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Abstract

We compute the matrix factorizations of all graded Maximal Cohen-Macaulay modules over the non-isolated hypersurface singularity $R = k[y_1, y_2, y_3]/(f)$, $f = y_1^3 - y_1^2 y_3 - y_2^2 y_3$, k an algebraically closed field with char k=0. They give an explicit description of the indecomposable coherent sheaves of rank one over the simple node $\mathrm{Proj} R \subset \mathbb{P}_k^2$. Using the classification of vector bundles on this node we describe also the rank two graded Maximal Cohen-Macaulay modules on R that have locally free sheafification.

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Contents

Introduction

Let be $R = k[y_1, y_2, y_3]/(f)$, $f = y_1^3 - y_1^2 y_3 - y_2^2 y_3$, k an algebraically closed field with char k=0.

If we consider the simple node $X = \text{Proj}R$ and a line bundle on it, namely $\mathcal{L} = \mathcal{L}(3P)$, where $P = (0 : 1 : 0)$ is a nonsingular point of the curve, by blowing down the vector bundle associated to $\mathcal{L}$ (see E.G.A. II 8.8) we obtain the affine non-isolated singular surface $C = \text{Spec}R$.

By a famous theorem of Grothendieck and Serre (see, for example, [11]) the coherent sheaves over the projective cone X correspond bijectively to the graded Maximal Cohen-Macaulay modules on the ring R. Every graded Maximal Cohen-Macaulay module on R is given uniquely (up to an isomorphism) by a matrix factorization of $f = y_1^3 - y_1^2 y_3 - y_2^2 y_3$ ([7]). In the first two sections we describe, by their matrix factorizations, all the rank one indecomposable graded Maximal Cohen-Macaulay modules on R, so, we give explicitly all indecomposable coherent sheaves of rank one on $X = \text{Proj } R$.

The vector bundles on the nodal curve X are nicely classified by Yu.Drozd and G.-M. Greuel in [3]. (see also [4]) The line bundles of degree d (denoted in the classification with $\mathcal{B}(d, 1, \lambda)$, $\lambda \in k^*$) are in bijection with the regular points of the curve. To the singular point correspond coherent sheaves that are not locally free. As we will see from their matrix factorizations they are kind of "completion" of the families of line bundles of the same degree.
Using the classification of rank two vector bundles, in the last section, we describe also the indecomposable graded Maximal Cohen-Macaulay modules of rank two on R whose sheafification is locally free.

After Eisenbud (see [7]), any graded MCM module over R, with no free direct summands is uniquely determined by a graded reduced matrix factorization of the polynomial $f = y_1^3 - y_1^2 y_3 - y_2^2 y_3$ (i.e. with homogeneous entries, of degree ≥ 1). Given such a module M, the corresponding matrix factorization (φ, ψ) determine a minimal free resolution of M:

$$\to R^n \xrightarrow{\varphi} R^n \xrightarrow{\psi} R^n \xrightarrow{\varphi} R^n \longrightarrow M \longrightarrow 0$$

By Herzog and Kühl ([6]), the minimal number of generators of M is bounded by $3 \cdot \text{rank}M$. So, a MCM module over R of rank 1 is minimally generated by two or three generators.

1 Rank one MCM modules over R with two generators

Let $s = (0 : 0 : 1)$ be the unique singular point of $V(f) \subset \mathbb{P}^2_k$ and denote $V(f)_{\text{reg}} = V(f) \backslash \{s\}$. Then $V(f)_{\text{reg}} = \{(\lambda_1 : \lambda_2 : 1), \lambda_1 \neq 0\} \cup \{(0 : 1 : 0)\}$.

For any $\lambda = (\lambda_1 : \lambda_2 : 1)$ in $V(f)$ denote:

$$\varphi_\lambda = \begin{pmatrix} y_1 - \lambda_1 y_3 & y_2 y_3 + \lambda_2 y_3^2 \\ y_2 - \lambda_2 y_3 & y_1^2 + (\lambda_1 - 1) y_1 y_3 + (\lambda_1^2 - \lambda_1) y_3^2 \end{pmatrix}$$

$$\psi_\lambda = \begin{pmatrix} y_1^2 + (\lambda_1 - 1) y_1 y_3 + (\lambda_1^2 - \lambda_1) y_3^2 & -(y_2 y_3 + \lambda_2 y_3^2) \\ -(y_2 - \lambda_2 y_3) & y_1 - \lambda_1 y_3 \end{pmatrix}$$

If $\lambda = (0 : 1 : 0)$ let be:

$$\varphi_\lambda = \begin{pmatrix} y_1 & y_1^2 + y_2^2 \\ y_3 & y_1^2 \end{pmatrix}$$

$$\psi_\lambda = \begin{pmatrix} y_1^2 & -(y_1^2 + y_2^2) \\ -y_3 & y_1 \end{pmatrix}$$

Theorem 1.1. $(\varphi_\lambda, \psi_\lambda)$ *is a matrix factorization for all* $\lambda \in V(f)$ *and the sets of graded MCM modules:*

$\mathcal{M}_{-1} = \{ \text{ Coker } \varphi_\lambda \mid \lambda \in V(f)_{\text{reg}} \}$, $\mathcal{M}_1 = \{ \text{ Coker } \psi_\lambda \mid \lambda \in V(f)_{\text{reg}} \}$ *and* $\underline{\mathcal{M}} = \{ \text{ Coker } \varphi_s, \text{Coker } \psi_s \}$

have the following properties:
1) Every two-generated non-free graded MCM R-module is isomorphic with one of the modules from $\mathcal{M}_{-1} \cup \mathcal{M}_1 \cup \underline{\mathcal{M}}$.
2) Every two different R-modules from $\mathcal{M}_{-1} \cup \mathcal{M}_1 \cup \underline{\mathcal{M}}$ *are not isomorphic.*
3) All the modules from $\mathcal{M}_{-1} \cup \mathcal{M}_1 \cup \underline{\mathcal{M}}$ *have rank 1.*
4) The modules from $\mathcal{M}_1$ *are the syzygies and also the duals of the modules from* $\mathcal{M}_{-1}$.
5) The coherent sheaves associated to the modules from $\mathcal{M}_{-1} \cup \mathcal{M}_1$ *are line bundles.*
6) The coherent sheaves associated to the modules from $\underline{\mathcal{M}}$ *are not locally free.*

Proof:

Clearly $\varphi_\lambda\psi_\lambda = \psi_\lambda\varphi_\lambda = f \cdot 1_2$ for any $\lambda \in V(f)$.
1) Let be M a two-generated non-free graded MCM R-module and (φ, ψ) the corresponding graded reduced matrix factorization. So $\varphi\psi = \psi\varphi = f \cdot 1_2$ and det $\varphi \cdot$ det $\psi = f^2$. Since f is irreducible, we have det $\varphi =$ det $\psi = f$. Because ψ is the adjoint of φ and f is irreducible it is sufficient to find

$$\varphi = \begin{pmatrix} \varphi_{11} & \varphi_{12} \\ \varphi_{21} & \varphi_{22} \end{pmatrix}$$

such that det $\varphi = f$ and φ_{11} and φ_{21} are two linearly independent linear forms. So, applying some elementary transformations on the matrix φ, we may suppose that:

$$\begin{cases} \varphi_{11} = y_1 - \lambda_1 y_3 \quad \text{and} \quad \varphi_{21} = y_2 - \lambda_2 y_3; \quad \lambda_1, \lambda_2 \in k \\ \qquad\qquad\qquad \text{or} \\ \varphi_{11} = y_1 - \lambda y_2 \quad \text{and} \quad \varphi_{21} = y_3; \qquad\qquad \lambda \in k \end{cases}$$

Let us consider the first case, when

$$\varphi = \begin{pmatrix} y_1 - \lambda_1 y_3 & \varphi_{12} \\ y_2 - \lambda_2 y_3 & \varphi_{22} \end{pmatrix}$$

with $\varphi_{12}, \varphi_{22}$ two-forms.
Notice that $(\det \varphi)(\lambda_1, \lambda_2, 1) = 0$. So $\lambda = (\lambda_1 : \lambda_2 : 1)$ is a point in $V(f)$. We will show that $\varphi \sim \varphi_\lambda$.
For this, consider the product $\psi_\lambda \cdot \varphi$ that has the form

$$\psi_\lambda \cdot \varphi = \begin{pmatrix} f & g \\ 0 & f \end{pmatrix}$$

with $g = (\ y_1^2 + (\lambda_1 - 1)y_1y_3 + (\lambda_1^2 - \lambda_1)y_3^2\ ) \cdot \varphi_{12} - (\ y_2y_3 + \lambda_2 y_3^2\ ) \cdot \varphi_{22}$. Since $g \cdot (y_1 - \lambda_1 y_3) = \varphi_{12} \cdot f - (y_2y_3 + \lambda_2 y_3^2) \cdot \det \varphi = f \cdot (\varphi_{12} - y_2y_3 - \lambda_2 y_3^2)$ and f is irreducible, we can write $g = f \cdot g_1$ with $g_1 \in k[y_1, y_2, y_3]$. Therefore, we have

$$\psi_\lambda\varphi = f \cdot \begin{pmatrix} 1 & g_1 \\ 0 & 1 \end{pmatrix}$$

Multiplying at left with φ_λ, we obtain

$$f \cdot \varphi = f \cdot \varphi_\lambda \cdot \begin{pmatrix} 1 & g_1 \\ 0 & 1 \end{pmatrix}$$

that means,

$$\varphi = \varphi_\lambda \cdot \begin{pmatrix} 1 & g_1 \\ 0 & 1 \end{pmatrix}$$

This equality induce the wished equivalence between φ and φ_λ.
The second case ($\varphi_{11} = y_1 - \lambda y_2$ and $\varphi_{21} = y_3; \lambda \in k$) can be treated exactly as above, replacing ψ_λ with ψ_{λ_0}, where λ_0 denotes the point (0:1:0). (! Since $f(\lambda, 1, 0) = 0$, we have only $\lambda = 0$). We obtain $\varphi \sim \varphi_{\lambda_0}$.

2) Because of the degrees of the entries of the matrices φ_λ and ψ_λ, no module from $\mathcal{M}_1 \cup \{\text{Coker } \psi_s\}$ is isomorphic with a module from $\mathcal{M}_{-1} \cup \{\text{Coker } \varphi_s\}$. For the rest, it is enough to consider the next fitting ideals:
$\mathcal{M}_{-1} \cup \mathcal{M}_1$:
$\text{Fitt}_1(\varphi_\lambda) = \text{Fitt}_1(\psi_\lambda) = \langle y_1 - \lambda_1 y_3,\ y_2 - \lambda_2 y_3,\ y_3^2 \rangle$, $\lambda = (\lambda_1 : \lambda_2 : 1) \in V(f)$
$\text{Fitt}_1(\varphi_{\lambda_0}) = \text{Fitt}_1(\psi_{\lambda_0}) = \langle y_1,\ y_3,\ y_2^2 \rangle$, $\lambda_0 = (0:1:0)$
$\underline{\mathcal{M}}$: $\text{Fitt}_1(\varphi_s) = \text{Fitt}_1(\psi_s) = \langle y_1,\ y_2 \rangle$

3) Follows from Corollary 6.4, [7].

4) By construction, the modules of $\mathcal{M}_1$ are the syzygies of the modules of $\mathcal{M}_{-1}$. Since

$$\varphi_\lambda^t = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \psi_\lambda \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Coker $\psi_\lambda \cong (\text{Coker } \varphi_\lambda)^\vee$

5) Since s is the only singular point of $V(f)$, it is sufficient to prove that for any module $M \in \mathcal{M}_{-1} \cup \mathcal{M}_1$, $M_{\langle y_1, y_2 \rangle} \simeq R_{\langle y_1, y_2 \rangle}$. By Prop. 1.3.8 ([2]) we have to check that $\text{Fitt}_1(\varphi_\lambda) R_{\langle y_1, y_2 \rangle} = \text{Fitt}_1(\psi_\lambda) R_{\langle y_1, y_2 \rangle} = R_{\langle y_1, y_2 \rangle}$.
This condition is fullfilled since y_3 is invertible in $R_{\langle y_1, y_2 \rangle}$.

6) Since $\text{Fitt}_1(\varphi_s) R_{\langle y_1, y_2 \rangle} = \text{Fitt}_1(\psi_s) R_{\langle y_1, y_2 \rangle} = \langle y_1, y_2 \rangle R_{\langle y_1, y_2 \rangle} \neq R_{\langle y_1, y_2 \rangle}$, the coherent sheaves associated to the modules from $\underline{\mathcal{M}}$ are not locally free.

