A Model for Spherical SH–Wave Propagation in Self–reinforced Linearly Elastic Media

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Abstract

We study a spherical wave propagating in radius– and latitude–direction and oscillating in latitude–direction in case of fibre–reinforced linearly elastic material. A function system solving Euler's equation of motion in this case and depending on certain Bessel and associated Legendre functions is derived.

Key Words: fibre–reinforced linearly elastic media, elasticity problem, sphere, Euler's equation of motion, Bessel functions, associated Legendre functions, hypergeometric functions

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1 Introduction

The propagation of seismic waves can be modelled by (the body force–free) Euler's equation of motion

$$\operatorname{div} \tau(u) = \rho \frac{\partial^2 u}{\partial t^2},$$

where u stands for the displacement function, ρ for the mass density, and $\tau(u)$ for the Cauchy stress tensor induced by u. The level of complexity depends on the elastic model for the discussed material, which ranges from the relatively(!) easy case of an isotropic material with 2 constants to various anisotropic models with e.g. 21 independent elastic constants (see, for instance, [13]). A comparably complex structure that has already been discussed intensively (see, for example, [2], [5], [6], [7], [9], [10], [14], [15], [16], and the references therein) is a fibre–reinforced linearly elastic medium with 7 independent constants, where various approaches like an inextensibility theory and a boundary layer theory are used. More degrees of freedom are, for instance, discussed in [3] for a homogeneous triclinic medium having 21 elastic constants.
The discussions of wave propagation in case of fibre–reinforced materials are often restricted to the planar case. Some other publications (see, for example, [2] and [7]) take different geometries, like cylinders, into account. In this paper we study theoretically Euler's equation of motion in spherical coordinates for a wave in fibre–reinforced materials propagating in ϑ–direction and r–direction and oscillating in ϑ–direction following the ansatz

$$u(r,\vartheta,\varphi,t) = A(r,\vartheta)\exp(i(\omega t - k_r r - k_\vartheta \vartheta)) \begin{pmatrix} -\sin\varphi \\ \cos\varphi \\ 0 \end{pmatrix}.$$

Using the spherical basis system and the corresponding representation of the relevant differential operators given by [11] we arrive at a solving function system for A consisting of Bessel functions for the radial part and a complicated expression depending on certain associated Legendre functions for the latitude part.

2 Spherical Nomenclature

The standard cartesian basis in $\mathbb{R}^3$

$$\varepsilon^1 := \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},\ \varepsilon^2 := \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix},\ \varepsilon^3 := \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

which is orthonormal with respect to the inner product

$$x \cdot y := x_1 y_1 + x_2 y_2 + x_3 y_3;$$

$x = (x_1, x_2, x_3)^{\mathrm{T}}, y = (y_1, y_2, y_3)^{\mathrm{T}} \in \mathbb{R}^3$, that induces the usual norm $|y| := \sqrt{y \cdot y}$, is apparently not appropriate for analyzing physical properties of a spherical Earth model, since it neither allows to distinguish tangential and normal phenomena nor is able to separate certain important differential operators. Those properties are satisfied by the orthonormal sytem

$$\varepsilon^r(\varphi,\vartheta) := \begin{pmatrix} \sin\vartheta\cos\varphi \\ \sin\vartheta\sin\varphi \\ \cos\vartheta \end{pmatrix}, \ \varepsilon^\varphi(\varphi) := \begin{pmatrix} -\sin\varphi \\ \cos\varphi \\ 0 \end{pmatrix}, \ \varepsilon^\vartheta(\varphi,\vartheta) := \begin{pmatrix} -\cos\vartheta\cos\varphi \\ -\cos\vartheta\sin\varphi \\ \sin\vartheta \end{pmatrix}.$$

(see e.g. [11]), where $\varphi \in [0, 2\pi[$ represents the longitude and $\vartheta \in [0,\pi]$ stands for the latitude. Note that ε^r is a normal vector field to the unit sphere

$$\Omega := \left\{ \xi \in \mathbb{R}^3 \,\middle|\, |\xi| = 1 \right\},$$

whereas ε^φ and ε^ϑ are tangential vector fields. We also have, for instance,

$$\frac{\mathrm{d}\varepsilon^\varphi}{\mathrm{d}\varphi}\cdot\varepsilon^\varphi = 0,\ \frac{\partial\varepsilon^\vartheta}{\partial\vartheta}\cdot\varepsilon^\vartheta = 0,\ \frac{\mathrm{d}\varepsilon^\varphi}{\mathrm{d}\varphi}\cdot\varepsilon^\vartheta = \cos^2\varphi\cos\vartheta + \sin^2\varphi\cos\vartheta = \cos\vartheta,\ \frac{\partial\varepsilon^r}{\partial\vartheta} = -\varepsilon^\vartheta,$$

$$\frac{\mathrm{d}\varepsilon^\varphi}{\mathrm{d}\varphi}\cdot\varepsilon^r = -\cos^2\varphi\sin\vartheta - \sin^2\varphi\sin\vartheta = -\sin\vartheta.$$

Furthermore, we obviously have $\varepsilon^r(\varphi,\vartheta) \wedge \varepsilon^\varphi(\varphi) = \varepsilon^\vartheta(\varphi,\vartheta)$, where "$\wedge$" denotes the vector product defined by

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \wedge \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} := \begin{pmatrix} x_2y_3 - x_3y_2 \\ x_3y_1 - x_1y_3 \\ x_1y_2 - x_2y_1 \end{pmatrix}.$$

