Resolutions and Moduli for Equivariant Sheaves over Toric Varieties

Markus Perling

Vom Fachbereich Mathematik der Universität Kaiserslautern
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1. Gutachter: Prof. Dr. Günther Trautmann, Universität Kaiserslautern

2. Gutachter: Prof. Dr. Klaus Altmann, Freie Universität Berlin

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CHAPTER 1

Introduction

1.1. Algebraic Groups, Equivariant Sheaves and Toric Varieties

The study of equivariant sheaves over toric varieties is one of those subjects which lie on the cross-roads of Geometry, Algebra and Combinatorics. The reason for this is the interplay of symmetry and geometry. In algebraic geometry, the most classical examples for this phenomenon are given by homogeneous spaces, such as Grassmannians, flag varieties, and more general quotients of algebraic groups by subgroups. Let us stay for a moment with these examples before we turn to toric varieties. The distinguishing property of a homogeneous space is that it is endowed with a transitive group action

$$\sigma : G \times X \longrightarrow X.$$

This action links the geometry of X with the representation theory of G and this way it opens the study of X for algebraic or combinatorial methods. It turns out that many vector bundles over X, such as the various tautological bundles, tangent and cotangent bundles and their tensor powers, are homogeneous. This means that they allow a continuation of the action of G to their total space; let $E \longrightarrow X$ be such a bundle together with its projection, then there exists a group action on E which commutes with the projection:

$$\begin{array}{ccc} G \times E & \longrightarrow & E \\ \downarrow & & \downarrow \\ G \times X & \longrightarrow & X \end{array}$$

and which is linear in the fibers of E. Since the classical works of Bott and Kostant ([**Bot57**], [**Kos61**]), homogeneous bundles have been completely described in terms of representation theory of the group G, and these bundles play a pivotal role in the study of vector bundles and sheaves over homogeneous spaces.

Toric varieties are examples for *quasi-homogeneous* varieties. This means that there exists an action of an algebraic group, an algebraic torus T, which has a dense orbit in X, but which in general does not coincide with X. Since their first appearance (see [**Dem70**], [**KKMS73**]), it was clear that the quasi-homogeneity of the torus action is still good enough to endow X with rich combinatorial and algebraic structures. However, if it comes to consider vector bundles, which are compatible with the T-action in some sense, the situation is more complicated. First, we can no longer speak of homogeneous bundles, as X is not a homogeneous space. The analog for such bundles is given by the notion of *equivariant* bundles. This notion comes from the observation that the action of T on the total space of some bundle E can be expressed in the way that for every

$t \in T$ there exists an isomorphism

$$t^* E \cong E$$

if t is considered as automorphism of X. Technically, this definition extends without problems to more general sheaves over X and incorporates, of course, the notion of homogeneous bundles. The next problem is that the more complicated orbit structure of X a priori allows an equivariant bundle to degenerate at certain places into a non-locally free sheaf. And finally, compared with such groups like GL_n, the representation theory of algebraic tori is quite trivial. So we can not expect a complete characterization of an equivariant bundle only in terms of representation theory.

In this work, we want to advocate the following point of view. We do not think of single equivariant sheaves as important, considered as objects in the category of coherent sheaves over a toric variety. Instead, we consider the whole subcategory of equivariant sheaves. Our aim is to develop methods which allow to understand this category as completely as possible. This means, in particular, that we want to find good invariants for equivariant sheaves which allow good descriptions for moduli and for studying degenerations. The distant hope is that the category of equivariant sheaves can play a similar role for the general theory of sheaves over toric varieties, as homogeneous bundles do for sheaves over homogeneous spaces.

So far, equivariant line bundles always have been part of the standard repertoire in the theory of toric varieties (see for instance the text books [**Ful93**], [**Oda88**]), but equivariant bundles of higher rank or more general equivariant sheaves are less known. The first characterization of equivariant bundles, over smooth toric varieties, was done by Kaneyama in [**Kan75**]; as an application, he gave the first classification of equivariant vector bundles of rank two over $\mathbb{P}_2$. His techniques were applied later in the thesis [**Beh86**] (very similar results were published in [**Kan88**]) to the classification of equivariant bundles on projective spaces, and more recently in [**LY00**] where the splitting types of certain bundles over toric surfaces were discussed. These techniques seem quite complicate to handle, and so far it is difficult to find more traces of them in the literature. The first truly general work on equivariant vector bundles was the seminal paper [**Kly90**] of Klyachko. This work contains a nearly complete account on equivariant bundles, discussing global resolutions, computation of cohomology and classification theory. Unfortunately, that paper is rather difficult to read, and although its existence is well known, the common toric geometer seems to avoid it. In the never published preprint [**Kly91**], Klyachko formulates an idea to generalize the constructions, and the results, from [**Kly90**] to more general torsion free sheaves. These two works, [**Kly90**] and [**Kly91**], form the starting point for our own work.

We give a short overview of the literature known to us on equivariant sheaves over toric varieties. Torus equivariant bundles on $\mathbb{P}_n$ were also subject of the work [**BE82**]. In [**BL94**] and [**Lun94**] a characterization of the equivariant derived category over a toric variety was given. Klyachko successfully applied his theory to Horn's conjecture ([**Kly98**], [**Kly02**], [**Ful98a**]). Klyachko's formalism had also an ephemeral episode in theoretical physics ([**KS98**], [**KS99**]), but in this regard, the last word is not yet spoken. It can be shown that exceptional sheaves , that is, simple sheaves $\mathcal{F}$ whose higher Ext-groups vanish, $\mathrm{Ext}^i(\mathcal{F}, \mathcal{F}) = 0$ for $i > 0$, are equivariant, and it would be a great success if the theory of equivariant sheaves could contribute to the classification problem for exceptional sheaves (see [**Zas96**], [**Dou01**]). Some works consider equivariant sheaves in terms of fine-graded modules over homogeneous coordinate rings (see [**BV97**], [**BC94**]), and there were some successes in computing cohomologies for equivariant sheaves this way (see [**Mus02**], [**Mus00**], [**EMS00**]). Equivariant bundles very recently appeared in [**Laf02**]. In the thesis [**Pen02**] a functorial equivalence between mixed Hodge structures and certain stable equivariant bundles on $\mathbb{P}_2$ is established. Some aspects of Klyachko's construction were recently extended to more general group actions in [**Kat02**].

To conclude, except for the fundamental work of Klyachko and some scattered contributions concerning different topics, the general theory of equivariant sheaves over toric varieties is not very developed so far. We hope that with this thesis we can contribute to this theory by giving some technical foundations.

Parts of this thesis have been submitted for publication ([**Per02a**], [**Per02b**]).

1.2. Ideas and Overview of the Results

1.2.1. Δ-Families. The key observation which is the basis for our work, was made by Klyachko ([**Kly90**]). Let $X = X_\Delta$ be a toric variety which is associated to some fan Δ. Consider an equivariant vector bundle $\mathcal{E}$ over X and choose some point $x_\sigma \in X$. The action of the stabilizer T_σ of x_σ over the fiber $\mathcal{E}(x_\sigma) \cong k^r$ then becomes a linear representation of T_σ. We obtain an eigenspace decomposition

$$\mathcal{E}(x_\sigma) = \bigoplus_{\chi \in X(T_\sigma)} \mathcal{E}(x_\sigma)_\chi$$

where T acts on $\mathcal{E}(x_\sigma)_\chi$ by multiplication with the character χ, that is, for every $e \in \mathcal{E}(x_\sigma)_\chi$:

$$t.e = \chi(t) \cdot e.$$

Klyachko proved for equivariant vector bundles the following:

Theorem ([**Kly90**])**:** *The category of equivariant vector bundles over a toric variety X_Δ is equivalent to the category of vector spaces with a family of filtrations $E^\rho(i)$ for each $\rho \in \Delta(1)$ which satisfy the following compatibility condition:*

Let E be a vector space with a family of filtrations

$$0 \subset \cdots \subset E^\rho(i) \subset E^\rho(i+1) \subset \cdots \subset E,$$

then for any $\sigma \in \Delta$ there exists a T_σ-eigenspace decomposition $E = \bigoplus_{\chi \in X(T_\sigma)} E_\chi$ such that

$$E^\rho(i) = \sum_{\langle \chi, n(\rho) \rangle \leq i} E_\chi$$

Here $n(\rho)$ is the primitive lattice vector of the ray ρ, which can be identified with some one-parameter subgroup of T, and $\langle \chi, n(\rho) \rangle \in \mathbb{Z}$ denotes the canonical pairing. Using this characterization, we obtain a formalism for equivariant bundles which makes the theory of equivariant bundles rather tractable, as the many results in [**Kly90**] show.

Our first result generalizes a proposal in [**Kly91**] of how this formalism can be extended to torsion free sheaves, and we obtain a new characterization of general equivariant sheaves over toric varieties. The basic idea is as follows. Denote $M = X(T) \cong \mathbb{Z}^n$

the character group of the torus T. Then it is a general fact that every affine toric variety over some algebraically closed field k is of the form $U_\sigma = \operatorname{spec}(k[\sigma_M])$. Here σ_M is a finitely generated subsemigroup of M and $k[\sigma_M]$ is a semigroup ring:

$$k[\sigma_M] = \bigoplus_{m \in \sigma_M} k \cdot \chi(m),$$

where we follow the convention that we write $\chi(m)$ if we want to write the group law in M multiplicatively, and m if we want to write it additively: $\chi(m+m) = \chi(m) \cdot \chi(m')$. Consider a T-equivariant coherent sheaf $\mathcal{E}$ over X, then the action of the torus T on $\Gamma(U_\sigma, \mathcal{E})$ induces a decomposition into T-eigenspaces:

$$\Gamma(U_\sigma, \mathcal{E}) = \bigoplus_{m \in M} \Gamma(U_\sigma, \mathcal{E})_m.$$

which in a natural way endows $\Gamma(U_\sigma, \mathcal{E})$ with the structure of an M-graded $k[\sigma_M]$-module. By this structure, for every $m, m' \in M$ such that $m' - m \in \sigma_M$ we have k-linear maps

$$\begin{aligned} \Gamma(U_\sigma, \mathcal{E})_m &\longrightarrow \Gamma(U_\sigma, \mathcal{E})_{m'} \\ e &\mapsto \chi(m' - m) \cdot e \end{aligned}$$

This way the set of vector spaces $\Gamma(U_\sigma, \mathcal{E})_m$ and the maps among them given by the characters $\chi(m)$ form a directed family of vector spaces. We will denote such data $\hat{E}^\sigma$ and call it a *σ-family.*

Theorem (4.5)**:** *The following three categories are equivalent:*

(i) equivariant quasicoherent sheaves over U_σ,
(ii) M-graded $k[\sigma_M]$-modules with morphisms of degree 0, and
(iii) σ-families

Given a set of σ-families for all $\sigma \in \Delta$, we obtain a system of sheaves $\mathcal{E}_\sigma$ over each U_σ which, if certain compatibility conditions are fulfilled, glue to a global sheaf $\mathcal{E}$ over the toric variety X_Δ. Such a compatible set of σ-families will be called a *Δ-family.*

Theorem (4.9)**:** *Let Δ be a fan. Then the category of Δ-families is equivalent to the category of quasicoherent equivariant sheaves over X.*

The first application is the following observation:

Theorem (4.14)**:** *Let X be any toric variety, then the Krull-Schmidt theorem holds for the category of equivariant coherent sheaves over X.*

The transformation of equivariant sheaves into Δ-families might not be very useful for general equivariant sheaves, but it allows an efficient characterization of *torsion free* equivariant sheaves. Namely, we can make use of the fact that a σ-family $\hat{E}^\sigma$ is a directed family, and we can consider a direct limit

$$\mathbf{E}^\sigma = \varinjlim_{m} E_m^\sigma$$

which is a finite dimensional vector space and which has the universal property that the following diagram commutes:

$$\begin{array}{ccc} E^\sigma_m & \xrightarrow{\cdot\chi(m'-m)} & E^\sigma_{m'} \\ & \searrow \quad \swarrow & \\ & \mathbf{E}^\sigma & \end{array}$$

In addition it can be shown that all maps in this diagram are injective. This turns a σ-family into a family of subvector spaces of the limit vector space $\mathbf{E}^\sigma$. We call such a family of vector spaces a *multifiltration* (see 4.19 for a precise definition). More generally, all vector spaces in a Δ-family become subvector spaces of some common limit vector space $\mathbf{E}^0$, and we obtain the notion of families of multifiltrations.

Theorem (4.20)**:** *The category of torsion free equivariant coherent sheaves is equivalent to the category of families of multifiltrations of finite-dimensional vector spaces.*

In the special case, for ρ a one-dimensional cone, a multifiltration degenerates to a filtration $E^\rho(i)$ of $\mathbf{E}^0$ in the usual sense. In the case where $\mathcal{E}$ is a reflexive sheaf, from structural theorems we have the following for every $\sigma \in \Delta$:

$$\Gamma(U_\sigma, \mathcal{E}) \cong \bigcap_{\rho \in \sigma(1)} \Gamma(U_\rho, \mathcal{E}).$$

where $\sigma(1)$ is the set of rays contained in σ. This implies in a natural way that the multifiltrations are completely determined by intersections of the filtrations $E^\rho(i)$ in $\mathbf{E}^0$:

$$E^\sigma_m = \bigcap_{\rho \in \sigma(1)} E^\rho(\langle m, n(\rho)\rangle).$$

Theorem (4.21)**:** *The category of equivariant reflexive sheaves on a toric variety X is equivalent to the category of vector spaces with filtrations $E^\rho(i)$ associated to each ray in $\Delta(1)$. The morphisms in this category are vector space homomorphisms which are compatible with the filtrations in the Δ-family sense.*

By this theorem one can describe the category of equivariant vector bundles as a subcategory of the category of reflexive sheaves and reprove Klyachko's explicit characterization. Any set of filtrations $E^\rho(i)$ automatically defines a reflexive sheaf over X, but not necessarily a locally free sheaf.

Another new point in our work is the incorporation of *homogeneous coordinates*, which are due to Cox ([**Cox95**]), into the formalism of filtrations. Consider the affine space $k^{\Delta(1)}$, where $\Delta(1)$ is the set of rays in Δ. On this space we have the canonical diagonal action of the torus $(k^*)^{\Delta(1)} =: \hat{T}$. Then one can construct a quasi-affine toric variety $\hat{X}$ which is an open, $\hat{T}$-invariant subset of $k^{\Delta(1)}$ such that there is a surjective, equivariant morphism $\phi : \hat{X} \longrightarrow X$, which represents a good quotient with respect to the action of some diagonalizable subgroup G of $\hat{T}$. The ring $S = k[x_\rho | \rho \in \Delta(1)]$ together with the $X(G)$-grading induced by the G-action, then is called the homogeneous coordinate ring of X. This is the toric analogue of the global homogeneous coordinate ring of projective spaces. This ring has the remarkable property that every sheaf $\mathcal{E}$ over X can be represented as the sheafification $\tilde{E}$ of some $X(G)$-graded S-module E. For instance, this module can always be chosen to be $\Gamma(\hat{X}, \phi^*\mathcal{E})$, considered as S-module. Moreover, if $\mathcal{E}$ is equivariant, E can be chosen such that, additionally to the $X(G)$-grading, it carries the natural $\mathbb{Z}^{\Delta(1)}$-grading of the ring S. In particular, an equivariant homomorphism

$$\bigoplus_{j=1}^{p} \mathcal{O}_X(D_j) \longrightarrow \bigoplus_{i=1}^{q} \mathcal{O}_X(D'_i)$$

of direct sums of invertible sheaves, where D_j, D'_i are T-invariant Cartier divisors (or more general, Weil divisors), can be translated into a homomorphisms of free, $\mathbb{Z}^{\Delta(1)}$-graded S-modules

$$\bigoplus_{j=1}^{p} S(\underline{m}_j) \xrightarrow{A} \bigoplus_{i=1}^{q} S(\underline{m}'_i).$$

Such a homomorphism is given by an $n \times m$-matrix $A = (a_{ij})$ whose entries are monomials $a_{ij} = \alpha_{ij} x^{\underline{m}_j - \underline{n}_i}$, where $\alpha_{ij} = 0$ whenever $\underline{m}_j - \underline{n}_i \notin \mathbb{N}^{\Delta(1)}$. The degree of the monomials is completely determined by the grading. As we will see in chapter 8, this allows the incorporation of technical tools which are not directly available if we consider only filtrations.

Finally, we want to mention the paper [**HS02**], where similar ideas of taking literally decompositions by gradings are developed, even though in a different flavour.

1.2.2. Resolutions for Equivariant Reflexive Sheaves. As we have seen above, equivariant vector bundles over toric varieties correspond to vector spaces which have filtrations satisfying certain compatibility conditions. Technically, in general it is not straightforward to check these compatibility conditions, and for this reason we are going consider the complete category of equivariant reflexive sheaves rather than only vector bundles. This is also motivated from the perspective of homogeneous coordinate rings. Every sheaf $\mathcal{E}$ over X is isomorphic to the sheafification $\Gamma(\hat{X}, \phi^*\mathcal{E})\tilde{}$. If $\mathcal{E}$ is locally free, the pullback $\phi^*\mathcal{E}$ over $\hat{X}$ is a locally free sheaf over $\hat{X}$, and $\Gamma(\hat{X}, \phi^*\mathcal{E}) \cong \Gamma(k^{\Delta(1)}, \phi^*\mathcal{E})$ is a continuation over $k^{\Delta(1)}$ as a reflexive S-module. This way the study of vector bundles over X is also the study of reflexive S-modules.

Now, by the results we have presented in the previous section, to a reflexive sheaf there is associated a set of filtrations

$$0 \subset \cdots \subset E^\rho(i) \subset E^\rho(i+1) \subset \cdots \subset \mathbf{E}^0$$

for every $\rho \in \Delta(1)$. The set of vector spaces $E^\rho(i)$ defines an *arrangement of vector spaces* in $\mathbf{E}^0$, which we denote $\mathcal{P}$. Following the discussion of the first section, we ask the following

question: Can one extract from $\mathcal{P}$ good combinatorial invariants of $\mathcal{E}$?

This question is consciously stated in a vague fashion, but this is okay because at present we would anyway not be able to give a precise answer. Our tentative answer is: it might be.

Let us explain this more concretely. Our strategy to find out which role is played by the set $\mathcal{P}$, is to check homological properties of $\mathcal{E}$, and the first problem in this respect is the computation of minimal resolutions of $\mathcal{E}$. For this purpose, we develop a new method to investigate properties of the arrangement $\mathcal{P}$. Assume for the moment that X is affine and $E := \Gamma(X, \mathcal{E})$ is a reflexive module. It turns out that linear dependence among generators of E is related to the question of linear dependence among the vector spaces of $\mathcal{P}$. Let $X_1, \dots, X_n$ be any subset of $\mathcal{P}$ and assume that $e_1, \dots, e_s$ is a set of vectors of $\mathbf{E}^0$ such that for every X_i there is a maximal subset subset $e_{i,1}, \dots, e_{i,s_i}$ which is contained in X_i, and we want that this subset is a basis for X_i. Then clearly the sum

$$\sum_{i=1}^{n} X_i \subset \mathbf{E}^0$$

is spanned by $e_1, \dots, e_s$, but in general, these are not linearly independent. The first question we solve is whether there exists a vector space F and a family of coordinate subvector spaces $\mathcal{F} = \{F^X \mid X \in \mathcal{P}\}$ such that $\mathcal{F}$ is isomorphic as poset to $\mathcal{P}$. We call such a family a *free representation* of $\mathcal{P}$.

Theorem (5.10)**:** *Let $\mathcal{P}$ be a family of subvector spaces of some vector space V. Then there exists a free representation of $\mathcal{P}$ by subvector spaces $\{F^X \mid X \in \mathcal{P}\}$ of some vector space F and a surjection of vector spaces*

$$\psi : F \twoheadrightarrow V$$

such that for all restrictions $\psi|_{F^X}$ we have surjections

$$\psi|_{F^X} \twoheadrightarrow X.$$

The combinatorial input for this construction is the poset

$$\mathcal{P}_\Sigma = \{\sum_{X \in I} X \mid I \subset \mathcal{P}\}.$$

To every poset we can associate a *directed graph*, whose vertices are the elements of $\mathcal{P}$ and whose edges are pairs (X, Y) such that $X \subsetneq Y$ and there exists no Z such that $X \subsetneq Z \subsetneq Y$. The construction of a free representation as in the theorem above involves the weighted counting of vertices which have only one *in*-edge (see chapter 5 for details).

For every F^X, we can consider the kernel K^X of ψ restricted to F^X, which is contained in the kernel K of ψ. These K^X form another arrangement $\mathcal{K} = \{K_1, \dots, K_n\}$ in K, and every K^X measures in a sense the overcounting of generators e_i for vector spaces $X \in \mathcal{P}$. Note that $\mathcal{K}$ on one hand, as a set is smaller than the original arrangement $\mathcal{P}$, but on the other hand, it is not a coordinate subspace arrangement. We can iterate the process of constructing coordinate subspaces, and we obtain a *free resolution* of $\mathcal{P}$:

Theorem (5.17)**:** *Every family of vector spaces $\mathcal{P}$ has a finite free resolution*

$$0 \longrightarrow F_n \xrightarrow{\psi_n} \cdots \xrightarrow{\psi_1} F_0 \xrightarrow{\psi_0} V \longrightarrow 0.$$

By combining these resolution with the notion of Δ-families, we obtain the following result:

Theorem (6.12)**:** *The arrangement $\mathcal{P}$ gives rise to an exact sequence*

$$0 \longrightarrow \mathcal{F}_s \longrightarrow \cdots \longrightarrow \mathcal{F}_0 \longrightarrow \mathcal{E} \longrightarrow 0 \tag{1}$$

where the $\mathcal{F}_i$ are isomorphic to direct sums of reflexive sheaves of rank one:

$$\mathcal{F}_i = \bigoplus_{(Y, K^X_{i-1}) \in b\big((\mathcal{P}_{i-1})_\Sigma\big)} \mathcal{O}\Big(- \sum_{\rho \in \Delta(1)} \kappa_{i-1}(K^X_{i-1}, \rho) D_\rho \Big)^{\mathrm{fd}^{i-1}(K^X_{i-1})},$$

where $\mathcal{P} =: \mathcal{P}_{-1}$.

Here the K_i^X are the vector spaces in the ith kernel arrangement $\mathcal{P}_i$. In the particular case, where X is a smooth, affine toric variety, we have:

Theorem (6.7)**:** *Let X be smooth and affine. Then sequence (1) is a minimal free resolution of $\mathcal{E}$.*

This way we have obtained quasi-closed expressions, which in some sense solve the problem of finding minimal resolutions. The remaining problem is to determine the posets $(\mathcal{P}_i)_\Sigma$ and the corresponding integers $\kappa_i(K_i^X, \rho)$ and $\mathrm{fd}^i(X)$ (see chapter 6 for definitions). However, the posets $(\mathcal{P}_i)_\Sigma$ can always be computed in an algorithmic manner just by means of solving systems of linear equations. This is an interesting statement from the point of view of computational commutative algebra. The problem of finding minimal resolutions for equivariant sheaves over smooth affine toric varieties essentially is equivalent to the problem of finding minimal resolutions for $\mathbb{Z}^n$-graded modules over the polynomial ring $k[x_1, \ldots, x_n]$. In turn finding closed expressions for such resolutions is an open – and important – problem in computational commutative algebra (see [**BM93**], [**Eis93**]).

We have seen that indeed the homological properties for reflexive equivariant sheaves depend on the arrangement $\mathcal{P}$, but we fall short to prove that they do this in a combinatorial manner. In computing examples one can see that this is very often the case, but we know that this can not be true in any naive sense. We show this in section 7.4, where we compute resolutions for monomial ideals by means of determining their first syzygy module, which is reflexive. It is well known that resolutions of these objects behave in a combinatorial manner, but at least also depend on characteristics (see section 7.4 and [**MP01**], section 5.3) for an example).

Open problem: find combinatorial or closed expressions for the numbers fd^i and $\kappa_i(K_i^X, \rho)$ above.

The theory of hyperplane arrangements has been a subject in the mathematical literature since a long time (see [**OT92**]). In [**GM88**], there was initiated a more general theory for subvector space arrangements (see also [**Bjö94**] for a survey). However, as far we can see, the techniques used in that field seem not immediately to be applicable or even relevant for our situation. It would be interesting to see if our notion of free resolution might be of any use. We think that the works [**BP00**], [**BP03**] are of direct relevance.

1.2.3. Moduli for Equivariant Vector Bundles of Rank Two over Toric Surfaces. Equivariant vector bundles of rank two on smooth complete toric surfaces are the simplest nontrivial case where we can study combinatorial invariants and degenerations. The first simple observation is that the filtrations $E^\rho(i)$ for such a bundle $\mathcal{E}$ are determined by numbers $i_1^\rho \leq i_2^\rho$ and one-dimensional subvector spaces E^ρ of a two dimensional vector space $\mathbf{E}^0$ such that:

$$E^\rho(i) = \begin{cases} 0 & \text{for } i < i_1^\rho, \\ E^\rho & \text{for } i_1^\rho \leq i < i_2^\rho, \\ \mathbf{E}^0 & \text{for } i \geq i_2^\rho, \end{cases}$$

So we see that a natural notion for moduli for these bundles is given by the configuration space of lines in $\mathbf{E}^0$ or, equivalently, by the configuration space of points in $\mathbb{P}\mathbf{E}^0$. To investigate this, let us define a combinatorial invariant for $\mathcal{E}$ as follows. We enumerate $\Delta(1) = \{\rho_0, \ldots, \rho_{n-1}\}$ clockwise by the cyclic group $\mathbb{Z}_n$ with respect to the positions of the ρ_i in the space $N_\mathbb{R} \cong \mathbb{R}^2$. Denote $\Pi \subset \Delta(1)$ the subset of rays ρ where $i_1^\rho < i_2^\rho$ and let $\mathfrak{P} = \{\Pi_1, \ldots, \Pi_s\}$ be the unique maximal partition of Π such that $\rho_i, \rho_{i+1} \in \Pi_k$ if and only if $E^{\rho_i} = E^{\rho_{i+1}}$. In other words, we group neighboured ρ_i together if their associated vector spaces E^{ρ_i} coincide. We shift the numbers i_1^ρ and i_2^ρ to positions $i^\rho = i_1^\rho - i_2^\rho$ and 0, which is harmless, and our first result is:

Theorem (8.2)**:** *Let $\mathcal{E}$ be an arbitrary equivariant vector bundle of rank 2 on a smooth complete toric surface X, defined by filtrations $\{(-i^\rho, 0, E_\rho)\}_{\rho\in\Delta(1)}$ of a two dimensional vector space $\mathbf{E}^0$. Let $\Pi = \{\rho \in \Delta(1) \mid i^\rho > 0\}$ and let $\mathfrak{P} = \{\Pi_1, \ldots \Pi_s\}$ be the partition of Π with respect to $\mathcal{E}$. If $s > 2$ then there exists a short exact sequence*

$$0 \longrightarrow \mathcal{O}^{s-2} \xrightarrow{A} \bigoplus_{i=1}^{s} \mathcal{O}(\sum_{\rho\in\Pi_i} i^\rho \cdot D_\rho) \longrightarrow \mathcal{E} \longrightarrow 0$$

where A is a matrix of monomials whose exponents are determined by the partition $\mathfrak{P}$. Moreover, the $(s-2)$-minors $A^{i,i+1}$ of A, $1 \leq i < s$, which consist of all rows of A except the i-th and the $(i+1)$-st, are of full rank. If $s \leq 2$, then $\mathcal{E}$ splits.

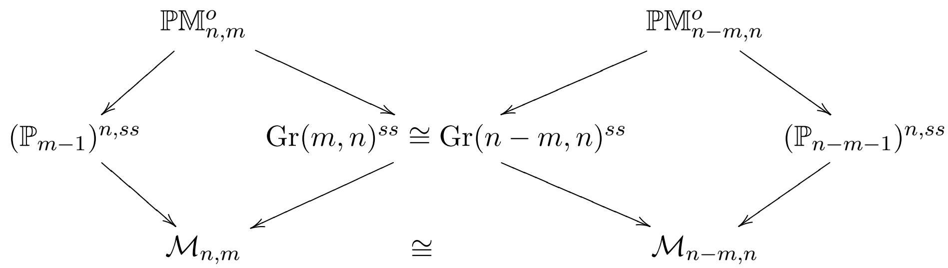
By this theorem we see that the partition $\mathfrak{P}$ might not be a good combinatorial invariant. If the family E^ρ varies, the corresponding partition changes, and the above resolution has another shape. So it is clear that some properties of $\mathcal{E}$ we would like to keep constant, such as Chern classes, do not remain constant if we naively consider the configuration space of points in $\mathbb{P}\mathbf{E}^0$. If we pass to the limits of the corresponding vector spaces, we obtain the following short exact sequence:

$$0 \longrightarrow k^{s-2} \xrightarrow{A} k^{\mathfrak{P}} \xrightarrow{A^\check{}} \mathbf{E}^0 \longrightarrow 0$$

where A is an $s \times (s-2)$ matrix and $A^\check{}$ a $2 \times s$ matrix. We observe that after a suitable choice of a basis we can write

$$A^\check{} = \begin{pmatrix} A_i & \ldots & A_s \end{pmatrix}$$

such that the column vectors A_i over k span the E^ρ for those ρ which are contained in Π_i. If we vary the matrix A in a way compatible with the equivariant structures, we get all resolutions for bundles $\mathcal{E}$ whose partition coincides with $\mathfrak{P}$. But there exist more possible cokernels of matrices A as in the above short exact sequence, which in general need not even be locally free. The question is, how are these related to our bundles? The variations of A and $A^{\vee}$ are related via the following diagram:



We explain the diagram. $\mathbb{PM}_{n,m}$ and $\mathbb{PM}_{n-m,n}$ are the spaces of $n \times m$ and $(n-m) \times n$ matrices, respectively, and $\mathcal{M}_{n,m}$, $\mathcal{M}_{n-m,n}$ are their quotients with respect to certain open subsets (denoted by $\ ^o$) by the groups $\mathrm{GL}_m \times T$ and $\mathrm{GL}_{n-m} \times T$ respectively, where $T \cong (k^*)^n$ is a torus. These quotients can be performed in two steps, either by first taking the quotient by GL_m (respectively GL_{n-m}) and then by T, or vice versa. The intermediate quotients then are Grassmannians or products of projective spaces, respectively. By using the duality isomorphism of the Grasmmannians $\mathrm{Gr}(m,n) \cong \mathrm{Gr}(n-m,n)$ we can conclude that both quotients are isomorphic. The two commuting squares are known als Gelfand-MacPherson correspondence (see [**GM82**] and [**Kap93**]), but we differ from their version, because we formulate this correspondence as GIT quotients (therefore the indices $\ ^{ss}$).

We can apply now this correspondence to our objects of study: variations of s points in $\mathbb{P}\mathbf{E}^0$ modulo automorphisms of $\mathbb{P}\mathbf{E}^0$ are parametrized by $2 \times s$ matrices $A^{\vee}$ modulo $\mathrm{GL}_2 \times (k^*)^s$, and sheaf homomorphisms A by $s \times (s-2)$ matrices modulo automorphisms of $\mathcal{O}^{s-2}$ and $\bigoplus_{i=1}^s \mathcal{O}(\sum_{\rho\in\Pi_i} i^\rho D_\rho)$ which coincide with GL_{s-2} and $(k^*)^s$, respectively. The isomorphism between the configuration spaces above then can explicitly be given as:

$$\mathcal{M}_{s,s-2} \longrightarrow \mathcal{M}_{s,2}, \quad \mathcal{E} \mapsto \mathcal{E}^{\vee\vee},$$

every cokernel $\mathcal{E}$ of some matrix A is mapped to its bidual, which is a locally free sheaf whose configuration of points E^ρ is given by the columns of the matrix $A^{\vee}$.

A more precise analysis of the equivalence classes in $\mathcal{M}_{s,s-2}$ via GIT stability leads to a new definition for stability for torsion free equivariant sheaves of rank two over toric surfaces, which justifies the denomination $\mathcal{M}_{\mathcal{P}} = \mathcal{M}_{s,s-2}$ the *moduli space of* $\mathcal{P}$*-(semi-)stable equivariant torsion free sheaves.*

Definition 1 (8.14)**:** Let $\mathcal{E}$ be a torsion free equivariant sheaf of rank 2 over X such that $\mathfrak{P}$ is a refinement of the coarse partition $\mathfrak{P}_{\mathcal{E}^{\vee\vee}}$ associated to the locally free sheaf $\mathcal{E}^{\vee\vee}$. Let $\mathcal{F} \subset \mathcal{E}$ be a torsion free equivariant subsheaf of rank 1. Then by 8.11 $\mathcal{F}^{\vee\vee} \cong \mathcal{O}(\sum_{\Pi\in\mathfrak{P}'} \sum_{\rho\in\Pi} i^\rho \cdot D_\rho)$ with a unique subset $\mathfrak{P}' \subset \mathfrak{P}$. We say that $\mathcal{E}$ is $\mathfrak{P}$-*stable* (respectively $\mathfrak{P}$-*semistable*) if for every equivariant torsion free subsheaf $\mathcal{F} \subset \mathcal{E}$ of rank 1, $\#\mathfrak{P}' < \frac{1}{2}\#\mathfrak{P}$ (respectively $\#\mathfrak{P}' \leq \frac{1}{2}\#\mathfrak{P}$).

The structural comparison is made in the following theorem:

Theorem (8.15)**:** *Let $i_\rho > 0$ for $\rho \in \Pi \subset \Delta(1)$ and let $\mathfrak{P} = \{\Pi_1, \ldots, \Pi_s\}$ be a partition of Π. Consider short exact sequences*

$$0 \longrightarrow \mathcal{O}^{s-2} \xrightarrow{A} \bigoplus_{i=1}^{s} \mathcal{O}(\sum_{\rho \in \Pi_i} i^\rho \cdot D_\rho) \longrightarrow \mathcal{E} \longrightarrow 0.$$

Then $\mathcal{E}$ is $\mathfrak{P}$-stable (respectively $\mathfrak{P}$-semistable) if and only if A represents a GIT-stable (respectively GIT-semistable) point in $(\mathbb{P}_{s-3})^s$ with respect to the action of GL_{s-2}.

And finally:

Theorem (8.17)**:** *Fix numbers $\{i^\rho \geq 0\}_{\rho \in \Delta(1)}$, let $\Pi = \{\rho \mid i^\rho > 0\} \subset \Delta(1)$ and let $\mathfrak{P} = \{\Pi_1, \ldots, \Pi_s\}$. Then $\mathcal{M}_{s,s-2}$ is the set of $\mathfrak{P}$-equivalence classes of $\mathfrak{P}$-semistable torsion free equivariant sheaves of rank 2 on X.*

In the situation of rank two bundle we are in the lucky situation that we can resort to a standard configuration space, the configuration space of points on the projective line and that the simple GIT quotients lead to a very natural notion of equivalence classes. It is not known to us if beyond configurations of points in Grassmannians there exist examples for more general configuration spaces of flags in the literature. We expect that for such spaces it would be an issue not only to consider GIT-like equivalence classes but rather to search for compactifications which are more combinatorial in flavour. Relevant might be considerations as in [**Kap93**] and [**Laf02**] (and references therein).

1.3. Overview of This Work

This work consists of four principal parts. The first part is formed by chapters 2 and 3, where we introduce the background on sheaf theory, algebraic groups and toric varieties needed for our later work.

Chapter 4 is the second main part, where we we develop our formalism of σ- and Δ-families and present some related results.

Chapters 5, 6 and 7 form the third part in which we construct resolutions for general equivariant reflexive sheaves over toric varieties.

The final part consists of chapter 8. In this part we give a complete classification of equivariant vector bundles of rank two over smooth complete toric surface and describe their combinatorial invariants and moduli.

This work finishes with two appendices. In appendix A we give a sketchy outlook concerning combinatorial invariants for equivariant bundles. Appendix B is based on appendix A. We give a complete classification of bundles of rank three over $\mathbb{P}_2$ and a computation of their global resolutions.

CHAPTER 2

Preliminaries

In order to keep this work as self-contained as possible, we present in this chapter the basic material which will form the background for the rest of this work. We assume general background from algebraic geometry as, e.g., from [**Har77**], and commutative algebra (as [**AM69**], [**BH94**]). Except if stated otherwise, we always work over an algebraically closed field k. When we talk about *varieties*, we always mean separable, but not necessarily reduced schemes of finite type over k.

This chapter consists of five sections. In section 2.1, we collect basic facts related to torsion free and reflexive sheaves over normal varieties. This material is mainly based on the references [**GD71**], [**Har80**], and [**Rei80**]. Section 2.2 is about algebraic groups, their representations, and actions of algebraic groups on varieties and sheaves. Standard references for algebraic groups are [**Bor91**] and [**Hum75**]. Concerning dual actions and linearizations of sheaves, we follow [**MFK94**]. In section 2.3 we present the basic notions for quotients by reductive algebraic groups. This material is from [**MFK94**] and [**New78**]. We have complemented the standard material by a proof on descent of certain linearized sheaves to good quotients. Section 2.4 presents *diagonalizable* groups and tori. We describe the functorial relation between those groups and their character groups; this material can essentially be found in chapter III, §11 of [**Bor91**]. We prove explicitly a characterization of linearized sheaves over affine schemes with an action of a diagonalizable group in terms of graded modules. Finally, in section 2.5 we present basic facts from the theory of graded rings. In particular, a graded version of Nakayama's lemma.

2.1. Torsion Free and Reflexive Sheaves

2.1.1. The sheaf of rational functions. Let X be an arbitrary scheme. Following [**Kle79**] and [**GD71**], we define for each open subscheme U of X:

$$S(U) := \{f \in \Gamma(U, \mathcal{O}_X) \text{ s.t. the germs } f_x \in \mathcal{O}_{X,x} \text{ of } f \text{ are nonzerodivisors } \forall x \in U\}.$$

Clearly, $S(U)$ is a multiplicatively closed subset of $\Gamma(U, \mathcal{O}_X)$ for all U, and we get a presheaf of rings given by $U \mapsto S(U)^{-1}\Gamma(U, \mathcal{O}_X)$. The sheaf $\mathcal{K}_X$ associated to this presheaf is called the *sheaf of rational functions*. For every open subset U of X, $\Gamma(U, \mathcal{K}_X)$ is isomorphic to the direct product $\prod_\xi \mathcal{O}_{X,\xi}$, running over all generic points of X which are contained in U.

If X is integral, then there exists a unique generic point ξ of X such that $\overline{\{\xi\}} = X$. The stalk $\mathcal{O}_{X,\xi} =: k(X)$ is a field, called *quotient field* of X. Every affine open subset $U = \text{spec}(R)$ of X contains ξ, and every stalk has a canonical identification $\mathcal{K}_{X,x} =$

$(\mathcal{O}_{X,x})_\xi = k(X)$ and $\mathcal{K}_X$ is a constant sheaf over X given by $U \mapsto k(X)$. Moreover, every $k(X)$-module, and thus every sheaf of $\mathcal{K}_X$-modules, is simple.

2.1.2. Torsion Sheaves and Torsion Free Sheaves. For every coherent sheaf $\mathcal{F}$ over a scheme X there is an exact sequence

$$0 \longrightarrow \mathcal{T} \longrightarrow \mathcal{F} \xrightarrow{f} \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{K}_X$$

where f is given by $f(s) = s \otimes 1$. The kernel $\mathcal{T}$ of this map is the *torsion subsheaf* of $\mathcal{F}$. If $\mathcal{T}$ is isomorphic to $\mathcal{F}$, then $\mathcal{F}$ is a *torsion sheaf*, and if $\mathcal{T}$ is zero, then $\mathcal{F}$ is *torsion free*. The image of $\mathcal{F}$ in $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{K}_X$ is isomorphic to $\mathcal{F}/\mathcal{T}$, which is torsion free.

Assume that X is integral, then $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{K}_X$ is isomorphic to $k(X)^r$, where $r \geq 0$ is called the *rank* of $\mathcal{F}$.

2.1.3. Reflexive sheaves. Assume now that X is noetherian, integral and normal, i.e. X is reduced and irreducible, and all local rings are integrally closed domains.

Definition 2.1: A coherent sheaf $\mathcal{F}$ on X is *reflexive* if $\mathcal{F}$ is isomorphic to its bidual $\mathcal{F}^{\check{}\check{}}$. A coherent sheaf $\mathcal{F}$ is *normal* if for every open set $U \subseteq X$ and every closed subset $Y \subset U$ of codimension ≥ 2, the restriction map $\mathcal{F}(U) \longrightarrow \mathcal{F}(U \setminus Y)$ is bijective.

In particular, if Y is any subvariety of X of codimension ≥ 2 and $\mathcal{F}$ is normal, then $H^0(X, \mathcal{F}) \cong H^0(X \setminus Y, \mathcal{F})$.

Proposition 2.2 ([**Har80**], proposition 1.6)**:** *Let $\mathcal{F}$ be a coherent sheaf on a normal integral scheme X. Then the following statements are equivalent:*

(i) $\mathcal{F}$ is reflexive
(ii) $\mathcal{F}$ is torsion free and normal
(iii) $\mathcal{F}$ is torsion free, and for each open $U \subset X$ and each closed subset $Y \subset U$ of codimension ≥ 2, $\mathcal{F}_U \cong j_*\mathcal{F}_{U \setminus Y}$, where $j : U \setminus Y \longrightarrow U$ is the canonical inclusion.

2.1.4. Reflexive Sheaves and Weil divisors: We assume that X is noetherian, integral, of dimension n, and normal. We follow [**Ful98b**] and denote $Z_{n-1}(X)$ the free abelian group generated by $(n-1)$-dimensional subvarieties of X, i.e. the group consisting of finite formal sums

$$\sum n_Y[Y]$$

where the sum runs over the closures $Y = \overline{\{y\}}$ of points $y \in X$ of height one. We call such a sum a *Weil divisor* on X. To every irreducible Y of codimension one in X there is associated a valuation $\upsilon_Y : \mathcal{O}_{X,y} \longrightarrow \mathbb{Z}$, which extends to $k(X)$ via $\upsilon_Y(\frac{f}{g}) = \upsilon_Y(f) - \upsilon_Y(g)$. Using this, we can associate to every $r \in k(X)$ its cycle

$$[\mathrm{div}(r)] = \sum_Y \upsilon_Y(r)[Y]$$

which is always a finite sum. Two cycles $\sum n_Y[Y]$ and $\sum m_Y[Y]$ are *rationally* equivalent, if $\sum(n_Y - m_Y)[Y] = [\mathrm{div}(r)]$ for some $r \in k(X)$. The $\mathbb{Z}$-module

$$A_{n-1}(X) = Z_{n-1}(X)/\text{rational equivalence}$$

is the $(n-1)$-st Chow group of X, respectively the group of Weil divisors modulo rational equivalence.

Let $D = \sum n_Y Y$ be a Weil divisor. One can define a presheaf $\mathcal{O}_X(D)$ associated to D by setting $U \mapsto \Gamma(U, \mathcal{O}_X(D)) := \{f \in k(X) \mid \upsilon_Y(f) \geq -n_Y \quad \forall\ Y = \overline{\{y\}}, y \in U \text{ and height}(y) = 1\}$. It is easy to see that this presheaf is a sheaf and when D is a Cartier divisor, this construction coincides with the standard construction as, e.g., in [**Har77**].

For the class of Cartier divisors, there is a one-to-one correspondence between the rational equivalence classes of divisors and isomorphism classes of invertible sheaves on X. Moreover, for any two Cartier divisors D_1, D_2 of X, the morphism $\mathcal{O}_X(D_1) \otimes_{\mathcal{O}_X} \mathcal{O}_X(D_2) \longrightarrow \mathcal{O}_X(D_1 + D_2)$ is an isomorphism. This is in general false if D_1 and D_2 are not Cartier, but instead there is an isomorphism

$$\big(\mathcal{O}_X(D_1) \otimes_{\mathcal{O}_X} \mathcal{O}_X(D_2)\big)^{\check{}\check{}} \longrightarrow \mathcal{O}_X(D_1 + D_2).$$

Let us denote by $X_{\text{reg}} \overset{i}{\hookrightarrow} X$ the open subset of regular points of X together with its canonical inclusion. Normality of X implies that $\operatorname{codim}_X X \setminus X_{\text{reg}} \geq 2$ such that all zero- and one-dimensional points of X are contained in X_{reg}. In particular, $k(X) = k(X_{\text{reg}})$ and $A_{n-1}(X) = A_{n-1}(X_{\text{reg}})$ (see also [**Har77**], prop. 6.5). For any open subset $U \subset X$ denote $U_{\text{reg}} = U \cap X_{\text{reg}}$. Then $(i_*\mathcal{O}_{X_{\text{reg}}}(D))(U) = \mathcal{O}_{X_{\text{reg}}}(D)(U_{reg}) = \mathcal{O}_X(D)(U_{\text{reg}}) = \mathcal{O}_X(D)(U)$, and using 2.2, we obtain:

Theorem 2.3: *(i) Let D be a Weil divisor. Then the sheaf $\mathcal{O}_X(D)$ associated to D is a reflexive sheaf of rank one.*

(ii) There is a one-to-one correspondence between Cartier divisors D on X_{reg} and Weil divisors i_*D on X. This extends to a one-to-one correspondence between invertible sheaves $\mathcal{O}_{X_{\text{reg}}}(D)$ on X_{reg} and reflexive sheaves of rank one. This correspondence is given by $i_*\mathcal{O}_{X_{\text{reg}}}(D)$.

(iii) (see [**Rei80**]*, Theorem (3)) There is a one-to-one correspondence between the set of classes in $A_{n-1}(X)$ and reflexive sheaves of rank one up to isomorphism.*

2.2. Algebraic Groups

Fundamental references for the theory of algebraic groups are the books [**Bor91**] and [**Hum75**]. Moreover, we use the book [**Kra85**]. For group actions on varieties and dual actions, references are [**MFK94**] [**New78**], and [**KSS89**].

2.2.1. Algebraic Groups and Group Actions. An *algebraic group* is a variety G whose set of closed points is endowed with the structure of a group such that

(i) group multiplication $\mu : G \times G \longrightarrow G$, $(g, h) \mapsto gh$,
(ii) inversion $\iota : G \longrightarrow G$, $g \mapsto g^{-1}$,

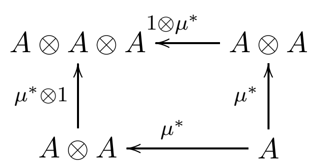
induce morphisms of varieties. Denote e the unit element of G. A *morphism* of algebraic groups is a group homomorphism $\rho : G \longrightarrow H$ which induces a morphism of varieties. If $H = \mathrm{GL}_n(k)$, then ρ is called a *rational representation* of G.

The following is a fundamental structure theorem for algebraic groups:

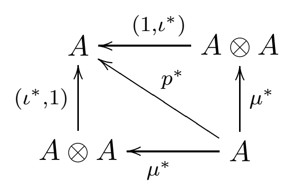
Theorem 2.4: *(i) Let G be an* affine *variety which has the structure of an algebraic group. Then G is isomorphic to a closed subgroup of some* $\mathrm{GL}_n(k)$.
(ii) Every closed subgroup of $\mathrm{GL}_n(k)$ *is an affine algebraic group.*

We will always assume that our algebraic groups are *affine* varieties, that is, we consider only *linear algebraic groups*. The structure of an algebraic group on an affine variety G is equivalent to the structure of a *Hopf algebra* on the coordinate ring $A = k[G]$. This means that there are morphisms μ, ι and p such that the following diagrams commute:

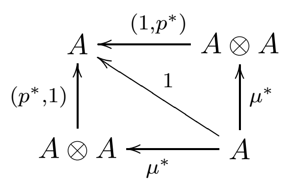
(1) Associativity:



(2) Inversion:



(3) Neutral element:



The morphisms μ and ι as above correspond to multiplication and inversion, respectively, and p is the constant morphism $G \longrightarrow e$.

A particular example of an algebraic group is the *multiplicative group* $\mathrm{G}_m = \mathbb{A}^1_k \backslash \{0\}$, whose underlying set of closed points and group structure can be identified with that of k^*, the multiplicative group of the base field k. In the sequel we will always write k^* instead of $\mathrm{G}_m = \mathbb{A}^1_k \setminus \{0\}$.

A *character* of G is a morphism of algebraic groups

$$\chi : G \longrightarrow k^*.$$

The set of characters of G is denoted $X(G)$ and has a natural group structure given by multiplication. Note that $X(k^*) \cong \mathbb{Z}$.

A *one-parameter subgroup* of G is a morphism of algebraic groups

$$p : k^* \longrightarrow G.$$

The set of one-parameter subgroups of G is denoted $X_*(G)$ and has also a natural group structure given by $(p{\cdot}q)(c) = p(c)q(c)$ in G. There is a natural pairing $X(G) \times X_*(G) \longrightarrow X(k^*) = \mathbb{Z}$ which is given by composition.

An *action* of G on a variety X is a morphism $\sigma : G \times X \longrightarrow X$, denoted for short $\sigma(g, x) = g.x$, such that:

(i) $g.(h.x) = (gh).x$, for all $g, h \in G$, $x \in X$,
(ii) $e.x = x$ for all $x \in X$.

Let G, H be two algebraic groups which act on varieties X and Y, respectively. An *equivariant morphism* from X to Y is a pair $(f, \phi) : (X, G) \longrightarrow (Y, H)$ such that the following diagram commutes:

$$\begin{array}{ccc} G \times X & \longrightarrow & X \\ {\scriptstyle \phi\times f}\downarrow & & \downarrow{\scriptstyle f} \\ H \times Y & \longrightarrow & Y \end{array}$$

Let $x \in X$, then the *stabilizer* $\operatorname{stab}(x)$, or G_x, of x is the subgroup of elements g in G such that $g.x = x$. The *orbit* $\operatorname{orb}(x)$ of x is the subvariety $G.x$. As varieties, the sets $\operatorname{orb}(x)$ and G/G_x are isomorphic.

Proposition 2.5 ([**Hum75**], §II.8.3)**:** *Let $G \times X \longrightarrow X$ be an action of an algebraic group G on some variety X. Each orbit is a smooth, locally closed subset of X, whose boundary is a union of orbits of strictly lower dimension. In particular, orbits of minimal dimension are closed.*

Let V be a vector space over k which is not necessarily finite dimensional. Then a *regular* action of G on V is a linear action of G on V such that for every finite dimensional subvector space $W \subset V$

(i) the set

$$G.W = \sum_{g \in G} g.W$$

is a finite dimensional subvector space of V and

(ii) G acts via a rational representation on $G.W$.

A *dual* action of G on V is a homomorphism of k-vector spaces

$$\sigma^* : V \longrightarrow k[G] \otimes_k V$$

such that

(i)

$$\begin{array}{ccccc} & & k[G] \otimes_k V & & \\ & \overset{\sigma^*}{\nearrow} & & \overset{\mu^* \otimes 1_V}{\searrow} & \\ V & & & & k[G] \otimes_k k[G] \otimes_k V \\ & \underset{\sigma^*}{\searrow} & & \underset{1_{k[G]} \otimes \sigma^*}{\nearrow} & \\ & & k[G] \otimes_k V & & \end{array}$$

commutes, and

(ii) $V \xrightarrow{\sigma^*} k[G] \otimes_k V \xrightarrow{\beta^*} V$ is the identity,

where $1_{k[G]}$ and 1_V are the identity homomorphisms on $k[G]$ and V, respectively, and $\beta^* : k[G] \to k$ is the homomorphism dual to $\beta : e \hookrightarrow G$, the inclusion of the unit element in G. A subvector space V' of V is G-invariant if $\sigma^*(V') \subset k[G] \otimes V'$, and an element $v \in V$ is G-invariant if $\sigma^*(v) = 1 \otimes v$. The subvector space of G-invariant elements of V is denoted V^G.

The most important example of a dual action is the one coming from an action $\sigma : G\times X \to X$ on an affine variety X. This action dually induces a ring homomorphism $\sigma^* : k[X] \longrightarrow k[G] \otimes_k k[X]$ which fulfills the axioms for a dual action. Likewise, one obtains for every G-invariant open subset U of a variety X a dual action via

$$\sigma^* : \mathcal{O}_X(U) \longrightarrow k[G] \otimes_k \mathcal{O}_X(U)$$

More explicitly, every $g \in G$ as automorphism of ringed spaces defines a map $g^\# : \mathcal{O}_X \to g_*\mathcal{O}_X$, where $(g^\# f)(x) = f(g^{-1}.x)$, and thus there is a linear action of G on $\mathcal{O}_X(U)$ for G-invariant U. The following is a fundamental and well-known fact:

Proposition 2.6: *Let X be an affine variety on which an algebraic group G acts. Then the dual action of G on $k[X]$ is regular.*

Slightly less well-known is the following generalization:

Proposition 2.7 (see [**MFK94**], Ch.1, §1)**:** *A dual action $\sigma^* : V \longrightarrow k[G] \otimes_k V$ of G on an arbitrary vector space V is regular.*

2.2.2. Equivariant Sheaves. Let $\sigma : G \times X \longrightarrow X$ be an action of an algebraic group G on a variety X and denote $p_2 : G \times X \longrightarrow X$ the projection onto the second factor. Following [**MFK94**], §3, we call a sheaf $\mathcal{E}$ on X *equivariant* or *G-linearized* if there exists an isomorphism

$$\Phi : \sigma^*\mathcal{E} \xrightarrow{\cong} p_2^*\mathcal{E}$$

such that

$$(\mu \times 1_X)^*\Phi = p_{23}^*\Phi \circ (1_G \times \sigma)^*\Phi,$$

where 1_X and 1_G are the identity morphisms of X and G, respectively, and $p_{23} : G \times G \times X \longrightarrow G \times X$ is the projection onto the second and third factors. Consider for every $g \in G$ the morphism

$$i_g : X \longrightarrow G \times X, \quad x \mapsto (g, x)$$

then for any two $g, g' \in G$ the identity

$$\sigma \circ i_{g'g} = \sigma \circ i_{g'} \circ \sigma \circ i_g$$

holds. This identity is the same as $(g'g).x = g'.(g.x)$, if g, g' are considered as automorphisms of X. Using this, we get for all $g \in G$ isomorphisms

$$\Phi_g := i_g^*\Phi : g^*\mathcal{E} = (\sigma \circ i_g)^*\mathcal{E} \xrightarrow{\cong} (p_2 \circ i_g)^* = \mathcal{E}.$$

Moreover, we get

$$\begin{aligned} (\mu \times 1_X) \circ (1_G, i_g) \circ i_{g'} &= i_{g'g} \\ p_{23} \circ (1_G, i_g) \circ i_{g'} &= i_g \\ (1_G \times \sigma) \circ (1_G, i_g) \circ i_{g'} &= i_{g'} \circ \sigma \circ i_g \end{aligned}$$

and by the cocycle condition, $\Phi_{g'g}$ factors as follows:

$$\begin{array}{ccccc} g^*g'^*\mathcal{E} & \xleftarrow{\;\cong\;} & (g'g)^*\mathcal{E} & \xrightarrow{\;\Phi_{g'g}\;} & \mathcal{E} \\ & \searrow^{g^*\Phi_{g'}} & \big\downarrow & \nearrow_{\Phi_g} & \\ & & g^*\mathcal{E} & & \end{array}$$

On a G-invariant open subset U of X, there exist natural dual actions on the cohomology groups of $\mathcal{E}$, which are given by

$$H^i(U,\mathcal{E}) \xrightarrow{\sigma^*} H^i(G\times U,\sigma^*\mathcal{E}) \xrightarrow{R^i\Phi} H^i(G\times U,p_2^*\mathcal{E}) \xrightarrow{\cong} k[G]\otimes_k H^i(U,\mathcal{E})$$

where the last isomorphism follows from flatness of p_2 (see [**Har77**], III, Prop. 9.3).

From proposition 2.7 follows that the dual action on the cohomologies is *regular*:

Proposition 2.8: *Let $\mathcal{E}$ be quasicoherent and let U be a G-invariant open subset of X. Then the dual action of G on $H^i(U,\mathcal{E})$ is regular.*

To write down the dual action on global sections more explicitly, let $s\in\Gamma(U,\mathcal{E})$ and denote g^*s the canonically lifted section of $g^*\mathcal{E}$. Then the dual action of G on $\Gamma(U,\mathcal{E})$ is defined as $g.s = \phi_{g^{-1}}(g^{-1*}s)$. This action is a linear representation of G on $\Gamma(U,\mathcal{E})$. Note that the following square is commutative by functoriality of pullback:

$$\begin{array}{ccc} \Gamma(U,g^{-1*}\mathcal{E}) & \xrightarrow{\;\Phi_{g^{-1}}\;} & \Gamma(U,\mathcal{E}) \\ \big\downarrow{\scriptstyle h^{-1*}} & & \big\downarrow{\scriptstyle h^{-1*}} \\ \Gamma(U,h^{-1*}g^{-1*}\mathcal{E}) & \xrightarrow[\;h^{-1*}\Phi_{g^{-1}}\;]{} & \Gamma(U,h^{-1*}\mathcal{E}) \end{array}$$

for all $g,h\in G$. Using this, we obtain $(hg).s = h.(g.s)$ for $s\in\Gamma(U,\mathcal{E})$:

$$\begin{aligned} (hg).s &= \Phi_{g^{-1}h^{-1}}\big(h^{-1*}g^{-1*}s\big) \quad \text{and} \\ h.(g.s) &= h.\big(\Phi_{g^{-1}}(g^{-1*}s)\big) \\ &= \Phi_{h^{-1}}\big(h^{-1*}\Phi_{g^{-1}}(g^{-1*}s)\big) \\ &= \Phi_{h^{-1}}\Big(\big(h^{-1*}\Phi_{g^{-1}}\big)(h^{-1*}g^{-1*}s)\Big) \quad \text{by the above square} \\ &= \Phi_{g^{-1}h^{-1}}(h^{-1*}g^{-1*}s) \quad \text{by the cocycle condition.} \end{aligned}$$

Lemma 2.9: *Let $(\Phi,f): G\times X\to H\times Y$ an equivariant morphism and let $\mathcal{E}$ be a H-equivariant sheaf over Y. Then $f^*\mathcal{E}$ is a G-equivariant sheaf over X.*

PROOF. Let $G\times X\xrightarrow{\sigma} X$ and $H\times Y\xrightarrow{\tau} Y$ be the actions of G and H and let $G\times X\xrightarrow{p_{2,X}} X$ and $H\times Y\xrightarrow{p_{2,Y}} Y$ be projections. We have $p_{2,Y}^*\mathcal{E}\cong\tau^*\mathcal{E}$ and want to show that $p_{2,X}^* f^*\mathcal{E} \cong \sigma^* f^*\mathcal{E}$. From the identity $p_{2,Y} \circ (\phi, f) = f \circ p_{2,X}$ follows:

$$\begin{aligned}\sigma^* f^*\mathcal{E} &\cong (\Phi, f)^*\tau^*\mathcal{E}\\ &\cong (\Phi, f)^* p_{2,Y}^*\mathcal{E}\\ &\cong (p_{2,Y} \circ (\phi, f))^*\mathcal{E}\\ &\cong (f \circ p_{2,X})^*\mathcal{E}\\ &\cong p_{2,X}^* f^*\mathcal{E}.\end{aligned}$$

We leave the proof of the cocycle condition to the reader. □

We state the following important fact:

Proposition 2.10 (see [**Tho87**], §1.3)**:** *Let $G \times X \to X$ be an action of an algebraic group on a variety X. Then the category of G-equivariant sheaves is abelian.*

2.3. Quotients and Descent of Sheaves

Let G be an algebraic group. Then the commutator $[\,,\,] : G \times G \longrightarrow G$ is defined as $[x, y] := xyx^{-1}y^{-1}$. The *derived series* of G is defined inductively as $D^0G := G$, $D^{i+1}G := [D^iG, D^iG]$. G is *solvable* if its derived series terminates in e. The *radical* $R(G)$ of G is the maximal connected solvable normal subgroup of G.