2 Rank one MCM modules over R with three generators

For any $\lambda = (\lambda_1 : \lambda_2 : 1)$ in $V(f)$ let be:

$$\alpha_\lambda = \begin{pmatrix} 0 & y_1 - \lambda_1 y_3 & y_2 - \lambda_2 y_3 \\ y_1 & y_2 + \lambda_2 y_3 & (\lambda_1^2 - \lambda_1) y_3 \\ y_3 & 0 & -y_1 - (\lambda_1 - 1) y_3 \end{pmatrix}$$

and β_λ the adjoint of α_λ.

Using the same notations as in the previous section ($s = (0:0:1)$, $\lambda_0 = (0:1:0)$, $V(f)_{\text{reg}} = V(f) \backslash \{s\}$), we have the following:

Theorem 2.1. *For all $\lambda = (\lambda_1 : \lambda_2 : 1) \in V(f)$, $(\alpha_\lambda, \beta_\lambda)$ is a matrix factorization of f and the set of three-generated graded MCM R-modules*

$$\mathcal{M}_0 = \{ \text{ Coker } \alpha_\lambda \mid \lambda \in V(f)_{\text{reg}} \backslash \{\lambda_0\} \ \}$$

has the following properties:
1) All the modules from $\mathcal{M}_0$ have rank 1.
2) Every two different modules from $\mathcal{M}_0$ are not isomorphic.
3) The coherent sheaves associated to the modules from $\mathcal{M}_0$ are line bundles.
4) Every three-generated, rank 1, non-free, graded MCM R-module is isomorphic with one of the modules from $\mathcal{M}_0$ or to Coker α_s.

Moreover, the coherent sheaf associated to Coker α_s *is not locally free and every three-generated, rank 2, with no free summands, graded MCM R-module is isomorphic with one of the* Coker β_λ, $\lambda = (\lambda_1 : \lambda_2 : 1) \in V(f)$.

Proof:
Clearly $\alpha_\lambda \beta_\lambda = \beta_\lambda \alpha_\lambda = f \centerdot 1_3$ for any $\lambda = (\lambda_1 : \lambda_2 : 1) \in V(f)$.

1) Since det $(\alpha_\lambda) = f$, by Corollary 6.4 ([7]) rank (Coker α_λ)=1.

2) Suppose that there exist two matrices U and V with entries in k of determinant 1 such that $U\alpha_\lambda = \alpha_\xi V$ for $\lambda, \xi \in V(f)$. With the help of computer (we use SINGULAR[GPS]) we obtain that $\lambda = \xi$;

```
LIB "matrix.lib";
option(redSB);
```

```
ring R=0,(u(1..9),v(1..9),y(1..3),a,b,c,d),(c,dp);
ideal I=a3-a2-b2,c3-c2-d2;
qring Q=std(I);
matrix A[3][3]=  0, y(1)-a*y(3), y(2)-b*y(3),
               y(1), y(2)+b*y(3), (a2-a)*y(3),
               y(3),      0,      -y(1)-(a-1)*y(3);
matrix B[3][3]=  0, y(1)-c*y(3), y(2)-d*y(3),
               y(1), y(2)+d*y(3), (c2-c)*y(3),
               y(3),      0,      -y(1)-(c-1)*y(3);
matrix U[3][3]=u(1..9);
matrix V[3][3]=v(1..9);
matrix C=U*A-B*V;
ideal I=flatten(C);
ideal J=ideal(det(U)-1);
J=J+transpose(coeffs(I,y(1)))[2];
J=J+transpose(coeffs(I,y(2)))[2];
J=J+transpose(coeffs(I,y(3)))[2];
ideal L=std(J);
L;
```

We get:

```
L[1]=b-d
L[2]=a-c
L[3]=v(8)
L[4]=v(7)
L[5]=v(6)
L[6]=v(5)-v(9)
L[7]=v(4)
L[8]=v(3)
L[9]=v(2)
L[10]=v(1)-v(9)
L[11]=u(9)-v(9)
L[12]=u(8)
L[13]=u(7)
L[14]=u(6)
L[15]=u(5)-v(9)
L[16]=u(4)
```

```
L[17]=u(3)
L[18]=u(2)
L[19]=u(1)-v(9)
L[20]=v(9)^3-1
```

Therefore a=c and b=d.

3) It follows immediately from Prop 1.3.8 ([2]). The same we get that the coherent sheaf associated to Coker α_s is not locally free.

4) Let be M a three-generated, rank one, non-free graded MCM R-module and (φ, ψ) the corresponding graded reduced matrix factorization. We can suppose det φ=f and det $\psi = f^2$. Since $f \in \langle y_1, y_3 \rangle$, by [8], φ has generalized zeros. Thus after some elementary transformations,

$$\varphi = \begin{pmatrix} 0 & \varphi_1 & \varphi_2 \\ \varphi_3 & a & b \\ \varphi_4 & c & d \end{pmatrix}$$

$\varphi_i (i = \overline{1,4}), a, b, c, d$ linear forms, $\{\varphi_1, \varphi_2\}$, $\{\varphi_3, \varphi_4\}$ linearly independent. As $f \in \langle \varphi_1, \varphi_2 \rangle \cap \langle \varphi_3, \varphi_4 \rangle$, we can suppose that φ_1 and φ_3 have non-zero coefficient of y_1. So, we have one of the following cases:

$$\begin{cases} \varphi_1 = y_1 - \lambda_1 y_3, \ \varphi_2 = y_2 - \lambda_2 y_3 & \text{or} \quad \varphi_1 = y_1 - \lambda y_2, \ \varphi_2 = y_3 \\ \varphi_3 = y_1 - \xi_1 y_2, \ \varphi_4 = y_2 - \xi_2 y_3 & \text{or} \quad \varphi_3 = y_1 - \xi y_2, \ \varphi_4 = y_3 \end{cases}$$

Since det $\varphi = f$, the points $(\lambda_1 : \lambda_2 : 0), (\xi_1 : \xi_2 : 0), (\lambda : 1 : 0), (\xi : 1 : 0)$ lay in $V(f)$. Therefore $\lambda = \xi = 0$.

Notice that φ can not have the form $\begin{pmatrix} 0 & y_1 & y_3 \\ y_1 & a & b \\ y_3 & c & d \end{pmatrix}$ because $f \notin \langle y_1^2, y_1 y_3, y_3^2 \rangle$.

