Hochschild- and Cyclic-Homology of LCNT-Spaces

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Abstract

We define a class of topological spaces (LCNT spaces) which come together with a nuclear Fréchet algebra. Like the algebra of smooth functions on a manifold, this algebra carries the differential structure of the object. We compute the Hochschild homology of this algebra and show that it is isomorphic to the space of differential forms. This is a generalization of a result obtained by Alain Connes in the framework of smooth manifolds.

Keywords: Hochschild homology, cyclic homology, stratified spaces, stratifolds, differential forms, non-commutative geometry

Mathematics Subjects Classification: 16E40, 57N80, 58A12

1 Introduction

In this paper we present a result about the Hochschild- and cyclic homology groups of certain algebras corresponding to certain singular spaces, which we will call locally coned topological spaces or in short “lcnt-spaces”. These algebras represent in some sense the ”smooth” functions on these spaces. The results are a slightly modified version of similar results about locally coned stratifolds (see [Ew], [Kr1] and [Kr2]) which were part of my PhD thesis. Some of the results have already been included in a sectional talk which I gave at the DMV-Jahrestagung 2000 in Dresden. The remaining results were obtained within the years 2000 and 2001 when I was a member of the Topology group at the University of Heidelberg.

Theorem 1. *Let $(X, \mathbb{C}^\infty_X)$ be a finite dimensional lcnt-space, then the antisymmetrization map*

$$\epsilon : \Omega^*_{\mathbb{C}^\infty_X} \to HH_*(\mathbb{C}^\infty_X)$$

is a topological isomorphism from the space of differential forms over the algebra $\mathbb{C}^\infty_X$ to its Hochschild homology.

Theorem 2. *If in addition to the conditions above X has finite dimensional homology groups. Then for all $n \in \mathbb{N}$ there is a natural topological isomorphism*

$$HC_n(\mathbb{C}^\infty_X) \cong \Omega^n_{\mathbb{C}^\infty_X}/d\Omega^{n-1}_{\mathbb{C}^\infty_X} \oplus H^{n-2}(X, \mathbb{C}) \oplus H^{n-4}(X, \mathbb{C}) \oplus H^{n-6}(X, \mathbb{C})...$$

These results are generalizations of results obtained by Alain Connes about the Hochschild- and cyclic homology of the algebra of smooth functions on a compact manifold (see [Co]). Related results in the context of Whitney-stratified spaces have been appeared in [Bra/Pf]. The result in this paper is in a quite general form, but can for example be applied on spaces which are locally coned stratifolds (see [Ew]). It is however not clear in what sense lcnt-spaces are stratified spaces and if Whitney stratified spaces provide examples for lcnt-spaces. The latter should be true under some mild extra conditions for the class of so called pseudo-manifolds, but we won't give a proof for this. Besides being a generalization of a famous result (Connes') the result was intended to be a first step toward a generalization of the concept of non-commutative geometry to non-smooth (”singular”) spaces which may or may not have applications in theoretical physics. I thank Prof. Dr. Matthias Kreck and Dr. Anna Grinberg who have both actively contributed to this work.

2 LCNT-Spaces

Given a topological space X, we consider the algebra $\mathbb{C}(X)$ of continuous complex valued functions on X together with the topology of uniform convergence on compact subsets of X.

Definition 2.1. *A sub-algebra A of $\mathbb{C}(X)$ is called* **ample***, if the following two conditions hold :*

1. *A is dense in $\mathbb{C}(X)$.*

2. *For any $x \in X$ and any open neighborhood V of x, there exists an element $a \in A$ such that $supp(a) \subset V$ and $a(x) \neq 0$.*

In case the space X is paracompact it is well known that the second condition above implies that for any open covering of X there exists a subordinated partition of unity. Any topological space considered in this paper is assumed to be paracompact and has a countable base of topology. Also, since we work in the categories of topological vector-spaces and algebras, homomorphism always means continuous homomorphism and and isomorphism is a continuous homomorphism which has a continuous inverse.

Definition 2.2. *A pair $(X, \mathbb{C}^\infty_X)$ consisting of a topological space and a unital, associative algebra $\mathbb{C}^\infty_X$ such that*

1. *$\mathbb{C}^\infty_X$ is a topological algebra and is nuclear Fréchet*[1]
2. *$\mathbb{C}^\infty_X$ is an ample sub-algebra of $\mathbb{C}(X)$ and the inclusion $i : \mathbb{C}^\infty_X \to \mathbb{C}(X)$ is continuous.*

is called a **nuclear topological space** *or in short "***nt-space***".*

Now let $(X, \mathbb{C}^\infty_X)$ be an nt-space, $U \subset X$ an open subset, $x \in X$ and $f \in \mathbb{C}(U)$. Then we say that f is of $\mathbb{C}^\infty$-type in x if there exists $\tilde{f} \in \mathbb{C}^\infty_X$ and a neighborhood U_x of X in U such that $f_{|U_x} = \tilde{f}_{|U_x}$. We say f is of $\mathbb{C}^\infty$-type on U if f is of $\mathbb{C}^\infty$-type in x for all $x \in U$.

Definition 2.3. *Let $(X, \mathbb{C}^\infty_X)$ be a nuclear topological space and $U \subset X$ an open subset. Then we define*

$$\mathbb{C}^\infty(U) := \{f \in \mathbb{C}(U) \mid f \text{ is of } \mathbb{C}^\infty\text{-type on } U\}.$$

The existence of partitions of unity in $\mathbb{C}^\infty_X$ implies that $\mathbb{C}^\infty_X = \mathbb{C}^\infty(X)$ and for any open $U \subset X$, the topology on $\mathbb{C}^\infty_X$ induces a locally convex topology on $\mathbb{C}^\infty(U)$ in the canonical way.

We will use the algebras $\mathbb{C}^\infty(U)$ corresponding to the nt-space $(X, \mathbb{C}^\infty_X)$ to prescribe the local structure of $(X, \mathbb{C}^\infty_X)$. The local structure within a locally coned nuclear topological space is that of a cone. The construction is as follows :

Let $cX = \frac{X \times [0,1)}{X \times \{0\}}$ denote the open cone over X. We consider $\mathbb{C}^\infty(X) \hat{\otimes} \mathbb{C}^\infty([0,1))$ as a sub-algebra of $\mathbb{C}(X \times [0,1))$. We define the **jet-map** :

$$\begin{aligned} J : \mathbb{C}^\infty(X) \hat{\otimes} \mathbb{C}^\infty([0,1)) &\to \mathbb{C}^\infty(X)[[z]] &(1)\\ f(x,t) &\mapsto \sum_{i=0}^{\infty} \frac{1}{i!} \frac{\partial^i}{\partial t^i} f(x,0) z^i &(2) \end{aligned}$$

In equation (1) $\mathbb{C}^\infty(X)[[z]]$ denotes the power series algebra with coefficients in $\mathbb{C}^\infty(X)$. We consider this object as a locally convex topological algebra. The topology is given by the product topology. It follows from the Leibniz rule, that

[1] see [Tr] page 85 for the definition of "Fréchet space", page 510 for the definition of "nuclear"

J is a continuous algebra homomorphism. Clearly $\mathbb{C}[[z]]$ is a closed subspace of $\mathbb{C}^{\infty}(X)[[z]]$. We define

$$\mathbb{C}^{\infty}_{cX} := J^{-1}(\mathbb{C}[[z]]). \tag{3}$$

This object is a closed sub-algebra of $\mathbb{C}^{\infty}(X)\hat{\otimes}\mathbb{C}^{\infty}([0,1))$ and hence a nuclear Fréchet algebra. We consider it as a subspace of

$$\mathbb{C}(cX) = \{f : X \times [0,1) \to \mathbb{C} | f(\cdot, 0) = const.\}$$

