard scenario with $\sigma_1 = 90\%, \sigma_2 = \sigma_3 = 50\%, \sigma_4 = 10\%$

Summary
The only method which is able to deal with inhomogenous volatilities is the method of Beisser. Levy's approach, which was up to now the best method, massively overprices. The prices of Gentle and Milevsky are getting worse with growing volatilities (see Figures 43, 45, 46 and 47), and besides that they seem to ignore changes in the volatilities, if there is one single stock with high volatilites (see Figures 48, 49 and 50).

5.4 Conclusion

The tests confirm that the approximations of Levy and Beisser are overall the best performing methods. With the exception of inhomogenous volatilities and out-of-the-money calls with inhomogenous forwards, Levy's approximation generally overprices slightly. Beisser's approximation underprices slightly in all cases. Apart from the above mentioned exceptions Levy's approximation is more closer to the real prices. The underpricing of Beisser's approach is not surprising since this method is essentially a lower bound on the true option price (see Beisser (1999)). Figures 51-53 shows, that Beisser's approach is the only method which is able to approximate the real distribution for the case of inhomogenous volatilities. In this case even Gentle's and Milevsky's method is better than Levy's approach. In the other cases, i.e. change of correlation, forwards and strikes, the density curves of Levy's and Beisser's approach lie almost on the Monte Carlo curve, where the methods of Gentle and Milevsky show evident deviations.

The performance of Milevsky's and Gentle's approach is poor, since they do not match the real distribution of the basket even in the case of the standard scenario with a homogenous set of parameters. A reason for the bad performance of Milevsky may be, that the sum of log-normally distributed random variables is distributed like the reciprocal gamma distribution *only* as $n \to \infty$. But as in our case where $n = 4$ or even in practice with $n = 30$ we are far away from infinity. The geometric mean used in Gentle's approach also seems to be an inappropriate approximation for the arithmetic mean. For instance, the geometric mean of the forwards equal to 1,2,3 and 4 would be without mean correction 2.21 instead of 2.5. This is corrected, but the variance is still wrong. The Levy method is the best approximation except for the case of inhomogenous volatilities. The reason for this drawback may be that all stocks are "thrown" together to one log-normal distribution. This is quite contrary to Beisser's approximation, where every single stock keeps a transformed log-normal distribution and the expected value of every stock is individually evaluated. This is probably the reason why this method is able to handle the case of inhomogenous volatilities.

An obvious advice to a practitioner would be to use Levy's method for homogenous volatilities and Beisser's for inhomogenous ones. But then the question occurs, how to define the switch exactly. So we suggest the following: Price the basket with Levy and Beisser: If relative difference is less 5% use Levy's price for an upper, and Beisser's price for a lower bound. If it is bigger than 5% run a Monte-Carlo simulation or if this is not suitable, keep the Beisser result.

6 Asian options on Average Spot with Average Strike

6.1 Introduction

Asian Options, also known as "Average Options", are options whose payments at maturity depends on a - in real world - discretely monitored average of stock prices. There are two basic types of Asian Options: Fixed Strike Options (syn. Average Price Options, Average Rate Options) and Floating Strike Options (syn. Average Strike Options). The first type pays at maturity the difference - if positive - between some arithmetic mean of the stock and a predetermined strike price. The second type pays at maturity the difference - if positive - between the stock price and the arithmetic mean at maturity. Asian Options are normally European-style options. It is not possible to derive an (exact) closed-form solution in the Black-Scholes model, since the sum of log-normally distributed random variables is not log-normal.

Numerical approximations for both types of options are among others:
Kemna&Vorst (1990) uses Monte-Carlo simulation with geometric mean as controle variate. Roger&Shi(1995), Zvan,Forsthy&Vetzal(1997) and Benhamou&Duguet(2000) solve the problem applying PDE-methods. Binomial and trinomial trees are not the appropriate methods to price these kind of options, since the branches would not recombine, and thus the calculation time would increase exponentially with the refinement of the tree.

Analytical approximations for Fixed Strike Options are (to name just a few):
Levy(1992) uses a log-normal distribution with matches the first two moments of the distribution of the arithmetic mean, not to confuse with Ruttiens&Vorst(1990), who simply use the geometric mean. Turnbull&Wakeman(1991) compute an edgeworth expansion to match the first four moments based on a log-normal distribution. Carverhill&Clewlow(1990) and Benhamou(2000) apply fourier-transformation.
Recently, Henderson&Wojakowski(2001) showed the equivalence of Floating and Fixed Strike Options. So all these methods can also be applied to Floating Strike Options.

We introduce here a quite new type of Asian Option, the so-called *Asian Option on Average Spot with Average Strike*. The payoff of this option depends on the ratio or the difference of two arithmetic averages of the stock prices. This is going to be specified in the next chapter.

6.2 Model

We are using the Black-Scholes Model with deterministic coefficients. The saving account reads as

$$dB(t) = B(t)r(t)dt, \quad B(0) = B_0,$$

where $r(t)$ is a deterministic and bounded function of time. The stock is given by

$$dS(t) = S(t)[(r(t) - d(t))dt + \sigma(t)dW(t)], \quad S(0) = S_0,$$

where $d(t)$ is the deterministic and bounded dividend yield and $\sigma(t)$ is the deterministic, bounded and piecewise continuous volatility.