For any $\lambda = (\lambda_1 : \lambda_2 : 0)$ in $V(f)$, we write $\varphi_{1\lambda} = y_1 - \lambda_1 y_3$, $\varphi_{2\lambda} = y_2 - \lambda_2 y_3$ and for $\lambda = (0 : 0 : 1)$ we write $\varphi_{1\lambda} = y_1, \varphi_{2\lambda} = y_3$. Then φ has the form:

$$\varphi = \begin{pmatrix} 0 & \varphi_{1\lambda} & \varphi_{2\lambda} \\ \varphi_{1\xi} & a & b \\ \varphi_{2\xi} & c & d \end{pmatrix}$$

with $a, b, c \in \langle y_2, y_3 \rangle_k$ and d linear forms. To finish the proof, we need two helping results:

Lemma 2.2. *Let M be a three-generated, rank one, graded MCM R-module and (φ, ψ) a matrix factorization of M, φ having the above form. Then there exists $\lambda' \in V(f) \backslash \{(0:1:0)\}$, a', b', c', d' linear forms such that the matrix*

$$\varphi' = \begin{pmatrix} 0 & \varphi_{1\lambda'} & \varphi_{2\lambda'} \\ y_1 & a' & b' \\ y_3 & c' & d' \end{pmatrix}$$

together with its adjoint matrix ψ' form another matrix factorization (φ', ψ') of M.

Proof:

We have to prove that after some linear transformation the matrix φ will become φ'. For this, we show that there exist two invertible 3×3 matrices U, V such that $U\varphi' = \varphi V$. It is sufficient to show that there exist two invertible 3×3 matrices U, V such that the first column of $U^{-1}\varphi V$ is $\begin{pmatrix} 0 \\ y_1 \\ y_3 \end{pmatrix}$.
Considering $U = (u_{ij})_{1\leq i,j\leq 3}$ and $V = (v_{ij})_{1\leq i,j\leq 3}$ we get the following system of equations:

$$\begin{cases} \varphi_{1\lambda}v_{21} + \varphi_{2\lambda}v_{31} = y_1u_{12} + y_3u_{13} \\ \varphi_{1\xi}v_{11} + av_{21} + bv_{31} = y_1u_{22} + y_3u_{23} \\ \varphi_{2\xi}v_{11} + cv_{21} + dv_{31} = y_1u_{32} + y_3u_{33} \end{cases}$$

In particular, $\varphi(0,1,0)\cdot \begin{pmatrix} v_{11} \\ v_{21} \\ v_{31} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$. Since $\det(\varphi(0,1,0)) = f(0,1,0) = 0$, we may choose a non-zero solution (v_{11}, v_{21}, v_{31}) which can be completed to an invertible matrix V and such that $(u_{12}, u_{13}, u_{22}, u_{23}, u_{32}, u_{33})$ can be also completed to an invertible matrix U.

Lemma 2.3. *Let M be a three-generated, rank one, graded MCM R-module and (φ, ψ) a matrix factorization of M, φ having the form:*

$$\varphi = \begin{pmatrix} 0 & \varphi_{1\lambda} & \varphi_{2\lambda} \\ y_1 & a & b \\ y_3 & c & d \end{pmatrix}$$

Then $(\alpha_\lambda, \beta_\lambda)$ is another matrix factorization of M.

Proof:

Observation: Since $\lambda \neq (0:1:0)$ we can consider that b is in $\langle y_3 \rangle_k$.
If $b = b_2y_2 + b_3y_3$, we subtract the first line multiplied with b_2 from the second line (to "kill" y_2 in b) and then add the first column multiplied with b_2 to the second column (to "kill" the new y_1b_2 in a).

So, instead of b we can write by_3 with $b \in k$.
Consider the following polynomials:

$\gamma = \begin{vmatrix} y_1 & by_3 \\ y_3 & d \end{vmatrix}$, $\delta = \begin{vmatrix} y_1 & a \\ y_3 & c \end{vmatrix}$, $\bar{\gamma} = \begin{vmatrix} y_1 & (\lambda_1^2 - \lambda_1)y_3 \\ y_3 & -y_1 - (\lambda_1 - 1)y_3 \end{vmatrix}$
and $\bar{\delta} = \begin{vmatrix} y_1 & y_2 + \lambda_2 y_3 \\ y_3 & 0 \end{vmatrix}$.

Since $f = -\varphi_{1\lambda}\gamma + \varphi_{2\lambda}\delta = -\varphi_{1\lambda}\bar{\gamma} + \varphi_{2\lambda}\bar{\delta}$, $\varphi_{1\lambda}(\bar{\gamma} - \gamma) = \varphi_{2\lambda}(\bar{\delta} - \delta)(*)$. So $\varphi_{1\lambda} \mid \bar{\delta} - \delta$. But $\bar{\delta} - \delta = -c(y_1 - \lambda_1 y_3) - y - 3(y_2 + \lambda_2 y_3 + \lambda_1 c - a)$ and $a, c \in \langle y_2, y_3 \rangle_k$. Therefore, $a = y_2 + \lambda_2 y_3 + \lambda_1 c$ and $\bar{\delta} - \delta = -c\varphi_{1\lambda}$. From $(*)$ we get $\bar{\gamma} - \gamma = -c(y_2 - \lambda_2 y_3)$. Since $\bar{\gamma} - \gamma = y_1(-y_1 - (\lambda_1 - 1)y_3) - d) - y_3^2(\lambda_1^2 - \lambda_1 - b)$ and $c \in \langle y_2, y_3 \rangle_k$ we obtain $d = -y_1 - (\lambda_1 - 1)y_3, b = \lambda_1^2 - \lambda_1$ and $c = 0$. Thus $\varphi \sim \alpha_\lambda$.

Using Lemma 2.2 and Lemma 2.3 the proof of point 4) of the theorem is finished.

Let us prove also the last part of the theorem.
Let be M a three-generated, rank two, with no free summands, graded MCM R-module and (φ, ψ) a matrix factorization of M. Then Coker ψ is a three-generated, rank one, non-free, graded MCM R-module. Therefore, Coker $\psi \in \mathcal{M}_0 \cup \{\text{Coker } \alpha_s\}$ and Coker $\varphi \in \{\text{Coker } \beta_\lambda, \lambda = (\lambda_1 : \lambda_2 : 1) \in V(f)\}$.