It is not hard to show that $\mathbb{C}^{\infty}_{cX}$ is an ample sub-algebra of $\mathbb{C}(cX)$ so that we are able to define the cone over the nt-space $(X, \mathbb{C}^{\infty}_{X})$ as

$$c(X, \mathbb{C}^{\infty}_{X}) := (cX, \mathbb{C}^{\infty}_{cX}). \tag{4}$$

and a locally coned nuclear topological space (in short lcnt-space) as an nt-space which has locally the structure of a product of an open ball and a cone over a another lcnt-space. More precisely :

Definition 2.4. *A nuclear topological space $(X, \mathbb{C}^{\infty}_{X})$ is called a* **locally coned nuclear topological space** *(***lcnt-space***) if for any $x \in X$ there exists an open neighborhood $U_x \subset X$ of x, $k \in \mathbb{N}$ and a homeomorphism $\phi : U_x \to B^k \times cL_x$, where $B^k \subset \mathbb{R}^k$ denotes the open unit ball and cL_x the cone over another lcnt-space L_x, such that the induced map*

$$\phi^* : \mathbb{C}^{\infty}(B^k)\hat{\otimes}\mathbb{C}^{\infty}_{cL_x} \to \mathbb{C}^{\infty}(U_x) \tag{5}$$

is an isomorphism.

Clearly, if $(X, \mathbb{C}^{\infty}_{X})$ is an lcnt-space, then $c(X, \mathbb{C}^{\infty}_{X})$ is also an lcnt-space. Hence the class of lcnt-spaces is closed under the cone construction. Also any smooth manifold M together with the algebra of smooth functions on M defines an lcnt-space $(M, \mathbb{C}^{\infty}(M))$. So the class of lcnt-spaces contains at least all spaces which are locally iterated cones over smooth manifolds. The concept of lcnt-spaces is related to that of pseudo-manifolds[2], but emphasizes the analytic structure of the underlying space.

To use inductive methods, we have to define some kind of (local) dimension. For this let $x \in X$ be an arbitrary point. We consider $\mathbb{C}$ as a module over $\mathbb{C}^{\infty}_{X}$ via the evaluation map $ev_x : \mathbb{C}^{\infty}_{X} \to \mathbb{C}$. We define the tangent space at x as

$$T^{\mathbb{C}}_x X := Der(\mathbb{C}^{\infty}_{X}, \mathbb{C}). \tag{6}$$

Definition 2.5. *Let $(X, \mathbb{C}^{\infty}_{X})$ be an lcnt-space and $x \in X$. We define the dimension of X at x as*

$$dim_x(X, \mathbb{C}^{\infty}_{X})) := dim_{\mathbb{C}}(T^{\mathbb{C}}_x X) - 1. \tag{7}$$

From the definition it is not clear that $dim_x(X)$ is a finite number. Nevertheless we have the following proposition :

[2] see [Bo] for a definition of "pseudo-manifold"

Proposition 2.1. *Let $(X, \mathbb{C}^\infty_X)$ be an lcnt-space, $x \in X$ and U_x a neighborhood of x such that according to Definition 2.4 we have $\mathbb{C}^\infty(B^k)\hat{\otimes}\mathbb{C}^\infty(cL_x) \cong \mathbb{C}^\infty(U_x)$. Then $dim_x(X, \mathbb{C}^\infty_X) = k$.*

Proof. From the second condition in Definition 2.1 it follows immediately that any derivation on the algebra $\mathbb{C}^\infty_X$ is a local operator. From this it follows that

$$\begin{aligned} T_x^{\mathbb{C}} X &= Der(\mathbb{C}^\infty_X, \mathbb{C}) &&= Der(\mathbb{C}^\infty(U_x), \mathbb{C}) \\ & &&= Der(\mathbb{C}^\infty(B^k)\hat{\otimes}\mathbb{C}^\infty_{cL_x}, \mathbb{C}) \\ & &&= Der(\mathbb{C}^\infty(B^k), \mathbb{C}) \oplus Der(\mathbb{C}^\infty_{cL_x}, \mathbb{C}) \end{aligned}$$

Hence the statement follows, if we show that $dim_{\mathbb{C}} Der(\mathbb{C}^\infty_{cL_x}, \mathbb{C}) = 1$. We have

$$Der(\mathbb{C}^\infty_{cL_x})\hat{\otimes}_{\mathbb{C}^\infty_{cL_x}} \mathbb{C} \quad = \quad Der(\mathbb{C}^\infty_{cL_x}, \mathbb{C}) = T_x^{\mathbb{C}}(cL_x)$$

where x is now considered as the peak of the cone. Let $D \in Der(\mathbb{C}^\infty_{cL_x})$ and $\sum_i \lambda_i f_i(y)\hat{\otimes} g_i(t)$ an element in $\mathbb{C}^\infty_{cL_x} \subset \mathbb{C}^\infty_{L_x}\hat{\otimes}\mathbb{C}^\infty([0,1))$.[3] Then we have

$$D(\sum_i \lambda_i f_i(y)\hat{\otimes} g_i(t)) \quad = \quad \sum_i \lambda_i (Df_i)(y)\hat{\otimes} g_i(t) + \sum_i \lambda_i f_i(y)\hat{\otimes}(Dg_i)(t)$$

Tensoring this with $\hat{\otimes}_{\mathbb{C}^\infty_{cL_x}} \mathbb{C}$ gives

$$(D\hat{\otimes}_{\mathbb{C}^\infty_{cL_x}} 1)(\sum_i \lambda_i f_i(y)\hat{\otimes} g_i(t)) \quad = \quad \underbrace{\sum_i \lambda_i (Df_i)(y)\hat{\otimes} g_i(0)}_{I} + \underbrace{\sum_i \lambda_i f_i(y)\hat{\otimes}(Dg_i)(0)}_{II}$$

By definition of $\mathbb{C}^\infty_{cL_x}$ the function $y \to \sum_i \lambda_i f_i(y)\hat{\otimes} g_i(0)$ is constant, hence

$$I = D(\sum_i \lambda_i f_i(y)\hat{\otimes} g_i(0)) = 0.$$

Hence any derivation $D \in T_x^{\mathbb{C}}(cL_x)$ is uniquely determined by its effect on the factor $\mathbb{C}^\infty([0,1))$ and we have

$$T_x^{\mathbb{C}}(cL_x) = Der(\mathbb{C}^\infty([0,1)), \mathbb{C})$$

where the right hand side is clearly of dimension 1 over $\mathbb{C}$.

□

The last proposition clearly justifies the somehow artificial term -1 in Definition 2.5. There is still one awkward thing, the dimension of an isolated point according to this definition is -1. However, if the point is non-isolated, then thinking of stratified spaces, the dimension is precisely what it should be. In fact one can decompose X with respect to $dim_x(X, \mathbb{C}^\infty_X)$ and gets something like a stratified (pre-stratified) space.

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[3] see [Tr] page 459 about the representation of elements in the projective tensor-product

Definition 2.6. *Let $(X, \mathbb{C}^\infty_X)$ be an lcnt-space. We define the (global) dimension of $(X, \mathbb{C}^\infty_X)$ as*

$$dim(X, \mathbb{C}^\infty_X) = \sup_{x \in X} dim_x(X, \mathbb{C}^\infty_X) \tag{8}$$

We conclude this section with a lemma, which reflects some metric aspects of nt-spaces and will later be used to localize the Hochschild complex of $\mathbb{C}^\infty_X$.

