Bills of Material and Linear Algebra

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MaMaEuSch‡-Report
Management Mathematics for European Schools

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‡MaMaEuSch has been carried out with the partial support of the European Community in the framework of the Socrates programme and with partial support of the state of Rheinland-Pfalz and with partial support of the VolkswagenStiftung. The content of the project does not necessarily reflect the position of the European Community, nor does it involve any responsibility on the part of the European Community.

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Chapter 1

Preliminary Considerations

1.1 Why Operational Planning and Decision Problems in School Mathematics ?

In the middle of the year 1997, the publication of the so called TIMMS-Study (*Third International Mathematics and Science Study)* has caused a monumental commotion in the German public. The reasons for this were the mathematical-scientific school achievements reached by the German pupils at the end of grade eight, which were merely in the international mid field, whereas especially in the mathematical field the majority of the northern, eastern and western European states which participated in TIMSS - not to speak of most of the asian countries - obtained perspicuously better results. All together not only a gap in school achievements of more than one school year compared to the European neighboring states and Germany's most important trade partners had to be certified to the German pupils, furthermore they showed an aggravating deficit in the field of conceptual-textual understanding of mathematics. This failing of the German pupils, not to be able to solve routine tasks and open conceptual formulations by independently applying what they have learned and transferring this into new contexts, is confirmed once again in the in 1999 published third part of the TIMMS-Study, which was mentioned above. Also in the new survey in the sixth form the results of the German pupils were still only in the lower mid field of the group of comparable countries, whereas the distances to the countries with great school achievements had even enlarged, the distances to the states with weak school achievements had diminished in comparison to secondary school. Especially from such exercises which leave the school context and which are regarded as symptomatical for the professional world monumental deficits arouse to some extent. So even among the pupils in the mathematical major courses only 10% are able to understand application-orientated problems, to structure them and to solve them successfully with already known mathematical procedures.

As a reaction to the humbling results of the TIMSS-Study the present mathematical-scientific education was criticized by the majority as too much centered on skills and too ostensible knowledge-orientated. As an instrument for an improvement of the current situation the concept of application-oriented school mathematics became more and more the center of the didactical discussion - since the mediocre German school performances had obviously only little to do with the different school forms, but much with the related forms of learning. The background of this approach is the scientifically proven knowledge that the furtherance of both the systematic and the situational learning are the essential requirements for the acquisition of intelligent, flexible applicable knowledge. In other words: Besides a well-organized disciplinary acquisition of knowledge, an application of the acquired knowledge in interdisciplinary and problem-orientated contexts, which are true to life, is required right from the beginning if the in principle available knowledge shall not remain dead, inactive and unused.

With this in mind the inclusion of applications in school mathematics pursues a double objective. On the one hand a better understanding of the situation, which would only be defectively understandable in essential parts without mathematical treatment, is aspired. On the other hand an essential insight into the significance of mathematics for the accomplishment of tasks of our everyday life is gained by demonstrating the tool-character of mathematics for solving real-world problems. But the frequently in this context existing reference to the ,,textual tasks", which are also popular in the present education, does not achieve the desired effect since here in the majority unrealistic pseudo-problems, false applications and problems with a strongly restricted contents, which are unmasked by the pupils as such tasks, are concerned.

Compared with this the present paper tries to develop the mathematical subject matter from concrete, at first sight quite ambitious and complex problems. Points of reference for this are real, in their complexity only minimally reduced applications from management mathematics, like they were already solved by the Arbeitsgruppe Mathematische Optimierung of the University of Kaiserslautern and the corresponding department of the Institut für Techno- und Wirtschaftsmathematik (ITWM) for customers from economics and management. Chapter 1.2 describes such an application.

In the present paper operational planning and decision problems, like they have to be solved in almost all companies, are centered. Thus one expects a greater interest from the pupils and an attendant improvement of the pupils' school achievements.

Further on this approach shows one of the rare possibilities to meet the claim of the curriculum for school mathematics concerning the trilogy *relevance of application, multidisciplinarity* and *mathematical modelling*. So for example the curriculum for the sixth form in Rhineland-Palatinate demands explicitly: ,,A further function of school mathematics is to give the pupils an understanding of the process of mathematics. Where mathematical methods can be used to structure a problem, to describe important aspects of a complex issue in a model and to search for a solution, interrelations between theory and practice can be experienced. (...) Pupils (...) shall find relations between a nonmathematical issue and mathematics, solve the problem with mathematical methods, interpret the solutions and judge them critically. Thereby also limits of specific procedures and limits of mathematics shall be discovered."

