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Hyperquasivarieties

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Abstract. We consider the notion of hyper-quasi-identities and hyperquasivarieties, as a common generalization of the concept of quasi-identity and quasivariety invented by A. I. Mal'cev, cf. [10], cf. [5] and hypervariety invented by the authors in [6].

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1. Notations

An identity is a pair of terms where the variables are bound by universal quantifiers. Let us take the following medial identity as an example

$$\forall u \forall x \forall y \forall w\, (u \cdot x) \cdot (y \cdot w) = (u \cdot y) \cdot (x \cdot w).$$

Let us look at the following hyperidentity

$$\forall F \forall u \forall x \forall y \forall w\, F(F(u,v), F(x,y)) = F(F(u,x), F(v,y)).$$

The hypervariable F is considered in a very specific way. Firstly every hypervariable is restricted to functions of a given arity. Secondly F is restricted to term functions of the given type. Let us take the variety $A_{n,0}$ of abelian groups of finite exponent n. Every binary term $t \equiv t(x,y)$ can be presented by $t(x,y) = ax + by$ with $a, b \in \mathbb{N}_0$. If we substitute the binary hypervariable F in the above hyperidentity by $ax + by$, leaving its variables unchanged, we get

$$a(au + bv) + b(ax + by) = a(au + bx) + b(av + by).$$

This identity holds for every term $t(x,y) = ax + by$ for the variety $A_{n,0}$. Therefore we say that the hyperidentity holds for the variety $A_{n,0}$.

2. Hyper-quasi-identities

In the sequel we use the definition of *hyperterm* from [6]. We accept the notation of [2], [8], [5] and [9].

We recall only our definitions of [6] of the fact that a hyperidentity is satisfied in an algebra of a given type and the notion of hypervariety:

Definition 2.1. An algebra $\mathbf{A}$ satisfies a hyperidentity $h_1 = h_2$ if for every substitution of the hypervariables by terms (of the same arity) of $\mathbf{A}$ leaving the variables unchanged, the identities which arise hold in $\mathbf{A}$. In this case, we write $\mathbf{A} \models (h_1 = h_2)$. A variety V satisfies a hyperidentity $h_1 = h_2$ if every algebra in the variety does; in this case, we write $V \models (h_1 = h_2)$.

Definition 2.2. A class V of a algebras of a given type is called a hypervariety if and only if V is defined by a set of hyperidentities.

The following was proved in [6]:

Theorem 2.3. *A variety V of type τ is defined by a set of hyperidentities if and only if $V = HSPD(V)$, i.e. V is a variety closed under derived algebras of type τ.*

We recall from [10] and [5]:

Definition 2.4. A quasi-identity e is an implication of the form:

$$(t_0 = s_0) \wedge ... \wedge (t_{n-1} = s_{n-1}) \rightarrow (t_n = s_n).$$

where $t_i = s_i$ are k-ary identities of a given type, for $i = 0, ..., n$.

A quasi-identity above is *satisfied in an algebra* $\mathbf{A}$ of a given type if and only if the following implication is satisfied in $\mathbf{A}$: given a sequence $a_1, ..., a_k$ of elements of A. If this elements satisfy the equations $t_i(a_1, ..., a_k) = s_i(a_1, ..., a_n)$ in $\mathbf{A}$, for $i = 0, 1, ..., n-1$, then the equality $t_n(a_1, ..., a_k) = s_n(a_1, ..., a_k)$ is satisfied in $\mathbf{A}$. In that case we write:

$$\mathbf{A} \models (t_0 = s_0) \wedge ... \wedge (t_{n-1} = s_{n-1}) \rightarrow (t_n = s_n).$$

A quasi-identity e is *satisfied in a class* V of algebras of a given type, if and only if it is satisfied in all algebras $\mathbf{A}$ belonging to V.

We modify the definition above in the following way:

Definition 2.5. A hyper-quasi-identity e^* is an implication of the form:

$$(T_0 = S_0) \wedge ... \wedge (T_{n-1} = S_{n-1}) \rightarrow (T_n = S_n).$$

where $T_i = S_i$ are hyperidentities of a given type, for $i = 0, ..., n$.