Example 2.11: An example of a solvable group is the group of *upper triangular matrices*, which is also known as *Borel subgroup* of $\mathrm{GL}_n(k)$. In that case $R(G) = G$.

Definition 2.12: Let G be an algebraic group, then G is

(i) *reductive* if the radical of G is a torus,
(ii) *linearly reductive* if every rational representation of G is completely reducible,
(iii) *geometrically reductive* if, whenever, $G \to \mathrm{GL}(V)$ is a rational representation and $0 \neq v \in V^G$, then for some $r > 0$, there is a polynomial $f \in (\mathrm{Sym}^r V^*)^G$ with $f(v) \neq 0$.

If $\mathrm{char}(k) = 0$, all three notions of reducibility coincide. If $\mathrm{char}(k) > 0$, a group is geometrically reductive if and only if it is reductive, and linearly reductive implies reductive (see [**MFK94**], Appendix 1.A, [**New78**]). Standard examples for reductive groups are the classical groups such as $\mathrm{GL}_n(k), \mathrm{SL}_n(k), \ldots$, and, of course, diagonalizable groups. The following statement about reductive groups is known as *Nagata's theorem*:

Theorem 2.13: *Let G be a reductive group acting rationally on a finitely generated k-algebra S. Then the ring of invariants S^G is finitely generated.*

Note that a regular representation in the sense of subsection 2.2.1 is also a rational action.

Nagata's theorem is an important building block for *geometric invariant theory*. The main problem there is to find objects in the category of varieties (or more general schemes) which share the universal properties of set-theoretical quotients (see also [**Bia01**] for a general treatment of this problem). There are several different notions of such quotients, the most prominent being *categorical*, *good* and *geometric quotients*:

Definition 2.14: Let X be a variety on which an algebraic group G acts. A pair (Y, ϕ), consisting of a variety Y and a morphism $X \to Y$ is a *categorical quotient* of X by G if

(i) $\phi \circ \sigma = \phi \circ p_2$,
(ii) for any pair (Z, ψ) for which (i) holds, there is a unique morphism $\chi : Y \to Z$ such that $\psi = \chi \circ \phi$.

(Y, ϕ) is a *good quotient* if

(i) $\phi \circ \sigma = \phi \circ p_2$,
(ii) ϕ is surjective,
(iii) ϕ is affine,
(iv) if U is open and affine in Y, then $\phi^* : \mathcal{O}_Y(U) \longrightarrow \phi_*\mathcal{O}_X(U)$ is an isomorphism of $\mathcal{O}_Y(U)$ onto $\left(\phi_*\mathcal{O}_X(U)\right)^G$,
(v) if W is a closed G-invariant subset of X, then $\phi(W)$ is closed,
(vi) if W_1, W_2 are disjoint closed invariant subsets of X, then $\phi(W_1) \cap \phi(W_2) = \emptyset$.

(Y, ϕ) is a *geometric quotient* if

(i) ϕ is a good quotient,
(ii) the fibers of ϕ are precisely the orbits of G, i.e. Y is an orbit space.

Definition 2.15: If Y is a good quotient of X by G, then it is also denoted $Y = X//G$.

Note that a good quotient is always a categorical quotient. In case G is reductive, Nagata's theorem allows the construction of good quotients of affine varieties:

Proposition 2.16 ([**New78**], Theorem 3.5)**:** *Let G be a reductive group which acts on a variety X. Let $Y := \mathrm{spec}(k[X]^G)$ and $\phi : X \longrightarrow Y$ the morphism induced by the inclusion $k[X]^G \hookrightarrow k[X]$. Then the pair (Y, ϕ) is a good quotient of X by G.*

We will need the following criterion:

Proposition 2.17 ([**MFK94**], p. 8, Remark (6))**:** *Let G be an algebraic group which acts on X, Y any variety on which G acts trivially and let $\phi : X \longrightarrow Y$ a G-equivariant map. Then (Y, ϕ) is a categorical quotient if*

(i) $\mathcal{O}_Y$ is the subsheaf of invariants of $\phi_*\mathcal{O}_X$,
(ii) if W is an invariant closed subset of X, then $\phi(W)$ is closed in Y; if W_i, $i \in I$, is as set of invariant closed subsets of X, then:

$$\phi\Big(\bigcap_{i\in I} W_i\Big) = \bigcap_{i\in I} \phi(W_i)$$

Moreover, if these conditions hold, then ϕ is submersive, i.e. a subset $U \subset Y$ is open if and only if $\phi^{-1}(U)$ is open.

If there exists a G-linearized line bundle on X, then the above construction of affine quotients can be used to glue quotients of projective varieties.

Definition 2.18: Let G be a reductive group which acts on a variety X and let $\mathcal{L}$ be an invertible G-linearized sheaf over X. Then a point $x \in X$ is *semistable* if, for some positive integer r, there exists an invariant section of L^r such that $f(x) \neq 0$ and the complement of $f = 0$, X_f, is affine. x is *stable* if x is semistable, $\dim G.x = \dim G$, and the action of G on X_f is closed. We denote $X^{ss}(\mathcal{L})$ (respectively, X^{ss}, if there is no confusion about $\mathcal{L}$), the set of semistable points with respect to $\mathcal{L}$ in X. We denote $X^s(\mathcal{L})$ (respectively X^s) the set of stable points in X.

Note that we use the notation $X^s(\mathcal{L})$ instead of $X_0^s(\mathcal{L})$ as in [**New78**].

Theorem 2.19 ([**New78**], Theorem 3.21)**:** *Let X be a variety on which a reductive group G acts and let $\mathcal{L}$ be an invertible sheaf over X. Then, for any G-linearization of $\mathcal{L}$*

(i) there exists a good quotient (Y, ϕ) of $X^{ss}(\mathcal{L})$ by G, and Y is quasi-projective,
(ii) there exists an open subset Y^s of Y such that $\phi^{-1}(Y^s) = X^s(\mathcal{L})$ and (Y^s, ϕ) is a geometric quotient of $X^s(\mathcal{L})$,
(iii) for $x_1, x_2 \in X^{ss}(\mathcal{L})$,

$$\phi(x_1) = \phi(x_2) \Leftrightarrow \overline{G.x_1} \cap \overline{G.x_2} \cap X^{ss}(\mathcal{L}) \neq \emptyset,$$

(iv) for $x \in X^{ss}(\mathcal{L})$, x is stable if and only if $\dim G.x = \dim G$ *and $G.x$ is closed in $X^{ss}(\mathcal{L})$.*

If X is projective and $\mathcal{L}$ is ample, then also Y is projective.

Remark 2.20: We want to point out that the above construction is a convenient way to produce good quotients, but it is by far not the only way. As we will see below, toric morphisms provide many examples of good quotients which are not even projective.

A technical tool is the *numerical criterion*: let X be a projective variety which is G equivariantly embedded in $\mathbb{P}H^0(X, \mathcal{L}) =: \mathbb{P}V$, where $\mathcal{L}$ is a very ample G-linearized invertible sheaf. Let $\lambda : k^* \rightarrowtail G$ be a one-parameter subgroup of G. We choose coordinates $x_0, \dots, x_n$ of $\mathbb{P}V$ such that $\lambda(G)$ is a diagonal subgroup of G with respect to this coordinates. The action $\lambda(G)$ then can be written as $\lambda(t).\langle x_0, \dots, x_n\rangle = \langle t^{r_0} x_0, \dots, t^{r_n} x_n\rangle$ for any point $x \in \mathbb{P}V$. We then define

$$\mu(x, \lambda) = \max\{-r_i \mid x_i \neq 0\}.$$

Theorem 2.21 ([**MFK94**], Theorem 2.1)**:** *Let G be a reductive group which acts on a projective variety X and let $\mathcal{L}$ be a G-linearized ample line bundle. Then if $x \in X$:*

$$x \in X^{ss}(L) \Leftrightarrow \mu(x, \lambda) \geq 0 \quad \textit{for all 1-psg's } \lambda$$
$$x \in X^{s}(L) \Leftrightarrow \mu(x, \lambda) > 0 \quad \textit{for all 1-psg's } \lambda$$

Definition 2.22: Let (Y, ϕ) be a good quotient of X by G and let $\mathcal{E}$ be a G-linearized sheaf on X. Then we say that $\mathcal{E}$ *descends* to Y if there exists a sheaf $\mathcal{E}'$ on Y such that there is an equivariant isomorphism $\mathcal{E} \cong \phi^* \mathcal{E}'$.

It is useful to have the following theorem, which was proven in [**Nev02**] for characteristic 0. We present a proof which works in any characteristic.

Proposition 2.23: *Let (Y, ϕ) be a good quotient of X by G and let $\mathcal{E}$ be a sheaf on Y. Then $\mathcal{E} \cong \left(\phi_* \phi^* \mathcal{E}\right)^G$.*

Proof. Because (Y, ϕ) is a good quotient, ϕ is affine and it thus suffices to show the statement for the case that $X = \operatorname{spec}(A)$ and $Y = \operatorname{spec}(A^G)$ for some finitely generated k-algebra A. In that case, denote $\Gamma(Y, \mathcal{E}) =: E$, and then $\Gamma(X, \phi^*\mathcal{E}) \cong E \otimes_{A^G} A$. The group G acts trivially over Y, so every sheaf $\mathcal{E}$ over Y is G-equivariant with respect to this trivial action. The corresponding dual action is given as:

$$E \longrightarrow k[G] \otimes_k E, \quad e \mapsto 1 \otimes e.$$

The pullback $\phi^*\mathcal{E}$ is a G-linearized sheaf over X. The associated dual action is given by

$$E \otimes_{A^G} A \longrightarrow k[G] \otimes_k \big(E \otimes_{A^G} A\big), \quad e \otimes a \mapsto \sum_i g_i \otimes (e \otimes f_i),$$

where the g_i and f_i are determined by the dual action of G on A:

$$A \mapsto k[G] \otimes_k A, \quad a \mapsto \sum_i g_i \otimes f_i$$

Using the isomorphism

$$k[G] \otimes_k \big(E \otimes_{A^G} A\big) \cong \Gamma(G \times X, p_2^* \phi^* \mathcal{E}) \cong \big(k[G] \otimes_k A\big) \otimes_{A^G} E,$$

which follows from flatness of p_2, we can write the dual action elementwise as

$$E \otimes_{A^G} A \ni e \otimes a \mapsto (\sum_i g_i \otimes f_i) \otimes e \in \big(k[G] \otimes_k A\big) \otimes_{A^G} E,$$

Then it follows that $e \otimes a$ is G-invariant if and only if the dual action on A maps a to $1 \otimes a$, that is, a is G-invariant. Now consider the canonical map $E \longrightarrow E \otimes_{A^G} A$, which is injective, then it follows that $e \otimes a$ is G-invariant if and only if $a \in A^G$ and thus $(E \otimes_{A^G} A)^G = E \otimes_{A^G} A^G \cong E$. □

The above proposition can be rephrased by saying that for any sheaf $\mathcal{E}$ over Y the pullback $\phi^*\mathcal{E}$ over X descends back to Y.

Corollary 2.24: *Let (Y, ϕ) be a good quotient of X by G. Then every sheaf on Y is a descent of a G-equivariant sheaf on X*

2.4. Diagonalizable Algebraic Groups and Tori

As a basic reference for diagonalizable groups we use [**Bor91**], in particular section III.8. A *diagonalizable* group is an algebraic group G which is isomorphic to a closed subgroup of a group of diagonal matrices in $\mathrm{GL}_n(k)$. We denote T_n the maximal subgroup of diagonal matrices in $\mathrm{GL}_n(k)$:

$$T_n = \left\{ \begin{pmatrix} * & & & \\ & * & 0 & \\ & 0 & \ddots & \\ & & & * \end{pmatrix} \in \mathrm{GL}_n(k) \right\}$$

In particular, the multiplicative group k^* of the field k has the structure of an algebraic group isomorphic to $T_1 \cong \mathrm{GL}_1(k)$.

A diagonalizable group D is uniquely described by its *character group* $X(D)$:

Theorem 2.25: *(i) $X(D)$ is a finitely generated $\mathbb{Z}$-module.*

(ii) *As a scheme, D is isomorphic to* $\mathrm{spec}(k[X(D)])$*, where $k[X(D)]$ is the group ring of $X(D)$ over k.*

(iii) $\text{max-spec}(D) = \mathrm{Hom}(X(D), k^*)$.

(iv) $X(D) = \mathrm{Mor}_{alg.gp.}(D, k^*)$.

(v) *The contravariant functor $D \mapsto X(D)$ is fully faithful from the category of diagonalizable groups over k and their morphisms as algebraic groups to the category of finitely generated $\mathbb{Z}$-modules. If* $\mathrm{char}(k) = 0$*, this functor is an equivalence of categories. If* $\mathrm{char}(k) = p > 0$*, this functor induces an equivalence of categories with the category of finitely generated $\mathbb{Z}$-modules without p-torsion.*

Definition 2.26: An *algebraic torus* is a diagonalizable group which is irreducible as a variety.

Theorem 2.27: *(i) Every algebraic torus is isomorphic to T_n for some $n \in \mathbb{N}$.*

(ii) *$X(T_n) \cong \mathbb{Z}^n$.*

(iii) *Every diagonalizable group D is isomorphic to $F \times T_n$, where F is a finite $\mathbb{Z}$-module. In particular, $X(D) \cong F \times \mathbb{Z}^n$.*

The following theorem is a basic ingredient for the combinatorial description of toric varieties:

Theorem 2.28 (Sumihiro's Theorem, [**Sum74**], Cor. 2)**:** *Let X be a normal variety on which a torus T acts. Then each $x \in X$ is contained in a T-invariant affine neighbourhood.*

Let T be an algebraic torus. Then we consider the pairing $X(T) \times X_*(T) \longrightarrow \mathbb{Z}$ given by $\langle \chi, \lambda \rangle = m$ if $(\chi \circ \lambda)(x) = x^m$. Then:

Proposition 2.29: *If T is an algebraic torus, then the canonical pairing*

$$X(T) \times X_*(T) \longrightarrow \mathbb{Z}$$

is a nondegenerate pairing over $\mathbb{Z}$.

If a diagonalizable group D acts rationally on a finite dimensional vector space V, then V decomposes into a direct sum of eigenspaces $V = \bigoplus_{\chi \in X(D)} V_\chi$. This is also true if V is not finite dimensional but has still a regular D-action. Then of course D acts diagonally on V with respect to a suitable basis of V. Let $v \in V$ be any element, then the set $D.(k \cdot v) =: D_v$ is a finite dimensional subvector space of V on which D acts rationally, such that D_v decomposes into D-eigenspaces: $D_v = \bigoplus_{\chi \in X(D)} (D_v)_\chi$. Then $v \in D_v$ decomposes into a finite sum of eigenvectors $v = \sum v_\chi$, $v_\chi \in (D_v)_\chi$ and we get:

Theorem 2.30 (The Complete Reducibility Theorem (CRT))**:** *Let D be a diagonalizable group which acts by a regular representation on a k-vector space V, which is not necessarily finite dimensional. Then V decomposes into a direct sum of eigenspaces:*

$$V = \bigoplus_{\chi \in X(D)} V_\chi$$

where $V_\chi = \{v \in V \mid g.v = \chi(g) \cdot v\}$.

The following generalizes [**Kan75**], Proposition 3.4.

Proposition 2.31: *Let $X = \operatorname{spec}(R)$ be an affine variety on which a diagonalizable group G acts. Then R has the structure of an $X(G)$-graded ring and there exists an equivalence of categories between $X(G)$-graded R-modules and G-equivariant quasicoherent sheaves over X.*

Proof. As we have seen in section 2.2, the dual action of G on R is locally finite, and thus R decomposes as k-vector space:

$$R = \bigoplus_{\chi \in X(G)} R_\chi.$$

This splitting is naturally equivalent to the structure of a $X(G)$-graded ring.

Now we make use of the fact that the Grothendieck functor $\tilde{}$ and the global section functor $\Gamma(X, .)$ define an equivalence of categories between R-modules and quasicoherent sheaves over X. Assume first that $\mathcal{E}$ is a G-linearized quasicoherent sheaf over X, then the dual action $k[X(G)] \to k[X(G)] \otimes_k \Gamma(X, \mathcal{E})$ induces a regular representation of G on $\Gamma(X, \mathcal{E})$, and thus we obtain an eigenspace decomposition

$$\Gamma(X, \mathcal{E}) = \bigoplus_{\chi \in X(G)} \Gamma(X, \mathcal{E})_\chi,$$

which naturally is compatible with the graded structure of R.

Now let $\mathcal{E} = \tilde{E}$ for some $X(G)$-graded $k[X]$-module E. Then we define an isomorphism

$$\Phi : \Gamma(G \times X, \sigma^* \mathcal{E}) \longrightarrow \Gamma(G \times X, p_2^* \mathcal{E})$$

as follows. Because X and G are affine, $G \times X$ is affine as well and there are isomorphisms

$$\Gamma(G \times X, \sigma^* \mathcal{E}) \cong (k[G] \otimes_k k[X]) \otimes_{k[X]} E, \quad \Gamma(G \times X, p_2^* \mathcal{E}) \cong k[G] \otimes_k E.$$

The vector spaces $k[G]$, $k[X]$, and E are $X(G)$-graded, and we set

$$\Phi\big((\chi \otimes f_{\chi'}) \otimes e_{\chi''}\big) = \chi\chi'' \otimes (f_{\chi'} \cdot e_{\chi''})$$

Φ is an isomorphism of $k[G \times X]$-modules, and, after composition with

$$\sigma^* : E \longrightarrow \Gamma(G \times X, \sigma^* \mathcal{E}), \quad e_\chi \mapsto (1 \otimes 1) \otimes e_\chi,$$

we get the dual action:

$$E \longrightarrow k[G] \otimes E, \quad \sum_\chi e_\chi \mapsto \sum_\chi \chi \otimes e_\chi.$$

The cocycle condition then is straightforwardly verified.

Let $\mathcal{E}$ and $\mathcal{F}$ be G-equivariant sheaves over X. Then a morphism $\phi : \Gamma(X, \mathcal{E}) \longrightarrow \Gamma(X, \mathcal{F})$ is G-equivariant if and only if $g.\phi(e)) = \phi(g.e))$ for every $g \in G$ and every $e \in \Gamma(X, \mathcal{E})$. Now $g.\phi(\sum_\chi e_\chi) = \phi(g. \sum_\chi e_\chi) = \phi(\sum_\chi \chi(g) \cdot e_\chi) = \sum_\chi \chi(g)\phi(e_\chi)$, which implies that a morphism ϕ is G-equivariant if and only if ϕ induces a graded homomorphism of graded $k[X]$-modules of degree zero. ∎

2.5. On Graded Rings

In this section we collect some standard facts on graded commutative rings and modules. The proofs of the statements below which are not explicitly given are more or less straightforwardly the same as their analogs in [**AM69**] and [**SS94**], [**SS88**].

Let G be an abelian group and let R be a G-*graded* ring. A module M over a G-graded ring R is called a *graded R-module* if M has a decomposition into a direct sum $M = \bigoplus_{g\in G} M_g$ of abelian subgroups M_g of M such that $r \cdot m \in M_{gg'}$ for all $r \in R_g, m \in M_{g'}$. If M is an ideal of R, it is also called *homogeneous*. We denote by $M(g)$ the degree shift such that $M(g)_{g'} = M_{gg'}$. Let M, N be two G-graded R-modules. A homomorphism $f : M \longrightarrow N$ of two G-graded R-modules M and N is a *graded* or *homogeneous homomorphism* if $f(M_g) \subset N_g$ for all $g \in G$. We can define a graded structure on the tensor product $M \otimes_R N$ as follows: denote by $T := M \otimes_{\mathbb{Z}} N$ the tensor product of M and N over $\mathbb{Z}$. If we consider $\mathbb{Z}$ as a trivially G-graded ring (i.e. $\mathbb{Z}_1 = \mathbb{Z}$ and $\mathbb{Z}_g = 0$ for all $1 \neq g \in G$) then T has a G-grading by setting T_g to be the additive group generated by the set $\{m \otimes n \mid m \in M_{g'}, n \in N_{g''}$ such that $g'g'' = g\}$ for all $g \in G$. Further let K be the G-graded submodule of T generated by $\{(rm) \otimes n - m \otimes (rn) \mid m \in M, n \in N, r \in R\}$. There is an isomorphism $T/K \cong M \otimes_R N$ such that $M \otimes_R N$ acquires the structure of a G-graded module.

Remark 2.32: If R is trivially G-graded then $M \otimes_R N$ can be endowed directly with a graded structure by setting $(M \otimes_R N)_g$ to be the additive group generated by $\{m \otimes_R n \mid m \in M_{g'}, n \in N_{g''}$ such that $g'g'' = g\}$.

A *morphism of graded rings* is a pair of morphisms $(\psi, \chi) : (R, G) \longrightarrow (R', G')$ such that $\psi(R_g) \subset R'_{\chi(g)}$. Such a morphism (ψ, χ) induces a G'-grading on R by setting $R_{g'} := \bigoplus_{g\in\chi^{-1}(g')} R_g$. Also for a given graded G-module M the grading can also be changed to a G'-grading. Thus:

Definition 2.33: Let $(\psi, \chi) : (R, G) \longrightarrow (R', G')$ be a morphism of commutative graded rings and let M be a G-graded R-module. Then we call the G'-graded R'-module $M \otimes_R R'$ the *graded scalar extension*.

Definition 2.34: Let R be a G-graded ring. A *maximal homogeneous ideal* is a homogeneous ideal of R which is not contained in any other homogeneous ideal of R. The intersection of all maximal homogeneous ideals is called the *homogeneous Jacobson radical* $\mathfrak{R}_G$.

Note that a maximal homogeneous ideal is not necessarily a maximal ideal.

Theorem 2.35 (Graded Version of Krull's Theorem)**:** *Let R be a graded ring. Then each proper homogeneous ideal is contained in a maximal homogeneous ideal.*

Proof. Clearly the set of homogeneous ideals is partially ordered by inclusion. Let $\{\mathfrak{g}_i\}_{i\in I}$ be a chain of homogeneous ideals. Then the ideal $\bigcup_{i\in I} \mathfrak{g}_i$ is also homogeneous. So the homogeneous ideals are even *inductively* ordered, hence we can apply Zorn's lemma and obtain that there exist maximal homogeneous ideals. □

As a consequence we have the

Corollary 2.36: *(i) Every homogeneous non-unit of R is contained in a maximal homogeneous ideal.*

(ii) If the zero ideal is the unique maximal homogeneous ideal of a G-graded ring R, then each finitely generated graded module over R is free.

(iii) Let R be a G-graded ring and assume that there exists a unique maximal homogeneous ideal $\mathfrak{m}$ in R. The ring $R/\mathfrak{m}$ is then also graded and each finitely generated graded module over $R/\mathfrak{m}$ is free.

Compare the following theorem to [**NvO82**], lemma A.I.7.5 and [**BPCM98**].

Theorem 2.37 (Graded Nakayama's lemma)**:** *Let R be a G-graded ring, and let M be a finitely generated graded R-module. Let $\mathfrak{a} \subset \mathfrak{R}_G$ be a graded ideal such that $\mathfrak{a}M = M$. Then $M = 0$.*

PROOF. Assume that $M \neq 0$, and let $u_1, \dots, u_n$ be a minimal set of generators of M. Then $u_n \in \mathfrak{a}M$, hence there exists an equation of the form $u_n = \sum_{i=1}^{n} a_i u_i$ where $a_i \in \mathfrak{a}$. Write $(1-a_n)u_n = \sum_{i=1}^{n-1} a_i u_i$. Since $a_n \in \mathfrak{a}$, by Corollary 2.36, $1-a_n$ is a unit in R. Hence $u_n \in \operatorname{span}_A\{u_i\}_{i=1,\dots,n-1}$ which contradicts the minimality of the generating set. □

Corollary 2.38: *Assume that there exists a unique maximal homogeneous ideal $\mathfrak{m}$ in R. Let M be a finitely generated graded R-module. If M is projective, then M is free.*

CHAPTER 3

Toric Varieties

A toric variety is a normal variety X together with an open and dense embedding of an algebraic torus

$$T \hookrightarrow X$$

such that the torus multiplication extends to an action of T on X:

$$\begin{array}{ccc} T \times T & \xrightarrow{\mu} & T \\ \big\downarrow & & \big\downarrow \\ T \times X & \xrightarrow{\sigma} & X. \end{array}$$

This chapter gives an exposition of the theory toric varieties. Our aim is to present this theory from the point of view of semigroup rings similarly as in older sources such as [**KKMS73**] and [**MO78**], but only as far as it is relevant for our work. This in particular excludes cohomologies and classification theory for toric varieties. Instead, we have included material on toric morphisms and quotient presentations which are not present in standard text books.

In the first three sections we introduce the basic notions of toric geometry. In section 3.1 we present the standard notions of convex geometry relevant for toric geometry. In section 3.2 we introduce affine toric varieties and describe their orbit structure, and in section 3.3 general toric varieties are introduced. Principal references for these sections are [**MO78**], [**Ful93**], [**Oda88**] and [**KKMS73**].

In section 3.4 we introduce toric morphisms and give a partial study of the question whether a surjective equivariant morphism of toric varieties represents a quotient. For surjective toric morphisms $X \longrightarrow Y$ we give criteria whether Y is a categorical, good or geometric quotient of X by some diagonal subgroup of the torus T acting on X. These criteria are motivated by the question which kind of quotient is represented by a quotient presentation for a toric variety (see section 3.6). This is related to and kind of inverse of the question whether there exist quotients of X by actions of subtori of T which can explicitly be constructed as toric varieties. This question has been studied in the literature before, see [**AH99a**], [**AH99b**] and [**AH00**], and [**AH03**] for a review.

In section 3.5 we briefly explain how orbit closures of T in X can be described as toric varieties.

Finally, in section 3.6 we introduce the general concept of quotient presentations for toric varieties. Historically, quotient presentations are attributed to Cox ([**Cox95**]) and Audin ([**Aud91**]) and some others (see references in [**Cox95**]). The advantage of quotient presentations is that they allow – as analog to projective spaces – the introduction of global coordinates and the treatment of sheaves over X as graded modules over some global coordinate ring. Quotient presentations in general are not unique, and there might be different good quotient presentations for different purposes (see for instance [**Kaj98**], [**Per02c**] and [**AHS02**]). However, it seems that Cox' original version of a quotient presentation is the most user friendly, at least for our purposes. So, later on we will only use Cox' version.

General conventions: in the theory of toric varieties it is standard to denote $M := X(T)$ the character group of T, $N = M\check{}$ the dual group group of one-parameter subgroups of T, $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$, $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$. The pairing $\langle\, .\, ,\, .\, \rangle$ between M and N extends naturally to a pairing between $M_{\mathbb{R}}$ and $N_{\mathbb{R}}$. For any character $m \in M$ we use the convention to write $\chi(m)$ if we want to write the group law in M multiplicatively, and m if we want to write it additively, i.e. $\chi(m + m') = \chi(m) \cdot \chi(m')$. In particular, we will use the notation $\chi(m)$ if we consider $\chi(m)$ as a regular function over the torus T. Recall that the coordinate ring of T is isomorphic to the group ring

$$k[T] = k[M] = \bigoplus_{m \in M} k \cdot \chi(m).$$

For a toric variety X, it is convention to consider the following dual action on the coordinate ring for any $f \in k[X]$ and $t \in T$:

$$(t.f)(x) = f(t.x).$$

instead of $(t.f)(x) = f(t^{-1}.x)$. This is well-defined because T is abelian. This convention is in order to be compatible with the group multiplication law of T: for any $t, t' \in T$ the characters $\chi(m)$ are regular functions over T, and by the above convention

$$(t.\chi(m))(t') = \chi(m)(t) \cdot \chi(m)(t') = \chi(m)(t \cdot t').$$

3.1. Convex Geometry

Let $V_{\mathbb{Z}} \cong \mathbb{Z}^n$ be a free $\mathbb{Z}$-module of finite rank, and let $V_{\mathbb{Q}} = V_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}$ and $V = V_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{R}$ the $\mathbb{Q}$-, respectively $\mathbb{R}$-vector spaces obtained by scalar extension, with the canonical inclusions

$$V_{\mathbb{Z}} \subset V_{\mathbb{Q}} \subset V.$$

A subset σ of V is called a *convex rational polyhedral cone (crpc)* if there exists a finite number of vectors $v_1, \ldots, v_s$ in $V_{\mathbb{Q}}$ such that $\sigma = \mathrm{span}_{\mathbb{R}_{\geq 0}}\{v_i | i = 1, \ldots, s\}$. σ is called a *strongly* convex rational polyhedral cone (*s*crpc) if $\sigma \cap -\sigma = \{0\}$. By definition, a crpc is a convex subset of V, and a scrpc is a crpc that does not contain any proper linear subspace of V. For any crpc σ there exists some minimal subvector space V_σ of V which contains σ. Then $\dim V_\sigma$ is the *dimension* of σ. The *interior* $\mathrm{int}\, \tau$ of τ is the interior of τ as a subset of V_τ.

Let $V\check{}$ be the vector space dual to V and let $v \in V\check{}$, then the *half space* H_v *orthogonal to* v is:

$$H_v := \{u \in V \mid v(u) \geq 0\}$$

A convex rational polyhedral cone is the intersection of finitely many half spaces.

A *face* of σ is the intersection $\sigma \cap \partial H_v$ where ∂H_v is the boundary hyperplane of a half space H_v containing σ. Every face of σ is also a crpc. Any cone is the disjoint

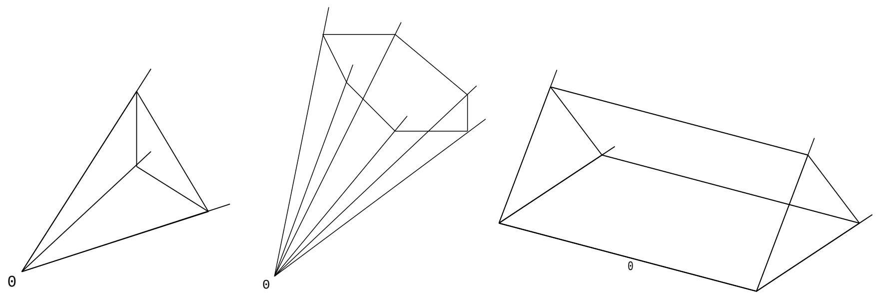


FIGURE 3.1. Various three-dimensional cones

union of the interiors of all of its faces:

$$\sigma = \dot{\bigcup_{\tau < \sigma}} \operatorname{int} \tau.$$

Proposition 3.1 ([**Oda88**], Proposition A.5)**:** *The set $\mathcal{F}(\sigma)$ of faces of a crpc σ is a finite partially ordered set with respect to the face relation $<$. σ is the largest element, while the smallest is $\sigma \cap (-\sigma)$, which is the largest $\mathbb{R}$-vector subspace of V contained in σ. Moreover, $\mathcal{F}(\sigma)$ is an abstract complex in the following sense: $\tau \in \mathcal{F}(\sigma)$ and $\rho < \tau$ imply $\rho \in \mathcal{F}(\sigma)$ and the intersection $\tau_1 \cap \tau_2$ of $\tau_1, \tau_2 \in \mathcal{F}(\sigma)$ is a face of τ_1 as well as of τ_2.*

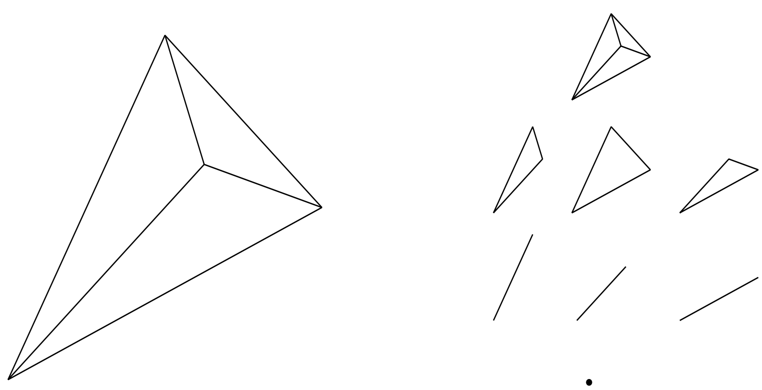


FIGURE 3.2. A three-dimensional cone and its cells.

Definition 3.2:
- If σ is a crpc, then $\sigma(i) := \{\tau < \sigma \mid \dim(\tau) = i\}$;
- scrpc's of dimension 1 are called *rays*;
- We call the first integral point on a ray ρ *primitive lattice vector* (or simply *primitive* of ρ) and denote it $n(\rho)$.

Note that the set of generators of a cone σ over $\mathbb{Q}$ can always be chosen as $\{n(\rho) \mid \rho \in \sigma(1)\}$.

Definition 3.3: Let σ be a crpc. Then

$$\check{\sigma} := \{w \in V^{\check{}} \mid \langle w, v\rangle \geq 0 \text{ for all } v \in \sigma\} = \bigcap_{\rho\in\sigma(1)} H_{n(\rho)}$$

is the *dual* cone of σ and

$$\sigma^{\perp} = \{w \in V^{\check{}} \mid \langle w, v\rangle = 0 \text{ for all } v \in \sigma\} = \bigcap_{\rho\in\sigma(1)} \partial H_{n(\rho)}$$

is the biggest linear subspace of $V^{\check{}}$ contained in $\check{\sigma}$.

We can interchange the roles of V and $V^{\check{}}$ and we get immediately that $\check{\check{\sigma}} = \sigma$. Note that if σ is strictly convex, then $\check{\sigma}$ has the same dimension as $V^{\check{}}$, and if σ has full dimension, then $\check{\sigma}$ is strictly convex. Dualising defines a one-to-one, order reversing correspondence between the faces of σ and $\check{\sigma}$ which is given by:

$$\sigma > \tau \mapsto \tau^{\perp} \cap \check{\sigma} = \left(\bigcap_{\rho\in\tau(1)} \partial H_{n(\rho)}\right) \cap \check{\sigma} < \check{\sigma}.$$

3.2. Affine Toric Varieties

Let X be an affine toric variety. The dual action of T on the coordinate ring $k[X]$ induces a decomposition

$$k[X] = \bigoplus_{m\in M} k[X]_m.$$

The torus T is densely and equivariantly contained in X, such that $k[X]$ becomes a graded subring of the group ring $k[T] = k[M]$:

$$k[X] = \bigoplus_{m\in\sigma_M} k \cdot \chi(m),$$

where the subset $\sigma_M \subset M$ is a finitely generated subsemigroup of M.

Proposition 3.4: *The coordinate ring of X is canonically isomorphic to a semigroup ring*

$$k[X] = k[\sigma_M]$$

generated by a subsemigroup σ_M of M.

The characters $\chi(m)$ for $m \in M$ are regular functions over $T \subset X$ and thus rational functions over X, as T is dense in X. The semigroup σ_M consists of precisely those characters which can be continued to X.

Definition 3.5: We denote U_σ an affine toric variety whose coordinate ring is isomorphic to the semigroup ring $k[\sigma_M]$.

Any commutative square

$$\begin{array}{ccc} T \times U_\sigma & \xrightarrow{(\phi,\psi)} & T' \times U_{\sigma'} \\ \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \mu'} \\ U_\sigma & \xrightarrow{\psi} & U_{\sigma'} \end{array}$$

induces a group homomorphism $\psi^\sharp : M' \longrightarrow M$ and a semigroup homomorphism $\phi^\sharp : \sigma'_{M'} \longrightarrow \sigma_M$ such that the following diagram commutes:

$$\begin{array}{ccc} M' & \xrightarrow{\psi^\sharp} & M \\ \downarrow{\scriptstyle \mu'^\sharp} & & \downarrow{\scriptstyle \mu^\sharp} \\ M' \times \sigma'_{M'} & \xrightarrow{(\psi^\sharp,\phi^\sharp)} & M \times \sigma_M \end{array}$$

where $\mu^\sharp$ and $\mu'^\sharp$ are the diagonal morphisms $\sigma_M \longrightarrow \sigma_M \times M$ and $\sigma'_{M'} \longrightarrow \sigma'_{M'} \times M'$, respectively, and $\phi^\sharp$ is the restriction of $\psi^\sharp$ to $\sigma'_{M'}$. Note that the diagonal morphisms $\mu^\sharp$ are compatible with the Hopf algebra structure corresponding to the group multiplication law of T.

Proposition 3.6: *There is an equivalence of categories between affine toric varieties U_σ together with equivariant maps, and semigroup rings $k[\sigma_M]$ with graded morphisms being induced by linear maps $M \longrightarrow M'$.*

We can describe a σ_M in terms of convex geometry:

Proposition 3.7 ([**Oda88**], §'s 1.1, 1.2)**:** *Let U_σ be an affine toric variety. Then the semigroup σ_M is of the form $\sigma_M = \check{\sigma} \cap M$, where $\check{\sigma} \subset M_{\mathbb{R}} := M \otimes_{\mathbb{Z}} \mathbb{R}$ is a convex rational polyhedral cone with* $\dim \check{\sigma} = \dim M_{\mathbb{R}}$. *This is equivalent to the following properties:*

(i) *σ_M is finitely generated as a semigroup.*
(ii) *σ_M generates M as a group, i.e. each element $m \in M$ is of the form $m = m_1 + m_2$ with $m_1 \in \sigma_M$ and $m_2 \in -\sigma_M$.*
(iii) *σ_M is saturated, i.e. if $c \cdot m \in \sigma_M$ with $0 < c \in \mathbb{N}$, and $m \in M$, then $m \in \sigma_M$.*

Remark 3.8: Saturatedness of σ_M is a consequence of normality of U_σ. More generally, one can consider not necessarily normal toric varieties, where the corresponding semigroup σ_M is not of the above form. However, it can be shown that σ_M then must be contained in a cone $\check{\sigma}$ and the inclusion $\sigma_M \hookrightarrow M' \cap \check{\sigma}$, where $M' \hookrightarrow M$ is the sublattice of M generated by σ_M, is a morphism of semigroups which induces a dominant morphism $\operatorname{spec}(k[\check{\sigma} \cap M']) \longrightarrow \operatorname{spec}(k[\sigma_M])$. The affine variety $\operatorname{spec}(k[\check{\sigma} \cap M'])$ is a toric variety on which the torus $T' = \operatorname{Hom}(M', k^*)$ acts, and the morphism then coincides with the normalization of $k[\sigma_M]$.

Let I be any semigroup ideal of σ_M, which by definition is a subset $I \subset \sigma_M$ such that $\sigma_M + I \subset I$. Then we can define a homogeneous ideal of $k[\sigma_M]$ by

$$\mathfrak{I} = \bigoplus_{m \in I} k \cdot \chi(m),$$

and conversely, any homogeneous ideal $\mathfrak{I}$ defines a semigroup ideal by its set of nonzero degrees:

$$I = \{m \in M \mid \mathfrak{I}_m \neq 0\}.$$

This way there is a one-to-one correspondence between homogeneous ideals in $k[\sigma_M]$ and semigroup ideals in σ_M. In terms of generators, $\mathfrak{I}$ is generated by characters

$\chi(m_1), \ldots, \chi(m_r)$ if and only if the corresponding semigroup ideal I is of the form $I = \{\sum_{i=1}^{r}(\sigma_M + m_i)\}$.

Let $\mathfrak{I}$ be a T-invariant ideal of $k[\sigma_M]$ and $k[\sigma_M]/\mathfrak{I}$ its quotient ring. Both objects have an induced M-grading:

$$\mathfrak{I} = \bigoplus_{m \in I} \mathfrak{I}_m, \quad k[\sigma_M]/\mathfrak{I} = \bigoplus_{m \in \sigma_M \setminus I} (k[\sigma_M]/\mathfrak{I})_m.$$

The pair $(\mathbf{V}(\mathfrak{I}), \mathcal{O}_{\mathbf{V}(\mathfrak{I})})$ is a T-invariant affine subscheme of U_σ whose ring of regular functions $\Gamma(\mathbf{V}(\mathfrak{I}), \mathcal{O}_{\mathbf{V}(\mathfrak{I})})$ is isomorphic to $k[\sigma_M]/\mathfrak{I}$.

If $\sigma_M \setminus I$ is a subsemigroup of σ_M then there is an inclusion of rings $k[V(\mathfrak{I})] \hookrightarrow k[\sigma_M]$ which commutes with the surjection of $k[\sigma_M]$ onto $k[V(\mathfrak{I})]$ and we have the following commutative equivariant inclusion/retraction diagram:

$$\mathbf{V}(\mathfrak{I}) \underset{\twoheadleftarrow}{\hookrightarrow} U_\sigma$$

There is a nice characterisation of subsemigroups of σ_M of the form $\sigma_M \setminus I$ for semigroup ideals I (see [**MO78**], §5):

Proposition 3.9: *Let $\mathfrak{I}$ be a homogeneous ideal of $k[\sigma_M]$. Then there exists a retraction $U_\sigma \twoheadrightarrow (\mathbf{V}(\mathfrak{I}), \mathcal{O}_{\mathbf{V}(\mathfrak{I})})$ if and only if $\mathfrak{I}$ is a prime ideal. Moreover $\mathfrak{I}$ is a prime ideal if and only if there exists a face F of $\check{\sigma}$ such that $\sigma_M \setminus I = F \cap M$.*

And consequently:

Corollary 3.10: *There is a one-to-one correspondence between the faces of $\check{\sigma}$ and irreducible T-invariant subvarieties of U_σ. These subvarieties inherit the partial ordering from the faces.*

For the study of general, non-affine toric varieties later on, it will be convenient not to work with cones $\check{\sigma}$ contained in $M_\mathbb{R}$, but rather to consider them as *duals* of cones σ contained in $N_\mathbb{R}$. As we have stated above, the $\check{\sigma}$ have full dimension in $M_\mathbb{R}$ and it follows that σ is strictly convex.

Definition 3.11: We denote

$$\sigma_M^\perp := \sigma^\perp \cap M$$

the maximal subgroup of σ_M.

Definition 3.12: Let $\tau < \sigma$, then we denote $V(\tau)$ the T-invariant closed subvariety of U_σ associated to the face $\tau^\perp \cap \check{\sigma}$ of $\check{\sigma}$.

We have $(\sigma \cap \tau^\perp) \cap M = \sigma_M \cap \tau_M^\perp$ and:

Proposition 3.13: *Let $\tau < \sigma$, then the coordinate ring of $V(\tau)$ can be described as*

$$k[V(\tau)] = k[\sigma_M \cap \tau_M^\perp].$$

In particular, $\dim V(\tau) = \operatorname{codim}_{N_\mathbb{R}} \tau$.

Theorem 3.14: *Let σ be a scrpc in $N_\mathbb{R}$. Then there is a one-to-one correspondence between affine toric open subvarieties of U_σ and the faces of σ. Moreover, the coordinate* *ring of* U_τ *for* $\tau < \sigma$ *is isomorphic to the localization* $k[\sigma_M]_{\chi(m_\tau)}$*, where* m_τ *is an integral element in the interior of* $\tau^\perp \cap \check{\sigma}$*. Equivalently, the semigroup* τ_M *is given by* $\sigma_M + \mathbb{Z}_{\geq 0}(-m_\tau)$*.*

U_σ has a unique minimal T-invariant closed subvariety, $V(\sigma)$, which by 2.5 is a minimal orbit of T in U_σ.

Definition 3.15: The minimal orbit of U_σ is denoted by $\mathrm{orb}(\sigma)$.

By theorem 3.14, we have thus an orbit decomposition of U_σ:

Corollary 3.16:

$$U_\sigma = \dot{\bigcup_{\tau<\sigma}} \mathrm{orb}(\tau)$$

and if $\rho < \tau$ *then* $\mathrm{orb}(\tau) \subset V(\rho)$*.*

Note that the complement of $\sigma_M^\perp$ in σ_M is precisely the set of non-units in the semigroup σ_M and thus $\sigma_M \backslash \sigma_M^\perp$ is the maximal semigroup ideal in σ_M. Correspondingly, $\bigoplus_{m\in\sigma_M\backslash\sigma_M^\perp} k \cdot \chi(m)$ is the maximal homogeneous ideal of $k[\sigma_M]$. Because $\sigma_M^\perp$ is a subgroup of σ_M, we have:

Proposition 3.17: $k[\sigma_M]$ *has a unique maximal homogeneous ideal which is prime.*

We have $k[\sigma_M^\perp] \cong k[\mathrm{orb}(\sigma)]$, i.e. $\mathrm{orb}(\sigma) \cong \mathrm{spec}(k[\sigma_M^\perp])$. From a decomposition $M = \sigma_M^\perp \oplus M/\sigma_M^\perp$ we obtain the decomposition $\sigma_M = \sigma_M^\perp \times \sigma_M'$ where σ_M' is a subsemigroup of $M/\sigma_M^\perp$ which is the image of the projection $M \twoheadrightarrow M/\sigma_M^\perp$, and thus by $k[\sigma_M^\perp \times \sigma_M'] \cong k[\sigma_M^\perp] \otimes k[\sigma_M']$ we get:

Proposition 3.18: *Let* U_σ *be an affine toric variety. Then* U_σ *is isomorphic to* $\mathrm{orb}(\sigma) \times U_{\sigma'} \cong (\mathbb{C}^*)^l \times U_{\sigma'}$ *where* l *is the rank of* $\sigma^\perp$ *and* $U_{\sigma'}$ *is the toric variety associated to the semigroup* σ_M' *which is the image of* σ_M *under the projection* $M_\mathbb{R} \twoheadrightarrow M_\mathbb{R}/\sigma^\perp$*.*

Proposition 3.19: *Let* U_σ *be an affine toric variety associated to a cone* σ*. Let* $U_\tau \subset U_\sigma$ *be an affine open toric subvariety associated to a face* $\tau < \sigma$*. Then* $U_\sigma \setminus U_\tau$ *is a hypersurface of* U_σ*.*

Proof. The complement of U_τ is of the form

$$U_\sigma \setminus U_\tau = \dot{\bigcup_{\rho<\sigma,\rho\not<\tau}} \mathrm{orb}(\rho)$$

Each face whose orbit is contained in this complement has at least one ray which is not contained in τ. Thus for each $\rho < \sigma$ with $\rho \not< \tau$ there exists a ray $\xi \in \rho(1) \setminus \sigma(1)$ such that $\mathrm{orb}(\rho) \subset V(\xi) \subset U_\sigma \setminus U_\tau$. Hence:

$$U_\sigma \setminus U_\tau = \bigcup_{\xi\in\sigma(1)\setminus\tau(1)} V(\xi)$$

which is a union of irreducible hypersurfaces. □

Theorem 3.20 ([**Oda88**], Theorem 1.10, affine version)**:** *The affine toric variety U_σ associated to a cone σ is nonsingular if and only if there exists a $\mathbb{Z}$-basis $n_1, \ldots, n_r$ of N and $s < r$ such that $\sigma = \sum_{i=1}^{s} \mathbb{R}_{\geq 0} n_i$.*

Definition 3.21: We say that a cone σ is *simplicial* iff $\dim \sigma = \#\sigma(1)$.

3.3. General Toric Varieties

Let X be a general toric variety. By Sumihiro's theorem (see 2.28) there exists an open cover of X by affine toric varieties such that for every point $x \in X$ there exists an open neighbourhood which is isomorphic to some U_σ for a convex rational polyhedral cone σ.

For any two T-invariant affine subsets $U_\sigma, U_{\sigma'}$ of X, their intersection $U_\sigma \cap U_{\sigma'}$ again is a T-invariant affine open subset U_τ of X which is associated to a cone τ. The cone τ is a face of both, σ and σ' and it is the intersection of σ and σ' in $N_\mathbb{R}$:

$$U_\sigma \cap U_{\sigma'} = U_{\sigma \cap \sigma'}$$

In general, a toric variety can thus be described by the following data:

Proposition & Definition 3.22: *Associated to a toric variety is a* finite *collection Δ of cones satisfying:*

(i) Every face of any $\sigma \in \Delta$ is contained in Δ.
(ii) For any $\sigma, \sigma' \in \Delta$ the intersection $\sigma \cap \sigma'$ is contained in Δ as well.

A collection of cones satisfying these properties is called a fan*. We denote by $|\Delta|$ the union of all cones in Δ. If we want to emphasize that a toric variety X is associated to a fan Δ, we denote this variety X_Δ.*

By gluing, on can construct for every fan Δ a toric variety X_Δ.

If there is a collection $\{\sigma_i\}$ of cones all contained in the same vector space V such that $\sigma_i \cap \sigma_j < \sigma_i, \sigma_j$, then the union of these cones and all their faces is also a fan. We speak of a fan that is *generated* by these cones.

Example 3.23: Figure 3.3 shows the fans for $\mathbb{P}_2$ and the Hirzebruch surface $\mathbb{F}_a \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}_1(\mathbb{C})} \oplus \mathcal{O}_{\mathbb{P}_1(\mathbb{C})}(a))$. Here $n_1 = (1, 0)$ and $n_2 = (0, 1)$ in $N \cong \mathbb{Z}^2$.

The orbit decomposition for affine toric varieties carries over to the general case:

Proposition 3.24 ([**Oda88**], Proposition 1.6)**:** *X_Δ is the disjoint union of the orbits associated to the cones in Δ:*

$$X_\Delta = \dot{\bigcup_{\sigma \in \Delta}} \operatorname{orb}(\sigma)$$

Proposition 3.25: *Let $\sigma \in \Delta$ be a cone. Then we denote by $V(\sigma) := \overline{\operatorname{orb}(\sigma)}$ the closure of the orbit associated to σ. $V(\sigma)$ is a closed subvariety of X_Δ and has the structure:*

$$V(\sigma) = \dot{\bigcup_{\tau > \sigma}} \operatorname{orb}(\tau).$$

Definition 3.26: Let Δ be a fan. Then denote:

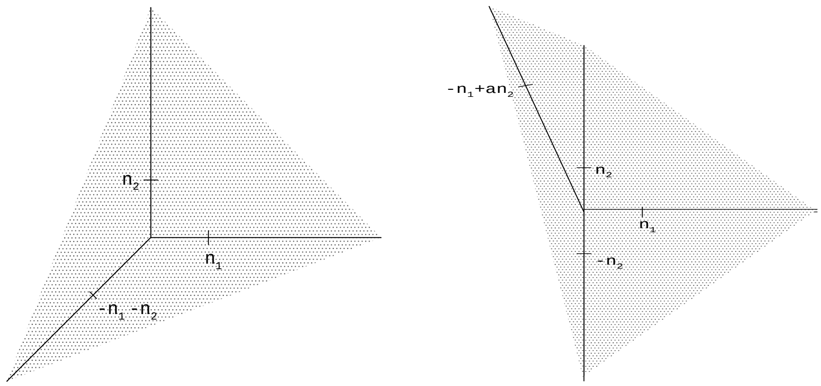


FIGURE 3.3. Fans for $\mathbb{P}_2$ and a Hirzebruch surface $\mathbb{F}_a$

- $\Delta(i) := \{\sigma \in \Delta \mid \dim \sigma = i\}$ the set of all cones of fixed dimension i.
- $\Delta_i := \bigcup_{j \leq i} \Delta(j)$ the union of all cones of dimension smaller or equal i.
- $\Delta_{\max} := \{\sigma \in \Delta \mid \sigma \not< \tau \; \forall \tau \in \Delta\}$ the set of maximal cones.
- $|\Delta| := \bigcup_{\sigma \in \Delta} \sigma$ the *support* of the fan,
- $D_\rho := V(\rho)$ for $\rho \in \Delta(1)$, the T-invariant *Weil divisor* associated to ρ,
- $\mathbb{Z}^{\Delta(1)}$ the group which is freely generated over the T-invariant Weil divisors D_ρ.

In the sequel, we will always assume that a fan $|\Delta|$ spans the whole space $N_\mathbb{R}$ over $\mathbb{R}$, except if stated otherwise. Note that if $|\Delta|$ spans a proper subspace $N'_\mathbb{R}$ of dimension k of $N_\mathbb{R}$, then the toric variety decomposes as $X_\Delta = (k^*)^{\dim T - k} \times X_{\Delta'}$, where $\Delta' = \{\sigma \cap N'_\mathbb{R}\}$ is the same fan considered to lie in N'.

Theorem 3.27 (cf. 3.20)**:** *A toric variety X associated to a fan Δ is nonsingular if each cone $\sigma \in \Delta$ is nonsingular in the sense of theorem 3.20.*

Theorem 3.28 ([**Oda88**], Theorem 1.11)**:** *A toric variety X_Δ is complete if and only if Δ is a finite and complete fan, i.e. Δ is finite and the support $|\Delta|$ is equal to $N_\mathbb{R}$.*

By [**MO78**], Proposition 6.1 and [**Ful93**], §3.4, for X an n-dimensional toric variety there exists a short exact sequence:

$$0 \longrightarrow M \longrightarrow \mathbb{Z}^{\Delta(1)} \longrightarrow A_{n-1}(X) \longrightarrow 0, \tag{2}$$

where the embedding $0 \longrightarrow M \longrightarrow \mathbb{Z}^{\Delta(1)}$ is defined by $m \mapsto (\langle m, n(\rho)\rangle)$. The group $A_{n-1}(X)$ is thus generated by the T-invariant Weil divisors. This sequence exists for all toric varieties whose fan is not contained in a proper subspace of $N_\mathbb{R}$.

3.4. Toric Morphisms and Quotients

Definition 3.29: Let N, N' be free $\mathbb{Z}$-modules and Δ, Δ' be fans in $N_\mathbb{R}$ and $N'_\mathbb{R}$. Then a *map of fans* $\phi : (N, \Delta) \longrightarrow (N', \Delta')$ is a homomorphism of $\mathbb{Z}$-modules $\phi : N \longrightarrow N'$ such that for each $\sigma \in \Delta$ there exists a $\sigma' \in \Delta'$ with $\phi_\mathbb{R}(\sigma) \subset \sigma'$, where $\phi_\mathbb{R} : N_\mathbb{R} \longrightarrow N'_\mathbb{R}$ is the vector space homomorphism induced by scalar extension.

Equivalently, a map of fans is defined by specifiying a homomorphism of the dual $\mathbb{Z}$-modules, $\check{\phi} : M' \longrightarrow M$ such that for each $\sigma' \in \Delta'$ there exists a $\sigma \in \Delta$ such that

$\check{\phi}_{\mathbb{R}}(\check{\sigma}') \subset \check{\sigma}$. This functorially defines an equivariant map of toric varieties $\phi_* : X_\Delta \longrightarrow X_{\Delta'}$ and $\phi_* |_T : T \longrightarrow T'$, content of

Theorem 3.30 ([**Oda88**], Theorem 1.13)**:** *A map of fans $\phi : (N, \Delta) \longrightarrow (N', \Delta')$ gives rise to an equivariant morphism*

$$(\phi_*, \phi_* |_T) : (X_\Delta, T) \longrightarrow (X_{\Delta'}, T')$$

Conversely, suppose $(f, f') : (X_\Delta, T) \longrightarrow (X_{\Delta'}, T')$ is an equivariant morphism of toric varieties such that $f' = f|_T$. Then there exists a unique map of fans $\phi : (N, \Delta) \longrightarrow (N', \Delta')$ such that $f = \phi_*$.

Corollary 3.31: *There is an equivalence of categories between the category of toric varieties with equivariant morphisms and the category of fans whose morphisms are maps of fans.*

Consider any map of fans $\phi : (N, \Delta) \longrightarrow (N', \Delta')$ and $\phi_* : X_\Delta \longrightarrow X_{\Delta'}$ the corresponding toric morphism. We obtain an exact sequence of abelian groups

$$0 \longrightarrow G \longrightarrow T \xrightarrow{\phi_*} T' \longrightarrow T'/\bar{T} \longrightarrow 0$$

where $\bar{T}$ denotes the image of T in T'. The kernel G is a diagonalizable, though in general not connected, subgroup of T. The character group of G is given by the corresponding dual homomorphism of $\mathbb{Z}$-modules and can be naturally identified as $X(G) = M/\bar{M}$, where $\bar{M}$ denotes the image of M' in M.

