SURFACE MEASURES ON PATHS IN AN EMBEDDED RIEMANNIAN MANIFOLD

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Abstract. We construct and study two surface measures on the space $C([0,1],M)$ of paths in a compact Riemannian manifold M imbedded into $\mathbb{R}^n$. The first one is induced by conditioning the usual Wiener measure on $C([0,1],\mathbb{R}^n)$ to the event that the Brownian particle does not leave the tubular ε-neighbourhood of M up to time 1, and passing to the limit. The second one is defined as the limit of the laws of reflected Brownian motions with reflection on the boundary of the tubular ε-neighbourhoods. We prove that the both surface measures exist and compare them with the Wiener measure $\mathbb{W}_M$ on $C([0,1],M)$. We show that the first one is equivalent to $\mathbb{W}_M$ and compute the corresponding density explicitly in terms of the scalar curvature and the mean curvature vector of M. Finally, we show that the second surface measure coincides with $\mathbb{W}_M$.

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Contents

Chapter 1

Introduction

1.1 Motivation and main results

The study of surface measures in infinite-dimensional spaces plays an important role both for functional analysis and for stochastic. The notion of surface measure is a natural generalization of the notion of the Lebesgue area of a surface in $\mathbb{R}^n$: for a measure μ on an infinite-dimensional space X and for a sufficiently smooth surface $S \subset X$ one constructs a measure μ^S concentrated on S and related to the original measure μ in a similar way as the Lebesgue volume and area of a surface are related in a finite-dimensional space. Such a measure μ^S is called a surface measure corresponding to the surface S.

There are different approaches to the construction of surface measures related to sufficiently smooth measures on infinite-dimensional spaces. The first general construction is based on the notion of differentiability of measures introduced by Fomin ([3]). Such surface measures were studied in works of Skorokhod ([24]) and Uglanov ([30]). Quite a different idea was proposed by Airault and Malliavin ([2]).

A common disadvantage of these approaches is that they work only in the case of finite-dimensional conditioning, i.e., for surfaces of finite codimension.

In this thesis we deal with infinite-dimensional conditioning. We consider the space $C_{a_0}([0,1],\mathbb{R}^n)$ of continuous paths in $\mathbb{R}^n$ starting at

a fixed point $a_0 \in \mathbb{R}^n$ as an ambient space. Then we fix a smooth compact m-dimensional Riemannian manifold $M \subset \mathbb{R}^n$ without boundary, containing a_0 and consider the path space $C_{a_0}([0,1],M)$ as a surface in $C_{a_0}([0,1],\mathbb{R}^n)$. Now there are (at least) two natural ways to define a surface measure on $C_{a_0}([0,1],M)$ corresponding to the Wiener measure $\mathbb{W}$ on the ambient space, i.e., to a standard Brownian motion on $\mathbb{R}^n$.

Let $\mathbb{M}_\varepsilon$ the tubular neighborhood of M, which consists of all points in $\mathbb{R}^n$ such that their distance to the manifold is less or equal ε, i.e.,

$$\mathbb{M}_\varepsilon = \{a \in \mathbb{R}^n : \ \mathrm{dist}(a,M) \le \varepsilon\},$$

where dist denotes the Euclidean distance in $\mathbb{R}^n$.

Let (b_t) be a standard Brownian motion in $\mathbb{R}^n$ starting at a_0. We denote by $\mathbb{W}_\varepsilon$ the probability measure on $C_{a_0}([0,1],\mathbb{R}^n)$ (concentrated on $C_{a_0}([0,1],\mathbb{M}_\varepsilon)$) that is the law of (b_t) conditioned on the event that the process does not leave the tubular ε-neighborhood of the manifold M, i.e.,

$$\mathbb{W}_\varepsilon = \frac{\mathbb{W}|_{C_{a_0}([0,1],\mathbb{M}_\varepsilon)}}{\mathbb{W}(C_{a_0}([0,1],\mathbb{M}_\varepsilon))}.$$

Definition 1. *The measure $\mathbb{S}_1$ on $C(M)$ defined as the weak limit*

$$\mathbb{S}_1 = (weak) \lim_{\varepsilon\to 0} \mathbb{W}_\varepsilon$$

is called the surface measure of the first type (or, the surface measure generated by conditioned Brownian motions).

The main result of this thesis is Theorem 1, which says that the surface measure $\mathbb{S}_1$ exists, it is absolutely continuous with respect to the Wiener measure $\mathbb{W}_M$ on $C_{a_0}([0,1],M)$ corresponding to a Brownian motion on M, and its density is given by

$$\frac{d\mathbb{S}_1}{d\mathbb{W}_M}(\omega) = \frac{\exp\left\{-\frac{1}{4}\int_0^1 R(\omega_t)dt + \frac{1}{8}\int_0^1 ||\sigma||^2(\omega_t)dt\right\}}{\mathbb{E}_{\mathbb{W}_M} \exp\left\{-\frac{1}{4}\int_0^1 R(\omega_t)dt + \frac{1}{8}\int_0^1 ||\sigma||^2(\omega_t)dt\right\}} \qquad (1.1)$$

where R is the scalar curvature and σ is the tension field of the embedding of M into $\mathbb{R}^n$ (which is equal to $(\dim M)\kappa$, where κ denotes the mean curvature vector field of M). In particular, if M has constant scalar and mean curvature then the surface measure $\mathbb{S}_1$ coincides with the Wiener measure $\mathbb{W}_M$, as was already announced in [25].

A surprising corollary of this theorem is that the surface measure $\mathbb{S}_1$ coincides with the surface measure studied in [26], which is defined in the following way. For any partition $P : 0 = t_0 < t_1 \cdots < t_n = 1$ one considers the law of a Brownian motion in $\mathbb{R}^n$ conditioned to be in M at times t_i (i.e., a sequence of Brownian bridges) and let the mesh of the partition go to zero; the limit measure is then called a surface measure.

It is also remarkable that the particular geometric potential in our main result (1.1) appears also in the context of the study of holonomic constraints in quantum mechanics in [13, p. 500].

There is another natural way to define a surface measure on the space $C_{a_0}([0,1],M)$ corresponding to a Brownian motion in the ambient space. Let (r_t^ε) be a Brownian motion in $\mathbb{R}^n$ starting at a_0 with reflection on the boundary $\partial\mathbb{M}_\varepsilon$. Denote the law of (r_t^ε) by $\mathbb{W}_\varepsilon^r$.

Definition 2. *The measure $\mathbb{S}_2$ on $C(M)$ defined as the weak limit*

$$\mathbb{S}_2 = (weak) \lim_{\varepsilon \to 0} \mathbb{W}_\varepsilon^r$$

is called the surface measure of the second type (or, the surface measure generated by reflected Brownian motions).

We prove in Theorem 2 that the surface measure $\mathbb{S}_2$ exists and, moreover, coincides with the Wiener measure $\mathbb{W}_M$.

We remark that these results also yield another constructions of the Wiener measure $\mathbb{W}_M$ which complement the classical ones (cf. e.g. [9]). According to Theorem 2 the Wiener measure $\mathbb{W}_M$ can be constructed directly as the limit of reflected Brownian motions in the surrounding space. On the other hand, let φ denote any continuous extension of the density given by formula (1.1) to the space $C(\mathbb{M}_\varepsilon)$ which is also bounded away from 0. Then by Theorem 1 the measures $\varphi^{-1}\mathbb{W}_\varepsilon$ converge to $\mathbb{W}_M$.

Certainly, the similar definitions of the surface measures can be given for the time interval $[0,T]$ with arbitrary $T > 0$ and similar theorems will be true. So we come to the natural question of convergence of such surface measures as $T \to \infty$. For the surface measures of the

second type the answer (as well as the question) is trivial: it is the Wiener measure on $C_{a_0}([0,\infty),M)$. The limit behaviour of surface measures of the first type $\mathbb{S}_1^T$ is investigated in Theorem 3 for one-dimensional manifolds in $\mathbb{R}^n$. The limit measure turns out to be the law of a drifted Brownian motion, which solves the stochastic differential equation

$$dy_t = db_t + \nabla \log \varphi(y_t) dt$$

with the initial condition $y_0 = a_0$, where φ is a uniquely determined positive eigenfunction of the operator $\frac{1}{2}\Delta_M + \frac{1}{8}\kappa^2$ and κ denotes the (mean) curvature of M.

1.2 Structure of the thesis and further results

The thesis is organized as follows.

In the rest of Chapter 1 we introduce the terminology and present main ideas of the proofs.

The first two sections of Chapter 2 are purely geometrical. In Section 2.1 we introduce the notion of a special coordinate system at a point in M and compute the basic geometrical invariants, such as curvature, second fundamental form, tension field, and the derivatives of some projections in terms of these special coordinates. In Section 2.2 we introduce two natural measures on a tubular neighborhood of M: the first one is just the Lebesgue measure induced by the embedding of $\mathbb{M}_\varepsilon$ into $\mathbb{R}^n$; the second one is called the reference measure and is induced by the flat measure on the normal bundle NM. Further, we define the shifting vector field corresponding to this pair of measures. In Section 2.3 we introduce the notion of stochastic parallel translation and define the Fermi decomposition of a continuous semimartingale with values in a tubular neighborhood.

Chapter 3 contains the proofs of the main results. In Section 3.1 we define the shifted process (y_t) and study its surface measure (of the first type). In Section 3.2 we compare the law of the shifted process $\mathcal{L}(y)$ with the Wiener measure $\mathbb{W}$ and compute some approximation for the density $d\mathbb{W}/d\mathcal{L}(y)$ on the paths from $\mathbb{M}_\varepsilon$. Finally, in Section 3.3 we prove Theorem 1 and Theorem 2.

In Chapter 4 we consider two particular cases. The first one is described in Section 4.1 and is devoted to closed curves without self-intersections in $\mathbb{R}^n$. We extend the notion of surface measure to arbitrary time intervals $[0,T]$, study the processes corresponding to such measures and find the limit of the surface measures as $T \to \infty$.

The second one is presented in Section 4.2 and describes surface measures on manifolds with the flat normal bundle. In this case most notions and constructions used in the proofs can be interpreted geometrically. Actually, the idea of the proof of Theorem 1 is prompted by this example.

Certainly, it would be interesting to generalize our results from a Brownian motion to general diffusion processes, and from smooth to non-smooth manifolds. In last Chapter 5 we make the first steps in these directions.

In Section 5.1 we study the simplest diffusion process that is not a Brownian motion – we consider a process (c_t) in $\mathbb{R}^2$, whose components are one-dimensional scaled Brownian motions with different variances. Even in this simple case we are only able to deal with the manifold that is a straight line passing through the origin. In Proposition 5.2 and Proposition 5.3 we show that the surface measures of the first and of the second type are the laws of scaled Brownian motions on the straight line with (different) variances, and we compute their variances in terms of the variances of the original processes (c_t) and the angle between the straight line and the x-axis.

In Section 5.2 we consider an example of a manifold with one singular point. Such a manifold can be approximated by smooth manifolds. We are interested in the convergence of the surface measures corresponding to the approximating manifolds. It turns out (see Proposition 9) that, if the limit measure exists then it can not be absolutely continuous with respect to the Wiener measure on the original manifold. Moreover, the process corresponding to the limit measure must come to the singular point before time 1 with probability 1.

1.3 Terminology

In this section we introduce the notation that will be used throughout the thesis.

In the previous sections we have fixed a smooth compact m-dimensional Riemannian manifold $M \subset \mathbb{R}^n$ without boundary and a point $a_0 \in M$. We have also defined tubular ε-neighborhoods $\mathbb{M}_\varepsilon$ of M.

Denote the codimension of M in $\mathbb{R}^n$ by $k = n - m$.

For any Borel set $X \subset \mathbb{R}^n$ containing a_0, we denote by $C(X)$ the space of continuous paths $\omega : [0,1] \to X$ such that $\omega_0 = a_0$.

For any $a \in M$, we denote by T_aM (respectively, by N_aM) the tangent (respectively, the normal) space to the manifold M at the point a. Since M is embedded into $\mathbb{R}^n$ they can (and will) be considered as linear subspaces of $\mathbb{R}^n$.

Now let us fix an orthonormal basis (e_i) in $\mathbb{R}^n$ such that its first m basis vectors span the tangent space $T_{a_0}M$ (and hence the last k ones span the normal space $N_{a_0}M$).

By the compactness and smoothness of M there exists $\varepsilon_0 > 0$ such that the orthogonal projection $\pi : \mathbb{M}_{\varepsilon_0} \to M$ given by the relation

$$\operatorname{dist}(a, \pi(a)) = \min_{x \in M} \operatorname{dist}(a, x)$$

is well-defined and uniquely determined. Moreover, π is smooth.

We denote by $gl(n)$ the linear space of all $n \times n$ matrices with real elements and by $o(n) \subset gl(n)$ the subset of all orthogonal matrices.

For each $a \in M$, let P_a denote the orthogonal projection of $\mathbb{R}^n$ onto T_aM and $Q_a = Id - P_a$ denote the orthogonal projection of $\mathbb{R}^n$ onto N_aM. Then P and Q are smooth functions from M to the vector space of linear maps from $\mathbb{R}^n$ to $\mathbb{R}^n$. Since we have fixed the orthonormal basis (e_i) in $\mathbb{R}^n$ we can also say that P and Q are smooth functions from M to $gl(n)$. In the sequel, we will always identify matrices with corresponding linear operators if there is no risk of confusion.

Further, let $\mathrm{pr}_1 : \mathbb{R}^n \to \mathbb{R}^m$ (respectively, $\mathrm{pr}_2 : \mathbb{R}^n \to \mathbb{R}^k$) be the linear operator that maps $u \in \mathbb{R}^n$ to its first m (respectively, to its last k) coordinates. Denote by $\mathrm{pr}_1^{-1} : \mathbb{R}^m \to \mathbb{R}^n$ (respectively, $\mathrm{pr}_2^{-1} : \mathbb{R}^k \to \mathbb{R}^n$) the right inverse of pr_1 (respectively, of pr_2) such that $\mathrm{pr}_1^{-1}\mathrm{pr}_1 = P_{a_0}$ (respectively, $\mathrm{pr}_2^{-1}\mathrm{pr}_2 = Q_{a_0}$).

We also use the Einstein summation convention: an index occurring twice in a product is to be summed from one up to the space dimension.

1.4 Outlines of the proofs

In this section we present the main ideas of the proofs of Theorems 1, 2, and 3. It should be mentioned that the proof of Theorem 2 is essentially easier then one of Theorem 1 since in the case of $\mathbb{S}_2$ there is no need to consider conditional distributions.

1.4.1 Surface measure $\mathbb{S}_1$: flat case

The idea of the proof of Theorem 1 is prompted by the following decomposition of the generator $\frac{1}{2}\Delta$ of the Brownian motion in $\mathbb{R}^n$, which is, however, only true for compact Riemannian manifolds with flat normal bundle (cf. Proposition 6)

$$\frac{1}{2}(\Delta u)(a) = \frac{1}{2}\left[\Delta_{M_a} u + \Delta_{N_a M} u\right](a) + \frac{m}{2}\langle \kappa, \nabla u\rangle(a), \tag{1.2}$$

where M_a is the manifold through $a \in \mathbb{M}_{\varepsilon_0}$ that is parallel to M, Δ_{M_a} is the corresponding Laplace-Beltrami operator, $\Delta_{N_a M}$ is the Laplace operator in the normal space $N_a M = N_{\pi(a)} M$, and $\kappa(a)$ is the mean curvature vector of the manifold M_a at the point a.

For $a \in \mathbb{M}_{\varepsilon_0}$, the parallel manifold M_a is roughly speaking an m-dimensional submanifold of $\mathbb{R}^n$ passing through a such that for all $b \in M_a$ the tangent space $T_b M_a$ coincides with the tangent space $T_{\pi(b)} M$. In Section 4.2 we show that such manifolds locally exist if and only if the normal bundle of M in $\mathbb{R}^n$ is flat (and hence the decomposition (1.2) makes sense only in this case). Although the parallel manifolds can not be extended to global parallel manifolds over M, the local existence is sufficient for the definition of the Laplace-Beltrami operator $[\Delta_{M_a} u](a)$ at the point a in (1.2).

It follows from the decomposition (1.2) that the process (y_t) shifted by the vector field $\frac{1}{2}m\kappa$, i.e., being the solution of the stochastic differential equation

$$\begin{cases} dy_t = db_t - \frac{1}{2}m\kappa(y_t)dt \\ y_0 = a_0 \end{cases}$$

has the generator $\frac{1}{2}[\Delta_{M_a} + \Delta_{N_a M}]$. This means that the orthogonal part of the process (y_t) is almost independent of its part along the

manifold, which is in turn almost a Brownian motion in M. Therefore it is naturally to expect that the surface measure corresponding to (y_t) is a Wiener measure on the manifold.

Finally, since the process (y_t) differs from a Brownian motion by a drift term, we could pass on to the surface measure of a Brownian motion using Girsanov's Theorem.

1.4.2 Surface measure $\mathbb{S}_1$: general case

It turns out that also in the case of non-flat normal vector bundle there is a vector field v on $\mathbb{R}^n$ such that the solution of the stochastic differential equation

$$\begin{cases} dy_t = db_t - \frac{1}{2}v(y_t)dt \\ y_0 = a_0 \end{cases}$$

yields in the surface limit a Brownian motion on M. Such a vector field reflects the difference between the Lebesgue measure $\lambda_{\mathbb{R}^n}$ and the reference measure $\lambda_\oplus$ on $\mathbb{M}_{\varepsilon_0}$ defined by

$$\lambda_\oplus(A) = \int_{\pi(A)} \lambda_{\mathbb{R}^k}(A_x)d\lambda_M(x),$$

where $A_x = \pi^{-1}(x)$ and $\lambda_{\mathbb{R}^k}$ and λ_M are the Lebesgue measures on $\mathbb{R}^k$ and M, respectively. The reference measure $\lambda_\oplus$ is equivalent to $\lambda_{\mathbb{R}^n}$ and the shifting vector field v is then defined by

$$v(a) = -\,Q_{\pi(a)}\left[\nabla \log \frac{d\lambda_\oplus}{d\lambda_{\mathbb{R}^n}}\right],$$

for $a \in \mathbb{M}_{\varepsilon_0}$ and extended smoothly to $\mathbb{R}^n$.

We prove in Proposition 2 that the surface measure corresponding to the process (y_t) is just the Wiener measure on the manifold. The idea of the proof is based on Fermi decomposition of the process (y_t), which is constructed in Section 2.3. Namely, we represent the process (y_t) by a pair of processes (x_t) and (z_t), where (x_t) is a process in M and (z_t) is a process in $\mathbb{R}^k$. The first one is just the projection of (y_t) to the manifold stopped while leaving $\mathbb{M}_{\varepsilon_0}$. To construct the second process, we fix an orthonormal basis in $N_{a_0}M$ and move it by stochastic parallel translation along the semimartingale (x_t) to the point x_t. So we get an orthonormal basis in $N_{x_t}M$ and define z_t by the coordinates of $y_t - x_t \in N_{x_t}M$ with respect to this basis up to the exit time of $\mathbb{M}_{\varepsilon_0}$. Then we prove in Lemma 10 and Lemma 11 that (z_t) is a k-dimensional Brownian motion independent of the m-dimensional Brownian motion driving the process (x_t). Using this fact, we show that the distribution of (x_t) under the condition that $||z_s|| \leq \varepsilon$ for all $0 \leq s \leq 1$ converges to the Wiener measure on the manifold.

It follows from Girsanov's Theorem (see Lemma 13) that the distribution μ of the process (y_t) is equivalent to $\mathbb{W}$, and the density is given by

$$\rho = \frac{d\mathbb{W}}{d\mu} = \exp\left\{\frac{1}{2}\int_0^1 \langle v(b_t), db_t\rangle + \frac{1}{8}\int_0^1 |v(b_t)|^2 dt\right\}.$$

If ρ were continuous and bounded we could find the density $d\mathbb{S}_1/d\mathbb{W}_M$ just by the normalized restriction of ρ to $C(M)$. Since ρ is not necessarily of this kind we approximate it by a continuous bounded function ρ_0 in such a way that the approximation is quite good on the paths from $C(\mathbb{M}_\varepsilon)$ (the precise definition is given in Section 3.2). In Proposition 3 we compute ρ_0 explicitly to

$$\rho_0 = \exp\left\{-\frac{1}{4}\int_0^1 R(x_t)dt + \frac{1}{8}\int_0^1 ||\sigma||^2(x_t)dt\right\},$$

where (x_t) is the first component of the Fermi decomposition of (b_t). Finally, we prove that the density $d\mathbb{S}_1/d\mathbb{W}_M$ coincides with the normalized restriction of ρ_0 to $C(M)$ and obtain the formula (1.1).

1.4.3 Surface measure $\mathbb{S}_2$

The proof of Theorem 2 is based on the fact that the reflected Brownian motion (r_t^ε) is a solution of the Skorokhod stochastic differential equation

$$dr_t^\varepsilon = db_t + \frac{1}{2}n(r_t^\varepsilon)dl_t^\varepsilon,$$

where (l_t^ε) is the local time of the process (r_t^ε) on the boundary $\partial\mathbb{M}_\varepsilon$, and $n(y)$ is the inward pointing unit normal vector at $y \in \partial\mathbb{M}_\varepsilon$. This

enables us to write down a stochastic differential equation for the projection $(\pi(r_t^\varepsilon))$ and to show that the coefficients of this equation converge to the coefficients of the stochastic differential equation for a Brownian motion on M. This implies the weak convergence of $\mathbb{W}_\varepsilon^r$ to $\mathbb{W}_M$ and $\mathbb{S}_2 = \mathbb{W}_M$.