Lemma 2.1. *Let $(X, \mathbb{C}^\infty_X)$ be an nt-space. Then there exists a function $\delta \in \mathbb{C}^\infty_X \hat{\otimes} \mathbb{C}^\infty_X$ which has values in $\mathbb{R}_+$ and has the property that $\sqrt{\delta(x,y)}$ is a metric on X which generates the topology of X.*

Proof. To show the existence of such a δ, we will modify the proof of the Urysohn metrization theorem, which states that every regular T_1 space with countable base of topology is metrizable. Using that $\mathbb{C}^\infty_X$ is ample one can easily proof that for any two disjoint and closed subsets A and B of X there is a function $f_{A,B} \in C^\infty_X$ and $\epsilon_{A,B} > 0$ such that $A \subset f^{-1}_{A,B}(0)$ and $B \subset f^{-1}_{A,B}(\epsilon_{A,B})$. Let us now consider a complete bounded family $\mathcal{F}$ of such function, that is for any two disjoint and closed subsets A and B of X there is $f_{A,B} \in \mathcal{F}$ as above and $\{f_{A,B}\}$ is bounded.[4] We can assume that $\mathcal{F}$ is countable and given by the family $\{f_n | n \in \mathbb{N}\}$. We define δ as follows :

$$\begin{array}{rcl} \delta : X \times X & \to & \mathbb{R}_+ \\ (x,y) & \mapsto & \displaystyle\sum_{i=1}^{\infty} \lambda_i \mu_i (f_i(x) - f_i(Y))^2. \end{array}$$

where $\lambda_i, \mu_i > 0$ are chosen such that $\lim \mu_i f_i = 0$ and $\sum_i \lambda_i < 1$. Then $\delta \in \mathbb{C}^\infty_X \hat{\otimes} \mathbb{C}^\infty_X$ (see [Tr], page 459). The map

$$\begin{array}{rcl} \psi : X & \to & [0,1]^{\mathcal{F}} \\ x & \mapsto & (f(x))_{f \in \mathcal{F}} \end{array}$$

is a topological embedding, where $[0,1]^{\mathcal{F}}$ is considered with the product topology (see [Ke], page 125). It is well known that this topology can be realized via the metric

$$d((x_i),(y_i)) = \sum_{i=1}^{\infty} \frac{1}{2^i} |x_i - y_i|.$$

and hence also via the map

$$\tilde{\delta}((x_i),(y_i)) = \sum_{i=1}^{\infty} \lambda_i \mu_i (x_i - y_i)^2.$$

We have $\delta = \tilde{\delta} \circ (\psi \times \psi)$ which shows that δ satisfies the conditions in the lemma. □

In a forthcoming paper we will study the geometric aspects of lcnt-spaces. It is possible to generalize most of the concepts of Riemannian geometry and also stochastic analysis to this class of spaces.

[4] to achieve this we have to allow the constants $\epsilon_{A,B}$ to be arbitrary small

3 Hochschild Homology and Differential Forms

In this section we will give a short introduction into Hochschild homology and differential forms. We work in the category of locally convex spaces and use the projective tensor product. A reference for this section is the original work [Co] as well as the articles [Wo] and [Bro/Lyk] about excision in Hochschild homology. From now on we assume that A is a locally convex topological algebra over $\mathbb{C}$. For any natural number $n \in \mathbb{N}$ let

$$C_n(A) = A^{\hat{\otimes}(n+1)} \tag{9}$$

be the $(n+1)$-fold completed tensor product. We define operators

$$\begin{aligned}
b'_n : C_n(A) &\to C_{n-1}(A) \\
b'_n(a_0 \hat{\otimes} ... \hat{\otimes} a_n) &= \sum_{i=0}^{n-1} (-1)^i a_0 \hat{\otimes} ... \hat{\otimes} a_i a_{i+1} \hat{\otimes} ... \hat{\otimes} a_n \\
b_n : C_n(A) &\to C_{n-1}(A) \\
b_n(a_0 \hat{\otimes} ... \hat{\otimes} a_n) &= b'_n(a_0 \hat{\otimes} ... \hat{\otimes} a_n) + (-1)^n a_n a_0 \hat{\otimes} a_1 \hat{\otimes} ... \hat{\otimes} a_{n-1}.
\end{aligned}$$

An easy computation shows $b_{n-1} \circ b_n = 0 = b'_{n-1} \circ b'_n = 0$, hence we get two chain complexes

$$C_*(A) = (C_n(A), b_n) \tag{10}$$

$$C_*^{bar}(A) = (C_n(A), b'_n) \tag{11}$$

The first complex is called the **Hochschild complex**, the second complex is called the **bar-complex**. Both complexes give rise to homology groups. The homology groups of the bar complex are only considered for technical reasons. In fact, the bar complex is acyclic, if the algebra A in case is unital. In general, the acyclicity of the bar-complex is related to the question of excision.

Definition 3.1. *Let A be a unital algebra. For $n \in \mathbb{N}$ we define the n-th Hochschild homology group of A as*

$$HH_n(A) = \frac{ker(b_n : C_n(A) \to C_{n-1}(A))}{im(b_{n+1} : C_{n+1}(A) \to C_n(A))}.$$

We consider these homology groups as topological vector-spaces but keep in mind that unless $im(b)$ is not closed, they are not Hausdorff. In general this can cause some problems. For the problem addressed in this paper though, they will turnout to be Hausdorff and everything is fine. Clearly the construction above is functorial, and any continuous algebra homomorphism induces a continuous homomorphism of the Hochschild homology groups. For computational reasons, it is important, that Hochschild homology can be represented as a derived functor, in fact as a special kind of Tor-product.

Definition 3.2. *A locally convex topological vector-space M which is also a topological module over A is called* **topological projective** *if it is a topological direct summand of a module of the form $N = A \hat{\otimes} E$, where E is a locally convex vector-space.*

Definition 3.3. *Let M be a locally convex topological module over A. A* **topological projective resolution** *of M is an exact sequence of topological projective A-modules and A-linear maps*

$$...M_2 \xrightarrow{b_2} M_1 \xrightarrow{b_1} M_0 \xrightarrow{b_0} M \ ,$$

which admits a $\mathbb{C}$-linear continuous contraction, i.e. for all $i \in \mathbb{N}$ maps s_i satisfying

$$\begin{aligned} s_i : M_i &\rightarrow M_{i+1}, \\ id &= b_{i+1}s_i + s_{i-1}b_i. \end{aligned}$$

Now let A^{op} denote the algebra A with the opposite multiplication and $B := A\hat{\otimes}A^{op}$. The algebra A itself becomes a topological B-module by setting

$$(a\hat{\otimes}b) \cdot c = acb.$$

The following proposition gives an answer on how to compute Hochschild homology groups using projective resolutions.

Proposition 3.1. *Let (M_n, b_n) be a topological projective resolution of A over B. Then the Hochschild homology groups of A coincide with the homology groups of the complex*

$$...M_3\hat{\otimes}_B A \xrightarrow{b_3} M_2\hat{\otimes}_B A \xrightarrow{b_2} M_1\hat{\otimes}_B A \xrightarrow{b_1} M_0\hat{\otimes}_B A \ .$$

Proof. (see [Co]) □

We will use this proposition to give two examples.