1.2 Example from Management Mathematics

In the operational reality the manufacturing methods are only occasionally linear. In the prevailing production structures the assembly does not proceed "stepped" in different production levels, but both certain preliminary products and certain intermediate products, which were made out of these preliminary products before, are included in the production.

Moreover even mutual relations can be found in the production process where then for example an intermediate product enters directly the final product as well as the manufacturing of other intermediate products. In this case one speaks of a so-called **joint production**.

The Institut für Techno- und Wirtschaftsmathematik (ITWM) in Kaiserslautern solved the following problem for a huge German software-company.

Figure 1.1 shows a cyclic production process, which can be found for example in refineries or chemical industries.

One can see in this figure that the products with the numbers 4, 5, 7, 8, 9, 12, 13, 16, 17 enter the production process (,,In"), but are not produced in this process. So one would better call these products **raw-materials** . The other

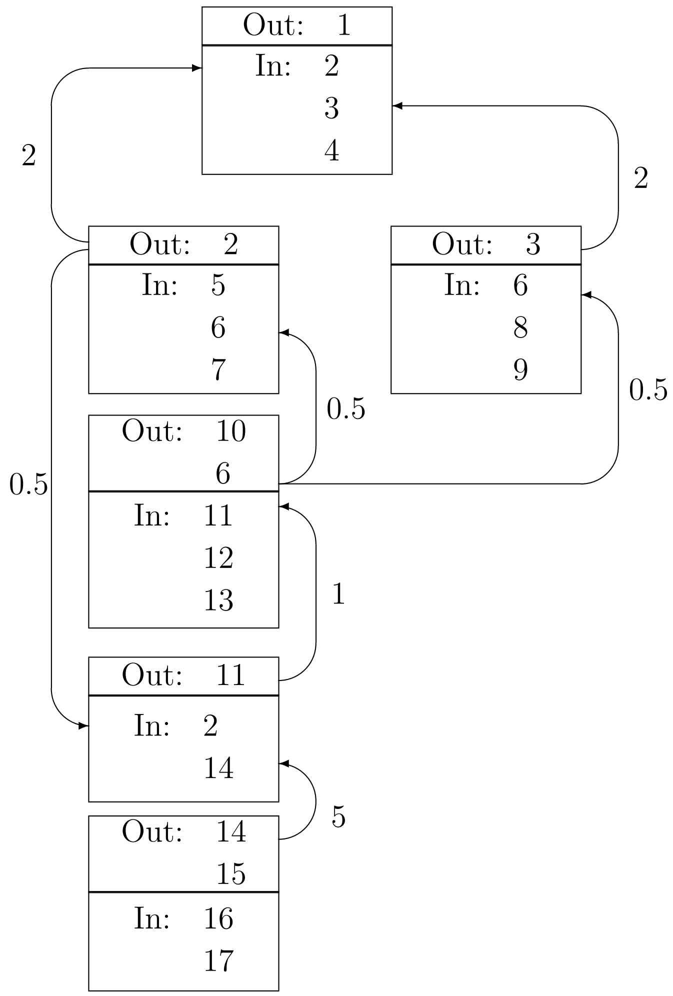

Figure 1.1: Cyclic production process

products are manufactured in different production stages („Out") and are used again. Considering product 2, which is needed in those production processes which are superior (production process of product 1) as well as inferior (production process of product 11) to it's own production process, it is easy to see that a cyclic relation occurs in this production process. The numbers next to the arrows between the different production processes indicate how many units of one product enter another production process.

The difficulty now is to calculate different parameters of the production process quickly. The different production processes and the storage of raw-materials, intermediate and end products cause costs. The raw-materials have to be bought. In contrast the selling of end and eventually also of intermediate products yields receipts. Taking into account all these and eventually even more factors, one tries for example to realize a maximum profit. It would also be possible to minimize the production time or the environmental pollution subject to a given production. The solution for the problem which was solved by the ITWM would surely be too complex in this part of the paper. Thus in the following first of all simplified problems will be discussed in order to be finally able to look at this concrete problem in detail.

