Linear Optimization in School Mathematics

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Contents

Chapter 1

Why Linear Optimization in School?

Mathematics in general is said to be not vivid and to exist only for mathematicians. The idea of mathematics among pupils is the idea of a science which serves only as its end in itself. It seems to be important to face the prejudice that mathematics is far from any practical use.

Mathematics is a science which provides services and whose help is needed in almost all fields of life. School mathematics should awaken the perception in the pupils' fields of life how mathematics works and how the search for the right theory for the solution of a whole class of problems makes it possible in the opposite way to act again practically. If it is e.g. even for today's powerful computers already difficult to solve the "travelling salesman"-problem for 25 places to visit, how much more necessary is it hence to have a suitable theory for this and similar problems. Here the mathematician is needed.

The motivation to develop materials for lessons of a different kind is also due to meet the demands of the school curriculum: "A further function of lessons in mathematics is to give pupils an understanding of the process of mathematics. Where mathematical methods can be used to structure a problem, to represent essential aspects of complex facts in a model and to search for a solution, correlations between theory and practice can be experienced. (...) Pupils (...) shall draw relations between non-mathematical facts and mathematics, work the problem with mathematical methods, interpret the solutions found and judge them critically. Moreover limits of the subject's specific methods and limits of mathematics shall be realized."[2]

Optimization is one of those themes whose practical relevance is obvious. Pupils "optimize" with the method "off the top of their heads" and obtain in many fields of everyday life on basis of their respective experiences quite useful results. But if one proceeds in this way in decisive fields of life, stranding will be preprogrammed. Namely if personal assessments and ratings influence the judgement of a situation, the whole insecurity, which is naturally present in human action, will also be included. If a problem is handled mathematically, this insecurity will not exist.

But before a problem can be formulated mathematically, a reduction to the essential, which is done by men, has to happen. This again entails that several persons extract various mathematical problems from one real-world problem, since they allow different questions with the same underlying information. This process, which is called modelling, will be elaborated among other things in section 3.1.

To begin with in chapter 2 it shall become clear what the term "linear optimization" means. For that purpose some problems from the real life, which can be solved with the aid of linear optimization, are specified. One of these problems will be considered closer and finally, after in chapter 3 and 4 procedures for the solution have been presented, it will be solved in chapter 5. With a further example in chapter 6 it shall be briefly explained how one obtains a solution for an integer optimization problem.

The present text is meant to be an assistance for teachers. It is clear to the authors that it is yet not suitable for pupils in its actual form, since still some mathematical terms, which in general are not introduced in school mathematics, are used. We hope that this text is understood by some teachers as a suggestion to work out a version which is "closer to the pupils" - as a joint work between university and school.

To the mathematical fields which are required in the present text or for whose introduction in school mathematics this work may serve as well belong the drawing of lines by means of equations of the straight line, the shift of terms of inequalities and their geometrical interpretation as well as the calculation with vectors and matrices as part of the linear algebra. Within the scope of the represented themes the introduction of the notion of a vector as ordered number-n-tuple is possible as well.

Chapter 2

Was does linear Optimization mean?

Linear optimization is a field of application of linear algebra and has a great relevance to the solution of optimization problems in economy, engineering and administration. Linear optimization deals with the maximization or minimization of a value subject to certain restrictive conditions. Thus an optimal value is no "extremum", but an "extremum with certain constraints".

If an enterpriser wants to find out how many units of diverse products have to be produced in order to maximize the profit subject to given selling-prices, the production facilities will be restricted by marketing conditions, capacity limitations and financial bottlenecks.

If e.g. hazardous goods shall be carried by a transportation enterpriser, then the number of carried goods shall be maximized. The company's capacities like size and number of lorries however restrict the number of goods which have to be carried. Besides the given safety regulations have to be kept. According to the hazard which originates from the material only fixed amounts are allowed to be carried at once. Some goods are not allowed to be carried together, since they become hazardous only in combination. Restrictive conditions can be derived likewise from this.

Another example of a real-world problem which can be solved with linear optimization is the design of a pipeline. The pipeline of a plant conveys e.g. a liquid with a fixed temperature. The occurring heat loss has to be compensated by heating before entering the next production step. The costs for heating are proportional to the heat loss. But the heat loss can be reduced by adding an

isolation, from which costs arise. If now the best possible compromise between the thickness of the isolation and the compensation of the heat loss shall be found, a method of linear optimization can be used.
But the heat loss does not only depend on the thickness of the isolation, but also on the diameter of the pipeline. The diameter of the pipeline fixes again the costs of investment of the pipeline as well as the running expenses of the pipeline system, since the expended hoisting capacity follows from the diameter of the pipeline by the pressure loss. Here as well a compromise between the hoisting capacity and the costs of investment can be found by linear optimization.

A discussion of further examples of situations where one can solve a real-world problem with linear optimization would certainly lead to far. A detailed example is represented now and shall be solved after the theory of linear optimization was discussed and the procedure for the solution was explained.

Example 2.1 *A big company for soft drinks wants to put a new product on the market. The new beverage shall be mixed out of three liquid ingredients, where the first ingredient costs 5 Euro per liter, the second ingredient 2 Euro per liter and the third ingredient 0,25 Euro per liter. Besides ingredient 1 contains 3g/l of sugar and 4 units/l of a flavor, while the second ingredient contains 7g/l of sugar and 8 units/l of the flavor and the third ingredient 20g/l of sugar and no flavor. Due to technical reasons at least 100 liters of the beverage have to be produced per production process.*

The market research found out that the beverage will be accepted by the target group, if the parameters are in the following interval.

The completed beverage shall contain at least 3g/l and at most 6g/l of sugar. At least 3 units of the flavor shall be in one liter of the beverage. Besides the beverage shall consist of at least 40% of ingredient 1, while ingredient 2 is allowed to amount at most 50% and ingredient 3 at most 30% of the new beverage.

Chapter 3

Translation of the Real-World Problem

3.1 Modelling

The assumptions of the real-world problem now have to be seized in a mathematical model. Therefore the variables x_1,x_2 and x_3, which stand for the amount of the respective liquid in liters, are introduced.

The soft drink company of course wants to keep the production costs low. The cost function

$$5 \cdot x_1 + 2 \cdot x_2 + 0.25 \cdot x_3$$

is the sum of the products of the respective liquid with its price and is called the **objective funktion**.

From the restrictions concerning the beverage's content of sugar the following **constraints**result:

$$\begin{aligned} 3 \cdot x_1 + 7 \cdot x_2 + 20 \cdot x_3 &\geq 3 \cdot (x_1 + x_2 + x_3) \\ 3 \cdot x_1 + 7 \cdot x_2 + 20 \cdot x_3 &\leq 6 \cdot (x_1 + x_2 + x_3) \end{aligned}$$

The constraint for the content of flavor is:

$$4 \cdot x_1 + 8 \cdot x_2 \geq 3 \cdot (x_1 + x_2 + x_3)$$

For the portion of each ingredient in the soft drink one obtains a constraint as well.

$$\begin{aligned} x_1 &\geq 0.4 \cdot (x_1 + x_2 + x_3) \\ x_2 &\leq 0.5 \cdot (x_1 + x_2 + x_3) \\ x_3 &\leq 0.3 \cdot (x_1 + x_2 + x_3) \end{aligned}$$

The minimum amount of 100 liters, which has to be produced per production process, yields:

$$x_1 + x_2 + x_3 \geq 100$$

Of course the portion of all ingredients has to be greater than zero. One obtains the **nonnegativity constraints**:

$$x_1, x_2, x_3 \geq 0$$

Since on the right-hand side there should not be any variables, some transformations are necessary. Finally one obtains the optimization problem[1]:

$$\begin{array}{lrcrcrcr} min & 5 \cdot x_1 & + & 2 \cdot x_2 & + & 0.25 \cdot x_3 & & \\ s.t. & & & 4 \cdot x_2 & + & 17 \cdot x_3 & \geq & 0 \\ & -3 \cdot x_1 & + & x_2 & + & 14 \cdot x_3 & \leq & 0 \\ & x_1 & + & 5 \cdot x_2 & - & 3 \cdot x_3 & \geq & 0 \\ & 0.6 \cdot x_1 & - & 0.4 \cdot x_2 & - & 0.4 \cdot x_3 & \geq & 0 \\ & -0.5 \cdot x_1 & + & 0.5 \cdot x_2 & - & 0.5 \cdot x_3 & \leq & 0 \\ & -0.3 \cdot x_1 & - & 0.3 \cdot x_2 & + & 0.7 \cdot x_3 & \leq & 0 \\ & x_1 & + & x_2 & + & x_3 & \geq & 100 \\ & & & & & x_1, x_2, x_3 & \geq & 0 \end{array}$$

3.2 Linear Programs

The optimization model found at the end of chapter 3.1 is called **linear program**. The **objective function** $\vec{c} \cdot \vec{x}$ is linear. Each solution $\vec{x}$ which satisfies all the constraints is called **feasible solution** of the LP[2] and $\vec{c} \cdot \vec{x}$ is called **objective value** of this solution.