Let us first introduce some notations:

T : maturity of option

K : strike price

α : strike in percent

N_s : total number of Spot-Fixings

N_k : total number of Strike-Fixings

$$A_s = \frac{1}{N_s}\sum_{i=1}^{N_s} S(t_i^s)$$

$$A_k = \frac{1}{N_k}\sum_{i=1}^{N_k} S(t_i^k)$$

$$\theta = \begin{cases} 1 & : \text{Call} \\ -1 & : \text{Put} \end{cases} \quad \text{(Call/Put Operator)}$$

The fixing dates are ordered, i.e. $0 < t_1^k < \ldots < t_{N_k}^k < t_1^s \ldots < t_{N_k}^s = T$. The payoffs read as:

(i)	Fixed Strike	: $(\theta\{A_s - K\})^+$
(ii)	Average Spot with Average Strike in Equity	: $(\theta\{A_s - \alpha A_k\})^+$
(iii)	Average spot with Average Strike in Performance	: $\left(\theta\left\{\frac{A_s}{A_k} - \alpha\right\}\right)^+$

Note that a Floating Strike Option is a special case of an Average Spot with Average Strike in Equity Option with $N_s = 1$.

6.3 Approximate Valuation

To our knowledge, up to now, the pricing of these options is rarely or even not treated in literature. There are four ways to price these options:

- *Approximation by geometric means*
- *Monte Carlo (with geometric mean as control variate)*
- *Approximation by bivariate log-normal distribution with matching moments*
- *PDE and FM (with dimension reduction)*

We implemented and tested the first three methods. Let us first describe the *Approximation by Geometric Means*: If we replace the arithmetic means by geometric means, we get a bivariate log-normal distribution. Then the problem is equivalent to an exchange option whose solution is well-known (e.g. see the Collector's book) and was already treated in the late 70's by Margrabe(1978). As usual we use this approximation with or without adjusted strike to match the expectation of the arithmetic means. Without adjusting the strike this approximation can be used as control variate for variance reduction in the Monte Carlo simulation.

We will concentrate here on the third method, namely the *Approximation by bivariate log-normal distribution with matching moments*. In order to ease the calculations we introduce some additional notation:

The T-forward price at time t is given by:

$$F_t(T) = S_t \exp\left(\int_t^T r(s) - d(s)\, ds\right)$$

Hence we can write the stock price given S_t as:

$$S_t(T) = F_t(T) \exp\left(-\int_t^T \frac{1}{2}\sigma^2(s)ds + \int_t^T \sigma(s)dW(s)\right)$$

Since short rates are not quoted, we work instead with discount factors of the form:

$$Df_t(T) = \exp\left(-\int_t^T r(s)ds\right)$$

To take observed prices into account, when we want to price the option during the lifetime, we have to distinguish the means of observed- and future quotes:

$$
\begin{aligned}
A_s(t) &= \frac{1}{N_s}\sum_{i:t_i^s>t} S(t_i^s) \text{ (sum of future spot-quotes)}\\
A_s'(t) &= \frac{1}{N_s}\sum_{i:t_i^s\leq t} S(t_i^s) \text{ (sum of observed spot-quotes)}\\
A_k(t) &= \frac{1}{N_k}\sum_{i:t_i^k>t} S(t_i^k) \text{ (sum of future strike-quotes)}\\
A_k'(t) &= \frac{1}{N_k}\sum_{i:t_i^k\leq t} S(t_i^k) \text{ (sum of observed strike-quotes)}
\end{aligned}
$$

Our idea is to approximate the arithmetic means by an appropriate bivariate log-normal distribution with matching moments. More precisely, we replace $A_s(t)$ and $A_k(t)$ by log-normal distributions $X_s = \exp(M_s + V_sY_s)$ and $X_k = \exp(M_k + V_kY_k)$, where Y_s, Y_k such that

$$
\begin{pmatrix} Y_s \\ Y_k \end{pmatrix} = N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}\right).
$$

Thus we choose the parameters such that the first and second moments are matching

$$
\begin{aligned}
E[A_s(t)] &\overset{!}{=} E[X_s] = \exp(M_s + 0.5V_s^2)\\
E[A_s(t)^2] &\overset{!}{=} E[X_s^2] = \exp(2M_s + 2V_s^2)\\
E[A_k(t)] &\overset{!}{=} E[X_k] = \exp(M_k + 0.5V_k^2)\\
E[A_k(t)^2] &\overset{!}{=} E[X_k^2] = \exp(2M_k + 2V_k^2)
\end{aligned}
$$

and the correlations coincide

$$
E[A_k(t)A_s(t)] \overset{!}{=} E[X_kX_s] = \exp(M_s + M_k + 0.5V_s^2 + 0.5V_k^2 + \rho V_sV_k)
$$

which yields

$$
\begin{aligned}
M &= 2\log E[A(t)] - \frac{1}{2}\log E[A(t)^2]\\
V &= \sqrt{\log E[A(t)^2] - 2\log E[A(t)]}\\
\rho &= \frac{\log\left(\frac{E[A_k(t)A_s(t)]}{E[A_k(t)]E[A_s(t)]}\right)}{V_sV_k}.
\end{aligned}
$$