A hyper-quasi-identity e^* is hyper-satisfied (holds) in an algebra $\mathbf{A}$ is and only if the following implication is satisfied:
If σ is a hypersubstitution of type τ and the elements $a_1, ..., a_n \in A$ satisfy the equalities $\sigma(T_i)(a_1, ..., a_k) = \sigma(S_i)(a_1, ..., a_k)$ in $\mathbf{A}$, for $n = 0, 1, ..., n-1$, then the equality $\sigma(T_n)(a_1, ..., a-k) = \sigma(S_n)(a_1, ..., a_k)$ holds in $\mathbf{A}$.

By other words, hyper-quasi-identity is a universally closed Horn $\forall x \forall \sigma$-formulas, where x vary over all sequences of individual variables (occuring in terms of the implication) and σ vary over all hypersubstitutions of a given type.

Remark. All hyper-quasi-identities and hyperidentities are written without quantifiers but they are considered as universally closed Horn $\forall$-formulas (cf. [10]). A syntactic side of the notions described here will be considered in a forthcoming paper.

3. Examples of hyper-quasi-identities

3.1. Hyper-quasi-identities for abelian algebras.

Definition 3.1. An algebra $\mathbf{A}$ is called abelian if for every $n > 1$ and every n-ary term operation f of $\mathbf{A}$ and for all $u, v, x_1, ..., x_{n-1}, y_1, ..., y_{n-1}$ the following equivalence holds:

$$f(u, x_1, ..., x_n) = f(u, y_1, ..., y_n) \leftrightarrow f(v, x_1, ..., x_n) = f(v, x_1, ..., x_n)$$

A variety V is called abelian, if each algebra of V is abelian.

It follows from [9, p. 40], [4, p. 290]:

Proposition 3.2. *An algebra* $\mathbf{A}$ *is abelian if and only if the following hyper-quasi-identity holds in* $\mathbf{A}$*:*

$$F(u, x_1, ..., x_n) = F(u, y_1, ..., y_n) \leftrightarrow F(v, x_1, ..., x_n) = F(v, y_1, ..., y_n).$$

Example. The variety RB of rectangular bands fulfills the hyperidentities of type $(1, 2, 3, ..., n, ...)$:
$F(x, ..., x) = x$, $F(F(x_{11}, ..., x_{1n}), x_2, ..., x_n) = F(x_{11}, x_2, ..., x_n)$,
$F(F(x_1, ..., x_{n-1}), F(x_{n1}, ..., x_{nn})) = F(x_1, ..., x_{n-1}, x_{nn})$.
We can derive by the associative hyperidentity: $F(x, F(y, z)) = (F(x, y), z)$, that the following hyper-quasi-identity holds in RB:
$F(u, x_1, ..., x_n) = F(u, y_1, ..., y_n) \to F(v, u, x_1, ..., x_{n-1}) = F(v, u, y_1, ..., y_{n-1}) \to F(v, x_1, ..., x_n) = F(v, y_1, ..., y_n)$, i.e. the variety RB is abelian.

Remark. For further examples of abelian varieties consult [9] and [13].

3.2. Semidistributive lattices.

A lattice is *joinsemidistributive* if it satisfies the following condition, cf. [9, p. 82], [5, p. 141]:

$$(SD_\vee)\ x \vee y = x \vee z \text{ implies } x \vee y = x \vee (y \wedge z)$$

The *meetsemidistributivity* is defined by duality:

$$(SD_\wedge)\ x \wedge y = x \wedge z \text{ implies } x \wedge y = x \wedge (y \vee z)$$

A lattice is *semidistributive* if it is simultaneously join and meet semidistributive.

Proposition 3.3. *Let* $\mathbf{L} = (L, \wedge, \vee)$ *be a lattice which is semidistributive. Then the following hyper quasi-identity is hypersatisfied in* $\mathbf{L}$*:*

$$(F(x, y) = F(x, z)) \to (F(x, y) = F(x, G(y, z)))$$

Proof. We consider all cases on hypervariables of a semidistributive lattice.

Case 1. $F(x, y) := x$.
Obviously $(x = x) \to (x = x)$.