1.3 Relations between the Parts in the Production Process

In chapter 2.1 the issue of the relations between the parts in a production process shall be discussed as a simplified problem, which is basically partitioned into two different problems:

- Assuming external given orders, the demand of raw material for the manufacturing of certain ordered products has to be calculated.
- If the above problem is reversed, the question about the required production volume which is necessary to use up the existing stocks comes up.

Chapter 2

Solving Operational Questions by Means of Linear Algebra

The tasks described in section 1.3 from the range of operational relations between the parts can be solved by means of fundamental knowledge from Linear Algebra.

2.1 Analysis of Operational Production Processes

In the following text the assumption that the production of single goods takes place in a linear multistage process is centered. So the company which has to be analyzed produces for example intermediate products by assembling different raw products in department A from which then in department B end products are manufactured. Such an assembly structure, which consists of only two production stages, illustrates figure 2.1.

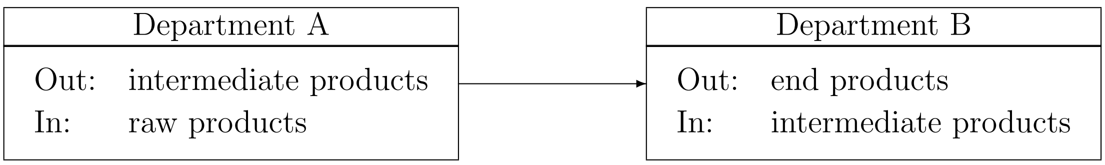

Figure 2.1: Linear, multistage production process

2.1.1 Computation of the Demand of Raw Material subject to Given Orders

Now the quantitative connection between the external given orders and the resulting demand of raw material for the production is needed.

The following slightly simplified example of the production structure in the fictive furniture company ABC-Furniture may serve as an illustration:

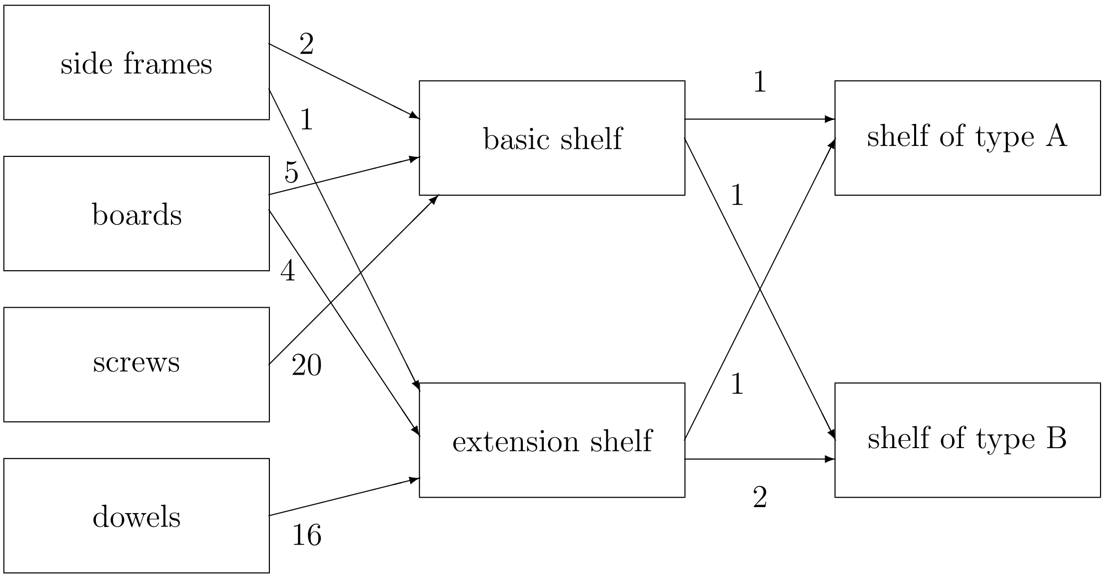

Figure 2.2: Material flow of the furniture company

Considering the material flow graph in figure 2.2 the demand of material for the production of ten shelves of type A and five shelves of type B can be determined by counting. So one needs for just the five shelves of type B $5{\cdot}2{+}5{\cdot}2 = 20$ side frames, $5 \cdot 5 + 5 \cdot 2 \cdot 4 = 65$ boards, $5 \cdot 20 = 100$ screws and $5 \cdot 2 \cdot 16 = 160$ dowel.