[1] s.t. = subject to

[2] linear program

Example 3.1 *(from[3]) Another linear program is:*

$$\begin{array}{llcccc} max & x_1 & & & & \\ s.t. & -x_1 & + & x_2 & \leq & 1 \\ & x_1 & + & x_2 & \leq & 3 \\ & & x_1, & x_2 & \geq & 0 \end{array}$$

Example 3.1 was chosen because it only has two variables x_1 and x_2. A linear program with only two variables can be solved in a graphical way.

3.2.1 The graphical Procedure for the Solution

To solve a LP with only two variables one can use the graphical procedure for the solution. For that purpose the variables x_1 and x_2 are drawn upon the axes of abscissae and ordinate of a co-ordinate system in which subsequently the constraints are drawn (see figure 3.1).

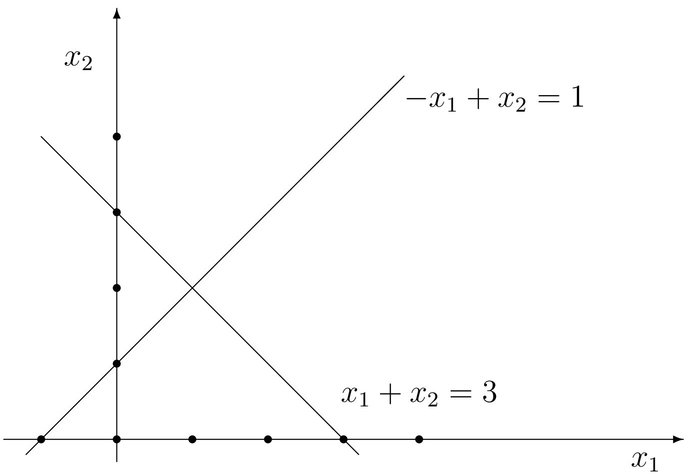

Figure 3.1: Graphical representation of the constraints from example 3.1

If one notes that the constraints are inequalities and that the nonnegativity constraints have to be fulfilled as well, one obtains the speckled region in figure 3.2 in which one has to search for the optimal solution. This region is called **feasible region** .

The objective function now has to be shifted to the right as far as possible [3]. In general the objective function however will be no line parallel to the ordinate.

[3] With minimization problems one shifts the objective function to the left.

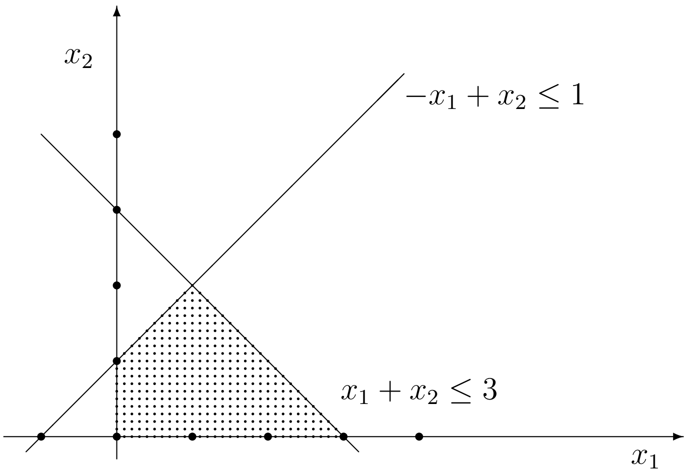

Figure 3.2: Graphical representation of the feasible region from example 3.1

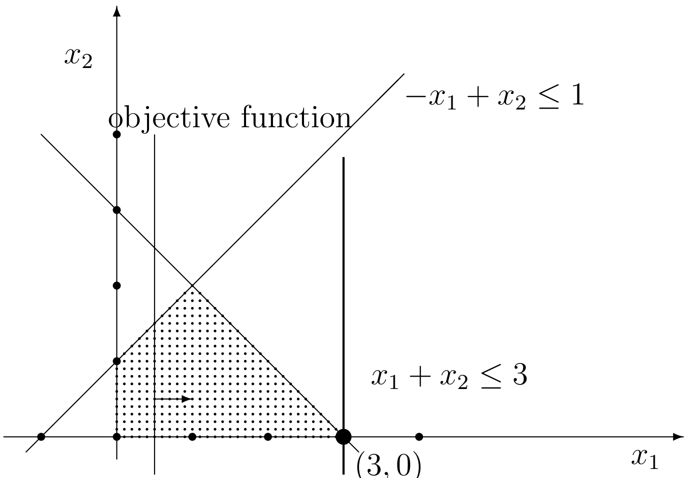

Figure 3.3: Graphical representation of the optimization problem from example 3.1

By a parallel shift of the objective function to greater or smaller objective values one finally obtains the optimal solution. In figure 3.3 it can be seen that after the shift the objective function is still adherent to the feasible region in point $(3, 0)$. Thus the optimal solution $x_1 = 3$ and $x_2 = 0$ is found.

Chapter 4

The Simplex Method

The idea of the simplex method, with which in contrast to the graphical procedure LPs with more than two variables can be considered as well, is to move from corner point to corner point of the feasible region and to improve thereby constantly the objective value. The procedure will end, if the objective value cannot be improves any more.

In Example 3.1 one would move from corner point $(0,0)$ to $(3,0)$ or via $(0,1)$ and $(1,2)$ to $(3,0)$, which can be seen in figure 3.2.

4.1 Standard Form

To solve a LP with the help of the simplex method, it has to be in **standard form**.

Definition 4.1 *A LP of the form*

$$\begin{array}{rrcll} min & \vec{c}\cdot\vec{x} & & & \\ s.t. & A\vec{x} & = & \vec{b} & \\ & x_i & \leq & 0 & \forall i \end{array}$$

is called LP in standard form, where $\vec{c}$ is the cost vector and $\vec{b}$ is the demand vector and A represents the coefficient matrix. One assumes that A is a $m \times n$-matrix with $rank(A)^1 = m$. Thus one leaves away the redundant constraints.

[1] see page 38

To transform am arbitrary LP into standard form, several transformations have to be done. These shall be illustrated in example 3.1.

The LP has the following form:

$$\begin{array}{llcccl} max & x_1 & & & & \\ s.t. & -x_1 & + & x_2 & \leq & 1 \\ & x_1 & + & x_2 & \leq & 3 \\ & & x_1, & x_2 & \geq & 0 \end{array}$$

This is a maximization problem. To obtain a minimization problem as required for the standard form, the objective function has to be multiplied by -1. One obtains:

$$\begin{array}{llcccl} -min & -x_1 & & & & \\ s.t. & -x_1 & + & x_2 & \leq & 1 \\ & x_1 & + & x_2 & \leq & 3 \\ & & x_1, & x_2 & \geq & 0 \end{array}$$

Now the constraints, which are given in terms of inequalities, shall be transformed into equalities. This is done by introducing so called **slack variables** and **surplus variables**. The slack variables are added to the $\leq$-equalities to generate equalities. Likewise the surplus variables are subtracted from the $\geq$-equalities. In the present example only $\leq$-equalities exist, so that only slack variables have to be introduced.

$$\begin{array}{llcccccccl} -min & -x_1 & & & & & & & & \\ s.t. & -x_1 & + & x_2 & + & x_3 & & & = & 1 \\ & x_1 & + & x_2 & & & + & x_4 & = & 3 \\ & & & x_1, & x_2, & x_3, & & x_4 & \geq & 0 \end{array}$$

In this example all variables x_1, x_2 are ≥ 0, so that referring to this no transformations have to be done. If in a LP there is a variable x_i which is not sign constrained, x_i will be replaced by $x_i^+ \geq 0$ and $x_i^- \geq 0$, where $x_i = x_i^+ - x_i^-$ will be valid.