Case 2. $F(x, y) := y$. Consider $G(y, z) := y, z, y \wedge z, y \vee z$.
Then the following quasi-identities are satisfied in $\mathbf{L}$:
$(y = z) \to (y = y)$, $(y = z) \to (y = z)$, $(y = z) \to (y = y \wedge z)$,
$(y = z) \to (y = y \vee z)$.

Case 3. $F(x, y) := x \wedge y$. Consider $G(y, z) := y, z, y \wedge z, y \vee z$.
Then the following quasi-identities hold in $\mathbf{L}$:

$(x \wedge y = x \wedge z) \to (x \wedge y = x \wedge y)$, $(x \wedge y = x \wedge z) \to (x \wedge y = x \wedge z)$, $(x \wedge y = x \wedge z) \to (x \wedge y = x \wedge (y \wedge z))$, $(x \wedge y = x \wedge z) \to (x \wedge y = x \wedge (y \vee z))$. The last one follows from the meet semidistributivity of $\mathbf{L}$.

Case 4. $F(x,y) := x \vee y$. Consider $G(y,z) := y, z, y \wedge z, y \vee z$.
Then the following quasi-identities hold in $\mathbf{L}$:
$(x \vee y = x \vee z) \to (x \vee y = x \vee y)$, $(x \vee y = x \vee z) \to (x \vee y = x \vee z)$, $(x \vee y = x \vee z) \to (x \vee y = x \vee (y \wedge z))$, $(x \vee y = x \vee z) \to (x \vee y = x \vee (y \vee z))$. The prelast one follows from the join semidistributivity of $\mathbf{L}$. □

4. **Derived algebras**

We recall from [6], cf. [2, p. 145] the notion of a *derived algebra* and the *derived class* of algebras. Given a type $\tau = (n_0, n_1, ..., n_\gamma, ...)$. An algebra $\mathbf{B}$ of type τ is called a derived algebra of $\mathbf{A} = (A, f_0, f_1, ..., f_\gamma, ...)$ if there exist term operations $t_0, t_1, ..., t_\gamma, ...$ of type τ such that $\mathbf{B} = (A, t_0, t_1, ..., t_\gamma, ...)$. For a class K of algebras of type τ we denote by $\mathbf{D}(K)$ the class of all derived algebras (of type τ) of K.
A class K is called *deriverably closed* if and only if $\mathbf{D}(K) \subseteq K$.

For a given algebra $\mathbf{A}$ we denote by $QId(\mathbf{A})$ and $HQId(\mathbf{A})$ the set of all quasi-identities and hyper-quasi-identities satisfied (hypersatisfied) in $\mathbf{A}$, respectively. Similarly for a class K, $QId(K)$ and $HQId(K)$ denote the set of all quasi-identities and hyper-quasi-identities satisfied (hypersatisfied) in K, respectively. Following [5] by $\mathbf{Q}(K)$ and $H\mathbf{Q}(K)$ we denote the class of all algebras of a given type satisfying (hypersatisfying) all the quasi-identities and hyper-quasi-identities of K, respectively. The transfomation T of an quasi-identity e into hyper-quasi-identity e^* is defined in a natural way. Similarly T^{-1} transform every hyper-quasi-identity e^* into the qasiidentity e. (cf. [6]).

Proposition 4.1. *Given a class K of algebras of type τ. Then the following equality holds:*

$$HQId(K) = T(QId(\mathbf{D}(K))).$$

Proof. To prove the inclusion $\subseteq$, given a hyper-quasi-identity e^* of K and an algebra a derived algebra $\mathbf{B} = \mathbf{A}^\sigma$ for $\mathbf{A} \in K$. Then by definition 2.5 $T^{-1}(e)$ is satisfied in $\mathbf{B}$ as a quasi-identity, i.e. $e^* \in T(QId(\mathbf{D}(K)))$.