1.4.4 Convergence of surface measures

We have already mentioned that Theorem 3, which proves the convergence of the surface measure $\mathbb{S}_1^T$ defined for the time interval $[0, T]$ to the law of a drifted Brownian motion, deals only with one-dimensional manifolds (and perhaps could deal with n-dimensional tori). The main reason is the fact that the existence and uniqueness of a positive eigenfunction of the operator $\frac{1}{2}\Delta_M + V$ with some potential V is proved only for tori.

We also use the Feynman-Kac formula in order to show that, for each T, the process (y_t^T) corresponding to the measure $\mathbb{S}_1^T$ is an inhomogeneous Markov process and to compute its transition density explicitly (Proposition 4).

Further, we use Girsanov's Theorem in order to find the stochastic differential equations solved by the processes (y_t^T) (Proposition 5). They show that these processes are drifted Brownian motions with time-space dependent drifts, which are computed explicitly.

Finally, using the existence and uniqueness of a positive eigenfunction of the operator $\frac{1}{2}\Delta_M + \frac{1}{8}\kappa^2$, we prove the convergence of the coefficients of these stochastic differential equations as $T \to \infty$ to the coefficients of the limit equation describing a drifted Brownian motion with an independent of the time drift, which is computed in terms of the positive eigenfunction.

Chapter 2

Geometry of tubular neighbourhoods

2.1 Special coordinates and geometric invariants of the manifold

In this section we introduce the notion of special coordinates at a given point a in the tubular neighbourhood. Roughly speaking, we consider orthogonal coordinates in $\mathbb{R}^n$ centered at the projection $\pi(a)$ such that the first m basis vectors of the coordinates generate the tangent space and the last ones the normal space to M at $\pi(a)$. Such coordinates can be described by an orthogonal matrix u and in such coordinates the manifold M can be locally described by a function f from $\mathbb{R}^m$ to $\mathbb{R}^k$. We compute the basic geometrical characteristics of the manifold and of the embedding of the manifold, such as curvature, second fundamental form, tension field, and derivatives of some projections in terms of the special coordinates, i.e., in terms of the matrix u, the function f, and its derivatives.

2.1.1 Special coordinate systems

For a given point $a \in \mathbb{M}_{\varepsilon_0}$, denote by $o_a(n) \subset o(n)$ the set of all orthogonal $n \times n$-matrices u such that

$$T_{\pi(a)}M = \langle ue_i : 1 \le i \le m \rangle$$
$$N_{\pi(a)}M = \langle ue_i : m+1 \le i \le n \rangle.$$

We will call such matrices tangent to M at the point a.

Definition 3. *Let $a \in \mathbb{M}_{\varepsilon_0}$ and $u \in o_a(n)$. The orthogonal coordinate system (y^i) in $\mathbb{R}^n$ centered at $\pi(a)$ with the basis (ue_i) is called the special coordinate system corresponding to the pair (a, u).*

Let us fix $a \in \mathbb{M}_{\varepsilon_0}$ and $u \in o_a(n)$. By the implicit function theorem, in the special coordinate system (y^i) corresponding to the pair (a, u) the manifold M can be represented locally in a neighborhood of the point $\pi(a)$ by a system of equations

$$y^{m+s} = f_s(y^1, \dots, y^m),\ 1 \le s \le k,$$

or, equivalently, by a system of equations

$$\varphi_s(y) = 0,\ 1 \le s \le k,$$

where the functions (φ_s) are given by

$$\varphi_s(y) = y^{m+s} - f_s(y^1, \dots, y^m).$$

Notice that $\nabla\varphi_s(0) = ue_{m+s}$ for all $1 \le s \le k$.
Further, for $1 \le s \le k$, denote

$$F_s = \mathrm{Hess} f_s(0)$$

and denote by $z \in \mathbb{R}^k$ the last k coordinates of the point a in the coordinate system (y^i) (notice that the first m coordinates of the point a are equal to zero).

It should be mentioned that F_s is actually a smooth function of $x = \pi(a)$ and u, and in the sequel we will sometimes use the notation $F_s(x, u)$.

Finally, denote by $\bar{\pi}$ the function from a neighborhood of zero in $\mathbb{R}^n$ to $\mathbb{R}^n$ that is the projection π written in special coordinates (y^i) corresponding to (a, u).

Notice that the function $\bar{\pi}$, the families of functions (f_s) and (φ_s), the family of $m \times m$-matrices (F_s) and the k-dimensional vector z are uniquely determined by the pair (a, u), which makes possible to give the following definition.

Definition 4. *The collection $\{(f_s), (\varphi_s), (F_s), z, \bar{\pi}\}$ is called the local representation of M corresponding to the pair (a, u).*

2.1.2 Derivative of the projection π

In this section we compute the first derivative operator of the projection π at a point in the ε_0-neighborhood of M.

Lemma 1. *Let $a \in \mathbb{M}_{\varepsilon_0}$ and $u \in o_a(n)$. Then the first derivative operator $D\pi(a)$ of the projection is given by the formula*

$$D\pi(a) = u\mathrm{pr}_1^{-1}[I - z^s F_s]^{-1}\mathrm{pr}_1 u^T, \tag{2.1}$$

or, equivalently, the first derivative operator $D\bar{\pi}(0, z)$ is given by the $n \times n$-matrix

$$D\bar{\pi}(0, z) = \begin{bmatrix} [I - z^s F_s]^{-1} & 0 \\ 0 & 0 \end{bmatrix}, \tag{2.2}$$

where $\bar{\pi}$, (F_s) and z are from the local representation of M corresponding to (a, u) and I denotes the $m \times m$ unit matrix.

Proof. Let us prove the second formula. Notice that

$$\partial_{m+s}\bar{\pi}(0, z) = 0, \text{ for all } 1 \le s \le k,$$

since the projection is constant along these directions. This proves that the both right blocks of the matrix (2.2) are equal to zero.

Further, differentiating $\varphi_s \circ \bar{\pi} = 0$ with respect to y_i and taking into account that

$$\partial_j \varphi_s(0) = \delta_{j,m+s}$$

we obtain the equality

$$\partial_i \bar{\pi}^{m+s}(0, z) = 0, \text{ for all } 1 \le i \le n \text{ and } 1 \le s \le k.$$

This means that the left lower block is also equal to zero.

It remains to prove the formula for the remaining block, which we denote by X. Since

$$y - \bar{\pi}(y) \in N_{\bar{\pi}(y)}M = \langle (\nabla\varphi_s \circ \bar{\pi})(y) : 1 \le s \le k \rangle$$

we have

$$y = \bar{\pi}(y) + \alpha^s(y)(\nabla\varphi_s \circ \bar{\pi})(y),$$

where α^s are some smooth functions with $\alpha^s(0) = z^s$. Differentiating with respect to y, we obtain

$$I_{n\times n} = D\bar{\pi} + (\nabla\varphi_s \circ \bar{\pi})D\alpha^s + \alpha^s \mathrm{Hess}\varphi_s D\bar{\pi}.$$

Evaluating at the point $(0, z)$, we get

$$\begin{bmatrix} \mathrm{I} & 0 \\ 0 & \mathrm{I} \end{bmatrix} = \begin{bmatrix} X & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \mathrm{I} \end{bmatrix} [D\alpha^s(0)] - z^s \begin{bmatrix} F_s & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} X & 0 \\ 0 & 0 \end{bmatrix},$$

and considering the left upper block, we obtain $X = [I - z^s F_s]^{-1}$.

It remains to notice that the expression (2.1) can be easily derived from (2.2) by the usual change of coordinates rule. □

2.1.3 Derivative of the projector P

In this section we compute the first derivative operator of the projector $P : M \to gl(n)$.

Lemma 2. *Let $a \in M$ and $u \in o_a(n)$. Then the first derivative operator $dP_a : T_aM \to gl(n)$ is given by the relations*

$$dP_a(ue_i) = \begin{bmatrix} 0 & \partial_i Df^T \\ \partial_i Df & 0 \end{bmatrix}(0), \ 1 \le i \le m,$$

where f is from the local representation corresponding to the pair (a, u).

Proof. Consider the first m coordinates of the coordinate system (y^i) corresponding to the pair (a,u) as local coordinates in M. Notice that

$$P\begin{bmatrix} \mathrm{I} & -Df^T \\ Df & \mathrm{I} \end{bmatrix} = \begin{bmatrix} \mathrm{I} & 0 \\ Df & 0 \end{bmatrix}$$

since the first m columns of the matrix on the left hand side generate the tangent space $T_{(y,f(y))}M$ and the last k columns generate the normal space $N_{(y,f(y))}M$. Differentiating with respect to y_i, we obtain

$$\partial_i P\begin{bmatrix} \mathrm{I} & -Df^T \\ Df & \mathrm{I} \end{bmatrix} + P\begin{bmatrix} 0 & -\partial_i Df^T \\ \partial_i Df & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ \partial_i Df & 0 \end{bmatrix}.$$

Evaluating at zero and using $Df(0)=0$, we get

$$\begin{aligned} dP_a(ue_i) = \partial_i P_a &= \begin{bmatrix} 0 & 0 \\ \partial_i Df & 0 \end{bmatrix}_0 - \begin{bmatrix} \mathrm{I} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & -\partial_i Df^T \\ \partial_i Df & 0 \end{bmatrix}_0 \\ &= \begin{bmatrix} 0 & \partial_i Df^T \\ \partial_i Df & 0 \end{bmatrix}_0 \end{aligned}$$

which proves the lemma. □

2.1.4 Second fundamental form in special coordinates

In this section we find a relation between the second fundamental form of the manifold M and the local representation of M corresponding to a given pair (a,u), where $a \in \mathbb{M}_{\varepsilon_0}$ and $u \in o_a(n)$.

First, let $a \in M$. The following definition is due to [11].

Definition 5. *The second fundamental form of M at the point a in the direction $\nu \in N_aM$ is the map $l_\nu : T_aM \times T_aM \to \mathbb{R}$ defined by*

$$l_\nu(v,w) = \langle \nabla_v^{\mathbb{R}^n} \nu, w\rangle.$$

In order to check that the right hand side of the formula is well-defined, one has to extend v and ν locally to a neighborhood of a in $\mathbb{R}^n$ and to show that the result is independent of the chosen extension. This is proved in [11].

Lemma 3. *Let $a \in \mathbb{M}_{\varepsilon_0}$ and $u \in o_a(n)$. Then*

$$(F_s)_{ij} = -l_{\partial_{m+s}}(\partial_i, \partial_j)|_{\pi(a)},$$

where $\partial_l = ue_l$, $1 \le l \le n$, and (F_s) is from the local representation corresponding to the pair (a,u).

Proof. Let

$$v = v^p\partial_p, w = w^q\partial_q, \text{ and } \nu = \nu^r\partial_r$$

be vector fields defined on a neighborhood of $\pi(a)$ in $\mathbb{R}^n$ such that

1) $v(x), w(x) \in T_xM$ and $\nu(x) \in N_xM$ locally for all $x \in M$;

2) $v(\pi(a)) = \partial_i$, $w(\pi(a)) = \partial_j$, and $\nu(\pi(a)) = \partial_{m+s}$.

Notice that we can take $\nu = \nabla\varphi_s$. By the definition of the second fundamental form we obtain

$$\begin{aligned} l_\nu(v,w)|_{\pi(a)} &= \langle \nabla_v^{\mathbb{R}^n}\nu, w\rangle|_{\pi(a)} \\ &= \langle \nabla_{v^p\partial_p}^{\mathbb{R}^n}\nu^r\partial_r, w^q\partial_q\rangle|_{\pi(a)} \\ &= v^p\partial_p\nu^r w^q\langle\partial_r,\partial_q\rangle|_{\pi(a)} \\ &= \partial_i\nu^j|_{\pi(a)} \\ &= \partial_{ij}\varphi_s|_0 \\ &= -(F_s)_{ij}, \end{aligned}$$

which completes the proof. □

2.1.5 Scalar and mean curvature and tension field in special coordinates

In this section we compute the scalar curvature R, the mean curvature vector κ, and the norm of the tension field σ of the manifold M at a point $a \in M$ in terms of the local representation of M corresponding to a given pair (a,u), where $u \in o_a(n)$.

Lemma 4. *The scalar curvature $R(a)$ of the manifold M at the point a is given by*

$$R(a) = \sum_{s=1}^{k}\left[(\mathrm{tr}F_s)^2 - \mathrm{tr}(F_s)^2\right],$$

where the matrices (F_s) correspond to some arbitrary $u \in o_a(n)$.

Proof. We consider the first m coordinates of the coordinate system (y^i) corresponding to (a,u) as local coordinates in M in a neighborhood of a. Then $g_{ij}(0)=\delta_{ij}$ and by the definition of the scalar curvature $R(a)=R_{ijij}(0)$. Now by the Gauss equation (see [11]) and by Lemma 3 we obtain

$$\begin{aligned} R(a) &= \langle R(\partial_i,\partial_j)\partial_j,\partial_i\rangle|_a \\ &= \sum_{s=1}^{k}[l_{\partial_{m+s}}(\partial_j,\partial_j)l_{\partial_{m+s}}(\partial_i,\partial_i)-l_{\partial_{m+s}}(\partial_i,\partial_j)l_{\partial_{m+s}}(\partial_i,\partial_j)]_a \\ &= \sum_{s=1}^{k}[(F_s)_{jj}(F_s)_{ii}-(F_s)_{ij}(F_s)_{ij}] = \sum_{s=1}^{k}[(\mathrm{tr}F_s)^2-\mathrm{tr}(F_s)^2], \end{aligned}$$

and the formula is proved. □

Lemma 5. *Denote by $\bar\kappa$ the coordinates of the mean curvature vector with respect to the special coordinate system corresponding to (a,u), where $u\in o_a(n)$ is arbitrary. Then*

$$\bar\kappa(a)=(0,\ldots,0,-\frac{1}{m}\mathrm{tr}F_1,\ldots,-\frac{1}{m}\mathrm{tr}F_k).$$

Further, the norm of the tension field σ of M at the point $a\in M$ is given by

$$||\sigma(a)||^2=\sum_{s=1}^{k}(\mathrm{tr}F_s)^2, \qquad (2.3)$$

where the matrices (F_s) are from the local representation of M corresponding to (a,u).

Proof. The first m coordinates of $\bar\kappa(a)$ are equal to zero by the choice of special coordinates since the mean curvature vector belongs to the normal space.

Further, by the definition of κ (see [11]) and using Lemma 3, we have

$$\bar\kappa_{m+s}(a)=\frac{1}{m}\sum_{i=1}^{m}l_{\partial_{m+s}}(\partial_i,\partial_i)=-\frac{1}{m}\mathrm{tr}F_s.$$

Finally, by [8] we have

$$||\sigma||=m||\kappa||,$$

which implies (2.3). □

2.2 Shifting vector field

2.2.1 Two measures on the tubular neighborhood

In this section we introduce two natural measures $\lambda_{\mathbb{R}^n}$ and $\lambda_\oplus$ on the tubular neighbourhood $\mathbb{M}_{\varepsilon_0}$. The first one is just the usual Lebesgue measure induced by the embedding $\mathbb{M}_{\varepsilon_0}\subset\mathbb{R}^n$. The second one is defined by

$$\lambda_\oplus(A)=\int_{\pi(A)}\lambda_{\mathbb{R}^k}(A_x)d\lambda_M(x), \quad A\subset\mathbb{M}_{\varepsilon_0} \text{ Borel},$$

where $A_x=\pi^{-1}(x)$ and $\lambda_{\mathbb{R}^k}$ and λ_M are the Lebesgue measures on $\mathbb{R}^k$ and M, respectively. We have used here the fact that $A_x\subset N_xM$ and that there is a linear isometry between N_xM and $\mathbb{R}^k$. Moreover, the Lebesgue measure $\lambda_{\mathbb{R}^k}$ is independent of the choice of such an isometry and hence $\lambda_\oplus$ is well-defined.

Definition 6. *$\lambda_\oplus$ is called reference measure.*

Lemma 6. *$\lambda_\oplus$ is equivalent to $\lambda_{\mathbb{R}^n}$, and the density is given by*

$$\frac{d\lambda_\oplus}{d\lambda_{\mathbb{R}^n}}(a)=\det[\mathrm{I}-z^sF_s]^{-1},$$

where z and (F_s) are from the local representation corresponding to (a,u) with an arbitrary $u\in o_n(a)$.

Proof. Let $a\in\mathbb{M}_{\varepsilon_0}$ and $u\in o_a(n)$ be fixed and let f, (F_s), and z be from the local representation corresponding to (a,u).

Let $V\subset M$ be a neighborhood of $\pi(a)$ and $U\subset\mathbb{R}^m$ be a neighborhood of zero such that the mapping $\psi_0:U\to V$ given by

$$\psi_0(x)=(x,f(x))$$

is a bijection. Let $\nu_s : U \to \mathbb{R}^n$ be smooth functions such that $(\nu_s(x))$ is an orthonormal basis of $T_{\psi_0(x)}M$ and $\nu_s(0)$ is the $(m+s)$-th basis vector of the coordinate system corresponding to u. Consider now the mapping $\psi : U \times B(\varepsilon_0) \to \pi^{-1}(V)$ given by

$$\psi(x,z) = \psi_0(x) + z^s \nu_s(x).$$

We have

$$D\psi(0,z) = \begin{pmatrix} \mathrm{I} - z^s F_s & 0 \\ * & \mathrm{I} \end{pmatrix}, \tag{2.4}$$

where the star denotes some $k \times m$ matrix. In fact,

$$\partial_{z^s}\psi(0,z) = \nu_s(0)$$

and therefore two right blocks of $D\psi(0,z)$ are 0 and I, respectively. In order to compute the left upper block notice that

$$\nu_s = \alpha_s^p \eta_p,$$

where $\alpha_s^p : U \to \mathbb{R}$ are some smooth functions such that $\alpha_s^p(0) = \delta_s^p$ and

$$\eta_p(x) = (\nabla\varphi_p) \circ \psi_0.$$

Then, for $i \le m$, we have using $\eta_p^i(0) = 0$

$$\begin{aligned}\partial_{x^j}\psi_i(0,z) &= \delta_{ij} + z^s \partial_{x^j}\alpha_s^p \eta_p^i|_0 + z^s \alpha_s^p \partial_{x^j}\eta_p^i|_0 \\ &= \delta_{ij} + z^s \delta_s^p \partial_{ij}\varphi_p(0) = \delta_{ij} - z^s (F_s)_{ij}.\end{aligned}$$

It remains to notice that by definition $\lambda_\oplus$ is locally the image measure of $\lambda_M \otimes \lambda_{\mathbb{R}^k}$ under the mapping ψ. Therefore

$$\frac{d\lambda_\oplus}{d\lambda_{\mathbb{R}^n}}(a) = \det[D\psi(0,z)]^{-1} = \det[\mathrm{I} - z^s F_s]^{-1},$$

and the statement is proved. □

2.2.2 Shifting vector field: definition and properties

Let us define a vector field v on $\mathbb{M}_{\varepsilon_0}$ by

$$v(a) = -\,Q_{\pi(a)}\left[\nabla \log \frac{d\lambda_\oplus}{d\lambda_{\mathbb{R}^n}}\right],$$

Definition 7. *The vector field v is called the shifting vector field.*

Lemma 7. *Let $a \in \mathbb{M}_{\varepsilon_0}$ and $u \in o_a(n)$. Denote the coordinates of v with respect to the coordinate system (y^i) corresponding to (a,u) by $\bar{v}$. Then*

$$\bar{v}(a) = \left(0,\ldots,0,-\mathrm{tr}(F_1[I - z^p F_p]^{-1}),\ldots,-\mathrm{tr}(F_k[I - z^p F_p]^{-1})\right),$$

where (F_s) and z are from the local representation corresponding to the pair (a,u).

Proof. The first m coordinates of $\bar{v}(a)$ are equal to zero by the definition of v. Further, by Lemma 6 we have

$$\begin{aligned} v(a) &= -\,Q_{\pi(a)}\left[\nabla \log \frac{d\lambda_\oplus}{d\lambda_{\mathbb{R}^n}}\right] \\ &= -\,Q_{\pi(a)}\left(\nabla \log \det[I - z^s(a,u)F_s(a,u)]^{-1}\right) \\ &= \quad Q_{\pi(a)}\left(\mathrm{tr}\nabla \log[I - z^s(a,u)F_s(a,u)]\right)\end{aligned}$$

and therefore

$$\bar{v}^{m+s}(a) = \mathrm{tr}\ \partial_{z^s}\log[I - z^p F_p] = -\mathrm{tr}(F_s[I - z^p F_p]^{-1}),$$

for all $1 \le s \le k$. □

Lemma 8. *For $a \in M$,*

$$\mathrm{div}\ v(a) = R(a),$$

where $R(a)$ is the scalar curvature of M at the point a.