The separation of the common differential operators into their normal (radial) part and their tangential (angular) part is well–known (see, for instance, [11]). We get, using the representation $x = r\xi$, $r = |x|$, $\xi \in \Omega$ for $x \in \mathbb{R}^3 \setminus \{0\}$, for the gradient

$$\operatorname{grad} = \nabla = \varepsilon^r\frac{\partial}{\partial r} + \frac{1}{r}\nabla^*,$$

where the angular part

$$\nabla^* = \varepsilon^\varphi\frac{1}{\sin\vartheta}\frac{\partial}{\partial\varphi} - \varepsilon^\vartheta\frac{\partial}{\partial\vartheta}$$

is called surface gradient, and for the Laplace operator

$$\Delta := \left(\frac{\partial}{\partial x_1}\right)^2 + \left(\frac{\partial}{\partial x_2}\right)^2 + \left(\frac{\partial}{\partial x_3}\right)^2 = \left(\frac{\partial}{\partial r}\right)^2 + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\Delta^*,$$

where

$$\Delta^* = \frac{1}{\sin\vartheta}\frac{\partial}{\partial\vartheta}\sin\vartheta\frac{\partial}{\partial\vartheta} + \frac{1}{\sin^2\vartheta}\left(\frac{\partial}{\partial\varphi}\right)^2$$

is called Beltrami operator.
The nabla operator ∇ allows the representation of the divergence, the curl, and the Jacobian

matrix as well as the Laplace operator formally by a scalar product, a vector product or a tensor product in the sense that

$$\begin{aligned} \operatorname{div} f &:= \nabla \cdot f, \\ \operatorname{curl} f &:= \nabla \wedge f, \\ \mathrm{D} f &:= (\nabla \otimes f)^{\mathrm{T}}, \\ \Delta F &:= \nabla \cdot \nabla F, \end{aligned}$$

provided that $f : S \to \mathbb{R}^3$ and $F : S \to \mathbb{R}$ ($S \subset \mathbb{R}^3$) are sufficiently often differentiable. Note that the tensor product is defined as

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \otimes \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} := \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} (y_1, y_2, y_3) = \begin{pmatrix} x_1y_1 & x_1y_2 & x_1y_3 \\ x_2y_1 & x_2y_2 & x_2y_3 \\ x_3y_1 & x_3y_2 & x_3y_3 \end{pmatrix}$$

or, briefly,

$$(x_i)_{i=1,2,3} \otimes (y_j)_{j=1,2,3} := (x_i y_j)_{i,j=1,2,3} \, .$$

In general, if $A = (a_{i_1 \ldots i_n})_{i_1,\ldots,i_n=1,2,3}$ and $B = (b_{j_1 \ldots j_m})_{j_1,\ldots,j_m=1,2,3}$ are tensors of order n and m, respectively, we denote the tensor product $A \otimes B$ by the tensor of order $n+m$ given by

$$A \otimes B := (a_{i_1 \ldots i_n} b_{j_1 \ldots j_m})_{i_1,\ldots,i_n,j_1,\ldots,j_m=1,2,3}$$

and the double–dot product $A : B$ by

$$A : B := \left(\sum_{k_1=1}^{3} \cdots \sum_{k_m=1}^{3} a_{i_1 \ldots i_{n-m} k_1 \ldots k_m} b_{k_1 \ldots k_m} \right)_{i_1,\ldots,i_{n-m}=1,2,3}$$

if $n \geq m$ and

$$A : B := \left(\sum_{k_1=1}^{3} \cdots \sum_{k_n=1}^{3} a_{k_1 \ldots k_n} b_{k_1 \ldots k_n j_1 \ldots j_{m-n}} \right)_{j_1,\ldots,j_{m-n}=1,2,3}$$

if $n < m$.

Obviously, if $c_1, \ldots, c_n, d_1, \ldots, d_n \in \mathbb{R}^3$ are arbitrary vectors, then

$$(c_1 \otimes \cdots \otimes c_n) : (d_1 \otimes \cdots \otimes d_n) = \prod_{i=1}^{n} (c_i \cdot d_i) \, .$$

For further details on the calculation with tensors, we refer to, for example, [8].

Moreover, the application of the divergence operator to a matrix–valued function is meant rowwise in the sense that

$$\operatorname{div} (F_{ij})_{i,j=1,2,3} := \left(\sum_{j=1}^{3} \frac{\partial F_{ij}}{\partial x_j} \right)_{i=1,2,3}$$

and the application of the Laplace operator to a vector field is meant componentwise.

3 Spherical Self–Reinforced Linearly Elastic Model

According to [15] the Cauchy stress tensor of a self–reinforced linearly elastic medium observing a motion $u(x,t)$ (x: space, t: time) can be modelled by

$$\tau_{ij} = \lambda \sum_{k=1}^{3} e_{kk}\delta_{ij} + 2\mu_{\mathrm{T}} e_{ij} + \alpha\left(\sum_{k=1}^{3}\sum_{m=1}^{3} a_k a_m e_{km}\delta_{ij} + \sum_{k=1}^{3} e_{kk} a_i a_j\right) \tag{1}$$
$$+2(\mu_{\mathrm{L}} - \mu_{\mathrm{T}})\left(a_i \sum_{k=1}^{3} a_k e_{kj} + a_j \sum_{k=1}^{3} a_k e_{ki}\right) + \beta \sum_{k=1}^{3}\sum_{m=1}^{3} a_k a_m e_{km} a_i a_j,$$

where $2e_{ij} = \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}$, $u_i := u \cdot \varepsilon^i$, $a = (a_1, a_2, a_3)^{\mathrm{T}} \in \Omega$ is the preferred direction which may depend on x, and the quantities $\lambda, \mu_{\mathrm{T}}, \mu_{\mathrm{L}}, \alpha, \beta \in \mathbb{R}$ are constants. More precisely, we can regard the Cauchy stress tensor τ as an operator acting on u and write $\tau_{ij}(u)$ in (1).
The governing equation in this context is Euler's equation of motion without body forces (see also [13]), i.e. the partial differential equation

$$\operatorname{div} \tau(u) = \rho \frac{\partial^2 u}{\partial t^2}, \tag{2}$$

where ρ denotes the mass density function.
Using the nabla nomenclature for the spatial derivatives we can reformulate (1) as