The moments are given by (the $A_s(t)$-moments are similar to those of $A_k(t)$):

$$\begin{aligned}
E[A_k(t)] &= \frac{1}{N_k}\sum_{i:t_i^k>t} F_t(t_i^k) \\
E[A_k(t)^2] &= \frac{1}{N_k^2}\sum_{i:t_i^k>t}\sum_{j:t_j^k>t} F_t(t_i^k)F_t(t_j^k)\exp\left(\int_t^{\min(t_i^k,t_j^k)}\sigma(s)ds\right) \\
E[A_s(t)A_k(t)] &= \frac{1}{N_kN_s}\sum_{i:t_i^k>t}\sum_{j:t_j^s>t} F_t(t_i^k)F_t(t_j^s)\exp\left(\int_t^{\min(t_i^k,t_j^s)}\sigma(s)ds\right)
\end{aligned}$$

Hence we achieve the appropriate distribution, but we are not finished yet, since in some cases the derivation of the expectation is not trivial. The next section will adress to this problem.

6.4 Final Computation

6.4.1 Fixed Strike Option

$$\begin{aligned}
V_{FixS}(t) &= Df_t(T) * E\left[\left(\theta\Big\{A_s(t) + A'_s(t) - K\Big\}\right)^+\right] \\
&\approx Df_t(T) * E\left[\left(\theta\Big\{\exp(M_s + V_sY_s) - (K - A'_s(t))\Big\}\right)^+\right]
\end{aligned}$$

The Average-so-far can be put in the strike price, hence the option can be treated as a vanilla option,

i) **Case:** $(K - A'_s(t)) > 0$

$$\begin{aligned}
V_{FixS}(t) &= Df_t(T)\theta\left\{\exp(M_s + 0.5V_s^2)N(\theta d_1) - (K - A'_s(t)) * N(\theta d_2)\right\}, \\
d2 &= (M_s - \log(K - A'_s(t))/V_s, \\
d1 &= d2 + V_s,
\end{aligned}$$

ii) **Case:** $(K - A'_s(t)) \leq 0$

$$V_{FixS}(t) = Df_t(T)\left(\theta\left\{\exp(M_s + 0.5V_s^2) - (K - A'_s(t))\right\}\right)^+.$$

In this case our method is equivalent to Levy(1992).

6.4.2 Average Spot with Average Strike in Equity

$$\begin{aligned} V_{AveSK}(t) &= Df_t(T) * E\left[\left(\theta\Big\{(A_s(t)+A'_s(t)) - (A_k(t)+A'_k(t))\Big\}\right)^+\right] \\ &\approx Df_t(T) * E\left[\left(\theta\Big\{\exp(M_s+V_sY_s) - \exp(M_k+V_kY_k) - (A'_k(t)-A'_s(t))\Big\}\right)^+\right] \end{aligned}$$

Therefore the option valuation depends on $A'_k(t) - A'_s(t)$, which we call here *Average-Strike-so-far*. We distinguish three cases:

i) **Case:** $(A'_k(t) - A'_s(t)) = 0$**:**
In this case we are concerned with an exchange option pricing problem, which has been solved by Margrabe(1978).

$$\hat{V} = \sqrt{V_s^2 + V_k^2 - 2\rho V_s V_k}$$

$$d1 = \Big(M_s - M_k + 0.5(V_s - V_k) + 0.5 * \hat{V}^2\Big)/\hat{V}$$

$$d2 = d1 - \hat{V}$$

$$V_{AveSK} = Df_t(T)\theta\Big(\exp(M_s + 0.5V_s)N(\theta d_1) - \exp(M_k + 0.5V_k)N(\theta d_2)\Big)$$

ii) **Case:** $(A'_k(t) - A'_s(t)) > 0$**:**
We are now concerned with a pricing problem which is equivalent to that of a spread option. This type of options has been treated in Shimko(1994) and Zhang(1997). We were able to reduce the two-dimensional integral to a one-dimensional one by applying conditional expectation techniques:

$$d = \frac{M_s - \log(A'_k(t) - A'_s(t))}{\sqrt{V_s}}$$

$$\phi(x) = -\frac{1}{\sqrt{V_s}} \log\left[1 + \exp(M_k + V_k x)/(A'_k(t) - A'_s(t))\right]$$

$$\begin{aligned} V_{AveSK}(t) = Df_t(T)\theta\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\Bigg[& e^{M_s+0.5V_s} N\left(\theta\frac{d+\rho x+\sqrt{V_s}+\phi(x+\rho\sqrt{V_s})}{\sqrt{1-p^2}}\right) \\ & - e^{M_k+0.5V_k} N\left(\theta\frac{d+\rho(x+\sqrt{V_k})+\phi(x+\sqrt{V_k})}{\sqrt{1-p^2}}\right) \\ & - (A'_k(t)-A'_s(t))N\left(\theta\frac{d+\rho x+\phi(x)}{\sqrt{1-p^2}}\right)\Bigg] e^{-\frac{x^2}{2}}dx \end{aligned}$$

The derivation of this formula can be found in Appendix 1.

iii) **Case:** $(A'_k(t) - A'_s(t)) < 0$**:**
Due to the payoff structure and our approximation, the value of the option is a function of $M_s, V_s, M_k, V_k, A'_k(t), A'_s(t), \rho$ and we have the following symmetric relationship:

$$V_{AveSK}(\theta, M_s, V_s, M_k, V_k, A'_k(t), A'_s(t), \rho) = V_{AveSK}(-\theta, M_k, V_k, M_s, V_s, A'_s(t), A'_k(t), \rho)$$

So we if have a Call(Put) Option and the *Average-Strike-so-far* is negative we can shuffle the parameters and price it as Put(Call) Option. In this new notation our *Average-Strike-so-far* will be positive, and we can price it as in the second case.