Proof. Let $u \in o_a(n)$. Since div$v(a)$ is independent of the choice of orthogonal coordinates it suffices to compute div$\bar{v}(0)$ in the coordinates (y^i) corresponding to the pair (a,u). Using the fact that $\bar{v}(y) \in N_{\pi(y)}M$ for all y, we have

$$\bar{v}(y) = \alpha^s(y)\eta_s(x(y)),$$

where α^s are some smooth functions with

$$\alpha^s(0) = -\mathrm{tr}F_s, \text{ for all } 1 \leq \leq k$$

since we have by Lemma 7)

$$\bar{v}^{m+s}(0) = -\mathrm{tr}F_s,$$

and the functions (x^i) are defined as

$$x^i(y) = (\psi^{-1})^i(y), \text{ for all } 1 \leq i \leq m.$$

It follows from (2.4) that

$$(D\psi)^{-1}(0,0) = \begin{pmatrix} \mathrm{I} & 0 \\ * & \mathrm{I} \end{pmatrix}$$

and therefore $\partial_{y^i}x^j(0) = \delta_i^j$. Then using $\partial_i f_s(0) = 0$ and the definition of η_i we obtain

$$\sum_{i=1}^m \partial_i \bar{v}^i(0) = -\sum_{i,s,j} \alpha^s \partial_{ij} f_s \partial_{y^i} x^j|_0$$
$$= \sum_{i,s} \mathrm{tr}F_s (F_s)_{ii} = \sum_{s=1}^k (\mathrm{tr}F_s)^2,$$
$$\sum_{s=1}^k \partial_{m+s}\bar{v}^{m+s}(0) = -\sum_{s=1}^k \partial_{z^s}\mathrm{tr}(F_s[I - z^p F_p]^{-1})(0)$$
$$= -\sum_{s=1}^k \mathrm{tr}(F_s)^2.$$

Finally, by Lemma 4 we get

$$\begin{aligned} \mathrm{div}\ v(a) &= \sum_{i=1}^m \partial_i \bar{v}^i(0) + \sum_{s=1}^k \partial_{m+s}\bar{v}^{m+s}(0) \\ &= \sum_{s=1}^k (\mathrm{tr}F_s)^2 - \sum_{s=1}^k \mathrm{tr}(F_s)^2 = R(a), \end{aligned}$$

which proves the assertion. □

2.3 Fermi decomposition of a continuous semimartingale

2.3.1 Stochastic parallel translation

The notion of the parallel translation of a vector from $\mathbb{R}^n$ along a smooth curve in M is studied in classical differential geometry. In this section we define stochastic parallel translation along a continuous M-valued semimartingale, whose paths are not necessarily smooth. The definition is taken from [7].

For $a \in M$ and $w \in T_aM$, we define the connection Γ by

$$\Gamma_a(w) = dQ_a(w)P_a + dP_a(w)Q_a \in gl(n). \tag{2.5}$$

Further, let (x_t) be a continuous M-valued semimartingale starting at a_0. Recall that $a_0 \in M$ and the orthonormal basis (e_i) were fixed in such a way that $I \in o_{a_0}(n)$, i.e.,

$$T_{a_0}M = \langle e_i : 1 \leq i \leq m\rangle \text{ and } N_{a_0}M = \langle e_i : m+1 \leq i \leq n\rangle.$$

Definition 8. *Given $\nu \in \mathbb{R}^n$, let $\nu_t = u_t\nu$, where (u_t) solves the Stratonovich stochastic differential equation*

$$\delta u_t + \Gamma_{x_t}(\delta x_t)u_t = 0 \quad \textit{with} \quad u_0 = I \in gl(n). \tag{2.6}$$

Then the $\mathbb{R}^n$-valued process (ν_t) is called stochastic parallel translation of ν along (x_t) and the $gl(n)$-valued process (u_t) is called the translation matrix.

Lemma 9. *1)* $u_t \in o_{x_t}(n)$ *for all* t.

2) The process (u_t^T) *solves the stochastic differential equation*

$$\delta u_t^T = u_t^T \Gamma_{x_t}(\delta x_t) \quad \textit{with} \quad u_0^T = I \in gl(n). \tag{2.7}$$

Proof. 1) By Theorem 3.18 from [7], the process (u_t) is orthogonal for all t and satisfies

$$P_{x_t} u_t = u_t P_{x_0}.$$

Hence, using $e_i \in T_{a_0}M$ for all $1 \le i \le m$, we obtain

$$P_{x_t} u_t e_i = u_t P_{x_0} e_i = u_t P_{a_0} e_i = u_t e_i,$$

and therefore $u_t e_i \in T_{x_t}M$ for all $1 \le i \le m$ and for all t. This means that $u_t \in o_{x_t}(n)$ for all t.

2) Since

$$\Gamma = dQP + dPQ$$

and P and Q are orthogonal projections and hence symmetric, the adjoint Γ^T of Γ is given by

$$\Gamma^T = PdQ + QdP.$$

Thus $\Gamma^T = -\Gamma$ because

$$PdQ = -dPQ \quad \text{and } QdP = -dQP,$$

which follow from the equality $PQ = 0$. Now we obtain from (2.6) that

$$\delta u_t^T = -[\Gamma_{x_t}(\delta x_t) u_t]^T = u_t^T(-\Gamma_{x_t}^T(\delta x_t)) = u_t^T \Gamma_{x_t}(\delta x_t),$$

which completes the proof. □

2.3.2 Fermi decomposition of an $\mathbb{M}_{\varepsilon_0}$-valued continuous semimartingale

Let (y_t) be an $\mathbb{M}_{\varepsilon_0}$-valued continuous semimartingale starting at the point a_0. In this section we construct a decomposition of such a process

to a pair of two processes (x_t) (with values in M) and (z_t) (with values in the ε_0-ball $B(\varepsilon_0) \subset \mathbb{R}^k$ around zero) in such a way that the first one is just the projection of the original process (y_t) to the manifold

$$x_t = \pi(y_t)$$

and the second one describes the orthogonal component $(y_t - x_t)$ of the process (y_t).

Notice that (x_t) is a M-valued continuous semimartingale starting at a_0 and therefore the stochastic parallel translation along (x_t) is well-defined. Denote the parallel translation of the basis vector e_i by $u_t e_i$. By Lemma 9 the system $(u_t e_i)$ is an orthonormal basis such that its last k basis vectors form an orthonormal basis of the normal space $N_{x_t}M$. Now the process (z_t) is defined as the coordinates of the vector $y_t - x_t \in N_{x_t}M$ with respect to the basis $(u_t e_i : m+1 \le i \le n)$. More precisely,

$$z_t = \mathrm{pr}_2 u_t^T (y_t - x_t).$$

It is easy to see that the process (z_t) is a $B(\varepsilon_0)$-valued continuous semimartingale starting at zero.

Definition 9. *The pair of the processes* (x_t) *and* (z_t) *is called Fermi decomposition of the process* (y_t).

Denote by $S_a(A)$ the set of all A-valued continuous semimartingales starting at $a \in A$.

Proposition 1. *The Fermi decomposition gives a one-to-one correspondence between the sets* $S_{a_0}(\mathbb{M}_{\varepsilon_0})$ *and* $S_{a_0}(M) \times S_0(B(\varepsilon_0))$.

Proof. It was already mentioned that for $(y_t) \in S_{a_0}(\mathbb{M}_{\varepsilon_0})$ holds

$$((x_t), (z_t)) \in S_{a_0}(M) \times S_0(B(\varepsilon_0)).$$

On the other hand, it follows from the definition of the Fermi decomposition that the process (y_t) can be uniquely reconstructed from the processes $(x_t) \in S_{a_0}(M)$ and $(z_t) \in S_0(B(\varepsilon_0))$ by the formula

$$y_t = x_t + u_t \mathrm{pr}_2^{-1}(z_t),$$

and such a process is obviously an $\mathbb{M}_{\varepsilon_0}$-valued continuous semimartingale starting at the point a_0. □

Finally, notice that for all t the z-component of the local representation of the manifold M corresponding to the pair (y_t, u_t) is just z_t, for all t.

Chapter 3

Surface measures

3.1 Shifted process (y_t)

3.1.1 Definition and properties

In this subsection we construct a stochastic process that is obtained from the standard n-dimensional Brownian motion by shifting in the direction of the vector field v defined in Section 2.2 and study its Fermi decomposition.

First, let us extend the vector field v from $\mathbb{M}_{\varepsilon_0}$ to $\mathbb{R}^n$ in such a way that the extension is smooth and has compact support (the choice of the extension is not essential for further considerations). We denote such an extension also by v. Then there exists a unique weak solution (y_t) of the stochastic differential equation

$$\begin{cases} dy_t = db_t - \frac{1}{2}v(y_t)dt, \\ y_0 = a_0, \end{cases} \tag{3.1}$$

where (b_t) is a standard n-dimensional Brownian motion starting at the point a_0.

Definition 10. *The process (y_t) is called the shifted process.*

Let τ be the exit time of the process (y_t) from the tubular neighborhood $\mathbb{M}_{\varepsilon_0}$ and consider the stopped shifted process $(y_{t\wedge\tau})$, which

is an $\mathbb{M}_{\varepsilon_0}$-valued continuous semimartingale. Denote by $((x_t),(z_t))$ its Fermi decomposition and by (u_t) the translation matrix corresponding to the semimartingale (x_t).

Furthermore, consider the process

$$\bar{b}_t = \int_0^t u_s^T db_s. \tag{3.2}$$

It is an n-dimensional Brownian motion by the Lévy's characterization theorem (see, for example, [20]): it is a continuous local martingale such that

$$(d\bar{b}_t)(d\bar{b}_t)^T = I dt$$

since the translation matrices u_s are orthogonal for all s. Finally, denote by $(\bar{b}'_t)$ (respectively, by $(\bar{b}''_t)$) the first m (respectively, the last k) components of $(\bar{b}_t)$.

Now we define linear operators $I_t : \mathbb{R}^n \to \mathbb{R}^m$ and $J_t : \mathbb{R}^n \to \mathbb{R}^k$ by

$$I_t = \mathrm{pr}_1 u_t^T \quad \text{and} \quad J_t = \mathrm{pr}_2 u_t^T.$$

In the following two lemmas we show that the process (x_t) is a solution of a stochastic differential equation driven by the m-dimensional Brownian motion $(\bar{b}'_t)$ and the process (z_t) coincides with the Brownian motion $(\bar{b}''_t)$, which is independent of $(\bar{b}'_t)$.

Lemma 10. *The Itô differential of the process (x_t) considered in $\mathbb{R}^n$ up to time τ is given by the formula*

$$dx_t = u_t\ \mathrm{pr}_1^{-1}[I - z_t^s F_s(x_t,u_t)]^{-1} d\bar{b}'_t + \frac{1}{2}\Delta\pi(y_t)dt.$$

Proof. Using Itô's formula, Lemma 1, and the equation (3.1) we obtain up to time τ

$$\begin{aligned} dx_t &= d\pi(y_t) \\ &= D\pi(y_t)dy_t + \frac{1}{2}DD\pi(y_t)dy_t dy_t \\ &= u_t \mathrm{pr}_1^{-1}[I - z_t^s F_s(x_t,u_t)]^{-1}\mathrm{pr}_1 u_t^T(db_t - \frac{1}{2}v(y_t)dt) + \frac{1}{2}\Delta\pi(y_t)dt \\ &= u_t \mathrm{pr}_1^{-1}[I - z_t^s F_s(x_t,u_t)]^{-1} d\bar{b}'_t + \frac{1}{2}\Delta\pi(y_t)dt \end{aligned}$$

since the observation that $v(y_t) \in N_{x_t}M$ for all t implies

$$\mathrm{pr}_1 u_t^T v(y_t) = 0,$$

and also it follow from (3.2) that

$$\mathrm{pr}_1 u_t^T db_t = d\bar{b}_t'.$$

The lemma is proved. □

Lemma 11. *$z_t = \tilde{b}_t''$ up to time τ.*

Proof. Let us show that the Stratonovich differentials of the processes (z_t) and $(\bar{b}_t'')$ coincide. Recall that for two continuous semimartingales a and b holds

$$a\delta b = adb + \frac{1}{2}\delta a \delta b. \tag{3.3}$$

Then, using the equalities $dQP = -QdP$ and $dPQ = -PdQ$, the definition of Γ, the relation (3.3), the equation of stochastic parallel translation (2.7), and, finally, Lemma 9, which implies $J_t P_{x_t} = 0$ and $J_t Q_{x_t} = J_t$, we obtain

$$\begin{aligned}
\delta\bar{b}_t'' &= d\bar{b}_t'' \\
&= \mathrm{pr}_2 u_t^T db_t \\
&= J_t \delta b_t - \frac{1}{2}\mathrm{pr}_2 \delta u_t^T \delta b_t \\
&= J_t \delta b_t - \frac{1}{2}\mathrm{pr}_2 u_t^T \Gamma_{x_t}(\delta x_t)\delta b_t \\
&= J_t \delta b_t - \frac{1}{2} J_t dQ_{x_t}(\delta x_t) P_{x_t} \delta b_t - \frac{1}{2} J_t dP_{x_t}(\delta x_t) Q_{x_t}\delta b_t \\
&= J_t \delta b_t + \frac{1}{2} J_t Q_{x_t} dP_{x_t}(\delta x_t)\delta b_t + \frac{1}{2} J_t P_{x_t} dQ_{x_t}(\delta x_t)\delta b_t \\
&= J_t \delta b_t + \frac{1}{2} J_t dP_{x_t}(\delta x_t)\delta b_t.
\end{aligned}$$

Analogously, using additionally the relation

$$y_t - \pi(y_t) \in N_{x_t}M$$

and Lemma 1, which implies

$$\mathrm{Im}[D\pi(y)] \subset T_{\pi(y)}M$$

and hence

$$J_t D\pi(y_t) = 0,$$

we compute

$$\begin{aligned}
dz_t &= \delta(\mathrm{pr}_2 u_t^T (y_t - \pi(y_t))) \\
&= \mathrm{pr}_2 \delta u_t^T (y_t - \pi(y_t)) + \mathrm{pr}_2 u_t^T \delta(y_t - \pi(y_t)) \\
&= \mathrm{pr}_2 u_t^T \Gamma_{x_t}(\delta x_t)(y_t - \pi(y_t)) + \mathrm{pr}_2 u_t^T \delta y_t - J_t D\pi(y_t)\delta y_t \\
&= J_t (dQ_{x_t}(\delta x_t) P_{x_t} + dP_{x_t}(\delta x_t) Q_{x_t})(y_t - \pi(y_t)) + J_t \delta y_t \\
&= J_t \delta y_t - J_t P_{x_t} dQ_{x_t}(\delta x_t)(y_t - \pi(y_t)) \\
&= J_t \delta b_t - \frac{1}{2} J_t v(y_t) dt.
\end{aligned}$$

It remains to show that

$$J_t dP_{x_t}(\delta x_t)\delta b_t = -J_t v(y_t) dt$$

or, equivalently, that the last k coordinates of the vectors $dP_{x_t}(\delta x_t)\delta b_t$ and $-v(y_t)dt$ with respect to the coordinate system $(u_t e_i)$ coincide (coordinates of vectors computed with respect to the coordinate system (ue_i) are marked by the bar). We compute them using Itô's formula, Lemma 1, Lemma 2, and Lemma 7.

$$\begin{aligned}
&(\overline{dP_{x_t}(\delta x_t)\delta b_t})^{m+s} \\
&= (\partial_i P_{x_t})_{m+s,k} (\mathrm{pr}_1 u_t^T \delta x_t)^i (u_t^T \delta b_t)^k \\
&= \partial_{ik} f_s(0) (\mathrm{pr}_1 u_t^T D\pi(y_t)\delta y_t)^i (u_t^T \delta b_t)^k \\
&= \partial_{ik} f_s(0) (\mathrm{pr}_1 u_t^T u_t \mathrm{pr}_1^{-1} [I - z_t^p F_p]^{-1} \mathrm{pr}_1 u_t^T \delta b_t)^i (u_t^T \delta b_t)^k \\
&= \partial_{ik} f_s(0) ([I - z_t^p F_p]^{-1} \mathrm{pr}_1)_{ir} (u_t^T)_{rq} (\delta b)^q (u_t^T)_{kl} (\delta b)^l \\
&= \partial_{ik} f_s(0) [I - z_t^p F_p]^{-1}_{ik} dt \\
&= \mathrm{tr}(F_s [I - z_t^p F_p]^{-1}) dt \\
&= -\bar{v}^{m+s}(y_t) dt,
\end{aligned}$$

where (f_s) and (F_s) are from the local representation of M corresponding to the pair (y_t, u_t). This implies $\delta z_t = \delta\bar{b}_t''$ and $z_t = \bar{b}_t''$ up to time τ since this stochastic differential equation is exact. □

3.1.2 Surface measure of the shifted process

Recall that in Introduction we have defined the surface measure of the first type corresponding to a Brownian motion. Actually, we can analogously define the surface measure of the first type corresponding to any diffusion process starting at a_0, in particular, corresponding to the shifted process (y_t).

More precisely, let μ be the distribution of (y_t). Denote by μ_ε the normalized restriction of μ to the set $C(\mathbb{M}_\varepsilon)$

$$\mu_\varepsilon = \frac{\mu|_{C(\mathbb{M}_\varepsilon)}}{\mu(C(\mathbb{M}_\varepsilon))}.$$

Then the weak limit of the family (μ_ε) as $\varepsilon \to 0$ is called the surface measure of the first type corresponding to the process (y_t).

Proposition 2. *$\mu_\varepsilon \to \mathbb{W}_M$ weakly, i.e., the surface measure (of the first type) corresponding to the shifted process (y_t) exists and is just the Wiener measure on the manifold M.*

Proof. We need to prove that the conditional law of the shifted process (y_t), given that (y_t) does not leave $\mathbb{M}_\varepsilon$ before time 1 converges to $\mathbb{W}_M$ as ε tends to zero.

First notice that for all $\varepsilon < \varepsilon_0$ we have

$$\begin{aligned}\mathcal{L}((y_t)|y_s \in \mathbb{M}_\varepsilon\ \forall s) &= \mathcal{L}((y_{t\wedge\tau})|y_{s\wedge\tau} \in \mathbb{M}_\varepsilon\ \forall s)\\ &= \mathcal{L}((y_{t\wedge\tau})|z_s \in B(\varepsilon)\ \forall s),\end{aligned}$$

where τ is the exit time of (y_t) from $\mathbb{M}_{\varepsilon_0}$ and $((x_t),(z_t))$ is the Fermi decomposition of the stopped process $(y_{t\wedge\tau})$.

Further, on the event $\{z_s \in B(\varepsilon)\ \forall s\}$ we have $|y_{t\wedge\tau} - x_t| \le \varepsilon$ for all t and hence

$$\lim_{\varepsilon\to 0} \mathcal{L}((y_{t\wedge\tau})|z_s \in B(\varepsilon)\ \forall s) = \lim_{\varepsilon\to 0} \mathcal{L}((x_t)|z_s \in B(\varepsilon)\ \forall s)$$

if one of the two (weak) limits exists.

By Lemma 10 and Lemma 11 up to time τ the process (z_t) is just a k-dimensional Brownian motion independent of the m-dimensional Brownian motion driving the process (x_t).

Consider now any probability space on which there is an n-dimensional Brownian motion (b_t^*) and, moreover, there is a family (z_t^ε) of processes such that each (z_t^ε) has the same law as (z_t) under μ_ε and the whole family $\{(z_t^\varepsilon) : \varepsilon < \varepsilon_0\}$ is independent of (b_t^*). On this probability space we consider the filtration $(\mathcal{F}_t)$ generated by $\{b_s^* : s \le t\}$ and all the processes (z_t^ε), $0 \le t \le 1$. Then (b_t^*) is an n-dimensional $(\mathcal{F}_t)$-Brownian motion, $(\mathrm{pr}_1 b_t^*)$ is an m-dimensional $(\mathcal{F}_t)$-Brownian motion and the coefficients in the system of stochastic differential equations

$$\begin{cases}\delta u_t^\varepsilon + \Gamma_{x_t^\varepsilon}(\delta x_t^\varepsilon)u_t^\varepsilon = 0\\ dx_t^\varepsilon = u_t^\varepsilon \mathrm{pr}_1^{-1}[I - z_t^{\varepsilon s}F_s(x_t^\varepsilon, u_t^\varepsilon)]^{-1}\mathrm{pr}_1 db_t^* + \frac{1}{2}\Delta\pi(x_t^\varepsilon + u_t^\varepsilon \mathrm{pr}_2^{-1} z_t^\varepsilon)dt\\ u_0^\varepsilon = I\\ x_0^\varepsilon = a_0\end{cases}$$

are adapted. It follows from $||z_t^\varepsilon|| \le \varepsilon$ that the coefficients are also bounded and hence there is a unique solution $(u_t^\varepsilon, x_t^\varepsilon)$ of this system for each ε. Further, it follows from Lemma 10 and Lemma 11 that the law of (x_t^ε) is the same as the law of (x_t) under μ_ε. Moreover, on this probability space the processes (z_t^ε) converge uniformly to zero. It follows now from Lemma 12 below that $(u_t^\varepsilon, x_t^\varepsilon)$ converges locally uniformly in probability to the solution $(\bar{u}_t, \bar{x}_t)$ of the system of stochastic differential equation

$$\begin{cases}\delta \bar{u}_t + \Gamma_{\bar{x}_t}(\delta \bar{x}_t)\bar{u}_t = 0,\\ d\bar{x}_t = \bar{u}_t P_{a_0} db_t^* + \frac{1}{2}\Delta\pi(\bar{x}_t)dt,\\ \bar{u}_0 = I,\\ \bar{x}_0 = a_0.\end{cases}$$

It remains to show that the process $(\bar{x}_t)$ is a Brownian motion on the manifold M. Since $\bar{u}_t \in o_{\bar{x}_t}(n)$ by Lemma 9, we obtain

$$\bar{u}_t P_{a_0} db_t^* = P_{\bar{x}_t}\bar{u}_t db_t^* = P_{\bar{x}_t} db_t^{**},$$

where (b_t^{**}) is another n-dimensional Brownian motion starting in a_0. Further, notice that $P_x = D\pi(x)$ for $x \in M$ by Lemma 1, and the Itô differential equation for the process $(\bar{x}_t)$ now looks like

$$\begin{cases}d\bar{x}_t = D\pi(\bar{x}_t)db_t^{**} + \frac{1}{2}\Delta\pi(\bar{x}_t)dt,\\ \bar{x}_0 = a_0.\end{cases}$$

Due to [21, Th.30.14] the drift c of the Stratonovich stochastic differential equation at the point $x \in M$ can be computed in local coordinates corresponding to x, and using

$$\begin{aligned} \partial_q \pi^j(x) &= \delta_q^j, \quad \text{for } j, q \le m, \\ \partial_q \pi^j(x) &= 0, \quad \text{otherwise } j, q \end{aligned}$$

we obtain

$$\begin{aligned} 2c^i &= \Delta\pi^i(x) - \sum_{q=1}^{n} \partial_q \pi^j \partial_j (\partial_q \pi^i)(x) \\ &= \Delta\pi^i(x) - \sum_{j=1}^{m} \partial_{jj}\pi^i(x,0) = \sum_{j=m+1}^{n} \partial_{jj}\pi^i(x) = 0 \end{aligned}$$

as π is constant in the normal directions. Now the Stratonovich stochastic differential equation for the process $(\bar{x}_t)$ looks like

$$\begin{cases} \delta\bar{x}_t = P_{\bar{x}_t}\delta b_t^{**}, \\ \bar{x}_0 = a_0. \end{cases}$$

Hence $\bar{x}_t$ is a Brownian motion on M. □

Lemma 12. $(x_t^\varepsilon, u_t^\varepsilon) \to (\bar{x}_t, \bar{u}_t)$ *locally uniformly in probability.*

Proof. Denote the processes $(x_t^\varepsilon, u_t^\varepsilon)$ and $(\bar{x}_t, \bar{u}_t)$ by (a_t^ε) and $(\bar{a}_t)$, respectively. Then the processes (a_t^ε) and $(\bar{a}_t)$ satisfy the stochastic differential equations

$$\begin{aligned} da_t^\varepsilon &= f_1(a_t^\varepsilon, z_t^\varepsilon)db_t^* + f_2(a_t^\varepsilon, z_t^\varepsilon)dt, \\ d\bar{a}_t &= f_1(\bar{a}_t, 0)db_t^* + f_2(\bar{a}_t, 0)dt \end{aligned}$$

respectively, with the same initial conditions, where f_i are short notations for the coefficients. It can be easily seen that

$$f_i(x,z) \to f_i(x,0) \text{ as } z \to 0$$

uniformly in x and the functions $f_i(x,0)$ are Lipschitz.