$$\begin{aligned} \tau \;=\; & \lambda(\operatorname{div} u)\mathbf{i} + \mu_{\mathrm{T}}\left((\nabla \otimes u)^{\mathrm{T}} + \nabla \otimes u\right) + \alpha\left(\left(a^{\mathrm{T}}(\nabla \otimes u)\, a\right)\mathbf{i} + (\operatorname{div} u) a \otimes a\right) \\ & +(\mu_{\mathrm{L}} - \mu_{\mathrm{T}})\left(a \otimes \left(\left((\nabla \otimes u)^{\mathrm{T}} + \nabla \otimes u\right) a\right) + \left(\left((\nabla \otimes u)^{\mathrm{T}} + \nabla \otimes u\right) a\right) \otimes a\right) \\ & +\beta\left(a^{\mathrm{T}}(\nabla \otimes u)\, a\right) a \otimes a, \end{aligned} \tag{3}$$

where $\mathbf{i} = (\delta_{ij})_{i,j=1,2,3}$ denotes the identity matrix in $\mathbb{R}^{3\times 3}$. Insertion into (2) yields

$$\begin{aligned} \rho\frac{\partial^2 u_i}{\partial t^2} = & \lambda \sum_{j=1}^{3} \frac{\partial}{\partial x_j} \operatorname{div} u\, \delta_{ij} + \mu_{\mathrm{T}} \sum_{j=1}^{3}\left(\frac{\partial^2 u_i}{\partial x_j^2} + \frac{\partial^2 u_j}{\partial x_j \partial x_i}\right) \\ & +\alpha\left(\sum_{j=1}^{3}\frac{\partial}{\partial x_j}\left(\sum_{k,l=1}^{3} a_k \frac{\partial u_l}{\partial x_k} a_l\right)\delta_{ij} + \sum_{j=1}^{3}\left(\frac{\partial}{\partial x_j}\operatorname{div} u\right) a_i a_j + \operatorname{div} u\left(\sum_{j=1}^{3}\left(\frac{\partial a_i}{\partial x_j} a_j + a_i \frac{\partial a_j}{\partial x_j}\right)\right)\right) \\ & +(\mu_{\mathrm{L}} - \mu_{\mathrm{T}})\left(a_i \sum_{j=1}^{3}\sum_{k=1}^{3}\left(\frac{\partial^2 u_j}{\partial x_j \partial x_k} + \frac{\partial^2 u_k}{\partial x_j^2}\right) a_k + \sum_{j=1}^{3}\sum_{k=1}^{3}\left(\frac{\partial^2 u_i}{\partial x_j \partial x_k} + \frac{\partial^2 u_k}{\partial x_j \partial x_i}\right) a_k a_j\right) \\ & +(\mu_{\mathrm{L}} - \mu_{\mathrm{T}})\left(\sum_{j=1}^{3}\sum_{k=1}^{3}\left(\frac{\partial u_j}{\partial x_k} + \frac{\partial u_k}{\partial x_j}\right)\frac{\partial}{\partial x_j}(a_i a_k) + \sum_{k=1}^{3}\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\right)\sum_{j=1}^{3}\frac{\partial}{\partial x_j}(a_k a_j)\right) \\ & +\beta \sum_{j,k,m=1}^{3} a_k a_m \frac{\partial^2 u_m}{\partial x_j \partial x_k} a_i a_j + \beta \sum_{k,m=1}^{3}\frac{\partial u_m}{\partial x_k}\left(\sum_{j=1}^{3}\frac{\partial}{\partial x_j}(a_k a_m a_i a_j)\right) \qquad \forall i \in \{1,2,3\} \end{aligned}$$

$$
\begin{aligned}
&\Leftrightarrow \rho\frac{\partial^2 u}{\partial t^2} = \lambda \operatorname{grad}\operatorname{div} u + \mu_{\mathrm{T}}(\Delta u + \operatorname{grad}\operatorname{div} u) \\
&\quad +\alpha\left(\nabla\left(a^{\mathrm{T}}(\nabla\otimes u)a\right) + (a\cdot\operatorname{grad}\operatorname{div} u)a + \operatorname{div} u\left((\nabla\otimes a)^{\mathrm{T}}a + (\operatorname{div} a)a\right)\right) \\
&\quad +(\mu_{\mathrm{L}}-\mu_{\mathrm{T}})((a\cdot\operatorname{grad}\operatorname{div} u)a + (a\cdot(\Delta u))a + (a\otimes a):(\nabla\otimes\nabla\otimes u) + (\nabla\otimes\nabla\otimes u):(a\otimes a)) \\
&\quad +(\mu_{\mathrm{L}}-\mu_{\mathrm{T}})\left((\nabla\otimes u):(a\otimes(\nabla\otimes a)) + a\left((\nabla\otimes u)^{\mathrm{T}}:(\nabla\otimes a)\right)\right) \\
&\quad +(\mu_{\mathrm{L}}-\mu_{\mathrm{T}})\left((\nabla\otimes u)^{\mathrm{T}}:(a\otimes(\nabla\otimes a)) + a((\nabla\otimes u):(\nabla\otimes a))\right) \\
&\quad +(\mu_{\mathrm{L}}-\mu_{\mathrm{T}})\left(\left((\nabla\otimes u)^{\mathrm{T}} + (\nabla\otimes u)\right)(\operatorname{div}(a\otimes a))\right) \\
&\quad +\beta((\nabla\otimes\nabla\otimes u):(a\otimes a\otimes a))a + \beta(\operatorname{div} a)(a\otimes a\otimes a):(\nabla\otimes u) \\
&\quad +\beta\left(\left(\nabla\otimes u + (\nabla\otimes u)^{\mathrm{T}}\right):((a(\nabla\otimes a))\otimes a)\right)a + \beta\left(\left((\nabla\otimes a)^{\mathrm{T}}a\right)\otimes a\otimes a\right):(\nabla\otimes u)\ . \qquad (4)
\end{aligned}
$$

4 A Wave Ansatz

Motivated by the results derived in [6] and [5] for the half–space we will use the ansatz

$$u(r,\vartheta,\varphi,t) := A(r,\vartheta)\exp(i(\omega t - k_r r - k_\vartheta \vartheta))\varepsilon^\varphi(\varphi); \qquad (5)$$