6.4.3 Average Spot with Average Strike in Performance

$$\begin{aligned} V_{Per}(t) &= Df_t(T) * E\left[\left(\theta\left\{\frac{A_s(t) + A'_s(t)}{A_k(t) + A'_k(t)} - \alpha\right\}\right)^+\right] \\ &\approx Df_t(T) * E\left[\left(\theta\left\{\frac{\exp(M_s + V_s Y_s) + A'_s(t)}{\exp(M_k + V_k Y_k) + A'_k(t)} - \alpha\right\}\right)^+\right] \end{aligned}$$

i) **Case:** $A'_k(t) = 0 = A'_s(t)$
Note that:

$$\frac{\exp(M_s + V_s Y_s)}{\exp(M_k + V_k Y_k)} \stackrel{d}{=} \exp\left(M_s - M_k + \sqrt{V_s^2 + V_k^2 - 2\rho V_s V_k} Y\right),$$

where Y is standard normally distributed. So we have again a vanilla option, since the bivariate log-normal distribution collapse to an one-dimensional normal distribution.

ii) **Case:** $A'_k(t) > 0$
We get a new type of option which we again have to price by numerical integration. Since the strike fixing dates are before the spot fixing dates, we know that the two events $A'_s(t) \neq 0$ and $A'_k(t) = 0$ cannot occur simultaneously.

$$\phi(x) = A'_k(t) + \exp(M_k + V_k\, x)$$

$$d_1(x) = \begin{cases} \dfrac{\log(\alpha\phi(x) - A'_s(t)) - M_s}{V_s} & : \ (\alpha\phi(x) - A'_s(t)) > 0 \\ -\infty & : \ (\alpha\phi(x) - A'_s(t)) \leq 0 \end{cases}$$

$$A(x) = N\left(\theta\frac{\rho x - d_1(x)}{\sqrt{1-\rho^2}}\right)\left[\alpha - \frac{A'_s(t)}{\phi(x)}\right]$$

$$B(x) = N\left(\theta\frac{\rho x + V_s(1-\rho^2) - d_1(x)}{\sqrt{1-\rho^2}}\right)\frac{1}{\phi(x)}\exp\left(M_s + \frac{V_s^2(1-\rho^2)}{2} + \rho x V_s\right)$$

where $N(.)$ is the standard normal distribution with $N(-\infty) = 0$ and $N(\infty) = 1$. Then the option value is given by the following one-dimensional integral which has to be evalutad by numerical integration:

$$V_{Per}(t) = Df_t(T)\theta \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Big(B(x) - A(x) \Big) e^{-\frac{1}{2}x^2} dx$$

The proof of this formula can be found in Appendix 2.

6.5 Numerical Results and Conclusions

Since the numerical methods for Fixed Strike Options were already compared in many other papers (e.g. Levy&Turnbull(1992)), we will concentrate on the Average Spot on Average Strike Options. We denote the moment matching method with **MM**, **Geo** the approximation by geometric mean (used as MC-control variate), **Geo Adj** geometric mean with adjusted strikes, and **MC** Monte Carlo.

The numerical results are shown in Table 1 to 4. The number in brackets is the Std Dev of Monte Carlo. The deviation in the last column is calculated as

$$Dev = \sqrt{\frac{1}{20} \sum_{i=1}^{20} (MC_i - V_i)^2},$$

and is used as a performance index for the different methods.

... and the winner is:
Overall **MM** seems to be the most efficient and accurate approximation method. In particular, if the valuation date is after first fixing (see table 2 and table 4) it clearly outperforms the other two approximation methods. With a slight exception in table 3 (and there only for high volatilities) it is extremely close to Monte Carlo. It seems to be extremely promissing to apply this method to related option types such as e.g. baskets.

Table 1:
Comparison of Average Spot with Average Strike in Equity Options,
valuation at starting date: $t = 0, N_k = N_s = 26, t_i^k = i/52, t_i^s = 4.5 + i/52, r = 5\%, d = 1\%, S = 100$.

σ	α	MC	Std Dev	MM	Geo	Geo Adj
0.10	0.50	54.87	(0.000)	54.87	54.84	54.87
	0.80	31.42	(0.000)	31.42	31.41	31.42
	1.00	17.43	(0.000)	17.43	17.43	17.43
	1.20	7.71	(0.000)	7.71	7.71	7.71
	1.50	1.55	(0.000)	1.55	1.55	1.55
0.30	0.50	56.22	(0.002)	56.22	56.01	56.19
	0.80	38.59	(0.002)	38.59	38.44	38.52
	1.00	29.80	(0.002)	29.80	29.69	29.71
	1.20	23.02	(0.001)	23.01	22.93	22.93
	1.50	15.72	(0.001)	15.72	15.66	15.64
0.50	0.50	61.43	(0.006)	61.42	60.79	61.14
	0.80	48.92	(0.007)	48.92	48.42	48.56
	1.00	42.56	(0.006)	42.56	42.13	42.19
	1.20	37.36	(0.006)	37.35	36.97	36.97
	1.50	31.14	(0.006)	31.13	30.82	30.76
0.70	0.50	67.89	(0.023)	67.88	66.52	67.01
	0.80	58.87	(0.022)	58.87	57.70	57.90
	1.00	54.18	(0.030)	54.18	53.10	53.19
	1.20	50.20	(0.022)	50.22	49.22	49.22
	1.50	45.34	(0.027)	45.28	44.38	44.29
Deviation:			(0.0128)	0.0141	0.6122	0.5217

Table 2:
Comparison of Average Spot with Average Strike in Equity Options,
valuation after starting date with observed prices equal to forwards: $t = 0.25, N_k = N_s = 26, t_i^k = i/52, t_i^s = 4.5 + i/52, r = 5\%, d = 1\%, S = 100$.