Consider the case, where $\operatorname{rk} N = \operatorname{rk} N'$, Δ and Δ' both are generated by one cone, σ and τ, respectively, and ϕ is an injective map from N to N' such that ϕ over $\mathbb{R}$ maps σ bijectively to τ. This implies that $N'/\phi(N) = \operatorname{coker} \phi$ is a finite group, and therefore we obtain a dual short exact sequence

$$0 \longrightarrow M' \xrightarrow{\phi^T} M \longrightarrow X(G) \longrightarrow 0$$

where $X(G)$ is finite (Note that $X(G) \cong G$). In particular, we obtain an injective map $k[\tau_M] \rightarrowtail k[\sigma_M]$ by which $k[\tau_M]$ can be identified with the degree zero part of the $k[\sigma_M]$ with respect to its induced $X(G)$-grading. This in turn can be interpreted as the invariant subring of $k[\sigma_M]$ with respect to the action of G on U_σ, and U_τ then can be interpreted as a good quotient of U_σ by G. Because $X(G)$ is finite, all its orbits in U_σ, are closed, and it follows that U_τ is even a geometric quotient of U_σ by G. The most interesting case is that where τ is a *simplicial* cone, which over $\mathbb{R}$ always is isomorphic to the positive orthant $(\mathbb{R}_{\geq 0})^n$ of some $\mathbb{R}^n$:

Proposition 3.32: *Let σ be a simplicial cone such that* $\dim \sigma = \operatorname{rk} N$. *Then U_σ is isomorphic to a geometric quotient k^n/G, where G is isomorphic to N modulo the sublattice generated by the primitive vectors $n(\rho)$, $\rho \in \sigma(1)$, over $\mathbb{Z}$.*

In general, the image of ϕ_* is a constructible, $\bar{T}$-invariant subset of $X_{\Delta'}$ which contains $\bar{T}$ as a dense open subset. Its closure coincides with the so-called *scheme theoretic image* of ϕ_* (see [**GD71**], §I,6.10 or [**Har77**], ex. II.3.11 (d)). We call the scheme theoretic image Y for the moment. Then Y is the unique closed subscheme of $X_{\Delta'}$ which has the universal property that if ϕ_* factorizes through any other closed subset Z of $X_{\Delta'}$, then the canonical inclusion of Y in $X_{\Delta'}$ factorizes through Z. The action of $\bar{T}$ extends to Y, as it is the closure of $\bar{T}$ in $X_{\Delta'}$ and thus Y becomes a toric variety.

Proposition 3.33: *The scheme theoretic image of ϕ_* is a (not necessarily normal) toric variety.*

Example 3.34: Consider the map

$$\phi : \mathbb{Z} \longrightarrow \mathbb{Z}^2$$

mapping 1 to $(2,3)$. Let $U_\sigma \cong \mathbb{A}^1_k$ and $U_{\sigma'} \cong \mathbb{A}^2_k$ be given by the cones $\sigma = \mathbb{R}_{\geq 0}$ and $\sigma' = (\mathbb{R}_{\geq 0})^2$, respectively. Then ϕ is a map of fans with respect to the fans generated by σ and σ' and the image of the corresponding toric morphism ϕ_* is a nonnormal curve in $\mathbb{A}^2_k$ given by the equation $x^2 - y^3 = 0$ for suitable coordinates x, y, in $\mathbb{A}^2_k$. The equivariant normalization of $\operatorname{im} \phi_*$ is naturally isomorphic to U_σ.

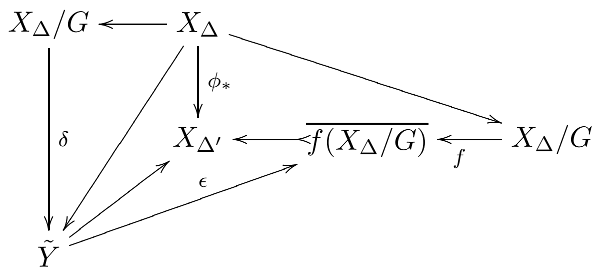
In order to avoid enlarging the category of normal toric varieties to include also nonnormal toric varieties. Instead, we consider the *normalization* of the scheme theoretic image. Recall that the normalization $\tilde{Y}$ of an integral scheme Y has the universal property that for every normal integral scheme Z, every dominant morphism $f : Z \longrightarrow Y$ factors uniquely through $\tilde{Y}$ (cf. [**Har77**], ex. II.3.9). Moreover, note that the normalization morphism is equivariant.

Definition 3.35: Let $\phi_* : X_\Delta \longrightarrow X_{\Delta'}$ be an equivariant morphism of toric varieties. Then the *toric scheme theoretic image* of ϕ_* is the normalization $\tilde{Y}$, where Y is the usual scheme theoretic image of ϕ_*.

Proposition 3.36: *Notations as in the definition. Then the toric scheme theoretic image $\tilde{Y}$ of ϕ_* has the following universal property. ϕ_* factors equivariantly through $\tilde{Y}$, and for every sequence of inclusions of tori $\bar{T} \subseteq T'' \subseteq T$ and every T''-invariant closed subset Y'' of X_Δ such that ϕ_* factors through Y'', and hence through the normalization $X_{\Delta''}$ of Y'', the map $\tilde{Y} \to X_{\Delta'}$ factors equivariantly through $X_{\Delta''}$.*

Proof. Clearly, by the universal property of the usual scheme theoretic image, for every equivariant morphism $X_\Delta \longrightarrow X_{\Delta''}$, the normalization of Y factors through $X_{\Delta''}$ by composition $\tilde{Y} \to Y \to X_{\Delta''}$, which is naturally equivariant. □

Assume that there exists a variety X_Δ/G which has the universal property of a categorical quotient of X_Δ by G. Then we have for the morphism ϕ_* the following G-invariant diagram induced by the universal properties of the scheme theoretic image of ϕ_* and of X_Δ/G:



The morphisms f and δ are induced by the universal property of categorical quotients because ϕ_* and its lift to $\tilde{Y}$ have G-invariant fibers. ϵ is induced by the existence of f and the universal property of the toric scheme theoretic image. Because the whole diagram is G-equivariant, the composition $\epsilon \circ \delta$ coincides with f, and thus f factorizes through $\tilde{Y}$.

Now we discuss under which condition a toric morphism $X_\Delta \longrightarrow X_{\Delta'}$ has the structure of a quotient with respect to the action of G on X_Δ. A similar question has been investigated earlier, see e.g. [**AH99a**], where suitable quotients of X_Δ by G were constructed. These quotients are toric varieties and one of the main problems treated in [**AH99a**] is how the fan of the quotient can explicitly be constructed. Here we ask the converse — given any map of fans $\phi : (N, \Delta) \to (N', \Delta')$ such that the corresponding toric morphism $\phi_* : X_\Delta \to X_{\Delta'}$ is surjective, under which conditions is $(X_{\Delta'}, \phi_*)$ a quotient - and what kind of quotient - of X_Δ by G?

Proposition 3.37: *Let $\sigma \subset N_\mathbb{R}$, $\tau \subset N'_\mathbb{R}$ be two cones and $\phi : N \longrightarrow N'$ a $\mathbb{Z}$-linear map which extends to an $\mathbb{R}$-linear surjective map $\phi : N_\mathbb{R} \longrightarrow N'_\mathbb{R}$ which induces a surjection of cones $\sigma \twoheadrightarrow \tau$. Then U_τ has the structure of a good quotient of U_σ by G. Moreover, U_τ is a geometric quotient of U_σ if and only if ϕ is a bijection of the cones σ and τ.*

Proof. Let G and $\bar{T}$ as above, then U_τ is an affine toric variety with respect to the action of $\bar{T}$. Consider the grading on $k[\sigma_M]$ induced by the action of G:

$$k[\sigma_M] = \bigoplus_{\chi \in X(G)} k[\sigma_M]_\chi.$$

The invariant ring $k[\sigma_M]^G$ then is the degree zero part of $k[\sigma_M]$. It suffices to consider the case that $\dim \tau = \operatorname{rk}_\mathbb{Z} N'$ and thus ϕ is surjective up to a finite cokernel and $\phi^\top$ is an inclusion. We have to show that $\phi(\tau_{M'}) = \sigma_M \cap \phi^\top(M')$. For this, we use the fact that

$$\langle m', \phi(n) \rangle = \langle \phi^\top(m'), n \rangle$$

for all $m' \in M'$ and all $n \in N$. It is clear that $\phi^\top(\tau_{M'}) \subset \sigma_M \cap \phi^\top(M')$. For any $m' \in M'$ for which $\langle \phi^\top(m'), n \rangle < 0$ for some $n \in \sigma$, follows $\langle m', \phi(n) \rangle < 0$, and thus $m' \notin (\phi(\sigma))\check{} = \tau_{M'}$.

To proof the second claim, we show that the G-orbits in U_σ are closed if and only if ϕ is a bijection of cones. Assume first that ϕ is not bijective. Then there exists a character $m \in \sigma_M$ such that $\chi(m)$ is contained in the maximal homogeneous ideal of $k[\sigma_M]$ and there is no unit m' in σ_M with the property that $m + m'$ is contained in the

$\mathbb{R}$-subvector space of M spanned by $\phi^\top(M')$. Then $\chi(m)$ is a nonconstant character of G which vanishes in the minimal orbit of U_σ. Now for any $t \in T \subset U_\sigma$, the limit $\lim_{t\to 0}\chi(m)(t)$ exists and is not contained in the G-orbit of t. Therefore, this orbit is not closed. Now assume that ϕ is bijective. Then U_σ splits:

$$U_\sigma \cong (k^*)^{\dim\sigma - \dim\tau} \times U_{\sigma'},$$

and ϕ_* factorizes as

$$\begin{array}{ccc} U_\sigma & \xrightarrow{\psi} & U_{\sigma'} \\ & \searrow^{\phi_*} & \downarrow \kappa \\ & & U_\tau \end{array}$$

The character group M/M' of G splits as $\mathbb{Z}^{\dim\sigma-\dim\tau}\oplus F$, where F is some finite abelian group. Then G can be identified with $(k^*)^{\dim\sigma-\dim\tau} \times F$ and ψ_* is the composition of the quotients by $(k^*)^{\dim\sigma-\dim\tau}$ and F, respectively. Note that because F is finite, the orbits of F are automatically closed and thus κ has closed orbits, and thus U_τ is a geometric quotient of $U_{\sigma'}$ by F. The projection ψ clearly is a geometric quotient and thus ψ_*, as composition of geometric quotients, is a geometric quotient as well. □

The preimage of some U_τ, $\tau \in \Delta'$ with respect to a map of fans $\phi : (N, \Delta) \longrightarrow (N', \Delta')$ is given by $X_{\Delta_\tau^{-1}}$, where

$$\Delta^{-1}(\tau) = \{\sigma \in \Delta \mid \phi(\sigma) \subset \tau\}.$$

Proposition 3.38: *Let $\phi : (N, \Delta) \longrightarrow (N', \Delta')$ be a map of fans such that the map $N_\mathbb{R} \to N_\mathbb{R}$ is surjective and induces a surjection $\Delta_{\max} \twoheadrightarrow \Delta'_{\max}$ such that every $\sigma \in \Delta_{\max}$ is mapped surjectively onto some $\tau \in \Delta'_{\max}$. Then $X_{\Delta'}$ is a categorical quotient of X_Δ with respect to the action of G.*

Proof. The affine $\bar{T}$-invariant subsets U_τ, where $\tau \in \Delta'_{\max}$, form an open cover of $X_{\Delta'}$ and it suffices to check that for every such τ the morphism

$$\phi_* : X_{\Delta^{-1}(\tau)} \longrightarrow U_\tau$$

is a categorical quotient. Because of the assumption on ϕ, we have that every $\sigma \in \Delta_{\max} \cap \Delta^{-1}(\tau)$ is mapped surjectively onto τ and moreover, by proposition 3.37, U_τ is a good quotient of U_σ by G. We have that

$$\Gamma(X_{\Delta^{-1}(\tau)}, \mathcal{O}_X) = \bigcap_{\sigma\in\Delta^{-1}(\tau)} \Gamma(U_\sigma, \mathcal{O}_X),$$

where the intersection is taken by the canonical inclusions $\Gamma(U_\sigma, \mathcal{O}_X) \rightarrowtail \Gamma(T, \mathcal{O}_X)$. This inclusion respects the G-grading, and by 3.37, $\Gamma(U_\sigma, \mathcal{O}_X)^G = \Gamma(U_\tau, \mathcal{O}_{X_{\Delta'}})$ for every $\sigma \in \Delta^{-1}(\tau) \cap \Delta_{\max}$, and thus

$$\Gamma(X_{\Delta^{-1}(\tau)}, \mathcal{O}_X)^G = \Gamma(U_\tau, \mathcal{O}_{X_{\Delta'}}).$$

We want to apply the criterion of proposition 2.17, so we have to show that for every G-invariant closed subset $\phi_*(W)$ is a closed subset of $X_{\Delta^{-1}(\tau)}$, and that for any family

W_i, $i \in I$, of G-invariant closed subsets of $X_{\Delta^{-1}(\tau)}$, we have:

$$\phi\big(\bigcap_{i\in I} W_i\big) = \bigcap_{i\in I} \phi(W_i).$$

We do induction on the number n of cones in $\Delta^{-1}(\tau) \cap \Delta_{\max}$. Let W, W_i, $i \in I$, be a family of G-invariant closed subsets of $X_{\Delta^{-1}(\tau)}$. For $n = 1$, we apply proposition 3.37. For $n > 0$, consider the fan Ξ generated by $\Delta^{-1}(\tau) \cap \Delta_{\max} \setminus \{\sigma\}$ for some $\sigma \in \Delta^{-1}(\tau) \cap \Delta_{\max}$, and denote the associated toric variety by X_Ξ. By induction, $\phi(X_\Xi \cap W)$ is closed in U_τ, and thus, $\phi(\overline{X_\Xi \cap W})$ is closed in U_τ, where the closure is taken in $X_{\Delta^{-1}(\tau)}$. The set $W' := W \setminus \overline{X_\Xi \cap W}$ is a closed subset of $U_\sigma \setminus X_\Xi$, and thus by proposition 3.37 its image is closed in U_τ. Thus $\phi(W) = \phi(\overline{X_\Xi \cap W}) \cup \phi(W')$ is a closed subset of U_τ. Similarly, by induction $\phi\big(\big(\bigcap_{i\in I} W_i\big) \cap X_\Xi\big) = \phi\big(\overline{\big(\bigcap_{i\in I} W_i\big) \cap X_\Xi}\big)$, and together with $\phi\big(\bigcap_{i\in I} W_i \setminus \overline{\big(\bigcap_{i\in I} W_i\big) \cap X_\Xi}\big) = \phi\big(\bigcap_{i\in I} W_i\big) \setminus \overline{\big(\bigcap_{i\in I} W_i\big) \cap X_\Xi}$, we obtain the result. ☐

Remark 3.39: We leave open whether the above condition on ϕ is necessary for inducing a categorical quotient. Note that in characteristic zero, one can show that the categorical quotient is *universal*, which means that for all morphisms $Y \longrightarrow X_{\Delta'}$ the pair (Y, ψ), where $\psi : X_\Delta \times_{X_{\Delta'}} Y \longrightarrow Y$ is pullback of ϕ_*, is a categorical quotient of $X_\Delta \times_{X_{\Delta'}} Y$ by G.

Definition 3.40: Let $\phi : (N, \Delta) \longrightarrow (N', \Delta')$ be a map of fans. For every $\tau \in \Delta'$ consider the preimage $\Delta^{-1}(\tau)$. If this set contains a unique maximal element, we say that ϕ is an *affine* map of fans.

Lemma 3.41: *Let $\phi : (N, \Delta) \longrightarrow (N', \Delta')$ be a map of fans. Then ϕ is affine if and only if ϕ_* is affine.*

Proof. The preimage of U_τ is $X_{\Delta^{-1}(\tau)}$ which is affine if and only if $\Delta^{-1}(\tau)$ is a fan generated by precisely one $\sigma \in \Delta$. ☐

Proposition 3.42: *Let $\phi : (N, \Delta) \longrightarrow (N', \Delta')$ be a map of fans. Then $X_{\Delta'}$ is a good quotient of X_Δ if and only if ϕ is affine and every $\sigma \in \Delta_{\max}$ is mapped surjectively onto some $\tau \in \Delta'_{\max}$.*

Proof. If $(X_{\Delta'}, \phi_*)$ is a good quotient of X_Δ, then ϕ_* must be affine. Then for every $\tau \in \Delta'_{\max}$, there is a unique cone σ in Δ, and even in $\Delta_{\max}$ such that $\phi_*^{-1}(U_\tau) = U_\sigma$ and by proposition 3.37, U_τ is a good quotient of U_σ if and only if ϕ maps σ surjectively onto τ. On the other hand, if the conditions on ϕ are fulfilled, then the maps $U_\sigma \xrightarrow{\phi_*} U_\tau$ glue to a good quotient of X_Δ. ☐

Combining propositions 3.37 and 3.42, we obtain:

Proposition 3.43: *Let $\phi : (N, \Delta) \longrightarrow (N', \Delta')$ be a map of fans. Then $X_{\Delta'}$ is a geometric quotient of X_Δ if and only if it induces a bijection of fans and every $\sigma \in \Delta$ is mapped surjectively onto some $\tau \in \Delta'$.*

3.5. Orbit Closures

Let $T^\sigma := T/T_\sigma$ the torus modulo the stabilizer of $\mathrm{orb}(\sigma)$. Then T^σ is dense in $V(\sigma)$ and $V(\sigma)$ becomes a toric variety with respect to the T^σ-action. We are going to construct explicitely the fan of $V(\sigma)$. Consider the short exact sequence of abelian groups:

$$0 \longrightarrow T_\sigma \longrightarrow T \longrightarrow T^\sigma \longrightarrow 0$$

This corresponds to the short exact sequences

$$0 \longrightarrow \sigma_M^\perp \longrightarrow M \longrightarrow M/\sigma_M^\perp \longrightarrow 0$$

and

$$0 \longrightarrow N_\sigma \longrightarrow N \longrightarrow N/N_\sigma \longrightarrow 0,$$

where N_σ is N intersected with $N_{\mathbb{R},\sigma}$, which is the vector space spanned by σ over $\mathbb{R}$.

Definition 3.44: Let $\sigma \in \Delta$, then we denote $\Delta(\sigma) := \{\tau \in \Delta \mid \sigma < \tau\}$.

By definition, σ is the intersection of the linear space spanned over $\mathbb{R}$ by σ, $N_{\sigma,\mathbb{R}}$, with every $\tau \in \Delta(\sigma)$ and the image of the surjection $N_\mathbb{R} \twoheadrightarrow N_\mathbb{R}/N_{\sigma,\mathbb{R}}$ of $\Delta(\sigma)$ defines a new fan in $N_\mathbb{R}/N_{\sigma,\mathbb{R}}$ whose cones are the images of the $\tau \in \Delta(\sigma)$, given by $\bar{\tau} = \tau + N_{\sigma,\mathbb{R}}/N_{\sigma,\mathbb{R}}$.

Definition 3.45: Let $\sigma \in \Delta$ be a cone, then we define the *star* of σ to be the set $\mathrm{star}(\sigma) = \{\bar{\tau} \mid \tau \in \Delta(\sigma)\}$.

$\mathrm{star}(\sigma)$ is a fan which defines a toric variety which is isomorphic to $V(\sigma)$, and the projection $N \twoheadrightarrow N_\sigma$ induces a surjective morphism of toric varieties $X_{\Delta(\sigma)} \twoheadrightarrow V(\sigma)$.

In the opposite direction, we obtain an equivariant inclusion from $(V(\sigma), T^\sigma) \longrightarrow (X, T)$ by choosing a section $\phi : T^\sigma \to T$, which is always possible because T and T^σ have no torsion subgroups. Then the pair (i, ϕ), where i is the canonical inclusion of $V(\sigma)$ in X, is the required morphism. Note that if $\sigma \neq 0$, this equivariant inclusion is not a morphism of toric varieties in the sense of the previous section, because ϕ is not the restriction of i to T^σ. Indeed, the image of i in general even has empty intersection with the torus T. So, this inclusion does *not* correspond to a map of fans $(N_\sigma, \mathrm{star}(\sigma)) \to (N, \Delta)$.

3.6. Quotient Presentations

Recall that projective space can be represented as quotient of a quasiaffine variety:

$$\mathbb{P}_n \cong \big(k^{n+1} \setminus 0\big)/k^*,$$

where k^* acts diagonally on k^{n+1}. The dual action then induces a $\mathbb{Z}$-grading on the coordinate ring $S = k[x_0, \ldots, x_n]$, and any homogeneous function in S defines a well-defined function on $\mathbb{P}_n$. The idea of a *quotient presentation* for a toric variety is a generalization of this fact. We first present the general idea from [**AHS02**] and then we describe the original version by Cox ([**Cox95**]), which in the sequel will be the most important for us.

3.6.1. General Quotient Presentations. Let $\hat{X} = X_{\hat{\Delta}} \xrightarrow{\phi} X = X_{\Delta}$ be a surjective equivariant morphism of toric varieties on which the tori $\hat{T}$ and T act, respectively. Then we obtain a $\mathbb{Z}$-linear homomorphism $\mathbb{Z}^{\Delta(1)} \longrightarrow \mathbb{Z}^{\hat{\Delta}(1)}$ as follows. Consider the open subvariety X_1 of X which is determined by the fan generated by the one-dimensional cones of Δ. This is a smooth toric variety and by restriction we obtain an isomorphism $\mathrm{CDiv}^T(X_1) \xrightarrow{\cong} \mathbb{Z}^{\Delta(1)}$, where $\mathrm{CDiv}^T(X_1)$ denotes the group of T-invariant Cartier divisors on X_1. Denote $\hat{U} := \phi^{-1}(X_1)$, then, again by pull-back, we get a map $\mathrm{CDiv}^T(X_1) \longrightarrow \mathrm{CDiv}^{\hat{T}}(\hat{U})$, and by extension (see theorem 2.3 we get a map $\mathrm{CDiv}^{\hat{T}}(\hat{U}) \longrightarrow \mathbb{Z}^{\hat{\Delta}(1)}$, and by composition we obtain the *strict transform* $\phi^{\#} : \mathbb{Z}^{\Delta(1)} \longrightarrow \mathbb{Z}^{\hat{\Delta}(1)}$.

Definition 3.46 ([**AHS02**], Def. 2.1)**:** A *quotient presentation* for X is a quasi-affine toric variety $\hat{X}$ and a surjective, affine morphism $\phi : \hat{X} \longrightarrow X$ such that the strict transform $\phi^{\#} : \mathbb{Z}^{\Delta(1)} \longrightarrow \mathbb{Z}^{\hat{\Delta}(1)}$ is bijective.

In [**AHS02**], the following characterization of quotient presentations has been given:

Theorem 3.47 ([**AHS02**], Theorem 2.3)**:** *Let $\phi : \hat{X} \longrightarrow X$ be an equivariant morphism of toric varieties. This morphism is a quotient presentation if and only if the following conditions hold:*

(i) The corresponding homomorphism of $\mathbb{Z}$-modules $\hat{N} \longrightarrow N$ has finite cokernel,
(ii) $\hat{\Delta}$ is a subfan of a fan generated by a strictly convex polyhedral cone in $\hat{N}_{\mathbb{R}}$,
(iii) ϕ induces bijections $\hat{\Delta}(1) \leftrightarrow \Delta(1)$ and $\hat{\Delta}_{\max} \leftrightarrow \Delta_{\max}$.
(iv) every primitive lattice vector $n(\hat{\rho})$ of some ray $\hat{\rho} \in \hat{\Delta}(1)$ is mapped to a primitive lattice vector $n(\rho)$ of some ray $\rho \in \Delta(1)$.

As in section 3.4, there is a short exact sequence of diagonalizable groups

$$0 \longrightarrow G \longrightarrow \hat{T} \longrightarrow T \longrightarrow 0$$

which is dual to

$$0 \longrightarrow M \longrightarrow \hat{M} \longrightarrow X(G) \longrightarrow 0$$

Combining the above theorem with propositions 3.42 and 3.43, we obtain:

Corollary 3.48: *Let $\hat{X} \longrightarrow X$ be a quotient presentation, then X is a good quotient of $\hat{X}$ by G. Moreover, it is a geometric quotient if and only if the corresponding map of fans induces a bijection $\hat{\Delta} \leftrightarrow \Delta$.*

Because X is a good quotient of $\hat{X}$ by G, we know by proposition 3.42 that there is a bijection between maximal cones in $\hat{\Delta}$ and maximal cones in Δ. Moreover, there is a surjective map $\hat{\Delta} \to \Delta$ and a bijection $\hat{\Delta}(1) \leftrightarrow \Delta(1)$. Theorem 3.47 implies that for every $\sigma \in \Delta$ there exists a unique $\hat{\sigma} \in \hat{\Delta}$ such that $\hat{\sigma}(1) = \{\hat{\rho} \mid \rho \in \sigma(1)\}$. The corresponding $U_{\hat{\sigma}}$ give a T-invariant cover of $\hat{X}$ which for our purposes suffices instead of considering the full cover defined by the fan $\hat{\Delta}$. So in the sequel we will use the convention that if we speak about the fan $\hat{\Delta}$, we only consider the subset $\{\hat{\sigma} \mid \sigma \in \Delta\} \subset \hat{\Delta}$.

The reason why quotient presentations are of interest is the fact that these allow to introduce the notion of *homogeneous coordinate rings* for toric varieties. As stated in theorem 3.47, the fan $\hat{\Delta}$ is a subfan of the fan generated by some cone, denoted C for the moment, in $\hat{N}$. The complement of $\hat{X}$ in U_C is contained in the complement of the toric subvariety of $\hat{X}$ which is defined by the subfan of $\hat{\Delta}$ which consists of $\{0\} \cup \hat{\Delta}(1)$. This subvariety contains all orbits of codimensions 1 and 0, and thus the complement of $\hat{X}$ in U_C has codimension at least two. Therefore, the global regular functions of $\hat{X}$ coincide with the coordinate ring $k[U_C]$. In the following, we will denote $S := k[U_C]$. The G-action induces an $X(G)$-grading of S:

$$S = \bigoplus_{\chi \in X(G)} S_\chi.$$

The complement of $\hat{X}$ in U_C is a G-invariant subset of U_C and is naturally described by a $X(G)$-graded radical ideal B.

Definition 3.49: The ideal B of S which describes the complement of $\hat{X}$ in U_C, is called the *irrelevant ideal*.

Cox' construction: the most important, and in some sense most natural, quotient presentation is the one originally given by Cox. Consider the $\mathbb{Z}$-module $\hat{N} := \mathbb{Z}^{\Delta(1)}$ which is freely generated over the set of rays $\Delta(1)$of Δ. Denote e_ρ, $\rho \in \Delta(1)$ the canonical basis for $\hat{N}$. The cone C then is given by the positive orthant $(\mathbb{R}_{\geq 0})^{\Delta(1)}$ of $\hat{N}_\mathbb{R}$, and the fan $\hat{\Delta}$ is the fan generated by the set $\{\hat{\sigma} \mid \sigma \in \Delta\}$, where $\hat{\sigma} = \sum_{\rho \in \sigma(1)} \mathbb{R}_{\geq 0} e_\rho$, that is, $\hat{\sigma}$ is spanned by exactly those basis vectors e_ρ with $\rho \in \sigma(1)$. Then a map of fans $(\hat{N}, \hat{\Delta}) \longrightarrow (N, \Delta)$ is induced by the map

$$\hat{N} \longrightarrow N, \quad e_\rho \mapsto n(\rho),$$

where $n(\rho)$ is the primitive lattice vector of ρ in N. The dual inclusion $M \to \hat{M} \cong \mathbb{Z}^{\Delta(1)}$ is given elementwise by

$$m \mapsto (\langle m, n(\rho) \rangle).$$

Comparing this with sequence (2) in section 3.3, we see that $X(G)$ coincides with the Chow group $A_{n-1}(X)$ of X.

The toric variety U_C in this construction can naturally be identified with the affine space $\mathbb{A}_k^{\Delta(1)} = k^{\Delta(1)}$, on which the torus $\hat{T}$ acts diagonally. The coordinate ring S coincides with the polynomial ring with variables in the rays of Δ, i.e. $S = k[x_\rho \mid \rho \in \Delta(1)]$. An affine $k^{\Delta(1)}$-invariant open subset $U_{\hat{\sigma}}$ of $\hat{X}$ is the complement in $k^{\Delta(1)}$ of the union of all $\hat{T}$-invariant divisors $D_{\hat{\rho}}$ such that $\hat{\rho}$ is not in $\hat{\sigma}(1)$. The defining equation for such a union of hypersurfaces is given by the monomial

$$x^{\hat{\sigma}} := \prod_{\rho \in \Delta(1) \setminus \sigma(1)} x_\rho,$$

and the coordinate ring of $U_{\hat{\sigma}}$ is the localization $S_{x^{\hat{\sigma}}}$. So the irrelevant ideal of $\hat{X}$ in $k^{\Delta(1)}$ is given as

$$B := \langle x^{\hat{\sigma}} \mid \hat{\sigma} \in \hat{\Delta} \rangle.$$

Remark 3.50: In the construction of Cox, X is a geometric quotient of $\hat{X}$ by G if and only if Δ is a simplical fan. It is natural to ask whether every toric variety is a geometric quotient quotient of a suitable quotient presentation. In fact, in [**Kaj98**] and, independently, in [**Per02c**] it was shown to be true for a very large class of toric varieties, so-called *divisorial* toric varieties.

3.6.2. Quotient Presentations and Graded Modules. Let $\phi : \hat{X} \longrightarrow X$ be a quotient presentation of X. Let F be an $X(G)$-graded S-module. By proposition 2.31, such a module corresponds to a G-linearized sheaf $\mathcal{F}'$ over U_C. The restriction $\mathcal{F}'|_{\hat{X}}$, which we denote $\mathcal{F}$, defines an equivariant sheaf over $\hat{X}$. For every $U_{\hat{\sigma}}$ we can consider the restriction $\mathcal{F}|_{U_{\hat{\sigma}}}$.

These restrictions again define G-equivariant sheaves over the $U_{\hat{\sigma}}$. Denote S_σ the coordinate ring of $U_{\hat{\sigma}}$, which has an induced $X(G)$-grading. Again by proposition 2.31, the modules

$$\Gamma(U_{\hat{\sigma}}, \mathcal{F}) =: F_\sigma$$

are $X(G)$-graded S_σ-modules. Let us denote $S_{(\sigma)}$ and $F_{(\sigma)}$ the degree zero components of S_σ and F_σ, respectively. From the fact that X is a good quotient of $\hat{X}$ by G it follows that $(S_\sigma)^G = S_{(\sigma)} \cong k[\sigma_M]$ for every $\sigma \in \Delta$, and thus we can consider $k[\sigma_M]$ as a subring of S_σ. Similarly, we can identify the module of invariants

$$\Gamma(U_{\hat{\sigma}}, \mathcal{F})^G = (F_\sigma)^G = F_{(\sigma)}.$$

This way the sheaf

$$(\phi_* \mathcal{F})^G$$

of G-invariants of $\phi_* \mathcal{F}$ in a natural way defines a quasicoherent sheaf over X.

Definition 3.51: Let F be a G-graded S-module and let $\mathcal{F}$ be the associated sheaf over $\hat{X}$, then we denote the sheaf $\tilde{F} := (\phi_* \mathcal{F})^G$ the *sheafification* of F.

On the other hand, let $\mathcal{F}$ be any quasicoherent sheaf over X, then the pullback $\phi^* \mathcal{F}$ is G-equivariant and the S-module which is given by $\Gamma(\hat{X}, \phi^* \mathcal{F})$, has a natural $X(G)$-grading. By proposition 2.23:

$$\mathcal{F} \cong (\phi_* \phi^* \mathcal{F})^G = \Gamma(\hat{X}, \phi^* \mathcal{F})\tilde{}.$$

Hence:

Proposition 3.52: *Let $\phi : \hat{X} \longrightarrow X$ be a quotient presentation. Then for every quasicoherent sheaf $\mathcal{F}$ over X, there is an isomorphism $\mathcal{F} = \Gamma(\hat{X}, \phi^* \mathcal{F})\tilde{}$, and every quasicoherent sheaf $\mathcal{F}$ is of the form $\tilde{F}$ for some G-graded S-module F.*

Sheaves and Cox' construction: Let $\alpha = \alpha(D) \in A_{n-1}(X)$ be a rational equivalence class of some Weil divisor D and let $\mathcal{O}_X(D)$ its associated reflexive sheaf of rank one. It was shown in [**Cox95**], that for the Cox construction, the graded component S_α of S is naturally isomorphic to the space of global sections of $\mathcal{O}_X(D)$ (see 2.1.4):

$$S_\alpha \cong \Gamma(X, \mathcal{O}_X(D)).$$

This isomorphism conveniently is compatible with the ring structure of S, which means that the natural map

$$\Gamma(X, \mathcal{O}_X(D)) \otimes_{\Gamma(X, \mathcal{O}_X)} \Gamma(X, \mathcal{O}_X(D')) \longrightarrow \Gamma(X, \mathcal{O}_X(D + D'))$$

coincides with the ring multiplication of S, given by $S_\alpha \otimes_{S_0} S_\beta \longrightarrow S_{\alpha+\beta}$, if α, β are the classes of D, D' in $A_{n-1}(X)$, respectively.

The free S-modules of rank 1 are given by modules $S(\alpha)$, for $\alpha \in A_{n-1}(X)$, where $S(\alpha)$ denotes the *degree shift* of S by α, i.e. $S(\alpha)_\beta = S_{\alpha+\beta}$. It turns out that $\widetilde{S(\alpha)} \cong \mathcal{O}_X(D)$.

Let us denote by $\mathcal{O}_X(\alpha) := \widetilde{S(\alpha)}$ the distinguished representative of the isomorphism class of sheaves $\mathcal{O}_X(D)$, where D is a representative of α. Then, for a quasicoherent sheaf $\mathcal{F}$ on X one can define a graded S-module

$$\Gamma_* \mathcal{F} := \bigoplus_{\alpha \in A_{n-1}(X)} \Gamma\big(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(\alpha)\big).$$

Γ_* is a functor from the category of quasi-coherent sheaves over X to the category of $A_{n-1}(X)$-graded S-modules. In the following theorem for reference we summarize some results relating $A_{n-1}(X)$-graded modules and quasicoherent sheaves.

Theorem 3.53 ([**Cox95**], [**Mus02**])**:** *(i) The map $F \mapsto \tilde{F}$ is a covariant additive exact functor from the category of $A_{n-1}(X)$-graded S-modules to the category of quasi-coherent $\mathcal{O}_X$-modules.*

(ii) If F is a finitely generated $A_{n-1}(X)$-graded S-module, then $\tilde{F}$ is coherent.

(iii) There exists a natural isomorphism $(\Gamma_* \mathcal{F})^\sim \cong \mathcal{F}$. In particular, every quasi-coherent $\mathcal{O}_X$–module $\mathcal{F}$ is of the form $\tilde{F}$ for some $A_{n-1}(X)$-graded S-module F. If $\mathcal{F}$ is coherent, then there exists a finitely generated $A_{n-1}(X)$-graded S-module F with $\tilde{F} \cong \mathcal{F}$.

(iii) has been proven by Cox for the case of simplicial toric varieties and by Mustata ([**Mus02**]) in general. We complement these results by the following remark:

Proposition 3.54: *Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Then $\Gamma_* \mathcal{F} \cong \Gamma(\hat{X}, \pi^* \mathcal{F})$.*

Proof. By restricting the projection π to any $U_{\hat{\sigma}}$ we get a map of affine varieties $\pi|_{U_{\hat{\sigma}}} : U_{\hat{\sigma}} \longrightarrow U_\sigma$. The induced map $\pi' : k[\sigma_M] \longrightarrow k[U_{\hat{\sigma}}]$ yields a homomorphism of graded rings $(\pi', 0) : (k[\sigma_M], M) \longrightarrow (k[U_{\hat{\sigma}}], A_{n-1}(X))$, where $0 : M \longrightarrow A_{n-1}(X)$ is just the zero map. Let us denote $\Gamma(U_\sigma, \mathcal{F})$ by F_σ, then $\Gamma(U_{\hat{\sigma}}, \pi^* \mathcal{F})$ is isomorphic to $k[U_{\hat{\sigma}}] \otimes_{k[\sigma_M]} F_\sigma$, where $k[U_{\hat{\sigma}}] \otimes_{k[\sigma_M]} F_\sigma$ can be considered as a graded scalar extension (see 2.33) with F_σ and $k[\sigma_M]$ trivially $A_{n-1}(X)$-graded. Thus the pullback decomposes into homogeneous components:

$$\begin{aligned}
\Gamma(U_{\hat{\sigma}}, \pi^* \mathcal{F}) &\cong \Big(\bigoplus_{\alpha \in A_{n-1}(X)} (S_{x^{\hat{\sigma}}})_\alpha \Big) \otimes_{k[\sigma_M]} F_\sigma \\
&\cong \bigoplus_{\alpha \in A_{n-1}(X)} \big((S_{x^{\hat{\sigma}}})_\alpha \otimes_{k[\sigma_M]} F_\sigma \big) \\
&\cong \bigoplus_{\alpha \in A_{n-1}(X)} \Gamma\big(U_\sigma, \mathcal{O}_X(\alpha)\big) \otimes_{k[\sigma_M]} \Gamma(U_\sigma, \mathcal{F}) \\
&\cong \bigoplus_{\alpha \in A_{n-1}(X)} \Gamma\big(U_\sigma, \mathcal{F} \otimes \mathcal{O}_X(\alpha)\big).
\end{aligned}$$

Gluing then yields the result. □

In the next chapter we will extend these results to a correspondence between $\mathbb{Z}^{\Delta(1)}$-graded S-modules and equivariant quasicoherent sheaves.

CHAPTER 4

Equivariant Sheaves over Toric Varieties

In sections 4.1 and 4.2 we carry out our basic formalism, that of σ- and Δ-families, for equivariant sheaves, which we already have motivated in chapter 1. In section 4.3 we state our first observation derived from this construction, namely that the category of coherent, equivariant sheaves over any toric variety is a Krull-Schmidt category. In sections 4.4, 4.5 and 4.6 we specialize our descriptions to torsion free, reflexive, and locally free sheaves and explain the relation of our constructions to earlier constructions of Klyachko in [**Kly90**] and [**Kly91**]. In particular, in section 4.6 we give an interpretation of the filtrations describing equivariant locally free sheaves as filtrations of certain geometric fibers of the associated vector bundle. Moreover, we show how to compute filtrations for pullbacks and restrictions to orbit closures of locally free sheaves. In section 4.7, we give a short description of how filtrations change after tensoring with an equivariant line bundle or a reflexive sheaf of rank one. In sections 4.8 and 4.9 we connect our formalism to Cox' construction of homogeneous coordinate rings. Section 4.9 contains a global version of the so-called *Rees construction*. The motivation for this construction comes from the thesis of Penacchio ([**Pen02**]), where it was used to construct equivariant vector bundles over $\mathbb{P}_2$. We present a global version of this construction with respect to the Cox quotient presentation $\phi : \hat{X} \to X$. This construction allows to associate to every reflexive equivariant sheaf $\mathcal{E}$ over X a reflexive, fine-graded S-module E such that $\mathcal{E} = \tilde{E}$. The application of this construction is that in general, the module $\Gamma(\hat{X}, \phi^*\mathcal{E})$ is not reflexive, and not even torsion free. The module E will be constructed as a reflexive submodule of $\Gamma(\hat{X}, \phi^*\mathcal{E})$ which essentially is defined by the same filtrations as $\mathcal{E}$. This allows later on to preserve the geometric information contained in the filtrations for $\mathcal{E}$ when passing to global coordinates.

Note that for a quasicoherent sheaf $\mathcal{E}$ and T-invariant open subsets U we use, as in chapter 3 (see page 30), for the induced dual actions on spaces $\Gamma(U, \mathcal{E})$ the following convention. For $s \in \Gamma(U, \mathcal{E})$ and $t \in T$ we write

$$t.s = \phi_t(t^* s)$$

instead of $t.s = \phi_{t^{-1}}((t^{-1})^* s)$. This is well-defined because T is abelian.

4.1. The Category of σ-Families

Let $\mathcal{E}$ be a quasi-coherent equivariant sheaf over an affine toric variety U_σ and denote by E^σ the $k[\sigma_M]$-module $\Gamma(U_\sigma, \mathcal{E})$. By proposition 2.31, there is an isotypical decomposition:

$$E^\sigma = \bigoplus_{m \in M} E^\sigma_m.$$

Let $\mathcal{F}$ be another quasi-coherent sheaf over U_σ and F^σ the corresponding M-graded $k[\sigma_M]$-module. Then an equivariant homomorphism between $\mathcal{E}$ and $\mathcal{F}$ is equivalent to a homogeneous morphism ϕ of degree 0 between E^σ and F^σ. Such a morphism decomposes into a sum

$$\phi = \bigoplus_{m \in M} \phi_m$$

of k-vector space homomorphisms $\phi_m : E^\sigma_m \longrightarrow F^\sigma_m$.

The graded components $k[\sigma_M]_m$ are one-dimensional k-vector spaces which are spanned by the characters $\chi(m)$, for $m \in \sigma_M$. Every such character represents a k-linear map between the homogeneous components

$$\begin{aligned} \chi(m) : E^\sigma_{m'} &\longrightarrow E^\sigma_{m+m'} \\ e &\mapsto \chi(m) \cdot e \end{aligned}$$

for all $m \in \sigma_M$ and all $m' \in M$. It is clear that the module structure of E^σ over $k[\sigma_M]$ is completely determined by these mappings. Moreover, if we break a homogeneous morphism ϕ as above into pieces, we get that the ϕ_m are k-linear vector space homomorphisms such that

$$\phi_{m+m'} \circ \chi(m) = \chi(m) \circ \phi_{m'}$$

for all $m \in \sigma_M$ and all $m' \in M$. What we are going to do now is just to formalize this a bit.

Observe first that there is such a linear mapping between some eigenspaces E^σ_m and $E^\sigma_{m'}$ if and only if $m' - m \in \sigma_M$. This motivates to introduce a relation on M as follows:

Definition 4.1: Let U_σ be an affine toric variety. Then we define a relation $\leq_\sigma$ on M by setting $m \leq_\sigma m'$ iff $m' - m \in \sigma_M$. We write $m <_\sigma m'$ if $m \leq_\sigma m'$ but not $m' \leq_\sigma m$.

It is easy to prove the following properties of $\leq_\sigma$:

(i) $\leq_\sigma$ defines a directed preorder on M.
(ii) $m \leq_\sigma m'$ and $m' \leq_\sigma m$ iff $m - m' \in \sigma^\perp_M$.
(iii) If $\tau \leq \sigma$, then $m \leq_\sigma m'$ implies $m \leq_\tau m'$.
(iv) If σ is of maximal dimension in N then $\leq_\sigma$ is a partial order.

Definition 4.2: Let $\{E^\sigma_m\}_{m \in M}$ be a family of k-vector spaces. For each relation $m \leq_\sigma m'$ let there be given a vector space homomorphism $\chi^\sigma_{m,m'} : E^\sigma_m \longrightarrow E^\sigma_{m'}$ such that $\chi^\sigma_{m,m} = 1$ for all $m \in M$ and $\chi^\sigma_{m,m''} = \chi^\sigma_{m',m''} \circ \chi^\sigma_{m,m'}$ for each triple $m \leq_\sigma m' \leq_\sigma m''$. We denote such data by $\hat{E}^\sigma$ and call it a *σ-family.*

Lemma 4.3: *Let $\hat{E}^\sigma$ be a σ-family. Let m, m' be given such that $m - m' \in \sigma^\perp_M$. Then the homomorphisms $\chi^\sigma_{m,m'}$ and $\chi^\sigma_{m',m}$ are isomorphisms.*

Proof. The lemma follows from the fact that $\chi^\sigma_{m,m'} \circ \chi^\sigma_{m',m} = \chi^\sigma_{m',m'} = 1$. □

Definition 4.4: Let $\hat{E}^\sigma$ and $\hat{F}^\sigma$ be two σ-families with vector space homomorphisms $\chi^\sigma_{m,m'}$ and $\psi^\sigma_{m,m'}$, respectively. Then a *morphism* $\hat{\phi}^\sigma$ from $\hat{E}^\sigma$ to $\hat{F}^\sigma$ is a set of vector space homomorphisms $\{\phi^\sigma_m : E^\sigma_m \longrightarrow F^\sigma_m\}_{m \in M}$ such that $\phi^\sigma_{m'} \circ \chi^\sigma_{m,m'} = \psi^\sigma_{m,m'} \circ \phi^\sigma_m$ for all $m, m' \in M$ with $m \leq_\sigma m'$.

It is clear that σ-families form a category. Moreover:

Theorem 4.5: *The following three categories are equivalent:*

(i) equivariant quasicoherent sheaves over U_σ*,*
(ii) M*-graded* $k[\sigma_M]$*-modules with morphisms of degree 0, and*
(iii) σ*-families*

Proof. The Grothendieck functor $\tilde{}$ and the global section functor $\Gamma(U_\sigma, .)$ establish an equivalence of categories between the category of quasicoherent sheaves over U_σ and the category of $k[\sigma_M]$-modules. It was shown in proposition 2.31 that the restrictions of these functors induce the equivalence between (i) and (ii).

We prove now the equivalence between (ii) and (iii). From the discussion above it is clear that each M-graded $k[\sigma_M]$-module E^σ gives rise to a σ-family $\hat{E}^\sigma$ via the family of vector spaces given by the homogeneous components E^σ_m and setting $\chi^\sigma_{m,m'}$ the map given by multiplication with $\chi(m' - m)$. Moreover, each homogeneous morphism ϕ by decomposition into homogeneous components gives rise to a morphism $\hat{\phi}$ of σ-families. On the other hand, given a σ-family $\hat{E}^\sigma$, we associate a graded $k[\sigma_M]$-module to $\hat{E}^\sigma$ by setting

$$E^\sigma := \bigoplus_{m \in M} E^\sigma_m$$

and for $m \in \sigma_M$ and a homogeneous element $e \in E^\sigma_{m'}$ we set $\chi(m) \cdot e := \chi^\sigma_{m,m+m'}(e)$. We then obtain the structure of a graded $k[\sigma_M]$-module by k-linear continuation. Analogously, morphisms are composed by forming direct sums. □

4.2. The Category of Δ-Families

Let $f : U_{\sigma'} \longrightarrow U_\sigma$ be an equivariant morphism of affine toric varieties. This means that its restriction to the torus $T' \subset U_{\sigma'}$ is a morphism of algebraic groups $T' \longrightarrow T \subset U_\sigma$ and that $f(t.x) = f(t).f(x)$ for all $t \in T$ and $x \in U_{\sigma'}$. Such an f induces contravariantly a homomorphism of rings $f^* : k[\sigma_M] \longrightarrow k[\sigma'_M]$ and a homomorphism $\check{f} : M \longrightarrow M'$ between the character groups of T and T', respectively. Both are compatible in the sense that the restriction $\check{f} : \sigma_M \longrightarrow \sigma'_{M'}$ of f is a homomorphism of semigroups. Then the pair $(f^*, \check{f})$ is a homomorphism of graded rings:

$$(f^*, \check{f}) : (k[\sigma_M], M) \longrightarrow (k[\sigma'_{M'}], M')$$

in the sense of Section 2.5. Let $\mathcal{E}$ be an equivariant quasicoherent sheaf on U_σ. Then by lemma 2.9 the sheaf $f^*\mathcal{E}$ is a T'-equivariant sheaf on $U_{\sigma'}$ and the $k[\sigma'_{M'}]$-module $\Gamma(U_{\sigma'} f^*\mathcal{E})$ has an M'-grading, where the natural isomorphism $\Gamma(U_{\sigma'} f^*\mathcal{E}) \cong \Gamma(U_\sigma, \mathcal{E}) \otimes_{k[\sigma_M]} k[\sigma'_{M'}]$ coincides with the graded scalar extension in the sense of Definition 2.33. The M'-graded $k[\sigma'_{M'}]$-module $\Gamma(U_{\sigma'} f^*\mathcal{E})$ thus corresponds to a σ'-family.

Definition 4.6: Let E be an M-graded $k[\sigma_M]$-module corresponding to a σ-family $\hat{E}^\sigma$. Further let $f : U_{\sigma'} \longrightarrow U_\sigma$ be an equivariant morphism of affine toric varieties. Then denote by $f^*\hat{E}^\sigma$ the σ'-family obtained by the graded scalar extension of E via the pair of homomorphisms $(f^*, \check{f}) : (k[\sigma_M], M) \longrightarrow (k[\sigma'_{M'}], M')$.

Proposition 4.7: *Given an equivariant morphism $f : U_{\sigma'} \longrightarrow U_\sigma$ of affine toric varieties. Pullback f^* defines a functor from the category of equivariant quasicoherent sheaves over U_σ to the category of equivariant quasicoherent sheaves over $U_{\sigma'}$. By equivalence of categories, f^* is up to natural equivalence a functor from the category of σ-families to the category of σ'-families, and there is the following functor diagram:*

$$\begin{array}{ccc} \mathcal{E} & \longmapsto & f^*\mathcal{E} \\ \updownarrow & & \updownarrow \\ \hat{E}^\sigma & \longmapsto & f^*\hat{E}^\sigma. \end{array}$$

Proof. $f^*\mathcal{E}$ is equivariant by lemma 2.9. Note that f^* in the category of sheaves is a functor only up to natural equivalence. □

We now paste together families of vector spaces associated to the cones of a fan:

Definition 4.8: Let Δ be a fan. A collection $\{\hat{E}^\sigma\}_{\sigma\in\Delta}$ of σ-families is called a Δ-*family*, denoted $\hat{E}^\Delta$, iff for each pair $\tau < \sigma$ with inclusions $i^\tau_\sigma : U_\tau \hookrightarrow U_\sigma$ there exists an isomorphism of families $\eta_{\tau\sigma} : i^{\tau *}_\sigma \hat{E}^\sigma \xrightarrow{\cong} \hat{E}^\tau$ such that for each triple $\rho < \tau < \sigma$ there is the equality $\eta_{\rho\sigma} = \eta_{\rho\tau} \circ i^{\rho *}_\tau \eta_{\tau\sigma}$. A *morphism* of Δ-families is a collection of morphisms $\{\hat{\phi}^\sigma : \hat{E}^\sigma \longrightarrow \hat{F}^\sigma\}_{\sigma\in\Delta}$ such that for all σ, τ and $\tau < \sigma$ the following diagram commutes:

$$\begin{array}{ccc} i^{\tau *}_\sigma \hat{E}^\sigma & \xrightarrow{i^{\tau *}_\sigma \hat{\phi}^\sigma} & i^{\tau *}_\sigma \hat{F}^\sigma \\ \downarrow{\scriptstyle \eta_{\tau\sigma}} & & \downarrow{\scriptstyle \eta'_{\tau\sigma}} \\ \hat{E}^\tau & \xrightarrow{\hat{\phi}^\tau} & \hat{F}^\tau \end{array}$$

With these definitions it is easy to see that the Δ-families form a category. Moreover:

Theorem 4.9: *Let Δ be a fan. Then the category of Δ-families is equivalent to the category of quasicoherent equivariant sheaves over X.*

Proof. Because the σ-families already have been shown to be equivalent to equivariant quasicoherent sheaves on the invariant open subvarieties U_σ of X, we have only to show that Definition 4.8 encodes the usual data for gluing the sheaves $\mathcal{E}_\sigma$ which correspond to the σ-families $\hat{E}^\sigma$ with respect to the open cover U_σ, $\sigma \in \Delta$. This means that there exists a family $\phi_{\sigma\sigma'} : \mathcal{E}_\sigma|_{U_{\sigma\cap\sigma'}} \xrightarrow{\cong} \mathcal{E}_{\sigma'}|_{U_{\sigma\cap\sigma'}}$ for all $\sigma, \sigma' \in \Delta$ such that $\phi_{\sigma\sigma} = \mathrm{id}$ and $\phi_{\sigma\sigma''} = \phi_{\sigma'\sigma''} \circ \phi_{\sigma\sigma'}$ over each triple intersection $U_{\sigma\cap\sigma'\cap\sigma''}$. Let $\sigma, \sigma' \in \Delta$ and set $\tau := \sigma \cap \sigma'$ define

$$\phi_{\sigma\sigma'} := \eta^{-1}_{\tau\sigma'} \circ \eta_{\tau\sigma}$$

Then the $\phi_{\sigma\sigma'}$ fulfill the cocycle condition up to natural equivalence and the $\mathcal{E}_\sigma$ glue to a quasi-coherent sheaf $\mathcal{E}$ on X. Furthermore it is then straightforward to verify that $\hat{E}^\Delta \mapsto \mathcal{E}$ defines a functor which induces an equivalence of categories. □

The following finiteness conditions of a Δ-family correspond to coherence:

Definition 4.10: We say that a σ-family $\hat{E}^\sigma$ is *finite* iff

(i) all the vector spaces E^σ_m are finite-dimensional,

(ii) for each chain $\cdots <_\sigma m_{i-1} <_\sigma m_i <_\sigma \cdots$ of characters in M there exists an $i_0 \in \mathbb{Z}$ such that $E^\sigma_{m_i} = 0$ for $i < i_0$,

(iii) there are only finitely many vector spaces E^σ_m such that the map

$$\bigoplus_{m' <_\sigma m} E^\sigma_{m'} \longrightarrow E^\sigma_m$$

defined by summation of the $\chi^\sigma_{m',m}$, is not surjective.

We say that a Δ-family is *finite* if all of its σ-families are finite.

Proposition 4.11: *A quasicoherent equivariant sheaf is coherent iff its associated Δ-family is finite.*

PROOF. Let $\hat{E}^\Delta$ be finite and let $\hat{E}^\sigma$ be any σ-family. Because the homomorphisms $\chi^\sigma_{m,m'}$ are isomorphisms iff $m \leq_\sigma m'$ and $m' \leq_\sigma m$, we assume without loss of generality that σ is maximal and thus $\leq_\sigma$ is a partial order. Consider the set $P = \{m \in M/\sigma^\perp_M$ such that $E_m \neq 0\}$ and let $P_{\min}$ be the set of minimal elements of P with respect to $\leq_\sigma$. Because of condition (iii), $P_{\min}$ must be finite, and by (ii), if the σ-family is nonempty, $P_{\min}$ is nonempty as well. Again by (iii), there exist only finitely many cokernels

$$\bigoplus_{m' <_\sigma m} E^\sigma_{m'} \longrightarrow E^\sigma_m \longrightarrow C_m \longrightarrow 0.$$

We denote $Q := \{m \in M \setminus P$ such that $C_m \neq 0\}$. The generators of the module $E^\sigma = \bigoplus_{m \in M} E^\sigma_m$ correspond to the k-vector space basis of $\bigoplus_{m \in P_{min}} E^\sigma_m \oplus \bigoplus_{m \in Q} C_m$ which by (i) is finite dimensional, hence E^σ is finitely generated. The converse follows straightforwardly by similar arguments. $\square$

Remark 4.12: Note that from Definition 4.10 follows immediately that for every chain $\cdots <_\sigma m_{i-1} <_\sigma m_i <_\sigma \cdots$ the sequence $E^\sigma_{m_i}$ becomes stationary, i.e. there exists an $i_1 \geq i_0$ such that the $\chi^\sigma_{m_i,m_{i+1}}$ become isomorphisms for all $i \geq i_1$.

4.3. The Krull-Schmidt Property

In this section we show that the Krull-Schmidt theorem holds for the category of equivariant coherent sheaves over any toric variety. Recall that the Krull-Schmidt theorem states the following. Let $\mathfrak{C}$ be any category in which direct sums exist. Then we say that the Krull-Schmidt theorem holds in $\mathfrak{C}$ if for every object A in $\mathcal{C}$ and for every two decompositions

$$A \cong X_1 \oplus X_2 \oplus \cdots \oplus X_n \cong Y_1 \oplus Y_2 \oplus \cdots \oplus Y_m$$

we have $m = n$ and there exists a permutation π of $\{1, \ldots, n\}$ such that $X_i \cong Y_{\pi(i)}$ for all i. In other words, the Krull-Schmidt theorem holds if direct sum decompositions are unique up to reordering of summands.

There is a classical result of Atiyah, which we want to apply below:

Proposition 4.13 ([**Ati56**], Corollary of Lemma 3)**:** *Let $\mathfrak{C}$ be an exact category over a field k such that* $\mathrm{Hom}(A, B)$ *is a finite dimensional k-vector space for every pair of objects (A, B) in $\mathfrak{C}$. Then the Krull-Schmidt theorem holds for $\mathfrak{C}$.*

It is well known that this result implies that in the category of coherent sheaves over a complete variety the Krull-Schmidt theorem holds. In our situation, we can drop the completeness condition on X:

Theorem 4.14: *Let X be any toric variety, then the Krull-Schmidt theorem holds for the category of equivariant coherent sheaves over X.*

Proof. According to proposition 4.13, it suffices to show that for every two equivariant sheaves $\mathcal{E}$ and $\mathcal{E}'$, the vector space $\operatorname{Hom}(\mathcal{E}, \mathcal{E}')^T$ of T-equivariant sheaf homomorphisms is finite-dimensional. Consider first the case where $X = U_\sigma$ is an affine toric variety and let $E = \Gamma(U_\sigma, \mathcal{E})$ and $E' = \Gamma(U_\sigma, \mathcal{E}')$. Denote $e_1, \dots, e_n$ some minimal set of homogeneous generators for E, then any $k[\sigma_M]$-linear homomorphism from E to E' is determined by the images of the e_i in E'. Possibly after reordering, we can group the generators as $\{e_1, \dots, e_{n_1}\}, \{e_{n_1+1}, \dots, e_{n_2}\}, \dots, \{e_{n_{r-1}+1}, \dots, e_{n_r}\}$, such that the e_i are in the same group if and only if their degrees in M coincide. Denote E_i the vector space spanned over k by the $\{e_{n_{i-1}+1}, \dots, e_{n_i}\}$, where $n_0 = 0$ and $n_r = n$. Then $E_i \subset E_{m_i}$ for some $m_i \in M$ and every homomorphism $E \to E'$ is given by an element

$$f \in \bigoplus_{i=1}^{r} \operatorname{Hom}_k(E_i, E'_{m_i}),$$

a direct sum of k-linear vector space homomorphisms. As we have seen in proposition 4.11, coherence implies that E'_{m_i} is finite dimensional for every i, and thus the space of graded module homomorphisms is finite dimensional as well.