If one increases the number of possible shelf types, the computational effort gets rapidly very high. The number of necessary operations even for this simple example indicates that here a suitable mathematical procedure, which displays the required raw materials based on the customers' diverse orders, would result in a significant saving of work.

What is missing is a **mathematical modelling** of the problem. It seems to be reasonable in the first step to write down the received orders in the form of a list, the so-called **bills of material** [1]. Besides the order vector $\vec{x}$ the intermediate and raw products can as well be represented analogously by the vectors $\vec{y}$ and $\vec{z}$.

To simplify the original task of the computation of the demand of parts, the following sub-problems can be derived:

- How many intermediate products of each kind (basic shelf, extension shelf) are required to comply with an order of end products (shelf of type A and B)?
- How much raw material of each kind (side frames, boards, screws and dowels) are needed for the production of a specific number of intermediate products?

To answer the first question one considers the fictive order vector $\vec{x} = \binom{10}{7}$. One takes the required demand of intermediate products from the material flow diagram in figure 2.2:

number of required basic shelves: $y_1 = 10 \cdot \underline{1} + 7 \cdot \underline{1} = 17$
number of required extension shelves: $y_2 = 10 \cdot \underline{1} + 7 \cdot \underline{2} = 24$

The underlined values can be combined in one matrix.

$$P_1 = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$$

The in this way developed so-called **production matrix** P_1 illustrates the events in the last production step.

Reverting to the scalar product and the new form of notation, the computation above can now be written in a very compact way:

$$\vec{y} = P_1 \cdot \vec{x} \tag{2.1}$$

Respectively with the numbers from the example above:

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \cdot \begin{pmatrix} 10 \\ 7 \end{pmatrix} = \begin{pmatrix} 17 \\ 24 \end{pmatrix}$$

The second question about the required raw materials subject to given intermediate products can be solved analogously. To begin with one determines the

[1] Thus it is easily possible to introduce the notion of a vector as an ordered number-n-tuple.

corresponding production matrix P_2 from the material flow diagram:

$$P_2 = \begin{pmatrix} 2 & 1 \\ 5 & 4 \\ 20 & 0 \\ 0 & 16 \end{pmatrix}$$

Thus for the demand of raw material in the first phase of the production

$$\vec{z} = P_2 \cdot \vec{y} \tag{2.2}$$

holds. With this follows for the example:

$$\vec{z} = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 5 & 4 \\ 20 & 0 \\ 0 & 16 \end{pmatrix} \cdot \begin{pmatrix} 17 \\ 24 \end{pmatrix} = \begin{pmatrix} 58 \\ 181 \\ 340 \\ 384 \end{pmatrix}$$

Thus 58 side frames, 181 boards, 340 screws and 348 dowel are required to comply the order.

From equation 2.1 and 2.2 follows:

$$\vec{z} = P_2 \cdot P_1 \cdot \vec{x} \tag{2.3}$$

One computes the production matrix P for the whole production:

$$P = P_1 \cdot P_2 = \begin{pmatrix} 2 & 1 \\ 5 & 4 \\ 20 & 0 \\ 0 & 16 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 3 & 4 \\ 9 & 13 \\ 20 & 20 \\ 16 & 31 \end{pmatrix}$$

2.1.2 Clearing the Existent Stocks

Basically the ideas above hold as well for the reversal of the question, if subject to a given stock of inventory the maximum possible production volume shall be determined. Here as well the pure computational effort is significant, whereas the task is even more complicated since possibly the existent inventory of raw materials cannot be assigned explicitly to the individual finished products. Thus normally at first some of the in principle possible alternatives have to be considered until one finally finds the optimum assignment in the sense of a minimal residual stock of inventory.

In this regard now the so far considered task of the computation of the demand of parts in a linear production structure can even be extended, whereby the task gets closer to reality.

Problem:
Since in the meantime the shelves which are produces in the company ABC-Furniture are no longer in great demand, the direction of the company decided to create a new design for the shelves, to produce the old ones no longer and to sell out the parts still in stock - except the screws and dowels, which can still be used in the future. In a stock-taking it was ascertained that 39 side frames and 120 boards are still in stock. If possible, the company management does not want to have any parts left over and thus searches for a combination of shelves which makes it possible to offer the remainder as a bargain so that all side frames and boards will be used up completely. The modified material flow diagram of the furniture company can be seen in figure 2.3.