Now, let

$$\varphi_\varepsilon(t) = \mathbb{E}\sup_{s\le t} ||a_s^\varepsilon - \bar{a}_s||^2.$$

It is sufficient to show that $\varphi_\varepsilon(1) \to 0$ as $\varepsilon \to 0$. Let $\delta > 0$. According to the uniform convergence of f_i, choose ε' such that for all $x \in M$ and for all z with $||z|| \le \varepsilon'$ holds

$$||f_i(a,z) - f_i(a,0)|| < \delta$$

(it does not matter what norm we use since they are equivalent).

Using the inequality

$$\begin{aligned} ||u-v||^2 &= ||u-w+w-v||^2 \\ &\le (||u-w|| + ||w-v||)^2 \\ &= ||u-w||^2 + 2||u-w||||w-v|| + ||w-v||^2 \\ &\le 2||u-w||^2 + 2||w-v||^2, \end{aligned}$$

which holds in every normed space we obtain for $||z|| < \varepsilon'$ and for all $a_1, a_2 \in M$

$$\begin{aligned} ||f_i(a_1,z) - f_i(a_2,0)||^2 &\le 2||f_i(a_1,z) - f_i(a_1,0)||^2 \\ &\quad + 2||f_i(a_1,0) - f_i(a_2,0)||^2 \\ &\le 2\delta^2 + 2c||a_1 - a_2||^2, \end{aligned}$$

where $\sqrt{c}$ is a Lipschitz constant for the both $f_i(x,0)$ simultaneously. Then, by Corollary 11.2.2 from [33], we obtain

$$\begin{aligned} \varphi_\varepsilon(t) &= \mathbb{E}\sup_{s\le t} || \int_0^s [f_1(a_u^\varepsilon, z_u^\varepsilon) - f_1(\bar{a}_u, 0)]db_u^* \\ &\qquad + \int_0^s [f_2(a_u^\varepsilon, z_u^\varepsilon) - f_2(\bar{a}_u, 0)]du||^2 \\ &\le c_1\mathbb{E}\int_0^t ||f_1(a_u^\varepsilon, z_u^\varepsilon) - f_1(\bar{a}_u, 0)||^2 du \\ &\quad + c_1\mathbb{E}\int_0^t ||f_2(a_u^\varepsilon, z_u^\varepsilon) - f_2(\bar{a}_u, 0)||^2 du \\ &\le c_2\delta^2 + c_3\mathbb{E}\int_0^t ||a_u^\varepsilon - \bar{a}_u||^2 du \\ &\le c_2\delta^2 + c_3\int_0^t \varphi_\varepsilon(u)du, \text{ for all } t \text{ and } \varepsilon < \varepsilon', \end{aligned}$$

where c_1, c_2, and c_3 are positive constants independent of δ. Now by Gronwall's lemma

$$\varphi_\varepsilon(t) \leq c_2\delta^2\varepsilon^{c_3 t}$$

and in particular

$$\varphi_\varepsilon(1) \leq c_2\delta^2\varepsilon^{c_3}$$

for all $\varepsilon < \varepsilon'$. Hence $\varphi_\varepsilon(1) \to 0$ and the processes $(x_t^\varepsilon, u_t^\varepsilon)$ converge to $(\bar{x}_t, \bar{u}_t)$ locally uniformly in probability. □

3.2 Relation between $\mathbb{W}$ and μ

3.2.1 Equivalence of $\mathbb{W}$ and μ. The density

We would like to study the relation between the families $(\mathbb{W}_\varepsilon)$ and (μ_ε). It can be derived from the relation between the measures $\mathbb{W}$ and μ. We prove in the following lemma that these two measures are equivalent and compute the corresponding density in terms of the shifting vector field v.

Lemma 13. *$\mathbb{W}$ is equivalent to μ and the density ρ is given by*

$$\rho(\omega) = \frac{d\mathbb{W}}{d\mu}(\omega) = \exp\left\{\frac{1}{2}\int_0^1 \langle v(\omega_t), d\omega_t\rangle + \frac{1}{8}\int_0^1 |v(\omega_t)|^2 dt\right\} \quad (3.4)$$

Proof. Recall that μ is the distribution of the process (y_t), which solves the stochastic differential equation (3.1). Hence the process (y_t) satisfies also

$$db_t = dy_t + \frac{1}{2}v(y_t)dt,$$

It follows now from Girsanov's theorem that the law $\mathbb{W}$ of a Brownian motion is equivalent to the distribution μ of (y_t) and the corresponding density is given by (3.4). □

It is easy to see that the density ρ is not necessarily continuous and bounded. In order to prove weak convergence of the family $\mathbb{W}_\varepsilon$ we will

approximate ρ by a continuous and bounded function ρ_0 in such a way that the approximation is quite good on the paths staying in $\mathbb{M}_\varepsilon$ for the whole time. In the remaining part of this section we describe the type of the approximation we need and investigate the approximation of stochastic differentials with respect to dt and dx_t, where (x_t) is the projection of the process (ω_t) stopped while leaving $\mathbb{M}_{\varepsilon_0}$. This enables us to find the approximation ρ_0 for the density ρ (Proposition 3) since the stochastic integrals in (3.4) partly can be reduced to integrals with respect to dt and dx_t and partly are approximated directly.

3.2.2 $O(\varepsilon)$-approximation

First, let us give the following definition.

Definition 11. *Let $\xi : C(\mathbb{R}^n) \to \mathbb{R}$ be a measurable function. We say that ξ is $O(\varepsilon)$ if there exists $c > 0$ such that*

$$\mathbb{E}_{\mu_\varepsilon}|\xi|^p \leq (pc\varepsilon)^p \text{ for all } p \in \mathbb{N}.$$

We say that a stochastic differential is $O(\varepsilon)$ if the corresponding stochastic integral from zero to one is $O(\varepsilon)$.

It follows from Minkowski's inequality that the sum of two $O(\varepsilon)$ is again $O(\varepsilon)$.

Lemma 14. *Let $\xi = O(\varepsilon)$, then $\mathbb{E}_{\mu_\varepsilon}|e^\xi - 1| \to 0$ as $\varepsilon \to 0$.*

Proof. We have by the definition of $O(\varepsilon)$

$$\mathbb{E}_{\mu_\varepsilon}|e^\xi - 1| \leq \mathbb{E}_{\mu_\varepsilon}\sum_{p=1}^\infty \frac{|\xi|^p}{p!} = \sum_{p=1}^\infty \frac{\mathbb{E}_{\mu_\varepsilon}|\xi|^p}{p!} \leq \sum_{p=1}^\infty \frac{p^p}{p!}(c\varepsilon)^p \to 0,$$

since the radius of convergence r of the power series $\sum_{p=1}^\infty \frac{p^p}{p!}z^p$ is positive

$$r = \lim_{p\to\infty}\frac{p^p}{p!}\cdot\frac{(p+1)!}{(p+1)^{p+1}} = \lim_{p\to\infty}(1+\frac{1}{p})^{-p} = e^{-1} > 0$$

□

Further, let (ω_t) be the coordinate process on the probability space $(C(\mathbb{R}^n), \mathcal{F}_t, \mu)$ and let τ be its exit time from $\mathbb{M}_{\varepsilon_0}$. Let $((x_t), (z_t))$ be the Fermi decomposition of the stopped process $(\omega_{t\wedge\tau})$ and let (u_t) be the corresponding translation matrix.

Lemma 15. *There are two smooth bounded functions*

$$\begin{aligned} g_{xx} &: o(n) \times M \times B_k(\varepsilon_0) \to gl(n) \text{ and} \\ g_{ux} &: o(n) \times M \times B_k(\varepsilon_0) \to \mathbb{R}^n \end{aligned}$$

such that up to the exit time τ

$$\begin{aligned} (\delta x_t)(\delta x_t)^T &= g_{xx}(u_t, x_t, z_t)dt, \\ (\delta u_t)(\delta x_t) &= g_{ux}(u_t, x_t, z_t)dt, \\ (\delta x_t)(\delta z_t)^T &= 0_{n\times k}. \end{aligned}$$

Proof. Notice that the coordinate process (ω_t) is a Brownian motion with respect to the Wiener measure $\mathbb{W}$, which is equivalent to the distribution μ of the shifted process. Hence

$$(\delta\omega_t)(\delta\omega_t)^T = Idt \tag{3.5}$$

for both μ and $\mathbb{W}$. Up to time τ we obtain

$$\begin{aligned} (\delta x_t)(\delta x_t)^T &= D\pi(\omega_t)\delta\omega_t(\delta\omega_t)^T D\pi(\omega_t)^T \\ &= (D\pi D\pi^T)(x_t + u_t \mathrm{pr}_2^{-1} z_t)dt \\ &= g_{xx}(u_t, x_t, z_t)dt, \end{aligned}$$

where the function g_{xx} is defined by the last line. Using this relation and the equation of the parallel translation (2.7) we obtain up to the exit time τ

$$\delta u_t \delta x_t = -\Gamma_{x_t}(\delta x_t)u_t\delta x_t = g_{ux}(u_t, x_t, z_t)dt.$$

Finally, using Lemma 1, the equation of the stochastic parallel translation (2.7), the expression (2.5) for Γ, and the equality

$$dPQ = -PdQ$$

we obtain up to time τ

$$\begin{aligned} (\delta x_t)(\delta z_t)^T &= D\pi(\omega_t)\delta\omega_t(\mathrm{pr}_2 \delta u_t^T(\omega_t - x_t) + \mathrm{pr}_2 u_t^T(\delta\omega_t - \delta x_t))^T \\ &= D\pi(\omega_t)\delta\omega_t(\mathrm{pr}_2 u_t^T \Gamma_{x_t}(\delta x_t)(\omega_t - \pi(\omega_t)))^T \\ &+ D\pi(\omega_t)\delta\omega_t(\mathrm{pr}_2 u_t^T(I - D\pi(\omega_t))\delta\omega_t)^T \\ &= D\pi(\omega_t)\delta\omega_t(\mathrm{pr}_2 u_t^T dQ_{x_t}(\delta x_t)P_{x_t}(\omega_t - \pi(\omega_t)))^T \\ &+ D\pi(\omega_t)\delta\omega_t(\mathrm{pr}_2 u_t^T dP_{x_t}(\delta x_t)Q_{x_t}(\omega_t - \pi(\omega_t)))^T \\ &+ D\pi(\omega_t)(\delta\omega_t)(\delta\omega_t)^T(I - D\pi(\omega_t))^T u_t \mathrm{pr}_2^T \\ &= -D\pi(\omega_t)\delta\omega_t(\mathrm{pr}_2 u_t^T P_{x_t} dQ_{x_t}(\delta x_t)(\omega_t - \pi(\omega_t)))^T \\ &+ D\pi(\omega_t)(I - D\pi(\omega_t))u_t \mathrm{pr}_2^T dt = 0 \end{aligned}$$

We have used in this chain of equalities three following observations. First,

$$P_{x_t}(\omega_t - \pi(\omega_t)) = 0$$

since $\omega_t - \pi(\omega_t) \in N_{x_t}M$. Secondly,

$$\mathrm{pr}_2 u_t^T P_{x_t} = \mathrm{pr}_2(P_{x_t}u_t)^T = \mathrm{pr}_2(u_t P_{a_0})^T = \mathrm{pr}_2 P_{a_0} u_t^T = 0$$

since $\mathrm{pr}_2 P_{a_0} = 0$. Finally,

$$D\pi(\omega_t)(I - D\pi(\omega_t))u_t \mathrm{pr}_2^T = 0$$

which follows from the fact that

$$D\pi(\omega_t)(I - D\pi(\omega_t))|_{N_{x_t}M} = 0.$$

The lemma is proved. □

In the next lemma we study approximations of some stochastic differentials.

Lemma 16. *Let* $f : o(n)\times\mathbb{R}^n\times\mathbb{R}^k \to \mathbb{R}$ *be a smooth bounded function. Then*

$$\begin{aligned} f(u_t, x_t, z_t)dt &= f(u_t, x_t, 0)dt + O(\varepsilon), \\ f(u_t, x_t, z_t)dx_t^i &= f(u_t, x_t, 0)dx_t^i + O(\varepsilon), \\ f(u_t, x_t, z_t)\delta x_t^i &= f(u_t, x_t, 0)\delta x_t^i + O(\varepsilon), \end{aligned}$$

for all i.

Proof. In order to prove the first statement notice that there is a constant c such that

$$|f(u,x,z) - f(u,x,0)| \leq c|z| \text{ for all } (u,x) \in o(n) \times M$$

by the compactness of $o(n) \times M$. Hence

$$\mathbb{E}_{\mu_\varepsilon}|\int_0^1 f(u_t,x_t,z_t)dt - \int_0^1 f(u_t,x_t,0)dt|^p \leq \mathbb{E}_{\mu_\varepsilon}(\int_0^1 c|z_t|dt)^p \leq (c\varepsilon)^p$$

Further, denote

$$\xi(\omega) = \int_0^1 f(u_t,x_t,z_t)dx_t^i.$$

It suffices to prove the last two formulas for the case when $f(u,x,0) = 0$ for all $x \in M$ and $u \in o_x(n)$.

Notice that μ_ε is absolutely continuous with respect to μ and the corresponding density is given by

$$\frac{d\mu_\varepsilon}{d\mu}(\omega) = \varphi_\varepsilon(||z||),$$

where $||\cdot||$ is the supremum norm on $C([0,1],\mathbb{R}^k)$ with respect to the Euclidean norm $|\cdot|$ in $\mathbb{R}^k$ and $\varphi_\varepsilon : \mathbb{R} \to \mathbb{R}$ is given by

$$\varphi_\varepsilon = \mu(C(\mathbb{M}_{\varepsilon_0}))\mathbf{1}_{[0,\varepsilon]}.$$

For each ε, let us approximate μ_ε by measures μ_ε^n that are not only absolute continuous but also equivalent to μ. Let (φ_ε^n) be a sequence of functions from $\mathbb{R}$ to $\mathbb{R}$ such that $\varphi_\varepsilon^n > 0$ and $\varphi_\varepsilon^n \downarrow \varphi_\varepsilon$. Denote

$$d_\varepsilon^n(\omega) = \frac{\varphi_\varepsilon^n(||z||)}{\mathbb{E}_\mu \varphi_\varepsilon^n(||z||)} > 0$$

and define μ_ε^n as the measure that is absolutely continuous with respect to μ with density d_ε^n. Then μ_ε^n is a probability measure, which is equivalent to μ since d_ε^n is positive and $\mathbb{E}_\mu d_\varepsilon^n = 1$. Therefore μ and μ_ε^n have the same semimartingales and, in particular, the process (x_t) is a semimartingale with respect to μ_ε^n.

Let $x = m + a$ and $x = m_\varepsilon^n + a_\varepsilon^n$ be the decompositions of (x_t) into a local martingale part and a part of bounded variation with respect to μ and μ_ε^n, respectively. Let us show that these decompositions coincide, for all n and ε. Consider

$$(d_\varepsilon^n)_t = \mathbb{E}_\mu[d_\varepsilon^n|\mathcal{F}_t]$$

and notice that

$$(d_\varepsilon^n)_t = \mathbb{E}_\mu[d_\varepsilon^n|\mathcal{F}_t'],$$

where $\mathcal{F}_t'$ is the natural filtration of (z_t). This follows from the observation that

$$\begin{aligned}(d_\varepsilon^n)_t &= \mathbb{E}_\mu[d_\varepsilon^n|\mathcal{F}_t] \\ &= \frac{1}{\mathbb{E}_\mu \varphi_\varepsilon^n(||z||)} \mathbb{E}_\mu[\varphi_\varepsilon^n(\max\{\sup_{0\leq s\leq t}|z_s|, \sup_{t\leq r\leq 1}|z_t + (z_r - z_t)|\})|\mathcal{F}_t] \\ &= \frac{1}{\mathbb{E}_\mu \varphi_\varepsilon^n(||z||)} \int_{C_0([0,1-t],\mathbb{R})} \varphi_\varepsilon^n(\max\{\sup_{0\leq s\leq t}|z_s|, \sup_{0\leq r\leq 1-t}|z_t + \tilde{\omega}_r|\}) d\mathbb{W}(\tilde{\omega})\end{aligned}$$

is measurable with respect to $\mathcal{F}_t'$ (the last equality is fulfilled since $z_r - z_t$ is a Brownian motion independent of $\mathcal{F}_t$). Hence (see [14]) the process d_ε^n is a stochastic integral with respect to the process z, i.e.

$$(d_\varepsilon^n)_t = \int_0^t h_s dz_s$$

for some k-dimensional process (h_t). Then by Girsanov's theorem and by Lemma 15 (3)

$$\begin{aligned}(m_\varepsilon^n)_t &= m_t - \int_0^t \frac{1}{(d_\varepsilon^n)_t} d[m_t, (d_\varepsilon^n)_t] \\ &= m_t - \int_0^t \frac{1}{(d_\varepsilon^n)_t} (dx_t)(dz_t)^T h_t^T \\ &= m_t - 0 = m_t.\end{aligned}$$

This means that the process (x_t) has the same semimartingale decomposition with respect to the measure μ and all measures μ_ε^n.

By definition of (x_t) and by Lemma 15 we have

$$(dm_t^i)^2 < c_1^2 dt \quad \text{and} \quad |da_t^i| < c_2 dt$$

with some constants c_1 and c_2. Now we can use Corollary 11.2.2 from [33] (notice that the constant c_p there can be chosen equal to $(2p)^p$)

$$\begin{aligned}\mathbb{E}_{\mu_\varepsilon^n}|\xi|^p \leq (2p)^p \mathbb{E}_{\mu_\varepsilon^n}\Bigg[\left(\int_0^1 (dx_t^i)^2\right)^{\frac{p}{2}-1}\int_0^1 |f(u_t,x_t,z_t)|^p (dx_t^i)^2 \\ + \int_0^1 |f(u_t,x_t,z_t)|^p |da_t^i| \left(\int_0^1 |da_t^i|\right)^{p-1}\Bigg] \\ \leq (2p)^p c_3^p \varepsilon^p (c_1^p + c_2^p) \leq (pc\varepsilon)^p,\end{aligned}$$

where c_3 is the Lipschitz constant for the function f with respect to z and $c = 4c_3(c_1+c_2)$.

By the monotone convergence theorem we have

$$\mathbb{E}_\mu \varphi_\varepsilon^n(||z||) \to \mathbb{E}_\mu \varphi_\varepsilon(||z||) = 1 \text{ as } n \to \infty$$

and, moreover,

$$\begin{aligned}\mathbb{E}_{\mu_\varepsilon}|\xi|^p = \mathbb{E}_\mu \varphi_\varepsilon(||z||)|\xi(\omega)|^p = \lim_{n\to\infty} \frac{\mathbb{E}_\mu \varphi_\varepsilon^n(||z||)|\xi(\omega)|^p}{\mathbb{E}_\mu \varphi_\varepsilon^n(||z||)} \\ = \lim_{n\to\infty} \mathbb{E}_\mu d_\varepsilon^n |\xi|^p = \lim_{n\to\infty} \mathbb{E}_{\mu_\varepsilon^n}|\xi|^p \leq (pc\varepsilon)^p\end{aligned}$$

by the previous estimate. This implies $f(u_t,x_t,z_t)dx_t^i = O(\varepsilon)$. The last statement follows now from the previous ones and from Lemma 15. In fact,

$$\begin{aligned}f(u_t,x_t,z_t)\delta x_t^i &= f(u_t,x_t,z_t)dx_t^i + \frac{1}{2}\delta f(u_t,x_t,z_t)\delta x_t^i \\ &= \frac{1}{2}D_u f(u_t,x_t,z_t)\delta u_t \delta x_t^i + \frac{1}{2}D_x f(u_t,x_t,z_t)\delta x_t \delta x_t^i \\ &+ \frac{1}{2}D_z f(u_t,x_t,z_t)\delta z_t \delta x_t^i + O(\varepsilon) = O(\varepsilon),\end{aligned}$$

since

$$D_u f(u_t,x_t,0) = 0 \quad \text{and} \quad D_x f(u_t,x_t,0) = 0,$$

and $\delta u_t \delta x_t^i$ and $\delta x_t \delta x_t^i$ are proportional to dt. □

3.2.3 Approximation of the density

Proposition 3. *The approximation of the density ρ is given by*

$$\rho = \rho_0 \exp(O(\varepsilon)),$$

where

$$\rho_0(\omega) = \exp\left\{-\frac{1}{4}\int_0^1 R(x_t)dt + \frac{1}{8}\int_0^1 ||\sigma||^2(x_t)dt\right\},$$

$R(a)$ is the scalar curvature, and $\sigma(a)$ is the tension field of M at the point $a \in M$.