Example 3.1. 1. *Consider $\mathbb{C}$ together with its natural topology. Then*

$$\mathbb{C}^{op}\hat{\otimes}\mathbb{C} \cong \mathbb{C}$$

via the multiplication map and

$$0 \longrightarrow \mathbb{C} \xrightarrow{id} \mathbb{C} \longrightarrow 0 \qquad (12)$$

is in fact a very short topological projective resolution of $\mathbb{C}$ over $\mathbb{C}^{op}\hat{\otimes}\mathbb{C}$. Hence we have

$$HH_n(\mathbb{C}) = \begin{cases} \mathbb{C} \ if \ n = 0 \\ 0 \ if \ n \neq 0 \end{cases}$$

2. *Consider the algebra $\mathbb{C}[[x]]$ of formal power series with product topology. Then we can identify*

$$\mathbb{C}[[x]]^{op}\hat{\otimes}\mathbb{C}[[x]] \cong \mathbb{C}[[x, y]]$$

and a topological projective resolution of $\mathbb{C}[[x]]$ over $\mathbb{C}[[x]]^{op}\hat{\otimes}\mathbb{C}[[x]]$ is given by

$$0 \longrightarrow \mathbb{C}[[x,y]] \xrightarrow{\cdot(x-y)} \mathbb{C}[[x,y]] \xrightarrow{x=y} \mathbb{C}[[x]] \longrightarrow 0$$

The $\mathbb{C}$-linear contraction can easily be found by a recursion on the coefficients of the corresponding power series. Tensoring this resolution with $\hat{\otimes}_{\mathbb{C}[[x,y]]}\mathbb{C}[[x]]$ results into the sequence

$$0 \longrightarrow \mathbb{C}[[x]] \xrightarrow{0} \mathbb{C}[[x]]$$

Computing the homology, we get

$$HH_n(\mathbb{C}[[x]]) = \begin{cases} \mathbb{C}[[x]] \; if \; n = 0 \\ \mathbb{C}[[x]] \; if \; n = 1 \\ 0 \quad if \; n \neq 0,1 \end{cases}$$

The concept of Hochschild homology is strongly related to the concept of differential forms. In fact it is a refinement. We will later show, that for lcnt-spaces the two concepts coincide. The next proposition is a first step into this direction.

Proposition 3.2. *For any $k \in \mathbb{N}$ we have*

$$HH_k(\mathbb{C}^\infty(B^n)) \cong \Omega^k(B^n) \tag{13}$$

where B^n denotes the open unit ball and Ω^k the complex-valued differential k-forms.

Proof. Corresponding to Proposition 3.1 we choose $A = \mathbb{C}^\infty(B^n)$ and construct an explicit projective resolution of A over $A\hat{\otimes}A^{op} = \mathbb{C}^\infty(B^n \times B^n)$. For each $k \in \mathbb{N}$ we define modules over $\mathbb{C}^\infty(B^n \times B^n)$ as

$$M_k := \mathbb{C}^\infty(B^n \times B^n, \Lambda^k(\mathbb{C}^{n*})).$$

Here $\mathbb{C}^{n*}$ denotes the space of linear forms on $\mathbb{C}^n$. Clearly

$$M_k \cong \mathbb{C}^\infty(B^n \times B^n)\hat{\otimes}\Lambda^k(\mathbb{C}^{n*}),$$

where completion is actually unnecessary, since the vector-space $\Lambda^k(\mathbb{C}^{n*})$ is finite dimensional. Nevertheless, it follows that each M_k is free, hence projective. Furthermore let γ denote the difference function

$$\begin{aligned} \gamma : B^n \times B^n &\rightarrow \mathbb{R}^n \subset \mathbb{C}^n \\ \gamma(a,b) &= b - a. \end{aligned}$$

This map induces maps

$$\begin{aligned} i_\gamma : M_{k+1} &\rightarrow M_k \\ i_\gamma\omega(a,b)(v_1,...,v_k) &= \omega(a,b)(\gamma(a,b),v_1,...v_k) = \omega(a,b)(b-a,v_1,...v_k). \end{aligned}$$

Here $\omega \in M_{k+1}$ denotes a form, a,b are points in $B^n \subset \mathbb{R}^n \subset \mathbb{C}^n$ and $v_1,...v_k$ are elements of $\mathbb{C}^n$. In other words, i_γ is contraction with the vector-field γ. Let us now consider the following sequence

$$\cdots \xrightarrow{i_\gamma} M_2 \xrightarrow{i_\gamma} M_1 \xrightarrow{i_\gamma} M_0 = \mathbb{C}^\infty(B^n \times B^n) \xrightarrow{\Delta^*} \mathbb{C}^\infty(B^n) \longrightarrow 0$$

where $\Delta : B^n \to B^n \times B^n$ denotes the diagonal map. Clearly $i_\gamma \circ i_\gamma = 0$. To show that the chain complex above is a topological projective resolution of $\mathbb{C}^\infty(B^n)$ over $\mathbb{C}^\infty(B^n \times B^n)$, we have to give a continuous $\mathbb{C}$-linear contraction. For this let $s_k : M_k \to M_{k+1}$ be defined as follows. Let $e_1^*, ..., e_n^*$ denote the dual basis of the standard canonical basis of $\mathbb{C}^n$, and let $\omega \in M_k$ be given as

$$\omega(x, y) = f(x, y) e_{i_1}^* \wedge ... \wedge e_{i_k}^*,$$

where $f \in \mathbb{C}^\infty(B^n \times B^n)$ is a smooth function on $B^n \times B^n$ and $i_1, ..., i_k \in \{1, ...n\}$. In this case we define

$$s_k\omega(a, b) := \sum_{j=1}^{n} \int_0^1 \frac{\partial f}{\partial y_j}(a, a + t(b-a)) t^k e_j^* \wedge e_{i_1}^* \wedge ... \wedge e_{i_k}^* dt.$$

In the following we suppress the subscript k and simply write $s\omega$. We have

$$\begin{aligned} (i_\gamma s\omega)(a, b) \quad &= \quad \sum_{j=1}^{n}\Big(\int_0^1 \frac{\partial f}{\partial y_j}(a, a + t(b-a)) t^k \\ &\cdot \quad \{\sum_{l=1}^{k} (-1)^{l+1} (b-a)_{i_l} e_j^* \wedge e_{i_1}^* \wedge ... \wedge \widehat{e_{i_l}^*} \wedge ... \wedge e_{i_k}^*\} \\ &+ \quad (b-a)_j e_{i_1}^* \wedge ... \wedge e_{i_k}^*\Big). \end{aligned}$$

From this we get the expression

$$(i_\gamma s\omega)(a, b) = \{\sum_{j=1}^{n} \int_0^1 \frac{\partial f}{\partial y_j}(a, a + t(b-a)) t^k (b-a)_j e_{i_1}^* \wedge ... \wedge e_{i_k}^* dt\}$$

$$+\{\sum_{l=1}^{k}\sum_{j=1}^{n} \int_0^1 (-1)^{l+1} \frac{\partial f}{\partial y_j}(a, a + t(b-a)) t^k (b-a)_{i_l} e_j \wedge e_{i_1}^* \wedge ... \wedge \widehat{e_{i_l}^*} \wedge ... \wedge e_{i_k}^* dt\}.$$

The chain-rule of differential calculus applied to the first sum gives

$$i_\gamma s\omega(a, b) = \int_0^1 \frac{d}{dt} f(a, a + t(b-a)) t^k e_{i_1}^* \wedge ... \wedge e_{i_k}^* dt$$

$$+\{\sum_{l=1}^{k}\sum_{j=1}^{n} \int_0^1 (-1)^{l+1} \frac{\partial f}{\partial y_j}(a, a + t(b-a)) t^k (b-a)_{i_l} e_j \wedge e_{i_1}^* \wedge ... \wedge \widehat{e_{i_l}^*} \wedge ... \wedge e_{i_k}^* dt\}.$$