In the general case, let $E^\sigma = \Gamma(U_\sigma, \mathcal{E})$ and $(E')^\sigma = \Gamma(U_\sigma, \mathcal{E}')$, the modules over the affine coordinate rings $k[\sigma_M]$ for all $\sigma \in \Delta$. Fix generators for the E^σ for every σ and define vector spaces E_i^σ of degrees m_i^σ as before. Then any sheaf homomorphism is given by some element

$$f \in \bigoplus_{\sigma \in \Delta} \bigoplus_{i=1}^{r} \operatorname{Hom}_k(E_i^\sigma, (E')^\sigma_{m_i^\sigma}).$$

This direct sum of vector spaces again is finite dimensional, and has $\operatorname{Hom}(\mathcal{E}, \mathcal{E}')^T$ as a subvector space. Note that the components f_σ of $f = (f_\sigma)_{\sigma \in \Delta}$ have to be compatible with respect to the gluing condition on the E^σ (see definition 4.8). Because the direct sum above already is finite dimensional, we refrain from making this explicit. □

4.4. Torsion Free Equivariant Sheaves

Let $\hat{E}^\Delta$ be a finite Δ-family. Each of its σ-families $\hat{E}^\sigma$ is preordered with respect to $\leq_\sigma$, so there exists for every $\sigma \in \Delta$ a direct limit

$$\varinjlim_{m} E_m^\sigma$$

which we denote by $\varinjlim \hat{E}^\sigma$ or $\mathbf{E}^\sigma$, respectively. This limit is even a *filtered direct limit* in the sense of [**Mac98**] (see also [**Eis95**], Appendix 6), which is an *exact* functor from the category of preordered families k-vector spaces into the category of k-vector spaces.

The direct limit has the universal property that for every two vector spaces $E^\sigma_m, E^\sigma_{m'}$ in $\hat{E}^\sigma$ with $m \leq_\sigma m'$, there is a commutative diagram

$$\begin{array}{ccc} E^\sigma_m & \xrightarrow{\chi^\sigma_{m,m'}} & E^\sigma_{m'} \\ & \searrow & \downarrow \\ & & \mathbf{E}^\sigma \end{array}$$

4.4.1. The Direct Limits for Torsion Free Equivariant Sheaves. Recall that a coherent sheaf $\mathcal{E}$ on a normal scheme X is *torsion free* iff there exists an injective homomorphism of $\mathcal{O}_X$-modules $\mathcal{E} \longrightarrow k(X)^r$ for some $r \geq 0$, where $k(X)$ is the field of rational functions over X. In our case $\mathcal{E}$ being a torsion free equivariant sheaf over a toric variety X, this implies that for all $\sigma \in \Delta$, $\Gamma(U_\sigma, \mathcal{E})$ is a torsion free $k[\sigma_M]$-module. It is easy to see that this is equivalent to the fact that $\chi(m) \cdot e \neq 0$ for all $m \in \sigma_M$ and all M-homogeneous elements $e \neq 0$ of $\Gamma(U_\sigma, \mathcal{E})$. This in turn implies:

Proposition 4.15: *Let $\mathcal{E}$ be an equivariant coherent sheaf on X and $\hat{E}^\Delta$ its Δ-family. $\mathcal{E}$ is torsion free iff for all $\sigma \in \Delta$ the maps in the above diagram are injective.*

If $\mathcal{E}$ is torsion free, then the restriction map $\Gamma(U_\sigma, \mathcal{E}) \longrightarrow \Gamma(U_\tau, \mathcal{E})$ for any $\tau < \sigma$ is injective. So there exists a natural inclusion of σ-families $\hat{E}^\sigma \hookrightarrow (i^\tau_\sigma)^* \hat{E}^\sigma$ for any $\tau < \sigma$. So there exists a natural inclusion of σ-families $\hat{E}^\sigma \hookrightarrow (i^\tau_\sigma)^* \hat{E}^\sigma$ for any $\tau < \sigma$. Therefore the composition

$$\hat{E}^\sigma \hookrightarrow (i^\tau_\sigma)^* \hat{E}^\sigma \xrightarrow[\cong]{\eta_{\tau\sigma}} \hat{E}^\tau \tag{3}$$

is injective. It is then easy to prove:

Proposition 4.16: *Let $\mathcal{E}$ be the coherent sheaf with corresponding Δ-family $\hat{E}^\Delta$. If $\mathcal{E}$ is torsion free then the homomorphisms $\hat{E}^\sigma \longrightarrow \hat{E}^\tau$ as in (3) are injections for any $\tau < \sigma$.*

For the rest of this subsection we will assume that all sheaves in question are torsion free. The injection $\hat{E}^\sigma \hookrightarrow \hat{E}^\tau$ of σ-families induces an injective map $\tilde{\eta}_{\tau\sigma} : \mathbf{E}^\sigma \hookrightarrow \mathbf{E}^\tau$. The system of vector spaces $\mathbf{E}^\sigma$ and homomorphisms $\tilde{\eta}_{\tau\sigma}$ forms a directed partially ordered family, $\mathbf{E}^\Delta$, with respect to the reversed partial order '$<$' among the cones of Δ. For any $\sigma_1, \sigma_2 \in \Delta$ we have diagrams

$$\begin{array}{ccc} \mathbf{E}^{\sigma_1} & & \\ & \searrow & \\ & & \mathbf{E}^{\sigma_1 \cap \sigma_2} \hookrightarrow \mathbf{E}^0 \\ & \nearrow & \\ \mathbf{E}^{\sigma_2} & & \end{array}$$

where 0 is the minimal cone in Δ. We obtain an identification of the direct limit $\varinjlim \mathbf{E}^\Delta$ with $\mathbf{E}^0$. By Proposition 2.36 any M-graded module over the ring $k[\sigma_M]$ is free, i.e. isomorphic to $k[M]^r$, for some $r \geq 0$. This means that every equivariant sheaf over the torus $T = U_0$ is free and r is the rank of this sheaf. Consequently, the morphisms $\chi^0_{m,m'}$ are all isomorphisms and $\mathbf{E}^0$ is an r-dimensional k-vector space. Recall that the restriction map $\Gamma(U_\sigma, \mathcal{E}) \hookrightarrow \Gamma(U_\tau, \mathcal{E})$ can be identified with the canonical map $\Gamma(U_\sigma, \mathcal{E}) \hookrightarrow \Gamma(U_\sigma, \mathcal{E})_{\chi(m_\tau)}$ into the localization with respect to some integral element m_τ in the interior of $\check{\sigma} \cap \tau^\perp$. By Remark 4.12 we know that for each chain $\cdots <_\sigma m + i \cdot m_\tau <_\sigma m + (i+1) \cdot m_\tau <_\sigma \cdots$ the sequence $E^\sigma_{m+i\cdot m_\tau}$ becomes stationary and because $m + (i+1) \cdot m_\tau - (m + i \cdot m_\tau) = m_\tau \in \tau^\perp_M$ the maps $\chi^\tau_{m+i\cdot m_\tau, m+(i+1)\cdot m_\tau}$ are isomorphisms for all $i \in \mathbb{Z}$. Therefore we have

Proposition 4.17: *Let $\tau < \sigma$ and $\hat{E}^\sigma$ a σ-family. Let m_τ be an integral element of the interior of $\check{\sigma} \cap \tau^\perp$ such that $\tau_M = \sigma_M + \mathbb{Z}_{\geq 0}(-m_\tau)$. For each $m \in M$ there is the chain $\cdots <_\sigma m + i \cdot m_\tau <_\sigma m + (i+1) \cdot m_\tau <_\sigma \cdots$. Then there is an $i_m \in \mathbb{N}$ such that $((i^\tau_\sigma)^*(\hat{E}^\sigma))_m \cong E^\sigma_{m+i\cdot m_\tau}$ for all $i \geq i_m$.*

Applying this to $0 < \sigma$ we get:

Corollary 4.18: *The inclusions $\mathbf{E}^\sigma \hookrightarrow \mathbf{E}^0$ are isomorphisms $\mathbf{E}^\sigma \cong \mathbf{E}^0$.*

4.4.2. The Category of Multifiltrations. Now we collect all the properties derived for equivariant, coherent torsion free sheaves:

Definition 4.19: Let Δ be a fan, V a finite-dimensional k-vector space, and let for each $\sigma \in \Delta$ a set of subvector spaces $\{E^\sigma_m\}_{m\in M}$ of V be given. We say that this system is a *family of multifiltrations of* V if:

(i) For $\sigma \in \Delta$ and $m \leq_\sigma m'$, E^σ_m is contained in $E^\sigma_{m'}$.
(ii) $V = \bigcup_{m\in M} E^\sigma_m$ for any $\sigma \in \Delta$.
(iii) For each chain $\cdots <_\sigma m_{i-1} <_\sigma m_i <_\sigma \cdots$ of characters in M there exists an $i_0 \in \mathbb{Z}$ such that $E^\sigma_{m_i} = 0$ for all $i \leq i_0$.
(iv) For every $\sigma \in \Delta$ there exist only finitely many vector spaces E^σ_m which are not contained in the union of all vector spaces $E^\sigma_{m'}$ with $m' <_\sigma m$.
(v) (compatibility condition) For each $\tau < \sigma$ with $\tau_M = \sigma_M + \mathbb{Z}_{\geq 0}(-m_\tau)$ we consider with respect to the preorder $\leq_\sigma$ the ascending chains $m + i \cdot m_\tau$ for $i \geq 0$. By condition (iv) and because V is finite dimensional the sequence of subvector spaces $E^\sigma_{m+i\cdot m_\tau}$ necessarily becomes stationary for some $i^\tau_m \in \mathbb{Z}$. We require that $E^\tau_m = E^\sigma_{m+i^\tau_m\cdot m_\tau}$ for all $m \in M$.

Families of multifiltrations are Δ-families which are realized as subvector spaces of the limit vector space $\mathbf{E}^0$.

A morphism of families of multifiltrations $\{E^\sigma_m\}_{\sigma\in\Delta, m\in M}$ and $\{F^\sigma_m\}_{\sigma\in\Delta, m\in M}$ then is equivalent to a homomorphism of vector spaces $\mathbf{E}^0 \longrightarrow \mathbf{F}^0$ which is compatible with these multifiltrations and so induces a morphism of Δ-families. This technical reformulation gives:

Theorem 4.20: *For any fan Δ, the category of torsion free equivariant coherent sheaves on X_Δ is equivalent to the category of families of multifiltrations of finite-dimensional vector spaces.*

Now we briefly explain how the above classification is related to the characterization of torsion free sheaves in [**Kly91**]. There it was stated that on a smooth complete toric variety X the category of equivariant torsion free sheaves is equivalent to the category of *multifiltrations* for finite-dimensional vector spaces. If X is smooth, for any r-dimensional cone σ, its primitive vectors $n(\rho)$, $\rho \in \sigma(1)$ form a part of a $\mathbb{Z}$-basis for N. Thus we can choose a basis $\{m(\rho_i)\}_{\rho_i \in \sigma(1)}$ of $M/\sigma_M^\perp$ dual to the minimal submodule of N which contains $\sigma \cap N$ and which is spanned by the $n(\rho)$. With respect to the dual basis we write for an element $m \in M$ its residual class $\overline{m} \in M/\sigma_M^\perp$ as $\overline{m} = \sum_{\rho_i \in \sigma(1)} i_{\rho_i} \cdot m(\rho_i)$. So by identifying $\overline{m}$ with the tuple $(i_{\rho_1}, \dots, i_{\rho_r})$, we identify $M/\sigma_M^\perp$ with $\mathbb{Z}^{\sigma(1)}$. Moreover, as subvector spaces of $\mathbf{E}^0$, we have that $E_m^\sigma = E_{m'}^\sigma$ whenever $m - m' \in \sigma_M^\perp$. So we can write $E^\sigma(i_{\rho_1}, \dots, i_{\rho_r})$ for E_m^σ. Then a multifiltration in the sense of [**Kly91**] associated to a cone σ is a set $\{E^\sigma(i_{\rho_1}, \dots, i_{\rho_r})\}_{(i_{\rho_1}, \dots, i_{\rho_r}) \subset \mathbb{Z}^{\sigma(1)}}$ of subvector spaces of some vector space $\mathbf{E}^0$ which is parametrized by indices i_{ρ_k} for each $\rho_k \in \sigma(1)$. In this formulation the conditions for the filtrations are:

- $E^\sigma(i_{\rho_1}, \dots, i_{\rho_k}, \dots, i_{\rho_r}) \subset E^\sigma(i_{\rho_1}, \dots, i_{\rho_k} + 1, \dots, i_{\rho_r})$ for each $\rho_k \in \sigma(1)$.
- $\bigcup_{i_{\rho_1}, \dots, i_{\rho_r}} E^\sigma(i_{\rho_1}, \dots, i_{\rho_r}) = V$.
- Let $\tau < \sigma$, then τ is spanned by the rays $\{\rho_{k_1}, \dots, \rho_{k_s}\} = \tau(1) \subset \sigma(1)$. Assume that $(k_1, \dots, k_s) = (1, \dots, s)$, then
$$E^\tau(i_{\rho_1}, \dots, i_{\rho_s}) = E^\sigma(i_{\rho_1}, \dots, i_{\rho_s}, \infty, \dots, \infty)$$
and analogously for $(k_1, \dots, k_s)$ different from $(1, \dots, s)$. Here ∞ abbreviates the choice of a suitable $i_m^\tau \in \mathbb{Z}$ as in Definition 4.19, (v).

These are the conditions of Klyachko ([**Kly91**]).

4.5. Reflexive Equivariant Sheaves

If $\mathcal{E}$ is a reflexive sheaf on a normal variety X, then $\Gamma(X, \mathcal{E}) = \Gamma(X \setminus Y, \mathcal{E})$ if Y is a closed subset of X of codimension at least two. If $\mathcal{E}$ is an equivariant reflexive sheaf on a toric variety $X = X_\Delta$, a natural choice for Y is the union of orbits in X which have codimension at least two, i.e. $\Gamma(X, \mathcal{E}) = \Gamma(X_{\Delta_1}, \mathcal{E})$, where $\Delta_1 = \Delta(0) \cup \Delta(1)$. In particular, on an affine toric variety we can write $\Gamma(U_\sigma, \mathcal{E}) = \Gamma\big(\bigcup_{\rho \in \sigma(1)} U_\rho, \mathcal{E}\big)$. So, if we consider the $\Gamma(U_\rho, \mathcal{E})$ as k-subvector spaces of $\Gamma(U_0, \mathcal{E})$, we have
$$\Gamma(U_\sigma, \mathcal{E}) \cong \bigcap_{\rho \in \sigma(1)} \Gamma(U_\rho, \mathcal{E}).$$
This implies that $\Gamma(U_\sigma, \mathcal{E})_m \cong \bigcap_{\rho \in \sigma(1)} \Gamma(U_\rho, \mathcal{E})_m$ for each graded component of degree m. We can translate this in a natural way to the intersection of multifiltrations in the limit vector space $\mathbf{E}^0$:
$$E_m^\sigma = \bigcap_{\rho \in \sigma(1)} E_m^\rho.$$
Hence a reflexive sheaf is completely determined by the multifiltrations E_m^ρ of V for $\rho \in \Delta(1)$. We know that $E_m^\rho = E_{m'}^\rho$ if $m - m' \in \rho_M^\perp$. Thus the multifiltrations are determined by the stabilizer of the minimal orbit of U_ρ, whose group of characters is

$M/\rho_M^\perp$ which canonically can be identified with $\mathbb{Z}$ by using the primitive lattice element $n(\rho)$. So we can pass to a filtration of $\mathbf{E}^0$:

$$0 \quad \ldots \subset E^\rho(i) \subset E^\rho(i+1) \subset \ldots \quad \mathbf{E}^0.$$

This filtration must be *full* in the sense that $E^\rho(i) = 0$ for i sufficiently small and $E^\rho(i) = \mathbf{E}^0$ for i sufficiently large. We have the identity:

$$E^\rho_m = E^\rho(\langle m, n(\rho)\rangle)$$

Using this identity we easily get back a Δ-family from a given set of filtrations $E^\rho(i)$ as follows. We set

$$E^\rho_m := E^\rho(\langle m, n(\rho)\rangle)$$

and this way obtain the ρ-families corresponding to free $k[\rho_M]$-modules, which then define a locally free sheaf over X_{Δ_1}. We form the reflexive continuation for all U_σ by setting:

$$E^\sigma_m := \bigcap_{\rho\in\sigma(1)} E^\rho_m = \bigcap_{\rho\in\sigma(1)} E^\rho(\langle m, n(\rho)\rangle).$$

Now we have a classification for equivariant reflexive sheaves on toric varieties:

Theorem 4.21: *The category of equivariant reflexive sheaves on a toric variety X is equivalent to the category of vector spaces with full filtrations associated to each ray in $\Delta(1)$. The morphisms in this category are vector space homomorphisms which are compatible with the filtrations in the Δ-family sense.*

4.6. Locally Free Sheaves

A locally free sheaf is reflexive, and thus an equivariant locally free sheaf $\mathcal{E}$ over a toric variety X can defined by a set of full filtrations $E^\rho(i)$ of a finite dimensional vector space $\mathbf{E}^0$. Recall that locally free means that for every $x \in X$ the stalk $\mathcal{E}_x$ is a free $\mathcal{O}_{X,x}$-module. This implies in particular, that for every affine open subset U_σ of X the global sections $\Gamma(U_\sigma, \mathcal{E})$ form a projective $k[\sigma_M]$-module. However, not every reflexive module is projective, and thus there must be conditions on the filtrations $E^\rho(i)$ such that they define a locally free sheaf. In this section we will derive these conditions – as they were originally stated by Klyachko in [**Kly90**] – from the formalism we have developed before, and we will give a geometric picture of the filtrations which will be derived from the induced representations of the torus T in the geometric fibers of the vector bundle associated to $\mathcal{E}$. Moreover, we will discuss how to compute filtrations for pullbacks by toric morphisms and for restrictions to orbit closures.

4.6.1. Compatibility Condition for Locally Free Sheaves.

Proposition 4.22: *Finitely generated projective M-graded $k[\sigma_M]$-modules are free.*

Proof. The homogeneous ideal of $k[\sigma_M]$ associated to the semigroup ideal $\sigma_M \setminus \sigma_M^\perp$ is the unique maximal homogeneous ideal of $k[\sigma_M]$, so by Corollary 2.38, projective M-graded $k[\sigma_M]$-modules are free. □

Remark 4.23: This is just another proof for the following fact: locally free equivariant sheaves over affine toric varieties are free. This observation was already made in [**Kly90**], Proposition 2.1 i) and in [**Kan75**], Theorem 3.5. More generally, it is even true that any locally free sheaf over an affine toric variety is free (see [**Gub99**]).

So an equivariant bundle $\mathcal{E}$ of rank r on U_σ corresponds to an M-graded free $k[\sigma_M]$-module

$$E^\sigma = \Gamma(U_\sigma, \mathcal{E}) = \bigoplus_{i=1}^{r} k[\sigma_M](-m_i) = \bigoplus_{i=1}^{r} \chi(m_i) \cdot k[\sigma_M]$$

i.e. it is a direct sum of graded free modules with respect to r characters $m_i \in M$. The latter equality comes from an equivariant embedding of E^σ into the quasi-coherent $k[\sigma_M]$-module $\Gamma(T, \mathcal{E}) \cong k[M]^r$. The direct limit is naturally compatible with splitting:

$$\lim_{\rightarrow} \bigoplus_{i=1}^{r} \hat{E}_i^\sigma = \bigoplus_{i=1}^{r} \lim_{\rightarrow} \hat{E}_i^\sigma$$

or, respectively,

$$\mathbf{E}^\sigma = \bigoplus_{i=1}^{r} \mathbf{E}_i^\sigma.$$

Proposition 4.24: *Let $\mathcal{E}$ be an equivariant reflexive sheaf of rank r over X with corresponding filtrations $E^\rho(i)$. Then $\mathcal{E}$ is locally free if and only if for any $\sigma \in \Delta$ there is an action of T_σ on $\mathbf{E}^0$ and a decomposition into T_σ-eigenspaces $\mathbf{E}^0 = \bigoplus_{m \in M/\sigma_M^\perp} \mathbf{E}_m^0$ such that*

$$E^\rho(i) = \bigoplus_{\substack{m \in M/\sigma_M^\perp \\ \langle m, n(\rho)\rangle \leq i}} \mathbf{E}_m^0$$

for any $\rho \in \sigma(1)$.

Proof. Let first $\mathcal{E}$ be locally free and let

$$E^\sigma = \bigoplus_{i=1}^{r} L_j^\sigma$$

be a decomposition as above with $L_j^\sigma = k[\sigma_M](-m_j)$. Then for every i:

$$\dim(L_i^\sigma)_m = \begin{cases} 0 & \text{if } m_j \not\leq_\sigma m \\ 1 & \text{if } m_j \leq_\sigma m. \end{cases}$$

If we consider the L_j^σ, we can pass to the limit vector spaces $\mathbf{L}_j^\sigma$, and we obtain isomorphisms $(L_j^\sigma)_m \cong \mathbf{L}_j^\sigma$ for $m \geq_\sigma m_j$. Consequently, the filtrations for L_j^σ are given by:

$$L^\rho(i) = \begin{cases} 0 & \text{if } i < \langle m_j, n(\rho)\rangle \\ \mathbf{L}_j^\sigma & \text{if } i \geq \langle m_j, n(\rho)\rangle \end{cases}$$

The vector space $\mathbf{L}_j^\sigma$ becomes a T-module by setting

$$T \times \mathbf{L}_j^\sigma \longrightarrow \mathbf{L}_j^\sigma, \quad (t, l) \mapsto \chi(m_j)(t) \cdot l.$$

By forming direct sums, we get:

$$\mathbf{E}^\sigma = \bigoplus_{j=1} \mathbf{L}_j^\sigma$$

and by this isomorphisms $\mathbf{E}^\sigma$ becomes a T-module:

$$T \times \mathbf{E}^\sigma \longrightarrow \mathbf{E}^\sigma, \quad (t,e) \longrightarrow \operatorname{diag}\big(\chi(m_1)(t), \dots, \chi(m_r)(t)\big) \cdot e.$$

Moreover,

$$E^\rho(i) = \bigoplus_{\langle m_j, n(\rho)\rangle \leq i} \mathbf{L}_j^\sigma.$$

The T-module structure induces by restriction automatically a T_σ-module structure: via the surjection

$$M \longrightarrow M/\sigma_M^\perp, m \mapsto \overline{m}$$

we have

$$T_\sigma \times \mathbf{E}^\sigma \longrightarrow \mathbf{E}^\sigma, \quad (t,e) \longrightarrow \operatorname{diag}\big(\chi(\overline{m}_1)(t), \dots, \chi(\overline{m}_r)(t)\big) \cdot e.$$

Moreover, we have $\langle m - m', n(\rho)\rangle = 0$ whenever $m - m' \in \sigma_M^\perp$, and thus we can equivalently write

$$E^\rho(i) = \bigoplus_{\langle \overline{m}_j, n(\rho)\rangle \leq i} \mathbf{L}_j^\sigma.$$

By restriction to the open torus T we get a diagram of isomorphisms:

$$\begin{array}{ccc} \mathbf{E}^\sigma & \xrightarrow{\cong} & \bigoplus_{i=1}^r \mathbf{L}_j^\sigma \\ \cong\downarrow & & \downarrow\cong \\ \mathbf{E}^0 & \xrightarrow{\cong} & \bigoplus_{i=1}^r \mathbf{L}_j^0 \end{array}$$

by which we transport the T_σ-action to $\mathbf{E}^0$.

Now assume conversely that for every $\sigma \in \Delta$ we can find a T_σ-eigenspace decomposition $\mathbf{E}^0 \cong \bigoplus_{j=1}^r \mathbf{E}_{m_j}^0$, $m_1, \dots, m_r \in M/\sigma_M^\perp$, such that

$$E^\rho(i) = \bigoplus_{\langle m_j, n(\rho)\rangle \leq i} \mathbf{E}_{m_j}^0.$$

Then for every $m \in M$

$$\begin{aligned} E_m^\sigma &= \bigcap_{\rho \in \sigma(1)} E^\rho(\langle m, n(\rho)\rangle) \\ &= \bigcap_{\rho \in \sigma(1)} \bigoplus_{\langle m_j, n(\rho)\rangle \leq \langle m, n(\rho)\rangle} \mathbf{E}_{m_j}^0 \\ &= \bigoplus_{\substack{\langle m_j, n(\rho)\rangle \leq \langle m, n(\rho)\rangle \\ \text{for all } \rho \in \sigma(1)}} \mathbf{E}_{m_j}^0 \end{aligned}$$

By comparison, we then obtain

$$E_m^\sigma \cong \bigoplus_{j=1}^r (L_j^\sigma)_m$$

where $L_j^\sigma \cong k[\sigma_M](-m_j)$, so that we have a splitting $E^\sigma = \bigoplus_{j=1}^r L_j^\sigma$. □

Remark 4.25: Proposition 4.24 is equivalent to the criterion of locally freeness of Klyachko:

Theorem ([**Kly90**], Theorem 2.2.1)**:** *The category of equivariant vector bundles over a toric variety X_Δ is equivalent to the category of vector spaces with a family of filtrations $E^\rho(i)$ for each $\rho \in \Delta(1)$ which satisfy the following compatibility condition:*

Let E be a vector space with a family of filtrations, then for any $\sigma \in \Delta$ there exists a T_σ-eigenspace decomposition $E = \bigoplus_{m \in M/\sigma_M^\perp} E_m$ such that

$$E^\rho(i) = \sum_{m, \langle m, n(\rho) \rangle \leq i} E_m$$

Note that here we are considering *increasing* filtrations instead of decreasing ones as in [**Kly90**]. Klyachko's original argument was by considering the induced T_σ-action in the geometric fibers of $\mathcal{E}$. We will show how this relates to our description above.

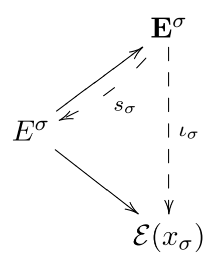
Let x_σ be a point in the minimal orbit of U_σ, and let T_σ be the stabilizer subgroup of T at the point x_σ. Consider the equivariant embedding $(i_\sigma, \mathrm{id}) : (x_\sigma, T_\sigma) \hookrightarrow (X, T)$. The pullback $i_\sigma^* \mathcal{E} =: \mathcal{E}(x_\sigma)$ to x_σ is an r-dimensional vector space over x_σ, which we call the *geometric fiber* of $\mathcal{E}$ over x_σ for the moment. There is – in sloppy notation – an isomorphism $\mathcal{E}(x_\sigma) \cong E^\sigma \otimes_{k[\sigma_M]} k$ and there is a natural morphism

$$E^\sigma \longrightarrow E^\sigma \otimes_{k[\sigma_M]} k, \quad e \mapsto e \otimes 1.$$

The geometric fiber is a representation space for the stabilizer T_σ, such that there is a splitting

$$\mathcal{E}(x_\sigma) = \bigoplus_{i=1}^{r} \mathcal{E}(x_\sigma)_{-\overline{m}_i}$$

where $\overline{m}_i$ are the images of the m_i in the character group $X(T_\sigma) = M/\sigma_M^\perp$. Taking also into account the homomorphism of k-vector spaces $E^\sigma \longrightarrow \mathbf{E}^\sigma$, which is defined as the sum of the homomorphisms $E_m^\sigma \to \mathbf{E}^\sigma$, we have a diagram



There is no canonical map between $\mathbf{E}^\sigma$ and $\mathcal{E}(x_\sigma)$, but if $\mathcal{E}$ is locally free, we can choose a section s_σ which maps $e \in \mathbf{E}_i^\sigma$ to its preimage in $(E_i^\sigma)_{m_i}$. By composition we obtain an isomorphism of vector spaces ι_σ such that $\iota_\sigma(\mathbf{E}_i^\sigma) \subset \mathcal{E}(x_\sigma)_{-\overline{m}_i}$. If we fix for every cone σ in Δ some point x_σ in the corresponding orbit, we can now use the ι_σ to lift the isomomorphisms $E^\sigma \to E^\tau$ for $\tau < \sigma$ to isomorphisms $\mathcal{E}(x_\sigma) \to \mathcal{E}(x_\tau)$. This way we have achieved a transport of the filtrations for $\mathbf{E}^0$ to filtrations of some general fiber $\mathcal{E}(x_0)$ over the open orbit of X.

4.6.2. Equivariant Pullbacks of Vector Bundles. Let $\mathcal{E}$ be a locally free sheaf over $X = X_\Delta$ and let $\phi : (N', \Delta') \longrightarrow (N, \Delta)$ be a map of fans which induces an equivariant morphism f from $X' = X_{\Delta'}$ to X. Denote $\mathcal{E}' := f^*\mathcal{E}$. In this section we want to compute the filtrations for the pullback $\mathcal{E}'$.

Let $\rho \in \Delta'$ and let $\sigma \in \Delta$ be the minimal cone such that $\phi(\rho) \subset \sigma$. Then there is the following diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \sigma_M^\perp & \longrightarrow & M & \longrightarrow & M/\sigma_M^\perp & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \alpha} & & \\ 0 & \longrightarrow & \rho_{M'}^\perp & \longrightarrow & M' & \longrightarrow & M/\rho_{M'}^\perp & \longrightarrow & 0 \end{array}$$

where α denotes the map between the character groups of the stabilizers T_σ and T'_ρ. Consider the T_σ-eigenspace decomposition

$$\mathbf{E}^\sigma = \bigoplus_{i=1}^{r} \mathbf{E}^\sigma_{m_i},$$

where $m_i \in M/\sigma_M^\perp$. We can transport this decomposition to a decomposition of $\mathbf{E}'^\rho$, which becomes via α a T'_ρ-eigenspace decomposition

$$\mathbf{E}'^\rho = \bigoplus_{i=1}^{r} \mathbf{E}^\sigma_{m_i} =: \bigoplus_{i=1}^{r} \mathbf{E}'^\rho_{\alpha(m_i)}.$$

Here we have set $\mathbf{E}'^\rho_{\alpha(m_i)} := \mathbf{E}^\sigma_{m_i}$ The induced filtrations then are given by:

$$E'^\rho(i) = \bigoplus_{m_j,\, \langle \alpha(m_j), n(\rho)\rangle \le i} \mathbf{E}^\sigma_{m_j} = \bigoplus_{m_j,\, \langle \alpha(m_j), n(\rho)\rangle \le i} \Bigg(\bigcap_{\tau \in \sigma(1)} E^\tau(\langle m_j, n(\tau)\rangle) \Bigg).$$

4.6.3. Restrictions of Vector Bundles. Let $X = X_\Delta$ and let $\sigma \in \Delta$. Consider the orbit closure $V(\sigma)$ corresponding to σ. We want to compute the restriction of a locally free sheaf $\mathcal{E}$ over X to $V(\sigma)$. This is similar to computing the pullback of a morphism of toric varieties as discussed in the last subsection, except that the new filtrations depend on the choice of a projection

$$M \overset{p}{\twoheadrightarrow} \sigma_M^\perp$$

which in general is not unique (see section 3.5). This projection corresponds to a section $s : T^\sigma \rightarrowtail T$, where $T^\sigma = T/T_\sigma$ is the quotient group of T which acts freely on $\operatorname{orb}(\sigma)$, and we consider the equivariant embedding

$$(\iota, s) : (V(\sigma), T^\sigma) \hookrightarrow (X, T).$$

Let $\tau > \sigma$ and $\tau \in \Delta(\dim \sigma + 1)$ such that the image $\bar{\tau}$ of τ in $N_\mathbb{R}/N_{\mathbb{R},\sigma}$ becomes a ray in $\operatorname{star}(\sigma)$ and consider the following diagram

$$\begin{array}{ccccccccc} & & & & M & \longrightarrow & M/\tau_M^\perp & \longrightarrow & 0 \\ & & & & \downarrow{\scriptstyle p} & \overset{\pi}{\searrow} & \downarrow & & \\ 0 & \longrightarrow & \tau_M^\perp & \longrightarrow & \sigma_M^\perp & \longrightarrow & \sigma_M^\perp/\tau_M^\perp & \longrightarrow & 0 \end{array}$$

The diagonalizable group $\mathrm{Hom}(\sigma_M^\perp/\tau_M^\perp, k^*)$ can naturally be identified with the image of the stabilizer subgroup T_σ of T in T^σ. Let

$$\mathbf{E}^\tau = \bigoplus_{i=1}^r E_{m_i}^\tau$$

be the eigenspace decomposition over U_τ, where $m_i \in M$. Then the filtrations for the restriction of $\mathcal{E}$ to $V(\sigma)$ are given by:

$$\begin{aligned} E^{\bar{\tau}}(k) &= \bigoplus_{\langle \pi(m_i), n(\bar{\tau})\rangle \le k} E_{m_i}^\tau \\ &= \bigoplus_{\langle \pi(m_i), n(\bar{\tau})\rangle \le k} \bigcap_{\rho\in\tau(1)} E^\rho\big(\langle m_i, n(\rho)\rangle\big) \end{aligned}$$

where $n(\bar{\tau})$ denotes the primitive vector of $\bar{\tau}$ in the lattice of N/N_σ.

Remark 4.26: Klyachko in [**Kly90**], Theorem 6.3.1, gave a similar characterization of restrictions.

4.7. Reflexive Equivariant Sheaves of Rank One and Twists

Let $D = \sum_{\rho\in\Delta(1)} i_\rho D_\rho$ be a T-invariant Weil divisor on X and let $\mathcal{O}_X(D)$ be its associated reflexive sheaf of rank one, which is naturally T-equivariant. The restriction of $\mathcal{O}_X(D)$ to some U_ρ, $\rho \in \Delta(1)$, is isomorphic to the sheaf $\mathcal{O}_{U_\sigma}(i_\rho D_\rho)$. The stabilizer T_ρ has character group $M/\rho_M^\perp \cong \mathbb{Z}$ and the ring $k[U_\rho] \cong k[\rho_M]$ is naturally $\mathbb{Z}$-graded via the surjection $M \twoheadrightarrow M/\rho_M^\perp$. Let $m_\rho \in M/\rho_M^\perp \cong \mathbb{Z}$ be the canonical generator, where $\langle m_\rho, n(\rho)\rangle = 1$, then $\Gamma(U_\rho, \mathcal{O}_X(D))$ is an $M/\rho_M^\perp$-graded module as follows:

$$\Gamma(U_\rho, \mathcal{O}_X(D)) \cong k[\rho_M](-i_\rho \cdot m_\rho)$$

and thus we can compute the corresponding filtrations $O^\rho(i)$ of the one dimensional vector space $\mathbf{O}^0$ for $\mathcal{O}_X(D)$ as:

$$O^\rho(i) = \begin{cases} 0 & \text{for } i < -i_\rho \\ \mathbf{O}^0 & \text{for } i \ge -i_\rho. \end{cases}$$

Now let $\mathcal{E}$ be any equivariant reflexive sheaf over X which corresponds to filtrations $E^\rho(i)$. We want to compute the filtrations $(E')^\rho(i)$ for the *"reflexive twist"*

$$\mathcal{E}' := \big(\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{O}_X(D)\big)^{\check{}\check{}}.$$

Again, it suffices to consider the restrictions of $\mathcal{E}$ and $\mathcal{O}_X(D)$ to T-invariant open subsets U_ρ, where $\rho \in \Delta(1)$. Over such an U_ρ both sheaves become locally free, and there is a natural isomorphism $\big(\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{O}_X(D)\big)^{\check{}\check{}}|_{U_\rho} \cong \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{O}_X(D)|_{U_\rho}$. The $k[\rho_M]$-module $\Gamma(U_\rho, \mathcal{E})$ splits as follows

$$\Gamma(U_\rho, \mathcal{E}) \cong \bigoplus_{k=1}^r k[\rho_M](-j_\rho^k m_\rho),$$

where m_ρ is as above and the j_ρ^k are some integers. For the tensor product we have:

$$\bigoplus_{k=1}^{r} k[\rho_M](-j_\rho^k m_\rho) \otimes_{k[\rho_M]} k[\rho_M](-i_\rho^k m_\rho) \cong \bigoplus_{k=1}^{r} k[\rho_M](-(i_\rho + j_\rho^k) m_\rho),$$

from which follows that

$$(E')^\rho(i) = E^\rho(i + i_\rho) \quad \text{for all } \rho \in \Delta(1).$$

This can be rephrased that twisting any reflexive sheaf by some equivariant reflexive sheaf $\mathcal{O}_X(\sum_{\rho\in\Delta(1)} i_\rho D_\rho)$ is the same as shifting its filtration $E^\rho(i)$ by $-i_\rho$.

Note that such shifting in general, in particular if X is not smooth, is by no means harmless, as $\mathcal{E}$ and $\mathcal{O}_X(\sum_{\rho\in\Delta(1)} i_\rho D_\rho)$ may not be locally free.

4.8. Equivariant Sheaves and Homogeneous Coordinate Rings

Let $\phi : \hat{X} \longrightarrow X$ be a quotient presentation of X such that X is a good quotient of $\hat{X}$ by some diagonalizable group G. Denote $S = \Gamma(\hat{X}, \mathcal{O}_{\hat{X}})$ the homogeneous coordinate ring. One may compare the following proposition with [**BC94**], Proposition 4.17.

Proposition 4.27:

(i) Let E be an $\hat{M}$-graded S-module. Then $\tilde{E}$ is a T-equivariant sheaf over X.
(ii) Every T-equivariant sheaf $\mathcal{E}$ is of the form $\tilde{E}$ for some $\hat{M}$-graded S-module E.

Proof. Consider the short exact sequence (see section 3.6)

$$0 \longrightarrow M \longrightarrow \hat{M} \longrightarrow X(G) \longrightarrow 0.$$

Locally, over every U_σ, the $k[\sigma_M]$-module $\Gamma(U_\sigma, \tilde{E})$ corresponds to the degree-zero component, with respect to its $X(G)$-grading, of the $k[\hat{\sigma}_M]$-module $\Gamma(U_{\hat{\sigma}}, \hat{\mathcal{E}})$, where $\hat{\mathcal{E}}$ is the sheaf over $\hat{X}$ associated to E. Because $\Gamma(U_{\hat{\sigma}}, \hat{\mathcal{E}})$ is $\hat{M}$-graded, it follows that the degree-zero component is M-graded, and part (i) is proved. According to Corollary 3.52, $\mathcal{E}$ is isomorphic to $\Gamma(\hat{X}, \pi^*\mathcal{E})\tilde{}$, and because ϕ is G-equivariant, $\pi^*\mathcal{E}$ is $\hat{T}$-equivariant, and thus $\Gamma(\hat{X}, \pi^*\mathcal{E})$ is $\hat{M}$-graded. □

We now specialize to Cox' homogeneous coordinate ring and denote $S = k[x_\rho \mid \rho \in \Delta(1)]$, C the positive orthant of $\mathbb{R}^{\Delta(1)}$, and $U_C \cong k^{\Delta(1)}$ the associated affine toric variety. The ring S has a natural structure of a $\mathbb{Z}^{\Delta(1)}$-graded ring where for every monomial $\deg x^{\underline{n}} = \underline{n}$. By the surjection $\mathbb{Z}^{\Delta(1)} \twoheadrightarrow A_{n-1}(X)$ this grading is compatible with the $A_{n-1}(X)$-grading the way that for every class $\alpha \in A_{n-1}(X)$

$$S_\alpha = \bigoplus_{\underline{n} \mapsto \alpha} S_{\underline{n}}.$$

Definition 4.28: We call the natural $\mathbb{Z}^{\Delta(1)}$-grading of S the *fine grading*.

As we have seen in section 3.6, all reflexive sheaves of rank one are isomorphic to a sheaf $\widetilde{S(\alpha)}$, where $S(\alpha)$ is the free, $A_{n-1}(X)$-graded S-module whose degrees are shifted by α. Analogously, one can pick some $\underline{n} = (n_\rho) \in \mathbb{Z}^{\Delta(1)}$ and consider the degree-shifted module $S(\underline{n})$ with respect to the fine-grading of S. By the surjection $\mathbb{Z}^{\Delta(1)} \twoheadrightarrow A_{n-1}(X)$, this degree shift implies a degree shift with respect to the $A_{n-1}(X)$-grading. It turns out that the sheafification $\widetilde{S(\underline{n})}$ is naturally isomorphic to $\mathcal{O}_X(-\sum_{\rho\in\Delta(1)} n_\rho D_\rho)$ together with its T-equivariant structure.

Remark 4.29: Theorem 1.1 of [**BV97**] states that, because S is a polynomial ring, for every graded S-module E, there exists a resolution

$$0 \longrightarrow F_r \longrightarrow \cdots \longrightarrow F_0 \longrightarrow E \longrightarrow 0$$

where the F_i are free, graded S-modules and $r \leq \#\Delta(1)$ by Hilbert's syzygy theorem. Then by sheafification we obtain a resolution

$$0 \longrightarrow \tilde{F}_r \longrightarrow \cdots \longrightarrow \tilde{F}_0 \longrightarrow \tilde{E} \longrightarrow 0$$

where $\tilde{F}_i \cong \bigoplus_{j=1}^{k_i} \mathcal{O}_X(D_{ij})$ for some Weil divisors D_{ij}. Thus every sheaf over X has a resolution of length $\leq \#\Delta(1)$ by direct sums of reflexive sheaves of rank one. In the equivariant case this resolution can be chosen in terms of T-linearized sheaves $\mathcal{O}_X(D_{ij})$ where the D_{ij} are T-invariant Weil divisors. In the following chapters we are going to refine this result.

A homogeneous homomorphism of fine-graded free modules

$$\bigoplus_{j=1}^{p} S(\underline{m}_j) \xrightarrow{A} \bigoplus_{i=1}^{q} S(\underline{n}_i)$$

is given by an $n \times m$-matrix $A = (a_{ij})$ whose entries are monomials $a_{ij} = \alpha_{ij} x^{\underline{m}_j - \underline{n}_i}$, where $\alpha_{ij} = 0$ whenever $\underline{m}_j - \underline{n}_i \notin \mathbb{N}^{\Delta(1)}$. The degree of the monomials is completely determined by the grading. Hence, it is equivalent to define such a morphism by a matrix (α_{ij}) with $\alpha_{ij} \in k$.

A fine-graded S-module E gives also rise to an equivariant quasicoherent sheaf over $k^{\Delta(1)}$ and is therefore equivalent to a family of vector spaces $\hat{E}^C$. One can easily see that $\hat{E}^C$ induces a $\hat{\Delta}$-family by restriction to $\hat{X}$. Over $U_{\hat{\sigma}}$ the module E localizes to the $k[\hat{\sigma}_M]$-module $E_{x^{\hat{\sigma}}}$. Its zero component $E_{(x^{\hat{\sigma}})}$ is an M-graded $S_{(x^{\hat{\sigma}})}$-module. By the inclusion $0 \longrightarrow M \xrightarrow{j} \mathbb{Z}^{\Delta(1)}$ from diagram (2) we can write

$$E_{(x^{\hat{\sigma}})} = \bigoplus_{m\in M} (E_{x^{\hat{\sigma}}})_{j(m)}$$

Thus the set of homogeneous components $\{(E_{(x^{\hat{\sigma}})})_m\}_{m\in M}$ can be considered as a subset of the set of homogeneous components of $E_{x^{\hat{\sigma}}}$. If we set $\chi^\sigma_{m,m'} := \chi^{\hat{\sigma}}_{j(m),j(m')}$ we obtain a σ-family as a subfamily of the $\hat{\sigma}$-family $\hat{E}^{\hat{\sigma}}$.

Thus we obtain a Δ-family associated to a quasicoherent equivariant sheaf over X as a subfamily of the $\hat{\Delta}$-family associated to the fine-graded module E. We can refine this for torsion free equivariant sheaves as follows. Let E be a fine-graded S-module such that $\tilde{E}$ is a torsion free sheaf on X. The torsion submodule $t(E)$ of E is fine-graded as well and the module $(E/t(E))$ gives rise to an equivariant sheaf $(E/t(E))\tilde{}$ over X isomorphic to $\tilde{E}$. Thus if we consider torsion free sheaves on X of the form $\tilde{E}$ for some fine-graded S-module E, we can assume that E is torsion free and thus equivalent to a family $E^{\hat{\Delta}}$ of multifiltrations for some vector space V associated to the fan $\hat{\Delta}$. By the above construction the Δ-family obtained as subfamily of the $\hat{\Delta}$-family defined by E is a collection of families of subvector spaces of V. These families are indeed multifiltrations of V as well:

Proposition 4.30: $\varinjlim \hat{E}^{\hat{\sigma}} = \varinjlim \hat{E}^{\sigma}$ *for all* $\sigma \in \Delta$.

PROOF. Because $\hat{E}^{\sigma}$ is a subfamily of $\hat{E}^{\hat{\sigma}}$ we know that $\varinjlim \hat{E}^{\sigma} \subset \varinjlim \hat{E}^{\hat{\sigma}}$. For the other inclusion it suffices to show that for each $\underline{n} \in \mathbb{Z}^{\Delta(1)}$ there exists $m \in M$ such that $\underline{n} <_{\hat{\sigma}} j(m)$. Denote by $\sup(\sigma) \in \mathbb{Z}^{\Delta(1)}$ the tuple whose ρth entry is 1 iff $\rho \in \sigma(1)$ and 0 else. Then the claim follows from the fact that for an $m \in \operatorname{int} \check{\sigma} \cap M$ and a suitable $n \in \mathbb{N}$ each component of $j(m) + n \cdot \sup(\sigma)$ is bigger than zero. Then for each $\underline{n} \in \mathbb{Z}^{\Delta(1)}$ there exists an $i \geq 0$ such that $\underline{n} <_{\hat{\sigma}} j(m) + (n+i) \cdot \sup(\sigma)$. It follows that $\underline{n} <_{\hat{\sigma}} j(m)$, because with respect to the preorder $\leq_{\hat{\sigma}}$ it holds that $\underline{n'} \leq_{\hat{\sigma}} \underline{n'} + k \cdot \sup(\sigma)$ and $\underline{n'} + k \cdot \sup(\sigma) \leq_{\hat{\sigma}} \underline{n'}$ for all $\underline{n'} \in \mathbb{Z}^{\Delta(1)}$ and all $k \in \mathbb{Z}$. □

Example 4.31: Consider a reflexive equivariant sheaf of rank 1 of the form $\widetilde{S(\underline{n})}$ for some $\underline{n} = (n_\rho) \in \mathbb{Z}^{\Delta(1)}$. $S(\underline{n})$ corresponds to a set of filtrations for the 1-dimensional vector space k:

$$E^{\hat{\rho}}(i) = \begin{cases} 0 & \text{if } i < -n_\rho \\ k & \text{if } i \geq -n_\rho \end{cases}$$

for all $\hat{\rho} \in \hat{\Delta}(1)$. Each equivariant reflexive sheaf of rank 1 then is determined by a function $\underline{n} : \Delta(1) \longrightarrow \mathbb{Z}$ (see also [**Kly90**], Example 2.3.1). We can identify these filtrations with the filtrations for the corresponding reflexive sheaf on X:

$$E^{\rho}(i) = \begin{cases} 0 & \text{if } i < -n_\rho \\ k & \text{if } i \geq -n_\rho \end{cases}$$

Example 4.32: Let E be a fine-graded S-module such that $\mathcal{E}$ is a torsion free equivariant sheaf of rank 1. As we have seen in the previous subsection, we can assume without loss of generality that E is a torsion free module of rank 1. Thus:

Proposition 4.33: *E is of the form $I(\underline{n})$, where I is an ideal of S generated by monomials.*

By this proposition the classification of torsion free equivariant sheaves of rank 1 is reduced to the classification of monomial ideals.

The one-to-one correspondence between invariant orbits of $\hat{X}$ and X implies:

Proposition 4.34: *Let $\sigma \in \Delta$, then the ideal sheaf $\mathcal{I}_{V(\sigma)}$ is of the form $\tilde{I}$ where I is the $\mathbb{Z}^{\Delta(1)}$-homogeneous ideal $\langle x_\rho \mid \rho \in \sigma(1) \rangle$ in S.*

4.9. The Rees Construction

Let $\mathcal{E} \cong \widetilde{E^{\sigma}}$ be a torsion free equivariant sheaf over U_σ and consider its σ-family $\hat{E}^{\sigma}$ and the limit $\varinjlim \hat{E}^{\sigma} = \mathbf{E}^{\sigma} \cong k^r$. In section 4.1 we have reconstructed the module E^{σ} as the direct sum of the vector spaces in the σ-family, $\bigoplus_{m \in M} E^{\sigma}_m$. If $\mathcal{E}$ is torsion free, there is another way to reconstruct the $k[\sigma_M]$-module E^σ from the σ-family, by means of identifying E^σ with a submodule of $\Gamma(T, \mathcal{E}|_T) \cong k[M]^r \cong k[M] \otimes_k \mathbf{E}^\sigma$ as follows:

Definition 4.35 (Rees construction)**:** Define the k-vector space

$$E^\sigma_R := \bigoplus_{m \in M} k[M]_m \otimes_k E^\sigma_m \subset k[M] \otimes_k \mathbf{E}^\sigma.$$

We give E^σ_R the structure of a $k[\sigma_M]$-module by setting

$$\chi(m) \cdot (\chi(m') \otimes e) := \chi(m + m') \otimes (\chi_{m', m+m'} \cdot e)$$

for every $m' \in M$.

From the definition follows immediately:

Proposition 4.36: $E^\sigma_R \cong E^\sigma$ *as M-graded $k[\sigma_M]$-modules.*

Now let X be any toric variety and let $\mathcal{E}$ be a *reflexive* sheaf over X. Consider the Cox quotient presentation

$$\hat{X} \xrightarrow{\pi} X$$

and the pullback $\pi^*\mathcal{E}$ on $\hat{X}$. While the sheaf $\pi^*\mathcal{E}$ is coherent, it might not be reflexive and not even torsion free over $\hat{X}$. Over U_ρ, however, $\mathcal{E}$ is locally free and so is $\pi^*\mathcal{E}$ over $\hat{U}_{\hat{\rho}}$. Using this, we use the data for $\pi^*\mathcal{E}$ to construct a reflexive equivariant sheaf $\hat{\mathcal{E}}$ over $\hat{X}$. Because $\hat{X} \subset k^{\Delta(1)}$ is a T-invariant open subset whose complement has codimension at least two, $\hat{\mathcal{E}}$ has a reflexive equivariant continuation to $k^{\Delta(1)}$ and by our construction below the module $\Gamma(\hat{X}, \hat{\mathcal{E}}) = \Gamma(k^{\Delta(1)}, \hat{\mathcal{E}})$ will be a finite, reflexive S-module.

Definition 4.37: Let $\mathcal{E}$ be a reflexive sheaf which is given by a set of filtrations $E^\rho(i)$ of a vector space $\mathbf{E}^0$ for each $\rho \in \Delta(1)$. The map $\hat{T} \longrightarrow T$ restricts for each ρ to a map of the stabiliser subgroups $\hat{T}_{\hat{\rho}} \longrightarrow T_\rho$ and thus to a map of character groups

$$\alpha_\rho : M/\rho^\perp_M \longrightarrow \hat{M}/\hat{\rho}^\perp_{\hat{M}}.$$

For each $\rho \in \Delta(1)$ the restriction of $\mathcal{E}$ to U_ρ is a locally free sheaf and thus if we restrict π to $U_{\hat{\rho}}$, the pullback

$$\hat{\mathcal{E}}^{\hat{\rho}} := (\pi|_{U_{\hat{\rho}}})^* \mathcal{E}|_{U_\rho}$$

is locally free over $U_{\hat{\rho}}$. Identify $\hat{M}/\hat{\rho}^\perp_{\hat{M}}$ with $\mathbb{Z}$ via $i \mapsto i \cdot \underline{\hat{m}}_{\hat{\rho}}$, then every $i \in \hat{M}/\hat{\rho}^\perp_{\hat{M}}$ lies in a unique intervall $\alpha_\rho(j) \leq i < \alpha_\rho(j+1)$ for some $j \in M/\rho^\perp_M \cong \mathbb{Z}$. $\hat{\mathcal{E}}^{\hat{\rho}}$ is given by the filtration

$$\hat{E}^{\hat{\rho}}(i) = E^\rho(j) \text{ for } \alpha_\rho(j) \leq i < \alpha_\rho(j+1)$$

With these filtrations, we use the Rees construction to define an S-module E^Δ as

$$E^\Delta := \bigoplus_{\underline{n} \in \mathbb{Z}^{\Delta(1)}} S_{\underline{n}} \otimes_k \left(\bigcap_{\rho \in \Delta(1)} \hat{E}^{\hat{\rho}}(n_\rho) \right) \subset S \otimes_k \mathbf{E}^\sigma,$$

where $\underline{n} = (n_\rho \mid \rho \in \Delta(1))$.

As above the S-module E^Δ by construction is reflexive. Moreover, we have

Proposition 4.38:

$$\mathcal{E} \cong (E^{\Delta})\tilde{}$$

PROOF. Because both $\mathcal{E}$ and $\hat{\mathcal{E}}$ are reflexive, we need to check only that for each $\rho \in \Delta(1)$:

$$E^{\Delta}_{(x^{\hat{\rho}})} \cong E^{\sigma}.$$

But this follows from

$$E^{\Delta}_{x^{\hat{\rho}}} \cong \bigoplus_{i=1}^{\dim \mathbf{E}^0} k[\hat{\rho}_{\hat{M}}](\alpha(m_i)) \cong \Gamma(\hat{U}_{\hat{\rho}}, \pi^* \mathcal{E})$$

and from the fact that taking degree zero (with respect to the $A_{n-1}(X)$-grading) commutes with intersection:

$$E^{\Delta}_{x^{\hat{\sigma}}} = \bigcap_{\hat{\rho} \in \hat{\sigma}(1)} E^{\Delta}_{x^{\hat{\rho}}}$$

and

$$(E^{\Delta}_{x^{\hat{\sigma}}})_0 = \Bigl(\bigcap_{\hat{\rho} \in \hat{\sigma}(1)} E^{\Delta}_{x^{\hat{\rho}}} \Bigr)_0 = \bigcap_{\hat{\rho} \in \hat{\sigma}(1)} E^{\Delta}_{(x^{\hat{\rho}})} = \bigcap_{\rho \in \sigma(1)} E^{\rho} = E^{\sigma}.$$

□

By this construction we see that for any reflexive equivariant sheaf over X we can associate a fine-graded S-module which is described by the same set of filtrations. This has occasionally the advantage that instead of considering a sheaf over some complicated X, we can instead consider it over the smooth, quasiaffine $\hat{X}$.

CHAPTER 5

Posets and Linear Representations

In this chapter we address the following question. Consider any vector space arrangement $\mathcal{W} = \{X_1, \dots, X_n\}$ contained in some vector space V. Does there exist a vector space F and an arrangement $\mathcal{F} = \{Y_1, \dots, Y_n\}$ of coordinate subvector spaces in F which is equivalent to $\mathcal{W}$ in the sense that the underlying partially ordered sets, which are given by inclusion of vector spaces, are isomorphic? This is in fact true and moreover, our construction yields surjections of vector spaces

$$\begin{array}{ccc} F & \overset{\psi}{\twoheadrightarrow} & V \\ \uparrow & & \uparrow \\ Y_i & \overset{\psi|_{Y_i}}{\twoheadrightarrow} & X_i \end{array}$$

for every $i \in \{1, \dots, n\}$. The idea behind this construction is as follows. Let $X_{i_1}, \dots, X_{i_s}$ be any subset of $\mathcal{W}$, where we assume for the moment that none of them is contained in some other, and let $e^1_{i_k}, \dots, e^{d_k}_{i_k}$ be basis vectors for X_{i_k}, then of course, the vectors $e^j_{i_k}$ span the vector space $\sum_{k=1}^s X_{i_k}$, but in general these do not form a basis of this space. But if we consider the vector spaces

$$Y_{i_k} = \bigoplus_{j=1}^{d_k} k \cdot e^j_{i_k} \quad \text{and } F = \bigoplus_{k=1}^{s} Y_{i_k},$$

where we formally consider the $e^j_{i_k}$ as free generators of F over k, we have clearly an arrangement of coordinate subspaces Y_{i_k} of F which has also the property that none of the Y_{i_k} is contained in some other Y_{i_l}. Moreover, we obtain the above mentioned surjections ψ just by mapping the $e^j_{i_k}$, considered as free generators of F, to their images in V. We will see below that this procedure works also for more complicate arrangements.

For every Y_i, we can consider the kernel K_i of ψ restricted to Y_i, which is contained in the kernel K of ψ. The K_i form another arrangement $\mathcal{K} = \{K_1, \dots, K_n\}$ in K, and every K_i measures in a sense the overcounting of generators $e^j_{i_k}$ for the subset $X_{i_1}, \dots, X_{i_s}$ of $\mathcal{W}$ where $X_{i_k} \subseteq X_i$. Note that $\mathcal{K}$ on one hand, as a set is smaller than the original arrangement $\mathcal{W}$, but on the other hand, it is not a coordinate subspace arrangement. If we iterate constructing coordinate subspaces, we end up with a finite exact sequence of vector spaces

$$0 \longrightarrow F^s \longrightarrow \cdots \longrightarrow F^0 \longrightarrow V \longrightarrow 0$$

such that every F^i contains a coordinate subspace arrangement $\mathcal{F}_i = \{Y^i_1, \dots, Y^i_n\}$. We call such a sequence of arrangements a *free resolution* of $\mathcal{W}$.