For a proof of the converse inclusion, given a quasi-identity e satisfied in $\mathbf{D}(K)$ and an algebra $\mathbf{A}$ of K. Let $a_1, ..., a_k \in A$. Given an hypersubstitution σ of type τ and consider $\sigma(e)$ in $\mathbf{A}$. As $\mathbf{A} \in \mathbf{D}(K)$, then $\sigma(e)$ may be considered as a quasi-identity of the derived algebra $\mathbf{A}^\sigma$ (cf. [12]). Assume that $p_i^{\mathbf{A}^\sigma}(a_1, , , a_k) = q_i^{\mathbf{A}^\sigma}(a_1, ..., a_k)$, therefore as $\mathbf{A} \in \mathbf{D}(K)$ we obtain that $p_n^{\mathbf{A}^\sigma}(a_1, ..., a_k) = q_n^{\mathbf{A}^\sigma}(a_1, ..., a_n)$, i.e. that the $T(e)$ holds in K as a hyper-quasi-identity, i.e. $T(e) \in HQId(K)$. □

The role of derived algebras in solvability questions may be visualized by the following:

Theorem 4.2. *Given a locally finite variety V. The class K of all locally solvable algebras in V is a hypervariety.*

Proof. By corollary 7.6, of [9] the class of all locally solvable algebras in a locally finite variety V is a variety. Due to our theorem 2.3 we need to show that K is deriverably closed. Given a derived algebra $\mathbf{B}$ of an algebra $\mathbf{A} \in K$ and its finite subalgebra $\mathbf{F}^*$. Then $\mathbf{F}^*$ is a derived subalgebra of a finite subalgebra $\mathbf{F}$ of $\mathbf{A}$, i.e. $\mathbf{F}^* = \mathbf{F}^\sigma$, for a hypersubstitution σ. From theorem 5.7 of [12] we conclude, that as $\mathbf{F}$ is solvable, therefore $\mathbf{F}^\sigma$ is solvable. □

Remark. The assumption that V is locally finite is essential, as for the variety of abelian groups AG, which are locally solvable, it is easy to notice that AG does not constitute a hypervariety, as the abelian law: $xy = yx$ is not satisfied in AG as a hyperidentity.

4.1. **Quasicompact classes.** We accept the definition of a *quasicompact classes* invented by V. A. Gorbunov [5, p. 77] and his theorem 2.3.1, which states:

Theorem 4.3. *For a class K, the prevariety* $\mathbf{SP}(K)$ *is a quasivariety if and only if K is quasicompact.*

We reformulate the definition of *prevariety* to a *hyperprevariety* and theorem above for the case of hyper-quasi-identities:

Definition 4.4. For a a class K of algebras, the class $\mathbf{SP}(\mathbf{D}(K))$ (for short $\mathbf{SPD}(K)$) will be called hyperprevariety generated by K.

A class K of algebras of a given type is hyper-quasi-compact provided that from the infinite implication, indexed by I and hyper-satisfied in K it follows that for some finite subset $F \subseteq I$ the finite implication (restricted to F) is hypersatisfied in K.

Theorem 4.5. *For a class K, the prevariety* $\mathbf{SP}(\mathbf{D}(K))$ *is a quasivariety if and only if* $\mathbf{D}(K)$ *is quasicompact if and only if K is hyper-quasi-compact.*

Proof. The first equivalence is the V. A. Gorbunov theorem of [5, p. 77].
The second equivalence follows from the proposition 4.1. □

5. **Hyper-quasi-varieties**

We reformulate the notion of *quasivariety* invented by A. I. Mal'cev in [10, p. 210] for the case of *hyper-quasi-identities* of a given type in a natural way:

Definition 5.1. A class K of algebras of type τ is called a hyperquasivariety if there is a set Σ of hyper-quasi-identities of type τ such that K conisits exactly of those algebras of type τ that hypersatisfy all the hyper-quasi-identities of Σ.

From proposition 4.1 we obtain immediately:

Theorem 5.2. *A quasivariety K of algebras given type is a hyperquasivariety if and only if it is deriverably closed.*

In the sequel we use the standard notation: $\mathbf{S}$ for the operator of creating *subalgebras*, $\mathbf{P}$, $\mathbf{P_s}$, $\mathbf{P_r}$, $\mathbf{P_u}$ and $\mathbf{P}\omega$ for *products, subdirect products, reduced products, ultraproducts* and *direct products of finite families of structures*, respectively. $\mathbf{L}$ and $\mathbf{L_s}$ will denote the operators of *direct limits* and *superdirect limits*.