After these necessary transformations the LP is in standard form with

$$\text{coefficient matrix} \quad A = \begin{pmatrix} -1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}$$

$$\begin{aligned}\text{demand vector} \quad \vec{b} &= \begin{pmatrix} 1 \\ 3 \end{pmatrix} \\ \text{cost vector} \quad \vec{c} &= (-1, 0, 0, 0)\end{aligned}$$

The coefficient matrix has $rank(A) = 2$. Two columns at a time are linearly independent. But if one adds to any combination of two columns a third one, the three columns will be linearly dependent:

$$\begin{aligned}1 \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix} - 1 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} + 2 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} &= 0 \\ 1 \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix} + 1 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} - 2 \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} &= 0 \\ 1 \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix} + 1 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} - 1 \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} &= 0 \\ 1 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} - 1 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} - 1 \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} &= 0\end{aligned}$$

4.2 Basic Representation

Definition 4.2 *A* ***basis*** *of A is a set $A_B = (A_{B(1)}, ..., A_{B(m)})$, where $A_{B(1)}, ..., A_{B(m)}$ are columns of A. A_B is a $m \times m$ sub-matrix of A with $rank(A_B) = m$. The correspondent variables $\vec{x}_B = (x_{B(1)}, ...x_{B(m)})^T$ are called* ***basic variables****. The remaining variables $\vec{x}_N = (x_{N(1)}, ...x_{N(n-m)})^T$ are called* ***non-basic variables*** *and the correspondent columns of the coefficient matrix are collected by $A_N = (A_{N(1)}, ..., A_{N(n-m)})$.*

If one considers example 3.1, several bases can be found, e.g.:

1. $B = (3, 4) \qquad A_B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
2. $B = (1, 2) \qquad A_B = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$
3. $B = (4, 1) \qquad A_B = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$

If now $\vec{x}$ is a solution of a LP in standard form, i.e. if $A \cdot \vec{x} = \vec{b}$ holds, then $A_B \cdot \vec{x}_B + A_N \cdot \vec{x}_N = \vec{b}$ will hold as well, and vice versa.

One can see this easily by writing $A \cdot \vec{x} = \vec{b}$ as
$x_1 \cdot A_1 + ... + x_n \cdot A_n = \vec{b}$, where $A_1, \ldots, A_n$ are the columns of A .

Example 4.1

$$\begin{pmatrix} -1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$
$$= \quad x_1 \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix} + x_2 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} + x_3 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + x_4 \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

Therefore it is clear that the summands can be exchanged in their sequence, and therefore they can be represented as $A_B \cdot \vec{x}_B + A_N \cdot \vec{x}_N = \vec{b}$.

For the basis $B = (3, 4)$ *it follows:*

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x_3 \\ x_4 \end{pmatrix} + \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

Therefore it holds:

$$\begin{aligned} & A \cdot \vec{x} = \vec{b} \\ \Longleftrightarrow \quad & A_B \cdot \vec{x}_B + A_N \cdot \vec{x}_N = \vec{b} \\ \Longleftrightarrow \quad & A_B \cdot \vec{x}_B = \vec{b} - A_N \cdot \vec{x}_N \\ \Longleftrightarrow \quad & \vec{x}_B = A_B^{-1} \cdot \vec{b} - A_B^{-1} \cdot A_N \cdot \vec{x}_N \end{aligned} \tag{4.1}$$

Equation 4.1 is the **basic representation** of $\vec{x}$ with respect to basis B. Based on the derivation it is clear that any solution can be represented in this form, if the inverse of the matrix A_B can be computed.

For $B = (3, 4)$ A_B is the identity matrix. Therefore $A_B = A_B^{-1}$.
For $B = (1, 2)$ $A_B = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$. For the computation of A_B^{-1} two systems of linear equations have to be solved:

$$\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Since the systems differ only on the right-hand side, the computations can be collected in one scheme:

$$\left(\begin{array}{cc|cc} -1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{cc|cc} -1 & 1 & 1 & 0 \\ 0 & 2 & 1 & 1 \end{array}\right) \longrightarrow$$

$$\left(\begin{array}{cc|cc} 1 & -1 & -1 & 0 \\ 0 & 2 & 1 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{cc|cc} 1 & 0 & -\frac{1}{2} & \frac{1}{2} \\ 0 & 1 & \frac{1}{2} & \frac{1}{2} \end{array}\right)$$

After the transformations the inverse matrix A_B^{-1} is on the right-hand side.

For the different bases from example 3.1 the basic representation can be computed.

1. $B = (3,4) \quad A_B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad A_B^{-1} = A_B = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$$\begin{aligned} \begin{pmatrix} x_3 \\ x_4 \end{pmatrix} = \vec{x}_B \quad &= I \cdot \vec{b} - I \cdot A_N \cdot \vec{x}_N \\ &= \begin{pmatrix} 1 \\ 3 \end{pmatrix} - \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ 3 \end{pmatrix} - \begin{pmatrix} -x_1 + x_2 \\ x_1 + x_2 \end{pmatrix} \\ &= \begin{pmatrix} 1 + x_1 - x_2 \\ 3 - x_1 - x_2 \end{pmatrix} \end{aligned}$$

2. $B = (1,2) \quad A_B = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \quad A_B^{-1} = \frac{1}{2} \cdot \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$

$$\begin{aligned} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \vec{x}_B \quad &= \frac{1}{2} \cdot \left[\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 3 \end{pmatrix} \right. \\ &\quad \left. - \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x_3 \\ x_4 \end{pmatrix} \right] \\ &= \frac{1}{2} \cdot \left[\begin{pmatrix} 2 \\ 4 \end{pmatrix} - \begin{pmatrix} -x_3 + x_4 \\ x_3 + x_4 \end{pmatrix} \right] \\ &= \begin{pmatrix} 1 \\ 2 \end{pmatrix} - \frac{1}{2} \cdot \begin{pmatrix} -x_3 - x_4 \\ x_3 + x_4 \end{pmatrix} \end{aligned}$$

3. $B = (4,1) \quad A_B = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} \quad A_B^{-1} = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$

$$\begin{aligned}
\begin{pmatrix} x_4 \\ x_1 \end{pmatrix} = \vec{x}_B &= \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 3 \end{pmatrix} \\
&- \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} x_2 \\ x_3 \end{pmatrix} \\
&= \begin{pmatrix} 4 \\ -1 \end{pmatrix} - \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} \cdot \begin{pmatrix} x_2 + x_3 \\ x_2 \end{pmatrix} \\
&= \begin{pmatrix} 4 \\ -1 \end{pmatrix} - \begin{pmatrix} 2 \cdot x_2 + x_3 \\ -x_2 - x_3 \end{pmatrix}
\end{aligned}$$

4.3 Basic Solution

Definition 4.3 *A solution $\vec{x}$ is called **basic solution** of $A \cdot \vec{x} = \vec{b}$, if $\vec{x}_N = \vec{0}$ and therefore $\vec{x}_B = A_B^{-1} \cdot \vec{b}$. If additionally $\vec{x}_B \geq 0$ holds, $\vec{x}$ is called **basic feasible solution** .*

In example 3.1 the solutions with respect to the bases $B = (3,4)$ and $B = (1,2)$ are basic feasible solutions.

1. $B = (3,4) \qquad \vec{x}_N = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \vec{x}_B = \begin{pmatrix} x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

2. $B = (1,2) \qquad \vec{x}_N = \begin{pmatrix} x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \vec{x}_B = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

3. $B = (4,1) \qquad \vec{x}_N = \begin{pmatrix} x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \vec{x}_B = \begin{pmatrix} x_4 \\ x_1 \end{pmatrix} = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$

   In this case $\vec{x}_B \not\geq \vec{0}$ and therefore $\vec{x}$ is no basic feasible solution.

By trying to represent these solutions in a graphical way (see figure 4.1), one can easily see why feasible respectively infeasible solutions are concerned.

The basic solution with respect to basis $B = (4,1)$ with $x_1 = -1$ and $x_2 = 0$ is not contained in the feasible region, while the solutions with respect to the bases $B = (3,4)$ and $B = (1,2)$ with $x_1 = 0$ and $x_2 = 0$ respectively $x_1 = 1$ and $x_2 = 2$ are contained in the feasible region.

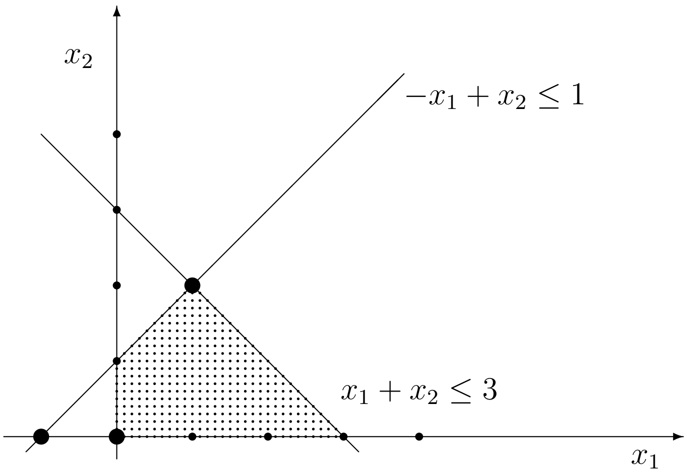

Figure 4.1: Graphical representation of feasible and infeasible solutions.

Moreover one sees in figure 4.1 that the basis solutions correspond to the corner points of the feasible region.

4.4 Optimality Test

From chapter 3.2.1 it is already known that the optimal solution of the LP from example 3.1 is $\vec{x} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$.

But how can starting from a known feasible solution the optimal solution be found ?
To begin with the objective value of the respective solution shall be considered. The objective value of the solution $\vec{x} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is $\vec{c} \cdot \vec{x} = (1,0) \cdot \begin{pmatrix} 0 \\ 0 \end{pmatrix} = 0$, while for the solution $\vec{x} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ it is $\vec{c} \cdot \vec{x} = (1,0) \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} = 1$.