3 The line bundles of ProjR

By a well-known theorem of Grothendieck and Serre (see for example [11]), the graded MCM R-modules are in 1:1 correspondence to the coherent sheaves of the projective cone $X = \mathrm{Proj}R$.
If M is a graded MCM R-module, the correspondent coherent sheaf is the so called sheafification $\widetilde{M}$.
The correspondent graded MCM R-module of a coherent sheaf $\mathcal{F}$ is $\Gamma_*(\mathcal{F}) = \oplus_{n\in\mathbb{Z}}\Gamma(X, \mathcal{F}(n))$, where $\mathcal{F}(n) = \mathcal{O}_X(n) \otimes_{\mathcal{O}_X} \mathcal{F}$.
We know that for any two graded R-modules M and N, $\widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N} \cong \widetilde{M \otimes_R N}$ and $\widetilde{\mathrm{Hom}_R(M, N)} \cong \mathcal{H}om(\widetilde{M}, \widetilde{N})$. If M and N are MCM, $\widetilde{M} \cong \widetilde{N}$ if and only if $M \cong N$.
By the classification of the vector bundles on the simple node X (see [4]), the line bundles of degree $d \in \mathbb{Z}$ are $\mathcal{B}(d, 1, \lambda)$ with $\lambda \in k^*$ (λ run over all regular points of the curve).
The tensor product of two line bundles is given by: $\mathcal{B}(d, 1, \lambda) \otimes \mathcal{B}(d', 1, \lambda') = \mathcal{B}(d + d', 1, \lambda \cdot \lambda')$.

Theorem 3.1. *The modules of $\mathcal{M}_1$, $\mathcal{M}_{-1}$ induce the bundles $\mathcal{B}(1, 1, \lambda)$, $\mathcal{B}(-1, 1, \lambda)$ and the modules of $\mathcal{M}_0 \cup \{R\}$ induce the bundles $\mathcal{B}(0, 1, \lambda)$, after some possible shifting. ($\lambda \in k^*$)*

Proof:

Any line bundle of degree one on X has the form $\mathcal{O}_X(P)$, with P regular point of X. Following the proof of Theorem 3.8 from ([1]), we obtain that the graded MCM R-module corresponding to $\mathcal{O}_X(P)$ is a module from $\mathcal{M}_1$, for any regular point P of X.
Since the modules of $\mathcal{M}_{-1}$ are the syzygies of the modules from $\mathcal{M}_1$, they induce the bundles $\mathcal{B}(-1, 1, \lambda)$, $\lambda \in k^*$.
Therefore, after some possible shifting, all other rank one graded MCM R-modules (the one from $\mathcal{M}_0 \cup \{R\}$) induce the bundles $\mathcal{B}(0, 1, \lambda)$, $\lambda \in k^*$.

The theorems 1.1, 2.1 and 3.1 give a description of the indecomposable coherent bundles of rank one on the simple node $X = \text{Proj}\left(\frac{k[y_1,y_2,y_3]}{\langle y_1^3 - y_1^2 y_3 - y_2^2 y_3\rangle}\right)$.

The line bundles of degree 3p-1: $\mathcal{M}_{-1} = \{\text{ Coker } \varphi_\lambda \mid \lambda \in V(f)_{\text{reg}} \}$

The line bundles of degree 3p+1: $\mathcal{M}_1 = \{\text{ Coker } \psi_\lambda \mid \lambda \in V(f)_{\text{reg}} \}$

The line bundles of degree 3p: $\mathcal{M}_0 = \{\text{ Coker } \alpha_\lambda \mid \lambda \in V(f)_{\text{reg}} \backslash \{\lambda_0\} \} \cup \{R\}$

The coherent sheaves that are not locally free: $\underline{\mathcal{M}} = \{\text{Coker } \varphi_s, \text{Coker } \psi_s, \text{Coker } \alpha_s\}$ where $s = (0:0:1)$ is the singular point of $X \subset \mathbb{P}^2_k$.

4 The rank two vector bundles on ProjR

There are two types of rank two vector bundles on Proj$R$:

•$\mathcal{B}(a, 2, \lambda)$, with $a \in \mathbb{Z}$ and $\lambda \in k^*$
•$\mathcal{B}(\mathbf{d}, 1, \lambda)$, with $\mathbf{d}$ a 2-cycle with entries in $\mathbb{Z}$ and $\lambda \in k^*$.($\mathbf{d} = (a, b), a \neq b$)

To generate the first type of rank two vector bundles it is sufficient to know the bundle $\mathcal{B}(0, 2, 1)$ and the line bundles, because:
$\mathcal{B}(a, 2, \lambda) \cong \mathcal{B}(a, 1, \lambda) \otimes \mathcal{B}(0, 2, 1)$.
Using the fact that the bundle $\mathcal{B}(0, 2, 1)$ is uniquely determined by the exact sequence

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{B}(0, 2, 1) \longrightarrow \mathcal{O}_X \longrightarrow 0$$

we will be able to determine the graded MCM R-module corresponding to it.

The second type of rank two vector bundles can be generated by the bundles $\mathcal{B}((0, n), 1, \lambda)$ and line bundles using the tensor product:
$\mathcal{B}((a, b), 1, \lambda) \otimes \mathcal{B}(c, 1, \mu) \cong \mathcal{B}((a + c, b + c), 1, \lambda\mu^2)$.

For $\mathbf{d} = (a, b)$ and $\mathbf{e} = (c, d)$ two 2-cycles with entries in $\mathbb{Z}$, we have:

$$\mathcal{B}(\mathbf{d}, 1, \lambda) \otimes \mathcal{B}(\mathbf{e}, 1, \mu) \cong \mathcal{B}(\mathbf{f}_1, 1, \lambda \cdot \mu) \oplus \mathcal{B}(\mathbf{f}_2, 1, \lambda \cdot \mu),$$

where $\mathbf{f}_1 = (a+c, b+d)$ and $\mathbf{f}_2 = (a+d, b+c)$. If $\mathbf{f}_i = (\alpha, \alpha)(i = 0 \text{ or } 1)$, then $\mathcal{B}(\mathbf{f}_i, 1, \lambda \cdot \mu)$ splits as: $\mathcal{B}(\mathbf{f}_i, 1, \lambda \cdot \mu) = \mathcal{B}(\alpha, 1, \sqrt{\lambda \cdot \mu}) \oplus \mathcal{B}(\alpha, 1, -\sqrt{\lambda \cdot \mu})$.