Formally, we will formulate our construction in terms of *representations* of posets, but note that in this chapter we do not intend to develop a new "theory" of representations of posets, but rather to provide a certain formal setting which allows us to concisely present our main results later on. At present, it is not clear to us whether our method might be useful for the study of general vector space arrangements, although we hope so. We remark that, if we consider the graph associated to a poset as a quiver, our representations of posets can be considered as certain kinds of quiver representations. This is a formal connection between the theory of posets and that of linear subspace arrangements.

In section 5.1, we introduce the notion of *linear representations* for finite posets. We explain how such a representation can be used to complete a poset to lattices. In section 5.2, we introduce technical notions associated to vector space arrangements, that of a *bottlenecks* and that of the *free dimension* of a vector space in an arrangement. Using these notions, in sections 5.3 and 5.4 we construct free representations and free resolutions of vector space arrangements.

Conventions on Posets: Our basic references for the theory of posets are [**Sta97**], [**Sta99**], [**Aig79**]. In this chapter, a *poset* always means a partially ordered set, where the partial order usually is denoted by '$\leq$'. If not explicitly stated otherwise, all posets are assumed to be *finite*. Two elements x, y in a poset $\mathcal{P}$ are called *comparable* if $x \leq y$ or $y \leq x$; otherwise, x and y are *incomparable*. We write $x < y$ if $x \leq y$ and $x \neq y$. A poset $\mathcal{Q}$ whose underlying set is a subset of the underlying set of $\mathcal{P}$ is a *subposet* if $x \leq y$ in $\mathcal{Q}$ if and only if $x \leq y$ in $\mathcal{P}$. A special type of subposet is the *interval* $[x, y] := \{z \in \mathcal{P} \mid x \leq z \leq y\}$. A *chain* is a poset in which any two elements are comparable. A subset C of $\mathcal{P}$ is called a *chain* if C is a chain regarded as a subposet of $\mathcal{P}$. A chain C of $\mathcal{P}$ is called *saturated* if there does not exists $z \in \mathcal{P} \setminus C$ such that $x < z < y$ for some $x, y \in C$ and such that $C \cup \{z\}$ is a chain. We define the *height* of an element $x \in \mathcal{P}$ to be the maximal length of saturated chains in $\mathcal{P}$ whose maximal element is x. If $x < y$ and $[x, y] = \{x, y\}$, then we say that y *covers* x. Any chain C can be ordered in such a way that $C = \{x_1, \ldots, x_n\}$, and x_{i+1} covers x_i. A *morphism* of posets is a map $f : \mathcal{Q} \longrightarrow \mathcal{P}$ such that $x \leq y$ implies $f(x) \leq f(y)$, that is, f is *order preserving*. The *graph* $\Gamma(\mathcal{P})$ of $\mathcal{P}$ is the directed graph whose vertices are the elements of $\mathcal{P}$ and whose edges are the pairs $(x, y) \in \mathcal{P} \times \mathcal{P}$ where y covers x. For every $x \in \mathcal{P}$, the set of *incoming* edges is the set $\operatorname{in}(x) := \{(y, x) \mid x \text{ covers } y\}$ and, the set of *outgoing* edges the set $\operatorname{out}(x) := \{(x, y) \mid y \text{ covers } x\}$. A drawing of $\Gamma(\mathcal{P})$ is called *Hasse diagram* of $\mathcal{P}$.

5.1. Representations of Posets

Definition 5.1: Let $\mathcal{P}$ be a poset and let X be any set. Then an *inclusion representation* of $\mathcal{P}$ in X is an order-preserving map

$$\rho : \mathcal{P} \longrightarrow 2^X$$

where the order on the power set 2^X of X is given by inclusion.

Every poset $\mathcal{P}$ has an inclusion representation: take X to be the set underlying $\mathcal{P}$ and set $\rho(x) = \{y \in \mathcal{P} \mid y \leq x\}$.

Definition 5.2: Let $\mathcal{P}$ be a finite poset and k any field. Then a *linear* representation of $\mathcal{P}$ over k is given by the following data:

(i) a *finite dimensional* vector space V together with a family of subvector spaces $\mathcal{W} := \{W_1, \ldots, W_n\}$,

(ii) a map $\rho : \mathcal{P} \longrightarrow \{1, \ldots, n\}$, by which we associate a vector space $W_{\rho(x)}$ to every $x \in \mathcal{P}$, such that $x \leq y$ if and only if $W_{\rho(x)} \subseteq W_{\rho(y)}$.

Note that a linear representation is also an inclusion representation of $\mathcal{P}$ in 2^V — ρ associates to each $x \in \mathcal{P}$ the subset $W_{\rho(x)}$ of the vector space V, considered as the set of its vectors. Every finite poset has a linear representation over any field k: let V be the vector space $k^{\mathcal{P}} \cong \bigoplus_{x \in \mathcal{P}} k$. Then we set $\rho(x) = k^{\{y \leq x\}} \subset k^{\mathcal{P}}$. This particular representation is an isomorphism of posets between of $\mathcal{P}$ and $\mathcal{W} = \{\rho(x) \mid x \in \mathcal{P}\}$. In the sequel, we will only consider linear representations and for simplicity we will usually omit the attribute 'linear' and speak only of *representations of posets*.

As we have seen, any poset can be isomorphically represented, so that the pure fact of existence of representations does not provide us with new insight for posets. However, we can use a representation to perform certain completion operations on the original poset:

Proposition & Definition 5.3: *Let $\mathcal{W} = \{W_1, \ldots, W_n\}$ be a set of subvector spaces of some finite dimensional vector space V, together with a partial order such that $W_i \leq W_j$ if and only if W_i is a subvector space of W_j. Then we can consider the following two posets generated from $\mathcal{W}$:*

(i) $\mathcal{W}_\cap = \{\bigcap_{i \in I} W_i \mid I \subset \{1, \ldots, n\}\} \cup \{0, V\}$ the set containing the zero vector space 0, V and all intersections of the vector spaces W_i,

(ii) $\mathcal{W}_\Sigma = \{\sum_{i \in I} W_i \mid I \subset \{1, \ldots, n\}\} \cup \{0, V\}$ the set containing the zero vector space 0, V, and all sums of the vector spaces W_i,

Then $\mathcal{W}_\cap$ and $\mathcal{W}_\Sigma$are lattices.

Proof. (i) For any two $X, Y \in \mathcal{W}_\cap$ we set

$$X \vee Y := \bigcap_{Z \geq X, Y} Z$$

and

$$X \wedge Y := X \cap Y.$$

Both vector spaces exist in $\mathcal{W}_\cap$. $X \vee Y$ is the unique vector space which is minimal over X and Y, and $X \wedge Y$ is the unique maximal vector space which is contained in X and Y.

(ii) For any two $X, Y \in \mathcal{W}_\Sigma$ we set analogously as before

$$X \vee Y := X + Y$$

and

$$X \wedge Y := \sum_{Z \leq X, Y} Z.$$

$X \vee Y$ is the unique vector space which is minimal over X and Y, and $X \wedge Y$ is the unique maximal vector space which is contained in X and Y.

Thus in both cases we obtain lattices. □

In general, $(\mathcal{W}_\cap)_\Sigma$ and $(\mathcal{W}_\Sigma)_\cap$ do not coincide:

Example 5.4: Let $V \cong k^3$ and denote e_1, e_2, e_3 a basis of V. Then consider the family $\mathcal{W}$ given by three vector spaces $\langle e_1\rangle$, $\langle e_2\rangle$, $\langle e_3, e_1 + e_2\rangle$. Then:

- $\mathcal{W}_\cap = \mathcal{W} \cup \{0\}$,
- $\mathcal{W}_\Sigma = \mathcal{W} \cup \{\langle e_1, e_2\rangle, V\}$,
- $(\mathcal{W}_\cap)_\Sigma = \mathcal{W}_\Sigma \cup \{0\}$,
- $(\mathcal{W}_\Sigma)_\cap = \mathcal{W}_\Sigma \cup \{\langle e_1 + e_2\rangle, 0\}$,

Note that we automatically get order preserving representations of the powerset $2^{\mathcal{W}}$ of $\mathcal{W}$, where the partial order on $2^{\mathcal{W}}$ is given by inclusion:

$$2^{\mathcal{W}} \longrightarrow \mathcal{W}_\Sigma, \quad \{W_{i_1}, \dots, W_{i_k}\} \mapsto \sum_{j=1}^{k} W_{i_j}.$$

Similarly, if we take the reverse order in $2^{\mathcal{W}}$, we obtain as representation

$$2^{\mathcal{W}} \longrightarrow \mathcal{W}_\cap, \quad \{W_{i_1}, \dots, W_{i_k}\} \mapsto \bigcap_{j=1}^{k} W_{i_j}.$$

Moreover, note that $\mathcal{W}'_\Sigma = \mathcal{W}_\Sigma$ for any $\mathcal{W} \subset \mathcal{W}' \subset \mathcal{W}_\Sigma$ and that $\mathcal{W}'_\cap = \mathcal{W}_\cap$ for any $\mathcal{W} \subset \mathcal{W}' \subset \mathcal{W}_\cap$.

5.2. Vector Space Arrangements and Bottlenecks

Definition 5.5: Let $X \geq Y \in \mathcal{P}$. We say that the pair (Y, X) is a *bottleneck* in $\Gamma(\mathcal{P})$ if X covers Y and $\{(Y, X)\} = \mathrm{in}(X)$. We denote $b(\mathcal{P}) \subset \mathcal{P} \times \mathcal{P}$ the set of *bottlenecks* of $\Gamma(\mathcal{P})$. For $Z \in \mathcal{P}$ we denote $b(Z)$ the bottlenecks $(Y, X) \in b(\mathcal{P})$ such that $X \leq Z$.

Definition 5.6: We call a function

$$w : b(\mathcal{P}) \longrightarrow \mathbb{Z}$$

bottleneck weight function, where we usually will write $w(X, Y)$ instead of $w((X, Y))$. Let $Z \in \mathcal{P}$, then we define for a given bottleneck weight function w:

$$wb(Z) := \sum_{(Y,X) \in b(Z)} w(Y, X)$$

the *weighted bottleneck sum*.

Now let $\mathcal{W} = \{W_1, \dots, W_n\}$ be an arrangement of subvector spaces of some finite dimensional vector space V and let $\rho : \mathcal{P} \longrightarrow \mathcal{W}$ be a linear representation of $\mathcal{P}$ which is an *isomorphism* of posets. In the sequel, we simply identify $\mathcal{P}$ and $\mathcal{W}$. We can thus define the poset $\mathcal{P}_\Sigma$, consider its graph $\Gamma(\mathcal{P}_\Sigma)$ and observe the following:

Let $(X, Y) \in b(\mathcal{P}_\Sigma)$, then

$$X = \sum_{\substack{Z \in \mathcal{P} \\ Z < Y}} Z$$

Moreover, note that always $Y \in \mathcal{P}$, but not necessarily $X \in \mathcal{P}$.

Thus there is a natural notion for a bottleneck weight, which is given by

$$w(X,Y) := \operatorname{codim}_Y X.$$

for all $(X,Y) \in b(\mathcal{P}_\Sigma)$.

Definition 5.7: Let $X \in \mathcal{P}_\Sigma$, then we define the *free dimension* of X as

$$\operatorname{fd}(X) := \begin{cases} w(Y,X) & \text{if } (Y,X) \text{ is a bottleneck in } \mathcal{P}_\Sigma \\ \dim X - wb(X) & \text{else.} \end{cases}$$

To illustrate this concept, consider the case of three one dimensional vector spaces L_1, L_2, L_3, contained in a three dimensional vector space $V \cong k^3$, such that the L_i are pairwise linearly independent but only span a two-dimensional subvector space of V. Figure 5.1 shows the projection of $V \setminus \{0\}$ to $\mathbb{P}V \cong \mathbb{P}_2$, the three lines being represented by points, lying on a one-dimensional linear subspace, here drawn as dashed line. The

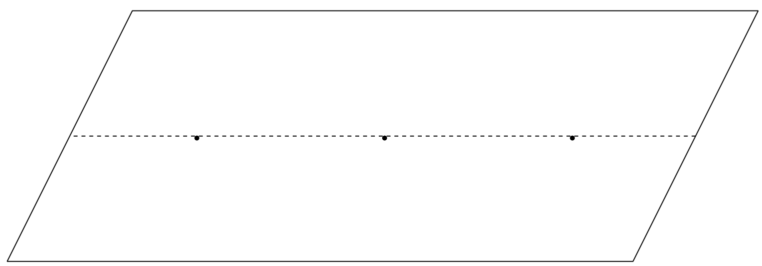


FIGURE 5.1. Three linearly dependent lines span a two dimensional vector space

Hasse diagrams representing the partial orders of $\mathcal{P}$ and $\mathcal{P}_\Sigma$ are shown in figure 5.2. In both graphs, we have added the zero vector space and V. We can see, that the

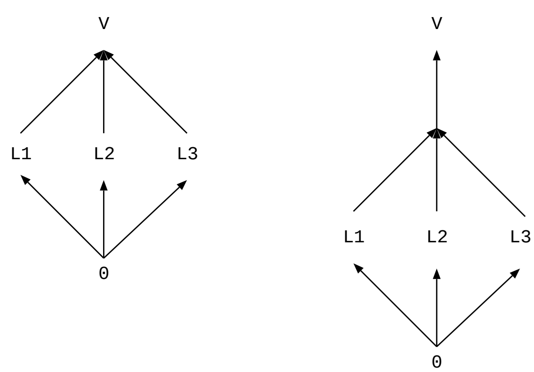


FIGURE 5.2. Hasse diagrams of $\mathcal{P}$ and $\mathcal{P}_\Sigma$ of figure 5.1

completion $\mathcal{P}_\Sigma$ has one additional bottleneck. The free dimensions of each of the three lines is one, the free dimension of V is one as well. The free dimension of the two dimensional vector space spanned by the three lines is minus one, which can intuitively be interpreted the way that we can remove one line and the remaining two still suffice to generate the two dimensional space.

A more complicated example is shown in figure 5.3, where there are seven points such that there are two lines each spanned over three points, and two additional points. The indicated lines have free dimension of minus one each. We have omitted to draw the lines spanned by two points. Figure 5.3 shows also the Hasse diagram associated to this configuration.

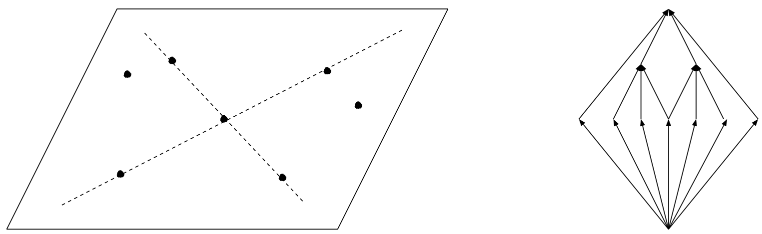


FIGURE 5.3. Picture and Hasse diagrams of seven lines

5.3. Free Representations of Posets

Definition 5.8: Let $\rho : \mathcal{P} \longrightarrow \mathcal{W} = \{W_1, \ldots, W_n\}$ be a linear representation in some vector space V. We call this representation *free* if there exists a basis $v_1, \ldots, v_r$ of V such that every $X \in \mathcal{W}_\Sigma$ is spanned by basis vectors $v_{i_1}, \ldots, v_{i_{\dim X}}$.

Remark 5.9: In the literature see (see for instance [**BP00**], [**BP03**]), such an arrangement sometimes is called *coordinate subspace arrangement*.

In the sequel, for simplicity we will talk about vector space arrangements and consider these as representation of their underlying partial ordered set of vector spaces without mentioning an explicit morphism.

Theorem 5.10: *Let $\mathcal{P}$ be a finite poset which is isomorphically represented by a family of subvector spaces $\mathcal{W} = \{W_1, \ldots, W_n\}$ of a finite dimensional vector space V. Then there exists a free representation of $\mathcal{P}$ by subvector spaces $\{F_1, \ldots, F_n\}$ of some vector space F and a surjection of vector spaces*

$$\psi : F \twoheadrightarrow V$$

such that all restrictions $\psi|_{F_i}$

$$\psi|_{F_i} : F_i \longrightarrow W_i$$

are surjective

PROOF. Let (Y, X) be a bottleneck in $\mathcal{W}_\Sigma$. Then $Y = \sum_{Z \in \mathcal{W}, Z < X} Z$ and there exists a short exact sequence of k-vector spaces:

$$0 \longrightarrow Y \longrightarrow X \underset{\psi_X}{\overset{\longrightarrow}{\dashleftarrow}} X/Y \longrightarrow 0,$$

where we have chosen a section ψ_X. Let $F_X := X/Y$ and set:

$$F := \bigoplus_{(Y,X)\in b(\mathcal{W}_\Sigma)} F_X$$

and

$$\psi := \sum_{(Y,X)\in b(\mathcal{W}_\Sigma)} \psi_X : F \longrightarrow V.$$

Moreover, for every $X \in \mathcal{W}_\Sigma$ we define a subvector space of F

$$F^X := \bigoplus_{\substack{(Z,Y)\in b(\mathcal{W}_\Sigma)\\ Y\leq X}} F_Y,$$

and denote

$$\psi^X := \psi|_{F^X}$$

in particular, $F^V = F$ and $\psi^V = \psi$. We claim that:

Lemma 5.11: *The restriction of ψ to every F^X is surjective onto X. In particular, ψ itself is surjective.*

Proof of lemma. We do induction over the maximal length of chains $[0, X]$, where we denote 0 the zero subvector space of V, the unique minimal element in $\mathcal{W}_\Sigma$. Let first X cover 0, then we have $\mathrm{fd}(X) = \dim X$ and $\psi|_X = \psi_X$, which is an isomorphism. Now let $Z := \sum_{Y<X} Y$. If $Z \neq X$, then $\mathrm{fd}(X) > 0$ and $\psi|_X = \psi|_X + \psi_X$, where by induction $\psi|_X$ is a surjection on Z and thus $\psi|_X$ is a surjection on X. If $Z = X$, then $\psi|_X = \sum_{Y<X} \psi_Y$. Its image coincides with $\sum_{Y<X} \psi|_Y$, where the $\psi|_Y$ by inductions are surjections onto the Y. Because the Y span X, the sum and thus $\psi|_X$ is a surjection onto X. $\square$

By construction, every F^X is direct sum of summands F_Y for some $Y \leq X$, and thus, by choosing bases for each F_Y, the vector space arrangement $\{F^X \mid X \in \mathcal{P}\}$ becomes a free arrangement. $\square$

Corollary 5.12: *Every finite poset has a free representation.*

Corollary 5.13: *Let $\mathcal{W} = \{W_1, \ldots, W_n\}$ be an arrangement of subvector spaces of some vector space V. Then $\mathrm{fd}(X) \geq 0$ for every $X \in \mathcal{W}_\Sigma$ if and only if $X \cong \bigoplus_{(Y,Z)\in b(X)} Z/Y$ for all $X \in \mathcal{W}_\Sigma$.*

In particular, $\mathrm{fd}(X) \geq 0$ for every $X \in \mathcal{W}_\Sigma$ implies that $\mathcal{W}$ is a free representation.

Proof. Assume first that $\mathrm{fd}(X) \geq 0$ for every $X \in \mathcal{W}_\Sigma$. Then there are two cases: (Y, X) is a bottleneck for some Y, or $\dim X \geq wb(X)$. In the first case, $X \cong Y \oplus X/Y$ and we need only to consider the second case. Now, from the proof of the theorem, we know that there is a surjection

$$\bigoplus_{(Y,Z)\in b(X)} Z/Y \twoheadrightarrow X,$$

and thus $\dim X \leq wb(X)$. Together with the assumption, we thus have $\dim X = wb(X)$, and the above surjection is a bijection.

In the other direction, if $X \cong \bigoplus_{(Y,Z)\in b(X)} Z/Y$, we know that $\dim X = wb(X)$. If there is no bottleneck (Y, X) for some Y, then $\mathrm{fd}(X) = \dim X - wb(X) = 0$, and if there exists a bottleneck (Y, X), then $\mathrm{fd}(X) > 0$ anyway. □

5.4. Free Resolutions of Posets

Let $\mathcal{P}$ be any poset isomorphically represented by an arrangement $\mathcal{W}$ of subvector spaces in V and let $\mathcal{F} = \{F^X \mid X \in \mathcal{W}\}$ be a free representation of $\mathcal{P}$ in F together with a surjection $\psi : F \twoheadrightarrow V$ and surjections $\psi^X = \psi|_{F^X} : F^X \twoheadrightarrow X$ as in theorem 5.10. Set $K := \ker\psi$ and $K^X := \ker\psi^X$ for all $X \in \mathcal{W}$, then the set $\mathcal{K} = \{K^X \mid X \in \mathcal{W}\}$ is an arrangement of subvector spaces of K such that $K^Y \subseteq K^X$ whenever $Y \subset X$. This automatically gives a new linear representation of $\mathcal{P}$, in K. We then have short exact sequences

$$0 \longrightarrow K \longrightarrow F \longrightarrow W \longrightarrow 0$$
$$0 \longrightarrow K^X \longrightarrow F^X \longrightarrow X \longrightarrow 0$$

for every $X \in \mathcal{W}$. We have the following obvious fact:

Lemma 5.14: $\dim K^X = wb(X) - \dim X$. *In particular, if (Y, X) is a bottleneck, then $K^X = K^Y$.*

Now we can iterate and find a free representation for $\mathcal{K}$, obtain a new kernel arrangement and so on. We end up with an exact sequence of vector spaces

$$\cdots \longrightarrow F_2 \xrightarrow{\psi_2} F_1 \xrightarrow{\psi_1} F_0 \xrightarrow{\psi_0} V \longrightarrow 0 \tag{4}$$

where each F_i contains a free representation $\mathcal{F}_i = \{F_i^X \mid X \in \mathcal{W}\}$ of $\mathcal{W}$ such that for every $X \in \mathcal{W}$ this sequence restricts to an exact sequence

$$\cdots \longrightarrow F_2^X \xrightarrow{\psi_2^X} F_1^X \xrightarrow{\psi_1^X} F_0^X \xrightarrow{\psi_0} X \longrightarrow 0 \tag{5}$$

Definition 5.15: Let $\mathcal{W}$ be an arrangement of vector spaces in some vector space V. Then we call a set of free representations $\mathcal{F}_i$ of $\mathcal{W}$, together with exact sequences as in diagrams 4 and 5, a *free resolution* of $\mathcal{W}$. The kernel configurations fitting into short exact sequences

$$0 \longrightarrow K_i^X \xrightarrow{\psi_{i+1}^X} F_i^X \xrightarrow{\psi_i^X} K_{i-1}^X \longrightarrow 0$$

are called *syzygy representations* of $\mathcal{W}$.

Our aim now is to show that every vector space arrangement has a *finite resolution*, that is, there exists some $n \geq 0$ such that $F_i = 0$ for every $i > 0$. As we have seen, the syzygy representations $\{K_i^X\}$ are representations of $\mathcal{P}$, that is, there exists a map of partially ordered sets $\mathcal{P} \longrightarrow \mathcal{K}_i, X \mapsto K_i^X$ for every $i \geq 0$. On the other hand, we have the property that $Y \subset X$ implies $K_i^Y \subset K_i^X$, but in general, K_i^Y and K_i^X may coincide. As a set, we can define a map $\mathfrak{i}_i : \mathcal{K} \to \mathcal{P}$ by setting

$$K_i^X \mapsto \min\{Y \mid Y \subset X, K_i^Y = K_i^X\}$$

This map is injective, and defines a *contraction* of the set $\mathcal{P}$ in the sense that the graph $\Gamma(\mathcal{K}_i)$ can be regarded as a contraction of $\Gamma(\mathcal{P})$. Note that we here always assume that the zero vector space is in $\mathcal{P}$.

Now let $X \in \mathcal{P}$ a minimal nonzero element, then in $\mathcal{P}_\Sigma$, the edge $(0, X)$ forms a bottleneck, and by construction, the kernel K_0^X of $F_0^X \twoheadrightarrow X$ is a zero dimensional vector space. Thus, all minimal elements $X \in \mathcal{P}$ are contracted to zero in $\mathcal{K}_0$.

Definition 5.16: We define $\mathrm{ht}(\mathcal{P})$ the *height* of $\mathcal{P}$ to be the height of V in the set $\mathcal{P} \cup \{0, V\}$.

From the discussion above we conclude that the height of K_i^X decreases at least by one in every step, and thus there is some $0 \leq n < \mathrm{ht}(\mathcal{P})$ such that $\mathfrak{i}_n(\mathcal{K}_n) = 0$. We have proved:

Theorem 5.17: *Every linear representation of some partially ordered set $\mathcal{P}$ has a finite free resolution*

$$0 \longrightarrow F_n \xrightarrow{\psi_n} \cdots \xrightarrow{\psi_1} F_0 \xrightarrow{\psi_0} V \longrightarrow 0$$

where $n \leq \mathrm{ht}(\mathcal{P}) - 1$.

This bound in general might be too big. For example, if $\mathcal{W} = \{0 \subset V_1 \subset \cdots \subset V_{n-1} \subset V_n\}$ is a flag, then $\mathcal{W}$ already is a free representation. On the other hand, the bound is sharp:

Example 5.18: Let $V \cong k^3$ and let $H_1, \ldots, H_4$ be an arrangement of hyperplanes in general position. Figure 5.4 shows the Hasse diagram of the partially ordered set

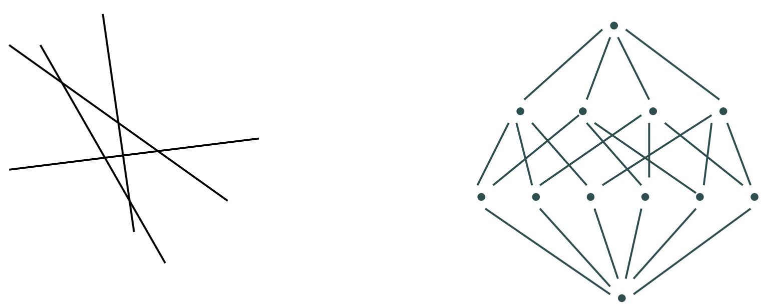


FIGURE 5.4. Four planes and Hasse diagram in k^3

$$\mathcal{P} = \{\bigcap_{i \in I} H_i \mid I \subset \{1, 2, 3, 4\}\} \cup \{0, V\}$$

We see that $\mathrm{ht}(V) = 3$ and the bottlenecks of this graph are precisely the one-dimensional intersection lines of the H_i, thus $F_0 \cong k^6$. The first syzygy consists precisely of the four vector spaces $K_0^{H_i}$, where $\dim K_0^{H_i} = 1$ for every i. Because $\dim K_0 = 3$, these four lines can not form a free representation. Thus we have to compute K_1 also, which then contains only a trivial configuration. Thus, we have a resolution of length 2:

$$0 \longrightarrow k \longrightarrow k^4 \longrightarrow k^6 \longrightarrow V \longrightarrow 0$$

Remark 5.19: The reader might have noticed that in above examples we did not really care to compute the set $\mathcal{P}_\Sigma$. In practice, this indeed is not always necessary, because in principle we are only interested in those sums of vector spaces such that bottlenecks are created. Formulated in a more algorithmical way, if we start to compute the sum of vector spaces $X_1, \dots, X_n$ contained in some Y, we do not need to proceed the computation for $X_1, \dots, X_n$ if already $\sum_{i=1}^{n-1} X_i = Y$. In the above example, and of course very often in low-dimensional examples, the computation of $\mathcal{P}_\Sigma$ yields *no* new bottlenecks, and hence we can stick with $\mathcal{P}$. Forming the set $\mathcal{P}_\Sigma$ is in principle an operation on the powerset $2^{\mathcal{P}}$ and thus combinatorially might be infeasible in the end, but we observe that in practice the complexity of forming resolutions can be much reduced. However, in this work we ignore the computational point of view except for this remark.

CHAPTER 6

Resolutions of Equivariant Reflexive Sheaves

In this chapter we want to apply the theory of linear representations of posets to describe a new method to construct minimal resolutions for reflexive, equivariant sheaves over toric varieties. That is, for every equivariant reflexive sheaf $\mathcal{E}$ over some toric variety X there exists an exact sequence of sheaves

$$0 \longrightarrow \mathcal{F}_s \longrightarrow \cdots \longrightarrow \mathcal{F}_0 \longrightarrow \mathcal{E} \longrightarrow 0$$

where the sheaves $\mathcal{F}_i$ are direct sums of equivariant reflexive sheaves of rank one. In particular, if X is smooth, the $\mathcal{F}_i$ are direct sums of line bundles. Klyachko in [**Kly90**] constructed the following kind of resolution for equivariant vector bundles over smooth, complete toric varieties:

$$0 \longrightarrow \mathcal{F}_0 \longrightarrow \bigoplus_{\sigma \in \Delta(1)} \mathcal{F}_\sigma \longrightarrow \cdots \longrightarrow \bigoplus_{\sigma \in \Delta(n)} \mathcal{F}_\sigma \longrightarrow \mathcal{E} \longrightarrow 0$$

where the $\mathcal{F}_\sigma$ are direct sums of line bundles whose filtrations are directly constructed from the filtrations of $\mathcal{E}$. The combinatorial input for this construction comes from the simplicial complex which is induced on a sphere S^{n-1} by intersecting it with the cones in the fan Δ. This construction is called by Klyachko the *canonical* resolution of $\mathcal{E}$, which is justified by the way how the bundles $\mathcal{F}_\sigma$ are constructed and its usefulness in studying structural properties of $\mathcal{E}$ in [**Kly90**]. In general, the canonical resolution is far from being minimal in the sense that we can find resolutions where $s < n - 1$ and the $\mathcal{F}_i$ are of smaller rank than the terms in the canonical resolution.

The difference of our approach is that our combinatorial input is not only the fan Δ itself, but instead we make use of the combinatorics of the arrangement of vector spaces $\mathcal{P}$ which is underlying the Δ-family of $\mathcal{E}$. Moreover, we will not restrict ourselves to the case of vector bundles over smooth toric varieties, but we will uniformly treat the case of general equivariant reflexive sheaves over general toric varieties. We will show that in the case of affine, smooth toric varieties, our resolutions indeed are minimal resolutions for reflexive modules in the sense of commutative algebra. In more general cases, there exists no well-defined notion of minimal resolutions. In the case where σ defines a nonsmooth toric variety, it is known that there exists no finite free resolutions. Instead, we consider here resolutions where the $\mathcal{F}_i$ are direct sums of reflexive modules of rank one. For such resolutions we do not know any uniqueness results. Similarly, global minimal resolutions are not uniquely defined, even in the smooth case.

We will present four kinds of resolutions:

(i) over affine toric varieties (theorem 6.7),
(ii) over general toric varieties using the homogeneous coordinate ring and the Rees construction (definition 6.10), which we call the *canonical global resolution*,

(iii) over general toric varieties using a direct construction (theorem 6.12), called *reflexive resolutions*,
(iv) a refinement of reflexive resolutions, called *minimal global resolutions* (theorem 6.16).

The second kind of resolution is a direct consequence of the construction over affine toric varieties. It gives rise to another notion of *canonical* resolution, because its combinatorial input is the vector space configuration $\mathcal{P}_\cap$ which is formed by the *complete* set of intersections of vector spaces $E^\rho(i)$ in $\mathcal{P}$. These resolutions are in general not minimal, because they are constructed over the homogeneous coordinate ring and thus their length essentially is bounded by the number of rays in Δ — these resolutions in general are even bigger than those constructed by Klyachko's canonical resolution. However, we can construct *directly* global resolutions over X by using the set

$$\mathcal{P} = \bigcup_{\sigma \in \Delta} \mathcal{P}^\sigma,$$

where the $\mathcal{P}^\sigma$ are the vector space arrangements underlying the σ-families $\hat{E}^\sigma$. This construction ultimately leads to the short resolutions we are looking for.

For comparison purposes, we briefly review Klyachko's canonical resolution in section 6.1. This section contains essentially Klyachko's original statement and proof except that we write the resolution in a reverse order. In section 6.2 we then start constructing resolutions for reflexive sheaves over affine toric varieties. We show that in the case of smooth affine toric varieties this construction yields minimal resolutions for reflexive, fine-graded modules in the sense of commutative algebra. In section 6.3 we apply this technique to produce global resolutions for general toric varieties using the homogeneous coordinate ring and the Rees construction. In section 6.4 we present another kind of global resolutions without using the homogeneous coordinate ring.

6.1. Klyachko's Canonical Resolution of Vector Bundles

Let $\mathcal{E}$ be a locally free sheaf over a smooth, complete n-dimensional toric variety X, given by filtrations $E^\rho(i)$ for $\rho \in \Delta(1)$ of some vector space $\mathbf{E}^0$. Let

$$\nu : \Delta(1) \longrightarrow \mathbb{Z}, \quad \nu(\rho) = \min\{i \mid E^\rho(i) = \mathbf{E}^0\},$$

then we define reflexive sheaves $\mathcal{F}_\sigma$ over X by the following filtrations in $\mathbf{E}^0$:

$$F_\sigma^\rho(i) = \begin{cases} E^\rho(i) & \text{if } \rho \in \sigma(1) \\ 0 & \text{if } \rho \notin \sigma(1) \text{ and } i < \nu(\rho) \\ \mathbf{E}^0 & \text{if } \rho \notin \sigma(1) \text{ and } i \geq \nu(\rho) \end{cases}$$

Lemma 6.1: *For every $\sigma \in \Delta$ the sheaf $\mathcal{F}_\sigma$ splits into a direct sum of line bundles and moreover, there exists an injective sheaf homomorphism $\mathcal{F}_\sigma \rightarrowtail \mathcal{E}$ whose restriction to U_σ is an isomorphism. Moreover, there are injective sheaf homomorphisms $\mathcal{F}_\tau \rightarrowtail \mathcal{F}_\sigma$ whenever $\tau < \sigma$ whose restriction to U_τ are isomorphisms.*

Proof. We know that the restrictions $\mathcal{F}_\sigma|_{U_\sigma}$ and $\mathcal{E}|_{U_\sigma}$ are defined by the same filtrations, $E^\rho(i)$ for $\rho \in \sigma(1)$, and thus isomorphic. Moreover, by lemma 4.22, the bundles split over U_σ, and thus there exists a basis $e_1, \ldots, e_r$ of $\mathbf{E}^0$ such that the $E^\rho(i)$

are spanned by elements of this basis for $\rho \in \sigma(1)$. Now for any other cone $\tau \in \Delta$, the corresponding filtrations are given by $E^\rho(i)$ for $\rho \in \tau(1) \cap \sigma(1)$ or by trivial filtrations for $\rho \in \tau(1) \setminus \sigma(1)$, so the basis all filtrations $F^\rho(i)$ are spanned by vectors of the basis $e_1, \ldots, e_n$, and thus $\mathcal{F}_\sigma$ splits into a sum of reflexive sheaves of rank one, hence into a direct sum of line bundles because X is smooth.

Now, because $F^\rho(i) \subset E^\rho(i)$ for every $\rho \in \Delta(1)$ and every i, the identity homomorphism $\mathbf{E}^0 \longrightarrow \mathbf{E}^0$ induces a morphism of filtered vector spaces and thus an injective homomorphism of sheaves $\mathcal{F}_\sigma \longrightarrow \mathcal{E}$. Analogously, we get injective sheaf homomorphisms $\mathcal{F}_\tau \rightarrowtail \mathcal{F}_\sigma$. □

Explicitly, the sheaves $\mathcal{F}_\sigma$ have the following form. Let

$$\Gamma(U_\sigma, \mathcal{E}) \cong \bigoplus_{i=1}^r k[\sigma_M](-m_i),$$

then:

$$\mathcal{F}_\sigma \cong \Big(\bigoplus_{i=1}^r \mathcal{O}\big(\sum_{\rho\in\sigma(1)} \langle m_i, n(\rho)\rangle D_\rho\big)\Big) \otimes_{\mathcal{O}} \mathcal{O}\big(-\sum_{\rho\in\Delta(1)\setminus\sigma(1)} \nu(\rho) D_\rho\big)$$

Now consider S^{n-1} the unit $(n-1)$-sphere in $N_\mathbb{R} \cong \mathbb{R}^n$. X is smooth and thus all cones $\sigma \in \Delta$ are simplicial, and the intersection of $S^{n-1} \cap \sigma$ cuts out a simplex in S^{n-1}. The set of all such simplices defines a simplicial decomposition of S^{n-1}. We denote Σ the corresponding simplicial complex and for every $\sigma \in \Delta$ we denote $\tilde{\sigma} := \sigma \cap S^{n-1}$ the associated simplex. We write $\Sigma(i) = \{\tilde{\sigma} \mid \dim \sigma = i+1\}$ Consider the augmented simplical cohomology complex $C^\cdot$ for Σ:

$$0 \longrightarrow \mathbb{Z} \longrightarrow \bigoplus_{\tilde{\sigma}\in\Sigma(0)} \mathbb{Z}\tilde{\sigma} \xrightarrow{\delta_0} \cdots \xrightarrow{\delta_{n-2}} \bigoplus_{\tilde{\sigma}\in\Sigma(n-1)} \mathbb{Z}\tilde{\sigma} \longrightarrow 0$$

This complex has cohomology only at the $(n-1)$st place, $H^{n-1}(C^\cdot) \cong \mathbb{Z}$. By tensoring with $\mathbf{E}^0$, we obtain a complex of vector spaces

$$(6) \quad 0 \longrightarrow \mathbb{Z} \otimes_\mathbb{Z} \mathbf{E}^0 \longrightarrow \bigoplus_{\tilde{\sigma}\in\Sigma(0)} (\mathbb{Z}\tilde{\sigma}) \otimes_\mathbb{Z} \mathbf{E}^0 \xrightarrow{\delta_0\otimes\mathrm{id}} \cdots \xrightarrow{\delta_{n-2}\otimes\mathrm{id}} \bigoplus_{\tilde{\sigma}\in\Sigma(n-1)} (\mathbb{Z}\tilde{\sigma}) \otimes_\mathbb{Z} \mathbf{E}^0 \longrightarrow 0$$

which has as its only cohomology $H^{n-1}(C^\cdot \otimes_\mathbb{Z} \mathbf{E}^0) \cong \mathbb{Z} \otimes_\mathbb{Z} \mathbf{E}^0 \cong \mathbf{E}^0$. We define filtrations $\tilde{F}^\rho_\sigma(i)$ for the vector spaces $(\mathbb{Z}\tilde{\sigma}) \otimes_\mathbb{Z} \mathbf{E}^0$ by:

$$\tilde{F}^\rho_\sigma(i) := (\mathbb{Z}\tilde{\sigma}) \otimes_\mathbb{Z} F^\rho_\sigma(i).$$

and as in lemma 6.1 we obtain for every i:

$$(7) \qquad 0 \longrightarrow \tilde{F}^\rho_0 \longrightarrow \bigoplus_{\tilde{\sigma}\in\Sigma(0)} \tilde{F}^\rho_\sigma(i) \xrightarrow{\delta_0\otimes\mathrm{id}} \cdots \xrightarrow{\delta_{n-2}\otimes\mathrm{id}} \bigoplus_{\tilde{\sigma}\in\Sigma(n-1)} \tilde{F}^\rho_\sigma(i) \longrightarrow C \longrightarrow 0.$$

Proposition 6.2: *Complex 7 is exact for every $\rho \in \Delta(1)$ and every $i \in \mathbb{Z}$. Moreover, $C = E^\rho(i)$.*

PROOF. We separate two cases: if $i \geq \nu(\rho)$, the above complex coincides with the complex 6, and the statement follows. If $i < \nu(\rho)$, we have for every term:

$$\bigoplus_{\tilde{\sigma} \in \Sigma(i)} \tilde{F}^{\rho}_{\sigma}(i) = \Big(\bigoplus_{\tilde{\sigma} \in \Sigma(i), \rho \in \sigma(1)} \tilde{F}^{\rho}_{\sigma}(i) \Big) \oplus \Big(\bigoplus_{\tilde{\sigma} \in \Sigma(i), \rho \notin \sigma(1)} 0 \Big)$$

This implies that complex 7 coincides with the simplicial cohomology complex of the complex formed by the $\sigma \cap S^{n-1}$, where $\sigma \in \Delta(\rho) = \{\tau \in \Delta \mid \rho < \tau\}$, tensored with $E^{\rho}(i)$. This complex is acyclic except for the nth cohomology, being $E^{\rho}(i)$. □

The complexes (7) are naturally compatible with the inclusion $F^{\rho}(i) \subset \mathbf{E}^0$, and thus altogether we obtain an exact sequence filtered vector spaces whose cokernel is the filtered vector space $\mathbf{E}^0$ with filtrations $E^{\rho}(i)$.

Theorem 6.3 ([**Kly90**], Theorem 3.1.1)**:** *Let X be an n-dimensional smooth complete toric variety and let $\mathcal{E}$ be a locally free sheaf over X. Then there is an exact sequence of sheaves*

$$0 \longrightarrow \mathcal{F}_0 \longrightarrow \bigoplus_{\sigma \in \Delta(1)} \mathcal{F}_{\sigma} \longrightarrow \cdots \longrightarrow \bigoplus_{\sigma \in \Delta(n)} \mathcal{F}_{\sigma} \longrightarrow \mathcal{E} \longrightarrow 0$$

such the $\mathcal{F}_{\sigma}$ are direct sums of invertible sheaves.

PROOF. We have to show that the above constructed complex of filtered vector spaces induces a complex of sheaves and that its has only nth cohomology, which is $\mathcal{E}$. By proposition 6.2 and the fact that

$$\bigoplus_{\sigma \in \Delta(i)} (F_{\sigma})^{\sigma}_{m} = \bigoplus_{\sigma \in \Delta(i)} \bigcap_{\rho \in \sigma(1)} F^{\rho}_{\sigma}(\langle m, n(\rho) \rangle)$$

it follows that on the level of τ-families we have exact sequences of vector spaces for every $m \in M$ and every $\tau \in \Delta$:

$$0 \longrightarrow (F_0)^{\tau}_{m} \longrightarrow \bigoplus_{\sigma \in \Delta(1)} (F_{\sigma})^{\sigma}_{m} \longrightarrow \cdots \longrightarrow \bigoplus_{\sigma \in \Delta(n)} (F_{\sigma})^{\tau}_{m} \longrightarrow E^{\tau}_{m} \longrightarrow 0,$$

which proves the claim. □

6.2. Resolutions over Affine Toric Varieties

Consider an affine toric variety U_{σ}. A σ-family $\hat{E}^{\sigma}$ for a torsion free sheaf $\mathcal{E}$ is equivalent to the following data:

(i) an arrangement $\mathcal{P}$ of subvector spaces of some vector space V,
(ii) a function $\varepsilon : M \longrightarrow \mathcal{P}$ which has the following properties.
  (a) ε is order preserving in the sense that $m \leq_{\sigma} m'$ for $m, m' \in M$ if and only if $\varepsilon(m) \subset \varepsilon(m')$,
  (b) $V = \bigcup_{m \in M} \varepsilon(m)$,
  (c) for every chain $\cdots <_{\sigma} m_{i-1} <_{\sigma} m_i <_{\sigma} \ldots$ there exist $i_0 < i_1$ such that $\varepsilon(m_i) = 0$ for $i \leq i_0$ and $\varepsilon(m_i) = V$ for $i \geq i_1$,
  (d) there exist only finitely many $m \in M$ such that $\varepsilon(m) \neq \sum_{m' <_{\sigma} m} \varepsilon(m')$.

This correspondence is explicitly given by setting

$$\mathcal{P} = \{E^\sigma_m \subset \mathbf{E}^\sigma\}_{m \in M},$$
$$\varepsilon(m) = E^\sigma_m$$

or, in the other direction

$$E^\sigma_m := \varepsilon(m).$$

As we have seen in section 4.2, these data correspond to a finite σ-family.

From now on we assume that $\mathcal{E}$ is reflexive and is defined by a set of filtrations $E^\rho(i)$ in $\mathbf{E}^\sigma$, every $X \in \mathcal{P}$ is of the following form:

$$X = \bigcap_{\rho \in \sigma(1)} E^\rho\big(\langle m, n(\rho)\rangle\big) \subseteq \mathbf{E}^\sigma,$$

for some $m \in M$. Let us define the following function

$$\kappa : \big(\mathcal{P} \setminus \{0\}\big) \times \sigma(1) \longrightarrow \mathbb{Z}, \quad \kappa(X, \rho) = \min\{i \mid X \subset E^\rho(i)\}.$$

Lemma 6.4: *For every $X \in \mathcal{P}$ we have for some $m \in M$:*

$$X = \bigcap_{\rho \in \sigma(1)} E^\rho\big(\langle m, n(\rho)\rangle\big) = \bigcap_{\rho \in \sigma(1)} E^\rho\big(\kappa(X, \rho)\big).$$

Moreover, $\kappa(X, \rho)) \leq \langle m, n(\rho)\rangle$ for all $m \in \varepsilon^{-1}(X)$ and for all $\rho \in \sigma(1)$.

Proof. For some $m \in M$ we have

$$X = E^\sigma_m = \bigcap_{\rho \in \sigma(1)} E^\rho\big(\langle m, n(\rho)\rangle\big)$$

and clearly, $\kappa(X, \rho) \leq \langle m, n(\rho)\rangle$ for every $\rho \in \sigma(1)$. Moreover, $X \subset \bigcap_{\rho \in \sigma(1)} E^\rho\big(\kappa(X, \rho)\big)$ and $\kappa(X, \rho) \leq \langle m, n(\rho)\rangle$ implies

$$E^\rho\big(\kappa(X, \rho)\big) \subseteq E^\rho\big(\langle m, n(\rho)\rangle\big),$$

for every $\rho \in \sigma(1)$, and thus

$$\bigcap_{\rho \in \sigma(1)} E^\rho\big(\kappa(X, \rho)\big) \subset \bigcap_{\rho \in \sigma(1)} E^\rho\big(\langle m, n(\rho)\rangle\big).$$

The last statement follows from the fact that $\varepsilon(m) = E^\sigma_m$ for every $m \in \varepsilon^{-1}(X)$ and $\kappa(E^\sigma_m, \rho) = \kappa(X, \rho)$ for every $m \in \varepsilon^{-1}(X)$ and every $\rho \in \sigma(1)$. □

Now let $X, Z \in \mathcal{P}$ and $X \subsetneq Z$, then there exists at least one $\rho \in \sigma(1)$ such that $\kappa(X, \rho) < \kappa(Z, \rho)$. This implies that $X \subset E^\sigma_m$ for all $m \in M$ with $\langle m, n(\rho)\rangle \geq \kappa(X, \rho)$ and all $\rho \in \sigma(1)$.

For some bottleneck $(Y, X) \in b(\mathcal{P}_\Sigma)$ we consider the following direct sum of reflexive sheaves of rank one:

$$\mathcal{O}\big(-\sum_{\rho \in \sigma(1)} \kappa(X, \rho) D_\rho\big)^{\mathrm{fd}(X)}$$

The corresponding filtrations are given by:

$$O^\rho(i) = \begin{cases} 0 & i < \kappa(X, \rho) \\ \mathbf{O}^\sigma & i \geq \kappa(X, \rho) \end{cases}$$

where $\mathbf{O}^\sigma \cong k^{\mathrm{fd}(X)}$, and the σ-family is given by:

$$O_m^\sigma = \begin{cases} \mathbf{O}^\sigma & \text{if } \langle m, n(\rho)\rangle \geq \kappa(X,\rho) \text{ for all } \rho \in \sigma(1) \\ 0 & \text{else.} \end{cases}$$

So, on the level of σ-families, we can define a homomorphism

$$\mathcal{O}\big(-\sum_{\rho\in\sigma(1)} \kappa(X,\rho)D_\rho\big)^{\mathrm{fd}(X)} \longrightarrow \mathcal{E}$$

by setting for m with $\langle m, n(\rho)\rangle \geq \kappa(X,\rho)$ for all $\rho \in \sigma(1)$:

$$O_m^\sigma \xrightarrow{\psi_X} E_m^\sigma$$

and the zero homomorphism else. Here we have identified $\mathbf{O}^\sigma$ with X/Y and use the fact that $X \subset E_m^\sigma$ implies that $\langle m, n(\rho)\rangle \geq \kappa(X,\rho)$ for all $\rho \in \sigma(1)$.

Now we can take a free representation of $\mathcal{P}$ as constructed in the previous chapter:

$$\mathbf{F}_0 \xrightarrow{\psi_0} \mathbf{E}^\sigma \longrightarrow 0$$

where ψ_0 restricts to:

$$\psi_0^X : F_0^X \longrightarrow X \longrightarrow 0$$

and

$$F_0^X = \bigoplus_{(Z,Y)\in b(X)} Y/Z, \quad \text{and} \quad F_0 = \bigoplus_{(Y,X)\in b(\mathcal{P})} X/Y$$

for every $X \in \mathcal{P}$. For every bottleneck (Y, X) we can define a sheaf homomorphism $\mathcal{O}\big(-\sum_{\rho\in\sigma(1)} \kappa(X,\rho)D_\rho\big)^{\mathrm{fd}(X)} \longrightarrow \mathcal{E}$ as above and obtain:

$$\mathcal{F}_0 \longrightarrow \mathcal{E}$$

where

$$\mathcal{F}_0 = \bigoplus_{(Y,X)\in b(\mathcal{P})} \mathcal{O}\big(-\sum_{\rho\in\sigma(1)} \kappa(X,\rho)D_\rho\big)^{\mathrm{fd}(X)}.$$

On the level of σ-families, we have in every degree $m \in M$:

$$(F_0^\sigma)_m \longrightarrow E_m^\sigma$$

where

$$(F_0^\sigma)_m = \bigoplus_{(Y,X)\in b(E_m^\sigma)} X/Y.$$

Denote $\mathcal{P}_0 := \{F^X | X \in \mathcal{P}\}$, which is a free representation of $\mathcal{P}$. By construction the above homomorphisms are surjective for every $m \in M$, and thus:

Proposition 6.5: *Let $\mathcal{E}$ be a equivariant reflexive sheaf and notation as above. Then there exists a short exact sequence of equivariant reflexive sheaves*

$$0 \longrightarrow \mathcal{K}_0 \longrightarrow \mathcal{F}_0 \longrightarrow \mathcal{E} \longrightarrow 0$$

where

$$\mathcal{F}_0 := \bigoplus_{(Y,X)\in b(\mathcal{P}_\Sigma)} \mathcal{O}\Big(-\sum_{\rho\in\sigma(1)} \kappa(X,\rho)D_\rho\Big)^{\mathrm{fd}(X)}.$$

PROOF. We have only to show the reflexiveness of $\mathcal{K}_0$. But this follows from the more general lemma below. □

Lemma 6.6: *Let X be any variety and let*

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{F} \longrightarrow \mathcal{E} \longrightarrow 0$$

be a short exact sequence of sheaves over X such that $\mathcal{E}$ and $\mathcal{F}$ are torsion free. Then $\mathcal{K}$ is reflexive.

PROOF. Consider the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{K} & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{E} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \\ \mathcal{C} & \longrightarrow & \mathcal{K}^{\check{}\check{}} & \longrightarrow & \mathcal{F}^{\check{}\check{}} & \longrightarrow & \mathcal{E}^{\check{}\check{}} & & \end{array}$$

The homomorphism α and β both are injective, and thus $\mathcal{K}$ maps injectively into $\mathcal{F}^{\check{}\check{}}$ and thus ϕ is injective as well. $\mathcal{C}$ is a torsion free sheaf subsheaf of $\mathcal{K}^{\check{}\check{}}$ of rank zero and thus $\mathcal{C}$ must be the zero sheaf. Applying the snake lemma then gives that ϕ is surjective. □

By iterating the above procedure, we obtain an exact sequence of equivariant reflexive sheaves:

$$0 \longrightarrow \mathcal{F}_s \xrightarrow{\psi_s} \dots \xrightarrow{\psi_1} \mathcal{F}_0 \xrightarrow{\psi_0} \mathcal{E} \longrightarrow 0$$

such that every $\mathcal{F}_i$ is isomorphic to a direct sum of reflexive sheaves

$$\mathcal{F}_i = \bigoplus_{(Y,K_{i-1}^X)\in b\left((\mathcal{P}_{i-1})_\Sigma\right)} \mathcal{O}\Big(- \sum_{\rho\in\sigma(1)} \kappa_{i-1}(K_{i-1}^X, \rho) D_\rho \Big)^{\mathrm{fd}^{i-1}(K_{i-1}^X)},$$

where $\mathcal{P} =: \mathcal{P}_{-1}$, $\mathcal{P}_i$ is the i-th syzygy representation of $\mathcal{P}$ given by $\{K_i^X\}$. κ_i is given by

$$\kappa_i : \left(\mathcal{P}_i \setminus \{0\}\right) \times \sigma(1) \longrightarrow \mathbb{Z}, \quad \kappa_i(K_i^X, \rho) = \min\{l \mid K_i^X \subset K_i^\rho(l)\},$$

and fd^i is the free dimension in the arrangement $(\mathcal{P}_i)_\Sigma$.

Note that $K_{i-1}^X \subset K_{i-1}^\rho(k)$ implies $K_i^X \subset K_i^\rho(k)$ and thus

$$\kappa_i(K_i^X, \rho) \leq \kappa_{i-1}(K_{i-1}^X, \rho).$$

In the case where U_σ is smooth, we have constructed a free resolution of the reflexive $k[\sigma_M]$-module $E := \Gamma(U_\sigma, \mathcal{E})$:

$$0 \longrightarrow F_s \xrightarrow{\psi_s} \dots \xrightarrow{\psi_1} F_0 \xrightarrow{\psi_0} E \longrightarrow 0 \tag{8}$$

where the F_i are fine-graded, free $k[\sigma_M]$-modules.

Theorem 6.7: *Let U_σ be smooth. Then sequence (8) defines a minimal free resolution of E as a $k[\sigma_M]$-module in the sense of commutative algebra.*

Proof. As we have seen in the proof of proposition 4.11, homogeneous generators of E which are contained in some E^σ_m, are elements which do not lie in the image of the maps $\chi^\sigma_{m',m}$ for $m' \leq_\sigma m$. Passing to the direct limit $\mathbf{E}^\sigma$, this image precisely coincides with the linear span of all $E^\sigma_{m'}$, $m' \leq_\sigma m$ in $\mathbf{E}^\sigma$, because $m' \leq_\sigma m$ implies that $\chi^\sigma_{m',m}\big(E^\sigma_{m'}\big) \subset E^\sigma_m$. On the other hand, we have for every E^σ_m that

$$E^\sigma_m = \bigcap_{\rho \in \sigma(1)} E^\rho\big(\kappa(E^\sigma_m, \rho)\big)$$

and $E^\sigma_{m'} \subset E^\sigma_m$ implies that

$$\bigcap_{\rho \in \sigma(1)} E^\rho\big(\kappa(E^\sigma_{m'}, \rho)\big) \subset \bigcap_{\rho \in \sigma(1)} E^\rho\big(\kappa(E^\sigma_m, \rho)\big)$$

and if $E^\sigma_{m'} \neq E^\sigma_m$, then this implies that $\kappa(E^\sigma_{m'}, \rho) < \kappa(E^\sigma_m, \rho)$ for at least one $\rho \in \sigma(1)$. So, we have that a homogeneous component E^σ_m contains a generator if and only if $\langle m, n(\rho)\rangle = \kappa(E^\sigma_m, \rho)$ for every $\rho \in \sigma(1)$ and if (Y, E^σ_m) is a bottleneck for some $Y \in \mathcal{P}_\Sigma$. □

Corollary 6.8: *Let $\mathcal{W}$ be a subvector space arrangement in some vector space V which is generated by the intersection of n flags. Then this arrangement has a free resolution of length at most $n - 2$.*

Proof. Let $F^1_i, \ldots, F^{k_i}_i$, $i = 1, \ldots, n$ be the n flags, then we can associate to it some reflexive equivariant module over the polynomial ring $k[x_1, \ldots, x_n]$ by choosing any set of numbers $j^1_i < \cdots < j^{k_i}_i$ and setting

$$E^i(k) = F^l_i \quad \text{for } j^l_i \leq k < j^{l+1}_i.$$

Then the statement follows by Hilbert's Syzygy Theorem and the fact that every reflexive module over $k[x_1, \ldots, x_n]$ is a second syzygy sheaf. □

On the other hand, we can see that the maximal length of a resolution is also bounded by the rank of $\mathcal{E}$:

Corollary 6.9: *Let $\mathcal{E}$ be a reflexive sheaf of rank r over U_σ. Then the maximal length of a reflexive resolution of $\mathcal{E}$ is at most $r - 1$.*

Proof. This follows because any chain $X_1 \subset X_2 \subset \ldots X_s$, such that $\dim X_i > \dim X_{i-1}$ can at most have length $\dim \mathbf{E}^\sigma = r$, and thus $\operatorname{ht} \mathbf{E}^\sigma \leq r$. □

6.3. Resolutions of Reflexive Sheaves Using the Rees Construction

Using the homogeneous coordinate ring, we now can construct global resolutions for reflexive sheaves. Let $\mathcal{E}$ be any reflexive sheaf over some toric variety X, which is given by filtrations $E^\rho(i)$, and let

$$\hat{X} \xrightarrow{\phi} X$$

be the Cox quotient presentation. Using the global Rees construction of section 4.9, there exists a reflexive sheaf $\hat{\mathcal{E}}$ over $\hat{X}$ such that $\Gamma\big(\hat{X}, \hat{\mathcal{E}}\big)\tilde{} \cong \mathcal{E}$. The corresponding filtrations are the same as the filtrations of $\mathcal{E}$, except that the filtrations may be 'stretched' a bit. This means, the mapping of character groups $M \longrightarrow \hat{M}$ induces an injective map

$$\alpha_\rho : M/\rho_M^\perp \longrightarrow \hat{M}/\hat{\rho}_{\hat{M}}^\perp$$

for every $\rho \in \Delta(1)$, and the filtrations $\hat{E}^{\hat{\rho}}(i)$ in $\mathbf{E}^0$ are given by

$$\hat{E}^{\hat{\rho}}(i) = E^\rho(j)$$

where j is such that i lies in the interval $\alpha_\rho(j) \leq i < \alpha_\rho(j+1)$. By continuation of $\hat{\mathcal{E}}$ to the whole affine space $k^{\Delta(1)}$, we obtain a reflexive S-module $\hat{E} := \Gamma(k^{\Delta(1)}, \hat{\mathcal{E}})$. To this module there corresponds the C-family $\hat{E}^C$ with respect to the cone $C = (\mathbb{R}_{\geq 0})^{\Delta(1)}$ in $\hat{N}_\mathbb{R}$. We denote the partially ordered set of vector spaces by $\hat{\mathcal{P}}$. Then there is a minimal resolution

$$0 \longrightarrow F_r \longrightarrow \cdots \longrightarrow F_0 \longrightarrow E \longrightarrow 0$$

by free, fine-graded S-modules, and by sheafification we obtain correspondingly a resolution of $\mathcal{E}$:

$$0 \longrightarrow \mathcal{F}_r \longrightarrow \cdots \longrightarrow \mathcal{F}_0 \longrightarrow \mathcal{E} \longrightarrow 0$$

where every $\mathcal{F}_i$ is isomorphic to a direct sum

$$\mathcal{F}_i = \bigoplus_{(Y, K_{i-1}^X) \in b\left((\hat{\mathcal{P}}_{i-1})_\Sigma\right)} \mathcal{O}\Big(- \sum_{\rho \in \sigma(1)} \kappa_{\alpha,i-1}(K_{i-1}^X, \rho) D_\rho \Big)^{\mathrm{fd}^{i-1}(K_{i-1}^X)},$$

where $\hat{\mathcal{P}} =: \hat{\mathcal{P}}_{-1}$, and

$$\kappa_{\alpha,i} : \big(\hat{\mathcal{P}}_i \setminus \{0\}\big) \times \Delta(1) \longrightarrow \mathbb{Z}, \quad \kappa_{\alpha,i}(K_i^X, \rho) = \min\{l \mid K_i^X \subset K_i^\rho(l)\}$$

and fd^i is the free dimension of the ith syzygy arrangement $\hat{\mathcal{P}}_i = \{K_i^X\}$.