Now one can use the basic representation of a basic feasible solution (see equation 4.1) to derive an optimality condition.

$$\begin{aligned} \vec{c} \cdot \vec{x} &= \vec{c}_B \cdot \vec{x}_B + \vec{c}_N \cdot \vec{x}_N \\ &\overset{(4.1)}{=} \vec{c}_B \cdot (A_B^{-1} \cdot \vec{b} - A_B^{-1} \cdot A_N \cdot \vec{x}_N) + \vec{c}_N \cdot \vec{x}_N \\ &= \vec{c}_B \cdot A_B^{-1} \cdot \vec{b} + (\vec{c}_N - \vec{c}_B \cdot A_B^{-1} \cdot A_N) \cdot \vec{x}_N \end{aligned}$$

Since for a basic solution $\vec{x}_N = 0$ holds, it follows: $\vec{c} \cdot \vec{x} = \vec{c}_B \cdot \vec{x}_B = \vec{c}_B \cdot A_B^{-1} \cdot \vec{b}$.

The question now is, if the objective value can be improved further.
The modification of the solution yields a change of the objective value by $(\vec{c}_N - \vec{c}_B \cdot A_B^{-1} \cdot A_N) \cdot \vec{x}_N$. Since so far $\vec{x}_N = \vec{0}$ holds, there is only the possibility to increase $\vec{x}_N$. Since moreover we always consider a minimization problem and therefore want to decrease the objective value, for a $j \in \{1, \ldots, n-m\}$ $(c_{N(j)} - \vec{c}_B \cdot A_B^{-1} \cdot A_{N(j)}) < 0$ has to hold in order to achieve an improvement of the objective value. This means:

Theorem 4.1 *The basic feasible solution $\vec{x}$ with respect to B is optimal, if* $(c_{N(j)} - \vec{c}_B \cdot A_B^{-1} \cdot A_{N(j)}) \geq 0 \quad \forall j \in \{1, \ldots, n-m\}$

Thus the values $\bar{c}_{N(j)} := (c_{N(j)} - \vec{c}_B \cdot A_B^{-1} \cdot A_{N(j)})$, which are called **reduced costs**, contain the information whether it is useful to increase the value of a non-basic variable $x_{N(j)}$ from 0 to a value $\delta > 0$.

Example 4.2 *In the following now once more the solutions with respect to the different bases shall be considered.*

1. $B = (3,4),\ N = (1,2)$

$$\begin{aligned} & \vec{c}_N - \vec{c}_B \cdot A_B^{-1} \cdot A_N \\ = \; & (-1,0) - (0,0) \cdot \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \\ = \; & (-1,0) - (0,0) \cdot \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \\ = \; & (-1,0) - (0,0) \\ = \; & (-1,0) \ngeq (0,0) \end{aligned}$$

The optimality condition is not satisfied.

2. $B = (1,2),\ N = (3,4)$

$$\begin{aligned} & \vec{c}_N - \vec{c}_B \cdot A_B^{-1} \cdot A_N \\ = \; & (0,0) - (-1,0) \cdot \frac{1}{2} \cdot \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ = \; & (0,0) - \frac{1}{2} \cdot (-1,0) \cdot \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \\ = \; & (0,0) - \frac{1}{2} \cdot (1,-1) \\ = \; & (-\frac{1}{2}, \frac{1}{2}) \ngeq (0,0) \end{aligned}$$

The optimality condition is not satisfied.

3. $B = (1,3),\ N = (2,4)$

$$\begin{aligned}
& \vec{c}_N - \vec{c}_B \cdot A_B^{-1} \cdot A_N \\
= \ & (0,0) - (-1,0) \cdot \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \\
= \ & (0,0) - (-1,0) \cdot \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} \\
= \ & (0,0) - (-1,-1) \\
= \ & (1,1) \geq (0,0)
\end{aligned}$$

Thus the basis belonging to B is optimal.

$$\begin{aligned}
\vec{x}_B &= \begin{pmatrix} x_1 \\ x_3 \end{pmatrix} = A_B^{-1} \cdot \vec{b} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix} \\
\vec{x}_N &= \begin{pmatrix} x_2 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\end{aligned}$$

As already found out graphically in chapter 3.2.1, $\vec{x} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$ [2] *is the optimal solution.*

4.5 Basis Exchange

As already mentioned in chapter 4.3, the basic feasible solutions correspond to the corner points of the feasible region. Corresponding to the idea of the simplex method to move from corner point to corner point as long as the objective value still improves, we will now move from a basic feasible solution to the next one as long as the optimality condition is not fulfilled.

But how does one get from a basic feasible solution to the next one ?

The situation subsists that the optimality condition does not hold. This means, $\exists \quad s \in \{1, \ldots, n-m\} : \bar{c}_{N(s)} = c_{N(s)} - \vec{c}_B \cdot A_B^{-1} \cdot A_{N(s)} < 0$

[2] With $\vec{x}$ it is meant $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$. As soon as a final solution is specified, the slack variables, surplus variables or other variables, which were only needed to transform the LP in standard form, are disregarded.

Till now $x_{N(s)} = 0$ held, but at present $x_{N(s)}$ is increased to a value $\delta > 0$ while all the other non-basic variables $x_{N(j)}$ remain the same.

What happens to the objective value if $x_{N(s)} = \delta$?

$$\begin{aligned} \vec{c} \cdot \vec{x} &= \vec{c}_B \cdot A_B^{-1} \cdot \vec{b} + (\vec{c}_N - \vec{c}_B \cdot A_B^{-1} \cdot A_N) \cdot \vec{x}_N \\ &= \vec{c}_B \cdot A_B^{-1} \cdot \vec{b} + \underbrace{(\vec{c}_{N(s)} - \vec{c}_B \cdot A_B^{-1} \cdot A_{N(s)})}_{<0} \cdot \delta \end{aligned}$$

i.e. the objective value $\vec{c} \cdot \vec{x}$ decreases since $\delta > 0$.

Subsequently the question occurs how large δ can be chosen. Of course δ shall be made as large as possible since the objective value shall be minimized.

For that purpose we consider the basic representation 4.1 of the solution

$$\vec{x}_B = A_B^{-1} \cdot \vec{b} + A_B^{-1} \cdot A_N \cdot \vec{x}_N$$

Since all non-basic variables except $x_{N(s)}$ remain equal to zero, it holds:

$$\begin{aligned} \vec{x}_B &= A_B^{-1} \cdot \vec{b} + A_B^{-1} \cdot A_N \cdot x_{N(s)} \\ &= A_B^{-1} \cdot \vec{b} + A_B^{-1} \cdot A_N \cdot \delta \end{aligned}$$

Since the new solution shall further remain feasible, every component of $\vec{x}_B$ has to be greater or equal to zero.

$$(\vec{x}_B)_i = (A_B^{-1} \cdot \vec{b})_i + (A_B^{-1} \cdot A_{N(s)})_i \cdot \delta \geq 0 \quad \text{für} \quad i = 1, \ldots, m$$

Since δ shall be chosen as large as possible, it follows:

$$\delta = x_{N(s)} := min \left\{ \frac{(A_B^{-1} \cdot \vec{b})_i}{(A_B^{-1} \cdot A_{N(s)})_i} : (A_B^{-1} \cdot A_{N(s)})_i > 0 \right\} \tag{4.2}$$

While computing δ with respect to equation 4.2, which is called **ratio rule**, two cases may occur.

Case 1:

$$\forall i = 1, \ldots, m : \quad (A_B^{-1} \cdot A_{N(s)})_i \leq 0$$

In this case the ratio rule yields no restriction for δ. Thus δ can be chosen arbitrarily large and therefore the objective value can be made arbitrarily small. In this case the LP is called **unbounded** .