Proof. The approximation of the second term in the exponents of the density (3.4) is given by Lemma 5, Lemma 7, and Lemma 16 (1)

$$\begin{aligned}|v(\omega_t)|^2 dt &= |v(x_t)|^2 dt + O(\varepsilon) \\ &= \sum_{l=1}^k (\mathrm{tr} F_s(x_t,u_t))^2 dt + O(\varepsilon) \\ &= ||\sigma||^2(x_t)dt + O(\varepsilon). \qquad (3.6)\end{aligned}$$

Consider now the first term in the exponents of the density (3.4). By the definition of the Fermi decomposition we have

$$\omega_t = x_t + z_t^s u_t e_{m+s} \text{ and } x_t = \pi(\omega_t)$$

up to the exit time from $\mathbb{M}_{\varepsilon_0}$. Using Itô's formula, the transformation rule from the Itô to the Stratonovich calculus, and Lemma 8, we obtain

$$\begin{aligned}\langle v(\omega_t), d\omega_t\rangle &= \langle v(\omega_t), \delta\omega_t\rangle - \frac{1}{2}\langle dv(\omega_t), d\omega_t\rangle \\ &= \langle v(\omega_t), \delta(\pi(\omega_t) + z_t^s u_t e_{m+s})\rangle - \frac{1}{2}\langle Dv(\omega_t)d\omega_t, d\omega_t\rangle \\ &= \langle v(\omega_t), D\pi(\omega_t)\delta\omega_t + \delta z^s u_t e_{m+s} + z^s \delta u_t e_{m+s}\rangle \\ &- \frac{1}{2}\mathrm{div} v(\omega_t)dt \\ &= \delta z^s \langle v(\omega_t), u_t e_{m+s}\rangle - z^s \langle v(\omega_t), \Gamma_{x_t}(\delta x_t)u_t e_{m+s}\rangle \\ &- \frac{1}{2}R(x_t)dt + O(\varepsilon). \qquad (3.7)\end{aligned}$$

We have used here the fact that

$$\mathrm{Im} D\pi(y) \perp N_y M$$

which implies

$$\langle v(\omega_t), D\pi(\omega_t)\delta\omega_t\rangle = 0$$

and the equality (3.5).

Now let us show that the first term in (3.7) is $O(\varepsilon)$. In order to do this consider the process

$$c_t = \mathrm{tr}\log[I - z_t^s F_s(x_t, u_t)]$$

and notice that

$$\delta c_t = O(\varepsilon)$$

since $\mathrm{tr}\log I = 0$. Using Itô's formula, Lemma 7, Lemma 16, and the equation of the parallel translation (2.6), we obtain

$$\begin{aligned}
O(\varepsilon) = \delta c_t \\
&= -\mathrm{tr}(F_s(x_t,u_t)[I - z_t^p F_p(x_t,u_t)]^{-1})\delta z_t^s \\
&- z_t^s \mathrm{tr}[I - z_t^p F_p(x_t,u_t)]^{-1}\delta F_s(x_t,u_t) \\
&= -\delta z^s\langle v(\omega_t), u_t e_{m+s}\rangle - z_t^s \mathrm{tr}[I - z_t^p F_p(x_t,u_t)]^{-1} \\
&\times (D_x F_s(x_t,u_t)\delta x_t + D_u F_s(x_t,u_t)\delta u_t) \\
&= -\delta z^s\langle v(\omega_t), u_t e_{m+s}\rangle - z_t^s \mathrm{tr}[I - z_t^p F_p(x_t,u_t)]^{-1} \\
&\times (D_x F_s(x_t,u_t)\delta x_t - D_u F_s(x_t,u_t)\Gamma_{x_y}(\delta x_t)u_t) \\
&= -\delta z^s\langle v(\omega_t), u_t e_{m+s}\rangle \\
&- z_t^s \mathrm{tr}[I - z_t^p F_p(x_t,u_t)]^{-1}\varphi(x_t,u_t)\delta x_t + O(\varepsilon) \\
&= -\delta z^s\langle v(\omega_t), u_t e_{m+s}\rangle + O(\varepsilon),
\end{aligned}$$

where φ is some smooth function. This implies

$$\delta z^s\langle v(\omega_t), u_t e_{m+s}\rangle = O(\varepsilon).$$

Further, notice that the second term in (3.7) is equal to zero. In fact, by the definition of Γ and the relation $dPQ = -PdQ$ we have

$$\begin{aligned}
&\langle v(\omega_t), \Gamma_{x_t}(\delta x_t)u_t e_{m+s}\rangle \\
&= \langle v(\omega_t), (dQ_{x_t}(\delta x_t)P_{x_t} + dP_{x_t}(\delta x_t)Q_{x_t})u_t e_{m+s}\rangle \\
&= -\langle v(\omega_t), P_{x_t} dQ_{x_t}(\delta x_t)u_t e_{m+s}\rangle = 0
\end{aligned}$$

since e_{m+s} and $v(\omega_t)$ belong to $N_{x_t}M$. Finally,

$$\begin{aligned}
\rho &= \exp\left\{\frac{1}{2}\int_0^1 \langle v(\omega_t), d\omega_t\rangle + \frac{1}{8}\int_0^1 |v(\omega_t)|^2 dt\right\} \\
&= \exp\left\{-\frac{1}{4}\int_0^1 R(x_t)dt + \frac{1}{8}\int_0^1 ||\sigma||^2(x_t)dt + O(\varepsilon)\right\}
\end{aligned}$$

□

3.3 Main theorems

3.3.1 Surface measure of the first type

Consider the function ρ_0 introduced in Proposition 3. It is defined on $C(M)$ and is continuous and bounded. Moreover, it can be extended to a continuous bounded function on $C(\mathbb{R}^n)$. In the sequel we understand under ρ_0 such an extension. It turns out that ρ_0 approximates the Girsanov density $\rho = d\mathbb{W}/d\mu$ near the manifold also in the following sense.

Lemma 17. $\mathbb{E}_{\mu_\varepsilon}|\rho - \rho_0| \to 0$ *as* $\varepsilon \to 0$.

Proof. Let ρ_0 be bounded by a constant c. Denote by ξ the $O(\varepsilon)$-part in the asymptotic representation of ρ. Then

$$\mathbb{E}_{\mu_\varepsilon}|\rho - \rho_0| = \mathbb{E}_{\mu_\varepsilon}|\rho_0(e^\xi - 1)| \le c\mathbb{E}_{\mu_\varepsilon}|e^\xi - 1| \to 0$$

by Lemma 14. □

Now we are able to prove the main result of the thesis.

Theorem 1. *Let $\mathbb{W}_\varepsilon$ be the normalized restriction of the flat Wiener measure $\mathbb{W}$ on $C_{a_0}([0,1],\mathbb{R}^n)$ to the set of the paths that do not leave the tubular ε-neighbourhood of the manifold M up to time 1. Then $\mathbb{W}_\varepsilon$*

converges weakly to a measure $\mathbb{S}_1$, which is absolutely continuous with respect to the Wiener measure $\mathbb{W}_M$ on the space of paths $C_{a_0}([0,1],M)$ on the manifold M, and the density is given by

$$\frac{d\mathbb{S}_1}{d\mathbb{W}_M}(\omega) = \frac{\exp\left\{-\frac{1}{4}\int_0^1 R(\omega_t)dt + \frac{1}{8}\int_0^1 ||\sigma||^2(\omega_t)dt\right\}}{\mathbb{E}_{\mathbb{W}_M}\exp\left\{-\frac{1}{4}\int_0^1 R(\omega_t)dt + \frac{1}{8}\int_0^1 ||\sigma||^2(\omega_t)dt\right\}}$$

where R is the scalar curvature and σ is the tension field of the embedding of M in $\mathbb{R}^n$ (which is equal to $(\dim M)\kappa$, where κ denotes the mean curvature vector field of M).

Proof. First, we prove that $\rho\mu_\varepsilon \to \rho_0\mathbb{W}_M$ weakly. Let $h : C(\mathbb{R}^n) \to \mathbb{R}$ be continuous and bounded. Then

$$|\mathbb{E}_{\rho\mu_\varepsilon}h - \mathbb{E}_{\rho_0\mathbb{W}_M}h| \le ||h||_\infty \mathbb{E}_{\mu_\varepsilon}|\rho - \rho_0| + |\mathbb{E}_{\mu_\varepsilon}h\rho_0 - \mathbb{E}_{\mathbb{W}_M}h\rho_0| \to 0,$$

where the first term tends to 0 by Lemma 17 and the second term tends to 0 due to the weak convergence of μ_ε to $\mathbb{W}_M$ and since $h\rho_0$ is continuous. Now we can compute

$$\begin{aligned}
\lim_{\varepsilon\to 0}\mathbb{E}_{\mathbb{W}_\varepsilon}h &= \lim_{\varepsilon\to 0}\frac{\mathbb{E}_{\mathbb{W}}\mathbf{1}_{C(\mathbb{M}_\varepsilon)}h}{\mathbb{E}_{\mathbb{W}}\mathbf{1}_{C(\mathbb{M}_\varepsilon)}} = \lim_{\varepsilon\to 0}\frac{\mathbb{E}_{\mu}\mathbf{1}_{C(\mathbb{M}_\varepsilon)}\rho h}{\mathbb{E}_{\mu}\mathbf{1}_{C(\mathbb{M}_\varepsilon)}\rho} \\
&= \lim_{\varepsilon\to 0}\frac{\mathbb{E}_{\mu}\mathbf{1}_{C(\mathbb{M}_\varepsilon)}h\rho}{\mathbb{E}_{\mu}\mathbf{1}_{C(\mathbb{M}_\varepsilon)}}\cdot\frac{\mathbb{E}_{\mu}\mathbf{1}_{C(\mathbb{M}_\varepsilon)}}{\mathbb{E}_{\mu}\mathbf{1}_{C(\mathbb{M}_\varepsilon)}\rho} = \lim_{\varepsilon\to 0}\frac{\mathbb{E}_{\mu_\varepsilon}\rho h}{\mathbb{E}_{\mu_\varepsilon}\rho} \\
&= \lim_{\varepsilon\to 0}\frac{\mathbb{E}_{\rho\mu_\varepsilon}h}{\mathbb{E}_{\rho\mu_\varepsilon}\mathbf{1}} = \frac{\mathbb{E}_{\rho_0\mathbb{W}_M}h}{\mathbb{E}_{\rho_0\mathbb{W}_M}\mathbf{1}} = \frac{\mathbb{E}_{\mathbb{W}_M}\rho_0 h}{\mathbb{E}_{\mathbb{W}_M}\rho_0} \\
&= \mathbb{E}_{\mathbb{W}_M}h\left[\frac{\rho_0}{\mathbb{E}_{\mathbb{W}_M}\rho_0}\right],
\end{aligned}$$

where the last line follows from the first step of the proof. This means that $\mathbb{W}_\varepsilon$ converges weakly to a measure $\mathbb{S}_1$ that is absolutely continuous with respect to the Wiener measure $\mathbb{W}_M$ with the density given above, and the theorem is proved. □

3.3.2 Surface measure of the second type

Let us now prove the main statement concerning the surface measure of the second type.

Theorem 2. *Let $\mathbb{W}^r_\varepsilon$ be the distribution of the process (r^ε_t), which is a Brownian motion in $\mathbb{R}^n$ starting at a_0 with reflection on the boundary $\partial\mathbb{M}_\varepsilon$ of the tubular ε-neighbourhood. Then $\mathbb{W}^r_\varepsilon$ converges weakly to the Wiener measure $\mathbb{W}_M$ as $\varepsilon \to 0$, i.e., the surface measure of the second type $\mathbb{S}_2$ exists and is just the Wiener measure on the space of paths $C_{a_0}([0,1],M)$ on the manifold M.*

Proof. In order to prove the weak convergence $\mathbb{W}^r_\varepsilon \to \mathbb{W}_M$ it suffices to show that the reflected Brownian motions (r^ε_t) converge locally uniformly in probability to a Brownian motion on the manifold M.

First notice that the process (r^ε_t) is a continuous semimartingale since it satisfies the Skorokhod equation (see [4])

$$dr^\varepsilon_t = db_t + \frac{1}{2}n(r^\varepsilon_t)dl^\varepsilon_t,$$

where (b_t) is a standard n-dimensional Brownian motion starting at a_0, (l^ε_t) is the local time of the process (r^ε_t) on the boundary $\partial\mathbb{M}_\varepsilon$, and $n(y)$ is the inward pointing unit normal vector at $y \in \partial\mathbb{M}_\varepsilon$. Hence the Fermi decomposition $((x^\varepsilon_t),(z^\varepsilon_t))$ of (r^ε_t) is well-defined. Let (u^ε_t) be the transition matrix corresponding to (x^ε_t).

Since

$$\operatorname{dist}(x^\varepsilon_t, y^\varepsilon_t) \le \varepsilon$$

for all t it suffices to show that (x^ε_t) converges to $(\bar{x}_t)$ locally uniformly in probability, where $(\bar{x}_t)$ is a Brownian motion on the manifold M starting at a_0.

Using Itô's formula, the Skorokhod equation, Lemma 1, and the fact that $n(y) \in N_{\pi(y)}M$ for all $y \in \partial\mathbb{M}_{\varepsilon_0}$ we obtain

$$\begin{aligned}
dx^\varepsilon_t &= d\pi(r^\varepsilon_t) \\
&= D\pi(r^\varepsilon_t)dr^\varepsilon_t + \frac{1}{2}DD\pi(r^\varepsilon_t)dr^\varepsilon_t dr^\varepsilon_t \\
&= u^\varepsilon_t \mathrm{pr}_1^{-1}[I - (z^\varepsilon_t)^s F_s(x^\varepsilon_t, u^\varepsilon_t)]^{-1}\mathrm{pr}_1(u^\varepsilon_t)^T \\
&\times (db_t + \frac{1}{2}n(r^\varepsilon_t)dl^\varepsilon_t) + \frac{1}{2}\Delta\pi(x^\varepsilon_t + u_t\mathrm{pr}_2^{-1}z^\varepsilon_t)dt \\
&= u^\varepsilon_t \mathrm{pr}_1^{-1}[I - (z^\varepsilon_t)^s F_s(x^\varepsilon_t, u^\varepsilon_t)]^{-1}\mathrm{pr}_1(u^\varepsilon_t)^T db_t \\
&+ \frac{1}{2}\Delta\pi(x^\varepsilon_t + u_t\mathrm{pr}_2^{-1}z^\varepsilon_t)dt.
\end{aligned}$$

Hence the process $(x_t^\varepsilon, u_t^\varepsilon)$ is a solution of the system of stochastic differential equations

$$\begin{cases} \delta u_t^\varepsilon + \Gamma_{x_t^\varepsilon}(\delta x_t^\varepsilon)u_t^\varepsilon = 0, \\ dx_t^\varepsilon = u_t^\varepsilon \mathrm{pr}_1^{-1}[I - (z_t^\varepsilon)^s F_s(x_t^\varepsilon, u_t^\varepsilon)]^{-1}\mathrm{pr}_1(u_t^\varepsilon)^T db_t \\ \qquad +\frac{1}{2}\Delta\pi(x_t^\varepsilon + u_t \mathrm{pr}_2^{-1} z_t^\varepsilon)dt, \\ u_0^\varepsilon = I, \\ x_0^\varepsilon = a_0 \end{cases}$$

Using the relations

$$\mathrm{pr}_1^{-1}\mathrm{pr}_1 = P_{a_0} \quad \text{and} \quad \bar{u}_t P_{a_0} = P_{\bar{x}_t}\bar{u}_t$$

(see Lemma 9) we obtain that the coefficients of this system converge to the coefficients of the system

$$\begin{cases} \delta \bar{u}_t + \Gamma_{\bar{x}_t}(\delta \bar{x}_t)\bar{u}_t = 0, \\ d\bar{x}_t = P_{\bar{x}_t} db_t + \frac{1}{2}\Delta\pi(\bar{x}_t)dt, \\ \bar{u}_0 = I, \\ \bar{x}_0 = a_0 \end{cases}$$

By Lemma 12 the processes $(x_t^\varepsilon, u_t^\varepsilon)$ converge to $(\bar{x}_t, \bar{u}_t)$ locally uniformly in probability. Finally, it was shown in the proof of Proposition 2 that the process $(\bar{x}_t)$ is a Brownian motion on the manifold M. This completes the proof. □

Chapter 4

Particular cases

4.1 One-dimensional manifolds

Let M be a one-dimensional compact manifold in $\mathbb{R}^n$. Every such a manifold is a closed smooth curve without self-intersections and there is a natural arc-length coordinate on M. The scalar curvature of every such a manifold equals zero since the curvature tensor being written with respect to the arc-length coordinate is equal to zero. Hence by Theorem 1 the surface measure of the first type in this case is given by the density

$$\frac{d\mathbb{S}_1}{d\mathbb{W}_M}(\omega) = \frac{\exp \frac{1}{8}\int_0^1 \kappa^2(\omega_t)dt}{\mathbb{E}_{\mathbb{W}_M} \exp \frac{1}{8}\int_0^1 \kappa^2(\omega_t)dt},$$

where κ is the (mean) curvature of the curve M.

In particular, if M is a circle then its curvature is constant and hence the surface measure $\mathbb{S}_1$ is just the Wiener measure on the paths on the circle.

4.1.1 Surface measures for arbitrary time intervals

In the previous chapters we studied the surface measures on the spaces of paths defined on the time interval $[0,1]$. Certainly, the similar definitions of the surface measures can be given for the time interval $[0,T]$

with arbitrary $T > 0$ and similar theorems will be true. In particular, the surface measure of the second type $\mathbb{S}_2^T$ defined as the weak limit of the reflected Brownian motions up to time T with reflection on the boundary $\partial\mathbb{M}_\varepsilon$ coincides with the Wiener measure on the space $C_{a_0}([0,T],M)$. Furthermore, we define the surface measure of the first type $\mathbb{S}_1^T$ as

$$\frac{d\mathbb{S}_1^T}{d\mathbb{W}_M^T}(\omega) = (\text{weak})\lim_{\varepsilon\to 0}\mathbb{W}_\varepsilon^T = (\text{weak})\lim_{\varepsilon\to 0}\frac{\mathbb{W}^T|_{C_{a_0}([0,T],\mathbb{M}_\varepsilon)}}{\mathbb{W}^T(C_{a_0}([0,T],\mathbb{M}_\varepsilon))},$$

where $\mathbb{W}^T$ is the Wiener measure in the ambient space $C_{a_0}([0,T],\mathbb{R}^n)$. Then the generalization of Theorem 1 can be formulated as following: for each T, the surface measure $\mathbb{S}_1^T$ is equivalent to the Wiener measure $\mathbb{W}_M^T$ on $C_{a_0}([0,T],M)$ and its density is given by

$$\frac{d\mathbb{S}_1^T}{d\mathbb{W}_M^T}(\omega) = \frac{\exp\int_0^T v(\omega_t)dt}{\mathbb{E}_{\mathbb{W}_M^T}\exp\int_0^T v(\omega_t)dt}, \qquad \text{where } v(x) = \frac{1}{8}\kappa^2(x).$$

In this section we study the process $(y_t^T), t \le T$, corresponding to the surface measure $\mathbb{S}_1^T$. We show that (y_t^T) is an inhomogeneous Markov process and compute its transition density (Proposition 4). Moreover, we find a stochastic differential equation for (y_t^T), which implies that it is a drifted Brownian motion with a time-space dependent drift (Proposition 5).the main result of this section is Theorem 3, where we prove the weak convergence of the family (y_t^T) as $T \to \infty$ to a homogeneous Markov process that is a drifted Brownian motion (with an independent of the time drift) .

Proposition 4. *For each $T > 0$, (y_t^T) is an inhomogeneous Markov process up to time T with the transition density given by*

$$p^T(s,t,x,y) = p_v(t-s,x,y)\frac{u(T-t,y)}{u(T-s,x)}, \tag{4.1}$$

where u is the solution of the equation

$$\partial_t u = \frac{1}{2}\Delta u + vu \tag{4.2}$$

with the initial condition $u(0,x) = 1$ and $p_v(\cdot,\cdot,y)$ is a solution of (4.2) for each y such that $p_v(t,x,y)dy \to \delta_x$ weakly as $t \to 0$, for every x.

Proof. It is easy to see that the process (y_t^T) can be interpreted as a process in $\mathbb{R}$ if we consider the real line as the universal covering of M. The equation (4.2) can be also considered in $\mathbb{R}$ and the potential v as a periodic function from $\mathbb{R}$ to $\mathbb{R}$. Finally, $\mathbb{S}_1^T$ and $\mathbb{W}^T$ are considered as measures on the space $C_0([0,T],\mathbb{R})$.