Now we perform partial integration with the first integral on the right side. This yields us to the following expression

$$i_\gamma s\omega(a,b) = \underbrace{f(a,a+t(b-a))t^k|_0^1 e_{i_1}^* \wedge ... \wedge e_{i_k}^*}_{=\omega(a,b)}$$

$$- \int_0^1 f(a,a+t(b-a))kt^{k-1} e_{i_1}^* \wedge ... \wedge e_{i_k}^* dt$$

$$+\{\sum_{l=1}^k \sum_{j=1}^n \int_0^1 (-1)^{l+1} \frac{\partial f}{\partial y_j}(a,a+t(b-a))t^k(b-a)_{i_l} e_j \wedge e_{i_1}^* \wedge ... \wedge \widehat{e_{i_l}^*} \wedge ... \wedge e_{i_k}^* dt\}.$$

The first term on the right side clearly coincides with $\omega(a,b)$. In total we get

$$i_\gamma s\omega(a,b) = \omega(a,b) + R(a,b),$$

where $R(a,b)$ denotes the rest, i.e. the integral and the double sum on the right side.

Now we calculate the expression $si_\gamma\omega(a,b)$. The definition give us

$$\begin{aligned} si_\gamma\omega(a,b) &= \sum_{l=1}^k (-1)^l \{\sum_{j=1}^n \int_0^1 \frac{\partial f}{\partial y_j}(a,a+t(b-a))(a+t(b-a)-a)_{i_l} \\ &- f(a,a+t(b-a))\delta_{j,i_l} t^{k-1} dt\ e_j^* \wedge e_{i_1}^* \wedge ... \widehat{e_{i_l}^*} \wedge ... \wedge e_{i_k}^*\}. \end{aligned}$$

Reordering terms and evaluation of the Kronecker symbol δ_{j,i_l} yields to

$$si_\gamma\omega(a,b) = (-1)^{l+1} \int_0^1 f(a,a+t(b-a))t^{k-1} dt\ e_{i_l}^* \wedge e_{i_1}^* \wedge ... \wedge \widehat{e_{i_l}^*} \wedge ... \wedge e_{i_k}^*$$

$$+ \sum_{l=1}^k (-1)^l \{\sum_{j=1}^n \int_0^1 \frac{\partial f}{\partial y_j}(a,a+t(b-a))(b-a)_{i_l} t^k dt\ e_j^* \wedge e_{i_1}^* \wedge ... \widehat{e_{i_l}^*} \wedge ... \wedge e_{i_k}^*\} \qquad .$$

Shuffling $e_{i_l}^*$ from the first to the i_l-th position in $e_{i_l}^* \wedge e_{i_1}^* \wedge ... \wedge \widehat{e_{i_l}^*} \wedge ... \wedge e_{i_k}^*$ changes the sign by the factor $(-1)^{l-1}$. This cancels with the factor $(-1)^{l+1}$ in front of the first term on the right side, and we see that this term is actually independent of the summation index l. Hence for this term summation over l is just multiplication with k. Taking a close look at the summands we can recognize that we end up with $-R(a,b)$, where $R(a,b)$ was defined on the previous page. So we get

$$si_\gamma\omega(a,b) + i_\gamma s\omega(a,b) = -R(a,b) + \omega(a,b) + R(a,b) = \omega(a,b).$$

This proves

$$si_\gamma + i_\gamma s = id.$$

It is not hard to see that s is continuous and $\mathbb{C}$-linear. So far we have constructed a topological projective Resolution of $\mathbb{C}^\infty(B^n)$ over $\mathbb{C}^\infty(B^n \times B^n)$.

We do now calculate the Hochschild homology of $\mathbb{C}^\infty(B^n)$ by tensoring the topological projective resolution from above over $\mathbb{C}^\infty(B^n \times B^n)$ with $\mathbb{C}^\infty(B^n)$. For any $k \in \mathbb{N}$ we have

$$\begin{aligned} M_k \hat{\otimes}_{\mathbb{C}^\infty(B^n \times B^n)} \mathbb{C}^\infty(B^n) &= (\mathbb{C}^\infty(B^n \times B^n) \hat{\otimes} \Lambda^k(\mathbb{C}^{n*})) \hat{\otimes}_{\mathbb{C}^\infty(B^n \times B^n)} \mathbb{C}^\infty(B^n) \\ &= \Lambda^k(\mathbb{C}^{n*}) \hat{\otimes} \mathbb{C}^\infty(B^n) = \Omega^k(B^n). \end{aligned}$$

Since γ restricted to the diagonal is zero, we have

$$i_\gamma \hat{\otimes}_{\mathbb{C}^\infty(B^n \times B^n)} id_{\mathbb{C}^\infty(B^n)} = 0.$$

Therefore the tensored complex has zero differentials and we get $\Omega^k(B^n)$ for the k-th homology group of this complex. □

To study the relationship between Hochschild homology and differential forms in general, we have to say what we mean by differential forms for a general Fréchet-algebra. There are various concepts of this, we will use the following one :[5]

The first Hochschild homology group of a locally convex algebra A is given as

$$HH_1(A) = \frac{ker(b_1 : A^{\hat{\otimes} 2} \to A)}{im(b_2 : A^{\hat{\otimes} 3} \to A^{\hat{\otimes} 2})}$$

The boundary maps b_1 and b_2 are explicitly given as

$$\begin{aligned} b_1(a \hat{\otimes} b) &= ab - ba & (14) \\ b_2(a \hat{\otimes} b \hat{\otimes} c) &= ab \hat{\otimes} c - a \hat{\otimes} bc + ac \hat{\otimes} b. & (15) \end{aligned}$$

We assume from now on that A is commutative. Then in equation (14) $b_1 = 0$. We denote with $[a \hat{\otimes} b]$ the class of $a \hat{\otimes} b$ in $HH_1(A)$. We have

$$\begin{aligned} [a \hat{\otimes} b] &= [a(1 \hat{\otimes} b - b \hat{\otimes} 1) + ab \hat{\otimes} 1] \\ &= a[1 \hat{\otimes} b - b \hat{\otimes} 1] + \underbrace{[ab \hat{\otimes} 1]}_{=*} \\ &= a[1 \hat{\otimes} b - b \hat{\otimes} 1] \end{aligned}$$

That $* = 0$ follows by taking $c = 1$ in (15). We define the formal differential of b as

$$db := [1 \hat{\otimes} b - b \hat{\otimes} 1].$$

An elementary computation shows that

[5] This version of differential forms works very good in the case that the underlying algebra is commutative, in case one is working with non-commutative algebras one usually uses a slightly different version.

$$d(ab) = adb + bda. \tag{16}$$

From above, we also see that $HH_1(A) = \overline{span}\{adb | a, b \in A\}$. This leads us to the following definition.