$k_r, k_\vartheta \in \mathbb{R}$ constant; for an (SH–)wave propagating in ϑ– and r–direction and oscillating in ϑ–direction. Inserting (5) into (4) yields the following identities for the expressions occurring in (4):

$$
\begin{aligned}
\operatorname{div} u &= \left(\varepsilon^r\frac{\partial}{\partial r} + \frac{1}{r}\left(\varepsilon^\varphi\frac{1}{\sin\vartheta}\frac{\partial}{\partial\varphi} - \varepsilon^\vartheta\frac{\partial}{\partial\vartheta}\right)\right)\cdot u = 0; \\
\Delta u &= \left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\left(\frac{1}{\sin\vartheta}\frac{\partial}{\partial\vartheta}\sin\vartheta\frac{\partial}{\partial\vartheta} + \frac{1}{\sin^2\vartheta}\frac{\partial^2}{\partial\varphi^2}\right)\right)u \\
&= \left(\frac{\partial^2 A}{\partial r^2} - 2ik_r\frac{\partial A}{\partial r} - k_r^2 A + \frac{2}{r}\frac{\partial A}{\partial r} - \frac{2}{r}Aik_r\right)E\varepsilon^\varphi \\
&\quad + \frac{1}{r^2\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\left(\frac{\partial A}{\partial\vartheta} - Aik_\vartheta\right)E\right)\varepsilon^\varphi + \frac{1}{r^2\sin^2\vartheta}AE\,(-\varepsilon^\varphi) \\
&= \left(\frac{\partial^2 A}{\partial r^2} - 2ik_r\frac{\partial A}{\partial r} - k_r^2 A + \frac{2}{r}\frac{\partial A}{\partial r} - \frac{2}{r}Aik_r\right)E\varepsilon^\varphi \\
&\quad + \frac{1}{r^2\sin\vartheta}\left(\cos\vartheta\left(\frac{\partial A}{\partial\vartheta} - Aik_\vartheta\right) + \sin\vartheta\left(\frac{\partial^2 A}{\partial\vartheta^2} - 2\frac{\partial A}{\partial\vartheta}ik_\vartheta - Ak_\vartheta^2\right)\right)E\varepsilon^\varphi \\
&\quad - \frac{1}{r^2\sin^2\vartheta}AE\varepsilon^\varphi, \\
\nabla\otimes u &= \left(\varepsilon^r\frac{\partial}{\partial r} + \frac{1}{r}\left(\varepsilon^\varphi\frac{1}{\sin\vartheta}\frac{\partial}{\partial\varphi} - \varepsilon^\vartheta\frac{\partial}{\partial\vartheta}\right)\right)\otimes u \\
&= \left(\frac{\partial A}{\partial r} - ik_r A\right)E\varepsilon^r\otimes\varepsilon^\varphi + \frac{1}{r\sin\vartheta}AE\varepsilon^\varphi\otimes\frac{\mathrm{d}\varepsilon^\varphi}{\mathrm{d}\varphi} - \frac{1}{r}\left(\frac{\partial A}{\partial\vartheta} - ik_\vartheta A\right)E\varepsilon^\vartheta\otimes\varepsilon^\varphi, \\
\nabla\otimes\nabla\otimes u &= \left(\varepsilon^r\frac{\partial}{\partial r} + \frac{1}{r}\left(\varepsilon^\varphi\frac{1}{\sin\vartheta}\frac{\partial}{\partial\varphi} - \varepsilon^\vartheta\frac{\partial}{\partial\vartheta}\right)\right)\otimes(\nabla\otimes u)
\end{aligned}
$$

$$
\begin{aligned}
= \; & \left(\frac{\partial^2 A}{\partial r^2} - 2ik_r\frac{\partial A}{\partial r} - k_r^2 A\right) E\varepsilon^r\otimes\varepsilon^r\otimes\varepsilon^\varphi \\
& + \left(-\frac{1}{r^2\sin\vartheta}\,A + \frac{1}{r\sin\vartheta}\,\frac{\partial A}{\partial r} - \frac{1}{r\sin\vartheta}\,Aik_r\right) E\varepsilon^r\otimes\varepsilon^\varphi\otimes\frac{\mathrm{d}\varepsilon^\varphi}{\mathrm{d}\varphi} \\
& + \left(\frac{1}{r^2}\left(\frac{\partial A}{\partial\vartheta} - ik_\vartheta A\right) - \frac{1}{r}\left(\frac{\partial^2 A}{\partial r\partial\vartheta} - ik_\vartheta\frac{\partial A}{\partial r} - ik_r\frac{\partial A}{\partial\vartheta} - k_rk_\vartheta A\right)\right) E\varepsilon^r\otimes\varepsilon^\vartheta\otimes\varepsilon^\varphi \\
& + \frac{1}{r\sin\vartheta}\left(\left(\frac{\partial A}{\partial r} - ik_r A\right) E\left(\varepsilon^\varphi\otimes\frac{\mathrm{d}\varepsilon^r}{\mathrm{d}\varphi}\otimes\varepsilon^\varphi + \varepsilon^\varphi\otimes\varepsilon^r\otimes\frac{\mathrm{d}\varepsilon^\varphi}{\mathrm{d}\varphi}\right)\right. \\
& \qquad + \frac{1}{r\sin\vartheta}\,AE\left(\varepsilon^\varphi\otimes\frac{\mathrm{d}\varepsilon^\varphi}{\mathrm{d}\varphi}\otimes\frac{\mathrm{d}\varepsilon^\varphi}{\mathrm{d}\varphi} - \varepsilon^\varphi\otimes\varepsilon^\varphi\otimes\varepsilon^\varphi\right) \\
& \qquad \left. - \frac{1}{r}\left(\frac{\partial A}{\partial\vartheta} - ik_\vartheta A\right) E\left(\varepsilon^\varphi\otimes\frac{\mathrm{d}\varepsilon^\vartheta}{\mathrm{d}\varphi}\otimes\varepsilon^\varphi + \varepsilon^\varphi\otimes\varepsilon^\vartheta\otimes\frac{\mathrm{d}\varepsilon^\varphi}{\mathrm{d}\varphi}\right)\right) \\
& - \frac{1}{r}\left(\left(\frac{\partial^2 A}{\partial\vartheta\partial r} - ik_r\frac{\partial A}{\partial\vartheta} - ik_\vartheta\frac{\partial A}{\partial r} - k_rk_\vartheta A\right) E\varepsilon^\vartheta\otimes\varepsilon^r\otimes\varepsilon^\varphi\right. \\
& \qquad + \left(\frac{\partial A}{\partial r} - ik_r A\right) E\varepsilon^\vartheta\otimes\frac{\mathrm{d}\varepsilon^r}{\mathrm{d}\vartheta}\otimes\varepsilon^\varphi \\
& \qquad + \left(-\frac{\cos\vartheta}{r\sin^2\vartheta}\,A + \frac{1}{r\sin\vartheta}\,\frac{\partial A}{\partial\vartheta} - ik_\vartheta\,\frac{1}{r\sin\vartheta}\,A\right) E\varepsilon^\vartheta\otimes\varepsilon^\varphi\otimes\frac{\mathrm{d}\varepsilon^\varphi}{\mathrm{d}\varphi} \\
& \qquad - \frac{1}{r}\left(\frac{\partial^2 A}{\partial\vartheta^2} - 2ik_\vartheta\frac{\partial A}{\partial\vartheta} - k_\vartheta^2 A\right) E\varepsilon^\vartheta\otimes\varepsilon^\vartheta\otimes\varepsilon^\varphi \\
& \qquad \left. - \frac{1}{r}\left(\frac{\partial A}{\partial\vartheta} - ik_\vartheta A\right) E\varepsilon^\vartheta\otimes\frac{\mathrm{d}\varepsilon^\vartheta}{\mathrm{d}\vartheta}\otimes\varepsilon^\varphi\right),
\end{aligned}
$$