Case 2:

$$\begin{aligned}\delta = x_{N(s)} &:= min\left\{\frac{(A_B^{-1}\cdot\vec{b})_i}{(A_B^{-1}\cdot A_{N(s)})_i} : (A_B^{-1}\cdot A_{N(s)})_i > 0\right\}\\ &= \frac{(A_B^{-1}\cdot\vec{b})_r}{(A_B^{-1}\cdot A_{N(s)})_r}\end{aligned}$$

Now it holds:

$$\begin{aligned}x_{N(j)} &= 0 \quad \forall j \neq s\\ x_{N(s)} &= \frac{(A_B^{-1}\cdot\vec{b})_r}{(A_B^{-1}\cdot A_{N(s)})_r}\\ x_{B(i)} &= (A_B^{-1}\cdot\vec{b})_i - (A_B^{-1}\cdot A_{N(s)})_i \cdot x_{N(s)}\\ &= (A_B^{-1}\cdot\vec{b})_i - (A_B^{-1}\cdot A_{N(s)})_i \cdot \frac{(A_B^{-1}\cdot\vec{b})_r}{(A_B^{-1}\cdot A_{N(s)})_r}\end{aligned}$$

A so called **basis exchange** happens. $B(r)$ leaves the basis, i.e. $x_{B(r)} = 0$, and $N(s)$ enters the basis, i.e. $x_{N(s)} > 0$. ([3])

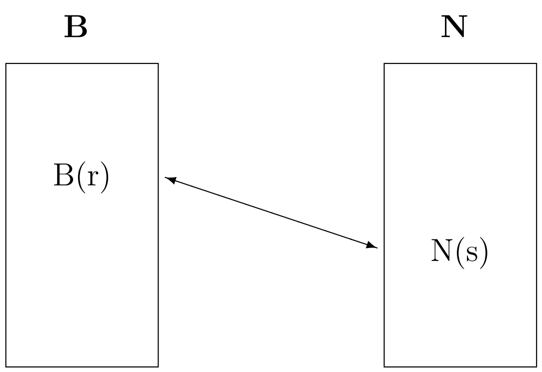

Figure 4.2: Basis exchange: $B(r)$ leaves the basis, $N(s)$ enters the basis.

Example 4.3 $B = (3,4),\ N = (1,2)$
As already found out in example 4.2 the basic solution, which belongs to this basis, is not optimal. $\vec{c}_N - \vec{c}_B \cdot A_B^{-1} \cdot A_N = (-1,0)$, *this means that by increasing* $x_{N(1)}$ *an improvement of the objective value is achieved.*

$$\begin{aligned} x_{N(1)} = \delta &= min\left\{ \frac{(A_B^{-1}\cdot\vec{b})_i}{(A_B^{-1}\cdot A_{N(1)})_i} : (A_B^{-1}\cdot A_{N(1)})_i > 0 \right\} \\ &= \left\{ \frac{(A_B^{-1}\cdot\vec{b})_2}{(A_B^{-1}\cdot A_{N(1)})_2} \right\} \\ &= \left\{ \frac{3}{1} \right\} = 3 = x_1 \end{aligned}$$

$$x_{N(2)} = x_2 = 0$$

$$x_{B(1)} = x_3 = (A_B^{-1}\cdot\vec{b})_1 - (A_B^{-1}\cdot A_{N(1)})_1 \cdot x_{N(1)} = 1 - (-1)\cdot 3 = 4$$

$$x_{B(2)} = x_4 = (A_B^{-1}\cdot\vec{b})_2 - (A_B^{-1}\cdot A_{N(1)})_2 \cdot x_{N(1)} = 3 - 1\cdot 3 = 0$$

The new basis now is $B' = (3,1),\ N' = (4,2)$. *By considering this graphically one finds out that one moved from the basic solution with respect to* $B = (3,4)$ $\vec{x} = (0,0)$ *to the basic solution with respect to* $B' = (3,1)$ $\vec{x} = (3,0)$ *(see figure 4.3).*

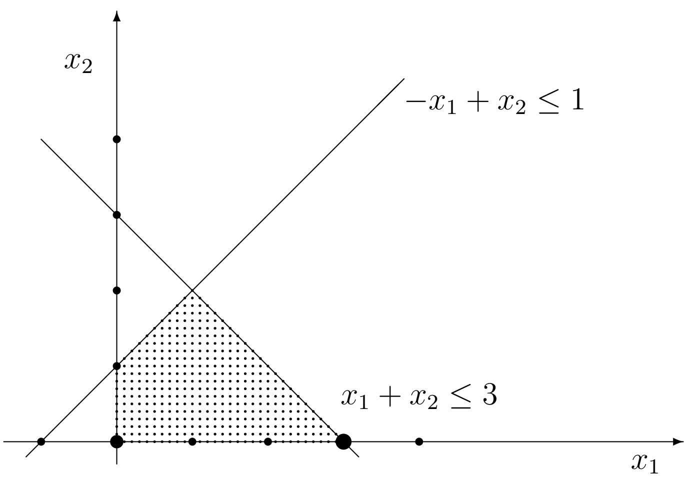

Figure 4.3: Graphical representation of the basic solutions with respect to $B = (3,4)$: $\vec{x} = (0,0)$ and with respect to $B' = (3,1)$: $\vec{x} = (3,0)$ as corner points of the feasible region.

4.6 Tableaus

Before in chapter 4.7 the simplex method is represented in a comprised form, the basis exchange shall be organized in an efficient way. This shall be done by storing the LP in so-called **tableaus** .

The objective function is rewritten as $-z + c_1 \cdot x_1 + \ldots + c_n \cdot x_n = 0$ and like the constraints it is also stored in a matrix which is written in tableau form as **starting tableau** $T = (t_{ij})$ with $i = 0, 1, \ldots, m$ and $j = 0, 1, \ldots, n, n+1$:

$$T = \begin{array}{c|ccc|c} -z & x_1 & \ldots & x_n & \\ \hline 1 & c_1 & \ldots & c_n & 0 \\ \hline 0 & a_{11} & \ldots & a_{1n} & b_1 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & a_{m1} & \ldots & a_{mn} & b_m \\ \hline \end{array} = \begin{array}{|c|c|c|} \hline 1 & \vec{c} & 0 \\ \hline \vec{0} & A & \vec{b} \\ \hline \end{array}$$

T represents a system of linear equations with m+1 equations. The 0^{th} column is associated with the variable $-z$, the i^{th} column with x_i $(i = 1, \ldots, n)$ and the $(n+1)$-st column contains the information about the right-hand sides.
For a basis B one denotes with T_B the non-singular $(m+1) \times (m+1)$ - matrix

$$T_B = \left(\begin{array}{c|c} 1 & \vec{c}_B \\ \hline 0 & \\ \vdots & A_B \\ 0 & \end{array} \right)$$

$$T_B^{-1} = \left(\begin{array}{c|c} 1 & -\vec{c}_B \cdot A_B^{-1} \\ \hline \vec{0} & A_B^{-1} \end{array} \right)$$

$$T_B^{-1} T = \left(\begin{array}{c|c|c} 1 & \vec{c} - \vec{c}_B \cdot A_B^{-1} \cdot A & -\vec{c}_B \cdot A_B^{-1} \cdot \vec{b} \\ \hline 0 & & \\ \vdots & A_B^{-1} \cdot A & A_B^{-1} \cdot \vec{b} \\ 0 & & \end{array} \right) =: T(B)$$

$T(B)$ is called the **simplex tableau** associated with the basis B:

- The first column is always the vector $(1, 0, \ldots, 0)^T$. This column only emphasizes the character of the 0^{th} row as an equation. Since this column does not change during the simplex method, it can be omitted.

- For $j = B(i) \in B$ $A_B^{-1} A_j = \vec{e}_i^{\,\mathrm{T}}$ (i^{th} unit vector with m components) holds. Furthermore $c_j - \vec{c}_B A_B^{-1} A_j = c_j - c_j = 0$ holds. Thus $T(B)$ contains in the column corresponding to the i^{th} basic variable $x_{B(i)}$ the value 0 in the 0^{th} row and then the i^{th} unit vector with m components.
- For $j = N(i) \in N$ the entry is $t_{0j} = c_j - \vec{c}_B A_B^{-1} A_j = \bar{c}_j$, i.e. the t_{0j} are the reduced costs of the non-basic variables x_j.
- In the last column $A_B^{-1} \cdot \vec{b}$ is the vector of the basic solution with respect to B and consequently $-\vec{c}_B \cdot A_B^{-1} \cdot \vec{b}$ is the negative of the objective value of the current basic solution.

Example 4.4 *Considering once more example 3.1 with basis $B = (1, 2)$ it follows:*

$T =$

$-z$	x_1	x_2	x_3	x_4	
1	−1	0	0	0	0
0	−1	1	1	0	1
0	1	1	0	1	3

Since:

$$
\begin{aligned}
A_B^{-1} &= \begin{pmatrix} -1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} \quad \textit{and} \\
\vec{c}_B \cdot A_B^{-1} &= (-1, 0) \cdot \begin{pmatrix} -1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} = \left(\frac{1}{2}, -\frac{1}{2}\right)
\end{aligned}
$$

one obtains:

$$
T_B^{-1} = \left(\begin{array}{c|cc} 1 & -1/2 & 1/2 \\ \hline 0 & -1/2 & 1/2 \\ 0 & 1/2 & 1/2 \end{array}\right)
$$

Thus the simplex tableau corresponding to the basis B is

$T(B) = T_B^{-1} T =$

$-z$	x_1	x_2	x_3	x_4	
1	0	0	−1/2	1/2	1
0	1	0	−1/2	1/2	1
0	0	1	1/2	1/2	2

Following the interpretation of $T(B)$ one takes from the 0^{th} row the reduced costs $\bar{c}_3 = -1/2, \quad \bar{c}_4 = 1/2$ of the non-basic variables and one can see that the optimality condition is not satisfied. Looking at the last column, we can see that $x_1 = 1$, $x_2 = 2$ are the values of the basic variables in the basic solution with objective value $-t_{0\,n+1} = -1$.