In general, the divisors $-\sum_{\rho \in \sigma(1)} \kappa_{\alpha,i-1}(K_{i-1}^X, \rho) D_\rho$ are only Weil divisors and the direct summands of the $\mathcal{F}_i$ thus are not invertible but rather reflexive sheaves.

Definition 6.10: We call the resolution of $\mathcal{E}$ as above the *canonical global resolution* of $\mathcal{E}$.

A particular case is where Δ is generated by a single cone σ. In this case, we have shown that for every fine-graded reflexive $k[\sigma_M]$-module E there exists a resolution of E by reflexive modules of rank one, and the length of this resolution by corollary 6.8 is bounded by the number $\#\sigma(1) - 2$. In general, it seems that this bound can not be reduced to, say $\dim U_\sigma$. This implies in particular that the canonical global resolution is of length at most $\#\Delta(1) - 2$.

Example 6.11: Consider the cone σ in $N_\mathbb{R} \cong \mathbb{R}^3$ which is spanned by the four rays $\rho_1, \ldots, \rho_4$ whose primitive vectors are $n_1 = (1,0,0), n_2 = (0,1,0), n_3 = (1,0,1), n_4 = (0,1,1)$. This cone defines a nonsmooth affine toric variety which has precisely one isolated singular point. Consider the arrangement as shown in figure 5.4 above. We define a reflexive sheaf $\mathcal{E}$ over U_σ by setting the filtrations

$$E^{\rho_k}(i) = \begin{cases} 0 & i < -1 \\ H_i & i = 0 \\ V & i > 0. \end{cases}$$

It is easy to see that we always can choose $m \in M$ such that $\langle m, n_i \rangle = \langle m, n_j \rangle = -1$ for some $i \neq j$ and $\langle m, n_k \rangle \geq 0$ for $k \notin \{i, j\}$. Thus, the intersection arrangement $\mathcal{P} = \{\bigcap_{i \in I} H_i \mid I \subset \{1,2,3,4\}\}$ coincides with the arrangement of the underlying σ-family and we obtain the following resolution of length 2:

$$0 \longrightarrow \mathcal{O} \longrightarrow \bigoplus_{i=1}^{4} \mathcal{O}(D_i) \longrightarrow \bigoplus_{i \neq j} \mathcal{O}(D_i + D_j) \longrightarrow \mathcal{E} \longrightarrow 0.$$

6.4. Minimal Global Resolutions

Now let X be any toric variety and let $\mathcal{E}$ be a reflexive sheaf over X and $\hat{E}^\Delta$ the corresponding Δ-family. Then for every $\sigma \in \Delta$, the σ-family $\hat{E}^\sigma$ corresponds to an arrangement $\mathcal{P}^\sigma$ of vector spaces in $\mathbf{E}^0$. We denote $\mathcal{P} := \bigcup_{\sigma \in \Delta_{\max}} \mathcal{P}^\sigma$ the union of all arrangements for all maximal cones, and let

$$\kappa : \big(\mathcal{P} \setminus \{0\}\big) \times \Delta(1) \longrightarrow \mathbb{Z}, \quad \kappa(X, \rho) := \min\{i \mid X \subset E^\rho(i)\}.$$

Theorem 6.12: *The arrangement $\mathcal{P}$ gives rise to an exact sequence*

$$0 \longrightarrow \mathcal{F}_s \longrightarrow \cdots \longrightarrow \mathcal{F}_0 \longrightarrow \mathcal{E} \longrightarrow 0 \tag{9}$$

where the $\mathcal{F}_i$ are isomorphic to direct sums of reflexive sheaves of rank one:

$$\mathcal{F}_i = \bigoplus_{(Y, K_{i-1}^X) \in b\left((\mathcal{P}_{i-1})_\Sigma\right)} \mathcal{O}\Big(- \sum_{\rho \in \Delta(1)} \kappa_{i-1}(K_{i-1}^X, \rho) D_\rho \Big)^{\mathrm{fd}^{i-1}(K_{i-1}^X)},$$

where $\mathcal{P} =: \mathcal{P}_{-1}$, $K_{-1}^X := X$ and $\mathcal{P}_i$ is the ith syzygy arrangement with free dimension fd^i and

$$\kappa_i : \big(\mathcal{P}_i \setminus \{0\}\big) \times \Delta(1) \longrightarrow \mathbb{Z}, \quad \kappa_i(K_i^X, \rho) = \min\{l \mid K_i^X \subset K_i^\rho(l)\}.$$

Proof. The exactness follows by restricting to U_σ and by comparison with proposition 6.5. $\square$

Corollary 6.13: *If X is smooth, resolution (9) gives a resolution of $\mathcal{E}$ by line bundles.*

Definition 6.14: Resolution (9) is called the *reflexive resolution* of $\mathcal{E}$.

Definition 6.15: Let $\hat{E}^\Delta$ be a Δ-family and let $\mathcal{P}$ be its underlying arrangement of subvector spaces. We call a maximal set $\mathcal{P}'$ such that

$$\mathcal{P} \subset \mathcal{P}' \subset \mathcal{P}_\cap$$

a *free intersection completion* if the following conditions hold:

(i) If $X \in \mathcal{P}$ and $\mathrm{fd}(X) \geq 0$, then $\mathrm{fd}'(X) \geq 0$;
(ii) if $X \in \mathcal{P}' \setminus \mathcal{P}$, then there exists some $Y \in \mathcal{P}$ with $X < Y$ and $\mathrm{fd}(Y) > 0$;

(iii) if $X \in \mathcal{P}' \setminus \mathcal{P}$, then $\mathrm{fd}(X) \geq 0$.
Here fd' is the free dimension of $\mathcal{P}'$.

Theorem 6.16: *Let $\hat{E}^\Delta$ be a Δ-family and $\mathcal{P}$ its underlying arrangement of vector spaces. Then a free intersection completion $\mathcal{P}'$ of $\mathcal{P}$ gives rise to an exact sequence of sheaves*

$$0 \longrightarrow \mathcal{F}_s \longrightarrow \cdots \longrightarrow \mathcal{F}_0 \longrightarrow \mathcal{E} \longrightarrow 0$$

where the $\mathcal{F}_i$ are isomorphic to direct sums of reflexive sheaves of rank one:

$$\mathcal{F}_i = \bigoplus_{(Y, K_{i-1}^X) \in b\left((\mathcal{P}'_{i-1})_\Sigma\right)} \mathcal{O}\Big(- \sum_{\rho \in \Delta(1)} \kappa_{i-1}(K_{i-1}^X, \rho) D_\rho \Big)^{\mathrm{fd}^{i-1}(K_{i-1}^X)},$$

where $\mathcal{P}' =: \mathcal{P}'_{-1}$, $\mathcal{P}'_i$ is the i-th syzygy arrangement with free dimension fd^i and

$$\kappa_i : \mathcal{P}'_i \setminus \{0\} \longrightarrow \mathbb{Z}, \quad \kappa_i(K_i^X, \rho) = \min\{l \mid K_i^X \subset K_i^\rho(l)\}.$$

Proof. We need only to consider the first step of the resolution and we show that we obtain a short exact sequence of sheaves

$$0 \longrightarrow \mathcal{K}_0 \longrightarrow \mathcal{F}_0 \longrightarrow \mathcal{E} \longrightarrow 0.$$

Consider some bottleneck $(Y, X) \in \mathcal{P}'_\Sigma$. Then as in proposition 6.5, over every U_σ, we have σ-families for $\mathcal{O}\big(- \sum_{\rho \in \Delta(1)} \kappa_{-1}(X, \rho) D_\rho)^{\mathrm{fd}^{-1}(X)}\big)$ given by:

$$O_m^\sigma = \begin{cases} \mathbf{O}^\sigma & \langle m, n(\rho) \rangle \geq \kappa_{-1}(X, \rho) \text{ for all } \rho \in \sigma(1) \\ 0 & \text{else.} \end{cases}$$

Moreover, X is contained in some E_m^σ if and only if $\langle m, n(\rho) \rangle \leq \kappa_{-1}(X, \rho)$ for every $\rho \in \sigma(1)$, and thus we obtain for every bottleneck (Y, X) morphism of σ-families given

$$O_m^\sigma \xrightarrow{\psi_X} E_m^\sigma.$$

So we can conclude analogously to proposition 6.5. ☐

Definition 6.17: Let $\mathcal{E}$ be a reflexive sheaf over X and let $\mathcal{P}$ be the arrangement of vector spaces underlying the associated Δ-family. Then the resolution defined by a free intersection completion $\mathcal{P}'$ is called *minimal global resolution* of $\mathcal{E}$.

The idea behind defining above notion of minimal resolution is the following. Consider the contractions

$$\mathfrak{i}_0 : \mathcal{K}_0 \to \mathcal{P}, \quad \mathfrak{i}_0(K_0^X) = \min\{Y \subset X \mid K_0^Y = K_0^X\}.$$

and

$$\mathfrak{i}'_0 : \mathcal{K}'_0 \to \mathcal{P}, \quad \mathfrak{i}_0((K')_0^X) = \min\{Y \subset X \mid (K')_0^Y = (K')_0^X\}.$$

By definition, elements X with $\mathrm{fd}(X) < 0$ in $\mathcal{P}'$ already are contained in $\mathcal{P}$, and $\mathrm{fd}'(X) \leq 0$. On the other hand, if $\mathrm{fd}'(X) > 0$, then $(K')_0^X$ will be contracted by $\mathfrak{i}'_0$. Thus we have that, as partially ordered sets, $\mathfrak{i}'_0(\mathcal{K}'_0) \subset \mathfrak{i}_0(\mathcal{K}_0)$ which is a proper inclusion whenever there exists some $X \in \mathcal{P}$ such that $\mathrm{fd}(X) < 0$ and $\mathrm{fd}'(X) = 0$. In many examples, this can lead to a resolution which is smaller than the reflexive resolution. As is shown in the example below, the choice of $\mathcal{P}'$ in general is not unique, and this indicates that there is no meaningful general notion of a *unique* minimal global resolution over a nonsmooth or nonaffine variety.

Example 6.18: Consider the toric surface whose fan is generated by the four rays $\rho_1, \ldots, \rho_4$, whose primitive vectors are $n_1 = (1,0)$, $n_2 = (0,1)$, $n_3 = (-1,0)$, $n_4 = (0,-1)$, and which contains no cones of dimension 2. Let $\mathbf{E}^0 \cong k^3$ and consider the filtrations

$$E^{\rho_k}(i) = \begin{cases} 0 & i < -1 \\ H_i & i = -1 \\ \mathbf{E}^0 & i \geq 0 \end{cases}$$

where the H_i, $i = 1, \ldots, 4$ are hyperplanes in general position. Then $\mathcal{P}_{\rho_i} = \{H_i\}$ for $i = 1, \ldots, 4$ and $\mathcal{P} = \{H_1, \ldots, H_4\}$ and the reflexive resolution of the associated sheaf $\mathcal{E}$ is given by:

$$0 \longrightarrow \mathcal{O}^5 \longrightarrow \bigoplus_{i=1}^{4} \mathcal{O}(D_i)^2 \longrightarrow \mathcal{E} \longrightarrow 0.$$

The canonical global resolution is given by the free resolution induced by the configuration of vector spaces in figure 5.4 and is given by:

$$0 \longrightarrow \mathcal{O} \longrightarrow \bigoplus_{i=1}^{4} \mathcal{O}(D_i) \longrightarrow \bigoplus_{i \neq j} \mathcal{O}(D_i + D_j) \longrightarrow \mathcal{E} \longrightarrow 0.$$

A valid minimal global resolution of $\mathcal{E}$ would be given by setting $\mathcal{P}' = \mathcal{P} \cup \{H_i \cap H_j \mid i \neq j \in \{1,2,3\}\}$. Then:

$$0 \longrightarrow \mathcal{O}^2 \longrightarrow \bigoplus_{i \neq j \in \{1,2,3\}} \mathcal{O}(D_i + D_j) \oplus \mathcal{O}(D_4)^2 \longrightarrow \mathcal{E} \longrightarrow 0.$$

Clearly the above choice is not unique, but this resolution is smaller than both, reflexive and canonical global resolutions. Another, even better, choice would be $\mathcal{P}' = \mathcal{P} \cup \{H_1 \cap H_2, H_2 \cap H_3, H_3 \cap H_4, H_1 \cap H_4\}$, which gives:

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}(D_1 + D_2) \oplus \mathcal{O}(D_2 + D_3) \oplus \mathcal{O}(D_3 + D_4) \oplus \mathcal{O}(D_1 + D_4) \longrightarrow \mathcal{E} \longrightarrow 0.$$

CHAPTER 7

Examples

In this chapter we present some examples for resolution of equivariant sheaves. In section 7.1 we remark that reflexive resolutions over toric varieties which have only two rays are trivial. In section 7.2, we construct resolutions for a general type of vector bundles over smooth complete toric surfaces. In section 7.3 we present resolutions for reflexive sheaves whose associated filtrations contain at most one nontrivial kind of vector spaces, which are lines. Finally, we show in section 7.4 that our methods are also applicable to a larger class of equivariant sheaves, notably to monomial ideals. We have included a classification of equivariant vector bundles of rank three over $\mathbb{P}_2$ in appendix B.

7.1. Reflexive Sheaves Defined by Two Filtrations

As already remarked by Klyachko in [**Kly90**], any two given filtrations can be made into a coordinate space arrangement by choosing a suitable basis. This implies that for a toric variety X whose fan contains only two rays, any reflexive sheaf over X decomposes into a direct sum of reflexive sheaves of rank one. We state Klyachko's example for the case of $\mathbb{P}_1$ and complement it for the only other case of interest, that of an affine toric surface.

Proposition 7.1: *(i) Let $X = \mathbb{P}_1$, then every equivariant vector bundle over X splits equivariantly into a direct sum of line bundles:*

$$\mathcal{E} \cong \bigoplus_{i=1}^{r} \mathcal{O}\big(k_1^i D_1 + k_2^i D_2\big)$$

*(see [**Kly90**], example 2.3.3)*

(ii) Let $X = U_\sigma$ be an affine toric surface, then every equivariant reflexive sheaf over X splits equivariantly into a direct sum of reflexive sheaves of rank one:

$$\mathcal{E} \cong \bigoplus_{i=1}^{r} \mathcal{O}\big(k_1^i D_1 + k_2^i D_2\big).$$

Here D_1, D_2 are the two divisors of X which correspond to the two rays in the fan of X, and k_1^i, k_2^i are any integers.

7.2. Some Vector Bundles over Toric Surfaces

In this section, let X be a smooth complete toric surface. Consider a locally free sheaf $\mathcal{E}$ of rank r over X which is given by a set of *complete* filtrations $E^\rho(i)$, that is, $\dim E^\rho(i) - \dim E^\rho(i-1)$ is either zero or one. For brevity, we represent one filtration irredundantly by a tuple $(i_i^\rho, \ldots, i_r^\rho, E_1^\rho, \ldots, E_{r-1}^\rho)$, where $i_1 < i_2 < \cdots < i_r$, $\dim E_k^\rho =$

k and $E_k^\rho = E^\rho(i)$ for $i_k \leq i < i_{k+1}$. Because X is smooth, our resolutions below will essentially not depend on the twist — this means that for any sheaf $\mathcal{E}$ and any line bundle $\mathcal{O}(D)$ we get the resolution of $\mathcal{E}(D)$ from that of $\mathcal{E}$ by twisting the whole resolution of $\mathcal{E}$ by $\mathcal{O}(D)$. As we have seen in section 4.7, twisting by some T-invariant divisor $D = \sum_{\rho \in \Delta(1)} i_\rho D_\rho$ corresponds to shifting the filtration by $-i_\rho$. So in the sequel we assume without loss of generality that $i_r^\rho = 0$ for all $\rho \in \Delta(1)$ and compute only resolutions for this case.

Theorem 7.2: *Let $\mathcal{E}$ be a locally free sheaf over X, whose associated filtrations $E^\rho(i)$, $\rho \in \Delta(1)$, are* complete, *that is,* $\dim E^\rho(i) - \dim E^\rho(i-1)$ *is either zero or one. Moreover assume, that for every two rays the corresponding filtrations are in general position, that is, $E^\rho(i) \cap E^{\rho'}(j)$ has the minimal possible dimension for all i, j. Denote $n := \#\Delta(1)$, then there are the following minimal resolutions:*

(i) $\operatorname{rk} \mathcal{E} \leq n$*:*

$$0 \longrightarrow \mathcal{F}_1 \longrightarrow \mathcal{F}_0 \longrightarrow \mathcal{E} \longrightarrow 0$$

where

$$\begin{aligned}\mathcal{F}_0 =& \left(\bigoplus_{\rho \in \Delta(1)} \mathcal{O}(i_1^\rho D_\rho) \right) \\ & \oplus \left(\bigoplus_{k=2}^{\lfloor \frac{r}{2} \rfloor} \bigoplus_{\substack{\sigma \in \Delta(2) \\ \{\rho_1, \rho_2\} = \sigma(1)}} \left(\mathcal{O}(i_k^{\rho_1} D_{\rho_1} + i_{r+1-k}^{\rho_2} D_{\rho_2}) \oplus \mathcal{O}(i_{r+1-k}^{\rho_1} D_{\rho_1} + i_k^{\rho_2} D_{\rho_2}) \right) \right)\end{aligned}$$

and

$$\mathcal{F}_1 = \mathcal{O}^{n-r} \oplus \left(\bigoplus_{\rho \in \Delta(1)} \bigoplus_{k=2}^{r-1} \mathcal{O}(i_k^\rho D_\rho) \right).$$

(ii) $\operatorname{rk} \mathcal{E} > n$*:*

$$0 \longrightarrow \mathcal{F}_2 \longrightarrow \mathcal{F}_1 \longrightarrow \mathcal{F}_0 \longrightarrow \mathcal{E} \longrightarrow 0$$

where

$$\begin{aligned}\mathcal{F}_0 =& \left(\bigoplus_{\rho \in \Delta(1)} \mathcal{O}(i_1^\rho D_\rho) \right) \\ & \oplus \left(\bigoplus_{k=2}^{\lfloor \frac{r}{2} \rfloor} \bigoplus_{\substack{\sigma \in \Delta(2) \\ \{\rho_1, \rho_2\} = \sigma(1)}} \left(\mathcal{O}(i_k^{\rho_1} D_{\rho_1} + i_{r+1-k}^{\rho_2} D_{\rho_2}) \oplus \mathcal{O}(i_{r+1-k}^{\rho_1} D_{\rho_1} + i_k^{\rho_2} D_{\rho_2}) \right) \right), \\ \mathcal{F}_1 =& \left(\bigoplus_{\rho \in \Delta(1)} \bigoplus_{k=2}^{r-1} \mathcal{O}(i_k^\rho D_\rho) \right)\end{aligned}$$

and

$$\mathcal{F}_2 = \mathcal{O}^{r-n}.$$

Proof. Consider first the restriction of $\mathcal{E}$ to U_σ for some $\sigma \in \Delta(2)$. Then the underlying vector space configuration $\mathcal{P}^\sigma$ is given by the intersections $E_k^{\rho_1} \cap E_l^{\rho_2}$ for

$\sigma(1) = \{\rho_1, \rho_2\}$ and arbitrary k, l. By assumption, the $E_k^{\rho_i}$ are in general position with respect to each other. In other words, the underlying flags $E_1^{\rho_1} \subset E_2^{\rho_1} \subset \cdots \subset E_{r-1}^{\rho_1}$ and $E_1^{\rho_2} \subset E_2^{\rho_2} \subset \cdots \subset E_{r-1}^{\rho_2}$ are in *opposite* position. This implies that $\mathbf{E}^\sigma$ splits as:

$$\mathbf{E}^\sigma = \bigoplus_{i=1}^{r} E^\sigma_{i,r+1-i}$$

where $E^\sigma_{ij} = E_i^{\rho_1} \cap E_j^{\rho_2}$, and $E_r^{\rho_i} = \mathbf{E}^\sigma$ for $i = 1, 2$. In particular, $\mathcal{E}|_{U_\sigma}$ splits as:

$$\begin{aligned}\mathcal{E}|_{U_\sigma} &\cong \bigoplus_{k=1}^{r} \mathcal{O}_{U_\sigma}\left(i_k^{\rho_1} D_{\rho_1} + i_{r+1-k}^{\rho_2} D_{\rho_2}\right) \\ &= \quad \mathcal{O}_{U_\sigma}(i_1^{\rho_1} D_{\rho_1}) \oplus \mathcal{O}_{U_\sigma}(i_1^{\rho_2} D_{\rho_2}) \\ &\quad \oplus \bigoplus_{k=2}^{\lfloor \frac{r}{2} \rfloor} \left(\mathcal{O}_{U_\sigma}(i_k^{\rho_1} D_{\rho_1} + i_{r+1-k}^{\rho_2} D_{\rho_2}) \oplus \mathcal{O}_{U_\sigma}(i_{r+k-1}^{\rho_1} D_{\rho_1} + i_k^{\rho_2} D_{\rho_2})\right)\end{aligned}$$

if we assume that $i_r^{\rho_1} = i_r^{\rho_2} = 0$. Now consider $\mathcal{P} = \bigcup_{\sigma \in \Delta(2)} \mathcal{P}^\sigma$. Because we have assumed that the filtrations are sufficient general, we have:

$$\mathcal{P}_{\min} = \bigcup_{\sigma \in \Delta(2)} \left(\bigcup_{i=1}^{r} E^\sigma_{i,r+1-i} \right).$$

The minimal elements of $\mathcal{P}$ automatically are bottlenecks, and moreover, forming $\mathcal{P}_\Sigma$ does not create some new bottlenecks. Thus the explicit description of $\mathcal{F}_0$ follows. The filtrations for $\mathcal{F}_0$ are:

$$F_0^\rho(i) = (i_1^\rho, \ldots, i_r^\rho, F_1^\rho, \ldots, F_{r-1}^\rho),$$

where $\dim F_0^\rho(i) - \dim F_0^\rho(i-1) = 0$ iff $i_k^\rho < i < i_{k+1}^\rho$ and

$$\begin{aligned} F_1^\rho &= E_1^\rho, \quad \text{and} \\ F_i^\rho &= F_{i-1}^\rho \oplus E^{\sigma_1}_{i,r+1-i} \oplus E^{\sigma_2}_{r+1-i,i} \quad \text{for } i > 1 \end{aligned}$$

where $\sigma_1, \sigma_2 \in \Delta(2)$ and $\{\rho\} = \sigma_1(1) \cap \sigma_2(1)$.

Thus, the kernel filtrations $(j_1^\rho, \ldots, j_{r-1}^\rho, K_1^\rho, \ldots, K_{r-2}^\rho)$ are given by

$$j_k^\rho = i_{k+1}^\rho$$

and

$$K_i^\rho = \ker\left(F_{i+1}^\rho \twoheadrightarrow E_{i+1}^\rho\right)$$

which implies that

$$\begin{aligned}\dim K_i^\rho &= \dim F_{i+1}^\rho - \dim E_{i+1}^\rho \\ &= (2i-1) - (i+1) \\ &= i.\end{aligned}$$

Moreover, we have $\operatorname{rk} \mathcal{F}_0 = n(r-1)$, hence $\operatorname{rk} \mathcal{K}_0 = n(r-1) - r$, and $n \cdot \dim K_{r-2}^\rho = n(r-2)$. Thus $\mathcal{K}_0$ is free iff $r \le n$ and the form of $\mathcal{F}_1$ and $\mathcal{F}_2$ follows immediately. □

7.3. Arrangements of Points

Assume that E is a reflexive fine-graded module of rank r over $S = k[x_1, \ldots, x_n]$, whose associated filtrations are given by

$$\{(-i^k, 0, E^k)\},$$

where $\dim E^k = 1$ and $i^k >= 0$. Let $\mathcal{S} \subset \{1, \ldots, n\}$ the subset where $k \in \mathcal{S}$ if and only if $i^k \neq 0$. Then there is a unique partition $P_1, \ldots, P_s$ of the set $\{1, \ldots, n\}$ which is given by the equivalence relation: $k = j$ if and only if $E^k = E^j$. Denote $\underline{n}^k = (n^k_j)$ the element of $\mathbb{Z}^n$ given by:

$$n^k_j = \begin{cases} i^j & j \in \mathcal{P}_k \\ 0 & \text{else.} \end{cases}$$

Denote $\mathbf{E}$ the limit vector space of the associated σ-family and $V := \sum_{k=1}^n E^k \subset \mathbf{E}$ the subvector space spanned by all lines with $\dim V =: t$.

Theorem 7.3: *The module specified by the above data has a minimal fine-graded resolution of the form*

$$0 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow E \longrightarrow 0$$

with

$$F_0 = \Big(\bigoplus_{k=1}^{s} S(\underline{n}^k)\Big) \oplus F_0'$$

where $F_0' = 0$ if $r = t$ and $F_0' = S^{r-t}$ else, and

$$F_1 = \begin{cases} S^{s-t} & \textit{if } s > t \\ 0 & \textit{else.} \end{cases}$$

Proof. The underlying configuration of vector spaces $\mathcal{P}$ consists of a set of lines in $\mathbf{E}^0$. The bottlenecks of $\mathcal{P}_\Sigma$ are these lines plus the pair $(V, \mathbf{E})$ if $\dim t < r$. So the explicit form of the resolution follows. □

7.4. Monomial Ideals

In principle, our techniques also allow resolutions of more general sheaves than just reflexive ones. However, it is not so straightforward to give compact expressions of the terms in the resolutions, as the relations between the vector spaces in the associated arrangements and the degrees of the generators are more complicated. A general idea is to translate the combinatorial structure of an equivariant sheaf $\mathcal{E}$ into that of a reflexive sheaf by computing its first or second syzygy, which in many cases can be done very straightforwardly. For instance, it is a general fact from commutative algebra that the first syzygy of a torsion free module over $k[\sigma_M]$ is a reflexive module. Thus if we can describe the syzygy

$$0 \longrightarrow K_0 \longrightarrow F_0 \longrightarrow E \longrightarrow 0$$

for some torsion free $k[\sigma_M]$-module E by means of a filtered vector space, we can apply the machinery of the previous chapter to compute the complete resolution.

In this section, we want to apply this idea to the case of *monomial ideals* in a polynomial ring. This class of objects has attracted certain interest in recent years, see for instance [**MP01**] and [**BPS98**] for an overview on this subject. It is still an open problem to write down in general a closed expression for the minimal resolution of such an ideal.

Consider the polynomial ring $S = k[x_1, \ldots, x_n]$ and let $T \subset \mathbb{N}^n \setminus \{0\}$ be a minimal set of generators for a semigroup ideal of $\mathbb{N}^n$. Denote $I = \langle x^\alpha \mid \alpha \in T \rangle$ the corresponding monomial ideal. Then the first step of the resolution of I is straightforwardly given by:

$$0 \longrightarrow K_0 \longrightarrow F_0 \longrightarrow I \longrightarrow 0$$

where

$$F_0 = \bigoplus_{\alpha \in T} S(-\alpha).$$

Passing to the limits of the corresponding σ-families (where $\sigma = (\mathbb{R}_{\geq 0})^n$), we obtain a short exact sequence of vector spaces:

$$0 \longrightarrow \mathbf{K}_0 \xrightarrow{B} \mathbf{F}_0 \xrightarrow{A} \mathbf{I} \longrightarrow 0$$

where $\dim \mathbf{I} = 1$, $\dim \mathbf{F}_0 = |T|$ and $\dim \mathbf{K}_0 = |T| - 1$. The matrix A can be explicitly written as the row matrix

$$A = \begin{pmatrix} 1 & 1 & \cdots & 1 \end{pmatrix}.$$

The vector space $\mathbf{F}_0$ splits into a direct sum of one-dimensional vector spaces:

$$\mathbf{F}_0 = \bigoplus_{\alpha \in T} F_\alpha,$$

where F_α denotes the direct limit of the σ-family which corresponds to the module $S(-\alpha)$.

To compute the σ-family explicitly, for $\underline{n} \in \mathbb{N}^n$ denote $C_{\underline{n}} = \{\underline{n}' \in \mathbb{N}^n \mid \underline{n}' \leq \underline{n}\}$. We call $C_{\underline{n}}$ the *backward cone* of $\underline{n}$ in $\mathbb{N}^n$.

Lemma 7.4: *(i) The dimension of a graded component of F_0 is as follows:*

$$\dim(F_0)_{\underline{n}} = |T \cap C_{\underline{n}}|;$$

(ii) the filtration $F_0^k(i)$ corresponding to the k-th coordinate x_k of S is given by

$$F_0^k(i) = \bigoplus_{\alpha \in T,\, \alpha_k \leq i} F_\alpha$$

where α_k is the k-th component of $\alpha = (\alpha_1, \ldots, \alpha_n)$.

Proof. The first two statements follow from the fact that we have to count the homogeneous components of degree $\underline{n}$ of the modules $S(-\alpha)$, which are nonzero if and only if $\alpha \leq \underline{n}$. The third statement follows immediately from the first. □

The σ-family corresponding to K_0 is computed by intersecting the elements of the σ-family of F_0 with the vector space $\mathbf{K}_0$. This is straightforward, because the σ-family of F_0 is a coordinate subspace arrangement and the vector space $\mathbf{K}_0$ is in general position to the coordinate subspaces — the inclusion B could for instance represented by a matrix of the form

$$\begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ -1 & 1 & 0 & \dots & 0 \\ 0 & -1 & 1 & \dots & 0 \\ \vdots & \vdots & -1 & \dots & \vdots \\ \vdots & & \vdots & & 1 \\ 0 & & & & -1 \end{pmatrix}.$$

Thus:

Proposition 7.5: *The dimension of a graded component of K_0 is as follows:*

$$\dim(K_0)_{\underline{n}} = \dim(F_0)_{\underline{n}} - 1 = |T \cap C_{\underline{n}}| - 1$$

and the filtration $K_0^k(i)$ corresponding to the k-th coordinate x_k of S is given by

$$K_0^k(i) = \Big(\bigoplus_{\alpha \in T,\, \alpha_k \leq i} F_\alpha \Big) \cap \mathbf{K}_0$$

where α_k is the k-th component of $\alpha = (\alpha_1, \dots, \alpha_n)$.

Thus the arrangement $\mathcal{K} = \{(K_0)_{\underline{n}}\}$ in $\mathbf{K}_0$ together with the map $\varepsilon : \mathbb{N}^n \longrightarrow \mathcal{P}$ defines a σ-family for which we can construct a minimal resolution by the methods of the previous chapter. However, although we have tried hard, for the moment we have nothing new to contribute to the question whether this resolution can be expressed in a combinatorial way, say, by looking for combinatorial expressions for computing the syzygy arrangements $\mathcal{P}_i$ and $(\mathcal{P}_i)_\Sigma$. Monomial ideals and subspace arrangements have been considered in several places, in particular because any monomial ideal defines a coordinate subspace arrangement; see for instance [**GPW01**], [**BP00**], [**BP03**]. It seems that these works are relevant for a more detailed study of our resolutions, even for more general equivariant reflexive sheaves.

It might be that our point of view can shed some light on the problem of the role that characteristics plays in the construction for resolutions. Let us exemplify this for the standard example of a monomial ideal which shows pathological homological behaviour in characteristic 2. This example is that of the Stanley-Reisner ideal corresponding to the triangulation of $\mathbb{RP}_2$ (see [**BH94**], section 5.3 for a picture), which is a monomial ideal in the polynomial ring of six variables, $S := k[x_1, \dots, x_6]$, given by:

$$I = \langle x_1x_2x_3, x_1x_2x_6, x_1x_3x_5, x_1x_4x_5, x_1x_4x_6, x_2x_3x_5, \\ x_2x_4x_5, x_2x_5x_6, x_3x_4x_6, x_3x_5x_6 \rangle$$

The explicit form was taken from [**MP01**], section 5.3. This is probably the easiest such example. In [**BH97**] it ways shown that for every monomial ideal in n variables, the 0th, 1th, $(n-2)$nd and $(n-1)$st Betti numbers do not depend on characteristics, which in particular implies that the Betti numbers for monomial ideals in up to five variables do not depend on characteristics. One can compute that the first syzygy representation in $\mathbf{K}_0$, given by

$$0 \longrightarrow \mathbf{K}_0 \longrightarrow \mathbf{F}_0 \longrightarrow \mathbf{I} \longrightarrow 0$$

is as follows. Of course, $\dim \mathbf{F}_0 = 10$ and thus, $\dim \mathbf{K}_0 = 9$. We have

$$\dim K^k(i) = \begin{cases} 0 & i < 0 \\ 4 & i = 0 \\ 9 & i > 0 \end{cases}$$

and $\dim K^k(0) \cap K^j(0) = 1$ for every $k \neq j \in \{1, \ldots, 6\}$. The corresponding graph $\mathcal{P}_0$ looks as shown in figure 7.1. Let us call $F^k := K^k(0)$ and $F^{kj} := K^k(0) \cap K^j(0)$. We see

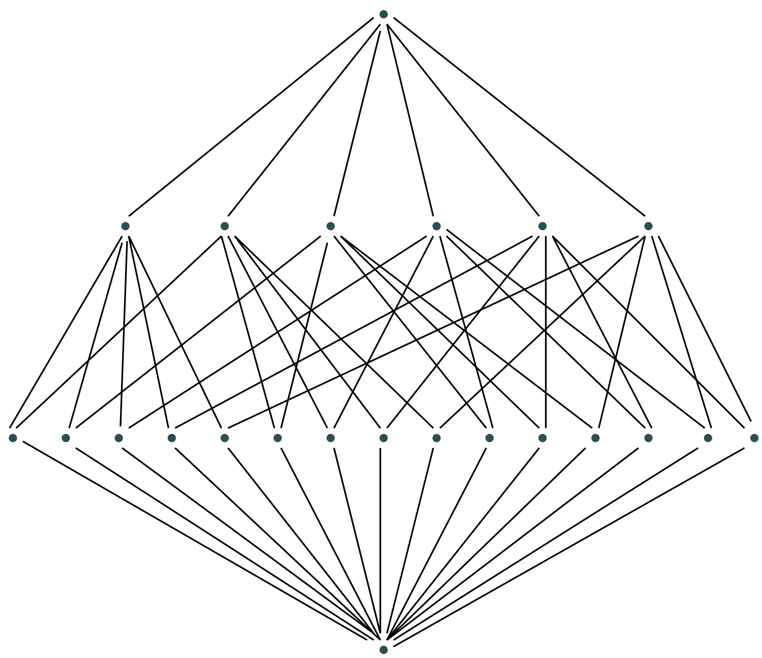


FIGURE 7.1. Hasse diagram for $\mathcal{P}_0$

that the one-dimensional spaces F^{kj} are the bottlenecks of $\mathcal{P}_0$ and that forming $(\mathcal{P}_0)_\Sigma$ does not create new bottlenecks. Thus we obtain:

$$\mathbf{F}_1 = \bigoplus_{j \neq k} F^{jk}.$$

Denote $G^k = \bigoplus_{j \neq k} F^{jk}$ and

$$A : \mathbf{F}_1 \longrightarrow \mathbf{K}_0.$$

Denote $H^k = \ker A|_{G^k}$, then we have an exact sequence of vector spaces

$$0 \longrightarrow \mathbf{K}_1 \longrightarrow \mathbf{F}_1 \longrightarrow \mathbf{F}_0 \longrightarrow \mathbf{I} \longrightarrow 0$$

where $\dim \mathbf{F}_0 = 10$, $\dim \mathbf{F}_1 = 15$, and $\dim \mathbf{K}_1 = 6$. This corresponds to an exact sequence of S-modules:

$$0 \longrightarrow K \longrightarrow S^{15} \longrightarrow S^{10} \longrightarrow I \longrightarrow 0.$$

The kernel K is a reflexive S-module to which the following filtrations belong:

$$K_1^k(i) = \begin{cases} 0 & i < 0 \\ H^k & i = 0 \\ \mathbb{K}_1 & i > 0 \end{cases}$$

where $\dim H^k = 1$ for every $k \in \{1, \ldots, 6\}$. Now we are left with six one-dimensional vector spaces H^k sitting inside a six-dimensional vector space $\mathbb{K}_1$, and it happens that $\dim \sum_{k=1}^{6} H^k = 6$ if and only if $\operatorname{char} k \neq 2$. In that case, the H^k are linearly independent and K_1 is a free module. If $\operatorname{char} k = 2$, we have $\dim \sum_{k=1}^{6} H^k = 5$, and thus the H^k are not linearly independent and we have to do one step more in the resolution. Altogether, we obtain in characteristics different from two:

$$0 \longrightarrow S^6 \longrightarrow S^{15} \xrightarrow{A} S^{10} \longrightarrow I \longrightarrow 0$$

and in characteristics two:

$$0 \longrightarrow S \longrightarrow S^7 \longrightarrow S^{15} \xrightarrow{A} S^{10} \longrightarrow I \longrightarrow 0.$$

This is due to the fact that the linear dependencies in the columns of A, considered as a vector space homomorphisms, are more pathological in characteristic two.

CHAPTER 8

Moduli for Equivariant 2-Bundles over Smooth Toric Surfaces

In this chapter we want to describe moduli for the particular case of equivariant rank two vector bundles over $X = X_\Delta$, a smooth, complete toric surface. This is in a sense the first nontrivial case one can consider, and it has the advantage that it can be reduced to the classical configuration spaces of points on a line. This comes from the fact that every filtration $E^\rho(i)$ of a two-dimensional vector space has at most one non-discrete degree of freedom, namely the position its one-dimensional vector space. So there is a natural notion of moduli for an equivariant vector bundle of rank two, namely that of the configuration of n one-dimensional subvector space in a two-dimensional vector space, where n is the number of rays in Δ. This is equivalent to the configuration space of n points on the projective line.

Consider $\mathcal{E}$ an equivariant bundle of rank two and filtrations $E^\rho(i)$ such that $E^\rho(i) = E^\rho$ for $i_1^\rho \leq i < i_2^\rho$ for some $i_1 < i_2$ and $\dim E^\rho = 1$. Then define a partition $\mathfrak{P} = \{\Pi_1, \ldots, \Pi_s\}$ of $\Delta(1)$ where the Π_i are maximal subsets of $\Delta(1)$ such that $\rho, \rho' \in \Pi_i$ if and only if $E^\rho = E^{\rho'}$. This partition plays the role of a *combinatorial* invariant of $\mathcal{E}$. With respect to this invariant, we can construct resolutions

$$0 \longrightarrow \mathcal{F}_1^{\mathfrak{P}} \stackrel{A}{\longrightarrow} \mathcal{F}_0^{\mathfrak{P}} \longrightarrow \mathcal{E} \longrightarrow 0.$$

It turns out that these resolutions really depend on $\mathfrak{P}$ and this implies that, for instance, Chern classes remain not constant if we vary the spaces E^ρ. So if we want to keep these constant, another way to obtain moduli spaces can be derived from the explicit form of the resolutions above. Namely, for any given $\mathcal{E}$, we take its resolution and vary the matrix A. The space of equivariant sheaf homomorphisms from $\mathcal{F}_1^{\mathfrak{P}}$ to $\mathcal{F}_0^{\mathfrak{P}}$ modulo automorphisms of $\mathcal{F}_1^{\mathfrak{P}}$ and $\mathcal{F}_0^{\mathfrak{P}}$ then results in a space $\mathcal{M}^{\mathfrak{P}}$ of equivariant sheaves which have the desired property of all having the same, say, Chern classes. It turns out that the space $\mathcal{M}^{\mathfrak{P}}$ however does not only contain locally free sheaves, but also degenerations of these, which are just torsion free.

The moduli space $\mathcal{M}^{\mathfrak{P}}$ obviously shares objects with the configuration space of s points, we call it $\mathcal{M}^s$ for the moment, and it turns out that indeed there exists a morphism

$$\mathcal{M}^{\mathfrak{P}} \longrightarrow \mathcal{M}^s$$

which is given by mapping a sheaf $\mathcal{E}$ to its double dual:

$$\mathcal{E} \mapsto \mathcal{E}^{\check{}\check{}}.$$

In fact, this morphism is even an isomorphism.

This way, we achieve two things: on the one hand we have identified the partition $\mathfrak{P}$ as a combinatorial invariant of vector bundles $\mathcal{E}$, and on the other hand we can give a complete description of moduli $\mathcal{M}^{\mathfrak{P}}$ with respect to this invariant. It is observed that, to have these moduli, it is really necessary also to study degenerations of equivariant vector bundles. Our hope is that this pattern can be extended for studying more general equivariant sheaves.

In Sections 8.1 and 8.2, we construct and analyze resolutions for general equivariant locally free sheaves of rank 2. Section 8.3 is devoted to duality of configurations of points in the GIT setting which we will use in Section 8.4 to give a GIT-classification of equivariant vector bundles of rank 2 on smooth complete toric surfaces.

8.1. Partitions and Resolutions of Equivariant Bundles of Rank Two

In this section we want to construct resolutions for general equivariant vector bundles of rank 2 on toric surfaces. Let $\mathcal{E}$ be such a bundle on X given by filtrations of a 2-dimensional vector space $\mathbf{E}^0$. Any such filtration $E^\rho(i)$ can be described as follows. There are two integers $i_1^\rho \leq i_2^\rho$ such that

$$E^\rho(i) = \begin{cases} 0 & \text{for } i < i_1^\rho, \\ E^\rho & \text{for } i_1^\rho \leq i < i_2^\rho, \\ \mathbf{E}^0 & \text{for } i \geq i_2^\rho, \end{cases}$$

where E^ρ is a 1-dimensional subvector space of $\mathbf{E}^0$. Thus in the case that $i_1^\rho < i_2^\rho$ a filtration can irredundantly be described by an ordered triple $(i_1^\rho, i_2^\rho, E^\rho)$. If $i_1^\rho = i_2^\rho$, the filtration is degenerate in the sense that at i_1^ρ the dimension jumps by two and there occurs no E^ρ.

Twisting $\mathcal{E}$ by an equivariant line bundle $\mathcal{O}(\underline{n})$ for some $\underline{n} = (n(\rho)) \in \mathbb{Z}^{\Delta(1)}$ has the effect that the numbers i_1^ρ, i_2^ρ are shifted to $i_1^\rho + n(\rho), i_2^\rho + n(\rho)$ for all ρ. In our further considerations such twists do not play any role and we may assume for simplicity that $i_1^\rho = -i^\rho$ and $i_2^\rho = 0$ for nonnegative integers i^ρ.

In order to obtain a complete classification and moduli, one has to consider the two ways how a set of filtrations may degenerate. Namely, the case where $i^\rho = 0$ for some $\rho \in \Delta(1)$ and $E^\rho = E^{\rho'}$ for two adjacent rays ρ and ρ'

To do this, we consider *partitions* of subsets $\Pi \subset \Delta(1)$ as follows. A partition of Π is a collection $\mathfrak{P} = \{\Pi_1, \ldots, \Pi_s\}$ of disjoint subsets of Π such that $\Pi = \coprod_{i=1}^s \Pi_i$. Among the partitions of Π we define a partial order $\leq$ as follows: given two partitions $\mathfrak{P} = \{\Pi_1, \ldots, \Pi_s\}$ and $\mathfrak{P}' = \{\Pi'_1, \ldots, \Pi'_{s'}\}$ we say $\mathfrak{P} \leq \mathfrak{P}'$ iff $\mathfrak{P}$ is a *refinement* of $\mathfrak{P}'$, i.e. there exists a map $\{1, \ldots, s\} \longrightarrow \{1, \ldots s'\}$, $i \mapsto j$ such that $\Pi_i \subset \Pi'_j$. We call the map given by a refinement

$$\pi : \mathfrak{P} \longrightarrow \mathfrak{P}', \quad \pi(\Pi_i) = \Pi'_j,$$

projection map, and any map $s : \mathfrak{P}' \longrightarrow \mathfrak{P}$ such that $\pi \circ s$ is the identity a *section* with respect to π. The partial order $\leq$ has unique minimal and maximal elements, namely the partitions $\{\{\rho\}\}_{\rho \in \Pi}$ and $\{\Pi\}$. For convenience, we call the partition $\{\{\rho\}\}_{\rho \in \Pi}$ the *fine partition* of Π.

Partitions associated to an equivariant bundle of rank two: for an equivariant bundle $\mathcal{E}$ of rank 2 we denote by $\Pi = \Pi(\mathcal{E}) \subset \Delta(1)$ the subset of those ρ for which $i^\rho > 0$. We assume that the rays $\{\rho_0, \dots \rho_{m-1}\} = \Pi$ are enumerated clockwise with respect to their circular order in the fan by the cyclic group $\mathbb{Z}_m$. On Π we can define a partition as follows.

Definition 8.1: Let $\Pi_1, \dots \Pi_s$ be the unique partition of Π with the following properties:

(i) if $\rho_i, \rho_j \in \Pi_k$ then $E^{\rho_i} = E^{\rho_j}$,
(ii) if for some $1 \leq k < s$ $\rho_i \in \Pi_k$ and $\rho_j \in \Pi_{k+1}$, or if $\rho_i \in \Pi_s$ and $\rho_j \in \Pi_1$, then $E^{\rho_i} \neq E^{\rho_j}$,
(iii) if Π_k contains ρ_i and ρ_{i+l} then it contains the interval $\rho_{i+1}, \dots, \rho_{i+l-1}$ or the interval $\rho_{i+l+1}, \dots, \rho_{i-1}$ or both.

One can think of this partition as the set of maximal intervals in the circularly ordered set Π on which the E^ρ coincide. We assume that the Π_i are enumerated clockwise. We denote the partition so defined $\mathfrak{P}_\mathcal{E}$ and call it *the coarse partition of* Π *with respect to* $\mathcal{E}$.

Theorem 8.2: *Let $\mathcal{E}$ be an arbitrary equivariant vector bundle of rank 2 on a smooth complete toric surface X, defined by filtrations $\{(-i^\rho, 0, E_\rho)\}_{\rho \in \Delta(1)}$ of a two dimensional vector space $\mathbf{E}^0$. Let $\Pi = \{\rho \in \Delta(1) \mid i^\rho > 0\}$ and let $\mathfrak{P}_\mathcal{E} = \{\Pi_1, \dots \Pi_s\}$ be the coarse partition of Π with respect to $\mathcal{E}$. If $s > 2$ then there exists a short exact sequence*

$$0 \longrightarrow \mathcal{O}^{s-2} \xrightarrow{A} \bigoplus_{i=1}^{s} \mathcal{O}(\sum_{\rho \in \Pi_i} i^\rho \cdot D_\rho) \longrightarrow \mathcal{E} \longrightarrow 0$$

where A is a matrix of monomials whose exponents are determined by the partition $\mathfrak{P}_\mathcal{E}$. Moreover, the $(s-2)$-minors $A^{i,i+1}$ of A, $1 \leq i < s$, which consist of all rows of A except the i-th and the $(i+1)$-st, are of full rank. If $s \leq 2$, then $\mathcal{E}$ splits.

Remark 8.3: The proof explains the precise relationship between A and the filtrations associated to $\mathcal{E}$, see also Proposition 8.4.

Proof. Let first $s \leq 2$. Then we can decompose the vector space $\mathbf{E}^0$ into a direct sum $\mathbf{E}^0 = E_1 \oplus E_2$ and the filtrations decompose into direct sums of filtrations for E_1 and E_2, respectively. Consequently, the associated bundle $\mathcal{E}$ splits into a direct sum of line bundles.

Now assume that $s > 2$. Consider the Cox quotient presentation $\hat{X} \longrightarrow X$ and let $\{n(\rho)\}_{\rho \in \Delta(1)}$ be the standard basis of the lattice $\hat{N} \cong \mathbb{Z}^{\Delta(1)}$. Let $\{i_1, \dots, i_{s-2}\} \subset \{1, \dots, s\}$, then we set

$$x^{\hat{\Pi}_i} := \prod_{\rho \in \Pi_i} x_\rho^{i^\rho}, \quad \text{and} \quad x^{\hat{\Pi}_{i_1 \dots i_{s-2}}} := \prod_{k=1}^{s-2} x^{\hat{\Pi}_{i_k}}.$$

We can define a morphism of fine-graded free S-modules

$$0 \longrightarrow S^{s-2} \xrightarrow{A} \bigoplus_{i=1}^{s} S(\sum_{\rho\in\Pi_i} i^\rho \cdot n(\rho))$$

which is an $s \times (s-2)$-matrix A with monomial entries:

$$A = (\alpha_{ij} \cdot x^{\hat{\Pi}_i})$$

where i runs from 1 to s and j from 1 to $s-2$. We require that, for $i = 1, \dots, s-1$, the $(s-2)$-minors $A^{i,i+1}$ of A which consist of all rows of A except the i-th and the $(i+1)$-st, have full rank over S. After applying the sheafification functor $\tilde{}$ to this sequence we obtain a short exact sequence of sheaves

$$0 \longrightarrow \mathcal{O}^{s-2} \xrightarrow{A} \bigoplus_{i=1}^{s} \mathcal{O}(\sum_{\rho\in\Pi_i} i^\rho \cdot D_\rho) \longrightarrow \mathcal{Q} \longrightarrow 0$$

where by abuse of notion we write A instead of $\tilde{A}$.

The matrix A defines an equivariant injective morphism of coherent sheaves, but it is not necessarily an injective vector bundle homomorphism. This is the case if and only if the rank of $A(x)$ equals $s-2$ at all points $x \in X$. This in turn means that A is an inclusion of vector bundles if and only if there exists a $k > 0$ such that $B^k \subset \mathrm{Fitt}_2(A)$, where B is the irrelevant ideal associated to the quotient presentation $\hat{X} \longrightarrow X$. If this is the case, then the cokernel $\mathcal{Q}$ is a vector bundle as well.

Let $\{i_1, \dots, i_{s-2}\} \subset \{1, \dots s\}$ and let $A^{i_1 \cdots i_{s-2}}$ be the $(s-2) \times (s-2)$-minor of A which contains the rows corresponding to $\{i_1, \dots i_{s-2}\}$. Moreover, let

$$A' := (\alpha_{i,j})$$

be the matrix of coefficients of A and $(A')^{i_1 \cdots i_{s-2}}$ the according minor. The second Fitting ideal $\mathrm{Fitt}_2(A)$ of A is generated by the determinants of all the $A^{i_1 \cdots i_{s-2}}$:

$$\mathrm{Fitt}_2(A) = \langle \det A^{i_1 \cdots i_{s-2}} \rangle = \langle \det(A')^{i_1 \cdots i_{s-2}} \cdot x^{\hat{\Pi}_{i_1 \cdots i_{s-2}}} \rangle$$

Thus $\mathrm{Fitt}_2(A)$ is a monomial ideal generated by the $x^{\hat{\Pi}_{i_1 \cdots i_{s-2}}}$. To show that $B^k \subset \mathrm{Fitt}_2(A)$ for some $k > 0$, it suffices to show that for each generator $x^{\hat{\sigma}}$ of B there exists a generator of $\mathrm{Fitt}_2(A)$ which divides some power of $x^{\hat{\sigma}}$. Without loss of generality we may assume that $i^\rho = 1$ for all $\rho \in \Pi$. Then the problem is equivalent to the question whether a given $x^{\hat{\sigma}}$ with $\sigma(1) = \{\rho_k, \rho_{k+1}\}$ is divided by some $x^{\hat{\Pi}_{i_1 \cdots i_{s-2}}}$ which in turn is equivalent to finding $i_1, \dots i_{s-2}$ such that $\Pi_{i_1} \cup \cdots \cup \Pi_{i_{s-2}}$ is contained in the interval $\{\rho_{k+2} \dots \rho_{k-1}\}$ (with indices modulo m). But because the complement of this interval is $\{\rho_k, \rho_{k+1}\}$, this complement intersects at most two of the intervals Π_i, say, after renumbering, Π_{s-1} and Π_s. Thus we choose $(i_1, \dots i_{s-2}) = (1, \dots, s-2)$ which is nonempty because $s > 2$. Moreover, $\Pi_1 \cup \cdots \cup \Pi_{s-2}$, is contained in $\{\rho_{k+2}, \dots, \rho_{k-1}\}$, and so $x^{\hat{\Pi}_{i_1, \dots, i_{s-2}}}$ divides $x^{\hat{\sigma}}$. Hence, chosing a matrix A as above ensures that the quotient $\mathcal{Q}$ of A is locally free.

Now we have to show that any $\mathcal{E}$ with associated coarse partition $\mathfrak{P}_{\mathcal{E}}$ can be resolved this way. We do this by explicitly writing down the filtrations for $\mathcal{O}^{s-2}$ and $\bigoplus_{i=1}^{s} \mathcal{O}(\sum_{\rho\in\Pi_i} i^\rho \cdot D_\rho)$ and by constructing with their help a homomorphism of the associated limit vector spaces which we will lift to a morphism of locally free sheaves. Denote $\mathbf{F}^0$ the $(s-2)$-dimensional filtered k-vector space associated to the vector bundle $\mathcal{O}^{s-2}$, and $\mathbf{G}^0$ the s-dimensional k-vector space associated to $\bigoplus_{i=1}^s \mathcal{O}(\sum_{\rho\in\Pi_i} i^\rho \cdot D_\rho)$. We will identify $\mathbf{G}^0$ with $k^{\mathfrak{P}_\mathcal{E}} \cong k^s$ and label its standard basis $e_1, \dots, e_s$. The filtrations are:

$$F^\rho(i) = \begin{cases} 0 & \text{for } i < 0 \\ F & \text{otherwise} \end{cases} \quad \text{and } G^\rho(i) = \begin{cases} 0 & \text{for } i < -i^\rho \\ k \cdot e_i & \text{for } -i^\rho \le i < 0 \text{ and } \rho \in \Pi_i \\ \mathbf{G}^0 & \text{otherwise} \end{cases}$$

The matrix A induces a vector space homomorphism from $\mathbf{F}^0$ to $\mathbf{G}^0$ which can be naturally identified with the matrix A'. We can define filtrations for the quotient vector space $\mathbf{E}^0 := \mathbf{G}^0/\mathbf{F}^0$ simply by taking the quotient filtrations

$$E^\rho(i) = G^\rho(i)/F^\rho(i)$$

with respect to A'. These filtrations are of the form

$$E^\rho(i) = (-i^\rho, 0, k \cdot \overline{e}_j)$$

where $\rho \in \Pi_j$ and $\overline{e}_j$ is the image of e_j in $\mathbf{E}^0$. If we assume that $B^k \subset \mathrm{Fitt}_2(A)$ for some $k > 0$, these filtrations become in a natural way the filtrations associated to the cokernel $\mathcal{E}$ of A. On the other hand, if we define a homomorphism from $\mathbf{G}^0$ to some 2-dimensional k-vector space $\mathbf{E}^0$ by fixing the images $\overline{e}_j \neq 0$ of the basis vectors e_j, $j = 1, \dots, s$, of $\mathbf{G}^0$, we immediately obtain a homomorphism of filtered vector spaces whose kernel is a filtered vector space $\mathbf{F}^0 \overset{A'}{\hookrightarrow} \mathbf{G}^0$. The corresponding matrix A with monomial entries then defines a sheaf homomorphism $0 \longrightarrow \mathcal{O}^{s-2} \overset{A}{\longrightarrow} \bigoplus_{i=1}^s \mathcal{O}(\sum_{\rho\in\Pi_i} i^\rho \cdot D_\rho)$. As we have seen before, the cokernel of A is locally free if and only if $\det(A)^{i,i+1} \neq 0$ for $i = 1, \dots, s-1$ and $\det(A)^{s,1} \neq 0$. Now it is a lemma from linear algebra that $\overline{e}_i$ and $\overline{e}_{i+1}$ are linearly independent if and only if $\det(A')^{i,i+1} \neq 0$. □

8.2. More on Partitions and Resolutions

Let us fix numbers $i^\rho > 0$ for $\rho \in \Pi \subset \Delta(1)$ and a partition $\mathfrak{P}$ of Π. In this section we consider short exact sequences of type

$$0 \longrightarrow \mathcal{O}^{s-2} \overset{A}{\longrightarrow} \bigoplus_{i=1}^{s} \mathcal{O}(\sum_{\rho\in\Pi_i} i^\rho \cdot D_\rho) \longrightarrow \mathcal{E} \longrightarrow 0$$

where A is given by a monomial matrix. By Theorem 8.2, there are conditions such that the cokernel $\mathcal{E}$ is a locally free sheaf whose associated coarse partition $\mathfrak{P}_\mathcal{E}$ coincides with $\mathfrak{P}$. In general, if A is arbitrary and has just maximal rank, we have the following as an immediate corollary from the constructions of the previous section:

Proposition 8.4: *Fix a set of numbers $I := \{i^\rho \geq 0\}_{\rho\in\Delta(1)}$, let $\Pi = \Pi_I = \{\rho \mid i^\rho > 0\} \subset \Delta(1)$ and let $\mathfrak{P} = \{\Pi_1, \dots, \Pi_s\}$, $s \geq 2$, be a partition of Π. Let $\mathcal{E} = \mathcal{E}(I, \mathfrak{P}, A)$ be a sheaf defined by a short exact sequence*

$$0 \longrightarrow \mathcal{O}^{s-2} \overset{A}{\longrightarrow} \bigoplus_{i=1}^{s} \mathcal{O}(\sum_{\rho\in\Pi_i} i^\rho \cdot D_\rho) \longrightarrow \mathcal{E} \longrightarrow 0.$$

Then $\mathcal{E}$ is a torsion free sheaf of rank 2 over X and we can consider the short exact sequence induced on the limit vector spaces

$$0 \longrightarrow k^{s-2} \xrightarrow{A^0} k^{\mathfrak{P}} \xrightarrow{\check{A}^0} \mathbf{E}^0 \longrightarrow 0,$$

with $\check{A}^0 = (A_1, \ldots, A_s)$ a $2\times s$-matrix. If $\mathcal{E}$ is locally free, then $\mathfrak{P} \leq \mathfrak{P}_{\mathcal{E}}$ is a refinement of the coarse partition associated to $\mathcal{E}$, and the filtrations for $\mathcal{E}$ are given by $\{(-i^\rho, 0, \langle A_i \rangle) \mid \rho \in \Pi_i\}_{\Pi_i \in \mathfrak{P}}$, where $\langle A_i \rangle$ denotes the 1-dimensional subvector space of $\mathbf{E}^0$ spanned by the i-th column of $\check{A}^0$.