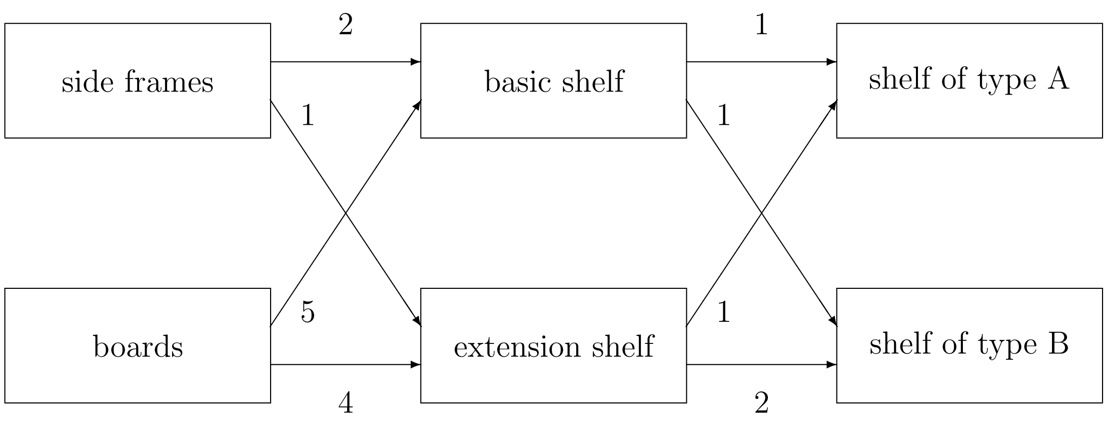

Figure 2.3: Modified material flow of the furniture company

Reverting to the knowledge already gained in chapter 2.1.1, the material flow is described in a new production matrix P.

$$P = P_1 \cdot P_2 = \begin{pmatrix} 2 & 1 \\ 5 & 4 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 3 & 4 \\ 9 & 13 \end{pmatrix}$$

In contrast to the example of the computation of the demand of parts, which was mentioned before, the problem now is reversed. Instead of determining the raw material vector $\vec{z}$ subject to a given order vector $\vec{x}$, now the former is known and $\vec{x}$ is in demand. The matrix-vector-product is:

$$\begin{pmatrix} 3 & 4 \\ 9 & 13 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 39 \\ 120 \end{pmatrix} \tag{2.4}$$

For the calculation of the inverse of $P = \begin{pmatrix} 3 & 4 \\ 9 & 13 \end{pmatrix}$ two systems of linear equations have to be solved:

$$\begin{pmatrix} 3 & 4 \\ 9 & 13 \end{pmatrix} \cdot \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 3 & 4 \\ 9 & 13 \end{pmatrix} \cdot \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Since the systems only differ on the right hand side, the calculations can be combined in one scheme with two right hand sides:

$$\left(\begin{array}{cc|cc} 3 & 4 & 1 & 0 \\ 9 & 13 & 0 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{cc|cc} 3 & 4 & 13 & -4 \\ 0 & 1 & -3 & 1 \end{array}\right) \longrightarrow$$

$$\left(\begin{array}{cc|cc} 3 & 0 & 13 & -4 \\ 0 & 1 & -3 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{cc|cc} 1 & 0 & \frac{13}{3} & \frac{-4}{3} \\ 0 & 1 & -3 & 1 \end{array}\right)$$

$$P^{-1} = \begin{pmatrix} \frac{13}{3} & \frac{-4}{3} \\ -3 & 1 \end{pmatrix}$$

After these transformations the inverse matrix is on the right hand side. So equation 2.4 can be rewritten as:

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{13}{3} & \frac{-4}{3} \\ -3 & 1 \end{pmatrix} \cdot \begin{pmatrix} 39 \\ 120 \end{pmatrix} = \begin{pmatrix} 9 \\ 3 \end{pmatrix}$$

Thus with the still existent stock of inventory - assuming the complete clearance of the stock - nine shelves of type A and three shelves of type B can be produced [2].

2.2 The Bills of Material Problem

As already described in the beginning, another production structure which is more consistent with the operational reality than the one in section 2.1, is centered here. In contrast to the material flow before, which was still divided into single production stages, now here in addition to certain preliminary products also intermediate products, which among other things contain these preliminary products, are needed for the assembly of a finished product.

But in this case the already introduced mathematical model for the description of multistage linear production processes yields incorrect results. So the aim of this chapter now is to identify concrete bills of material, which are required for the execution of a certain order, for companies which are structured in such a way (**bills of material problem**).