where we used the abbreviation

$$E(r,\vartheta,t) := \exp(i(\omega t - k_r r - k_\vartheta\vartheta)).$$

Thus, the choice of $a = \varepsilon^\vartheta$ as preferred direction yields

$$
\begin{aligned}
a\cdot\Delta u &= 0, \\
(\nabla\otimes\nabla\otimes u) : (a\otimes a) &= \frac{1}{r^2\sin^2\vartheta}\,AE\cos^2\vartheta\,\varepsilon^\varphi - \frac{1}{r^2\sin\vartheta}\left(\frac{\partial A}{\partial\vartheta} - ik_\vartheta A\right) E\cos\vartheta\,\varepsilon^\varphi, \\
(a\otimes a) : (\nabla\otimes\nabla\otimes u) &= \frac{1}{r}\left(\frac{\partial A}{\partial r} - ik_r A\right) E\varepsilon^\varphi + \frac{1}{r^2}\left(\frac{\partial^2 A}{\partial\vartheta^2} - 2ik_\vartheta\frac{\partial A}{\partial\vartheta} - k_\vartheta^2 A\right) E\varepsilon^\varphi, \\
(\nabla\otimes\nabla\otimes u) : (a\otimes a\otimes a) &= 0, \\
a^{\mathrm{T}}(\nabla\otimes u)a &= (\nabla\otimes u) : (a\otimes a) = (a\otimes a) : (\nabla\otimes u) = 0, \\
\nabla\otimes a &= \left(\varepsilon^r\frac{\partial}{\partial r} + \frac{1}{r}\left(\varepsilon^\varphi\,\frac{1}{\sin\vartheta}\,\frac{\partial}{\partial\varphi} - \varepsilon^\vartheta\,\frac{\partial}{\partial\vartheta}\right)\right)\otimes\varepsilon^\vartheta \\
&= \frac{1}{r\sin\vartheta}\,\varepsilon^\varphi\otimes\frac{\partial\varepsilon^\vartheta}{\partial\varphi} - \frac{1}{r}\varepsilon^\vartheta\otimes\frac{\partial\varepsilon^\vartheta}{\partial\vartheta} \\
&= -\frac{\cos\vartheta}{r\sin\vartheta}\,\varepsilon^\varphi\otimes\varepsilon^\varphi - \frac{1}{r}\,\varepsilon^\vartheta\otimes\varepsilon^r,
\end{aligned}
$$

$$
\begin{aligned}
(\nabla\otimes a)^{\mathrm{T}}a &= -\frac{1}{r}\,\varepsilon^r,\\
(\nabla\otimes u):\Big(\big((\nabla\otimes a)^{\mathrm{T}}a\big)\otimes a\Big) &= (\nabla\otimes u):\left(-\frac{1}{r}\,\varepsilon^r\otimes\varepsilon^\vartheta\right)=0,\\
(\nabla\otimes u)^{\mathrm{T}}:\Big(\big((\nabla\otimes a)^{\mathrm{T}}a\big)\otimes a\Big) &= 0,\\
(\nabla\otimes u):(a\otimes\nabla\otimes a) &= \frac{\cot\vartheta}{r^2}\left(\frac{\partial A}{\partial\vartheta}-ik_\vartheta A\right)E\varepsilon^\varphi,\\
(\nabla\otimes u)^{\mathrm{T}}:(a\otimes\nabla\otimes a) &= -\frac{\cos^2\vartheta}{r^2\sin^2\vartheta}\,AE\varepsilon^\varphi=-\frac{\cot^2\vartheta}{r^2}\,AE\varepsilon^\varphi,\\
(\nabla\otimes u)^{\mathrm{T}}:(\nabla\otimes a) &= 0,\\
(\nabla\otimes u):(\nabla\otimes a) &= 0,\\
\nabla\cdot a &= \left(\varepsilon^r\frac{\partial}{\partial r}+\frac{1}{r}\left(\varepsilon^\varphi\,\frac{1}{\sin\vartheta}\,\frac{\partial}{\partial\varphi}-\varepsilon^\vartheta\,\frac{\partial}{\partial\vartheta}\right)\right)\cdot\varepsilon^\vartheta\\
&= \frac{1}{r\sin\vartheta}\,\varepsilon^\varphi\cdot\frac{\partial\varepsilon^\vartheta}{\partial\varphi}-\frac{1}{r}\,\varepsilon^\vartheta\cdot\frac{\partial\varepsilon^\vartheta}{\partial\vartheta}=-\,\frac{\cot\vartheta}{r},\\
\operatorname{div}(a\otimes a) &= \left(\sum_{j=1}^{3}\frac{\partial}{\partial x_j}\,(a_ia_j)\right)_{i=1,2,3}=\left(\sum_{j=1}^{3}\left(\frac{\partial a_i}{\partial x_j}a_j+a_i\frac{\partial a_j}{\partial x_j}\right)\right)_{i=1,2,3}\\
&= (\nabla\otimes a)^{\mathrm{T}}a+a\operatorname{div}a=-\,\frac{1}{r}\,\varepsilon^r-\frac{\cot\vartheta}{r}\varepsilon^\vartheta,\\
(\nabla\otimes u)^{\mathrm{T}}(\operatorname{div}(a\otimes a)) &= -\frac{1}{r}\left(\frac{\partial A}{\partial r}-ik_rA\right)E\varepsilon^\varphi+\frac{\cot\vartheta}{r^2}\left(\frac{\partial A}{\partial\vartheta}-ik_\vartheta A\right)E\varepsilon^\varphi,\\
(\nabla\otimes u)(\operatorname{div}(a\otimes a)) &= \frac{1}{r^2}\,AE\varepsilon^\varphi-\frac{\cot^2\vartheta}{r^2}AE\varepsilon^\varphi.
\end{aligned}
$$