σ	α	MC	Std Dev	MM	Geo	Geo Adj
0.10	0.50	55.56	(0.000)	55.56	55.52	55.56
	0.80	31.81	(0.000)	31.81	31.78	31.81
	1.00	17.64	(0.000)	17.64	17.61	17.64
	1.20	7.79	(0.000)	7.79	7.77	7.79
	1.50	1.56	(0.000)	1.56	1.56	1.56
0.30	0.50	56.92	(0.002)	56.92	56.60	56.88
	0.80	39.04	(0.001)	39.04	38.78	38.97
	1.00	30.13	(0.001)	30.13	29.91	30.04
	1.20	23.25	(0.001)	23.26	23.07	23.17
	1.50	15.87	(0.001)	15.87	15.72	15.78
0.50	0.50	62.17	(0.006)	62.17	61.33	61.88
	0.80	49.48	(0.006)	49.49	48.75	49.11
	1.00	43.03	(0.006)	43.04	42.37	42.65
	1.20	37.75	(0.007)	37.76	37.14	37.36
	1.50	31.46	(0.007)	31.45	30.91	31.06
0.70	0.50	68.73	(0.023)	68.70	67.04	67.80
	0.80	59.53	(0.022)	59.56	58.04	58.55
	1.00	54.80	(0.024)	54.81	53.36	53.77
	1.20	50.76	(0.022)	50.79	49.42	49.74
	1.50	45.74	(0.021)	45.78	44.50	44.74
Deviation:			(0.0116)	0.0151	0.8104	0.5297

Table 3:
Comparison of Average Spot with Average Strike in Performance Options,
valuation at starting date: $t = 0, N_k = N_s = 26, t_i^k = i/52, t_i^s = 4.5 + i/52, r = 5\%, d = 1\%, S = 100, Notional = 1000$.

σ	α	MC	Std Dev	MM	Geo	Geo Adj
	0.50	542.22	(0.002)	542.22	542.22	542.22
	0.80	310.18	(0.002)	310.18	310.18	310.18
0.10	1.00	171.88	(0.002)	171.88	171.88	171.86
	1.20	75.89	(0.001)	75.89	75.89	75.86
	1.50	15.23	(0.001)	15.23	15.23	15.21
	0.50	549.72	(0.020)	549.70	549.73	549.43
	0.80	376.15	(0.018)	376.09	376.12	375.44
0.30	1.00	289.86	(0.017)	289.83	289.86	289.06
	1.20	223.49	(0.016)	223.43	223.46	222.62
	1.50	152.25	(0.015)	152.21	152.24	151.45
	0.50	590.34	(0.066)	590.21	590.41	587.81
	0.80	468.44	(0.064)	468.30	468.51	465.05
0.50	1.00	406.94	(0.070)	406.63	406.83	403.16
	1.20	356.47	(0.068)	356.20	356.40	352.68
	1.50	296.32	(0.062)	296.16	296.35	292.71
	0.50	638.44	(0.202)	637.38	638.11	629.94
	0.80	551.67	(0.229)	550.77	551.49	542.12
0.70	1.00	506.82	(0.219)	505.89	506.61	496.96
	1.20	468.73	(0.224)	468.08	468.78	459.06
	1.50	421.59	(0.194)	421.05	421.73	412.12
Deviation:			(0.1122)	0.4334	0.1081	5.0306

Table 4:
Comparison of Average Spot with Average Strike in Performance Options,
valuation after starting date with observed prices equal to forwards: $t = 0.25, N_k = N_s = 26, t_i^k = i/52, t_i^s = 4.5 + i/52, r = 5\%, d = 1\%, S = 100, Notional = 1000$.

σ	α	MC	Std Dev	MM	Geo	Geo Adj
0.10	0.50	549.41	(0.002)	549.41	549.10	549.41
	0.80	314.42	(0.002)	314.42	314.12	314.42
	1.00	174.25	(0.001)	174.25	173.99	174.22
	1.20	76.88	(0.001)	76.88	76.71	76.84
	1.50	15.38	(0.001)	15.38	15.33	15.36
0.30	0.50	559.69	(0.015)	559.68	557.15	559.37
	0.80	383.27	(0.014)	383.28	381.07	382.57
	1.00	295.49	(0.014)	295.49	293.55	294.64
	1.20	227.84	(0.013)	227.84	226.19	226.96
	1.50	155.23	(0.013)	155.24	153.95	154.41
0.50	0.50	605.73	(0.057)	605.75	599.17	603.08
	0.80	481.28	(0.064)	481.24	475.33	477.71
	1.00	418.01	(0.059)	418.12	412.66	414.37
	1.20	366.39	(0.059)	366.46	361.41	362.64
	1.50	304.80	(0.055)	304.87	300.39	301.13
0.70	0.50	661.05	(0.167)	661.40	648.86	653.22
	0.80	572.66	(0.212)	572.33	560.69	562.95
	1.00	526.03	(0.218)	526.09	514.99	516.42
	1.20	486.87	(0.191)	487.06	476.47	477.31
	1.50	438.38	(0.193)	438.47	428.55	428.79
Deviation:			(0.1030)	0.1240	6.2819	4.9723