Following A. I. Mal'cev [10, p. 153, 215] for a given class K of algebras of a given type, by $\mathbf{S}(K)$, $\mathbf{P}(K)$, $\mathbf{P_s}(K)$, $\mathbf{P_r}(K)$, $\mathbf{P_u}(K)$ we denote the class of algebras isomorphic to all possible subalgebras, direct products, subdirect products, reduced (filtered) products or ultraproducts of algebras of K, respectively. $\mathbf{P}_\omega$ is the class of algebras isomorphic to the direct products of finite families of structures of K. Similarly, $\mathbf{L}(K)$ and $\mathbf{L_s}(K)$ will be the class of algebras isomorphic to direct and superdirectlimits of algebras of K, respectively (cf. [5, p.21].

Adding a trivial system to K we obtain the class K_0.

We reformulate the resut of [6]:

Proposition 5.3. *Given a class K of algebras, then the following inclusions holds:*
1) $\mathbf{DS}(K) \subseteq \mathbf{SD}(K)$*;*
2) $\mathbf{DP}(K) \subseteq \mathbf{PD}(K)$*;*
3) $\mathbf{DP}_\omega(K) \subseteq \mathbf{P}_\omega\mathbf{D}(K)$*;*
4) $\mathbf{DP_s}(K) \subseteq \mathbf{P_s}\mathbf{D}(K)$*;*
5) $\mathbf{DP_r}(K) \subseteq \mathbf{P_r}\mathbf{D}(K)$*;*
6) $\mathbf{DP_u}(K) \subseteq \mathbf{P_u}\mathbf{D}(K)$*;*
7) $\mathbf{DL}(K) \subseteq \mathbf{LD}(K)$*;*
8) $\mathbf{DL_s}(K) \subseteq \mathbf{L_s}\mathbf{D}(K)$*.*

Proof. Obviously an isomorphism respects the inclusions above.
The inclusions 1) and 2) were proved in [6].
3) and 4) immediately follows from 2).

To show 5) assume that an algebra $\mathbf{B} \in \mathbf{DP_r}(K)$. Let $\mathbf{B} = (A, t_0^{\mathbf{A}}, t_1^{\mathbf{A}}, ..., t_\gamma^{\mathbf{A}}, ...)$ is a derived algebra of a reduced product $\mathbf{A} = (A, f_0^{\mathbf{A}}, f_1^{\mathbf{A}}, ..., f_\gamma^{\mathbf{A}}, ...)$, where $A = (\Pi_{i\in I} A_i)/\sim_F$, for a set I, a family $(\mathbf{A}_i)_{i\in I}$ of algebras of type τ from K and a filter F over I, i.e. $\mathbf{A} = (\Pi_{i\in I}\mathbf{A}_i)/F$, where for any n-ary functional symbol f the following holds (cf. [5, p. 13]):

$$f^{\mathbf{A}}(a_0/F, ..., a_{n-1}/F) = a_n/F \text{ if and only if } \{i \in I : f^{\mathbf{A}_i}(a_0, ..., a_{n-1}) = a_n\} \in F.$$

Note that by the induction on the complexity of a term t of type τ one may show, that the following holds for any n-ary polynomial t of type τ:

$$t^{\mathbf{A}}(a_0/F, ..., a_{n-1}/F) = a_n/F \text{ if and only if } \{i \in I : t^{\mathbf{A}_i}(a_0, ..., a_{n-1}) = a_n\} \in F.$$

From the above equality it follows that the algebra $\mathbf{B} = \Pi_{i\in I}\mathbf{A}_i^\sigma/F$, where $\mathbf{A}_i^\sigma = (A_i, t_0^{\mathbf{A}_i}, t_1^{\mathbf{A}_i}, ..., t_\gamma^{\mathbf{A}_i}, ...)$, for $i \in I$ is a derived algebra of $\mathbf{A}_i$, i.e. $\mathbf{B} \in \mathbf{P_r}\mathbf{D}(K)$.

The inclusion 6) is an immediate consequence of 5).