Hence, Equation (4) becomes in case of the ansatz in (5) with $a=\varepsilon^\vartheta$:

$$
\begin{aligned}
\rho(-\omega^2)A\varepsilon^\varphi = \;& \mu_{\mathrm{T}}\left(\frac{\partial^2A}{\partial r^2}-2ik_r\frac{\partial A}{\partial r}-k_r^2A+\frac{2}{r}\frac{\partial A}{\partial r}-\frac{2}{r}ik_rA\right.\\
&\left.+\frac{\cot\vartheta}{r^2}\left(\frac{\partial A}{\partial\vartheta}-Aik_\vartheta\right)+\frac{1}{r^2}\left(\frac{\partial^2A}{\partial\vartheta^2}-2\frac{\partial A}{\partial\vartheta}\,ik_\vartheta-Ak_\vartheta^2\right)-\frac{1}{r^2\sin^2\vartheta}\,A\right)\varepsilon^\varphi\\
&+(\mu_{\mathrm{L}}-\mu_{\mathrm{T}})\left(\frac{1}{r}\left(\frac{\partial A}{\partial r}-ik_rA\right)+\frac{1}{r^2}\left(\frac{\partial^2A}{\partial\vartheta^2}-2ik_\vartheta\frac{\partial A}{\partial\vartheta}-k_\vartheta^2A\right)\right)\varepsilon^\varphi\\
&+(\mu_{\mathrm{L}}-\mu_{\mathrm{T}})\,\frac{1}{r^2\sin\vartheta}\left(A\,\frac{\cos^2\vartheta}{\sin\vartheta}-\frac{\partial A}{\partial\vartheta}\cos\vartheta+ik_\vartheta A\cos\vartheta\right)\varepsilon^\varphi\\
&+(\mu_{\mathrm{L}}-\mu_{\mathrm{T}})\,\frac{\cot\vartheta}{r^2}\left(\frac{\partial A}{\partial\vartheta}-ik_\vartheta A-\cot\vartheta\,A\right)\varepsilon^\varphi\\
&+(\mu_{\mathrm{L}}-\mu_{\mathrm{T}})\,\frac{1}{r}\left(ik_rA-\frac{\partial A}{\partial r}+\frac{\cot\vartheta}{r}\left(\frac{\partial A}{\partial\vartheta}-ik_\vartheta A\right)+\frac{1}{r}\,A-\frac{\cot^2\vartheta}{r}\,A\right)\varepsilon^\varphi.
\end{aligned}
$$

Collecting the terms with respect to the different derivatives of A we get

$$
\begin{aligned}
0 \;=\;& \mu_{\mathrm{T}}\,\frac{\partial^2 A}{\partial r^2} + \left(\frac{\mu_{\mathrm{T}}}{r^2} + \frac{\mu_{\mathrm{L}}-\mu_{\mathrm{T}}}{r^2}\right)\frac{\partial^2 A}{\partial \vartheta^2} \\
&+ \left(-2\mu_{\mathrm{T}} i k_r + 2\,\frac{\mu_{\mathrm{T}}}{r} + \frac{\mu_{\mathrm{L}}-\mu_{\mathrm{T}}}{r} - \frac{\mu_{\mathrm{L}}-\mu_{\mathrm{T}}}{r}\right)\frac{\partial A}{\partial r} \\
&+ \left(\mu_{\mathrm{T}}\,\frac{\cot\vartheta}{r^2} - 2i\mu_{\mathrm{T}}\,\frac{k_\vartheta}{r^2} - 2ik_\vartheta\frac{\mu_{\mathrm{L}}-\mu_{\mathrm{T}}}{r^2} - \frac{\mu_{\mathrm{L}}-\mu_{\mathrm{T}}}{r^2}\,\cot\vartheta \right.\\
&\qquad \left. + \frac{\mu_{\mathrm{L}}-\mu_{\mathrm{T}}}{r^2}\,\cot\vartheta + \frac{\mu_{\mathrm{L}}-\mu_{\mathrm{T}}}{r^2}\,\cot\vartheta\right)\frac{\partial A}{\partial\vartheta} \\
&+ \left(\rho\omega^2 - \mu_{\mathrm{T}}\left(k_r^2 + \frac{2}{r}\,ik_r + \frac{\cot\vartheta}{r^2}\,ik_\vartheta + \frac{k_\vartheta^2}{r^2} + \frac{1}{r^2\sin^2\vartheta}\right)\right.\\
&\qquad -(\mu_{\mathrm{L}}-\mu_{\mathrm{T}})\left(i\,\frac{k_r}{r} + \frac{k_\vartheta^2}{r^2} - \frac{\cot^2\vartheta}{r^2} - ik_\vartheta\,\frac{\cot\vartheta}{r^2} + ik_\vartheta\,\frac{\cot\vartheta}{r^2} + \frac{\cot^2\vartheta}{r^2}\right.\\
&\qquad\qquad \left.\left. -i\,\frac{k_r}{r} + ik_\vartheta\,\frac{\cot\vartheta}{r^2} - \frac{1}{r^2} + \frac{\cot^2\vartheta}{r^2}\right)\right) A \\
=\;& \mu_{\mathrm{T}}\frac{\partial^2 A}{\partial r^2} + \frac{\mu_{\mathrm{L}}}{r^2}\,\frac{\partial^2 A}{\partial\vartheta^2} + 2\mu_{\mathrm{T}}\left(\frac{1}{r} - ik_r\right)\frac{\partial A}{\partial r} \\
&+ \frac{\mu_{\mathrm{L}}}{r^2}\left(\cot\vartheta - 2ik_\vartheta\right)\frac{\partial A}{\partial\vartheta} \\
&+ \left(\rho\omega^2 - \mu_{\mathrm{T}}\left(k_r^2 + \frac{2}{r}\,ik_r + \frac{1}{r^2\sin^2\vartheta}\right) - \mu_{\mathrm{L}}\,\frac{k_\vartheta}{r^2}\left(i\cot\vartheta + k_\vartheta\right) + \frac{\mu_{\mathrm{L}}-\mu_{\mathrm{T}}}{r^2}\left(1-\cot^2\vartheta\right)\right) A.
\end{aligned}
$$