Remark 8.5: Observe that for A and $\mathcal{E}$ as in Proposition 8.4 and $\mathcal{E}$ locally free, and for $\{\Pi'_1, \ldots \Pi'_{s'}\} = \mathfrak{P}' \leq \mathfrak{P}$ any refinement with corresponding projection π, we can write the filtrations as $\{(-i^\rho, 0, \langle A_{\pi(i)} \rangle) \mid \rho \in \Pi'_i\}_{\Pi'_i \in \mathfrak{P}'}$.

Using the fact that every torsion free sheaf $\mathcal{E}$ embeds into its bidual, $0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}^{\check{}\check{}}$, we can now completely describe torsion free equivariant sheaves as in 8.4 without explicitly considering Δ-families:

Theorem 8.6: *Let $\mathcal{E}' = \mathcal{E}'(I, \mathfrak{P}', B)$ be a cokernel*

$$0 \longrightarrow \mathcal{O}^{s'-2} \xrightarrow{B} \bigoplus_{i=1}^{s'} \mathcal{O}\Big(\sum_{\rho \in \Pi'_i} i^\rho \cdot D_\rho\Big) \longrightarrow \mathcal{E}' \longrightarrow 0.$$

and let $\check{B}^0$ be defined by

$$0 \longrightarrow k^{s'-2} \xrightarrow{B^0} k^{\mathfrak{P}'} \xrightarrow{\check{B}^0} k^2 \longrightarrow 0,$$

Let then $\mathcal{E}$ be the bundle defined by the filtrations $\{-i^\rho, 0, E^\rho\}_{\rho \in \Pi}$ associated to $\check{B}^0$ by

$$E^\rho = \begin{cases} 0 & \rho \notin \Pi \\ \langle B_i \rangle & \rho \in \Pi'_i \end{cases}$$

Then $\mathcal{E} \cong \mathcal{E}'^{\check{}\check{}}$, $\mathfrak{P}' \leq \mathfrak{P}_{\mathcal{E}}$, and we have an exact diagram

$$\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}^{s'-2} & \xrightarrow{B} & \bigoplus_{i=1}^{s'} \mathcal{O}(\sum_{\rho \in \Pi'_i} i^\rho \cdot D_\rho) & \xrightarrow{\check{B}} & \mathcal{E}' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle \pi} & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}^{s-2} & \xrightarrow{A} & \bigoplus_{i=1}^{s} \mathcal{O}(\sum_{\rho \in \Pi_i} i^\rho \cdot D_\rho) & \xrightarrow{\check{A}} & \mathcal{E} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & \longrightarrow & \mathcal{C} & \longrightarrow & \mathcal{C} & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

The cokernel sheaf $\mathcal{C}$ is a skyscraper sheaf whose support is contained in the set of 0-dimensional orbits of X. More precisely,

$$\operatorname{supp}(\mathcal{C}) = \dot{\bigcup_{\sigma \in \delta(2)}} \operatorname{orb}(\sigma)$$

where $\delta(2) \subset \Delta(2)$ is the set of cones $\sigma \in \Delta(2)$ such that for $\{\rho_i, \rho_{i+1}\} = \sigma(1)$, it is true that $\rho_i \in \Pi'_i$, $\rho_{i+1} \in \Pi'_j$, for some $i \neq j$, and $\pi(\Pi'_i) = \pi(\Pi'_j)$.

PROOF. Denote $\mathfrak{P} := \mathfrak{P}_{\mathcal{E}}$ and let $\mathfrak{P}' \leq \mathfrak{P}$ be any refinement with projection π. Using a section $t : \mathfrak{P}' \longrightarrow \mathfrak{P}$ we fix a choice of elements in the preimage of π. We define a matrix $\check{A}^0 := (B_{t(1)}, \ldots, B_{t(s)})$ and $\hat{\pi} : k^{\mathfrak{P}'} \longrightarrow k^{\mathfrak{P}}$ the morphism induced by π over k. This way we obtain in the category of k-vector spaces a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & k^{s'-2} & \xrightarrow{B^0} & k^{\mathfrak{P}'} & \xrightarrow{\check{B}^0} & \mathbf{E}^0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle \tilde{\pi}} & & \downarrow{\scriptstyle \mathrm{id}} & & \\
0 & \longrightarrow & k^{s-2} & \xrightarrow{A^0} & k^{\mathfrak{P}} & \xrightarrow{\check{A}^0} & \mathbf{E}^0 & \longrightarrow & 0
\end{array}$$

where id^0 is the identity homomorphism on $\mathbf{E}^0$ and A^0 the kernel homomorphism of $\check{A}^0$.

The morphisms in the left square of the diagram can immediately be lifted to morphisms of locally free sheaves by considering them as matrices of coefficients of the entries of matrices of monomials. So we obtain the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{O}^{s'-2} & \xrightarrow{B} & \bigoplus_{i=1}^{s'} \mathcal{O}(\sum_{\rho \in \Pi'_i} i^{\rho} \cdot D_{\rho}) & \xrightarrow{\check{B}} & \mathcal{E}' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle \pi} & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}^{s-2} & \xrightarrow{A} & \bigoplus_{i=1}^{s} \mathcal{O}(\sum_{\rho \in \Pi_i} i^{\rho} \cdot D_{\rho}) & \xrightarrow{\check{A}} & \mathcal{E} & \longrightarrow & 0
\end{array}$$

where we interprete the matrices $\check{A}^0$ and $\check{B}^0$ as sheaf homomorphisms. The injectivity of the homomorphism $\mathcal{E}' \longrightarrow \mathcal{E}$ is an immediate consequence of the fact that after restriction to the open sets U_ρ, $\rho \in \Delta(1)$, it induces the identity homomorphism. It follows that the cokernel $\mathcal{C}$ is a skyscraper sheaf whose support must be contained in the set of 0-dimensional orbits of X, and its description is immediate. □

8.3. Duality for Configuration Spaces of Points in Projective Spaces

Let $m < n$, $T_n \cong (k^*)^n$ the n-dimensional algebraic torus, GL_m the group of automorphisms of k^m and denote $G := \mathrm{GL}_m \times T_n$, which is a reductive group. Denote $\mathbb{M}_{n,m}$ the space $\mathrm{Hom}_k(k^m, k^n)$ of $n \times m$-matrices over k and let G act on $\mathbb{M}_{n,m}$ by $(g,t).A := t \circ A \circ g^{-1}$.

We first want to consider the actions of the two subgroups GL_m and T_n of G separately. Because the representations of GL_m and T_n in $\mathrm{GL}(\mathbb{M}_{n,m})$ both contain the homotheties, their actions induce actions on the projective space $\mathbb{P}\mathbb{M}_{n,m}$ and linearizations of the ample line bundle $\mathcal{O}_{\mathbb{P}\mathbb{M}_{n,m}}(1)$, so that we are able to perform GIT-quotients of $\mathbb{P}\mathbb{M}_{n,m}$ by GL_m and T_n, respectively. GL_m acts from the right on the matrices $\mathbb{M}_{n,m}$, and the set of semistable points in $\mathbb{PM}_{n,m}$ is precisely the set of points represented by matrices which have maximal rank m:

$$\mathbb{PM}^{ss}_{n,m}(\mathrm{GL}_m) = \{\langle A\rangle \mid \operatorname{rank} A = m\}$$

Furthermore, GL_m acts freely on this set, so that $\mathbb{PM}^{s}_{n,m}(\mathrm{GL}_m) = \mathbb{PM}^{ss}_{n,m}(\mathrm{GL}_m)$, and there exists the geometric quotient

$$\mathbb{PM}^{ss}_{n,m}(\mathrm{GL}_m)//\,\mathrm{GL}_m \cong \mathrm{Gr}(m,n)$$

where $\mathrm{Gr}(m,n)$ is the Grassmannian of m-dimensional linear subspaces of k^n. Similarly, with help of the eigenspace decomposition of the left action of T_n on $\mathbb{M}_{n,m}$, it easy to see that

$$\mathbb{PM}^{ss}_{n,m}(T_n) = \{\langle A\rangle \mid \text{ no row of } A \text{ is zero }\}.$$

T_n acts freely on this set, so that stable and semistable points of $\mathbb{PM}_{n,m}$ coincide and we obtain a geometric quotient

$$\mathbb{PM}^{ss}_{n,m}(T_n)/T_n \cong (\mathbb{P}_{m-1})^n$$

which is given by the map $\langle A\rangle \mapsto (\langle A_1\rangle, \dots, \langle A_n\rangle)$, where A_i denote the row vectors of the matrix A.

The action of the group G descends to actions of the groups $G/\,\mathrm{GL}_m \cong T_n$ on $\mathrm{Gr}(m,n)$ and $G/T_n \cong \mathrm{GL}_m$ on $(\mathbb{P}_{m-1})^n$, respectively. Both actions are textbook examples from GIT and there are the following criteria for stability:

Proposition 8.7 ([**MFK94**], Proposition 4.3)**:**

(1) An n-tuple $(p_1, \dots, p_n)$ of points in $(\mathbb{P}_{m-1})^n$ is (semi-)stable with respect to the diagonal action of GL_m if and only if for every proper linear subspace L of k^m

$$\#\{i \mid p_i \in L\} < \frac{m}{n} \dim L \tag{10}$$

(respectively $\leq$).

(2) Consider the action of GL_n on $\mathrm{Gr}(m,n)$. Then a point $A \in \mathrm{Gr}(m,n)$ is (semi-)stable with respect to this action if and only if, for every proper linear subspace L of k^n,

$$\dim(A \cap L) < \frac{m}{n} \dim L \tag{11}$$

(respectively $\leq$).

We need to modify the second statement only slightly for the case of the action of a maximal subtorus of GL_n on $\mathrm{Gr}(m,n)$:

Corollary 8.8: *Consider the action of a maximal subtorus T_n of GL_n on $\mathrm{Gr}(m,n)$. Then a point $A \in \mathrm{Gr}(m,n)$ is (semi-)stable with respect to this action if and only if, for every proper linear subspace L of k^n which is spanned by eigenspaces of the action of T_n on k^n, inequality (11) holds.*

These results imply that the preimages of $(\mathbb{P}_{m-1})^{n,ss}(\mathrm{GL}_m)$ and $\mathrm{Gr}(m,n)^{ss}(T_n)$ in $\mathbb{PM}_{n,m}$ coincide and we denote this set by $\mathbb{PM}^o_{n,m}$. Then the sets $(\mathbb{P}_{m-1})^{n,ss}(\mathrm{GL}_m)$ and $\mathrm{Gr}(m,\, n)^{ss}(T_n)$ both are geometric quotients of $\mathbb{PM}^o_{n,m}$ and their quotients

$$\mathrm{Gr}(m,n)^{ss}(T_n)//T_n \quad \text{and} \quad (\mathbb{P}_{m-1})^{n,ss}(\mathrm{GL}_m)//\,\mathrm{GL}_m\,.$$

are good quotients of $\mathbb{PM}^o_{n,m}$ as each is a good quotient of a good quotient. By the universal property of good quotients, these two spaces coincide with the good quotient $\mathbb{PM}^o_{n,m}//G$, and thus are isomorphic. In particular, there is a commutative diagram consisting of good quotients:

$$\begin{array}{ccc} & \mathbb{PM}^o_{n,m} & \\ \swarrow & & \searrow \\ (\mathbb{P}_{m-1})^{n,ss}(\mathrm{GL}_m) & & \mathrm{Gr}(m,n)^{ss}(T_n) \\ \searrow & & \swarrow \\ & \mathcal{M}_{n,m} & \end{array}$$

We want to extend this correspondence using the well-known isomorphism

$$\mathrm{Gr}(m,n) \cong \mathrm{Gr}(n-m,n)$$

which can be interpreted as saying that an $n \times m$-matrix A of rank m representing a point in $\mathrm{Gr}(m,n)$ is mapped to an $(n-m) \times n$-matrix $\check{A}$ representing a point in $\mathrm{Gr}(n-m,m)$ such that both matrices fit into a short exact sequence

$$0 \longrightarrow k^m \xrightarrow{A} k^n \xrightarrow{\check{A}} k^{n-m} \longrightarrow 0$$

This correspondence is compatible with the action of the torus T_n on both sides:

Lemma 8.9: *Let T_n be a maximal subtorus of GL_n. Consider the actions of T_n on $\mathrm{Gr}(m,n)$ and $\mathrm{Gr}(n-m,n)$, induced by its natural actions on k^n and the dual vector space $(k^n)^\vee$, respectively. Then the canonical isomorphism between $\mathrm{Gr}(m,n)$ and $\mathrm{Gr}(n-m,n)$, which is induced by the canonical isomorphism between k^n and $(k^n)^\vee$, is T_n-equivariant and maps the (semi-)stable points as specified in Proposition 8.7, to (semi-)stable points. There exists a natural isomorphism*

$$\mathrm{Gr}(m,n)^{ss}(T_n)//T_n \cong \mathrm{Gr}(n-m,n)^{ss}(T_n)//T_n$$

Proof. A little bit of linear algebra shows that for some $A \in \mathrm{Gr}(m,n)$ and for all linear subspaces $L \subset k^n$ the following holds,

$$\dim A \cap L < \frac{m}{n} \dim L \quad \text{if and only if} \quad \dim A^\vee \cap L^\vee < \frac{n-m}{n} \dim L^\vee$$

(respectively $\leq$), where $A^\vee$ and $L^\vee$ are the annihilators of A and L in $(k^n)^\vee$. $\square$

Because of this we can extend our correspondences to the following diagram:

$$
\begin{array}{ccccccc}
 & \mathbb{PM}^o_{n,m} & & & & \mathbb{PM}^o_{n-m,n} & \\
 \swarrow & & \searrow & & \swarrow & & \searrow \\
(\mathbb{P}_{m-1})^{n,ss} & & & \mathrm{Gr}(m,n)^{ss} \cong \mathrm{Gr}(n-m,n)^{ss} & & & (\mathbb{P}_{n-m-1})^{n,ss} \\
 \searrow & & \swarrow & & \searrow & & \swarrow \\
 & \mathcal{M}_{n,m} & & \cong & & \mathcal{M}_{n-m,n} &
\end{array}
$$

8.4. Moduli of Equivariant Sheaves

Let us fix a tuple of nonnegative numbers $I = (i^\rho \mid \rho \in \Delta(1))$ and a partition $\mathfrak{P} = \{\Pi_1, \ldots, \Pi_s\}$ of the set $\Pi = \{\rho \in \Delta(1) \mid i^\rho > 0\}$ where $s \geq 2$. In Section 8.1 we have identified such data as a set of typical discrete parameters for equivariant vector bundles of rank 2 on a toric surface X.

We have shown in Theorem 8.2 that for each such bundle $\mathcal{E}$ whose equivariant first Chern class in $\mathbb{Z}^{\Delta(1)}$ and whose coarse partition $\mathfrak{P}_\mathcal{E}$ coincide with I and $\mathfrak{P}$, respectively, there exists a short exact sequence of the form

$$0 \longrightarrow \mathcal{O}^{s-2} \xrightarrow{A} \bigoplus_{i=1}^{s} \mathcal{O}(\sum_{\rho \in \Pi_i} i^\rho \cdot D_\rho) \xrightarrow{\check{A}} \mathcal{E} \longrightarrow 0$$

which corresponds to a short exact sequence of vector spaces

$$0 \longrightarrow k^{s-2} \xrightarrow{A^0} k^s \xrightarrow{\check{A}^0} k^2 \longrightarrow 0.$$

In order to obtain moduli spaces, we ask for spaces which parametrize isomorphism classes of equivariant vector bundles $\mathcal{E}$ of rank 2 with fixed coarse partition $\mathfrak{P}_\mathcal{E} = \mathfrak{P}$. The conditions on A in Theorem 8.2 imply that the set of matrices A whose cokernel is such a vector bundle is dense in $\mathbb{M}_{s,s-2}$. So by varying matrices A we have a natural candidate for a parameter space of vector bundles $\mathcal{E}$ with fixed I and $\mathfrak{P}$ which is given by $\mathbb{M}_{s,s-2}$ modulo the equivariant automorphisms of $\mathcal{O}^{s-2}$ and $\bigoplus_{i=1}^s \mathcal{O}(\sum_{\rho\in\Pi_i} i^\rho \cdot D_\rho)$, GL_{s-2} and T_s, respectively.

Another natural parameter space is the set of configurations of s points in $\mathbb{P}\mathbf{E}^0 \cong \mathbb{P}_1$ which can be given by the columns of the matrix $\check{A}^0$. In that case equivariant isomorphism classes of bundles are determined by configurations modulo linear transformations by GL_2. This is the sort of moduli space which has already been suggested by Klyachko in [**Kly90**].

By the results of the previous section both spaces can be compared in terms of the GIT-quotients $\mathcal{M}_{s,s-2}$ and $\mathcal{M}_{2,s}$. By Theorem 8.6 the isomorphism $\mathcal{M}_{s,s-2} \xrightarrow{\cong} \mathcal{M}_{2,s}$ which is given by the map $A \mapsto \check{A}$, respectively $A^0 \mapsto \check{A}^0$, can be interpreted as the map

$$\mathcal{E} \mapsto \mathcal{E}^{\check{}\check{}}$$

which is defined for any cokernel $\mathcal{E}$ represented by some GIT-semistable matrix A in $\mathbb{PM}_{s,s-2}$.

Let us now investigate the semistable points of $\mathbb{PM}_{s,s-2}$ and $\mathbb{PM}_{2,s}$. Recall from proposition 8.7 that a point $(p_1, \ldots, p_s) \in (\mathbb{P}_1)^s$ is properly semistable with respect to the action of GL_2 iff precisely $\frac{s}{2}$ of the p_i coincide. Thus properly semistable points exist only in the case $s = 2t$ even.

Proposition 8.10: *Let $\mathcal{E} = \mathcal{E}(I, \mathfrak{P}, A)$ be a torsion free sheaf given by a short exact sequence as above. Let A_{i_1} the i_1-th column of $\check{A}$, let $\mathfrak{P}_1 = \{\Pi_{i_1}, \dots \Pi_{i_r}\}$ be the maximal subset of $\mathfrak{P}$ with $\langle A_{i_k} \rangle = \langle A_{i_1} \rangle$ for $1 \leq k \leq r$, and let $\mathfrak{P}_2$ be the complement of $\mathfrak{P}_1$ in $\mathfrak{P}$. Then the torsion free sheaf $\mathcal{E}$ defined by A is an extension*

$$0 \longrightarrow \mathcal{E}_1 \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}_2 \longrightarrow 0$$

where $\mathcal{E}_1$ and $\mathcal{E}_2$ are torsion free sheaves of rank 1 with

$$\mathcal{E}_1{}^{\check{}\check{}} \cong \mathcal{O}(\sum_{\Pi \in \mathfrak{P}_1} \sum_{\rho \in \Pi} i^\rho \cdot D_\rho) \quad \textit{and} \quad \mathcal{E}_2{}^{\check{}\check{}} \cong \mathcal{O}(\sum_{\Pi \in \mathfrak{P}_2} \sum_{\rho \in \Pi} i^\rho \cdot D_\rho).$$

PROOF. We obtain this extension by partition of the matrix $\check{A}$ via the following diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}^{r-1} & \xrightarrow{A_1} & \bigoplus_{\Pi \in \mathfrak{P}_1} \mathcal{O}(\sum_{\rho \in \Pi} i^\rho \cdot D_\rho) & \xrightarrow{\check{A}_1} & \mathcal{E}_1 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}^{s-2} & \xrightarrow{A} & \bigoplus_{\Pi \in \mathfrak{P}} \mathcal{O}(\sum_{\rho \in \Pi} i^\rho \cdot D_\rho) & \xrightarrow{\check{A}} & \mathcal{E} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}^{s-r-1} & \xrightarrow{A_2} & \bigoplus_{\Pi \in \mathfrak{P}_2} \mathcal{O}(\sum_{\rho \in \Pi} i^\rho \cdot D_\rho) & \xrightarrow{\check{A}_2} & \mathcal{E}_2 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where A_1 is represented by the submatrix of A consisting of the rows corresponding to $\mathfrak{P}_1$. □

Corollary 8.11: *Let $\mathcal{E}$ and $\mathfrak{P}$ be as above and let $\mathcal{F} \subset \mathcal{E}$ be any torsion free equivariant subsheaf of rank 1. Then there exists a subset $\mathfrak{P}' \subset \mathfrak{P}$ such that $\mathcal{F}^{\check{}\check{}} \cong \mathcal{O}(\sum_{\Pi \in \mathfrak{P}'} \sum_{\rho \in \Pi} i^\rho \cdot D_\rho)$.*

Corollary 8.12: *Let $s = 2t$ be even, let $\check{A}$ represent a properly semistable point in $(\mathbb{P}_1)^{ss}$ and let $0 \longrightarrow \mathcal{E}_1 \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}_2 \longrightarrow 0$ be the corresponding extension. Then the image of A in $\mathcal{M}_{s,s-2}$ represents all matrices whose corresponding extensions are in $\mathrm{Ext}^1(\mathcal{E}_2, \mathcal{E}_1)$ or $\mathrm{Ext}^1(\mathcal{E}_1, \mathcal{E}_2)$, i.e. $\mathcal{E}$ is GIT-equivalent to the direct sum $\mathcal{E}_1 \oplus \mathcal{E}_2$.*

PROOF. Each orbit in $(\mathbb{P}_1)^s$ which contains a point $(p_1, \dots, p_s)$ such that some t points $p_{i_1}, \dots p_{i_t}$ coincide contains in its closure the points of the form $(p_{i_1}, \dots, p_{i_t}, \dots p, \dots p)$ for some $p_{i_1} \neq p \in \mathbb{P}_1$. □

In the generic case, we have in particular:

Corollary 8.13: *Let $n = \#\Delta(1)$ be even and $i^\rho > 0$ for all $\rho \in \Delta(1)$, let $\mathfrak{P} = \{\{\rho\}\}_{\rho\in\Delta(1)}$ be the fine partition of $\Delta(1)$, and let $n = \#\mathfrak{P}$. Then there exists precisely one point in $\mathcal{M}_{n,n-2}$ which can be represented by a direct sum $\mathcal{E}_1 \oplus \mathcal{E}_2$ such that $\mathcal{E}_1$ and $\mathcal{E}_2$ are locally free.*

Proof. There exists precisely one partition $\Pi_1 \dot\cup \Pi_2 = \Delta(1)$ such that the Π_i do not contain two adjacent rays and which is given by $\Pi_1 = \{\{\rho_{2\cdot i}\} | 1 \leq i \leq \frac{n}{2}\}$. □

The observation made in Proposition 8.10 motivates the following definition:

Definition 8.14: Let I and $\mathfrak{P}$ as before and let $\mathcal{E}$ be a torsion free equivariant sheaf of rank 2 over X such that $\mathfrak{P}$ is a refinement of the coarse partition $\mathfrak{P}_{\mathcal{E}^{\check{}\check{}}}$ associated to the locally free sheaf $\mathcal{E}^{\check{}\check{}}$. Let $\mathcal{F} \subset \mathcal{E}$ be a torsion free equivariant subsheaf of rank 1. Then by 8.11 $\mathcal{F}^{\check{}\check{}} \cong \mathcal{O}(\sum_{\Pi\in\mathfrak{P}'} \sum_{\rho\in\Pi} i^\rho \cdot D_\rho)$ with a unique subset $\mathfrak{P}' \subset \mathfrak{P}$. We say that $\mathcal{E}$ is $\mathfrak{P}$-*stable* (respectively $\mathfrak{P}$-*semistable*) if for every equivariant torsion free subsheaf $\mathcal{F} \subset \mathcal{E}$ of rank 1 $\#\mathfrak{P}' < \frac{1}{2}\#\mathfrak{P}$ (respectively $\#\mathfrak{P}' \leq \frac{1}{2}\#\mathfrak{P}$).

Theorem 8.15: *Let $i_\rho > 0$ for $\rho \in \Pi \subset \Delta(1)$ and let $\mathfrak{P} = \{\Pi_1, \ldots, \Pi_s\}$ be a partition of Π. Consider short exact sequences*

$$0 \longrightarrow \mathcal{O}^{s-2} \xrightarrow{A} \bigoplus_{i=1}^{s} \mathcal{O}(\sum_{\rho\in\Pi_i} i^\rho \cdot D_\rho) \longrightarrow \mathcal{E} \longrightarrow 0.$$

Then $\mathcal{E}$ is $\mathfrak{P}$-stable (respectively $\mathfrak{P}$-semistable) if and only if A represents a GIT-stable (respectively GIT-semistable) point in $(\mathbb{P}_{s-3})^s$ with respect to the action of GL_{s-2}.

Proof. This follows from the fact that we can represent A by a configuration $(p_1, \ldots, p_s)$ of points in $(\mathbb{P}_1)^s$. Then Definition 8.14 is equivalent to the fact that at most $\frac{s}{2}$ of the points p_i coincide. □

Now we can define an equivalence relation on the set of $\mathfrak{P}$-semistable sheaves as follows:

Definition 8.16: Let $\mathcal{E}$ and $\mathcal{E}'$ be $\mathfrak{P}$-semistable sheaves. Then we say that $\mathcal{E}$ and $\mathcal{E}'$ are $\mathfrak{P}$-equivalent iff one of the following conditions holds:

(i) $\mathcal{E}$ and $\mathcal{E}'$ both are $\mathfrak{P}$-stable and equivariantly isomorphic, $\mathcal{E} \cong \mathcal{E}'$,

(ii) $\mathcal{E}$ and $\mathcal{E}'$ both are $\mathfrak{P}$-semistable and the following holds. Let $\Psi \in \mathfrak{P}_{\mathcal{E}}$ such that $\#\Psi = \frac{s}{2}$. Then either $\Psi \in \mathfrak{P}_{\mathcal{E}'}$ or $\Pi \setminus \Psi \in \mathfrak{P}_{\mathcal{E}'}$.

The last condition implies that if $0 \longrightarrow \mathcal{E}_1 \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}_2 \longrightarrow 0$ is the extension of $\mathcal{E}$ corresponding to Ψ as in Proposition 8.10, then by Corollary 8.12 $\mathcal{E}$ is $\mathfrak{P}$-equivalent to $\mathcal{E}_1 \oplus \mathcal{E}_2$. From this definition follows

Theorem 8.17: *Fix numbers $\{i^\rho \geq 0\}_{\rho\in\Delta(1)}$, let $\Pi = \{\rho \mid i^\rho > 0\} \subset \Delta(1)$ and let $\mathfrak{P} = \{\Pi_1, \ldots, \Pi_s\}$. Then $\mathcal{M}_{s,s-2}$ is the set of $\mathfrak{P}$-equivalence classes of $\mathfrak{P}$-semistable torsion free equivariant sheaves of rank 2 on X.*

Definition 8.18: If $\mathfrak{P}$ is fixed, we denote $\mathcal{M}_{\mathfrak{P}} := \mathcal{M}_{s,s-2}$ and call $\mathcal{M}_{\mathfrak{P}}$ *moduli space of* $\mathfrak{P}$-*equivalence classes.*

Remark 8.19: There is the following result of Klyachko:

Proposition 8.20 ([**Kly90**], Corollary 1.2.5)**:** *Let $\mathcal{E}$ and $\mathcal{E}'$ be two equivariant vector bundles over a smooth complete toric variety. If there exists an arbitrary isomorphism $\mathcal{E} \cong \mathcal{E}'$ of vector bundles, then there is an $m \in M$ such that there is an equivariant isomorphism $\mathcal{E} \cong \mathcal{E} \otimes \mathcal{O}(\chi(m))$, where $\mathcal{O}(\chi(m))$ denotes the structure sheaf endowed with the action by the character $\chi(m)$.*

This means that in our situation, where X is complete, equivariant isomorphism classes and isomorphism classes of vector bundles coincide up to a twist with a character. So after fixing numbers i^ρ, $\rho \in \Delta(1)$, the subspace $\mathcal{M}'_{\mathfrak{P}} \subset \mathcal{M}_{\mathfrak{P}}$ consisting of isomorphism classes of vector bundles even classifies non-equivariant isomorphism classes.

Remark 8.21: We want to point out that our moduli depend only on the *combinatorial* structure of the underlying toric variety, that is, the number of rays $\Delta(1)$ in the fan of X, but not on the concrete realization of the fan Δ inside the lattice N.

Example 8.22: Let $X = \mathbb{P}_2(k)$, then we have to consider the quotient of $\mathbb{PM}_{3,1}$ by the group $G \cong \mathrm{GL}_1 \times T_3 \cong k^* \times (k^*)^3$. This quotient is just a point, i.e. the set of equivariant isomorphism classes of indecomposable equivariant vector bundles of rank 2 on $\mathbb{P}_2(k)$ is *discrete*. This reproduces the original result of Kaneyama ([**Kan75**]).

Example 8.23: Let $a \geq 0$ and $X = \mathbb{F}_a$ a Hirzebruch surface. Assume that the rays $\rho_1, \ldots, \rho_4$ are enumerated clockwise. The set $(\mathbb{P}_1)^{4,s}$ of stable points of $(\mathbb{P}_1)^4$ with respect to the diagonal action of GL_2 is

$$\{(p_1, \ldots, p_4) \subset (\mathbb{P}_1)^4 \mid p_i \neq p_j \text{ for all } i \neq j\},$$

i.e. the set of four-point configurations in $\mathbb{P}_1$ no two points of which coincide. There is an isomorphism $(\mathbb{P}_1)^{4,s} \cong \mathrm{PGL}_2 \times (\mathbb{P}_1 \setminus \{0, 1, \infty\})$ which we choose to be

$$(p_1, p_2, p_3, p_4) \mapsto (g, g.p_4),$$

where $g \in \mathrm{PGL}_2$ is the unique element which moves the points p_1, p_2, p_3 to the positions 0, 1, and ∞, respectively. The inverse map is given by

$$(g, p) \mapsto (g^{-1}0, g^{-1}1, g^{-1}\infty, g^{-1}p).$$

The quotient $\mathbb{PM}^{ss}_{4,2}(\mathrm{GL}_2 \times T_2) // \mathrm{GL}_2 \times T_2$ has a completion by semistable points:

$$(\mathbb{P}_1)^{4,ss} = \{(p_1, p_2, p_3, p_4) | \text{ such that no three points } p_i \text{ coincide}\}$$

In terms of 4×2 matrices this means that each semistable but not stable matrix can be brought into one of six standard forms with at most one zero in a row and two zeros in a column:

$$A_{34} = \begin{pmatrix} * & * \\ * & * \\ 0 & * \\ 0 & * \end{pmatrix}, \quad A_{24} = \begin{pmatrix} * & * \\ 0 & * \\ * & * \\ 0 & * \end{pmatrix}, \quad \ldots \quad, A_{12} = \begin{pmatrix} 0 & * \\ 0 & * \\ * & * \\ * & * \end{pmatrix}.$$

The image of a matrix A_{ij} in $\mathcal{M}_{4,2}$ represents the $\mathfrak{P}$-equivalence class of $\mathcal{E}_1 \oplus \mathcal{E}_2$ where $\mathcal{E}_1\check{}\check{} \cong \mathcal{O}(i^{\rho_i} \cdot D_{\rho_i} + i^{\rho_j} \cdot D_{\rho_j})$ and $\mathcal{E}_2\check{}\check{} \cong \mathcal{O}(i^{\rho_k} \cdot D_{\rho_k} + i^{\rho_l} \cdot D_{\rho_l})$ with $\{i, j, k, l\} = \{0, 1, 2, 3\}$. By our choice of coordinates the matrices of types A_{13} and A_{24} with locally free cokernels are mapped to the point $1 \in \mathbb{P}_1$.

Remark 8.24: Let $s > 4$, let X_{ij}, $1 \leq i \leq s$, $1 \leq j \leq s-2$ be the coordinates of $\mathbb{M}_{s,s-2}$ and let $\mathcal{X} = (X_{ij})$. Then the determinants of the minors $\mathcal{X}^{ij}$ of $\mathcal{X}$ describe $\frac{1}{2}s(s-1)$ hypersurfaces $T^{ij}_{\mathfrak{P}}$ in $\mathbb{M}_{s,s-2}$. Via the duality of Section 8.3, the minors $\mathcal{X}^{ij}$ describe precisely the configurations of points $(p_1, \ldots, p_s)$ in $\mathbb{P}_1$, where the points p_i and p_j coincide. Denote $\mathcal{T}^{ij}_{\mathfrak{P}}$ the image of $T^{ij}_{\mathfrak{P}}$ in $\mathcal{M}_{\mathfrak{P}}$. $\mathcal{M}_{\mathfrak{P}}$ is a good quotient of $\mathbb{P}\mathbb{M}_{s,s-2}$, and thus $\mathcal{T}^{ij}_{\mathfrak{P}}$ is a closed subset of $\mathcal{M}_{\mathfrak{P}}$. Moreover, because $s > 4$, by Proposition 8.7 we see that each $\mathcal{T}^{ij}_{\mathfrak{P}}$ contains a dense subset whose preimage in $\mathbb{P}\mathbb{M}_{s,s-2}$ consists of stable points. Thus, the $\mathcal{T}^{ij}_{\mathfrak{P}}$ describe $\frac{1}{2}s(s-1)$ hypersurfaces of $\mathcal{M}_{\mathfrak{P}}$ and the locus in $\mathcal{M}_{\mathfrak{P}}$ consisting of torsion free sheaves which are not locally free is described by the s divisors $\mathcal{T}^{s,1}$ and $\mathcal{T}^{i,i+1}$ for $1 \leq i \leq s-1$.

Example 8.25: Let X be a toric surface which has six rays. The space $\mathcal{M}_{2,6}$ has been calculated in [**Dol94**] and is a cubic hypersurface in $\mathbb{P}_4$ defined by the equation $X_1X_2X_4 - X_3X_0X_4 + X_3X_1X_2 + X_3X_0X_1 + X_3X_0X_2 - X_3X_0^2 = 0$. This hypersurface has ten nodes representing precisely the ten $\mathfrak{P}$-equivalence classes of $\mathfrak{P}$-semistable but not $\mathfrak{P}$-stable sheaves.

Remark 8.26: We do not show that the spaces $\mathcal{M}_{\mathfrak{P}}$ are moduli spaces of suitably defined $\mathfrak{P}$-families, e.g. in the sense of [**New78**]. A detailed treatment of this problem would require to generalize all our constructions to families. A more detailed study of the spaces $\mathcal{M}_{\mathfrak{P}}$ is beyond the scope of this work. For a study of these spaces, see for instance [**KLW87**] and [**Kly94**].

APPENDIX A

Perspectives: Combinatorial Invariants

In this appendix we want to give an outlook on some topics concerning combinatorial invariants for equivariant bundles over toric varieties. In particular, we want to give examples where equivariant Chern classes take the role of such invariants, and we want to consider equivariant vector bundles over $\mathbb{P}_2$ in some more detail. Our aim is to both give a survey of the relevant topics and to document some ideas which we could not follow in the main body of this work. Therefore our treatment will be rather sketchy and we will not present any deep proofs.

In section A.1 we introduce equivariant cohomology for toric varieties and equivariant Chern classes for equivariant bundles. Basic references for this are [**Bri98**], [**Bri97**] and [**BDP90**]. In section A.2 we show that for the case of equivariant vector bundles of rank two over toric surfaces, their equivariant Chern classes indeed play the role of combinatorial invariants. In section A.3 we give a short sketch on two applications of the theory of equivariant bundles over $\mathbb{P}_2$. The first is the work of Klyachko on Horn's conjecture ([**Kly98**]), from which we extract a criterion for Mumford-Takemoto stability for equivariant bundles, and the second is a recent construction of Penacchio ([**Pen02**]), relating the theory of equivariant bundles over $\mathbb{P}_2$ to that of mixed Hodge structures. We want to point out that even for the case of $\mathbb{P}_2$ the theory of equivariant bundles leads to nontrivial applications. In section A.4 we present a combinatorial classification for configurations of three complete flags, which is derived from the so-called Gelfand-Serganova stratification of flag varieties ([**GS87**]). We will exemplify this stratification for the case of equivariant vector bundles of rank three over $\mathbb{P}_2$ whose filtrations form complete flags in appendix B.

In order to be compatible with existing literature, in this chapter we fix our base field to be $\mathbb{C}$.

A.1. Equivariant Cohomology

For every topological group G there exists a universal classifiying bundle E_G over an universal classifying space B_G:

$$E_G \longrightarrow B_G$$

which is a G-principal bundle over B_G (see [**Hus75**]). This space has the property that for any space X and any G-principal bundle P over X there exists a (continuous, differentiable, ...) map $\phi : X \longrightarrow B_G$ such that $\phi^* E_G \cong P$. In the case where $G = (\mathbb{C}^*)^n$ is a complex algebraic torus, we have:

$$E_G = (\mathbb{C}^\infty \setminus \{0\})^n \longrightarrow (\mathbb{P}^\infty_{\mathbb{C}})^n$$

where we can consider $\mathbb{C}^\infty$ as limit of $\mathbb{C}^n$ for $n \to \infty$.

To every G-space X we can associate the *equivariant* cohomology ring of X with respect to G by setting:

$$H_G^*(X) := H^*(X \times_G E_G)$$

where

$$X \times_G E_G = (X \times E_G)/G$$

where G acts diagonally on $X \times E_G$. Here, we consider the cohomology with coefficients in $\mathbb{Z}$.

Because B_G in general is infinite dimensional, the ring $H_G^*(X)$ in general is not artinian and thus has infinitely many degrees $i \geq 0$ such that $H_G^i(X) \neq 0$. For example, if $X = \{\mathrm{pt}\}$ and $G = (\mathbb{C}^*)^n =: T$, we have

$$H_T^*(\mathrm{pt}) = H^*\big((\mathbb{P}_{\mathbb{C}}^\infty)^n\big) \cong \mathbb{Z}[x_1, \ldots, x_n]$$

the polynomial ring in n variables over $\mathbb{Z}$. In the case where X is a toric manifold which is defined by some fan Δ, there is a nice description of $H_T^*(X)$. Namely, consider the ring $\mathbb{Z}[x_\rho \mid \rho \in \Delta(1)]$ and let $SR(\Delta) \subset \mathbb{Z}[x_\rho \mid \rho \in \Delta(1)]$ be the so-called Stanley-Reisner ideal, which is defined as follows:

$$SR(\Delta) = \langle \prod_{i=1}^{s} x_{\rho_i} \mid \text{ where } \{\rho_1, \ldots, \rho_s\} \neq \sigma(1) \text{ for all } \sigma \in \Delta \rangle.$$

and the equivariant cohomology ring then is:

$$H_T^*(X) = \mathbb{Z}[x_\rho \mid \rho \in \Delta(1)]/SR(\Delta).$$

For example, if X is a toric surface, then $SR(\Delta)$ is generated by all monomials $x_\rho x_{\rho'}$ where $\{\rho, \rho'\} \neq \sigma(1)$ for all $\sigma \in \Delta(2)$.

Let E be an equivariant vector bundle over X, then the map

$$E \times_G E_G \longrightarrow X \times_G E_G$$

defines a vector bundle over $X \times_G E_G$, and one can define the *equivariant Chern classes* of E in $H_G^*(X)$ by

$$c_i^G(E) := c_i(E \times_G E_G) \in H_G^{2i}(X).$$

One can check that the equivariant Chern classes have the same functorial properties as the usual Chern classes, such as multiplicativity of c_t and splitting principle. But note that $c_i^G(E) = 0$ for $i > 2\operatorname{rk} E$, but, because $H_G^i(X)$ in general might be nonzero for degrees bigger than $\dim X$, $c_i^G(E)$ can be nonzero for $2\operatorname{rk} E \geq i > 2\dim X$.

Note that for X a toric variety and $E = \mathcal{O}(D)$ an equivariant line bundle with $D = \sum_{\rho \in \Delta(1)} i^\rho D_\rho$:

$$c_1^T(E) = \sum_{\rho \in \Delta(1)} i^\rho \cdot x_\rho.$$

A.2. Equivariant Cohomology and Equivariant Vector Bundles of Rank Two

Let X be a smooth compact toric surface and $\mathcal{E}$ an equivariant vector bundle of rank two over X. Consider a subset $\Pi \subset \Delta(1)$ and a partition $\mathfrak{P}_{\mathcal{E}} = \{\Pi_1, \dots, \Pi_s\}$ of Π, the coarse partition with respect to $\mathcal{E}$. Then we have a resolution

$$0 \longrightarrow \mathcal{O}^{s-2} \xrightarrow{A} \bigoplus_{i=1}^{s} \mathcal{O}(\sum_{\rho \in \Pi_i} i^\rho \cdot D_\rho) \longrightarrow \mathcal{E} \longrightarrow 0$$

and thus the equivariant Chern classes of $\mathcal{E}$ are:

$$\begin{aligned} c_0^T &= 1 \\ c_1^T &= \sum_{\rho \in \Pi} i^\rho \cdot x_\rho \\ c_2^T &= \sum_{\substack{\sigma \in \delta(2) \\ \sigma(1) = \{\rho, \rho'\}}} i^\rho i^{\rho'} \cdot x_\rho x_{\rho'}. \end{aligned}$$

This shows that we can reconstruct Π and the partition $\mathfrak{P}$ from the equivariant Chern classes by setting:

$$\Pi := \{\rho \mid i^\rho \neq 0\}$$

and

$$\delta(2) := \{\sigma \in \Delta(2) \mid i^\rho i^{\rho'} \cdot x_\rho x_{\rho'} \text{ is a summand in } c_2^T(\mathcal{E}) \text{ and } \sigma(1) = \{\rho, \rho'\}\}.$$

So the moduli spaces $\mathcal{M}_{\mathfrak{P}}$ can as well be written as $\mathcal{M}(2, c_T^1, c_T^2)$ and thus:

Theorem: *Let X be a compact toric surface and let $\mathcal{E}$ an equivariant vector bundle of rank two over X. Then the equivariant Chern classes $c_i^T(\mathcal{E})$ are combinatorial invariants of $\mathcal{E}$.*

A.3. Equivariant Vector Bundles on $\mathbb{P}_2$

Any three partial flags $F^0_\cdot, F^1_\cdot, F^2_\cdot$ in some vector space V can be made into a filtrations of V by assigning to every F_j^k an integer i_j^k such that $F_j^k \subset F_l^k$ implies $i_j^k < i_l^k$. So every such set of filtrations gives rise to some equivariant vector bundle over $\mathbb{P}_2$. This way in principle every question about three filtrations can be reformulated to a question about equivariant vector bundles over $\mathbb{P}_2$.

A.3.1. Mumford-Takemoto Stability. A by now famous example is Klyachko's proof of the Horn conjecture (see [**Kly98**], [**Ful98a**]). We do not review this proof, but want to take a look at the notion of stability which was involved. We say a bundle $\mathcal{E}$ is Mumford-Takemoto (semi-)stable if, for any proper subsheaf $\mathcal{F} \subset \mathcal{E}$ the following inequality holds:

$$\frac{c_1(\mathcal{F})}{\operatorname{rk} \mathcal{F}} < \frac{c_1(\mathcal{E})}{\operatorname{rk} \mathcal{E}} \quad \text{(respectively } \leq).$$

The quantity $\frac{c_1(\mathcal{E})}{\operatorname{rk} \mathcal{E}}$ is called the *slope* of $\mathcal{E}$. Let $\mathcal{E}$ be an equivariant bundle given by filtrations $E^k(i)$, $k = 0, 1, 2$, of some vector space $\mathbf{E}^0$, and denote $F^k_\cdot$ the partial flags of vector spaces underlying the $E^k(i)$. Let $\mathbb{F}_k$ be the partial flag manifolds of flags of the same type as $F^k_\cdot$ and denote ω^k certain dominant weights of $\mathrm{SL}(\mathbb{E}^0)$ (see [**Kly98**], section 3.3 for explanation). Then there is the following observation:

Theorem ([**Kly98**], observation 3.3.1)**:** *An equivariant vector bundle over* $\mathbb{P}_2$ *given by filtrations* $E^k(i)$*,* $k = 0, 1, 2$*, is (semi-)stable if and only if the corresponding triple of flags* $(F^0_\cdot, F^1_\cdot, F^2_\cdot) \subset \mathbb{F}_0 \times \mathbb{F}_1 \times \mathbb{F}_2$ *is GIT-stable with respect to the diagonal action of* $\mathrm{SL}(\mathbf{E}^0)$ *and a polarization*

$$\mathcal{L}(\omega^0) \boxtimes \mathcal{L}(\omega^1) \boxtimes \mathcal{L}(\omega^2)$$

where the $\mathcal{L}(\omega^k)$ *are the line bundles over* $\mathbb{F}_k$ *corresponding to the weights* ω^k*.*

If $\mathcal{E}$ has rank two and is indecomposable, then it is $\mathfrak{P}$-stable with respect to the partition $\mathfrak{P} = \{\{\rho_0\}, \{\rho_1\}, \{\rho_2\}\}$ of $\Delta(1) = \{\rho_0, \rho_1, \rho_2\}$, and $\mathbb{F}_k \cong \mathbb{P}_1$ for all k. The quotient of $\mathbb{P}_1 \times \mathbb{P}_1 \times \mathbb{P}_1$ by $\mathrm{SL}(\mathbf{E}^0) = \mathrm{SL}_2$ (which is the same as the quotient by GL_2) is a point and GIT-stability does not depend on any twists $\mathcal{L}(\omega^k)$. Thus we have with the notions from the previous chapter:

Corollary A.1: *Any indecomposable equivariant vector bundle of rank two over* $\mathbb{P}_2$ *is Mumford-Takemoto stable.*

We have not checked if this pattern persists for rank two bundles over general toric surfaces.

A.3.2. Mixed Hodge Structures and Equivariant Bundles. Let $H_{\mathbb{Z}}$ be a free $\mathbb{Z}$-module, $H_{\mathbb{Q}} := H_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}$, and $H := H_{\mathbb{C}} = H_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C}$. Fix an nonnegative integer n and for all nonnegative integers p, q such that $p + q = n$ pick a complex linear subspace $H^{p,q} \subset H$. A *pure Hodge structure of weight* n consists of the data $\{H_{\mathbb{Z}}, H^{p,q}\}$ such that the following conditions are satisfied.

$$H = \bigoplus_{p+q=n} H^{p,q}, \quad H^{p,q} = \overline{H^{q,p}}.$$

For every pure Hodge structure one can define decreasing filtrations, the *Hodge filtrations*:

$$F^p = H^{p,0} \oplus H^{p-1,1} \oplus \cdots \oplus H^{p,n-p}, \quad F^{n+1} = 0$$

These filtrations satisfy:

$$H = F^p \oplus \overline{F^{n-p+1}}, \quad , H^{p,q} = F^p \cap \overline{F^q}.$$

A *mixed Hodge structure* is defined by the following data:

(i) an increasing filtration $W_\cdot$ of H, called *weight filtration*,
(ii) a decreasing filtration $F^\cdot$ of H which induces a pure Hodge structure of weight n on $\mathrm{Gr}^W_n H = W_n / W_{n-1}$.

The filtration $F^\cdot \mathrm{Gr}^W_n H$ on $\mathrm{Gr}^W_n H$ induced by $F^\cdot$ is:

$$F^p \mathrm{Gr}^W_n H = (F^p \cap W_n + W_{n-1}) / W_{n-1}.$$

Note that we differ from the usual definition by that we consider W not as a complexification of a filtration of $H_{\mathbb{Q}}$ (see [**Pen02**] for details). One can define a category of mixed Hodge structures by introducing an appropriate notion of morphisms, which we call $\mathrm{Cat}_{\mathbb{C}-mhs}$.

Consider a fixed embedding $\mathbb{P}_1 \hookrightarrow \mathbb{P}_2$. A bundle $\mathcal{E}$ on $\mathbb{P}_2$ is called $\mathbb{P}_1$-(semi-) stable if for every subbundle $\mathcal{F} \subset \mathcal{E}|_{\mathbb{P}_1}$ over $\mathbb{P}_1$:

$$\frac{c_1(\mathcal{F})}{\operatorname{rk}\mathcal{F}} < \frac{c_1(\mathcal{E}|_{\mathbb{P}_1})}{\operatorname{rk}\mathcal{F}|_{\mathbb{P}_1}} \quad \text{(respectively } \leq).$$

We denote the slope $\frac{c_1(\mathcal{E}|_{\mathbb{P}_1})}{\operatorname{rk}\mathcal{E}|_{\mathbb{P}_1}}$ by $\mu(\mathcal{E})$. It can be shown that $\mathbb{P}_1$-(semi-)stability implies Mumford-Takemoto stability.

To every mixed Hodge structure $(W^{\cdot}, F., H)$, one associate an equivariant vector bundle by associating the filtrations $F.$, $\overline{F}.$ and the opposite filtration, defined by $W_{-p} := W^p$ to the rays of $\mathbb{P}_2$ (in any order). In [**Pen02**], the following theorem is proven:

Theorem A.2 ([**Pen02**], theorem 5)**:** *There is an equivalence of categories*

$$\mathrm{Cat}_{\mathbb{C}-mhs} \leftrightarrow \mathbb{P}_1 - \textit{semistable vector bundles over } \mathbb{P}_2 \textit{ with } \mu = 0.$$

In [**Pen02**] there are proven some more equivalences, depending if W is defined as filtration of $H_{\mathbb{Z}}$, $H_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{R}$ or H.

The special interest with this correspondence is the fact that this way the notion of *variation of mixed Hodge structures* is tied to the theory of moduli for vector bundles over $\mathbb{P}_2$. In particular, one would like to determine *limits of mixed Hodge structures* over 0 with respect to families over the pointed disc in terms of limit vector bundles.

A.4. Gelfand-Serganova Stratification and Equivariant Vector Bundles on $\mathbb{P}_2$

Consider some vector space $V \cong \mathbb{C}^r$ together with its standard basis $e_1, \ldots, e_r$. Then we call *standard flag* the flag of vector spaces

$$F^1_{\cdot} : \langle e_1 \rangle \subset \langle e_1, e_2 \rangle \subset \cdots \subset \langle e_1, \ldots, e_r \rangle = V,$$

where $\langle . \rangle$ denotes the span over $\mathbb{C}$. Let $\mathbb{F}_r$ the flag variety of complete flags in V, then there exists a cell decomposition (see [**Ful97**])

$$\mathbf{F}_r = \bigcup_{w \in S_r} X_w^{\circ}$$

where S_r is the rth symmetric group, $X_w^{\circ} \cong \mathbb{C}^{\mathrm{length}(w)}$. For every cell

$$X_w^{\circ} = \{E. \in \mathbb{F}_r \mid \dim(E_p \cap F_q) = \#\{i \leq p \mid w(i) \leq q\} \text{ for } 1 \leq p, q \leq r\}.$$

In particular, X_w° contains as a distinguished point the permuted flag

$$F^w_{\cdot} : \langle e_{w(1)} \rangle \subset \langle e_{w(1)}, e_{w(2)} \rangle \subset \cdots \subset \langle e_{w(1)}, \ldots, e_{w(r)} \rangle = V$$

and it can be shown that every cell X_w° is the orbit of the left action of the group B of upper triangular matrices on $\mathbb{F}_r$. It is easy to see that the flags $F^w_{\cdot}$ are precisely the fixed points of the action of the torus $T = (\mathbb{C}^*)^r$ on $\mathbb{F}_r$.

The above cell decomposition gives an answer to the following question. Consider two complete flags $E^1_\cdot$ and $E^2_\cdot$ in V, then what are the possible configurations two flags can have with respect to each other? The answer is that one can always choose a basis $e_1, \dots, e_r$ such that $E^1_\cdot = F^1_\cdot$ and $E^2_\cdot = F^w_\cdot$ for some $w \in S_r$.

Now, what about three flags $E^1_\cdot$, $E^2_\cdot$, $E^3_\cdot$? The combinatorial characterization can be derived from the Gelfand-Serganova classification (see [**GM82**], [**GGMS87**], [**GS87**], and for a recent treatment [**Laf02**]) of torus strata in flag varieties. To explain this, we first give a classification of torus strata in Grassmannians $\mathrm{Gr}(k, r)$. For this we introduce the following notion:

Definition A.3: $\Sigma^{k,r} = \{\underline{n} \in \mathbb{Z}^r \mid n_i \in \{0,1\}$ and $n_i \neq 0$ k times$\}$. $\Sigma^{k,r}$ is called *hypersimplex*.

Definition A.4: Let $\dim V = r$ and $V = \sum_{i=1}^r V_i$ with $\dim V_i = 1$ for all i. Let $W \subset V$ be a k-dimensional subvector space. Then we define the polyhedron associated to W:

$$P(W) := \{\underline{n} \in \Sigma^{k,r} \mid \sum_{i \in I} n_i \geq \dim W \cap \oplus_{i \in I} V_i \text{ for all } I \subset \{1 \dots, r\}\}.$$

Example A.5: The first example of a hypersimplex which is not a simplex is $\Sigma^{2,4}$. This hypersimplex consists of the six points $(1,1,0,0)$, $(1,0,1,0)$, $(1,0,0,1)$, $(0,1,1,0)$, $(0,1,0,1)$, $(0,0,1,1)$. The possible types of polyhedra $P(W)$ up to symmetry of $\Sigma^{2.4}$ are shown in figure A.1.

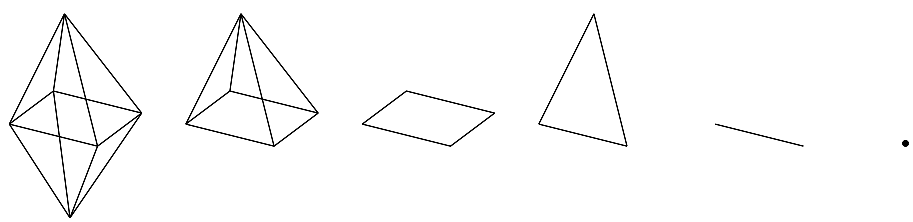


FIGURE A.1. Subpolyhedra of $\Sigma^{2,4}$

Definition A.6: We say that *two points* $W_1, W_2 \in \mathrm{Gr}(k, r)$ *are in the same stratum* if $P(W_1) = P(W_2)$.

Remark A.7: These strata are also called *thin Schubert cells.*

The resulting stratification of $\mathrm{Gr}(k, r)$ can be rephrased in the language of matroid theory. A *matroid* of rank k is a function

$$d : 2^{\{1,\dots,r\}} \longrightarrow \mathbb{N}$$

such that

$$\begin{aligned} d(\emptyset) &= 0 \\ d(\{1, \dots, r\}) &= k \\ d(I) + d(J) &\leq d(I \cap J) + d(I \cup J) \quad \text{for all } I, J \subset \{1, \dots, r\}. \end{aligned}$$

For more information about matroids we refer to [**Wel76**] and [**Aig79**]. With notation as in definition A.4, the function

$$d_W : 2^{\{1,\dots,r\}} \longrightarrow \mathbb{N}, \quad I \mapsto \dim W \cap \bigoplus_{i\in I} V_i$$

defines a matroid, and we obtain a stratification of $\mathrm{Gr}(k,r)$ such that two points $W_1, W_2 \in \mathrm{Gr}(k,r)$ are in the same stratum if $d_{W_1} = d_{W_2}$.

Now the formalism can be applied to the classification of torus strata in $\mathrm{Gr}(k,r)$. The action of the torus T which is diagonal with respect to a basis $e_1, \dots, e_r$ of V on $\mathrm{Gr}(k,r)$ has precisely $\binom{r}{k}$ fixed points, namely these subvector spaces of V which are spanned by k-subsets of the basis $e_1, \dots, e_r$. One can see that these fix points correspond naturally with the vertices of the hypersimplex $\Sigma^{k,r}$.

Definition A.8: Let $W_1, W_2 \in \mathrm{Gr}(k,r)$ and consider the orbits $T.W_1$ and $T.W_2$. We say that these two orbits are of the same type if the orbit closures $\overline{T.W_1}$ and $\overline{T.W_2}$ contain the same set of fixed points.