If $t_{0j} < 0$ for some $j \in \{1, \ldots, n\}$, the optimality condition will not be satisfied and one will try to move the non-basic variable into the basis. Using the simplex tableau, the value of δ can be easily computed with the ratio rule:

$$\delta = x_j = \min \left\{ \frac{t_{i\,n+1}}{t_{ij}} \; : \; t_{ij} > 0 \right\}.$$

Thus one recognizes an unbounded objective function by the fact that a column corresponding to a non-basic variable x_j with $t_{0j} < 0$ contains only entries ≤ 0. If $\delta = \frac{t_{r\,n+1}}{t_{rj}}$, one will do a so-called **pivot operation** with the element $t_{rj} > 0$, i.e. one will transform the j^{th} column of $T(B)$ into an unit vector using elementary row operations. The resulting tableau is the simplex tableau $T(B')$ with respect to the new basis B'.

Example 4.5 *We continue example 4.4 . Since $t_{03} = -1/2$, x_3 shall be moved into the basis. The ratio rule yields $\delta = x_3 = \frac{t_{25}}{t_{23}} = \frac{2}{1/2} = 4$, thus the last tableau of example 4.4 is pivoted with the element $t_{23} = \frac{1}{2}$.*

$T(B)$:

1	0	0	−1/2	1/2	1
0	1	0	−1/2	1/2	1
0	0	1	[1/2]	1/2	2

$\longrightarrow$

$T(B')$:

1	0	1	0	1	3
0	1	1	0	1	3
0	0	2	1	1	4

In $T(B')$ all reduced costs t_{0j} are non-negative, thus the corresponding basic solution $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$ is optimal.

If $t_{0j} \geq 0 \quad \forall\, j = 1, \ldots, n$ and $t_{i\,n+1} \geq 0 \quad \forall\, i = 1, \ldots, m$, $T(B)$ is called an **optimal (simplex-) tableau**. ([3])

4.7 The Simplex Algorithm

The procedure obtained in the previous sections shall now be formulated in terms of an algorithm.

Simplex Algorithm

Problem: $min\{\vec{c}\vec{x} : A \cdot \vec{x} = \vec{b},\ \vec{x} \geq \vec{0}\}$

(INPUT) Basic feasible solution $\vec{x} = (\vec{x}_B, \vec{x}_N)$ with respect to a basis B.

(1) Compute the simplex tableau $T(B)$.

(2) If $t_{0j} \geq 0 \ \ \forall\ j = 1, \ldots, n$

(**STOP**) $\vec{x} = (\vec{x}_B, \vec{x}_N)$ with $x_{B(i)} = t_{i\,n+1}$ $(i = 1, \ldots, m)$ and $\vec{x}_N = \vec{0}$ is the optimal solution of the LP with objective value $-t_{0\,n+1}$

(3) Choose j with $t_{0j} < 0$.

(4) If $t_{ij} \leq 0 \ \ \forall\ i = 1, \ldots, m$

(**STOP**) The LP is unbounded.

(5) Determine $r \in \{1, \ldots, m\}$ with $\frac{t_{r\,n+1}}{t_{rj}} = \min\left\{\frac{t_{i\,n+1}}{t_{ij}} : t_{ij} > 0\right\}$
and pivot with t_{rj}.
Goto (2).

Chapter 5

Example: Soft Drinks

Now one can return to example 2.1 and determine an optimal solution using the simplex method.

5.1 Standard Form

The LP now has to be transformed into standard form. After the introduction of slack variables and surplus variables it follows:

$$
\begin{array}{llllllllllllll}
min & 5x_1+ & 2x_2 + 0.25x_3 & & & & & & & & \\
u.d.N. & & 4x_2+ \; 17x_3 & -x_4 & & & & & & & = & 0 \\
& -3x_1+ & x_2+ \; 14x_3 & & +x_5 & & & & & & = & 0 \\
& x_1+ & 5x_2- \; 3x_3 & & & -x_6 & & & & & = & 0 \\
& 0.6x_1 - & 0.4x_2 - \; 0.4x_3 & & & & -x_7 & & & & = & 0 \\
& -0.5x_1 + & 0.5x_2 - \; 0.5x_3 & & & & & +x_8 & & & = & 0 \\
& -0.3x_1 - & 0.3x_2 + \; 0.7x_3 & & & & & & +x_9 & & = & 0 \\
& x_1+ & x_2+ \; x_3 & & & & & & & -x_{10} & = & 100 \\
& & & & & & & & & x_i & \geq & 0 & i = 1, \ldots, 10
\end{array}
$$

As it is required for the algorithm, the problem now is in standard form with

$$\vec{c} = (5, 2, 0.25, 0, 0, 0, 0, 0, 0, 0)$$

$$\vec{b}^T = (0, 0, 0, 0, 0, 0, 100)$$

$$A = \begin{pmatrix} 0 & 4 & 17 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -3 & 1 & 14 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 5 & -3 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0.6 & -0.4 & -0.4 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ -0.5 & 0.5 & -0.5 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ -0.3 & -0.3 & 0.7 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

5.2 Simplex Method

As (INPUT) a basic feasible solution with respect to a basis B is required. $B = (1, 4, 5, 6, 7, 8, 9)$ is a basis. Now it has to be checked, whether the corresponding basic solution is feasible, whether $\vec{x}_B = A_B^{-1} \cdot \vec{b} \geq \vec{0}$ holds.

$$A_B = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ -3 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0.6 & 0 & 0 & 0 & -1 & 0 & 0 \\ -0.5 & 0 & 0 & 0 & 0 & 1 & 0 \\ -0.3 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$A_B^{-1} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0.6 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0.5 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0.3 \end{pmatrix}$$

$$A_B^{-1} \cdot \vec{b} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0.6 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0.5 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0.3 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 100 \end{pmatrix} = \begin{pmatrix} 100 \\ 0 \\ 300 \\ 100 \\ 60 \\ 50 \\ 30 \end{pmatrix}$$

Thus the basis $B = (1, 4, 5, 6, 7, 8, 9)$ is feasible. Consequently the algorithm can start.

(1) Computation of $T(B)$:

- $$A_B^{-1} \cdot A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0.6 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0.5 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0.3 \end{pmatrix} \cdot \begin{pmatrix} 0 & 4 & 17 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -3 & 1 & 14 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 5 & -3 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0.6 & -0.4 & -0.4 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ -0.5 & 0.5 & -0.5 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ -0.3 & -0.3 & 0.7 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & -4 & -17 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 17 & 0 & 1 & 0 & 0 & 0 & 0 & -3 \\ 0 & -4 & 4 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & -0.6 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -0.5 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & -0.3 \end{pmatrix}$$

- $$\vec{c}_B \cdot A_B^{-1} \cdot A = (5, 0, 0, 0, 0, 0, 0) \cdot \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & -4 & -17 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 17 & 0 & 1 & 0 & 0 & 0 & 0 & -3 \\ 0 & -4 & 4 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & -0.6 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -0.5 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & -0.3 \end{pmatrix}$$
$$= (5, 5, 5, 0, 0, 0, 0, 0, 0, -5)$$

- $\vec{c}_B \cdot A_B^{-1} \cdot \vec{b} = (5,0,0,0,0,0,0) \cdot \begin{pmatrix} 100 \\ 0 \\ 300 \\ 100 \\ 60 \\ 50 \\ 30 \end{pmatrix} = 500$

- $\vec{c} - \vec{c}_b \cdot A_B^{-1} \cdot A = (5,2,0.25,0,0,0,0,0,0,0) - (5,5,5,0,0,0,0,0,0,-5)$
  $= (0,-3,-\frac{19}{4},0,0,0,0,0,0,5)$

$$\Longrightarrow T(B) = \left[\begin{array}{c|ccccccccccc|c} 1 & 0 & -3 & -\frac{19}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 5 & -500 \\ \hline 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 100 \\ 0 & 0 & -4 & -17 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 17 & 0 & 1 & 0 & 0 & 0 & 0 & -3 & 300 \\ 0 & 0 & -4 & 4 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 100 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & -0.6 & 60 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -0.5 & 50 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & -0.3 & 30 \end{array}\right]$$

In the following the first column can be omitted, as already explained on page 23.

(2) $t_{02} < 0$ and $t_{03} < 0 \Longrightarrow$ The solution is not optimal yet.

(3) Let $j = 2$ with $t_{02} = -3 < 0$.

(4) $t_{12}, t_{32}, t_{52}, t_{62} > 0 \Longrightarrow$ The LP is not unbounded.