Therefore, inductively, we can obtain all $\mathcal{B}((0, n), 1, \lambda)$, $n \in \mathbb{N}^*$, if we know the bundles $\mathcal{B}((0, 1), 1, \lambda)$. By duality, $(\mathcal{B}(\mathbf{d}, 1, \lambda)^\vee \cong \mathcal{B}(-\mathbf{d}, 1, \lambda^{-1}))$ so we obtain also $\mathcal{B}((0, n), 1, \lambda)$ with n negative.
Using the fact that the bundles $\mathcal{B}((0, 1), 1, \lambda)$ are uniquely determined by the existence of the exact sequences

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{B}((0, 1), 1, \lambda) \longrightarrow \mathcal{B}(1, 1, -\lambda) \longrightarrow 0,$$

we will give the graded MCM R-module corresponding to them. So, inductively, we can obtain all rank two graded MCM R-module corresponding to vector bundles on Proj$R$.

Let us determine in the following the module M_2 corresponding to $\mathcal{B}(0,2,1)$.

Lemma 4.1. *Let be* $\rho = \begin{pmatrix} y_1^2 - y_1y_3 & -y_2 & -y_3 & 0 \\ -y_2y_3 & y_1 & 0 & -y_3 \\ 0 & 0 & y_1 & y_2 \end{pmatrix}$,
$\psi = \begin{pmatrix} y_1 & y_2 & y_3 & 0 \\ y_2y_3 & y_1^2 - y_1y_3 & 0 & y_3 \\ 0 & 0 & y_1^2 - y_1y_3 & -y_2 \\ 0 & 0 & -y_2y_3 & y_1 \end{pmatrix}$, $\gamma = \begin{pmatrix} 0 & 0 & y_2y_3 & y_1^2 - y_1y_3 \end{pmatrix}$
and $\varphi = \begin{pmatrix} \rho \\ \gamma \end{pmatrix}$. *Then* (ψ, φ) *is a matrix factorization of* $\Omega^1_R(m)$, *where* m *is the unique graded maximal ideal of* R, $m = \langle y_1, y_2, y_3 \rangle$. *More, the following exact sequence*

$$\xrightarrow{\psi} R(-3) \oplus R(-2)^3 \xrightarrow{\rho} R(-1)^3 \xrightarrow{(\mathrm{y}_1, \mathrm{y}_2, \mathrm{y}_3)} m \longrightarrow 0 \qquad (1)$$

is a minimal free graded resolution of m. *In particular,* $\Omega^1_R(m)$ *has no free summands.*

Proof:

Clearly $\varphi\psi = \psi\varphi = f \cdot \mathrm{Id}_4$. The above sequence is a complex since ρ is a part of φ and $\begin{pmatrix} y_1 & y_2 & y_3 & 0 \end{pmatrix}$ is the first row of ψ.

Let $u_1, u_2, u_3 \in R$ such that $\sum_{i=1}^3 y_i u_i = 0$. We show that $\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \in \mathrm{Im}\rho$.

Subtracting multiples of the columns 2 and 3 of ρ from $u = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$ we may suppose that u_1 depends only on y_1. As the maps are graded, we may suppose that u is graded, so $u_1 = a y_1^s, a \in k, s \in \mathbb{N}$. Since $\sum_{i=1}^3 y_i u_i = 0$, if $a \neq 0$, $s + 1 \geq 3$, so, subtracting from u multiples of $\begin{pmatrix} y_1^2 \\ -y_2y_3 \\ -y_1^2 \end{pmatrix} \in \mathrm{Im}\rho$, we reduce to the case $u_1 = 0$. Then $y_2u_2 + y_3u_3 = 0$ and since $\{y_2, y_3\}$ is a regular sequence in R we see that u is a multiple of the column 4 of ρ.

To show that $\mathrm{Ker}\rho \subset \mathrm{Im}\psi$, it is enough to show that $\mathrm{Ker}\rho \subset \mathrm{Ker}\varphi$ ($\mathrm{Im}\psi$=$\mathrm{Ker}\varphi$). Let be ρ_3 the third row of ρ and ν an element of $\mathrm{Ker}\rho$. Since $y_1\gamma = y_2y_3\rho_3$ and $\rho_3\nu = 0$, $y_1(\gamma\nu) = 0$. But y_1 is a non-divisor in R and so $\gamma\nu = 0$. Therefore, $\nu \in \mathrm{Ker}\varphi$. Because no entry of φ or ψ is unite, $\Omega^1_R(m)$ has no free summands.

Proposition 4.2. *There exists a graded exact sequence:*

$$0 \longrightarrow R \xrightarrow{\ i\ } \Omega^2_R(m) \otimes R(3) \xrightarrow{\ \pi\ } m \longrightarrow 0 \qquad (2)$$

and $\Omega^2_R(m) \otimes R(3)$ *corresponds to the bundle* $\mathcal{B}(0,2,1)$.

Proof:

1) The existence of the exact sequence:

Define $i : R \longrightarrow \Omega^2_R(m) \otimes R(3)$ by $i(1) = \begin{pmatrix} 0 \\ y_3 \\ -y_2 \\ y_1 \end{pmatrix}$ (the fourth column of ψ) and $\pi : \Omega^2_R(m) \otimes R(3) \longrightarrow m$ the first projection. Since $\Omega^2_R(m) \otimes R(3) = \mathrm{Im}\psi \otimes R(3) \subset R \oplus R(1)^3$, i and π are graded morphisms and, clearly, i is injective, π is surjective and $\mathrm{Im}i \subset \mathrm{Ker}\pi$. We prove that $\mathrm{Ker}\pi \subset \mathrm{Im}i$.

Let be $\psi \cdot \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$ an element in $\mathrm{Ker}\pi$. Then $\begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathrm{Im}\rho$. Denote by ψ' the matrix 4×3 obtained from ψ by eliminating the last column.

Then $\psi'\rho \equiv \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & y_3(y_1^2+y_2^2) & 0 & -y_1^2y_3 \\ 0 & -y_2(y_1^2+y_2^2) & 0 & -y_1^2(-y_2) \\ 0 & y_1(y_1^2+y_2^2) & 0 & -y_1^2 \cdot y_1 \end{pmatrix}$.

Therefore, $\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \in R \cdot \begin{pmatrix} 0 \\ y_3 \\ -y_2 \\ y_1 \end{pmatrix} = \mathrm{Im}i$.