Definition 3.4. *Let A be a locally convex topological algebra and $k \in \mathbb{N}$. We define the locally convex topological vector-space of* **differential k-forms on** *A as*

$$\Omega_A^k := \Lambda_A^k HH_1(A)$$

where Λ_A^k denotes the k-th exterior product over A.[6]

By definition we have $\Omega_A^0 = A$ and $\Omega_A^1 := HH_1(A)$. In general we have the **antisymmetrization map**

$$\begin{array}{rcl} \epsilon : \Omega_A^k & \to & HH_k(A) \\ a_0 da_1 \wedge ... \wedge da_k & \to & \sum_{\sigma \in \Sigma_k} sign(\sigma)(a_0 \otimes a_{\sigma^{-1}(1)} ... \otimes a_{\sigma^{-1}(k)}). \end{array}$$

This map is well defined, continuous and has a continuous left inverse (see [Lo], page 27ff). Hence it is always injective and Ω_A^k is even a topological direct summand in $HH_k(A)$. The question is now, when is it also surjective or what can Hochschild homology see, what differential forms do not see. In the case of lcnt-spaces we will answer this question in the next section.

To use inductive methods it is very useful to have at hand certain long exact sequences of Hochschild homology groups associated to short exact sequences of certain algebras. Those long exact sequences do not always exist. The question of their existence is referred to as the Excision-problem in Hochschild homology, because it somehow resembles the situation of the Mayer-Vietoris sequence in ordinary homology. In general the obstruction is a property called H-unitality ([Lo], page 32). By definition A is H-unital, if $C_*^{bar}(A)$ is acyclic. For a non-unital algebra I such as for example an ideal in a bigger unital algebra, the Hochschild homology groups are defined as $coker(i_*) : HH_n(\mathbb{C}) \to HH_n(I^+)$ whereI^+ denotes the unitalization of I and i the canonical inclusion. By definition I is called H unital, if $C_*^{bar}(I)$ is acyclic. The following proposition can be found in [Bro/Lyk].

Proposition 3.3. *Let $0 \to I \to A \to A/I \to 0$ be an exact sequence of nuclear Fréchet algebras such that A is unital and I is H-unital. Then there is a long exact sequence of continuous Hochschild homology groups*

$$\cdots \longrightarrow HH_n(I) \longrightarrow HH_n(A) \longrightarrow HH_n(A/I) \xrightarrow{\delta} HH_{n-1}(I) \longrightarrow \cdots.$$

The task to determine whether an algebra is H-unital or not can turn out to be quite difficult. We will later make use of the following proposition.

[6] in the category of locally convex topological vector-spaces [He],[Ew] page 64

Proposition 3.4. *Let A be a unital nuclear Fréchet algebra and*

$$\mathbb{C}_0^\infty([0,1)) = ker(res : \mathbb{C}^\infty((-1,1)) \to \mathbb{C}^\infty(-1,0)) \tag{17}$$

be the algebra of smooth function on $[0,1)$ vanishing of any order at zero. Then the non-unital nuclear Fréchet algebra $\mathbb{C}_0^\infty([0,1))\hat{\otimes}A$ is H-unital.

Proof. We have to show that the corresponding bar-complex (11) is acyclic. Let

$$\alpha = \sum_i \lambda_i (f_0^i \hat{\otimes} b_0^i)\hat{\otimes}...\hat{\otimes}(f_n^i \hat{\otimes} b_n^i) \in C_n(\mathbb{C}_0^\infty([0,1))\hat{\otimes}A)$$

be an element in the bar complex. Here λ_i is a sequence of complex numbers such that $\sum_{i=0}^\infty |\lambda_i| < 1$ and f_j^i respectively b_j^i converge to zero as i goes to infinity. A factorization theorem (see [Vo], Thm. 3.4) applied to $\mathbb{C}_0^\infty([0,1))$ and the sequence (f_0^i) says there are functions $g^i \in \mathbb{C}_0^\infty([0,1))$ for all $i \in \mathbb{N}$ and $h \in \mathbb{C}_0^\infty([0,1))$ with the following properties :

1. $f_0^i = h \cdot g^i$ forall $i \in \mathbb{N}$
2. $g^i \in \overline{\mathbb{C}_0^\infty([0,1)) \cdot (f_0^i | i \in \mathbb{N})}$

The expression in condition 2 denotes the closure of the ideal in $\mathbb{C}_0^\infty([0,1))$ which is generated by the functions f_0^i. Let us define $\beta \in C_n(\mathbb{C}_0^\infty([0,1))\hat{\otimes}A)$ as

$$\beta = \sum_{i=0}^\infty \lambda_i (g^i \hat{\otimes} b_0^i)\hat{\otimes}...\hat{\otimes}(f_n^i \hat{\otimes} b_n^i).$$

From condition 2 above, it follows that

$$\beta \in \overline{\mathbb{C}_0^\infty([0,1))\hat{\otimes}A) \cdot \alpha} \subset C_n(\mathbb{C}_0^\infty([0,1))\hat{\otimes}A). \tag{18}$$

Here the term in the middle denotes the closure of the ideal generated by α. A simple calculation shows, that

$$\alpha = b'((h\hat{\otimes}1_A)\hat{\otimes}\beta) + (h\hat{\otimes}1_A)\hat{\otimes}b'(\beta). \tag{19}$$

Let us now assume that α is a cycle in the bar complex. Then $b'(\alpha) = 0$. Hence by continuity and $\mathbb{C}_0^\infty([0,1))\hat{\otimes}A$ linearity of b' it follows from (18) that $b'(\beta) = 0$. Hence by (19) we have that

$$\alpha = b'((h\hat{\otimes}1_A)\hat{\otimes}\beta)$$

is a boundary in the continuous bar complex and the bar complex is acyclic. □

4 Hochschild Homology of lcnt-Spaces

The following theorem is the main result of this work. It generalizes a result of A. Connes[7] about Hochschild homology in the case, where the underlying space is a smooth manifold to the case where the underlying space is an lcnt-space. We will not use Connes' result in the proof but Proposition 3.2 and a localization method introduced in [Te].

[7] see [Co] Lemma 45 and Theorem 46 (Connes works in the cohomological setting)

Theorem 4.1. *Let $(X, \mathbb{C}^\infty_X)$ be a finite dimensional lcnt-space, then the antisymmetrization map*

$$\epsilon : \Omega^*_{\mathbb{C}^\infty_X} \to HH_*(\mathbb{C}^\infty_X)$$

is an isomorphism.

The proof of the theorem will follow after a couple of lemmas. The first step is :

Lemma 4.1. *The antisymmetrization map*

$$\epsilon : \Omega^k_{\mathbb{C}[[x]]} \to HH_k(\mathbb{C}[[x]])$$

is an isomorphism for all $k \in \mathbb{N}$.

Proof. By definition for $k = 0, 1$ there is nothing to show. On the other side it follows from (16) that $d(x^n) = n \cdot x^{n-1} dx$. Since obviously $dx \wedge dx = 0$ we have $\Omega^k_{\mathbb{C}[[x]]} = 0$ for all $k \geq 2$ which together with Example 3.1 (2) shows the rest. $\square$

Lemma 4.2. *Let $(X, \mathbb{C}^\infty_X)$ be an lcnt-space such that the antisymmetrization map*

$$\epsilon : \Omega^*_{\mathbb{C}^\infty_X} \to HH_*(\mathbb{C}^\infty_X)$$

is an isomorphism and furthermore $HH_*(\mathbb{C}^\infty_X)$ is Hausdorff then the antisymmetrization map

$$\epsilon : \Omega^*_{\mathbb{C}^\infty_{cX}} \to HH_*(\mathbb{C}^\infty_{cX})$$

is also an isomorphism and $HH_*(\mathbb{C}^\infty_{cX})$ is also Hausdorff.