(5) $\delta = \frac{t_{r\,n+1}}{t_{r2}} = \min\left\{\frac{t_{i\,n+1}}{t_{i2}} : t_{i2} > 0\right\} = \min\left\{100, \frac{300}{4}, 60, 50\right\} = 50 \Rightarrow r = 6$
Now one has to pivot with t_{62}:

$$T(B) = \left[\begin{array}{cccccccccc|c} 0 & \boxed{-3} & -\frac{19}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 5 & -500 \\ \hline 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 100 \\ 0 & -4 & -17 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 17 & 0 & 1 & 0 & 0 & 0 & 0 & -3 & 300 \\ 0 & -4 & 4 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 100 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & -0.6 & 60 \\ 0 & \boxed{1} & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -0.5 & 50 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & -0.3 & 30 \end{array}\right]$$

$\Longrightarrow$

0	0	$-\frac{19}{4}$	0	0	0	0	3	0	3.5	−350
1	0	1	0	0	0	0	−1	0	−0.5	50
0	0	−17	1	0	0	0	4	0	−2	200
0	0	17	0	1	0	0	−4	0	−1	100
0	0	4	0	0	1	0	4	0	−3	300
0	0	1	0	0	0	1	−1	0	−0.1	10
0	1	0	0	0	0	0	1	0	−0.5	50
0	0	1	0	0	0	0	0	1	−0.3	30

Now the column 2 has entered the basis, while the eighth column left the basis. The new basis is $B = (1, 2, 4, 5, 6, 7, 9)$.
$\longrightarrow$ (2)

(2) $t_{03} < 0 \Longrightarrow$ The solution is not optimal yet.

(3) Let $j = 3$ with $t_{02} = -\frac{19}{4} < 0$.

(4) $t_{13}, t_{33}, t_{43}, t_{53}, t_{73} > 0 \Longrightarrow$ The LP is not unbounded.

(5) $\delta = \frac{t_{r\,n+1}}{t_{r3}} = \min\left\{\frac{t_{i\,n+1}}{t_{i3}} : t_{i3} > 0\right\} = \min\left\{50, \frac{100}{17}, \frac{300}{4}, 10, 30\right\} = \frac{100}{17} \Rightarrow r = 3$
Now one has to pivot with t_{33}:

$T(B) =$

0	0	$\boxed{-\frac{19}{4}}$	0	0	0	0	3	0	3.5	−350
1	0	1	0	0	0	0	−1	0	−0.5	50
0	0	−17	1	0	0	0	4	0	−2	200
0	0	$\boxed{17}$	0	1	0	0	−4	0	−1	100
0	0	4	0	0	1	0	4	0	−3	300
0	0	1	0	0	0	1	−1	0	−0.1	10
0	1	0	0	0	0	0	1	0	−0.5	50
0	0	1	0	0	0	0	0	1	−0.3	30

$\Longrightarrow$

0	0	0	0	$\frac{19}{68}$	0	0	$\frac{32}{17}$	0	$\frac{219}{68}$	$-\frac{5475}{17}$
1	0	0	0	$-\frac{1}{17}$	0	0	$-\frac{13}{17}$	0	$-\frac{15}{34}$	$\frac{750}{17}$
0	0	0	1	1	0	0	0	0	−3	300
0	0	1	0	$\frac{1}{17}$	0	0	$-\frac{4}{17}$	0	$-\frac{1}{17}$	$\frac{100}{17}$
0	0	0	0	$-\frac{4}{17}$	1	0	$\frac{84}{17}$	0	$-\frac{47}{17}$	$\frac{4700}{17}$
0	0	0	0	$-\frac{1}{17}$	0	1	$-\frac{13}{17}$	0	$-\frac{7}{170}$	$\frac{70}{17}$
0	1	0	0	0	0	0	1	0	−0.5	50
0	0	0	0	$-\frac{1}{17}$	0	0	$\frac{4}{17}$	1	$-\frac{41}{170}$	$\frac{410}{17}$

The new basis is $B = (1, 2, 3, 4, 6, 7, 9)$.
$\longrightarrow$ (2)

(2) $t_{0j} \geq 0 \quad \forall j = 1, \ldots, n$ (STOP)

$$\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_{10} \end{pmatrix} = \begin{pmatrix} 750/17 \\ 50 \\ 100/17 \\ 300 \\ 0 \\ 4700/17 \\ 70/17 \\ 0 \\ 410/17 \\ 0 \end{pmatrix}$$

is optimal with objective value $-t_{0n+1} = \frac{5475}{17} \approx 322$

From the optimal tableau the values of $x_1, \ldots, x_{10}$ can be taken as follows:

Non-basic variables have the value of zero, i.e. in this case $x_5 = 0$, $x_8 = 0$ and $x_{10} = 0$.

The values of the basic variables are written in the last column. In the first column the unit vector with 1 in the first row is written. Therefore the basic variable x_1 is associated with the value of $\frac{750}{17}$ which is written in the last column of the first row.
In the second column the unit vector with 1 in the sixth row can be found. Consequently $x_2 = 50$, since 50 is written in the last column in the sixth row. The values of the other basic variables have been equally taken from the tableau.

This procedure is easy to understand, if one recalls that the non-basic variables are equal to zero and that the tableaus represent a system of linear equations.

Chapter 6

Example: Gardening-machines

In this section another example shall be considered, on which several limits and difficulties of the simplex method are illustrated.

Example 6.1 *A company produces and sells four different gardening-machines: shredders, lawn-mowers, small tractors and reaping-machines. Per shredder 1500 Euro profit are realized, while per lawn-mower 3500 Euro, per small tractor 3000 Euro and per reaping-machine 4000 Euro are gained. The company of course wants to maximize its profit.*

The production happens in a three-step process:
Step 1: Production of components
Step 2: Improvement of the surface
Step 3: Assembly
For the single production steps defined production times per production unit are given. Moreover the production capacities in the single production steps are bounded. The following schedule represents the conditions:

Product	*Shredder*	*Lawn-mower*	*Tractor*	*Reaping-machine*	*Capacity*
Step 1	*3.0*	*1.0*	*3.0*	*4.0*	*315*
Step 2	*1.0*	*2.0*	*2.7*	*4.0*	*270*
Step 3	*2.0*	*5.0*	*5.5*	*3.0*	*400*

It is expected that a maximum of 30 shredders is saleable. Moreover due to operational reasons at least twelve lawn-mowers, 20 small tractors and ten reaping-machines shall be sold.

6.1 Solution with the Simplex Method

For example 6.1 the following optimization model results:

$$\begin{array}{lrcrcrcrcl}
max & 1.5x_1 & + & 3.5x_2 & + & 3.0x_3 & + & 4.0x_4 & & \\
s.t. & 3.0x_1 & + & 1.0x_2 & + & 3.0x_3 & + & 4.0x_4 & \leq & 315 \\
 & 1.0x_1 & + & 2.0x_2 & + & 2.7x_3 & + & 4.0x_4 & \leq & 270 \\
 & 2.0x_1 & + & 5.0x_2 & + & 5.5x_3 & + & 3.0x_4 & \leq & 400 \\
 & x_1 & & & & & & & \leq & 30 \\
 & & & x_2 & & & & & \geq & 12 \\
 & & & & & x_3 & & & \geq & 20 \\
 & & & & & & & x_4 & \geq & 10 \\
 & & & & & & & x_i & \geq & 0 \quad \forall i = 1, \ldots, 4
\end{array}$$

After the transformation into standard form and the application of the simplex method one obtains the following solution: $x_1 = 0, x_2 = 36,5714, x_3 = 20, x_4 = 35,7143$[1]. The now arising problem is easy to see. The solution is not integer. In example 2.1 this was not a problem, for it is not difficult to measure $\frac{750}{17}l \approx 44.12l$ of a liquid, but it is problematic now. There are only *whole* gardening-machines.

6.2 Integer Optimization

Problems, whose solution has to be integer, are considered in integer optimization. Integer optimization shall not be discussed here as in detail as the simplex method. Nevertheless an insight into how one can obtain an integer solution shall be given.

6.2.1 Problems

We consider once again the solution which we have obtained for example 6.1:
$x_1 = 0, x_2 = 36,571438, x_3 = 20, x_4 = 35,71429$
This solution does not really solve the problem of the enterpriser who wants to optimize the production of his gardening-machines. He needs an integer solution.

How can one proceed to obtain an integer solution starting from the optimal solution?

[1] In the internet one finds e.g. under [4] software with which one can among other things solve linear programs.

It is obvious that by rounding the optimal solution up or down an integer solution is obtained. Consequently for example 6.1 one obtains $x_1 = 0$, $x_2 = 37$, $x_3 = 20$, $x_4 = 36$ as a solution. But this solution is infeasible, since it violates the second and the third constraint of the LP.
There are cases in which one obtains a feasible but very bad integer solution by rounding the solution.

Thus one sees that the obvious method to generate an integer solution by rounding quickly leads to bad or even infeasible solutions. In the following a better method to generate an integer solution shall be presented briefly.

6.2.2 Solution in the two-dimensional Case

Based on the possibility of the graphical representation, the method to generate integer solutions is introduced by an example with two variables.