2) Let be $M_2 = \Omega^2_R(m) \otimes R(3)$. We prove that it is an indecomposable module. If it would decompose, then it would be isomorphic to Coker θ, where θ has the form $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ with A, B quadratic matrices of size two,

with determinant equal to f. More, rank(Coker A) = rank(Coker B)=1. By the Corollary 6.4 from [7], they define rank 1, graded MCM R-modules, so they are of one of the forms $\varphi_\lambda, \psi_\lambda$, with $\lambda \in V(f)$.(see the previous sections) Since $\theta \sim \varphi$, their fitting ideals are equal, so $\mathrm{Fitt}_2(\theta) = \mathrm{Fitt}_2(\varphi) = m^2$.

But the elements of degree 2 from $\mathrm{Fitt}_2(\theta)$ are given just by $l_1^A \cdot l_1^B$, $l_1^A \cdot l_2^B$, $l_2^A \cdot l_1^B$, $l_2^A \cdot l_2^B$ where l_1^A, l_2^A respectively l_1^B, l_2^B are the entries of A and B of degree 1. The ideal generated by them is not m^2 since m^2 is minimally generated by 6 elements. Therefore, M_2 is indecomposable.

3) We prove that the sheafification of M_2 is a vector bundle. For this it is sufficient to notice that $\mathrm{Fitt}_2(\varphi)R_{\langle y_1,y_2\rangle} = R_{\langle y_1,y_2\rangle}$ and $\mathrm{Fitt}_3(\varphi) = 0$ (see Proposition 1.3.8, [2]).

4) From the exact sequence (2) we get the following exact sequence of vector bundles on X =Proj$R$:

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \widetilde{M_2} \longrightarrow \mathcal{O}_X \longrightarrow 0$$

Since $\widetilde{M_2}$ is an indecomposable vector bundle of rank 2, it has to be isomorphic to $\mathcal{B}(0,2,1)$.

Using this result we have the following lemma:

Lemma 4.3. *For any $\lambda \in \mathbf{k}^*$, the modules corresponding to*

$$\begin{cases} \mathcal{B}(0,2,\lambda), & \textit{have 6 generators;} \\ \mathcal{B}(-1,2,\lambda), & \textit{have 4 generators;} \\ \mathcal{B}(1,2,\lambda), & \textit{have 4 generators;} \end{cases}$$

Proof

We know that $\mathcal{B}(a,2,\lambda) \cong \mathcal{B}(a,1,\lambda) \otimes \mathcal{B}(0,2,1)$. Therefore the modules corresponding to the bundles $\mathcal{B}(0,2,\lambda)$ are given by $(M_2 \otimes \text{Coker } \alpha_\lambda)^{\vee\vee}$ and we can compute them using the computer. We use the procedures that compute the reflexive hull, the tensor product in the category of Cohen-Macaulay modules and the module M2 from [1].

```
LIB"matrix.lib";
option(redSB);

 proc reflexivHull(matrix M)
{
module N=mres(transpose(M),3)[3];
N=prune(transpose(N));
return(matrix(N));
}

 proc tensorCM(matrix Phi, matrix Psi)
{
int s=nrows(Phi);
int q=nrows(Psi);
matrix A=tensor(unitmat(s),Psi);
matrix B=tensor(Phi,unitmat(q));
matrix R=concat(A,B);
return(reflexivHull(R));
}

 proc M2(ideal I)
{
matrix A=syz(transpose(mres(I,3)[3]));
return(transpose(A));
}




ring R=0,(y(1..3)),(c,dp);
ideal i=y(1)^3-y(1)^2*y(3)-y(2)^2*y(3);
qring S=std(i);
ideal I=maxideal(1);
matrix C=M2(I);
ring R1=0,(y(1..3),a,b),(c,dp);
ideal I=y(1)^3-y(1)^2*y(3)-y(2)^2*y(3),a3-a2-b2;
qring S1=std(I);
```

```
matrix A[3][3]=  0, y(1)-a*y(3), y(2)-b*y(3),
                y(1), y(2)+b*y(3), (a2-a)*y(3),
                y(3),      0,     -y(1)-(a-1)*y(3);
matrix C=imap(S,C);
nrows(tensorCM(C,A));
6
matrix B[2][2]=y(1)-a*y(3), y(2)*y(3)+b*y(3)^2,
                y(2)-b*y(3), y(1)^2+(a-1)*y(1)*y(3)+(a2-a)*y(3)^2;
nrows(tensorCM(C,B));
4
B=y(1),y(1)^2+y(2)^2,
  y(3),y(1)^2;
nrows(tensorCM(C,B));
4
```

Lemma 4.4. *Let be $\xi = (1 : 0 : 1)$ and $\lambda_0 = (0 : 1 : 0)$. Denote $P_0 = Coker\ \alpha_\xi$, $P_1 = Coker\ \psi_\xi$, $P_{-1} = Coker\ \varphi_\xi$, $N_1 = Coker\ \psi_{\lambda_0}$, $N_{-1} = Coker\ \varphi_{\lambda_0}$. Then:*

1)$(N_1 \otimes N_1 \otimes N_1)^\vee \cong R$;
2)$(P_0 \otimes P_0)^\vee \cong R$ and $\widetilde{P_0} \cong \mathcal{B}(0, 1, -1)$;
3)$(P_0 \otimes P_1)^{\vee\vee} \cong N_1$;
4)$(P_0 \otimes P_{-1})^{\vee\vee} \cong N_{-1}$;
5)If $\widetilde{N_1} \cong \mathcal{B}(1, 1, -\mu_0)$, then $\widetilde{P_1} \cong \mathcal{B}(1, 1, \mu_0)$.

Proof:

1)

```
setring S;
matrix N1[2][2]=y(1)^2, -(y(1)^2+y(2)^2),
                 -y(3), y(1);
tensorCM(N1,tensorCM(N1,N1));
_[1,1]=0
```

This means that $(N_1 \otimes N_1 \otimes N_1)^\vee \cong R$.

2)

```
matrix P0[3][3]= 0, y(1)-y(3), y(2),
                 y(1),   y(2),     0,
                 y(3),     0,    -y(1);
tensorCM(P0,P0);
_[1,1]=0
```

This means that $(P_0 \otimes P_0)^\vee \cong R$. It is clear that exists a graded isomorphism between the module P_0 and its dual,(see their matrices) therefore, $\widetilde{P_0} \cong \mathcal{B}(0,1,-1)$.