Proof. Let us first consider the following short exact sequence of algebras

$$0 \longrightarrow \mathbb{C}^\infty_X \hat{\otimes} \mathbb{C}^\infty_0([0,1)) \xrightarrow{incl.} \mathbb{C}^\infty_X \hat{\otimes} \mathbb{C}^\infty([0,1)) \xrightarrow{J} \mathbb{C}^\infty_X[[z]] \longrightarrow 0 \ , \quad (20)$$

where $\mathbb{C}^\infty_0([0,1))$ is given by (17). It follows from Proposition 3.4 that $\mathbb{C}^\infty_X \hat{\otimes} \mathbb{C}^\infty_0([0,1))$ is H-unital. Hence we have the following commutative diagram with an exact upper row

$$\begin{tikzcd}
HH_k(\mathbb{C}^\infty_X \hat{\otimes} \mathbb{C}^\infty_0([0,1))) \arrow[r, "incl._*"] & HH_k(\mathbb{C}^\infty_X \hat{\otimes} \mathbb{C}^\infty([0,1))) \arrow[r, "J_*"] & HH_k(\mathbb{C}^\infty_X[[z]]) \\
\Omega^k_{\mathbb{C}^\infty_X \hat{\otimes} \mathbb{C}^\infty_0([0,1))} \arrow[u, "\epsilon"] \arrow[r, "incl._*"] & \Omega^k_{\mathbb{C}^\infty_X \hat{\otimes} \mathbb{C}^\infty([0,1))} \arrow[u, "\epsilon", "\cong"'] \arrow[r, "J_*"] & \Omega^k_{\mathbb{C}^\infty_X[[z]]} \arrow[u, "\epsilon", "\cong"']
\end{tikzcd}$$

The two antisymmetrization maps on the right side are in fact isomorphisms. This follows from $\mathbb{C}^\infty_X[[z]] \cong \mathbb{C}^\infty_X \hat{\otimes} \mathbb{C}[[z]]$, Lemma 4.1 and Proposition 3.2 (slightly modified, to cover half open intervals) once there is a Kuenneth-Theorem for Hochschild homology at hand. Such a theorem indeed exists, if the corresponding Hochschild homology groups of the involved algebras are Hausdorff (see [Ka]). This is the case.[8] It is not hard to see that the second map in the lower sequence is always surjective. Therefore by commutativity and our assumption, the second map in the upper sequence is also surjective. Since the upper sequence is actually a part of a long exact sequence, the first map in the upper sequence is in addition injective and the upper sequence is in fact a short exact sequence. It follows from this information and that ϵ is in general split injective via a small diagram chase, that

$$\epsilon : \Omega^*_{\mathbb{C}^\infty_X \hat{\otimes} \mathbb{C}^\infty_0([0,1))} \quad \rightarrow \quad HH_*(\mathbb{C}^\infty_X \hat{\otimes} \mathbb{C}^\infty_0([0,1)))$$

is also an isomorphism. Let us now consider the short exact sequence of algebras

$$0 \longrightarrow \mathbb{C}^\infty_X \hat{\otimes} \mathbb{C}^\infty_0([0,1)) \xrightarrow{incl.} \mathbb{C}^\infty_{cX} \xrightarrow{J} \mathbb{C}[[z]] \longrightarrow 0\,. \tag{21}$$

As before this sequence induces a commutative diagram

$$\begin{tikzcd} HH_k(\mathbb{C}^\infty_X \hat{\otimes} \mathbb{C}^\infty_0([0,1))) \arrow[r, "incl._*"] & HH_k(\mathbb{C}^\infty_{cX}) \arrow[r, "proj._*"] & HH_k(\mathbb{C}[[z]]) \\ \Omega^k_{\mathbb{C}^\infty_X \hat{\otimes} \mathbb{C}^\infty_0([0,1))} \arrow[u, "\epsilon", "\cong"'] \arrow[r, "incl._*"] & \Omega^k_{\mathbb{C}^\infty_{cX}} \arrow[u, "\epsilon"] \arrow[r, "proj._*"] & \Omega^k_{\mathbb{C}[[z]]} \arrow[u, "\epsilon", "\cong"'] \end{tikzcd}$$

where the upper row is exact. Furthermore the right and the left antisymmetrization maps are isomorphisms. This follows from Lemma 4.1 and the first part. Also again, it is not hard to see that the second map in the lower sequence is always surjective. It follows from this and split injectivity , via a small diagram chase that the map in the middle

$$\epsilon : \Omega^k_{\mathbb{C}^\infty_{cX}} \rightarrow HH_k(\mathbb{C}^\infty_{cX}))$$

is also an isomorphism.

□

Lemma 4.3. *Let $(X, \mathbb{C}^\infty_X)$ be an nt-space, such that there exists a locally finite open covering $(U_i | i \in I)$ of X such that any of the antisymmetrization maps*

$$\epsilon : \Omega^*_{\mathbb{C}^\infty(U_i)} \rightarrow HH_*(\mathbb{C}^\infty(U_i))$$

is an isomorphism, then the antisymmetrization map

$$\epsilon : \Omega^*_{\mathbb{C}^\infty_X} \rightarrow HH_*(\mathbb{C}^\infty_X)$$

is also an isomorphism.

Proof. We consider the spaces $C_k(\mathbb{C}^\infty_X) = (\mathbb{C}^\infty_X)^{\hat{\otimes}(k+1)}$ in the Hochschild complex of $\mathbb{C}^\infty_X$ as subspaces of the space of continuous complex valued functions on the $(k+1)$-fold Cartesian product of X and use a method first presented in [Te] to localize the Hochschild complex around the diagonal. First Let

[8] for more details about the Kuenneth-Theorem in Hochschild homology see [Ew], page 86

$\lambda : [0,\infty) \to [0,1]$ be a smooth function, such that $supp(\lambda) \subset [0,1]$ and $\lambda_{|[0,1/2]} \equiv 1$. For $t > 0$ we define

$$\begin{aligned} \lambda_t : [0,\infty) &\to [0,1] \\ \lambda_t(s) &:= \lambda(s/t). \end{aligned}$$

These functions have the following properties :

1. $supp(\lambda_t) \subset [0,t]$
2. $\lambda_{t|[0,t/2]} \equiv 1$

Now for any $k \in \mathbb{N}$ let us define functions ρ_k as

$$\begin{aligned} \rho_k : X^{k+1} &\to [0,\infty) \\ \rho_k(x_0, x_1, ..., x_k) &= \delta(x_0,x_1) + \delta(x_1,x_2) + ... + \delta(x_k,x_0). \end{aligned}$$

Here δ denotes the function on $X \times X$ of Lemma 2.1 . ρ_k measures the distance of a point in X^{k+1} from the diagonal with respect to the "distance-function" δ. Clearly $\rho_k \in (\mathbb{C}_X^\infty)^{\hat{\otimes}(k+1)}$. Let

$$U_{t,k} := \{(x_0,x_1,...,x_k) | \rho_k(x_0,...,x_k) < t\}$$

be the t-neighborhood of the diagonal $\Delta_{k+1} \subset X^{k+1}$. Let $C_*^t(\mathbb{C}_X^\infty)$ be the sub-complex of the Hochschild complex $C_*(\mathbb{C}_X^\infty)$ where $C_k^t(\mathbb{C}_X^\infty)$ contains the elements of $(\mathbb{C}_X^\infty)^{\hat{\otimes}(k+1)}$ vanishing on $U_{t,k}$. Let

$$C_*^0(\mathbb{C}_X^\infty) = \varinjlim C_*^t(\mathbb{C}_X^\infty)$$

where the limit is for t to zero. The complex $C_*^0(\mathbb{C}_X^\infty)$ consists of the chains vanishing in an arbitrary neighborhood of the diagonal. We will now show that this complex is acyclic. We define an operator

$$\begin{aligned} E_t : C_k(\mathbb{C}_X^\infty) &\to C_{k+1}(\mathbb{C}_X^\infty) \\ E_t(F)(x_0,...,x_{k+1}) &= \lambda_t(\delta(x_0,x_1)^2) \cdot F(x_1,...,x_{k+1}), \forall F \in C_k(X) \end{aligned}$$