To prove 7), let $\mathbf{B} \in \mathbf{DL}(K)$, i.e. given a derived algebra $\mathbf{B}$ of an algebra $\mathbf{A}$, where $\mathbf{A} = (A, f_0^{\mathbf{A}}, f_1^{\mathbf{A}}, ..., f_\gamma^{\mathbf{A}}, ...)$ is a direct limit of a triple $\Lambda = (I, \mathbf{A}_i, g_{ij})$, where $I = (I, \leq)$ is an up-directed set, $(\mathbf{A}_i)_{i\in I}$ is a family of algebras of a given type τ and $\{g_{ij} : i, j \in I, i \leq j\}$ is a family of homomorphisms of $\mathbf{A}_i$ into $\mathbf{A}_j$ called a *direct* *spectrum* over $(I, \leq)$, cf. [5, p. 17]. For a given direct spectrum $\Lambda = (\mathbf{A}_i, g_{ij})$, we consider the quotient set $A = \bigcup_{i \in I} A_i \times \{i\}/\equiv$, where

$$(a, i) \equiv (b, j) \text{ if and only if } (\exists k \in I)(i, j \leq k, g_{ik}(a) = g_{jk}(b)).$$

Let $< a, i >$ denotes the equivalence class by $\equiv$ containing (a, i). The operations on A are defined by setting for any operation symbol of type τ:

$$f^{\mathbf{A}}(< a_0, i_0 >, ..., < a_{n-1}, i_{n-1} >) =< f^{\mathbf{A}_j}(g_{i_0 j}(a_0), ..., g_{i_{n-1} j}(a_{n-1})), j >.$$

Note, that as g_{ij} are homomorphisms for all $i \leq j$, $i, j \in I$, therefore for any polynomial symbol p of type τ, the following holds:

$$p^{\mathbf{A}}(< a_0, i_0 >, ..., < a_{n-1}, i_{n-1} >) =< p^{\mathbf{A}_j}(g_{i_0 j}(a_0), ..., g_{i_{n-1} j}(a_{n-1})), j >.$$

As $\mathbf{B} = \mathbf{A}^\sigma$ for a hypersubstitution σ of type τ, therefore $\mathbf{B} \in \mathbf{LD}(K)$, namely $\mathbf{B}$ is a direct limit of the triple $\Lambda^\sigma = (I, \mathbf{A}_i^\sigma, g_{ij})$ of derived algebras of K, i.e. of a derived spectrum of Λ.

To prove 8), recall [5, 17], that a direct spectrum $\Lambda = (I, \mathbf{A}_i, gij)$ is called *superdirect* if the mappings $g_{ij} : A_i \to A_j$ are surjective. The family $(\mathbf{A}_i)_{i \in I}$ is referred to as *superdirect family.* The direct limit of a superdirect spectrum is called the *superdirect limit.* Note, that the derived spectrum Λ^σ of a superdirect spectrum Λ is superdirect. This together with 7) proves 8). □

Via corollary 2.3.4 [5, p. 79] and proposition 5.1 we obtain another characterization of hyperquasivarieties:

Proposition 5.4. *For any class K of algebras, the following assertions hold:*
(1) $H\mathbf{Q}(K) = \mathbf{SPP_uD}(K)$*;*
(2) $H\mathbf{Q}(K) = \mathbf{SP_uPD}(K)$*;*
(3) $H\mathbf{Q}(K) = \mathbf{SP_uP}_\omega\mathbf{D}(K)$*;*
(4) $H\mathbf{Q}(K) = \mathbf{SL_sPD}(K) = \mathbf{L_sSPD}(K) = \mathbf{L_sP_sSD}(K)$*.*

Via proposition 4.1 we obtain the following reformulation of Mal'cev theorems of [10, p. 214 - 215]:

Theorem 5.5. *A class K of algebras of a given type is a hyperquasivariety if and only if K*
i) is ultraclosed;
ii) is heraditery;
iii) is multiplicatively closed;
iv) contains a trivial system;
v) is deriverably closed.

Theorem 5.6. *For every class K of algebras of a given type we have*

$$H\mathbf{Q}(K) = \mathbf{SP_rD}(K_0)$$

Remark. If we accept, that the direct product of an empty family of algebras of a given type is a trivial algebra of a given type, we may remove condition iv) of theorem 5.4 and substitute K_0 by K in theorem 5.4. (cf. Corollary 2.3. of [5, p. 78]).

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