Example 6.2 *A transportation company wants to transport several goods which are classified into different hazard rates. A unit of good* 1 *has a hazard rate of* 9 *on a scale from* -10 *to* $+10$*, while a unit of good* 2 *has a hazard rate of* -4*. Besides one unit of good* 1 *requires one unit of space in the transporter and gains a profit of* 2 *million Euro. One unit of good* 2 *yields a profit of* 7 *million Euro, but requires* 4 *units of space.*

The total capacity of a transporter is 14 *units of space and the maximum hazard value, which is not allowed to be exceeded, is* 36*.*

Since the transportation company wants to place as many goods as possible in a transporter, the following optimization problem results:

$$\begin{array}{llcrll} max & 2 \cdot x_1 & + & 7 \cdot x_2 & & \\ s.t. & 1 \cdot x_1 & + & 4 \cdot x_2 & \leq & 14 \\ & 9 \cdot x_1 & - & 4 \cdot x_2 & \leq & 36 \\ & & & x_1, x_2 & \geq & 0 \\ & & & x_1, x_2 & & integer \end{array}$$

In figure 6.1 one sees that the optimal solution of this problem is not integer. Indeed $x_1 = 5$ is an integer number, but with $x_2 = 2,25$ the transportation enterpriser cannot do a lot.

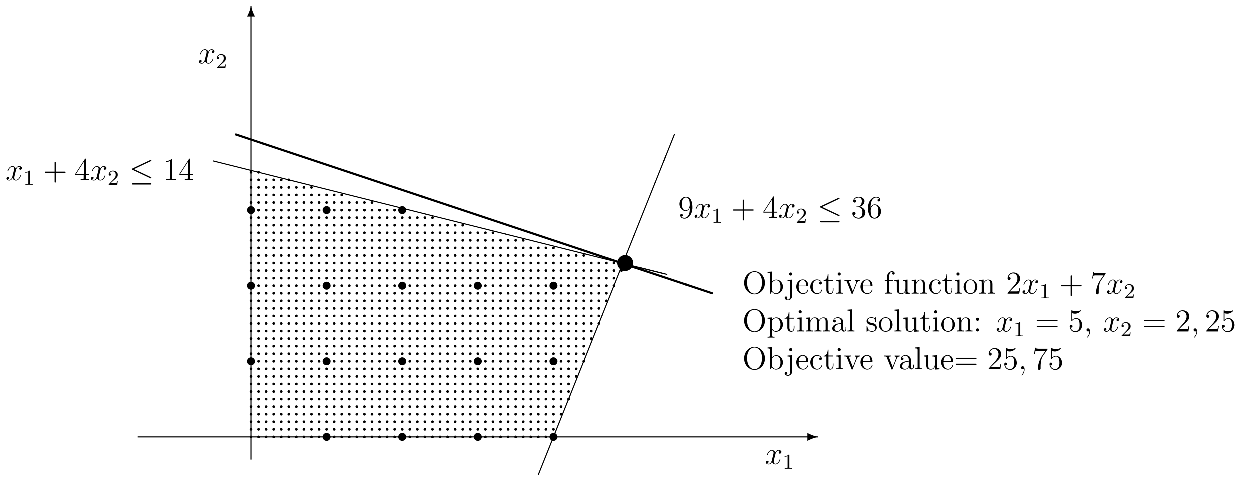

Figure 6.1: Graphical representation of the integer optimization problem from example 6.2 with non-integer optimal solution

Now either $x_2 \leq 2$ or $x_2 \geq 3$ has to hold. These two cases now have to be considered separately.

If one shifts the objective function separately in both parts of figure 6.2, one will obtain for each of the subproblems an optimal solution with an objective value which is smaller than the objective value of the original solution. In this case one obtains for $x_1 = 4,\bar{8}$, $x_2 = 2$ an objective value of $23,\bar{7}$ and for $x_1 = 2$, $x_2 = 3$ an objective value of 25. Since the greater one of the objective values is associated with an integer solution, the problem is solved. If this is not the case, so if the better value is associated with a non-integer value, one will have to repeat the method and to compare always all objective values.

6.2.3 Solution in the more-dimensional Case

The method from section 6.2.2 can be applied to problems with more than two variables. The subproblems are handled and solved like a LP. The optimal solution is checked on integrality and if necessary, the problem will be divided further.

We return once more to example 6.1. The solution obtained by the simplex method is $x_1 = 0, x_2 = 36,571438, x_3 = 20, x_4 = 35,71429$.
Since x_2 and x_4 are not integral, four cases $x_2 \leq 36$ and $x_4 \leq 35$, $x_2 \leq 36$ and $x_4 \geq 36$, $x_2 \geq 37$ and $x_4 \leq 35$ as well as $x_2 \geq 37$ and $x_4 \geq 36$ have to be considered . If one respectively puts these inequalities as additional constraints in the LP and solves it with the simplex method, one will obtain:

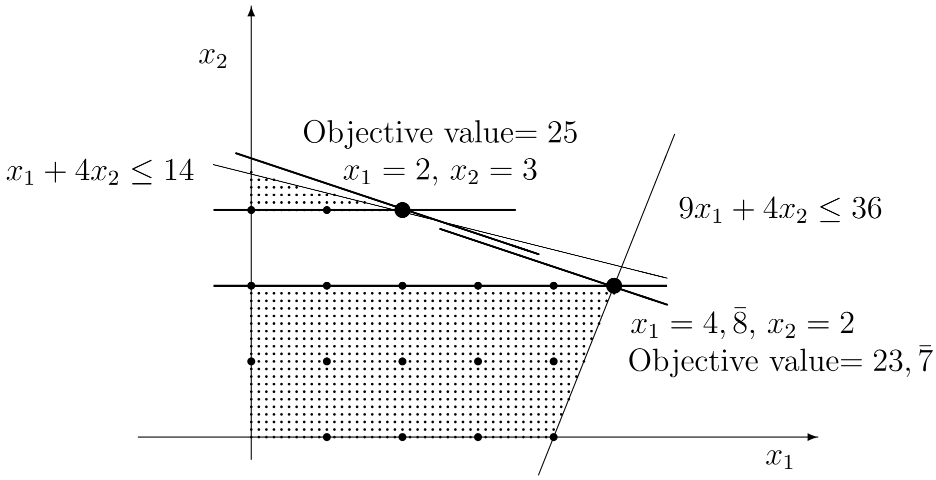

Figure 6.2: Partition of the optimization problem from example 6.2 into two subproblems.

$x_2 \leq 36$ $x_4 \leq 35$	$x_2 \leq 36$ $x_4 \geq 36$	$x_2 \geq 37$ $x_4 \leq 35$
$x_1 = 10$	$x_1 = 0$	$x_1 = 0$
$x_2 = 33$	$x_2 = 36$	$x_2 = 37$
$x_3 = 20$	$x_3 = 20$	$x_3 = 20$
$x_4 = 35$	$x_4 = 36$	$x_4 = 35$
$\vec{c} \cdot \vec{x} = 330,5$	$\vec{c} \cdot \vec{x} = 330$	$\vec{c} \cdot \vec{x} = 329,5$

For $x_2 \geq 37$ and $x_4 \geq 36$ an infeasible problem results.
$x_2 \geq 37$, $x_4 \geq 36$ and $x_3 \geq 20$ contradicts the constraint $x_1 + 2x_2 + 2.7x_3 + 4x_4 \leq 270$. Since a maximization problem is concerned, the greatest objective value $\vec{c} \cdot \vec{x} = 330,5$ is the best and $x_1 = 10$, $x_2 = 33$, $x_3 = 20$ and $x_4 = 35$ is the optimal integer solution.

There are still further methods of integer optimization which can be looked up in [1].

Appendix A

Rank of a Matrix A

To explain the rank of a matrix A, the notion of the linear dependence of vectors is required.

Definition A.1 (Linear Dependence) *The vectors* $(a_1, a_2, \ldots, a_n)$ *are called* ***linearly dependent****, if there are* $\alpha_1, \alpha_2, \ldots, \alpha_n \in I\!R$ *which are not equal to zero and for which*

$$\alpha_1 \cdot a_1 + \ldots + \alpha_n \cdot a_n = 0$$

holds, this means, if $a_1, \ldots, a_n$ *represent zero in a non-trivial way.*

The vectors $(a_1, a_2, \ldots, a_n)$ *are called* ***linearly independent****, if they are not linearly dependent, this means, if*

$$\alpha_1 \cdot a_1 + \ldots + \alpha_n \cdot a_n = 0, \qquad \alpha_1, \alpha_2, \ldots, \alpha_n \in I\!R \quad \alpha_1 = \ldots = \alpha_n = 0$$

holds.

Rank of a Matrix A

A $m \times n$-matrix A has **rank** r, if and only if among the column vectors of A

(i) there are r linearly independent vectors and

(ii) $r + 1$ vectors at a time are linearly dependent.

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