This operator maps $C_k^s(\mathbb{C}_X^\infty)$ into $C_{k+1}^{s/4}(\mathbb{C}_X^\infty)$ which can be verified by computation. Another computation shows that

$$b \circ E_t + E_t \circ b = 1 - N_t,$$

where N_t is defined as

$$N_t(F)(x_0,...,x_k) = (-1)^k \lambda_t(\delta(x_0,x_1)) \cdot \{F(x_1,x_2,...x_k,x_0) - F(x_1,x_2,...x_k,x_1)\}$$

$\forall F \in C_k(\mathbb{C}_X^\infty)$. Again by computation $b \circ N_t = N_t \circ b$. Let's consider the k-th power of N_t. For $F \in C_k(\mathbb{C}_X^\infty)$ we get

$$(N_t)^k F(x_0, ..., x_k) = \prod_{i=0}^{k-1} \lambda_t(\delta(x_i, x_{i+1})) \cdot G(x_0, ..., x_k),$$

where $G(x_0, ..., x_k)$ is a linear combination of functions built out of F by restricting to certain diagonals and permutation of some arguments. For the product in front of G not to be zero, we must have $\delta(x_i, x_{i+1}) < t$ for each $0 \leq i \leq k-1$. The triangle equation shows that in this case we also have $\sqrt{\delta(x_0, x_k)} < kt^{1/2}$. Thus we have

$$\rho_k(x_0, ..., x_k) = \sum_{i=0}^{k-1} \delta(x_i, x_{i+1}) + \delta(x_k, x_0) < kt + k^2 t.$$

Hence for $F \in C_k^{(k+k^2)t}(\mathbb{C}_X^\infty)$ we have that $(N_t)^k(F) = 0$. Let's define another operator

$$K_t := E_t \cdot \sum_{r=0}^{k-1} (N_t)^r : C_k^{(k+k^2)t}(\mathbb{C}_X^\infty) \to C_{k+1}^{(k+k^2)4^{-(k+1)}t}(\mathbb{C}_X^\infty).$$

By construction this operator satisfies

$$b \circ K_t + K_t \circ b = 1,$$

which proves the acyclicity of $C_*^0(\mathbb{C}_X^\infty)$ by taking the direct limit where t goes to zero. This in fact shows that any Hochschild-homology class can be represented by a Hochschild-cycle $F \in C_k(\mathbb{C}_X^\infty)$ which has support in an arbitrary small neighborhood of the diagonal $X \approx \Delta \subset X^{k+1}$. Now let $(U_i | i \in I)$ be an open covering of X which satisfies the requirements of the proposition. Let $F \in C_k(\mathbb{C}_X^\infty)$ be an arbitrary Hochschild-cycle , then by the discussion above we can assume that

$$supp(F) \subset \bigcup_{i \in I} U_i^{k+1}.$$

Now we can use the $\mathbb{C}_X^\infty$-module structures on $C_k(\mathbb{C}_X^\infty)$ and $HH_k(\mathbb{C}_X^\infty)$ as well as a partition of unity in $\mathbb{C}_X^\infty$ subordinated to the covering $(U_i | i \in I)$ to see that

$$[F] = \sum_{i \in I} [F_i]$$

where $supp(F_i) \subset U_i^{k+1}$. It does now follow from the surjectivity assumption on the antisymmetrization map corresponding to each U_i that each of the $[F_i]$ is in the image of the antisymmetrization map corresponding to X. Hence $[F]$ is also in the image of the antisymmetrization map. This shows that the antisymmetrization map is surjective. Since it always has a continuous left inverse it is in fact an isomorphism. □

With these tools we are able to proof Theorem 4.1.

Proof. Let $n \in \mathbb{N}$. We have to show that the antisymmetrization map ϵ is surjective in equal any Hochschild cycle in $C_n(\mathbb{C}^\infty_X)$ is homologous to an antisymmetric Hochschild cycle. From Lemma 4.3. this follows if we show that this is locally the case. Hence we can assume that our lcnt-space is given as in (5) by $(B^k \times cL, \mathbb{C}^\infty(B^k)\hat{\otimes}\mathbb{C}^\infty_{cL})$ where B^k denotes the open unit ball of dimension k and cL denotes the cone over an lcnt-space of dimension less than the dimension of X (see (3),(4),(7) and Proposition 2.1) . Using induction on the dimension, we can assume that all antisymmetrization maps corresponding to L are isomorphisms and all Hochschild homology groups of L are Hausdorff. Therefore by Lemma 4.2 the same holds for $(cL, \mathbb{C}^\infty_{cL})$. Furthermore by Proposition 3.2 the antisymmetrization maps corresponding to B^k are isomorphisms. Hence the corresponding Hochschild-homology groups are also Hausdorff and Theorem 4.1 follows as in the proof of Lemma 4.2 from the Kuenneth Theorem in Hochschild-homology (see [Ka]) □

5 Cyclic Homology of lcnt-Spaces

In this section we will say something about the cyclic homology groups corresponding to an lcnt-space $(X, \mathbb{C}^\infty_X)$. The result we will present is a straight forward consequence of the general relationship between Hochschild homology and cyclic homology and Theorem 4.1. Using Theorem 4.1 the proof is identical to the proof in [Co]. To shorten this article, we keep this section very informal.

If we divide out a cyclic action from the Hochschild complex (10), in equal identifying cycles, which arise from another by cyclic permutation, we get another complex, which is sometimes called Connes' complex. The homology groups of this complex are called **cyclic homology groups** and will be denoted by $HC_*(A)$. These groups are related to the Hochschild homology groups by the so called Connes' exact sequence

$$...HH_n(A) \xrightarrow{I} HC_n(A) \xrightarrow{S} HC_{n-2}(A) \xrightarrow{B} HH_{n-1}(A) \xrightarrow{I} \cdots .$$

The operator S is the so called Connes periodicity operator and corresponds via some identifications to the Bott periodicity operator in K-theory. In the commutative case it is not hard to show that via the antisymmetrization map, up to a factor the operator

$$B \circ I : HH_n(A) \to HH_{n+1}(A)$$

exactly corresponds to the operator

$$\begin{array}{rcl} d : \Omega^n_A & \to & \Omega^{n+1}_A \\ a_0 da_1 \wedge ... \wedge da_n & \mapsto & da_0 \wedge da_1 \wedge ... \wedge da_n . \end{array}$$

Theorem 5.1. *Let $(X, \mathbb{C}^\infty_X)$ be an lcnt-space with finite dimensional homology groups. Then for all $n \in \mathbb{N}$ there is a natural topological isomorphism*

$$HC_n(\mathbb{C}_X^\infty) \cong \Omega^n_{\mathbb{C}_X^\infty}/d\Omega^{n-1}_{\mathbb{C}_X^\infty} \oplus H^{n-2}(X,\mathbb{C}) \oplus H^{n-4}(X,\mathbb{C}) \oplus H^{n-6}(X,\mathbb{C})...$$

Here $H^k(X,\mathbb{C})$ stands for singular cohomology with coefficients in $\mathbb{C}$. There is also a de Rham theorem for lcnt-spaces[9] and these singular cohomology groups can be identified with the corresponding de Rham cohomology groups.

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