For $x \in \mathbb{R}$ and $t > 0$, denote by $\mathbb{W}_x^t$ the Wiener measure on $C_x([0,t],\mathbb{R})$ corresponding to a real Brownian motion starting at x and by $\mathbb{W}_{xy}^t$ the measure on the same space corresponding to a Brownian bridge starting at x and ending at y. Then we have by the Feynman-Kac formula

$$u(t,x) = \mathbb{E}_{\mathbb{W}_x^t} \exp\left[\int_0^t v(\omega_r)dr\right]. \tag{4.3}$$

Further, let us prove that

$$p_v(t,x,y) = p(t,x,y)\mathbb{E}_{\mathbb{W}_{xy}^t} \exp\left[\int_0^t v(\omega_r)dr\right], \tag{4.4}$$

where $p(t,x,y)$ is the transition density of a standard Brownian motion. It suffices to show that p_v defined as above solves (4.2) and $p_v(t,x,y)dy \to \delta_x$. Indeed, let f be a continuous bounded function. Consider a function φ defined by

$$\begin{aligned}\varphi(t,x) &= \int_{\mathbb{R}} f(y)p_v(t,x,y)dy \\ &= \int_{\mathbb{R}} f(y)p(t,x,y)\mathbb{E}_{\mathbb{W}_{xy}^t} \exp\left[\int_0^t v(\omega_r)dr\right]dy \\ &= \mathbb{E}_{\mathbb{W}_x^t} f(\omega_t)\exp\left[\int_0^t v(\omega_r)dr\right].\end{aligned}$$

By the Feynman-Kac formula φ is a solution of the equation (4.2) with the initial condition $\varphi(0,x) = f(x)$. Differentiating the equality

$$\varphi(t,x) = \int_{\mathbb{R}} f(y)p_v(t,x,y)dy$$

with respect to t and x, substituting to (4.2), and using the observation that the equality must be true for all continuous and bounded f we

obtain, for all y,

$$\partial_t p_v(t,x,y) = \frac{1}{2}\Delta_x p_v(t,x,y) + v(x)p_v(t,x,y).$$

The weak convergence $p_v(t,x,y)dy \to \delta_x$ follows now from the observation that

$$\int_{\mathbb{R}} f(y)p_v(t,x,y)dy = \varphi(t,x) \to f(x)$$

as $t \to 0$, for all f.

Further, notice that for all $0 < s < t \leq T$ and for all x holds

$$\begin{aligned}u(t,x) &= \mathbb{E}_{\mathbb{W}_x^t} \exp\left[\int_0^t v(\omega_r)dr\right] \\ &= \int_{\mathbb{R}} p(s,x,y)\mathbb{E}_{\mathbb{W}_{xy}^t} \exp\left[\int_0^s v(\omega_r)dr\right]\mathbb{E}_{\mathbb{W}_y^{t-s}} \exp\left[\int_0^t v(\omega_r)dr\right] \\ &= \int_{\mathbb{R}} p_v(s,x,y)u(t-s,y)dy.\end{aligned} \tag{4.5}$$

Now consider the coordinate process (ω_t) on the probability space $(C_0([0,T],\mathbb{R}),\mathcal{F}_t,\mathbb{S}_1^T)$ with the natural filtration. Let C be a Borel set in $\mathbb{R}^n$ and $0 = t_0 < t_1 < \cdots < t_n < T$. Let $x_0 = 0$ and denote $\Delta t_i = t_i - t_{i-1}$. Then, using (4.3), (4.4), and (4.5), we obtain

$$\begin{aligned}&\mathbb{S}_1^T\{(\omega_{t_1},\ldots,\omega_{t_n}) \in C\} \\ &\quad= \int_C \left[\prod_{i=1}^n p(\Delta t_i,x_{i-1},x_i)\mathbb{E}_{\mathbb{W}_{x_{i-1}x_i}^{\Delta t_i}} \exp\left[\int_0^{\Delta t_i} v(\omega_r)dr\right]\right] \\ &\qquad\quad \times \frac{\mathbb{E}_{\mathbb{W}_{x_n}^{T-t_n}} \exp\left[\int_0^{T-t_n} v(\omega_r)dr\right]dx_1\ldots dx_n}{u(T,0)} \\ &\quad= \int_C \frac{\left[\prod_{i=1}^n p_v(\Delta t_i,x_{i-1},x_i)\right]u(T-t_n,x_n)dx_1\ldots dx_n}{u(T,0)} \\ &\quad= \int_C \prod_{i=1}^n \left[p_v(\Delta t_i,x_{i-1},x_i)\frac{u(T-t_i,x_i)}{u(T-t_{i-1},x_{i-1})}\right]dx_1\ldots dx_n \\ &\quad= \int_C \prod_{i=1}^n p^T(t_{i-1},t_i,x_{i-1},x_i)dx_1\ldots dx_n,\end{aligned}$$

which implies (see, for example, [34]) that (y_t^T) is an inhomogeneous Markov process with the transition density given by (4.1). □

Proposition 5. *The process* $(y_t^T), t \le T$, *is a solution of the stochastic differential equation*

$$dy_t^T = db_t + (\nabla \log u)(T-t, y_t^T)dt \tag{4.6}$$

with the initial condition $y_0^T = a_0$.

Proof. As in the previous lemma, we consider the process (y_t^T) as a coordinate process (ω_t) on the probability space $(C_0([0,T],\mathbb{R}), \mathcal{F}_t, \mathbb{S}_1^T)$. Since it is a Brownian motion with respect the Wiener measure $\mathbb{W}^T$, which is equivalent to $\mathbb{S}_1^T$, we can apply Girsanov's Theorem, which claims that

$$d\omega_t = db_t + \frac{dr_t d\omega_t}{r_t},$$

where

$$\begin{aligned} r_t = \mathbb{E}\left[d\mathbb{S}_1^T/d\mathbb{W}^T|\mathcal{F}_t\right] &= \frac{\mathbb{E}\left[\exp\left(\int_0^T v(\omega_s)ds\right)|\mathcal{F}_t\right]}{u(T,0)} \\ &= \frac{\exp\left(\int_0^t v(\omega_s)ds\right)\mathbb{E}\left[\exp\left(\int_t^T v(\omega_s)ds\right)|\mathcal{F}_t\right]}{u(T,0)} \\ &= \frac{\exp\left[\int_0^t v(\omega_s)ds\right]u(T-t,\omega_t)}{u(T,0)} \end{aligned}$$

and hence

$$\begin{aligned} \frac{dr_t d\omega_t}{r_t} &= \frac{\exp\left[\int_0^t v(\omega_s)ds\right]\nabla u(T-t,\omega_t)dt}{u(T,0)} \\ &: \frac{\exp\left[\int_0^t v(\omega_s)ds\right]u(T-t,\omega_t)}{u(T,0)} \\ &= \nabla \log u(T-t,\omega_t)dt, \end{aligned}$$

which gives us the drift term in the equation (4.6). □

4.1.2 Convergence at infinity

In what follows we investigate the convergence of the family $(\mathbb{S}_1^T)$ (respectively, of the family of the processes (y_t^T)) as $T \to \infty$. The main result of this section is Theorem 3, where we prove the weak convergence of the family (y_t^T) to a homogeneous Markov process that is a drifted Brownian motion (with an independent of the time drift) .

It should be mentioned that the measures $(\mathbb{S}_1^T)$ are defined on different spaces $C_{a_0}([0,T],M)$ for different T. In order to be able to study their convergence we will extend them in an arbitrary way to σ-additive measures on the common space $C_{a_0}([0,\infty],M)$.

Lemma 18. *There is a unique real constant c such that the operator* $\frac{1}{2}\Delta + v$ *on M has a positive eigenfunction to the eigenvalue c. In that case such an eigenfunction is unique up to a multiplicative constant.*

Proof. Again we prefer to consider the periodic case in $\mathbb{R}$. Then the positive eigenfunction we are looking for must be also periodic.

Denote $L = \frac{1}{2}\Delta + v$. Due to [18] (see also [1]) let

$$C_L(\mathbb{R}) = \{u \in C^2(\mathbb{R}) : Lu = 0 \text{ and } u > 0\}$$

denote the cone of positive harmonic functions for L in $\mathbb{R}$. Further, for every $\lambda \in \mathbb{R}$ denote

$$\begin{aligned} \Gamma_\lambda = \{\nu \in \mathbb{R} : \exists u \in C_{L-\lambda}(\mathbb{R}) \text{ of the form } u(t) = e^{\nu t}\psi_\nu(t), \\ \text{where } \psi_\nu \text{ is periodic}\}, \\ K_\lambda = \{\nu \in \mathbb{R} : \exists u \in C^2(\mathbb{R}) \text{ satisfying } (L-\lambda)u \le 0 \text{ and } u > 0 \\ \text{of the form } u(t) = e^{\nu t}\psi_\nu(t), \text{ where } \psi_\nu \text{ is periodic}\}. \end{aligned}$$

By Theorem A1 from [18] there exists a real number λ^* such that

1) if $\lambda = \lambda^*$ then $\Gamma_\lambda = K_\lambda = \{\nu_0\}$, for some $\nu_0 \in \mathbb{R}$;
2) if $\lambda < \lambda^*$ then $\Gamma_\lambda = K_\lambda = \emptyset$;
3) if $\lambda > \lambda^*$ then K_λ is a strictly convex compact set and $\Gamma_\lambda = \partial K_\lambda$;
4) $K_{\lambda_1} \subsetneq K_{\lambda_2}$, for $\lambda^* \le \lambda_1 < \lambda_2$;
5) $K_\lambda^* = -K_\lambda$, where K^* corresponds to L^*.

Further, by Corollary A3 from [18], for each $u \in C_{L-\lambda}(\mathbb{R})$ there exists a unique finite measure μ_u on Γ_λ such that

$$u(t) = \int_{\Gamma_\lambda} e^{\nu t}\psi_\nu(t)\mu_u(d\nu),$$

where, for each $\nu \in \Gamma_\lambda$, ψ_ν is periodic.

Since our operator L is self-adjoint we have $K_{\lambda^*} = -K_{\lambda^*}$ and hence $\nu_0 = 0$ by (5).

First, consider $c = \lambda^*$. Let $u \in C_{L-c}(\mathbb{R})$; it follows from the corollary and properties (1) and (5) that

$$u(x) = K\psi_{\nu_0}(x),$$

where K is some positive constant. Hence the set of positive eigenfunction of the operator $\frac{1}{2}\Delta + v$ to the eigenvalue $c = \lambda^*$ consists of one function (up to a multiplicative constant), which is periodic.

Secondly, if $c < \lambda^*$ then $\Gamma_c = \emptyset$ by (2) and $C_{L-c}(\mathbb{R})$ is empty by the corollary. Hence there are no positive (even non-periodic) eigenfunctions of $\frac{1}{2}\Delta + v$ to the eigenvalues $c < \lambda^*$.

Finally, consider $c > \lambda^*$. It follows from Lemma 3 in [18] that

$$\Gamma_\lambda = \{\nu \in \mathbb{R} : \lambda_0(\nu) = \lambda\},$$

where λ_0 is some continuous function. Hence $0 = \nu_0 \notin \Gamma_c$. It follows from (3) and (4) that $K_c = [a_c, b_c]$ with $a_c < 0 < b_c$ and $\Gamma_c = \{a_c, b_c\}$. Let $u \in C_{L-c}(\mathbb{R})$; by the corollary

$$u(t) = e^{a_c t}\psi_{a_c}(t)\mu(\{a_c\}) + e^{b_c t}\psi_{b_c}(t)\mu(\{b_c\}),$$

which is periodic only if $\mu \equiv 0$ and then not positive. Hence there are no positive periodic eigenfunctions of $\frac{1}{2}\Delta + v$ to the eigenvalues $c > \lambda^*$ as well and the lemma is proved. □

Denote such positive eigenfunction by φ.

Lemma 19. *$u(t,x) = e^{ct}\varphi(x)w(t,x)$, where $w(t,x)$ is a solution of the equation*

$$\partial_t w = Lw, \text{ where } L = \frac{1}{2}\Delta + (\nabla \log \varphi)\nabla. \tag{4.7}$$

Proof. Consider the substitution $u(t,x) = e^{ct}\varphi(x)w(t,x)$. Then, using the equation (4.2), which holds for u we have

$$\begin{aligned} &ce^{ct}\varphi(x)w(t,x) + e^{ct}\varphi(x)\partial_t w(t,x) \\ &= e^{ct}\left[\frac{1}{2}\Delta\varphi(x)w(t,x) + \nabla\varphi(x)\nabla w(t,x)\right. \\ &+ \left.\frac{1}{2}\varphi(x)\Delta w(t,x) + v(x)\varphi(x)w(t,x)\right]. \end{aligned}$$

Using the observation that φ is an eigenfunction of $\frac{1}{2}\Delta + v$ to the eigenvalue c, we obtain the desired equation for w. □

Now let (y_t) be a process on M with the generator L. It is a solution of the stochastic differential equation

$$dy_t = db_t + \nabla \log \varphi(y_t)dt. \tag{4.8}$$

Denote by P_x the law of such (y_t) starting at x.

Lemma 20. *Let w be a solution of (4.7) with some smooth initial function f. Then*

1) $w(t,x) \to c$ as $t \to \infty$ uniformly in x and if $f > 0$ then $c \neq 0$;
2) $\nabla w(t,x) \to 0$ as $t \to \infty$ uniformly in x;
3) if $f > 0$ then $\nabla \log w(t,x) \to 0$ as $t \to \infty$ uniformly in x.

Proof. 1) As a diffusion process on a circle (y_t) has an invariant measure μ with $\mu(M) = 1$ (see [18, p. 88]). Then, by Theorem 3.4 from [15] we have

$$w(t,x) \to \int_M f(x)\mu(dx) = c,$$

and $c > 0$ for positive f since μ is a positive measure.

In order to show that the convergence is uniform let us fix some $x_0 \in M$ and let τ be the time when the process (y_t) hits x_0. Denote

$$g(t,x) = P_x(\tau > t).$$

Notice that g is continuous and, moreover,

$$g(t_2,x) \le g(t_1,x) \text{ for } t_2 > t_1.$$

Using Lemma 5.2 from [15] we can easily show that the process (y_t) is recurrent since the equation $Lu = 0$ has the unique bounded solution

$$u(x) = u(a) + [u(b) - u(a)]\frac{\int_a^x \varphi^{-2}(y)dy}{\int_a^b \varphi^{-2}(y)dy}$$

on any interval $[a,b] \in \mathbb{R}$ with given boundary values $u(a)$, $u(b)$. Hence $g(t,x) \to 0$ as $t \to \infty$ and by the monotonicity of g we obtain that the convergence is uniform in x.

We have

$$w(t,x) = \mathbb{E}_x f(y_t).$$

Let $T : \mathbb{R}_+ \to \mathbb{R}_+$ be a monotone increasing function such that $T(t) \to \infty$ and $t/T(t) \to \infty$, for example, $T(t) = \log t$.

Let $\varepsilon > 0$; by the pointwise convergence of w there is $T_1 > 0$ such that for all $t > T_1$ holds

$$|w(t,x_0) - c| < \varepsilon/2.$$

Further, by the uniform convergence

$$g(T(t),x) = P_x\{\tau > T(t)\} \to 0$$

there exists $T_2 > 0$ such that for all $t > T_2$ and for all x holds

$$P_x\{\tau > T(t)\} < \varepsilon/2(c + ||f||),$$

where by $||\cdot||$ is denoted the supremum norm on the space of continuous functions on M. Finally, let $T_3 > 0$ be chosen in such a way that for all $t > T_3$ holds $t - T(t) > T_1$. Choose

$$T_0 = \max\{T_1, T_2, T_3\}$$

and denote the density of the distribution of τ under P_x by $p_x(\theta)$. Then we have for all $t > T_0$ and for all x

$$\begin{aligned}
|w(t,x) - c| &= |\mathbb{E}_x \mathbf{1}_{\{\tau > T(t)\}} f(y_t) + \int_0^{T(t)} p_x(\theta)\mathbb{E}_{x_0} f(x_{t-\theta})d\theta - c| \\
&\le (||f|| + c)P_x\{\tau > T(t)\} + \int_0^{T(t)} p_x(\theta)|w(t-\theta, x_0) - c|d\theta \\
&\le (||f|| + c)\frac{\varepsilon}{2(||f|| + c)} + \frac{\varepsilon}{2} = \varepsilon,
\end{aligned}$$

which implies the uniform convergence $w(t,x) \to c$ as $t \to \infty$.

2) Using the previous statement we get

$$\partial_t w(t,x) = (Le^{tL}f)(x) = (e^{tL}Lf)(x) \to c'$$

uniformly in x. Since we also have $w(t,x) \to c$ we get $c' = 0$.

Now we multiply the both parts of the equality (4.7) by $2\varphi^2$. Then it can be written in the form

$$\nabla(\varphi^2 \nabla w) = 2\varphi^2 \partial_t w.$$

and hence

$$\nabla w(t,x) = \frac{c(t)}{\varphi^2(x)} + \frac{2}{\varphi^2(x)} \int_{x_0}^{x} \varphi^2(y)\partial_t w(t,y)dy,$$

where $c(t)$ is a constant depending on t.

Using the necessary condition

$$\int_M \nabla w(t,x)dx = 0$$

for all t, boundedness of φ from zero and from infinity, and the uniform convergence $\partial_t w(t,y) \to 0$ we obtain $c(t) \to 0$ and $\nabla w(t,x) \to 0$ as $t \to \infty$ uniformly in x.

3) It is a trivial consequence of (1) and (2). □

Theorem 3. *The surface measures* $(\mathbb{S}_1^T)$ *converge (as* $T \to \infty$*) weakly to the law of the drifted Brownian motion* (y_t)*, which is a solution of the stochastic differential equation*

$$dy_t = db_t + \nabla \log \varphi(y_t)dt$$

with the initial condition $y_0 = a_0$.

The transition density of the (homogeneous) limit process (y_t) *is given by*

$$p^\infty(t,x,y) = \frac{p_v(t,x,y)}{e^{ct}} \frac{\varphi(y)}{\varphi(x)}$$

Proof. Recall that by Proposition 5 the process (y_t^T) with the law $\mathbb{S}_1^T$ solves the stochastic differential equation (4.6). Let us show that the coefficients of (4.6) converge to the coefficients of (4.8) in the sense of Theorem 11.1.4 from [29]. By Lemmata 19 and 20 we have

$$\begin{aligned}
\nabla \log u(t,x) &= \frac{\nabla[e^{ct}\varphi(x)w(t,x)]}{e^{ct}\varphi(x)w(t,x)} \\
&= \nabla \log \varphi(x) + \nabla \log w(t,x) \to \nabla \log \varphi(x)
\end{aligned}$$

as $t \to \infty$ uniformly in x. Hence $\nabla \log u(t,x)$ is uniformly bounded for all t and x. Furthermore, for every $T_0 > 0$ holds

$$\lim_{T\to 0}\int_0^{T_0} \sup_{x\in M} |\nabla \log u(T-t,x) - \nabla \log \varphi(x)| dt$$
$$= \lim_{T\to 0}\int_{T-T_0}^{T} \sup_{x\in M} |\nabla \log w(t,x)| dt = 0.$$

Thus, both conditions of Theorem 11.1.4 from [29] are fulfilled and we can conclude that $\mathbb{S}_1^T \to \mathcal{L}(y)$ weakly.

Finally, we compute the limit transition density using Lemma 20:

$$\begin{aligned} p^\infty(t,x,y) &= \lim_{T\to\infty} p^T(0,t,x,y) \\ &= \lim_{T\to\infty} p_v(t,x,y)\frac{u(T-t,y)}{u(T,x)} \\ &= \lim_{T\to\infty} p_v(t,x,y)\frac{e^{c(T-t)}\varphi(y)w(T-t,y)}{e^{cT}\varphi(x)w(T,x)} \\ &= \frac{p_v(t,x,y)}{e^{ct}}\frac{\varphi(y)}{\varphi(x)}, \end{aligned}$$

which completes the proof. □

4.2 The case of the flat normal bundle

The idea of the proof of Theorem 1 was prompted by the observation that the generator $\frac{1}{2}\Delta$ of the flat Brownian motion has the decomposition

$$\frac{1}{2}(\Delta u)(a) = \frac{1}{2}\left[\Delta_{M_a} u + \Delta_{N_a M} u\right](a) + \frac{m}{2}\langle \kappa, \nabla u\rangle(a), \qquad (4.9)$$

where M_a is the manifold through a that is parallel to M, Δ_{M_a} is the corresponding Laplace-Beltrami operator, and $\kappa(a)$ is the mean curvature vector of the manifold M_a at the point a. Roughly speaking the Laplace operator in $\mathbb{R}^n$ can be decomposed into the sum of the Laplace-Beltrami operator in the manifold direction, the Laplace

operator in the orthogonal direction, and a correction operator of the first order.

This observation is true only for an embedding $M \subset \mathbb{R}^n$ such that the normal bundle NM is flat since only in this case the parallel manifolds M_a exist. Then the shifted process (y_t) defined in Section 3.1 is nothing else but the process with generator $\frac{1}{2}\left[\Delta_{M_a} + \Delta_{N_a M}\right]$ and the shifting vector field is just $m\kappa$. Therefore the fact that the surface measure of the first type corresponding to the process (y_t) is just the Wiener measure on M (see Proposition 2) becomes intuitively clear.

In this section we explain the notion of a parallel manifold in more detail and prove the crucial decomposition formula (4.9).

4.2.1 Parallel manifolds

The aim of this part is to define and to construct m-dimensional manifolds parallel to the manifold M.

Let E be the space of all m-dimensional linear subspaces of $\mathbb{R}^n$ with its natural topology. The following definitions and the Frobenius theorem are due to [32].

Definition 12. *A smooth mapping $D : \mathbb{M}_{\varepsilon_0} \to E$ is called an m-dimensional distribution on $\mathbb{M}_{\varepsilon_0}$. A vector field u on $\mathbb{M}_{\varepsilon_0}$ is said to belong to the distribution D ($u \in D$) if $u(x) \in D(x)$ for each $x \in \mathbb{M}_{\varepsilon_0}$. A smooth distribution D is called involutive if $[u,w] \in D$ whenever u and w are smooth vector fields belonging to D.*

Definition 13. *An m-dimensional manifold $N \subset \mathbb{M}_{\varepsilon_0}$ is an integral manifold of a distribution D on $\mathbb{M}_{\varepsilon_0}$ if $T_y N = D(y)$ for each $y \in N$.*

Theorem 4 (Frobenius). *Let D be an m-dimensional involutive distribution on $\mathbb{M}_{\varepsilon_0}$ and let $y \in \mathbb{M}_{\varepsilon_0}$. Then there exists an integral manifold of D passing through y. Indeed, there exists a cubic coordinate system (U,φ) which is centered at y, with coordinate functions $x_1,\dots,x_n$ such that the slices*

$$x_i = const, \quad \text{for all } m+1 \le i \le n$$

are integral manifolds of D; and if $N \subset U$ is a connected integral manifold of D, then N lies in one of these slices.