Furthermore as an addition and a further approach to the operational reality not only finished products, but also certain spare parts can be ordered. One thinks in this context for example of the automobile industry, where licensed dealers order vehicles of different types as well as large quantities of diverse spare parts. The ordered end products and spare parts as well as all component parts which are needed for the assembly then appear on the bills of material, such that one can see the complete material requirements, which are necessary for the compliance of an order.

Example:
The baby-cradle ,,Sofie" (see figure 2.4) is produced. The corresponding material flow diagram illustrates the no longer linear, multistage production process without mutual relations (see figure 2.5).

From the production structure, which is shown in figure 2.5, one can see that the production - represented as vector $\vec{z}$ - now has to satisfy both the internal demand $\vec{a}$ and the external demand $\vec{x}$, i.e. the customers' orders. Thus it follows:

$$\vec{z} = \vec{a} + \vec{x} \tag{2.5}$$

[2] If this problem is discussed in secondary school, solving systems of linear equations is well to the fore. Here in the first place the substitution, equation and elimination methods, which are scheduled in secondary school, are appropriate procedures.

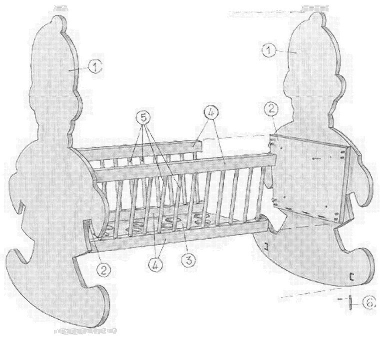

Figure 2.4: Baby-cradle ,,Sofie" (1 =rocker sides, 2 = front boards, 3 =bed base, 4 =wood strips, 5 =rungs, 6 =stopper)

Regarding the entries of the vectors $\vec{z}$, $\vec{a}$ and $\vec{x}$, it is obvious to take up the production process and to let a component appear in the vector not until all parts and assembly units which are contained in it are already listed.

An example for such a **technological order** of the vectors would be:

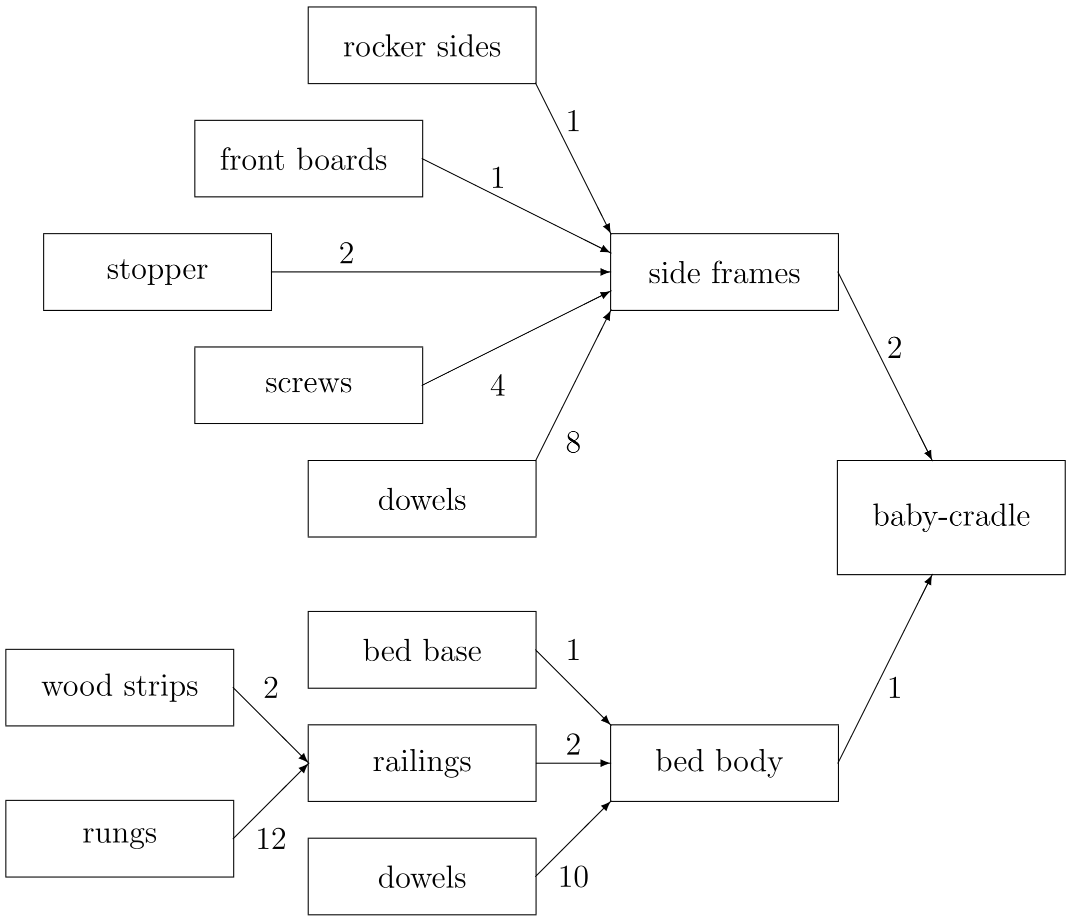

Figure 2.5: Material flow of the baby-cradle ,,Sofie"

$$\begin{pmatrix} \text{number of screws} \\ \text{number of dowels} \\ \text{number of rocker sides} \\ \text{number of front boards} \\ \text{number of stopper} \\ \text{number of wood strips} \\ \text{number of rungs} \\ \text{number of bed bases} \\ \text{number of railings} \\ \text{number of bed bodies} \\ \text{number of side frames} \\ \text{number of baby-cradles} \end{pmatrix}$$

But in general a technological order of the vectors, which is generated in this way, is not unique. So in the example above the order of the first eight entries is arbitrary, since the products which are listed there enter the production as raw materials.

Furthermore obviously a linear relationship exists between the internal material demand $\vec{a}$ and the given production $\vec{z}$, i.e.:

$$\vec{a} = P \cdot \vec{z} \tag{2.6}$$

At first the entries of the 12×12-matrix P are still unknown, but because of the linearity property of matrices

$$\begin{aligned} \vec{a} = P \cdot \vec{z} &= P \cdot (z_1\vec{e_1} + z_2\vec{e_2} + ... + z_{12}\vec{e_{12}}) \\ &= x_1 P \cdot \vec{e_1} + x_2 P \cdot \vec{e_2} + ... + x_{12} P \cdot \vec{e_{12}} \end{aligned}$$

holds, where $\vec{e_i}$ are the standard basic vectors.

Which meanings do the $P\vec{e_i}$ have?

$\vec{e_i}$ is the production vector for "exactly one part of sort i". So $\vec{e_{10}}$ stands for the production of one bed body.
$P\vec{e_i}$ indicates the internal material demand, which is required to produce one part of sort i. But this is already known from the material flow diagram.
The result of the multiplication $P\vec{e_i}$ is the i^{th} column of the matrix P. Thus one obtains the matrix P by regarding the $P\vec{e_1}$,...,$P\vec{e_{12}}$ as columns of P and determining these again from the material flow diagram.
This matrix P is also referred to as **amount matrix** . For the case above it is:

$$P = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 10 & 8 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 12 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

If one now chooses for example the ninth column for a closer analysis, one can see the material demand for the construction of one railing, namely two wood strips and twelve rungs.

Properties of the amount matrix P:
All entries on the main diagonal are zero, since for the production of one part this part itself is not required.
All entries which are below the main diagonal are zero since based on the technological order only the parts with a smaller index can be used and furthermore joint productions do not occur.

The matrix in this example contains many zeros, since the production process is very simple. In general the matrix will contain more entries which are different from zero, but it will always be a sparse matrix.

For the computation of the production volume $\vec{z}$ after an order $\vec{x}$ was received it follows from equation 2.5 and 2.6 that:

$$\begin{aligned} \vec{z} &= P\vec{z} + \vec{x} \quad \text{i.e.} \\ E\vec{z} &= P\vec{z} + \vec{x} \quad \text{, where } E \text{ is the unit matrix.} \end{aligned}$$

$$\Rightarrow \quad (E - P)\vec{z} = \vec{x} \tag{2.7}$$

The computational effort for solving this system of linear equations turns out to be very simple, since the matrix $(E-P)$ is already an upper triangular matrix, but the obtained procedure would be very laborious in practical use, since the system of equations 2.7 would have to be solved for every order.
For this reason the following transformation is aspired:

$$\vec{z} = (E - P)^{-1}\vec{x} \tag{2.8}$$

After the computation of $(E - P)^{-1}$ the effects of different orders on the production process could be simulated very easily.