Appendix 1

We will derive here the formula of a standard spread call option. The put option formula can be derived by call-put parity, and the formula of our particular problem by some simple parameter transformations. We apply here a conditioning technique to turn the two-dimensional integral to a single one.

$$\begin{aligned} S_1(T) &= S_1 \exp((\mu_1 - \sigma_1^2/2)(T) + \sigma_1 W_1(T)) \\ S_2(T) &= S_2 \exp((\mu_2 - \sigma_2^2/2)(T) + \sigma_2 W_2(T)) \end{aligned}$$

where $\mu_1 = r - d_1, \mu_2 = r - d_2$ and $Corr(W_1(.), W_2(.)) = \rho$. The PAYOFF of a spread option is given by:

$$\max(S_1(T) - S_2(T) - K, 0)$$

The density of a bivariate standard normaldistribution reads as follows:

$$\begin{aligned} f(x,y) &= \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left(-\frac{x^2 - 2\rho xy + y^2}{2(1-\rho^2)}\right) \text{ or written as a product} \\ f(y|x)f(x) &= \frac{1}{\sqrt{2\pi}\sqrt{1-\rho^2}} \exp\left(-\frac{(y-\rho x)^2}{2(1-\rho^2)}\right) \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) \end{aligned}$$

We calculate the price via conditional expectation:

$$\begin{aligned} & E[\max(S_1(T) - S_2(T) - K, 0)] \\ = \; & E\left[E[\max(S_1(T) - S_2(T) - K, 0)] | S_2\right] \\ = \; & \int_{-\infty}^{+\infty} \left[\int_{\{y:(S_1(y)>K+S_2(x)\}} \left[S_1 \exp\left((\mu_1 - \sigma_1^2/2)T + \sigma_1\sqrt{T}y\right) - \right.\right. \\ & \left.\left. S_2 \exp\left((\mu_2 - \sigma_2^2/2)T + \sigma_2\sqrt{T}x\right) - K \right] f(y|x)dy \right] f(x)dx \end{aligned}$$

First the conditional expectation is calculated. Determination of the integration-range for $K > 0$:

$$\begin{aligned} \tilde{d} &= -\frac{\log\left[S_1/\left(K + S_2 \exp\left((\mu_2 - \sigma_2^2/2)t + \sigma_2\sqrt{T}x\right)\right)\right] + (\mu_1 - \sigma_1^2/2)t}{\sigma_1\sqrt{T}} \\ &= -d - \phi(x) \text{ with} \\ d &= \frac{\log(S_1/K) + (\mu_1 - \sigma_1^2/2)t}{\sigma_1\sqrt{T}} \\ \phi(x) &= -\frac{1}{\sigma_1\sqrt{T}} \log\left[1 + \frac{S_2}{K} \exp\left((\mu_2 - \sigma_2^2/2)t + \sigma_2\sqrt{T}x\right)\right] \end{aligned}$$

Computation of the integrals over K:

$$\begin{aligned}
\int_{-\infty}^{+\infty}\left[\int_{\tilde{d}}^{\infty}\Big[-K\Big]f(y|x)dy\right]f(x)dx &= \int_{-\infty}^{+\infty}\left[\int_{\frac{\tilde{d}-\rho x}{\sqrt{1-\rho^2}}}^{\infty}\Big[-K\Big]f(y)dy\right]f(x)dx \\
&= -\int_{-\infty}^{+\infty}\left[\int_{-\infty}^{\frac{d+\rho x+\phi(x)}{\sqrt{1-\rho^2}}}Kf(y)dy\right]f(x)dx \\
&= -K\int_{-\infty}^{+\infty}N\left(\frac{d+\rho x+\phi(x)}{\sqrt{1-\rho^2}}\right)f(x)dx =: A_3
\end{aligned}$$

Computation of the integrals over S_2:

$$\begin{aligned}
&-\int_{-\infty}^{+\infty}\left[\int_{-\infty}^{\frac{d+\rho x+\phi(x)}{\sqrt{1-\rho^2}}}S_2(0)\exp\left((\mu_2-\sigma_2^2/2)T+\sigma_2\sqrt{T}x\right)f(y)dy\right]f(x)dx \\
=\;& -S_2(0)\exp(\mu_2 T)\int_{-\infty}^{+\infty}\left[\int_{-\infty}^{\frac{d+\rho x+\phi(x)}{\sqrt{1-\rho^2}}}\exp\left((-\sigma_2^2/2)T-\sigma_2\sqrt{T}x\right)f(y)dy\right]f(x)dx \\
=\;& -S_2(0)\exp(\mu_2 T)\int_{-\infty}^{+\infty}\left[\int_{-\infty}^{\frac{d+\rho x+\phi(x)}{\sqrt{1-\rho^2}}}f(y)dy\right]f(x-\sigma_2\sqrt{T})dx \\
=\;& -S_2(0)\exp(\mu_2 T)\int_{-\infty}^{+\infty}N\left(\frac{d+\rho(x+\sigma_2\sqrt{T})+\phi(x+\sigma_2\sqrt{T})}{\sqrt{1-\rho^2}}\right)f(x)dx =: A_2
\end{aligned}$$