MAPLE derives the following system of solutions

$$
\begin{aligned}
A(r,\vartheta) =\;& r^{-1/2}e^{irk_r}\left(\gamma_1 J_{\frac{\sqrt{\mu_{\mathrm{T}}+4c_1}}{2\sqrt{\mu_{\mathrm{T}}}}}\left(\sqrt{\frac{\rho\omega^2}{\mu_{\mathrm{T}}}}\,r\right) + \gamma_2 Y_{\frac{\sqrt{\mu_{\mathrm{T}}+4c_1}}{2\sqrt{\mu_{\mathrm{T}}}}}\left(\sqrt{\frac{\rho\omega^2}{\mu_{\mathrm{T}}}}\,r\right)\right) \\
&\times e^{ik_\vartheta\vartheta} F\left(-z_{1,2}+\frac{3}{2}, \frac{3}{2}+z_{1,2}; 2; \frac{1}{2}\left(1+\cos\vartheta\right)\right)\left(i\sin\vartheta\right)^{3/2} \\
&\times \left(\gamma_3 + i\,\gamma_4 \int \left(\left(F\left(-z_{1,2}+\frac{3}{2}, \frac{3}{2}+z_{1,2}; 2; \frac{1}{2}\left(1+\cos\theta\right)\right)\right)^2 i\,\sin^3\theta\right)^{-1}\mathrm{d}\theta\right)\left(\sin\vartheta\right)^{-1/2},
\end{aligned}
$$

where $z_{1,2}$ stands for the roots of the polynomial

$$4\mu_{\mathrm{L}}z^2 - 9\mu_{\mathrm{L}} - 4c_1 + 8\mu_{\mathrm{T}},$$

given by

$$z_{1,2} = \pm\sqrt{\frac{9\mu_{\mathrm{L}} + 4c_1 - 8\mu_{\mathrm{T}}}{4\mu_{\mathrm{L}}}}.$$

Here, $\gamma_1, \gamma_2, \gamma_3, \gamma_4, c_1$ are constants, J_ν and Y_ν, respectively are the Bessel functions of 1st and 2nd kind, respectively, and order ν, and $F(a,b;c;x)$ is the hypergeometric function given by

$$F(a,b;c;x) = \sum_{k=0}^{\infty}\frac{\Gamma(a+k)\Gamma(b+k)\Gamma(c)}{\Gamma(a)\Gamma(b)\Gamma(c+k)}\,\frac{x^k}{k!}$$

with Γ denoting the Gamma function. For further details on those special functions we refer to, for example, [1] and [12]. According to [12], p. 53, the identity

$$F\left(a,b;\frac{a+b+1}{2};x\right)=\Gamma\left(\frac{a+b+1}{2}\right)\left(x-x^2\right)^{\frac{1}{4}-\frac{a}{4}-\frac{b}{4}}P^{\frac{1}{2}-\frac{a+b}{2}}_{\frac{a}{2}-\frac{b+1}{2}}(1-2x)$$

holds for $0<x<1$ where P^{μ}_{ν} stands for the associated Legendre functions. In our case we, therefore, get

$$\begin{aligned}
&F\left(-z_{1,2}+\frac{3}{2},\frac{3}{2}+z_{1,2};2;\frac{1}{2}\left(1+\cos\theta\right)\right)\\
&= \Gamma(2)\left(\frac{1}{2}(1+\cos\theta)-\left(\frac{1}{2}(1+\cos\theta)\right)^2\right)^{\frac{1}{4}-\frac{3}{4}}P^{-1}_{-z_{1,2}-\frac{1}{2}}(-\cos\theta)\\
&= \left(\frac{1}{2}+\frac{1}{2}\cos\theta-\frac{1}{4}-\frac{1}{2}\cos\theta-\frac{1}{4}\cos^2\theta\right)^{-1/2}P^{-1}_{-z_{1,2}-\frac{1}{2}}(-\cos\theta)\\
&= \left(\frac{1}{4}-\frac{1}{4}\cos^2\theta\right)^{-1/2}P^{-1}_{-z_{1,2}-\frac{1}{2}}(-\cos\theta)\\
&= \frac{2}{\sin\theta}P^{-1}_{-z_{1,2}-\frac{1}{2}}(-\cos\theta).
\end{aligned}$$