Remark 1. It can be proved (see [32]) that if the integral manifolds exist in the sense of the Frobenius theorem then the distribution is involutive, i.e. this is not only a sufficient but also a necessary condition.

Let us now introduce the notion of a "parallel manifold" to M. We define it as an integral manifold of the "parallel" distribution D defined by $D(y) = T_{\pi(y)}M$. Or, equivalently,

Definition 14. *An m-dimensional manifold $N \subset \mathbb{M}_{\varepsilon_0}$ is parallel to M if $T_yN = T_{\pi(y)}M$ for each $y \in N$.*

Lemma 21. *The parallel distribution D is involutive if and only if the normal bundle of M is flat.*

Proof. Let $u, w \in D$ be two vector fields. Let us find necessary and sufficient conditions for the Lie-bracket $[u, w]$ to belong to D.

Let $a \in \mathbb{M}_{\varepsilon_0}$. There are k linear independent vector fields $l_1, \dots, l_s$ in $V = \pi^{-1}(U)$, where U is a neighborhood of the point $\pi(a)$ such that

$$D(y) = \langle l_1(y), \dots, l_k(y)\rangle^{\perp}, \text{ for all } y \in V.$$

Since $u, w \in D$ we have

$$\langle u, l_s\rangle = 0 \text{ and } \langle w, l_s\rangle = 0, \text{ for all } s.$$

Differentiating these equalities with respect to any orthogonal coordinates in U, we obtain

$$\langle \partial_j u\,, l_s\rangle + \langle u\,, \partial_j l_s\rangle = 0 \text{ and} \tag{4.10}$$
$$\langle \partial_j w, l_s\rangle + \langle w, \partial_j l_s\rangle = 0, \text{ for all } j, s. \tag{4.11}$$

The Lie-bracket is given in local coordinates by

$$[u, w] = (u^j\partial_j w^i - w^j\partial_j u^i)\partial_i$$

(see, for example, [11]). Hence by (4.10) and (4.11) we obtain

$$\begin{aligned}\langle [u, w], l_s\rangle &= u^j\langle \partial_j w, l_s\rangle - w^j\langle \partial_j u, l_s\rangle \\ &= -u^j\langle w, \partial_j l_s\rangle + w^j\langle u, \partial_j l_s\rangle \\ &= -u^j w^i\partial_j l_s^i + u^i w^j\partial_j l_s^i \\ &= u^i w^j(\partial_j l_s^i - \partial_i l_s^j).\end{aligned}$$

Now consider a special coordinate system (y_i) corresponding to a. Notice that in this system the last k coordinates of $u(a)$ and $w(a)$ are equal to zero and the first m coordinates run over $\mathbb{R}^m$ for an appropriate choice of the vector fields u and w. Hence the condition

$$\langle [u, w], l_s\rangle(a) = 0$$

is equivalent to the condition

$$(\partial_j l_s^i - \partial_i l_s^j)(a) = 0,$$

for all $1 \le i, j \le m$,$1 \le s \le k$. This means that the matrices L_s defined by

$$(L_s)_{ij} = \partial_j l_s^i(a)$$

must be symmetrical. Let us compute them explicitly.

Notice that we can take $l_s = \nabla\varphi_s \circ \pi$. Then, using Lemma 1, we obtain

$$L_s = [\text{Hess}\varphi_s(0)D\bar{\pi}(0, z)]_{m\times m} = -F_s[I - z^pF_p]^{-1}.$$

Recall that the product of two symmetrical matrices is symmetrical if and only if they can be diagonalized simultaneously. Hence this holds for the matrices F_s and $[I - z^pF_p]^{-1}$, for all z. This holds if and only if all F_s commute. Now we obtain using Lemma 3 that the necessary and sufficient condition for all L_s to be symmetrical (and hence for the parallel distribution to be involutive) is that there is a basis of $T_{\pi(a)}M$ which diagonalizes all of l_ν simultaneously, for all a. On the other hand, the last condition is equivalent to the assumption that the normal bundle of M is flat (see [6, p.137]). □

The next theorem follows immediately from the Frobenius theorem, Remark 1, and the previous lemma.

Theorem 5. *The following two conditions are equivalent*

a) For any $a \in \mathbb{M}_{\varepsilon_0}$, there exists a manifold M_a parallel to M passing through a. Indeed, there exists a cubic coordinate system (U, φ) which is centered at a, with coordinate functions $x_1, \dots, x_n$ such that the slices

$$x_i = const, \quad \textit{for all } i \in m+1, \dots, n$$

are manifolds parallel to M; and if $N \subset U$ is a connected manifold parallel to M, then N lies in one of these slices.

b) The normal bundle of M is flat.

4.2.2 Decomposition of the Laplace operator

For $a \in \mathbb{M}_{\varepsilon_0}$, let N_aM denote the normal space $N_{\pi(a)}M$.

Proposition 6. *Let $u : \mathbb{M}_{\varepsilon_0} \to \mathbb{R}$ be a twice differentiable function and let $a \in \mathbb{M}_{\varepsilon_0}$. Then*

$$(\Delta u)(a) = [\Delta_{M_a} u + \Delta_{N_aM} u](a) + m\langle \kappa, \nabla u\rangle(a),$$

where Δ_{M_a} is the Laplace-Beltrami operator corresponding to M_a and Δ_{N_aM} is the Laplace operator corresponding to N_aM.

Proof. First, let $a \in M$ and let (y_i) be a special coordinate system corresponding to a. Let g be the metric on M induced from $\mathbb{R}^n$. In local coordinates it is given by

$$g_{ij} = \delta_{ij} + \sum_{s=1}^{k} \partial_i f_s \partial_j f_s,$$

where f_s are from the local representation of M at the point a. Denote $g = \det(g_{ij})$. Notice that

$$g_{ij}(0) = g^{ij}(0) = \delta_{ij} \text{ and } g(0) = 1.$$

It follows from the proof of Lemma 4 that

$$\partial_r g_{ij}(0) = 0,\ \partial_r g^{ij}(0) = 0,\ \text{and } \partial_r g(0) = 0.$$

This implies

$$\partial_r (g^{\frac{1}{2}} g^{ij})|_0 = 1/2 g^{-\frac{1}{2}} \partial_r g g^{ij}|_0 + g^{\frac{1}{2}} \partial_r g^{ij}|_0 = 0$$

Now, using the local representation of the Laplace-Beltrami operator,

Lemma 5, and the previous computations, we obtain

$$\begin{aligned}
(\Delta_{M_a} u)(a) &= g(y)^{-\frac{1}{2}} \partial_i (g(y)^{\frac{1}{2}} g^{ij}(y) \partial_j u(y, f(y)))|_0 \\
&= \partial_i (g^{\frac{1}{2}} g^{ij})|_0 \partial_j u(y, f(y))|_0 + \delta^{ij} \partial_{ij} u(y, f(y))|_0 \\
&= \delta^{ij} \partial_i \left(u_j(y, f(y)) + u_{m+s}(y, f(y)) \partial_j f_s(y) \right)|_0 \\
&= \delta^{ij} u_{ij}(a) + u_{i,m+s}(a) \partial_i f_s(0) \\
&+ \partial_{i,m+s} u(a) \partial_i f_s(0) + u_{m+s}(a) \Delta f_s(0) \\
&= \Delta_{T_aM} u(a) + u_{m+s}(a) \mathrm{tr} F_s \\
&= \Delta_{T_aM} u(a) - \langle m\kappa, \nabla u\rangle(a) \\
&= \Delta_{T_aM} u(a) + \langle v, \nabla u\rangle(a),
\end{aligned}$$

where Δ_{T_aM} denotes the Laplace operator corresponding to T_aM and the lower indices of u denote the partial derivatives of u with respect to the corresponding arguments. Finally,

$$\begin{aligned}
(\Delta u)(a) &= \Delta_{T_aM} u(a) + \Delta_{N_aM} u(a) \\
&= (\Delta_{M_a} u)(a) + \Delta_{N_aM} u(a) - \langle v, \nabla u\rangle(a),
\end{aligned}$$

and the proposition is proved for $a \in M$.

For $a \in \mathbb{M}_{\varepsilon_0}$, we apply the same arguments to the manifold M_a and the statement is proved. □

In the next lemma we show that in the case of the embedding $M \subset \mathbb{R}^n$ with the flat normal bundle the vector field $m\kappa$ coincides with the shifting vector v.

Lemma 22. *Let $a \in \mathbb{M}_{\varepsilon_0}$ and $u \in o_a(n)$. Denote the coordinates of $m\kappa$ with respect to the coordinate system (y^i) corresponding to (a, u) by $m\bar{\kappa}$. Then*

$$m\bar{\kappa}(a) = \left(0, \ldots, 0, -\mathrm{tr}(F_1[I - z^p F_p]^{-1}), \ldots, -\mathrm{tr}(F_k[I - z^p F_p]^{-1})\right),$$

where (F_s) and z are from the local representation corresponding to the pair (a, u). On other words, the vector field $m\kappa$ is equal to the shifting vector field v.

Proof. The first m coordinates of $m\bar{\kappa}$ are equal to zero by the definition of the mean curvature vector and by the definition of special coordinates.

Further, let (φ_s) (respectively, (φ_s^a)) be from the local representation of M (respectively, of M_a). Then we have

$$\begin{aligned}\Delta\varphi_s^a &= \sum_{i=1}^{n} \partial_{ii}\varphi_s^a = \sum_{i=1}^{m} \partial_i[\partial_i\varphi_s^a] \\ &= \sum_{i=1}^{m} \partial_i[(\partial_i\varphi_s)\circ\pi] = \sum_{i=1}^{m} [(\partial_{ij}\varphi_s)\circ\pi]\partial_i\pi^j\end{aligned}$$

and therefore

$$\Delta\varphi_s^a(0) = [\mathrm{Hess}\varphi_s(0)]_{ij}[D\pi]_{ji}(a) = -\mathrm{tr}(F_s[I - z^p F_p]^{-1}).$$

Now by Lemma 5 we have

$$m\bar{\kappa}_{m+s}(a) = \mathrm{tr}[\mathrm{Hess}\varphi_s^a(0)] = \Delta\varphi_s^a(0) = -\mathrm{tr}(F_s[I - z^p F_p]^{-1}),$$

for all $1 \leq s \leq k$. Comparing this results with Lemma 7, we complete the proof of the assertion. □

Chapter 5

Miscellaneous

5.1 Introduction to general diffusions

In this section we consider the simplest generalization of a two-dimensional Brownian motion. Namely, we consider a two-dimensional process (c_t) such that its components are independent scaled Brownian motions starting at zero with different variances $\sigma_1 \neq \sigma_2$ (we will call such a process two-dimensional scaled Brownian motion).

Let M be a straight line in $\mathbb{R}^2$ passing through the origin

$$M = \left\{(x, y) \in \mathbb{R}^2 : y = x \tan(\alpha)\right\},$$

where $-\pi/2 < \alpha < \pi/2$. Though M is not compact we can define the surface measures $\mathbb{S}_1$ and $\mathbb{S}_2$ on the paths in M in the same way as before. It turns out that the surface measures exist and are the laws of scaled Brownian motions on M (with scaling parameter depending on σ_1, σ_2, and α and different for $\mathbb{S}_1$ and $\mathbb{S}_2$).

First, let $A : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear mapping given by the matrix

$$A = \begin{pmatrix} \sigma_1^{-1} & 0 \\ 0 & \sigma_2^{-1} \end{pmatrix},$$

which is chosen in such a way that the process (Ac_t) becomes a standard one-dimensional Brownian motion. It is easy to see that the

image of the manifold M under the mapping A is given by

$$AM = \left\{(x, y) \in \mathbb{R}^2 : y = x\frac{\sigma_1}{\sigma_2}\tan(\alpha)\right\}$$

and the image of the tubular ε-neighbourhood $\mathbb{M}_\varepsilon$ is the $k(\sigma_1, \sigma_2, \alpha)\varepsilon$-neighborhood of AM in $\mathbb{R}^2$, where the coefficient $k(\sigma_1, \sigma_2, \alpha)$ is independent of ε and can be computed explicitly.

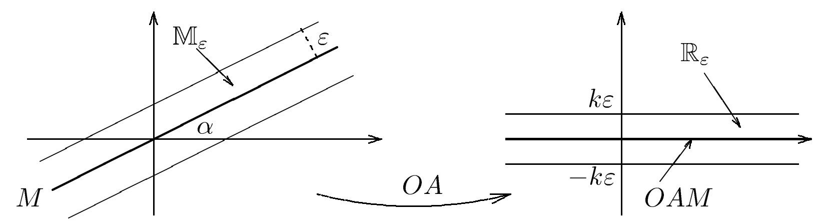

Figure 1: Mapping OA

Further, let $O : \mathbb{R}^2 \to \mathbb{R}^2$ be an orthogonal mapping given by the matrix

$$O = \begin{pmatrix} \cos(\beta) & \sin(\beta) \\ -\sin(\beta) & \cos(\beta) \end{pmatrix},$$

where the angle β is chosen in such a way that

$$OAM = \{(x, 0) : x \in \mathbb{R}\}.$$

This means that

$$\tan(\beta) = \frac{\sigma_1}{\sigma_2}\tan(\alpha). \tag{5.1}$$

Let us also compute

$$\sin^2(\beta) = \frac{1}{1+\cot^2(\beta)} = \frac{1}{1+\cot^2(\alpha)\sigma_2^2/\sigma_1^2} = \frac{\sigma_1^2\sin^2(\alpha)}{\sigma_1^2\sin^2(\alpha)+\sigma_2^2\cos^2(\alpha)}$$

$$\cos^2(\beta) = \frac{1}{1+\tan^2(\beta)} = \frac{1}{1+\tan^2(\alpha)\sigma_1^2/\sigma_2^2} = \frac{\sigma_2^2\cos^2(\alpha)}{\sigma_1^2\sin^2(\alpha)+\sigma_2^2\cos^2(\alpha)}$$

Notice also that the image $OA\mathbb{M}_\varepsilon$ is the $k(\sigma_1,\sigma_2,\alpha)\varepsilon$-neighborhood of the x-axis and the process (OAc_t) is a standard two-dimensional Brownian motion.

Further, for any continuous linear mapping $U:\mathbb{R}^2\to\mathbb{R}^2$ we denote by $[U]:C(\mathbb{R}^2)\to C(\mathbb{R}^2)$ the mapping

$$[U]:\omega\mapsto\{t\mapsto U\omega_t\}.$$

For any measure μ on $C(\mathbb{R}^2)$ we denote by $[U]\circ\mu$ the image measure of μ under the mapping $[U]$.

Lemma 23. *Let (b_t) be a one-dimensional scaled Brownian motion with variance σ_0. Then the process $(A^{-1}O^{-1}(b_t,0)^T)$ is a scaled Brownian motion on M with variance*

$$\sigma = \frac{\sigma_1\sigma_2}{\sqrt{\sigma_1^2\sin^2(\alpha)+\sigma_2^2\cos^2(\alpha)}}\sigma_0.$$

Proof. Since (b_t) is a scaled Brownian motion with variance σ_0 it can be written as $b_t=\sigma_0 b_t^0$, where b_t^0 is a standard one-dimensional Brownian motion. We consider the natural coordinate (x) on M defined by the distance to the point $(0,0)$ taken with the plus sign for the right row and with the minus sign for the left row. By the construction $(A^{-1}O^{-1}(b_t,0)^T)$ is an M-valued continuous process and in natural

coordinates it has the form

$$x_t = \frac{(A^{-1}O^{-1}(b_t,0)^T)_1}{\cos(\alpha)} = \frac{\sigma_1\cos(\beta)}{\cos(\alpha)}b_t = \frac{\sigma_1\sigma_2}{\sqrt{\sigma_1^2\sin^2(\alpha)+\sigma_2^2\cos^2(\alpha)}}\sigma_0 b_t^0,$$

which completes the proof. □

Proposition 7. $\mathbb{S}_1$ *is the law of a scaled Brownian motion on M starting at zero with variance*

$$\sigma = \frac{\sigma_1\sigma_2}{\sqrt{\sigma_1^2\sin^2(\alpha)+\sigma_2^2\cos^2(\alpha)}}. \tag{5.2}$$

Proof. Denote by ν the law of the scaled two-dimensional Brownian motion (c_t). Then by definition

$$\mathbb{S}_1 = (\text{weak})\lim_{\varepsilon\to0}\frac{\nu|_{C(\mathbb{M}_\varepsilon)}}{\nu(C(\mathbb{M}_\varepsilon))} = [A^{-1}O^{-1}]\circ(\text{weak})\lim_{\varepsilon\to0}\frac{[OA]\circ\nu|_{C(\mathbb{R}_{k\varepsilon})}}{[OA]\circ\nu(C(\mathbb{R}_{k\varepsilon}))},$$

where $\mathbb{R}_{k\varepsilon}$ is the $k(\sigma_1,\sigma_1,\alpha)\varepsilon$-tubular neighborhood of the real axis in $\mathbb{R}^2$. As it was already mentioned the process (OAc_t) is a standard two-dimensional Brownian motion and hence $[OA]\circ\nu$ is just the Wiener measure on $C(\mathbb{R}^2)$. By Theorem 1 the limit on the right hand side of the last formula exists and the limit measure corresponds to a standard Brownian motion b_t^0 on the real axis. It remains to apply Lemma 23, which implies that the measure $\mathbb{S}_1$ is the law of the process $(A^{-1}O^{-1}b_t^0)$, which is a Brownian motion on M with variance given by 5.2. □

Proposition 8. $\mathbb{S}_2$ *is the law of a scaled Brownian motion on M starting at zero with variance*

$$\sigma = \sqrt{\sigma_1^2\cos^2(\alpha)+\sigma_2^2\sin^2(\alpha)}. \tag{5.3}$$

Proof. Denote by (c_t^ε) a scaled Brownian motion in $\mathbb{R}^2$ starting at zero with reflection on the boundary $\partial\mathbb{M}_\varepsilon$. Then (c_t^ε) satisfies the Skorokhod equation

$$dc_t^\varepsilon = dc_t + \frac{1}{2}n(c_t^\varepsilon)dl_t^\varepsilon, \tag{5.4}$$

where (l_t^ε) is the local time of the process (c_t^ε) on the boundary $\partial\mathbb{M}_\varepsilon$ and $n(y)$ is the inward pointing unit normal vector at $y \in \partial\mathbb{M}_\varepsilon$. Notice that

$$n(y) = \mathrm{sgn}(y_2 - y_1\tan(\alpha))(\sin(\alpha), \cos(\alpha))^T.$$

Denote by ν_ε the law of the process (c_t^ε). By definition we have

$$\mathbb{S}_2 = (\text{weak}) \lim_{\varepsilon\to 0} \nu_\varepsilon = [A^{-1}O^{-1}](\text{weak}) \circ \lim_{\varepsilon\to 0} OA \circ \nu_\varepsilon.$$

Denote $b_t = OAc_t$ and $b_t^\varepsilon = OAc_t^\varepsilon$ and recall that (b_t) is a standard two-dimensional Brownian motion. Further, it follows from (5.4) that the process (b_t^ε) solves the stochastic differential equation

$$db_t^\varepsilon = db_t + \frac{1}{2}OA\bar{n}(b_t^\varepsilon)dl_t^\varepsilon, \tag{5.5}$$

where $\bar{n}(y) = n(A^{-1}O^{-1}y)$.

Denote by (x_t^ε) and (z_t^ε) the first and the second coordinate of (b_t^ε), respectively. By construction, the process (b_t^ε) takes values in the $k(\sigma_1, \sigma_2, \alpha)\varepsilon$-neighborhood of the real axis in $\mathbb{R}^2$ and hence

$$|z_t^\varepsilon| \leq k(\sigma_1, \sigma_2, \alpha)\varepsilon \text{ for all } t.$$

Notice that

$$OA\bar{n}(b_t^\varepsilon) = \mathrm{sgn}(z_t^\varepsilon)(n_1, n_2)^T,$$

where

$$n_1 = \frac{\sin(\alpha)\cos(\beta)}{\sigma_1} - \frac{\cos(\alpha)\sin(\beta)}{\sigma_2} \tag{5.6}$$

$$n_2 = -\frac{\sin(\alpha)\sin(\beta)}{\sigma_1} - \frac{\cos(\alpha)\cos(\beta)}{\sigma_2}. \tag{5.7}$$

Further, it follows from (5.5) that

$$z_t^\varepsilon = b_t'' + \frac{1}{2}n_2 \int_0^t \mathrm{sgn}(z_s^\varepsilon)dl_s^\varepsilon$$

and hence

$$x_t^\varepsilon = b_t' + \frac{1}{2}n_1 \int_0^t \mathrm{sgn}(z_s^\varepsilon)dl_s^\varepsilon = b_t' + \frac{n_1}{n_2}(z_t^\varepsilon - b_t''),$$

where $b_t = (b_t', b_t'')$.