Computation of the integrals over S_1:

$$\begin{aligned}
&\int_{-\infty}^{+\infty}\left[\int_{-d-\phi(x)}^{\infty}S_1(0)\exp\left((\mu_1-\sigma_1^2/2)T+\sigma_1\sqrt{T}y\right)f(y|x)dy\right]f(x)dx \\
=\;& S_1(0)e^{\mu_1 T}\int_{-\infty}^{+\infty}\left[\int_{-d-\phi(x)}^{\infty}\exp\left(-\frac{\sigma_1^2}{2}T+\sigma_1\sqrt{T}y\right)f(y|x)dy\right]f(x)dx \\
=\;& S_1(0)e^{\mu_1 T}\int_{-\infty}^{+\infty}\left[\int_{-d-\phi(x)}^{\infty}\exp\left(\sigma_1\sqrt{T}y-\frac{\sigma_1^2}{2}T(1-p^2)-\rho x\sigma_1\sqrt{T}+\rho x\sigma_1\sqrt{T}-\frac{\sigma_1^2}{2}Tp^2\right)\right. \\
&\qquad\left. f(y|x)dy\right]f(x)dx \\
=\;& S_1(0)e^{\mu_1 T}\int_{-\infty}^{+\infty}\int_{-d-\phi(x)}^{\infty}\frac{1}{\sqrt{2\pi}\sqrt{1-\rho^2}}\exp\left(-\frac{(y-\sigma_1\sqrt{T}(1-\rho^2)-\rho x)^2}{2(1-\rho^2)}\right)dy \\
&\qquad\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{(x-\sigma_1\sqrt{T}\rho)^2}{2}\right)dx
\end{aligned}$$

$$= S_1(0)e^{\mu_1 T}\int_{-\infty}^{+\infty}\int_{-d-\phi(x+\sigma_1\sqrt{T}\rho)}^{\infty}\frac{1}{\sqrt{2\pi}\sqrt{1-\rho^2}}$$

$$\exp\left(-\frac{\left(y-\sigma_1\sqrt{T}(1-\rho^2)-\rho(x+\sigma_1\sqrt{T}\rho)\right)^2}{2(1-\rho^2)}\right)dy\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{x^2}{2}\right)dx$$

$$= S_1(0)e^{\mu_1 T}\int_{-\infty}^{+\infty}\int_{-\infty}^{\frac{d+\rho x+\sigma_1\sqrt{T}+\phi(x+\sigma_1\sqrt{T}\rho)}{\sqrt{1-\rho^2}}} f(y)dyf(x)dx$$

$$= S_1(0)e^{\mu_1 T}\int_{-\infty}^{+\infty} N\left(\frac{d+\rho x+\sigma_1\sqrt{T}+\phi(x+\rho\sigma_1\sqrt{T})}{\sqrt{1-\rho^2}}\right)f(x)dx =: A_1$$

Putting all together:

$$V_{Call}(S_1,S_2,K) = e^{-rT}(A_1+A_2+A_3)$$

and

$$V_{Put}(S_1,S_2,K) = V_{Call}(S_1,S_2,K) - e^{-rT}(S_1e^{\mu_1 T}+S_2e^{\mu_2 T}+K)$$

Note, for $K<0: V_{Call}(S_1,S_2,K) = V_{Put}(S_2,S_1,-K)$.

Translation in Forward-Notation: Let $X_i = N(0,1)$ iid:

$$\begin{aligned} M_i &= \log(S_i(0)) + (\mu_i - \sigma_i/2)T \\ V_i &= \sigma_i^2 T \\ S_i(T) &= \exp(M_i + V_i X_i) \end{aligned}$$

Thus:

$$\begin{aligned} d &= \frac{M_1-\log(K)}{\sqrt{V_1}} \\ \phi(x) &= -\frac{1}{\sqrt{V_1}}\log\left[1+\exp(M_2+V_2x)/K\right] \end{aligned}$$

Hence:

$$V_{Call}(S_1,S_2,K) = e^{-rT}\int_{-\infty}^{\infty}\Bigg[e^{M_1+0.5V_1}N\left(\frac{d+\rho x+\sqrt{V_1}+\phi(x+\rho\sqrt{V_1})}{\sqrt{1-p^2}}\right)$$

$$- e^{M_2+0.5V_2}N\left(\frac{d+\rho(x+\sqrt{V_2})+\phi(x+\sqrt{V_2})}{\sqrt{1-p^2}}\right)$$

$$- KN\left(\frac{d+\rho x+\phi(x)}{\sqrt{1-p^2}}\right)\Bigg]e^{-\frac{x^2}{2}}dx$$

Appendix 2

Let the prices of two stocks be given by:

$$F_1(T) = \exp(M_1 + V_1 Y)$$

$$F_2(T) = \exp(M_2 + V_2 X)$$

where

$$\begin{pmatrix} X \\ Y \end{pmatrix} = N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}\right)$$