The derived solution system now has the form

$$\begin{aligned}
u(r,\vartheta,\varphi,t) &= r^{-1/2}e^{irk_r}\left(\gamma_1 J_{\frac{\sqrt{\mu_{\mathrm{T}}+4c_1}}{2\sqrt{\mu_{\mathrm{T}}}}}\left(\sqrt{\frac{\rho\omega^2}{\mu_{\mathrm{T}}}}r\right)+\gamma_2 Y_{\frac{\sqrt{\mu_{\mathrm{T}}+4c_1}}{2\sqrt{\mu_{\mathrm{T}}}}}\left(\sqrt{\frac{\rho\omega^2}{\mu_{\mathrm{T}}}}r\right)\right)\\
&\quad\times e^{ik_\vartheta\vartheta}\frac{2}{\sin\vartheta}P^{-1}_{-z_{1,2}-\frac{1}{2}}(-\cos\vartheta)(i\sin\vartheta)^{3/2}\\
&\quad\times\left(\gamma_3+\gamma_4\int\left(\left(\frac{2}{\sin\theta}P^{-1}_{-z_{1,2}-\frac{1}{2}}(-\cos\theta)\right)^2\sin^3\theta\right)^{-1}d\theta\right)(\sin\vartheta)^{-1/2}\\
&\quad\times\exp(i(\omega t-k_r r-k_\vartheta\vartheta))\varepsilon^{\varphi}(\varphi)\\
&= 2r^{-1/2}e^{i(\omega t+3\pi/4)}\left(\gamma_1 J_{\frac{\sqrt{\mu_{\mathrm{T}}+4c_1}}{2\sqrt{\mu_{\mathrm{T}}}}}\left(\sqrt{\frac{\rho\omega^2}{\mu_{\mathrm{T}}}}r\right)+\gamma_2 Y_{\frac{\sqrt{\mu_{\mathrm{T}}+4c_1}}{2\sqrt{\mu_{\mathrm{T}}}}}\left(\sqrt{\frac{\rho\omega^2}{\mu_{\mathrm{T}}}}r\right)\right)\\
&\quad\times P^{-1}_{\pm\sqrt{\frac{9\mu_{\mathrm{L}}+4c_1-8\mu_{\mathrm{T}}}{4\mu_{\mathrm{L}}}}-\frac{1}{2}}(-\cos\vartheta)\\
&\quad\times\left(\gamma_3+\gamma_4\int\left(\left(2P^{-1}_{\pm\sqrt{\frac{9\mu_{\mathrm{L}}+4c_1-8\mu_{\mathrm{T}}}{4\mu_{\mathrm{L}}}}-\frac{1}{2}}(-\cos\theta)\right)^2\sin\theta\right)^{-1}d\theta\right)\varepsilon^{\varphi}(\varphi)
\end{aligned}$$

Note that the obtained solution system is independent of k_r and k_ϑ.

5 Conclusions

We have discussed spherical waves propagating in radius– and latitude–direction and oscillating in latitude–direction in case of fibre–reinforced linearly elastic material. Using the notations and the spherical decomposition of the relevant differential operators from [11] we have modified Euler's equation of motion in the discussed case. The derived equation has been analysed with MAPLE yielding a function system depending on certain Bessel and associated Legendre functions. The obtained result is certainly of a pure primary theoretical interest. However, it shows the difficulty of transferring such problems to more complicated geometries than a plane.

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References

[1] L.C. Andrews: *Special Functions for Engineers and Applied Mathematicians*, Macmillan Publishing Company, New York, London, 1985.

[2] A.J. Belfield, T.G. Rogers, A.J.M. Spencer: Stress in Elastic Plates Reinforced by Fibres Lying in Concentric Circles, *J. Mech. Phys. Solids*, **31** (1983), 25-54.

[3] A. Chattopadhyay: Wave Reflection and Refraction in Triclinic Crystalline Media, *Technomathematics Report at the Department of Mathematics of the University of Kaiserslautern*, No. 254, 2003.

[4] A. Chattopadhyay, S. Choudhury: Propagation, Reflection and Transmission of Magnetoelastic Shear Waves in a Self–Reinforced Medium, *Int. J. Eng. Sci.*, **28** (1990), 485–495.

[5] A. Chattopadhyay, S. Choudhury: Magnetoelastic Shear Waves in an Infinite Self–Reinforced Plate, *International Journal for Numerical and Analytical Methods in Geomechanics*, **19** (1995), 289-304.

[6] A. Chattopadhyay, R.L.K. Venkateswarlu, S. Saha: Reflection of quasi–P and quasi–SV Waves at the free and rigid boundaries of a fibre–reinforced medium, *Sādhanā*, **27** (2002), 613-630.

[7] R.M. Christensen, K.H. Lo: Solutions for Effective Shear Properties in Three Phase Sphere and Cylinder Models, *J. Mech. Phys. Solids*, **27** (1979), 315-330.

[8] F.A. Dahlen, J. Tromp: *Theoretical Global Seismology*, Princeton University Press, Princeton, New Jersey, 1998.

[9] A.H. England, T.G. Rogers: Plane Crack Problems for Ideal Fibre–Reinforced Materials, *Q. Jl. Mech. appl. Math.*, **26** (1973), 303-320.

[10] G.C. Everstine, A.C. Pipkin: Boundary Layers in Fiber–Reinforced Materials, *J. appl. Mech.*, **40** (1973), 518-522.

[11] W. Freeden, T. Gervens, M. Schreiner: *Constructive Approximation on the Sphere - With Applications to Geomathematics*, Oxford University Press, Oxford, 1998.

[12] W. Magnus, F. Oberhettinger, R.P. Soni: *Formulas and Theorems for the Special Functions of Mathematical Physics*, Springer–Verlag, New York, 1966.

[13] J.E. Marsden, T.J.R. Hughes: *Mathematical Foundations of Elasticity*, Dover Publications, New York, 1983.

[14] A.C. Pipkin, T.G. Rogers: Plane Deformations of Incompressible Fiber–Reinforced Materials, **38** (1971), 634-640.

[15] A.J.M. Spencer: *Deformations of Fibre–Reinforced Materials*, Oxford University Press, London, 1972.

[16] A.J.M. Spencer: Boundary Layers in Highly Anisotropic Plane Elasticity, *Int. J. Solids Structures*, **10** (1974), 1103-1123.