It is easy to see now that the processes (x_t^ε) converge locally uniformly in probability to the process

$$\bar{x}_t = b_t' - \frac{n_1}{n_2}b_t'',$$

which is a scaled Brownian motion since (b_t') and (b_t'') are independent standard Brownian motions. Moreover, we can compute the variance of $(\bar{x}_t)$ using (5.7), (5.1), and the relation $||OAn|| = ||An||$. We have

$$\begin{aligned}
\mathrm{var}(\bar{x}_t) &= \frac{\sqrt{n_1^2 + n_2^2}}{n_2} \\
&= \sqrt{\frac{\sin^2(\alpha)}{\sigma_1^2} + \frac{\cos^2(\alpha)}{\sigma_2^2}} : \left[\frac{\sin(\alpha)\sin(\beta)}{\sigma_1} + \frac{\cos(\alpha)\cos(\beta)}{\sigma_2}\right] \\
&= \frac{\sqrt{\sigma_1^2\cos^2(\alpha) + \sigma_2^2\sin^2(\alpha)}\sqrt{\sigma_1^2\sin^2(\alpha) + \sigma_2^2\cos^2(\alpha)}}{\sigma_1\sigma_2}.
\end{aligned}$$

Finally, by Lemma 23 the process $(A^{-1}O^{-1}\bar{x}_t)$ is a scaled Brownian motion on M with variance

$$\begin{aligned}
\sigma &= \frac{\sigma_1\sigma_2}{\sqrt{\sigma_1^2\sin^2(\alpha) + \sigma_2^2\cos^2(\alpha)}}\mathrm{var}(\bar{x}_t) \\
&= \sqrt{\sigma_1^2\cos^2(\alpha) + \sigma_2^2\sin^2(\alpha)}.
\end{aligned}$$

□

5.2 Introduction to non-smooth manifolds

In the previous chapters we studied how the surface measures on smooth compact manifolds look like. Let us now discuss manifolds with singular points. We would like to consider just a right angle $\{(x,y) \in \mathbb{R}^2 : y = |x|\}$ but in order to be precise we need to use only compact manifolds. Therefore let M be the smooth manifold with the only singular point $(0,0)$ like on Picture 2.

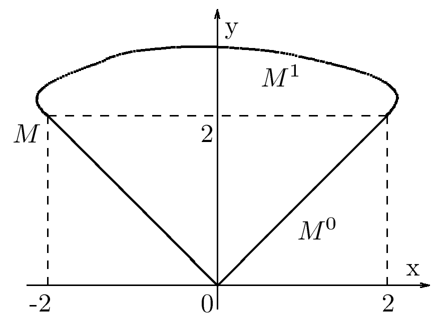

Figure 2: Manifold M

We denote the "lower" part of M corresponding to the right angle by M^0 and the remaining part by M^1.

Let $\varphi : [-2,2] \to \mathbb{R}$ be a smooth function approximating $y = |x|$ like on Picture 3. Let, further,

$$\varphi_\varepsilon(x) = \begin{cases} \varepsilon\varphi(\frac{x}{\varepsilon}) & , \quad x \in [-\varepsilon, \varepsilon] \\ |x| & , \quad \text{otherwise,} \end{cases}$$

and let

$$M_\varepsilon^0 = \{(x,y) \in \mathbb{R}^2 : y = \varphi_\varepsilon(x), |x| \le 2\}.$$

Then $M_\varepsilon = M_\varepsilon^0 \cup M^1$ is a smooth manifold approximating M. For any set $A \in \mathbb{R}^2$, we denote by $C(A)$ the set $C_{(1,1)}([0,1], A)$.

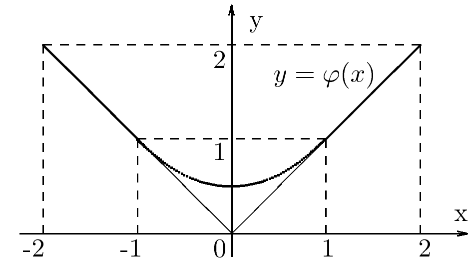

Figure 3: Approximating function φ

For every $\varepsilon > 0$, M_ε is a smooth compact Riemannian manifold embedded into $\mathbb{R}^2$ and hence by Theorem 1 there exists the surface measure η_ε of the first type on $C(M_\varepsilon)$ corresponding to a two-dimensional Brownian motion starting at $(1,1)$. The natural question we are interested in is the weak convergence of the family η_ε and properties of the limit measure η (if it exists).

Lemma 24. *For any $\delta > 0$*

$$\eta_\varepsilon\{\omega \in C(\mathbb{R}^2) : \inf_{0 \le t \le 1} |\omega(t)| > \delta\} \to 0 \ \textit{as} \ \varepsilon \to 0.$$

Proof. Denote by ν_ε the Wiener measure on M_ε corresponding to a Brownian motion on M_ε starting at $(1,1)$. Let δ be given. It follows from Theorem 1 that η_ε is absolutely continuous with respect to ν_ε and the density is given by

$$\rho_\varepsilon(\omega) = \frac{\exp \frac{1}{8}\int_0^1 \kappa_\varepsilon^2(\omega_t)dt}{\mathbb{E}_{\nu_\varepsilon} \exp \frac{1}{8}\int_0^1 \kappa_\varepsilon^2(\omega_t)dt},$$

where κ_ε is the mean curvature of M_ε.

First, let us prove that the denominator converges to infinity as $\varepsilon \to 0$. In fact, let $\alpha \in (0,1)$; then, for every $x \in [-\alpha\varepsilon, \alpha\varepsilon]$,

$$\begin{aligned} |\kappa_\varepsilon(x, \varphi_\varepsilon(x))| &= \left| \frac{\varphi''_\varepsilon(x)}{\sqrt{1+\varphi'_\varepsilon(x)^2}^3} \right| \\ &= \left| \frac{\frac{1}{\varepsilon}\varphi''(\frac{x}{\varepsilon})}{\sqrt{1+\varphi'(\frac{x}{\varepsilon})^2}^3} \right| \\ &\geq \frac{1}{\varepsilon\sqrt{2}^3} \inf_{-\alpha\leq x\leq\alpha} |\varphi''(x)| \end{aligned}$$

and hence

$$\frac{1}{8}\kappa_\varepsilon^2(x, \varphi_\varepsilon(x)) \geq \frac{1}{8\varepsilon^2\sqrt{4}^3} \inf_{-\alpha\leq x\leq\alpha} |\varphi''(x)|^2 = \frac{c_\alpha}{\varepsilon^2}.$$

Further, let T_A (respectively, $T_A^{(2)}$) be the time spent by a one-dimensional (respectively, two-dimensional) Brownian motion in the set A and denote by $\mathbb{W}$ the usual Wiener measure on $C_0(\mathbb{R})$. Then

$$\begin{aligned} &\mathbb{E}_{\nu_\varepsilon} \exp \frac{1}{8}\int_0^1 \kappa_\varepsilon^2(\omega_t)dt \\ &\geq \mathbb{E}_{\nu_\varepsilon} \exp\left(\frac{c_\alpha}{\varepsilon^2} T^{(2)}_{\{(x,y):x\in[-\varepsilon\alpha,\varepsilon\alpha],y=\varphi_\varepsilon(x)\}}\right) \\ &= \mathbb{E}_{\mathbb{W}} \exp\left(\frac{c_\alpha}{\varepsilon^2} T_{[(1-\varepsilon)\sqrt{2}+l_\varepsilon-d_\varepsilon,(1-\varepsilon)\sqrt{2}+l_\varepsilon+d_\varepsilon]}\right) \\ &= \mathbb{E}_{\mathbb{W}} \exp\left(\frac{c_\alpha}{\varepsilon^2} T_{[\sqrt{2}-\varepsilon(\sqrt{2}-l_1+d_1),\sqrt{2}+\varepsilon(l_1+d_1-\sqrt{2})]}\right), \end{aligned} \tag{5.8}$$

where (see Picture 4)

$$\begin{aligned} l_\varepsilon &= \int_0^\varepsilon \sqrt{1+\varphi'_\varepsilon(x)^2}dx = \int_0^\varepsilon \sqrt{1+\varphi'(\frac{x}{\varepsilon})^2}dx \\ &= \varepsilon\int_0^1 \sqrt{1+\varphi'(y)^2}dy = \varepsilon l_1 \end{aligned}$$

and analogously $d_\varepsilon = \varepsilon d_1$.

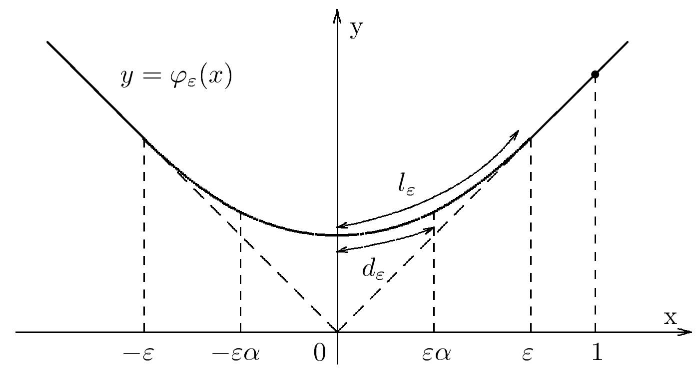

Figure 4: Approximating function φ_ε

Now let us symmetrize the interval in (5.8). Notice that d_1 depends on α and $d_1(\alpha) \to l_1$ as $\alpha \to 1$. If we choose α close enough to l_1 then $\sqrt{2} - l_1 + d_1$ becomes close to $\sqrt{2} > 1/2$ and $l_1 + d_1 - \sqrt{2}$ becomes close to $2l_1 - \sqrt{2} > 2 - \sqrt{2} > 1/2$. Precisely, there exists $\alpha \in (0,1)$ such that

$$\sqrt{2} - l_1 + d_1 > 1/2 \text{ and } l_1 + d_1 - \sqrt{2} > 1/2.$$

Then we have

$$[\sqrt{2} - \varepsilon(\sqrt{2} - l_1 + d_1), \sqrt{2} + \varepsilon(l_1 + d_1 - \sqrt{2})] \supset [\sqrt{2} - \varepsilon/2, \sqrt{2} + \varepsilon/2]$$

and we can continue the chain of inequalities (5.8)

$$\begin{aligned} &\mathbb{E}_W \exp\left(\frac{c_\alpha}{\varepsilon^2} T_{[\sqrt{2}-\varepsilon(\sqrt{2}-l_1+d_1),\sqrt{2}+\varepsilon(l_1+d_1-\sqrt{2})]}\right) \\ &\geq \mathbb{E}_{\mathbb{W}} \exp\left(\frac{c_\alpha}{\varepsilon^2} T_{[\sqrt{2}-\varepsilon/2,\sqrt{2}+\varepsilon/2]}\right). \end{aligned} \tag{5.9}$$

According to the properties of a local time

$$\frac{1}{\varepsilon} T_{[\sqrt{2}-\varepsilon/2,\sqrt{2}+\varepsilon/2]} = \frac{1}{\varepsilon}\int_{\sqrt{2}-\varepsilon/2}^{\sqrt{2}+\varepsilon/2} l(1,x)dx \to l(1,\sqrt{2}) \text{ a.s.},$$

where $l(t,x)$ is the occupation density of the Brownian motion, which is finite. This implies that the expectation in the formula (5.9) converges to infinity. In fact, let $\Delta > 0$ and $\bar{\delta}$ be such that

$$\mathbb{W}\{\omega : l(1,\sqrt{2}) > \Delta\} = \bar{\delta} > 0.$$

Now by Egorov's Theorem (see [16]) there exists a set $E_{\bar\delta}$ such that $\mathbb{W}(E_{\bar\delta}) > 1-\bar\delta/2$ and

$$\frac{1}{\varepsilon}T_{[\sqrt{2}-\varepsilon/2,\sqrt{2}+\varepsilon/2]} \to l(1,\sqrt{2}) \text{ uniformly on } E_{\bar\delta}.$$

Notice that by the choice of $\bar\delta$ and $E_{\bar\delta}$ we have

$$\mathbb{W}(E_{\bar\delta}\cap\{\omega : l(1,\sqrt{2}) > \Delta\}) \geq \bar\delta/2.$$

Due to the uniform convergence $\exists\varepsilon_0 > 0$ s.th. $\forall\varepsilon < \varepsilon_0$

$$\left|\frac{1}{\varepsilon}T_{[\sqrt{2}-\varepsilon/2,\sqrt{2}+\varepsilon/2]} - l(1,\sqrt{2})\right| < \frac{\Delta}{2}$$

on $E_{\bar\delta}\cap\{\omega : l(1,\sqrt{2}) > \Delta\}$ and hence

$$\frac{1}{\varepsilon}T_{[\sqrt{2}-\varepsilon/2,\sqrt{2}+\varepsilon/2]} \geq \frac{\Delta}{2}$$

on this set. Finally, we obtain

$$\begin{aligned}
&\mathbb{E}_{\mathbb{W}}\exp\left(\frac{c_\alpha}{\varepsilon^2}T_{[\sqrt{2}-\varepsilon/2,\sqrt{2}+\varepsilon/2]}\right)\\
&\geq \mathbb{E}_{\mathbb{W}}\mathbf{1}_{E_{\bar\delta}\cap\{\omega:l(1,\sqrt{2})>\Delta\}}\exp\left(\frac{c_\alpha}{\varepsilon^2}T_{[\sqrt{2}-\varepsilon/2,\sqrt{2}+\varepsilon/2]}\right)d\mathbb{W}\\
&\geq \mathbb{W}(E_{\bar\delta}\cap\{\omega : l(1,\sqrt{2}) > \Delta\})\exp\left(\frac{c_\alpha\Delta}{2\varepsilon}\right)\\
&\geq \frac{\bar\delta}{2}\exp\left(\frac{c_\alpha\Delta}{2\varepsilon}\right) \to \infty \text{ as } \varepsilon\to 0
\end{aligned}$$

since Δ, $\bar\delta$, and c_α are independent of ε.

Let us now prove the assertion of the lemma. Notice that we have for $\varepsilon < \delta/\sqrt{2}$

$$\rho_\varepsilon(\omega) \leq \frac{c}{\mathbb{E}_{\nu_\varepsilon}\exp\frac{1}{8}\int_0^1\kappa_\varepsilon^2(\omega_t)dt},$$

where

$$c = \exp\frac{1}{8}\sup_{x\in M^1}\kappa^2(x) < \infty$$

for all paths that do not hit the ball of radius δ centered at the origin. Now it easy to see that

$$\begin{aligned}
&\eta_\varepsilon\{\omega\in C(\mathbb{R}^2) : \inf_{0\leq t\leq 1}|\omega_t| > \delta\}\\
&= \mathbb{E}_{\nu_\varepsilon}\mathbf{1}_{\{\omega\in C(\mathbb{R}^2):\inf_{0\leq t\leq 1}|\omega_t|>\delta\}}\rho_\varepsilon(\omega)\\
&\leq \frac{c}{\mathbb{E}_{\nu_\varepsilon}\exp\frac{1}{8}\int_0^1\kappa_\varepsilon^2(\omega_t)dt} \to 0 \text{ as } \varepsilon\to 0.
\end{aligned}$$

and the statement is proved □

Proposition 9. *Suppose that the weak limit η of η_ε exists. Then, with η-probability 1, the limiting process visits the origin before time 1. More precisely,*

$$\eta\{\omega\in C(\mathbb{R}^2) : \exists t\in[0,1] \ s.th. \ \omega_t = (0,0)\} = 1.$$

Proof. For arbitrary $\delta > 0$, define $\varphi_\delta : \mathbb{R}\to\mathbb{R}$ by

$$\varphi_\delta(x) = \begin{cases} 1, & |x|\geq\delta,\\ 0, & |x|\leq\delta/2,\\ \text{smoothly between 0 and 1.} \end{cases}$$

Define $\Phi_\delta : C(\mathbb{R}^2)\to\mathbb{R}$ by

$$\Phi_\delta(\omega) = \varphi_\delta(\inf_{0\leq t\leq 1}|\omega_t|).$$

Obviously, Φ_δ is bounded and continuous. By definition of weak convergence and due to Lemma 24

$$\begin{aligned}
0 \leq \mathbb{E}_\eta\Phi_\delta(\omega) &= \lim_{\varepsilon\to 0}\mathbb{E}_{\eta_\varepsilon}\Phi_\delta(\omega)\\
&\leq \lim_{\varepsilon\to 0}\eta_\varepsilon\{\omega\in C(\mathbb{R}^2) : \inf_{0\leq t\leq 1}|\omega_t| > \delta/2\} = 0.
\end{aligned}$$

So we obtain

$$\mathbb{E}_\eta\Phi_\delta(\omega)d\eta = 0$$

and since $\Phi_\delta\geq 0$ we get also $\eta\{\omega : \Phi_\delta(\omega) > 0\} = 0$. This implies

$$\eta\{\omega\in C(\mathbb{R}^2) : \inf_{0\leq t\leq 1}|\omega_t|\geq\delta\} = 0$$

and, finally,

$$\eta\{\omega \in C(\mathbb{R}^2) : \exists t \in [0,1] \text{ s.th. } \omega_t = (0,0)\}$$
$$= \eta\{\bigcap_{n=1}^{\infty} \{\omega \in C(\mathbb{R}^2) : \inf_{0 \le t \le 1} |\omega_t| < \frac{1}{n}\}\} = 1$$

completes the proof. □

List of Symbols

General notations:

$\langle\cdot,\cdot\rangle$	scalar product in $\mathbb{R}^n$
$\|\|\cdot\|\|_\infty$	supremum-norm on spaces of continuous functions
Δ, ∇, div, tr	Laplacian in $\mathbb{R}^n$, gradient, divergence, trace
Hess	Hessian
I	unit matrix
δ_{ij}, δ_i^j	Kroneker's symbols
$\perp$	orthogonal

Notations from stochastic:

$\mathcal{L}$, $\mathcal{L}(\cdot\|\cdot)$	distribution, conditional distribution
$\mathbb{E}$, $\mathbb{E}[\,\cdot\,\|\,\cdot\,]$	expectation, conditional expectation
d, δ	Itô and Stratonovich stochastic differentials
τ	exit time of $\mathbb{M}_{\varepsilon_0}$

Notations from differential geometry:

M	compact Riemannian manifold in $\mathbb{R}^n$
$\mathbb{M}_\varepsilon$	tubular ε-neighbourhood of M in $\mathbb{R}^n$
a_0	fixed point in M
(e_i)	orthonormal basis in $\mathbb{R}^n$
m, k	$\dim M$, codim$M$
R, κ, σ	scalar curvature, mean curvature vector, tension field
l_ν	second fundamental form in the direction ν
Γ	connection
$[u,w]$	Lie-Bracket of vector fields u and w
M_a	parallel manifold through a
$n(\cdot)$	inward pointing normal vector on $\partial\mathbb{M}_\varepsilon$

T_aM, N_aM	tangent and normal spaces to M at a
NM	normal bundle of M in $\mathbb{R}^n$
P_a, Q_a	orthogonal projectors from $\mathbb{R}^n$ to T_aM and to N_aM
π	orthogonal projection from $\mathbb{M}_{\varepsilon_0}$ to M
ε_0	positive number such that π is well-defined
D	Distribution
Δ_M	Laplace-Beltrami operator in M
v	shifting vector field (ex. S. 4.1), potential (in S. 4.1)

Sets:

$C_a([0,T],A)$	$=\{f:[0,T]\to A: f(0)=a\}$
$C(A)$	$=C_{a_0}([0,1],A)$
$gl(n)$	space of $n\times n$ real matrices
$o(n)$	set of $n\times n$ orthogonal matrices
$o_a(n)$	set of tangent $n\times n$ matrices at a
$B(\varepsilon)$	ball of radius ε in $\mathbb{R}^k$ centered at zero
$S_a(A)$	space of continuous A-valued semimartingales starting at $a\in A$

Stochastic processes:

(b_t)	n-dimensional Brownian motion
(y_t)	shifted process (except S. 4.1), limit process (in S. 4.1)
(y_t^T)	the process corresponding to the measure $\mathbb{S}_1^T$
(ω_t)	coordinate process
(c_t)	two-dimensional scaled Brownian motion
(r_t^ε)	reflected Brownian motion with reflection on $\partial\mathbb{M}_\varepsilon$
(l_t^ε)	local time of (r_t^ε) on $\partial\mathbb{M}_\varepsilon$
(u_t)	translation matrix
(x_t), (z_t)	first and second components of Fermi decomposition

Measures:

$\mathbb{W}$, $\mathbb{W}^T$	Wiener measures on $C_{a_0}([0,1],\mathbb{R}^n)$, $C_{a_0}([0,T],\mathbb{R}^n)$
$\mathbb{W}_M$, $\mathbb{W}_M^T$	Wiener measure on $C_{a_0}([0,1],M)$, $C_{a_0}([0,T],M)$
$\mathbb{S}_1$, $\mathbb{S}_2$	surface measure of the first and of the second type
$\mathbb{S}_1^T$, $\mathbb{S}_2^T$	surface measures of the 1st and 2nd t. up to time T
μ	the law of the shifted process (y_t)
$\mathbb{W}_\varepsilon^r$	the law of the process (r_t^ε)

$\mathbb{W}_\varepsilon$, $\mathbb{W}_\varepsilon^T$, μ_ε	conditioned measures corresponding to $\mathbb{W}$, $\mathbb{W}^T$, μ
$\lambda_{\mathbb{R}^n}$, λ_M	Lebesgue measures on $\mathbb{R}^n$ and M
$\lambda_\oplus$	reference measure on $\mathbb{M}_{\varepsilon_0}$

Special functions:

ρ, $\bar\rho$	density of $\mathbb{W}$ with respect to μ and its approximation
pr_1, pr_2	projection of $\mathbb{R}^n$ to its first m (last k) coordinates
pr_1^{-1}	right inverse of pr_1 s.t. $\mathrm{pr}_1^{-1}\mathrm{pr}_1 = P_{a_0}$
pr_2^{-1}	right inverse of pr_2 s.t. $\mathrm{pr}_2^{-1}\mathrm{pr}_2 = Q_{a_0}$
$O(\varepsilon)$	type of approximation
(f_s), (φ_s)	local description of M
(F_s)	local description of the second fundamental form of M
φ	positive eigenfunction of $\frac{1}{2}\Delta_M + v$
p^T, p^∞	transition probability of (y_t^T) and (y_t)

Index

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