The option value is:

$$V = Df(T) * E\left[\left(\theta\left\{\frac{K_1 + F_1(T)}{K_2 + F_2(T)} - K\right\}\right)^+\right]$$

or

$$V = Df(T) * E\left[E\left[\left(\theta\left\{\frac{K_1 + F_1(T)}{K_2 + F_2(T)} - K\right\}\right)^+ \middle| F_2(T)\right]\right]$$

The conditional mean is given by an analytical solution, the outer integral is done through numerical integration. Note, if $K_1 = 0 = K_2$ the fraction collapses to lognormal distribution and the calculation can be done analytically by a Black-Scholes-Type formula. We derive the formula for a call, the derivation for a put is equivalent. The density of bivariate standard normal distribution is given by:

$$\begin{aligned} f(x,y) &= \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left(-\frac{x^2 - 2\rho xy + y^2}{2(1-\rho^2)}\right) \\ f(y|x)f(x) &= \frac{1}{\sqrt{2\pi}\sqrt{1-\rho^2}} \exp\left(-\frac{(y-\rho x)^2}{2(1-\rho^2)}\right) \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) \end{aligned}$$

Let $K_2 > 0$:

$$\phi(x) = K_2 + \exp(M_2 + V_2\, x)$$

Then:

$$\int_{-\infty}^{\infty} \int_{\{y|\exp(M_1+V_1 y) > K\phi(x) - K_1\}} \left[\frac{1}{\phi(x)} \exp(M_1 + V_1 y) - (K - \frac{K_1}{\phi(x)})\right] f(y|x) dy f(x) dx$$

$$= \int_{-\infty}^{\infty} \int_{d_1(x)}^{\infty} \left[\frac{1}{\phi(x)} \exp(M_1 + V_1 y) - (K - \frac{K_1}{\phi(x)})\right] f(y|x) dy f(x) dx$$

with

$$d_1(x) = \begin{cases} \frac{\log(K\phi(x) - K_1) - M_1}{V_1} & : \ (K\phi(x) - K_1) > 0 \\ -\infty & : \ (K\phi(x) - K_1) \le 0 \end{cases}$$

Integration of the constant part:

$$\begin{aligned}
A &= \int_{-\infty}^{\infty}\int_{d_1(x)}^{\infty}\left[K-\frac{K_1}{\phi(x)}\right]\frac{1}{\sqrt{2\pi}\sqrt{1-\rho^2}}\exp\left(-\frac{(y-\rho x)^2}{2(1-\rho^2)}\right)dyf(x)dx \\
&= \int_{-\infty}^{\infty}\int_{\frac{d_1(x)-\rho x}{\sqrt{1-\rho^2}}}^{\infty}\left[K-\frac{K_1}{\phi(x)}\right]\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{y^2}{2}\right)dyf(x)dx \\
&= \int_{-\infty}^{\infty}N\left(-\frac{d_1(x)-\rho x}{\sqrt{1-\rho^2}}\right)\left[K-\frac{K_1}{\phi(x)}\right]f(x)dx
\end{aligned}$$

Integration of the non-constant part:

$$\begin{aligned}
B &= \int_{-\infty}^{\infty}\int_{d_1(x)}^{\infty}\frac{1}{\phi(x)}\exp\left(M_1+\frac{V_1(1-\rho^2)y}{1-\rho^2}\right)\frac{1}{\sqrt{2\pi}\sqrt{1-\rho^2}}\exp\left(-\frac{(y-\rho x)^2}{2(1-\rho^2)}\right)dyf(x)dx \\
&= \int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}\sqrt{1-\rho^2}}\int_{d_1(x)}^{\infty}\frac{1}{\phi(x)}\exp\left(M_1+\frac{V_1^2(1-\rho^2)}{2}+\rho x V_1\right) \\
&\quad * \exp\left(-\frac{(y-\rho x-V_1(1-\rho^2))^2}{2(1-\rho^2)}\right)dyf(x)dx \\
&= \int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}\int_{\frac{d_1(x)-\rho x-V_1(1-\rho^2)}{\sqrt{1-\rho^2}}}^{\infty}\frac{1}{\phi(x)}\exp\left(M_1+\frac{V_1^2(1-\rho^2)}{2}+\rho x V_1\right)\exp\left(-\frac{y^2}{2}\right)dyf(x)dx \\
&= \int_{-\infty}^{\infty}N\left(-\frac{d_1(x)-\rho x-V_1(1-\rho^2)}{\sqrt{1-\rho^2}}\right)\frac{1}{\phi(x)}\exp\left(M_1+\frac{V_1^2(1-\rho^2)}{2}+\rho x V_1\right)f(x)dx
\end{aligned}$$

Putting these calculations together yields:

$$V = Df(T)(B-A)$$

and for a put option we get:

$$\begin{aligned}
V_{Put} &= Df(T)*\int_{-\infty}^{\infty}\left\{N\left(\frac{d_1(x)-\rho x}{\sqrt{1-\rho^2}}\right)\left[K-\frac{K_1}{\phi(x)}\right]-\right. \\
&\quad \left. N\left(\frac{d_1(x)-\rho x-V_1(1-\rho^2)}{\sqrt{1-\rho^2}}\right)\frac{1}{\phi(x)}\exp\left(M_1+\frac{V_1^2(1-\rho^2)}{2}+\rho x V_1\right)\right\}f(x)dx
\end{aligned}$$

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