The type of torus orbits indeed is compatible with the above defined stratification:

Theorem A.9 ([**GGMS87**], Theorem 2.4)**:** *$W_1, W_2 \in \mathrm{Gr}(k,r)$ are in the same stratum if and only if the two torus orbits $T.W_1$ and $T.W_2$ are of the same type.*

We have reformulated this theorem a bit in order to avoid a more elaborate setting. The cited paper employs the moment map of the T-action on $\mathrm{Gr}(k,r)$.

Via the canonical embedding $\mathbb{F}_r \hookrightarrow \mathrm{Gr}(1,r) \times \mathrm{Gr}(2,r) \times \cdots \times \mathrm{Gr}(r-1,r)$ the torus stratifications can be generalized to the case of flag manifolds. Every flag $F_\cdot$ can be classified by the thin Schubert cells in the grassmannians $\mathrm{Gr}(k,r)$ in which the vector F_k lies.

Definition A.10: Let $d_1, \dots, d_{r-1}$ be matroids on the set $\{1, \dots, r\}$ such that d_j has rank k for every $k = 1, \dots, r-1$. The series $d_1, \dots, d_{r-1}$ is *concordant* if there exists a flag $F_\cdot$ in V such that

$$d_k(I) = \dim F_k \cap \bigoplus_{i\in I} V_i$$

with respect to some direct sum decomposition $V = \bigoplus_{i=1}^r V_i$ into one-dimensional vector spaces.

Definition A.11: Let $F_\cdot^1$, $F_\cdot^2$ be two points in $\mathbb{F}_r$. Then $F_\cdot^1$ and $F_\cdot^2$ are *in the same stratum* if $\overline{T.F_\cdot^1}$ and $\overline{T.F_\cdot^2}$ have the same fixed points.

These strata are what we call *Gelfand-Serganova stratification*, or simply thin Schubert cells.

Theorem A.12 ([**GS87**], §5, §6.2 Theorem 1 and §9.3)**:** *Two flags $F_\cdot^1$, $F_\cdot^2$ are in the same stratum if and only if their concordant flags of matroids coincide.*

For a concordant flag of matroids one can also consider the set of polytopes $P(F_1)$, $\dots, P(r-1)$ in the hypersimplices $\Sigma^{1,r}, \dots, \Sigma^{r-1,r}$. and the *sum polytope* $P(F_\cdot) := P(F_1) + \cdots + P(F_{r-1})$. The sum polytope is a convex subset of the sum of hypersimplices

$\Sigma^{\cdot,r} := \Sigma^{1,r} + \cdots + \Sigma^{r-1,r}$. This object is called *permutohedron*, a convex polytope which is contained in the hyperplane $H^t = \{\underline{n} = (n_1, \ldots, n_r) \in \mathbb{N}^r \mid \sum_{i=1}^r n_i = t\}$, where $t = \sum_{i=1}^{r-1} i$, whose set of vertices is precisely given by the points $\{(w(1)-1, \ldots, w(r)-1) \mid w \in S_r\}$. The set of vertices of $P(F_\cdot)$ is a subset of the vertices of $\Sigma^{\cdot,r}$. Figure A.2 shows the permuhedron for S_3 and the possible typs of polyhedra up to symmetry of the permutohedron.

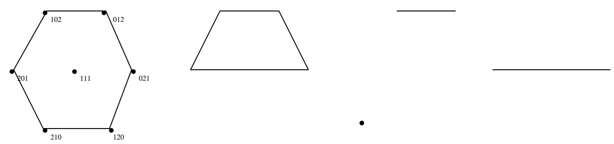


FIGURE A.2. Permutohedron and concordant convex subsets for flags in $\mathbb{C}^3$

Let us explain how this applies to the combinatorial classification of three flags. Consider the two configurations of three complete flags in $\mathbf{C}^3$ in figure A.3. We have drawn the images of these flags in $\mathbb{P}_2$ – the dots denote the one-dimensional spaces, the lines the two-dimensional spaces; two of the flags, denote $F_\cdot^1$ and $F_\cdot^{(321)}$ are supposed to be in general position with each other, that means, after choice of a basis, $F_\cdot^1$ is the standard flag and $F_\cdot^{(321)}$ is the permutation of the standard flag by (321). We

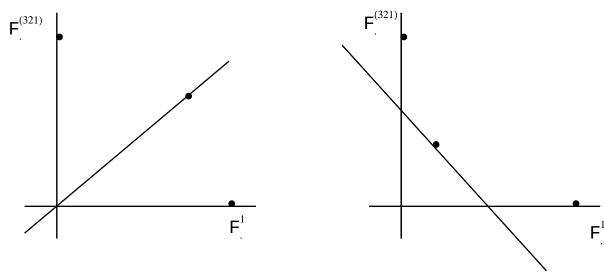


FIGURE A.3. Two configurations of flags in $\mathbb{C}^3$

can see that the third flag is in general position with respect to each of the other two flags, but nevertheless combinatorially it is clear that one would like to consider the two configurations as different, as in the first example the third flag meets the intersection of the other two flags, whereas in the second example it does not.

By using thin Schubert cells, we can distinguish these configurations. Note that, because the flags $F_\cdot^1$ and $F_\cdot^{(321)}$ are opposite to each other, the maximal subgroup of $\mathrm{GL}(V)$ which stabilizes both flags is an algebraic torus T. The action of this torus has three one-dimensional eigenspaces, $V_1 := F_1^1$, $V_2 := F^{(321)}$, and $V_3 := F_2^1 \cap F_2^{(321)}$, and we choose a basis such that the action of $T \cong (\mathbb{C}^*)^3$ becomes a diagonal action and which respects the decomposition $V = V_1 \oplus V_2 \oplus V_3$.

Denote the third flag $E_\cdot$. For the first configuration the polytopes which correspond to the concordant flags of matroids of $E_\cdot$ with respect to the stratification of the action of T, are given by:

$$P(E_1) = \{(1,0,0),(0,1,0),(0,0,1)\}$$
$$P(E_2) = \{(1,1,0),(0,1,1)\}$$
$$P(E.) = \{(2,1,0),(1,2,0),(0,2,1),(0,1,2),(1,1,1)\}$$

These polytopes are shown in figure A.4.

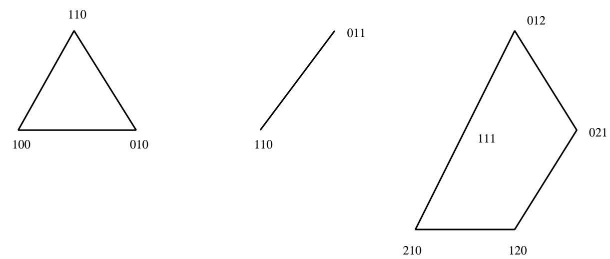


FIGURE A.4. Concordant polytopes for the first configuration in figure A.2

For the second configuration, we have:

$$P(E_1) = \{(1,0,0),(0,1,0),(0,0,1)\}$$
$$P(E_2) = \{(1,1,0),(0,1,1),(1,0,1)\}$$
$$P(E.) = \{(2,1,0),(1,2,0),(0,2,1),(0,1,2),(1,1,1),(2,0,1),(1,0,2)\}$$

These polytopes are shown in figure A.5.

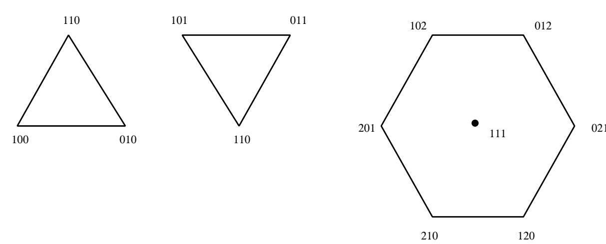


FIGURE A.5

One can already guess the general principle. The size of a polyhedra $P(E_1)$, $P(E_2)$, and $P(E.)$ measures how far the flag $E.$ sits in general position. If all spaces E_i are spanned by eigenspaces of T, the polyhedra are just points.

What if the second flag is different from $F^{(321)}$, say F^w for some $w \in S_r$? In that case, the subgroup B^w of $\mathrm{GL}(V)$ which stabilizes the two flags is of the form $wBw^{-1} \cap B$, where B is the Borel subgroup of $\mathrm{GL}(V)$ of upper triangular, nonsingular matrices and w is considered as permutation matrix. This group can explicitly be described as

$$B^w = \left\{ (b_{ij}) \mid \begin{array}{c} b_{ij} = 0 \text{ if } j < i \text{ or if} \\ i < j \text{ and } w(i) > w(j) \end{array} \right\}$$

the subgroup of B whose entries b_{ij} are zero if w produces a transposition of i and j. In the special case where $w = (321)$, we have $B^w = T$ as we have seen before, and in general, of course, $T \subset B^w \subset B$. Moreover, we expect that the set of fixed points of B^w is a subset of the set of fixed points of T. By definition, B^w has at least two fixed points, namely $F_\cdot^1$ and $F_\cdot^w$. To describe the other fixed points, we introduce the followingt notion.

Definition A.13 (see [**Man01**] §2.1.2)**:** We define a relation on the set S_r, which is called *weak Bruhat order*. Denote $s_i \in S_r$ the simple transposition $(i, i+1) \mapsto (i+1, i)$. Then we say that $v \in S_r$ *precedes* $w \in S_r$ if $\text{length}(w) = \text{length}(v) + 1$ and there exists a transposition s_i such that $w = vs_i$. We write $v \preceq w$ if v and w are connected by a chain of permutations where each element precedes the next.

It is well known that every permutation w in S_r can be (non-uniquely) represented by an irredundant sequence of transpositions $s_{i_1} \cdots s_{i_m}$, $m = \text{length}(w)$, called a *reduced decomposition* of w. Then $v \preceq w$ if and only if there exists a reduced decomposition $s_{i_1} \cdots s_{i_m}$ of w such that $s_{i_1} \cdots s_{i_k}$, $k < m$ is a reduced decomposition of v.

Lemma A.14: *The set of fixed points of B^w is given by the flags $F_\cdot^v$ where $v \preceq w$ in the weak Bruhat order.*

Sketch of proof. We do induction the length of w. If $w = 1$, then $B^1 = B$ and the only fixed point is the standard flag $F_\cdot^1$. For and reduced expression $s_{i_1} \cdots s_{i_m}$ of w we have by induction that the fixed points of $v = s_{i_1} \cdots s_{i_{m-1}}$ are precisely the flags $F_\cdot^u$ where $u \preceq v$. Now $w = vs_{i_m}$ and B^w is a subgroup of B^v, hence the set of fixed points of B^v is contained in the set of fixed points of B^w. Together with the fact that by definition $F_\cdot^w$ is a fixed point of B^w, we have then that the set $F_\cdot^u$, where $u \preceq w$, is contained in the fixed point set of B^w.

On the other hand, if $v \npreceq w$, then there exist $i < j$ such that $w(i) < w(j)$ and $v(i) > v(j)$. But there exists a matrix $b \in B^w$ such that $b_{ij} \neq 0$, and thus can not stabilize $F_\cdot^v$. □

Now for any $w \in S_r$ we can define strata in $\mathbb{F}_r$ with respect to the action of B^w by saying that two flags $E_\cdot^1, E_\cdot^2$ are in the same stratum if the B^w- fixed points of $\overline{B^w.E_\cdot^1}$ and $\overline{B^w.E_\cdot^1}$ coincide. For any $E_\cdot \in \mathbb{F}^r$ we can define a polyhedron $P^w(E_\cdot)$ as subpolyhedron of $\Sigma^{\cdot,r}$ as follows. Let $P(E_\cdot)$ be the polyhedron as defined above. Then we remove all vertices $(v(1)-1, \ldots, v(r)-1)$ where $v \npreceq w$ and take $P^w(E_\cdot)$ as the convex hull of the remaining vertices.

We will give examples in appendix B.

APPENDIX B

Resolutions for Equivariant Vector Bundles of Rank 3 on $\mathbb{P}_2$

In this appendix we want to apply the classification of strata from appendix A for the case of vector bundles of rank three on $\mathbb{P}_2$ defined by three complete filtrations. Denote these filtrations $E^0(i)$, $E^1(i)$, $E^2(i)$, then every filtration $E^k(i)$ is equivalent to a tuple $(-i_1^k, -i_2^k, E_\cdot^k)$, where $i_2^k < i_1^k$ and, $E_\cdot^k = E_1^k \subset E_2^k \subset \mathbf{E}^0$ is a complete flag. The filtrations then are given as:

$$E^k(i) = \begin{cases} 0 & i < -i_1^k \\ E_1^k & -i_1^k \le i < -i_2^k \\ E_2^k & -i_2^k \le i < 0 \\ \mathbf{E}^0 & 0 \le i. \end{cases}$$

Here we assume that by twisting with a suitable line bundle the filtrations are shifted to these standard positions, instead of introducing a third integer for every filtration.

Below we will give a complete list of all such vector bundles together with their combinatorial characterization using the Gelfand-Serganova stratification, their global minimal resolutions and their equivariant Chern polynomials c_t^T.

In the data below we will describe the combinatorial data as follows. We assume that the first flag is the standard flag $F_\cdot^1$, and the second flag is $F_\cdot^w$ for some $w \in S_3$. Every $w \in S_3$ has its own subsection. For every pair of flags $F_\cdot^1$, $F_\cdot^w$, we give a list of all combinatorial possibilities the third flag can have according to the decomposition of $\mathbb{F}_3$-stratification by B^w. For this, we specify the two subpolyhedra $P_1 \subset \Sigma^{1,3}$ and $P_2 \subset \Sigma^{2,3}$.

Conjecture The Gelfand-Serganova stratification gives a combinatorial invariant for equivariant vector bundles over $\mathbb{P}_2$.

One can see that for the rank three case, the combinatorial data from the Gelfand-Serganova stratification is slightly finer than the equivariant Chern classes. Consider the example in figure B.1 for the case $w = (321)$. The picture shows the third flag with $P_2 = \{(1,1,0),(0,1,1)\}$ in both cases. In the first case is $P_1 = \{(1,0,0),(0,1,0),(0,0,1)\}$, in the second $P_1 = \{(1,0,0),(0,0,1)\}$. Indeed, in the first case we have a resolution (see section B.0.1, no. 20 below)

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}(i_1^0 D_0) \oplus \mathcal{O}(i_1^1 D_1) \oplus \mathcal{O}(i_1^2 D_2) \oplus \mathcal{O}(i_2^0 D_0 + i_2^1 D_1 + i_2^2 D_2) \longrightarrow \mathcal{E} \longrightarrow 0$$

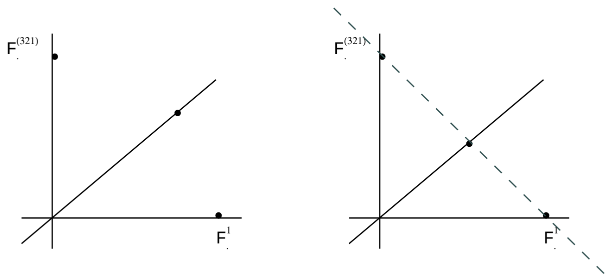


FIGURE B.1. Two configurations of flags in $\mathbb{C}^3$

and in the second case, we see that the three one-dimensional vector spaces are linearly dependent, and the bundle splits:

$$\mathcal{E} \cong \mathcal{E}' \oplus \mathcal{O}(i_2^0 D_0 + i_2^1 D_1 + i_2^2 D_2)$$

and the resolution for $\mathcal{E}'$ is:

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}(i_1^0 D_0) \oplus \mathcal{O}(i_1^1 D_1) \oplus \mathcal{O}(i_1^2 D_2) \longrightarrow \mathcal{E}' \longrightarrow 0,$$

whereas the Chern polynomials are the same:

$$c_t^T(\mathcal{E}) = (1 + (y_0 + y_1 + y_2)t)(1 + x_0 t)(1 + x_1 t)(1 + x_2 t).$$

Abbreviations: $x_k := i_1^k D_k$, $y_k := i_2^k D_k$

B.0.1. 321.

1. $P_1 = \{(1, 0, 0)\}, P_2 = \{(1, 1, 0)\}$

$$\mathcal{E} \cong \mathcal{O}(i_1^0 D_0 + i_1^1 D_1) \oplus \mathcal{O}(i_1^2 D_2) \oplus \mathcal{O}(i_2^0 D_0 + i_2^1 D_1 + i_2^2 D_2)$$

2. $P_1 = \{(1, 0, 0)\}, P_2 = \{(1, 0, 1)\}$

$$\mathcal{E} \cong \mathcal{O}(i_1^0 D_0 + i_1^1 D_1) \oplus \mathcal{O}(i_2^0 D_0 + i_1^2 D_2) \oplus \mathcal{O}(i_2^1 D_1 + i_2^2 D_2)$$

3. $P_1 = \{(1, 0, 0)\}, P_2 = \{(1, 1, 0), (1, 0, 1)\}$

$$0 \longrightarrow \mathcal{O}(i_2^2 D_2) \longrightarrow \begin{matrix} \mathcal{O}(i_1^2 D_2) \oplus \mathcal{O}(i_1^0 D_0 + i_1^1 D_1) \oplus \\ \mathcal{O}(i_2^0 D_0 + i_2^2 D_2) \oplus \mathcal{O}(i_2^1 D_1 + i_2^2 D_2) \end{matrix} \longrightarrow \mathcal{E} \longrightarrow 0$$

$$c_t^T(\mathcal{E}) = (1 + (x_0 + x_1)t)(1 + x_2 t)(1 + (y_0 + y_1 + y_2)t + y_0 y_1 t^2)$$

4. $P_1 = \{(0, 1, 0)\}, P_2 = \{(1, 1, 0)\}$

$$\mathcal{E} \cong \mathcal{O}(i_1^0 D_0 + i_2^1 D_1 + i_2^2 D_2) \oplus \mathcal{O}(i_1^1 D_1 + i_2^0 D_0) \oplus \mathcal{O}(i_1^2 D_2)$$

5. $P_1 = \{(0, 1, 0)\}, P_2 = \{(0, 1, 1)\}$

$$\mathcal{E} \cong \mathcal{O}(i_1^0 D_0 + i_2^1 D_1 + i_2^2 D_2) \oplus \mathcal{O}(i_1^2 D_2 + i_2^0 D_0) \oplus \mathcal{O}(i_1^1 D_1)$$

6. $P_1 = \{(0, 1, 0)\}, P_2 = \{(1, 1, 0), (0, 1, 1)\}$

$$\mathcal{E} \cong \mathcal{E}' \oplus \mathcal{O}(i_1^0 D_0 + i_2^1 D_1 + i_2^2 D_2)$$

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}(i_2^0 D_0) \oplus \mathcal{O}(i_1^1 D_1) \oplus \mathcal{O}(i_1^2 D_2) \longrightarrow \mathcal{E}' \longrightarrow 0$$

$$c_t^T(\mathcal{E}) = (1 + (x_0 + y_1 + y_2)t)(1 + y_0 t)(1 + x_1 t)(1 + y_2 t)$$

7. $P_1 = \{(0,0,1)\}, P_2 = \{(0,1,1)\}$

$$\mathcal{E} \cong \mathcal{O}(i_1^0 D_0 + i_1^2 D_2) \oplus \mathcal{O}(i_1^1 D_1) \oplus \mathcal{O}(i_2^0 D_0 + i_2^1 D_1 + i_2^2 D_2)$$

8. $P_1 = \{(0,0,1)\}, P_2 = \{(1,0,1)\}$

$$\mathcal{E} \cong \mathcal{O}(i_1^0 D_0 + i_1^2 D_2) \oplus \mathcal{O}(i_2^0 D_0 + i_1^1 D_1) \oplus \mathcal{O}(i_2^1 D_1 + i_2^2 D_2)$$

9. $P_1 = \{(0,0,1)\}, P_2 = \{(0,1,1),(1,0,1)\}$

$$0 \longrightarrow \mathcal{O}(i_2^1 D_1) \longrightarrow \begin{array}{c} \mathcal{O}(i_1^1 D_1) \oplus \mathcal{O}(i_1^0 D_0 + i_1^2 D_2) \oplus \\ \mathcal{O}(i_2^0 D_0 + i_2^1 D_1) \oplus \mathcal{O}(i_2^1 D_1 + i_2^2 D_2) \end{array} \longrightarrow \mathcal{E} \longrightarrow 0$$

$$c_t^T(\mathcal{E}) = (1 + x_1 t)(1 + (y_0 + y_1)t)(1 + (x_0 + y_1 + y_2)t + x_0 y_1 t^2)$$

10. $P_1 = \{(1,0,0),(0,1,0)\}, P_2 = \{(1,1,0)\}$

$$\mathcal{E} \cong \mathcal{E}' \oplus \mathcal{O}(i_1^2 D_2)$$

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}(i_2^0 D_0 + i_2^1 D_1 + i_2^2 D_2) \oplus \mathcal{O}(i_1^0 D_0) \oplus \mathcal{O}(i_1^1 D_1) \longrightarrow \mathcal{E}' \longrightarrow 0$$

$$c_t^T(\mathcal{E}) = (1 + x_0 t)(1 + (y_0 + y_1 + y_2)t)(1 + x_0 t)(1 + x_1 t)$$

11. $P_1 = \{(1,0,0),(0,1,0)\}, P_2 = \{(1,0,1),(0,1,1)\}$

$$\mathcal{E} \cong \mathcal{E}' \oplus \mathcal{O}(i_1^2 D_2 + i_2^0 D_0)$$

$$0 \longrightarrow \mathcal{O}(i_2^1 D_1) \longrightarrow \mathcal{O}(i_1^0 D_0 + i_2^1 D_1) \oplus \mathcal{O}(i_1^1 D_1) \oplus \mathcal{O}(i_2^1 D_1 + i_2^2 D_2) \longrightarrow \mathcal{E}' \longrightarrow 0$$

$$c_t^T(\mathcal{E}) = (1 + (x_2 + y_0)t)(1 + x_1 t)(1 + (x_0 + y_1 + y_2)t + x_0 y_2 t)$$

12. $P_1 = \{(1,0,0),(0,1,0)\}, P_2 = \{(1,1,0),(1,0,1),(0,1,1)\}$

$$0 \longrightarrow \mathcal{O}(i_2^1 D_1) \oplus \mathcal{O}(i_2^2 D_2) \longrightarrow \begin{array}{c} \mathcal{O}(i_1^1 D_1) \oplus \mathcal{O}(i_1^2 D_2) \oplus \mathcal{O}(i_1^0 D_0 + i_2^1 D_1) \\ \oplus \mathcal{O}(i_2^0 D_0 + i_2^2 D_2) \oplus \mathcal{O}(i_2^1 D_1 + i_2^2 D_2) \end{array} \longrightarrow \mathcal{E} \longrightarrow 0$$

$$\begin{gathered} c_t^T(\mathcal{E}) = 1 + (x_0 + x_1 + x_2 + y_0 + y_1 + y_2)t + \\ (x_0 x_1 + x_0 x_2 + x_0 y_0 + x_0 y_2 + x_1 x_2 + x_1 y_0 + x_1 y_1 + \\ x_1 y_2 + x_2 y_0 + x_2 y_1 + x_2 y_2 + y_0 y_1)t^2 + \\ (x_1 x_2 y_2 + x_0 x_1 y_0 + x_0 x_2 y_0 + x_0 x_2 y_2 + x_1 x_2 y_1 + x_1 x_2 y_2)t^3 \end{gathered}$$

13. $P_1 = \{(0,1,0),(0,0,1)\}, P_2 = \{(0,1,1)\}$

$$\mathcal{E} \cong \mathcal{E}' \oplus \mathcal{O}(i_1^1 D_1)$$

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}(i_2^0 D_0 + i_2^1 D_1 + i_2^2 D_2) \oplus \mathcal{O}(i_1^0 D_0) \oplus \mathcal{O}(i_1^1 D_1) \longrightarrow \mathcal{E} \longrightarrow 0$$

$$c_t^T(\mathcal{E}) = (1 + x_0 t)(1 + x_1 t)(1 + (y_0 + y_1 + y_2)t)$$

14. $P_1 = \{(0,1,0),(0,0,1)\}, P_2 = \{(1,0,1),(0,1,1)\}$

$$\mathcal{E} \cong \mathcal{E}' \oplus \mathcal{O}(i_1^1 D_1 + i_2^0 D_0)$$

$$0 \longrightarrow \mathcal{O}(i_2^1 D_1) \longrightarrow \mathcal{O}(i_1^0 D_0 + i_2^1 D_1) \oplus \mathcal{O}(i_1^1 D_1) \oplus \mathcal{O}(i_2^1 D_1 + i_2^2 D_2) \longrightarrow \mathcal{E} \longrightarrow 0$$

$$c_t^T(\mathcal{E}) = (1 + x_1 t)(1 + (x_0 + y_1 + y_2)t + x_0 y_2 t^2)$$

15. $P_1 = \{(0,1,0),(0,0,1)\}, P_2 = \{(1,1,0),(1,0,1),(0,1,1)\}$

$$0 \longrightarrow \mathcal{O}(i_2^1 D_1) \oplus \mathcal{O}(i_2^2 D_2) \longrightarrow \begin{matrix} \mathcal{O}(i_1^1 D_1) \oplus \mathcal{O}(i_1^2 D_2) \oplus \mathcal{O}(i_1^0 D_0 + i_2^2 D_2) \\ \oplus \mathcal{O}(i_2^0 D_0 + i_2^1 D_1) \oplus \mathcal{O}(i_2^1 D_1 + i_2^2 D_2) \end{matrix} \longrightarrow \mathcal{E} \longrightarrow 0$$

$$c_t^T(\mathcal{E}) = 1 + (x_0 + x_1 + x_2 + y_0 + y_1 + y_2)t +$$
$$(x_0x_1 + x_0x_2 + x_0y_0 + x_0y_1 + x_1x_2 + x_1y_0 + x_1y_1 +$$
$$x_1y_2 + x_2y_0 + x_2y_1 + x_2y_2 + y_0y_2)t^2 +$$
$$(x_0x_1y_0 + x_0x_2y_0 + x_0x_1y_1 + x_1x_2y_1 + x_1x_2y_2 + x_2y_0y_2)t^3$$

16. $P_1 = \{(1,0,0),(0,0,1)\}, P_2 = \{(1,0,1)\}$

$$\mathcal{E} \cong \mathcal{E}' \oplus \mathcal{O}(i_2^1 D_1 + i_2^2 D_2)$$

$$0 \longrightarrow \mathcal{O}(i_2^0 D_0) \longrightarrow \mathcal{O}(i_1^0 D_0) \oplus \mathcal{O}(i_1^2 D_2 + i_2^0 D_0) \oplus \mathcal{O}(i_1^1 D_1 + i_2^0 D_0) \longrightarrow \mathcal{E}' \longrightarrow 0$$
$$c_t^T(\mathcal{E}) = (1 + (y_1 + y_2)t)(1 + x_0t)(1 + (x_1 + x_2 + y_0)t + x_1x_2t^2)$$

17. $P_1 = \{(1,0,0),(0,0,1)\}, P_2 = \{(1,1,0),(0,1,1)\}$

$$\mathcal{E} \cong \mathcal{E}' \oplus \mathcal{O}(i_2^0 D_0 + i_2^1 D_1 + i_2^2 D_2)$$

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}(i_1^0 D_0) \oplus \mathcal{O}(i_1^1 D_1) \oplus \mathcal{O}(i_1^2 D_2) \longrightarrow \mathcal{E}' \longrightarrow 0$$
$$c_t^T(\mathcal{E}) = (1 + (y_0 + y_1 + y_2)t)(1 + x_0t)(1 + x_1t)(1 + x_2t)$$

18. $P_1 = \{(1,0,0),(0,0,1)\}, P_2 = \{(1,1,0),(1,0,1),(0,1,1)\}$

$$0 \longrightarrow \bigoplus_{k=0,1,2} \mathcal{O}(i_2^k D_k) \longrightarrow \bigoplus_{k=0,1,2} \mathcal{O}(i_1^k D_k) \bigoplus_{k=0,1,2} \mathcal{O}(i_2^k D_k + i_2^{k+1} D_{k+1}) \longrightarrow \mathcal{E} \longrightarrow 0$$

$$c_t^T(\mathcal{E}) = (1 + x_0t)(1 + x_1t)(1 + x_2t)(1 + (y_0 + y_1 + y_2)t)$$

19. $P_1 = \{(1,0,0),(0,1,0),(0,0,1)\}, P_2 = \{(1,1,0),(1,0,1)\}$

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}(i_1^0 D_0) \oplus \mathcal{O}(i_1^2 D_2) \oplus \mathcal{O}(i_1^1 D_1 + i_2^0 D_0) \oplus \mathcal{O}(i_2^1 D_1 + i_2^2 D_2) \longrightarrow \mathcal{E} \longrightarrow 0$$
$$c_t^T(\mathcal{E}) = (1 + x_0t)(1 + x_2t)(1 + (x_1 + y_0)t)(1 + (y_1 + y_2)t)$$

20. $P_1 = \{(1,0,0),(0,1,0),(0,0,1)\}, P_2 = \{(1,1,0),(0,1,1)\}$

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}(i_1^0 D_0) \oplus \mathcal{O}(i_1^1 D_1) \oplus \mathcal{O}(i_1^2 D_2) \oplus \mathcal{O}(i_2^0 D_0 + i_2^1 D_1 + i_2^2 D_2) \longrightarrow \mathcal{E} \longrightarrow 0$$
$$c_t^T(\mathcal{E}) = (1 + x_0t)(1 + x_1t)(1 + x_2t)(1 + (y_0 + y_1 + y_2)t)$$

21. $P_1 = \{(1,0,0),(0,1,0),(0,0,1)\}, P_2 = \{(1,0,1),(0,1,1)\}$

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}(i_1^0 D_0) \oplus \mathcal{O}(i_1^1 D_1) \oplus \mathcal{O}(i_1^2 D_2 + i_2^0 D_0) \oplus \mathcal{O}(i_2^2 D_2 + i_2^1 D_1) \longrightarrow \mathcal{E} \longrightarrow 0$$
$$c_t^T(\mathcal{E}) = (1 + x_0t)(1 + x_1t)(1 + (x_2 + y_0)t)(1 + (y_2 + y_1)t)$$

22. $P_1 = \{(1,0,0),(0,1,0),(0,0,1)\}, P_2 = \{(1,1,0),(1,0,1),(0,1,1)\}$

$$0 \longrightarrow \bigoplus_{k=0,1,2} \mathcal{O}(i_2^k D_k) \longrightarrow \bigoplus_{k=0,1,2} \mathcal{O}(i_1^k D_k) \bigoplus_{k=0,1,2} \mathcal{O}(i_2^k D_k + i_2^{k+1} D_{k+1}) \longrightarrow \mathcal{E} \longrightarrow 0$$

$$c_t^T(\mathcal{E}) = (1 + x_0t)(1 + x_1t)(1 + x_2t)(1 + (y_0 + y_1 + y_2)t)$$

B.0.2. 231.

1. $P_1 = \{(1,0,0)\},\ P_2 = \{(1,1,0)\}$

$$\mathcal{E} \cong \mathcal{O}(i_1^0 D_0 + i_1^1 D_1) \oplus \mathcal{O}(i_1^2 D_2 + i_1^0 D_0 + i_1^1 D_1) \oplus \mathcal{O}(i_2^2 D_2)$$

2. $P_1 = \{(1,0,0)\},\ P_2 = \{(1,0,1)\}$

$$\mathcal{E} \cong \mathcal{O}(i_1^0 D_0 + i_1^1 D_1) \oplus \mathcal{O}(i_1^2 D_2 + i_2^1 D_1) \oplus \mathcal{O}(i_2^0 D_2 + i_2^2 D_2)$$

3. $P_1 = \{(0,1,0)\},\ P_2 = \{(1,1,0)\}$

$$\mathcal{E} \cong \mathcal{O}(i_1^0 D_0 + i_1^2 D_2 + i_2^1 D_1) \oplus \mathcal{O}(i_1^1 D_1 + i_2^0 D_0) \oplus \mathcal{O}(i_2^2 D_2)$$

4. $P_1 = \{(0,1,0)\},\ P_2 = \{(0,1,1)\}$

$$\mathcal{E} \cong \mathcal{O}(i_1^0 D_0 + i_1^2 D_2 + i_2^1 D_1) \oplus \mathcal{O}(i_1^1 D_1) \oplus \mathcal{O}(i_2^0 D_0 + i_2^2 D_2)$$

5. $P_1 = \{(0,1,0)\},\ P_2 = \{(1,1,0),(0,1,1)\}$

$$\mathcal{E} \cong \mathcal{E}' \oplus \mathcal{O}(i_1^0 D_0 + i_1^2 D_2 + i_2^1 D_1)$$

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}(i_2^0 D_0) \oplus \mathcal{O}(i_1^1 D_1) \oplus \mathcal{O}(i_1^2 D_2) \longrightarrow \mathcal{E}' \longrightarrow 0$$

$$c_t^T(\mathcal{E}) = (1 + x_1 t)(1 + x_2 t)(1 + y_2 t)(1 + (x_0 + x_2 + y_1)t)$$

6. $P_1 = \{(0,0,1)\},\ P_2 = \{(0,1,1)\}$

$$\mathcal{E} \cong \mathcal{O}(i_1^1 D_1) \oplus \mathcal{O}(i_1^0 D_0 + i_2^2 D_2) \oplus \mathcal{O}(i_1^2 D_0 + i_2^0 D_0)$$

7. $P_1 = \{(0,0,1)\},\ P_2 = \{(1,0,1)\}$

$$\mathcal{E} \cong \mathcal{O}(i_1^1 D_1 + i_2^0 D_0) \oplus \mathcal{O}(i_1^2 D_2 + i_2^1 D_1) \oplus \mathcal{O}(i_1^0 D_0 + i_2^2 D_2)$$

8. $P_1 = \{(0,0,1)\},\ P_2 = \{(1,0,1),(0,1,1)\}$

$$\mathcal{E} \cong \mathcal{E}' \oplus \mathcal{O}(i_1^0 D_0 + i_2^2 D_2)$$

$$0 \longrightarrow \mathcal{O}(i_2^1 D_1) \longrightarrow \mathcal{O}(i_1^1 D_1) \oplus \mathcal{O}(i_1^2 D_2 + i_2^1 D_1) \oplus \mathcal{O}(i_2^0 D_0 + i_2^1 D_1) \longrightarrow \mathcal{E}' \longrightarrow 0$$

$$c_t^T(\mathcal{E}) = (1 + (x_0 + y_2)t)(1 + x_1 t)(1 + (x_2 + y_0 + y_1)t + x_2 y_0 t^2)$$

9. $P_1 = \{(1,0,0),(0,1,0)\},\ P_2 = \{(1,1,0)\}$

$$\mathcal{E} \cong \mathcal{E}' \oplus \mathcal{O}(i_2^2 D_2)$$

$$0 \longrightarrow \mathcal{O}(i_2^0 D_0 + i_2^1 D_1) \longrightarrow \begin{matrix} \mathcal{O}(i_1^0 D_0 + i_2^1 D_1) \oplus \mathcal{O}(i_1^1 D_1 + i_2^0 D_0) \\ \oplus \mathcal{O}(i_1^2 D_2 + i_2^0 D_0 + i_2^1 D_1) \end{matrix} \longrightarrow \mathcal{E}' \longrightarrow 0$$

$$c_t^T(\mathcal{E}) = (1 + (x_0 + x_1 + x_2 + y_0 + y_1)t + (x_0 x_2 + x_1 x_2)t^2)(1 + y_2 t)$$

10. $P_1 = \{(1,0,0),(0,1,0)\},\ P_2 = \{(1,0,1),(0,1,1)\}$

$$\mathcal{E} \cong \mathcal{E}' \oplus \mathcal{O}(i_2^2 D_2 + i_2^0 D_0)$$

$$0 \longrightarrow \mathcal{O}(i_2^1 D_1) \longrightarrow \mathcal{O}(i_1^2 D_2 + i_2^1 D_1) \oplus \mathcal{O}(i_1^0 D_0 + i_2^1 D_1) \oplus \mathcal{O}(i_1^1 D_1) \longrightarrow \mathcal{E}' \longrightarrow 0$$

$$c_t^T(\mathcal{E}) = (1 + x_1 t)(1 + (x_0 + x_2 + y_1)t + x_0 x_2 t^2)(1 + (y_0 + y_2)t)$$

11. $P_1 = \{(1,0,0),(0,0,1)\},\ P_2 = \{(1,1,0),(0,1,1)\}$

$$\mathcal{E} \cong \mathcal{E}' \oplus \mathcal{O}(i_1^2 D_2 + i_2^1 D_1 + i_2^0 D_0)$$

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}(i_1^0 D_0) \oplus \mathcal{O}(i_1^1 D_1) \oplus \mathcal{O}(i_2^2 D_2) \longrightarrow \mathcal{E}' \longrightarrow 0$$

$$c_t^T(\mathcal{E}) = (1 + (x_2 + y_0 + y_1)t)(1 + x_0 t)(1 + x_1 t)(1 + y_2 t)$$

12. $P_1 = \{(1,0,0),(0,0,1)\},\ P_2 = \{(1,0,1)\}$

$$\mathcal{E} \cong \mathcal{E}' \oplus \mathcal{O}(i_1^2 D_2 + i_2^1 D_1)$$

$$0 \longrightarrow \mathcal{O}(i_2^0 D_0) \longrightarrow \mathcal{O}(i_1^0 D_0) \oplus \mathcal{O}(i_1^1 D_1 + i_2^0 D_0) \oplus \mathcal{O}(i_2^0 D_0 + i_2^2 D_2) \longrightarrow \mathcal{E}' \longrightarrow 0$$

$$c_t^T(\mathcal{E}) = (1 + (x_2 + y_1)t)(1 + x_0 t)(1 + (x_2 + y_0 + y_2)t + x_1 x_2 t^2)$$

13. $P_1 = \{(1,0,0),(0,0,1)\},\ P_2 = \{(1,1,0),(1,0,1),(0,1,1)\}$

$$0 \longrightarrow \mathcal{O}(i_2^0 D_0) \longrightarrow \begin{matrix} \mathcal{O}(i_1^0 D_0) \oplus \mathcal{O}(i_1^1 D_1) \oplus \mathcal{O}(i_1^2 D_2 + i_2^1 D_1) \\ \scriptstyle \oplus \mathcal{O}(i_2^0 D_0 + i_2^2 D_2) \oplus \mathcal{O}(i_2^0 D_0 + i_2^1 D_1) \end{matrix} \longrightarrow \mathcal{E} \longrightarrow 0$$

$$c_t^T(\mathcal{E}) = (1 + x_0 t)(1 + x_1 t)(1 + (x_2 + y_1)t)(1 + (y_0 + y_1 + y_2)t + y_1 y_2 t^2)$$

B.0.3. 312.

1. $P_1 = \{(1,0,0)\},\ P_2 = \{(1,1,0)\}$

$$\mathcal{E} \cong \mathcal{O}(i_2^0 D_0 + i_2^1 D_1) \oplus \mathcal{O}(i_1^0 D_0 + i_1^1 D_1 + i_2^2 D_2) \oplus \mathcal{O}(i_1^2 D_2)$$

2. $P_1 = \{(1,0,0)\},\ P_2 = \{(1,0,1)\}$

$$\mathcal{E} \cong \mathcal{O}(i_1^0 D_0 + i_1^1 D_1 + i_2^2 D_2) \oplus \mathcal{O}(i_2^1 D_1) \oplus \mathcal{O}(i_2^2 D_2 + i_2^0 D_0)$$

3. $P_1 = \{(1,0,0)\},\ P_2 = \{(1,1,0),(1,0,1)\}$

$$\mathcal{E} \cong \mathcal{E}' \oplus \mathcal{O}(i_1^0 D_0 + i_1^1 D_1 + i_2^2 D_2)$$

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}(i_1^2 D_2) \oplus \mathcal{O}(i_2^0 D_0) \oplus \mathcal{O}(i_2^1 D_1) \longrightarrow \mathcal{E}' \longrightarrow 0$$

$$c_t^T(\mathcal{E}) = (1 + (x_0 + x_1 + y_2)t)(1 + x_2 t)(1 + y_0 t)(1 + y_1 t)$$

4. $P_1 = \{(0,1,0)\},\ P_2 = \{(1,1,0)\}$

$$\mathcal{E} \cong \mathcal{O}(i_1^0 D_0 + i_2^1 D_1) \oplus \mathcal{O}(i_1^1 D_1 + i_2^0 D_0) \oplus \mathcal{O}(i_1^2 D_2)$$

5. $P_1 = \{(0,1,0)\},\ P_2 = \{(0,1,1)\}$

$$\mathcal{E} \cong \mathcal{O}(i_1^0 D_0 + i_2^1 D_1) \oplus \mathcal{O}(i_1^1 D_1 + i_2^2 D_2) \oplus \mathcal{O}(i_1^2 D_2 + i_2^0 D_0)$$

6. $P_1 = \{(0,1,0)\},\ P_2 = \{(1,1,0),(0,1,1)\}$

$$\mathcal{E} \cong \mathcal{E}' \oplus \mathcal{O}(i_1^0 D_0 + i_2^1 D_1)$$

$$0 \longrightarrow \mathcal{O}(i_2^2 D_2) \longrightarrow \mathcal{O}(i_1^1 D_1 + i_2^2 D_2) \oplus \mathcal{O}(i_1^2 D_2) \oplus \mathcal{O}(i_2^2 D_2 + i_2^0 D_0) \longrightarrow \mathcal{E}' \longrightarrow 0$$

$$c_t^T(\mathcal{E}) = (1 + (x_0 + y_1)t)(1 + x_2 t)(1 + (x_1 + y_0 + y_2)t + x_1 y_0 t^2)$$

7. $P_1 = \{(0,0,1)\},\ P_2 = \{(1,0,1)\}$

$$\mathcal{E} \cong \mathcal{O}(i_1^0 D_0 + i_1^2 D_2) \oplus \mathcal{O}(i_1^1 D_1 + i_2^0 D_0 + i_2^2 D_2) \oplus \mathcal{O}(i_2^1 D_1)$$

8. $P_1 = \{(0,0,1)\},\ P_2 = \{(0,1,1)\}$

$$\mathcal{E} \cong \mathcal{O}(i_1^0 D_0 + i_1^2 D_2) \oplus \mathcal{O}(i_1^1 D_1 + i_2^2 D_2) \oplus \mathcal{O}(i_2^0 D_0 + i_2^1 D_1)$$

9. $P_1 = \{(1,0,0),(0,0,1)\},\ P_2 = \{(1,0,1)\}$

$$\mathcal{E} \cong \mathcal{E}' \oplus \mathcal{O}(i_2^1 D_1)$$

$$0 \longrightarrow \mathcal{O}(i_2^0 D_0 + i_2^2 D_2) \longrightarrow \begin{matrix} \mathcal{O}(i_1^1 D_1 + i_2^0 D_0 + i_2^2 D_2) \oplus \\ \mathcal{O}(i_1^0 D_0 + i_2^2 D_2) \oplus \mathcal{O}(i_1^2 D_2 + i_2^0 D_0) \end{matrix} \longrightarrow \mathcal{E}' \longrightarrow 0$$

$$c_t^T(\mathcal{E}) = (1 + (x_0 + x_1 + x_2 + y_0 + y_2)t + (x_0 x_1 + x_1 x_2)t^2)(1 + y_2 t)$$

10. $P_1 = \{(1,0,0),(0,0,1)\},\ P_2 = \{(1,1,0),(0,1,1)\}$

$$\mathcal{E} \cong \mathcal{E}' \oplus \mathcal{O}(i_2^0 D_0 + i_2^1 D_1)$$

$$0 \longrightarrow \mathcal{O}(i_2^2 D_2) \longrightarrow \mathcal{O}(i_1^1 D_1 + i_2^2 D_2) \oplus \mathcal{O}(i_1^0 D_0 + i_2^2 D_2) \oplus \mathcal{O}(i_1^2 D_2) \longrightarrow \mathcal{E}' \longrightarrow 0$$

$$c_t^T(\mathcal{E}) = (1 + (y_0 + y_1)t)(1 + x_2 t)(1 + (x_0 + y_2)t + x_0 x_1 t^2)$$

11. $P_1 = \{(0,1,0),(0,0,1)\},\ P_2 = \{(1,1,0),(0,1,1)\}$

$$\mathcal{E} \cong \mathcal{E}' \oplus \mathcal{O}(i_1^1 D_1 + i_2^0 D_0 + i_2^2 D_2)$$

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}(i_1^0 D_0) \oplus \mathcal{O}(i_2^1 D_1) \oplus \mathcal{O}(i_1^2 D_2) \longrightarrow \mathcal{E}' \longrightarrow 0$$

$$c_t^T(\mathcal{E}) = (1 + (x_1 + y_0 + y_2)t)(1 + x_0 t)(1 + y_1 t)(1 + x_2 t)$$

12. $P_1 = \{(0,1,0),(0,0,1)\},\ P_2 = \{(1,0,1)\}$

$$\mathcal{E} \cong \mathcal{E}' \oplus \mathcal{O}(i_1^2 D_2 + i_2^1 D_1)$$

$$0 \longrightarrow \mathcal{O}(i_2^0 D_0) \longrightarrow \mathcal{O}(i_1^0 D_0) \oplus \mathcal{O}(i_1^2 D_2 + i_2^0 D_0) \oplus \mathcal{O}(i_2^0 D_0 + i_2^1 D_1) \longrightarrow \mathcal{E}' \longrightarrow 0$$

$$c_t^T(\mathcal{E}) = (1 + (x_2 + y_1)t)(1 + x_0 t)(1 + (y_0 + y_1 + y_2)t + y_1 y_2 t^2)$$

13. $P_1 = \{(0,1,0),(0,0,1)\},\ P_2 = \{(1,1,0),(1,0,1),(0,1,1)\}$

$$0 \longrightarrow \mathcal{O}(i_2^0 D_0) \longrightarrow \begin{array}{c}\mathcal{O}(i_1^0 D_0) \oplus \mathcal{O}(i_1^2 D_2) \oplus \mathcal{O}(i_1^1 D_1 + i_2^2 D_2) \\ \oplus \mathcal{O}(i_2^0 D_0 + i_2^2 D_2) \oplus \mathcal{O}(i_2^0 D_0 + i_2^1 D_1)\end{array} \longrightarrow \mathcal{E} \longrightarrow 0$$

$$c_t^T(\mathcal{E}) = (1 + x_0 t)(1 + x_2 t)(1 + (x_1 + y_2)t)(1 + (y_0 + y_1 + y_2)t + y_1 y_2 t^2)$$

B.0.4. 213.

1. $P_1 = \{(1,0,0)\},\ P_2 = \{(1,1,0)\}$

$$\mathcal{E} \cong \mathcal{O}(i_1^0 D_0 + i_1^1 D_1 + i_2^2 D_2) \oplus \mathcal{O}(i_1^2 D_2 i_2^0 D_0 + i_2^1 D_1) \oplus \mathcal{O}$$

2. $P_1 = \{(1,0,0)\},\ P_2 = \{(1,0,1)\}$

$$\mathcal{E} \cong \mathcal{O}(i_1^0 D_0 + i_1^1 D_1 + i_2^2 D_2) \oplus \mathcal{O}(i_1^2 D_2 + i_2^1 D_1) \oplus \mathcal{O}(i_2^0 D_0)$$

3. $P_1 = \{(0,1,0)\},\ P_2 = \{(1,1,0)\}$

$$\mathcal{E} \cong \mathcal{O}(i_1^0 D_0 + i_1^2 D_2 + i_2^1 D_1) \oplus \mathcal{O}(i_1^1 D_1 + i_2^0 D_0 + i_2^2 D_2) \oplus \mathcal{O}$$

4. $P_1 = \{(0,1,0)\},\ P_2 = \{(0,1,1)\}$

$$\mathcal{E} \cong \mathcal{O}(i_1^0 D_0 + i_1^2 D_2 + i_2^1 D_1) \oplus \mathcal{O}(i_1^1 D_1 + i_2^2 D_2) \oplus \mathcal{O}(i_2^0 D_0)$$

5. $P_1 = \{(0,0,1)\},\ P_2 = \{(1,0,1)\}$

$$\mathcal{E} \cong \mathcal{O}(i_1^1 D_1 + i_2^0 D_0) \oplus \mathcal{O}(i_1^2 D_2 + i_2^1 D_1) \oplus \mathcal{O}(i_1^0 D_0)$$

6. $P_1 = \{(0,0,1)\},\ P_2 = \{(0,1,1)\}$

$$\mathcal{E} \cong \mathcal{O}(i_2^1 D_2 + i_2^0 D_0) \oplus \mathcal{O}(i_1^1 D_1 + i_2^2 D_2) \oplus \mathcal{O}(i_1^0 D_0)$$

7. $P_1 = \{(0,0,1)\},\ P_2 = \{(1,0,1),(0,1,1)\}$

$$\mathcal{E} \cong \mathcal{E}' \oplus \mathcal{O}(i_1^0 D_0)$$

$$0 \longrightarrow \mathcal{O}(i_2^1 D_1 + i_2^2 D_2) \longrightarrow \begin{array}{c}\mathcal{O}(i_1^1 D_1 + i_2^2 D_2) \oplus \mathcal{O}(i_1^2 D_2 + i_2^1 D_1) \\ \oplus \mathcal{O}(i_2^0 D_0 + i_2^1 D_1 + i_2^2 D_2)\end{array} \longrightarrow \mathcal{E}' \longrightarrow 0$$

$$c_t^T(\mathcal{E}) = (1 + x_0 t)(1 + (x_1 + x_2 + y_0 + y_1 + y_2)t + (x_1 + x_2)(y_0 + y_1 + y_2)t^2)$$

8. $P_1 = \{(1,0,0),(0,1,0)\},\ P_2 = \{(1,1,0)\}$

$$\mathcal{E} \cong \mathcal{E}' \oplus \mathcal{O}$$

$$0 \longrightarrow \mathcal{O}(i_2^0 D_0 + i_2^1 D_1 + i_2^2 D_2) \longrightarrow \begin{array}{c} \mathcal{O}(i_1^1 D_1 + i_2^0 D_0 + i_2^2 D_2)\oplus \\ \mathcal{O}(i_1^2 D_2 + i_2^0 D_0 + i_2^1 D_1)\oplus \\ \mathcal{O}(i_1^0 D_0 + i_2^1 D_1 + i_2^2 D_2) \end{array} \longrightarrow \mathcal{E}' \longrightarrow 0$$

$$c_t^T(\mathcal{E}) = (1+(x_0-y_0)t)(1+(x_1-y_1)t)(1+(x_2-y_2)t)(1+(y_0+y_1+y_2)t)$$

9. $P_1 = \{(1,0,0),(0,1,0)\},\ P_2 = \{(1,0,1),(0,1,1)\}$

$$\mathcal{E} \cong \mathcal{E}' \oplus \mathcal{O}(i_2^0 D_0)$$

$$0 \longrightarrow \mathcal{O}(i_2^1 D_1 + i_2^2 D_2) \longrightarrow \begin{array}{c} \mathcal{O}(i_1^2 D_2 + i_2^1 D_1) \oplus \mathcal{O}(i_1^1 D_1 + i_2^2 D_2) \\ \oplus \mathcal{O}(i_1^0 D_0 + i_2^1 D_1 + i_2^2 D_2) \end{array} \longrightarrow \mathcal{E}' \longrightarrow 0$$

$$c_t^T(\mathcal{E}) = (1+y_0 t)(1+(x_2+y_1)t)(1+(x_1+y_2)t+(x_1+x_2)x_0 t^2)$$

B.0.5. 132.

1. $P_1 = \{(1,0,0)\},\ P_2 = \{(1,1,0)\}$

$$\mathcal{E} \cong \mathcal{O}(i_1^0 D_0 + i_1^1 D_1 + i_1^2 D_2) \oplus \mathcal{O}(i_2^0 D_0 + i_2^1 D_1) \oplus \mathcal{O}(i_2^2 D_2)$$

2. $P_1 = \{(1,0,0)\},\ P_2 = \{(1,0,1)\}$

$$\mathcal{E} \cong \mathcal{O}(i_1^0 D_0 + i_1^1 D_1 + i_1^2 D_2) \oplus \mathcal{O}(i_2^0 D_0 + i_2^2 D_2) \oplus \mathcal{O}(i_2^1 D_1)$$

3. $P_1 = \{(1,0,0)\},\ P_2 = \{(1,1,0),(1,0,1)\}$

$$\mathcal{E} \cong \mathcal{E}' \oplus \mathcal{O}(i_1^0 D_0 + i_1^1 D_1 + i_1^2 D_2)$$

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}(i_2^0 D_0) \oplus \mathcal{O}(i_2^1 D_1) \oplus \mathcal{O}(i_2^2 D_2) \longrightarrow \mathcal{E}' \longrightarrow 0$$

$$c_t^T(\mathcal{E}) = (1+(x_0+x_1+x_2)t)(1+y_0 t)(1+y_1 t)(1+y_2 t)$$

4. $P_1 = \{(0,1,0)\},\ P_2 = \{(1,1,0)\}$

$$\mathcal{E} \cong \mathcal{O}(i_1^0 D_0 + i_2^1 D_1 + i_2^2 D_2) \oplus \mathcal{O}(i_1^1 D_1 + i_1^2 D_2 + i_2^0 D_0) \oplus \mathcal{O}(i_2^2 D_2)$$

5. $P_1 = \{(0,1,0)\},\ P_2 = \{(0,1,1)\}$

$$\mathcal{E} \cong \mathcal{O}(i_1^0 D_0 + i_2^1 D_1) \oplus \mathcal{O}(i_1^1 D_1 + i_1^2 D_2) \oplus \mathcal{O}(i_2^0 D_0 + i_2^2 D_2)$$

6. $P_1 = \{(0,0,1)\},\ P_2 = \{(1,0,1)\}$

$$\mathcal{E} \cong \mathcal{O}(i_1^0 D_0 + i_2^2 D_2) \oplus \mathcal{O}(i_1^1 D_1 + i_1^2 D_2 + i_2^0 D_0) \oplus \mathcal{O}(i_2^1 D_1)$$

7. $P_1 = \{(0,0,1)\},\ P_2 = \{(0,1,1)\}$

$$\mathcal{E} \cong \mathcal{O}(i_1^0 D_0 + i_2^2 D_2) \oplus \mathcal{O}(i_1^1 D_1 + i_1^2 D_2) \oplus \mathcal{O}(i_2^0 D_0 + i_2^1 D_1)$$

8. $P_1 = \{(0,0,1)\},\ P_2 = \{(1,1,0),(0,1,1)\}$

$$\mathcal{E} \cong \mathcal{E}' \oplus \mathcal{O}(i_1^1 D_1 + i_1^2 D_2 + i_2^0 D_0)$$

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}(i_1^0 D_0) \oplus \mathcal{O}(i_2^1 D_1) \oplus \mathcal{O}(i_2^2 D_2) \longrightarrow \mathcal{E}' \longrightarrow 0$$

$$c_t^T(\mathcal{E}) = (1+(x_0+x_1+x_2)t)(1+x_0 t)(1+y_1 t)(1+y_2 t)$$

9. $P_1 = \{(0,1,0)\},\ P_2 = \{(0,1,1)\}$

$$\mathcal{E} \cong \mathcal{E}' \oplus \mathcal{O}(i_1^1 D_1 + i_1^2 D_2)$$
$$0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}(i_1^0 D_0) \oplus \mathcal{O}(i_2^0 D_0 + i_2^1 D_1) \oplus \mathcal{O}(i_2^0 D_0 + i_2^2 D_2) \longrightarrow \mathcal{E}' \longrightarrow 0$$
$$c_t^T(\mathcal{E}) = (1 + (x_1 + x_2)t)(1 + x_0 t)(1 + (y_0 + y_1)t)(1 + (y_0 + y_2)t)$$

B.0.6. 123. The first two filtrations coincide, so the third is determined by a permutation.

1. (123)

$$\mathcal{E} \cong \mathcal{O}(i_1^0 D_0 + i_1^1 D_1 + i_1^2 D_2) \oplus \mathcal{O}(i_2^0 D_0 + i_2^1 D_1 + i_2^2 D_2) \oplus \mathcal{O}$$

2. (213)

$$\mathcal{E} \cong \mathcal{O}(i_1^1 D_1 + i_1^2 D_2 + i_2^0 D_0) \oplus \mathcal{O}(i_1^0 D_0 + i_2^1 D_1 + i_2^2 D_2) \oplus \mathcal{O}$$

3. (132)

$$\mathcal{E} \cong \mathcal{O}(i_1^0 D_0 + i_1^1 D_1 + i_1^2 D_2) \oplus \mathcal{O}(i_2^1 D_1 + i_2^2 D_2) \oplus \mathcal{O}(i_2^0 D_0)$$

4. (231)

$$\mathcal{E} \cong \mathcal{O}(i_1^1 D_1 + i_1^2 D_2) \oplus \mathcal{O}(i_1^0 D_0 + i_2^1 D_1 + i_2^2 D_2) \oplus \mathcal{O}(i_2^0 D_0)$$

5. (312)

$$\mathcal{E} \cong \mathcal{O}(i_1^1 D_1 + i_1^2 D_2 + i_2^0 D_0) \oplus \mathcal{O}(i_2^1 D_1 + i_2^2 D_2) \oplus \mathcal{O}(i_1^0 D_0)$$

6. (321)

$$\mathcal{E} \cong \mathcal{O}(i_1^1 D_1 + i_1^2 D_2) \oplus \mathcal{O}(i_2^1 D_1 + i_2^2 D_2 + i_2^0 D_0) \oplus \mathcal{O}(i_1^0 D_0)$$

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Wissenschaftlicher Werdegang

Markus Perling

Geboren am 7. Oktober 1973 in Daun

- 1993 Abitur am Geschwister Scholl Gymnasium, Daun
- Oktober 1993 – November 2000: Studium der Physik, Mathematik und Informatik an der Universität Kaiserslautern
- Oktober 1995: Vordiplome in Mathematik and Physik
- November 2000: Diplom in Physik
- Seit November 2000: wissenschaftlicher Assistent am Fachbereich Mathematik der Universität Kaiserslautern