But how can one compute $(E - P)^{-1}$?

If the inverse of a matrix was already discussed in the lessons, this operation is quite unproblematic, apart from a quite high computational effort. Otherwise one uses the correspondence between matrices and numbers with regard to their arithmetic operations:
For a real number a with $| a | < 1$ holds:

$$\begin{aligned} (1 - a)^{-1} &= 1 + a + a^2 + a^3 + ... + a^n + ... \quad \text{(geometric series)} \\ \text{or} \qquad 1 &= (1 - a) \cdot (1 + a + a^2 + a^3 + ... + a^n + ...) \end{aligned}$$

Does now $E = (E - P) \cdot (E + P + P^2 + ... + P^n + ...)$ *hold analogously?*

One can easily see that P is nilpotent, i.e. there exists an $n_0 \in \mathbb{N}$ such that $P^k = 0 \quad \forall k \geq n_0$. It follows:

$$\begin{aligned}
&(E - P) \cdot (E + P + ... + P^{n_0 - 1}) = \\
&E + P + P^2 + ... + P^{n_0 - 1} - (P + P^2 + ... + P^{n_0}) = \\
&E - P^{n_0} = E
\end{aligned}$$

Thus the matrix $(E - P)$ is invertible and for its inverse $(E - P)^{-1} = E + P + P^2 + ... + P^{n_0 - 1}$ holds. With this the equation 2.8 can be rewritten as:

$$\vec{z} = (E + P + ... + P^{n_0 - 1}) \cdot \vec{x} \tag{2.9}$$

Case Study:
Supposed that with a certain order in addition to four cradles three wood strips and eleven rungs would be ordered. In this case the order vector $\vec{x}$ would have the following form:

$$\vec{x} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 3 \\ 11 \\ 0 \\ 0 \\ 0 \\ 0 \\ 4 \end{pmatrix}$$

By using equation 2.9 one obtains for $\vec{z}$ [3] :

$$\vec{z} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 8 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 10 & 8 & 26 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 2 & 4 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 12 & 24 & 0 & 24 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 3 \\ 11 \\ 0 \\ 0 \\ 0 \\ 0 \\ 4 \end{pmatrix} = \begin{pmatrix} 32 \\ 104 \\ 8 \\ 8 \\ 16 \\ 19 \\ 107 \\ 4 \\ 8 \\ 4 \\ 8 \\ 4 \end{pmatrix}$$

2.3 Solution for the Example

With the mathematical models and techniques developed in section 2.1 and 2.2 the example from section 1.2 shall be solved.

From figure 1.1 one obtains the amount matrix P by not considering the non-self-produced articles 4, 5, 7, 8, 9, 12, 13, 16, and 17. Furthermore the products 10 and 15 are produced, but they are not used anymore in the production process [4]. Thus they can be left out of consideration in the amount matrix.

$$P = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0.5 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.5 & 0.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 & 0 \end{pmatrix}$$

[3] In this example $n_0 = 3$.

[4] In reality one may think of the products 10 and 15 as waste products, which result from the production processes of the products 6 and 14.

$$E - P = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 & -0.5 & 0 \\ -2 & 0 & 1 & 0 & 0 & 0 \\ 0 & -0.5 & -0.5 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -5 & 1 \end{pmatrix}$$

$$(E - P)^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 3.\bar{3} & 1.\bar{3} & 0.\bar{3} & 0.\bar{6} & 0.\bar{6} & 0 \\ 2 & 0 & 1 & 0 & 0 & 0 \\ 2.\bar{6} & 0.\bar{6} & 0.\bar{6} & 1.\bar{3} & 0.\bar{3} & 0 \\ 2.\bar{6} & 0.\bar{6} & 0.\bar{6} & 1.\bar{3} & 1.\bar{3} & 0 \\ 13.\bar{3} & 3.\bar{3} & 3.\bar{3} & 6.\bar{6} & 6.\bar{6} & 1 \end{pmatrix}$$

To supply for example the demand for one unit of product 1, one needs for the production:

$3.\bar{3}$	units of product 2
2	units of product 3
$2.\bar{6}$	units of product 10
$2.\bar{6}$	units of product 11
$13.\bar{3}$	units of product 14