3)

```
matrix P1[2][2]= y(1)^2, -y(2)*y(3),
                   -y(2),  y(1)-y(3);
matrix M=tensorCM(P0,P1);
print(M);
y(1), -y(3),
y(2)^2,-y(1)^2+y(1)*y(3)
```

This matrix is isomorphic to Coker ψ_{λ_0}.

4) Follows from 3) by duality.

5) Let $\widetilde{P_1} = \mathcal{B}(1,1,\theta)$. From 3) and 2) we get: $\mathcal{B}(0,1,-1) \otimes \mathcal{B}(1,1,\theta) \cong \mathcal{B}(1,1,-\mu_0)$. Therefore, $\theta = \mu_0$.

In a similar way, we get the following relations:

Lemma 4.5. *Let be* $\lambda_0 = (0:1:0)$. *For any* $\lambda = (\lambda_1 : \lambda_2 : 1) \in V(f)$, *we have:*

$$\begin{cases} \text{Coker } \alpha_{(\lambda_1:\lambda_2:1)} \cong (\text{Coker } \varphi_{\lambda_0} \otimes \text{Coker } \psi_{(\lambda_1:\lambda_2:1)})^{\vee\vee} \\ \text{Coker } \alpha_{(\lambda_1:-\lambda_2:1)} \cong (\text{Coker } \alpha_{(\lambda_1:\lambda_2:1)})^t \cong (\text{Coker } \psi_{\lambda_0} \otimes \text{Coker } \varphi_{(\lambda_1:\lambda_2:1)})^{\vee\vee} \end{cases}$$

In the following, we determine the modules corresponding to the bundles $\mathcal{B}((0,1),1,\lambda)$ with $\lambda \in k^*$. Consider the module $\Omega^1_R(M_2)$, given by Coker ψ.

Lemma 4.6. *For $\lambda \in V(f)$, we have:*
$(\Omega^1_R(M_2) \otimes$ *Coker* $\alpha_\lambda)^{\vee\vee}$ *has*

$$\begin{cases} \textit{4 generators}, & \textit{for } \lambda = (1:0:1) \\ \textit{3 generators}, & \textit{otherwise;} \end{cases}$$

$(\Omega^1_R(M_2) \otimes$ *Coker* $\varphi_\lambda)^{\vee\vee}$ *have 5 generators;*
$(\Omega^1_R(M_2) \otimes$ *Coker* $\psi_\lambda)^{\vee\vee}$ *have 5 generators;*

Proof:

We define the module $\Omega^1_R(M_2)$ by:

```
matrix D=transpose(syz(C));
```

and use the procedure `tensorCM` as before.

The matrix corresponding to the module $(\Omega^1_R(M_2) \otimes \mathrm{Coker}\varphi_\xi)^{\vee\vee}$ for $\xi = (1:0:1)$ is

$$A = \left(\begin{array}{c|cc} \multirow{3}{*}{α'_ξ} & y_2 & 0 \\ & y_1 & y_2 \\ & 0 & 0 \\ \hline 0 & \multicolumn{2}{c}{\psi_\xi} \end{array}\right), \text{ where } \alpha'_\xi = \begin{pmatrix} 0 & y_1 & y_3 \\ y_1 - y_3 & y_2 & 0 \\ y_2 & 0 & -y_1 \end{pmatrix} \text{ and } \alpha'_\xi \sim \alpha_\xi.$$

The matrix A has the adjoint: $$B = \left(\begin{array}{c|cc} \multirow{3}{*}{α'^*_ξ} & y_1 & -y_1y_2 \\ & -y_2 & 0 \\ & y_2 & -y_2^2 \\ \hline 0 & \multicolumn{2}{c}{\varphi_\xi} \end{array}\right), \text{ where}$$

$$\alpha'^*_\xi = \begin{pmatrix} -y_1y_2 & {y_1}^2 & -y_2y_3 \\ y_1(y_1 - y_3) & -y_2y_3 & y_3(y_1 - y_3) \\ -{y_2}^2 & y_1y_2 & -y_1(y_1 - y_3) \end{pmatrix}$$

Lemma 4.7. *There exists a graded exact sequence:*

$$0 \longrightarrow \text{Coker } \alpha_\xi \longrightarrow (\Omega^1_R(M_2) \otimes \text{Coker } \varphi_\xi)^{\vee\vee} \otimes R(2) \longrightarrow \text{Coker } \psi_\xi \longrightarrow 0$$

Proof:

1) We construct a graded exact sequence:

$$0 \longrightarrow \text{Im } \alpha_\xi'^{*} \xrightarrow{i} \text{Im } B \xrightarrow{\pi} \text{Im } \varphi_\xi \longrightarrow 0 \qquad (3)$$

with $i\left(\begin{pmatrix} a \\ b \\ c \end{pmatrix}\right) = \begin{pmatrix} a \\ b \\ c \\ 0 \\ 0 \end{pmatrix}$; $\pi\left(\begin{pmatrix} a \\ b \\ c \\ d \\ e \end{pmatrix}\right) = \begin{pmatrix} d \\ e \end{pmatrix}$. It is clear that (3) is a complex. We prove that Ker $\pi \subset$Im i.

Denote by $u_i, i = \overline{1,5}$ the i-th column of the matrix B. Since we have syz(φ_ξ)=Im ψ_ξ, it is sufficient to prove that Im$((u_4 \quad u_5)\cdot\psi_\xi)\subset$ Im $((u_1 \quad u_2 \quad u_3))$. But $(u_4 \quad u_5) \cdot \psi_\xi$=$(y_1u_2 - y_2u_1 \quad - y_2u_2)$. So the sequence (3) is exact. Since Im $\alpha_\xi'^{*} \simeq$ Coker $\alpha'_\xi \simeq$ Coker α_ξ, Im$B \simeq (\Omega^1_R(M_2) \otimes \text{Coker } \varphi_\xi)^{\vee\vee} \otimes R(l)$, with $l \in \mathbb{Z}$ and Im $\varphi_\xi \simeq$ Coker ψ_ξ, we have:

$$0 \longrightarrow \text{Coker } \alpha_\xi \longrightarrow (\Omega^1_R(M_2) \otimes \text{Coker } \varphi_\xi)^{\vee\vee} \otimes R(l) \longrightarrow \text{Coker } \psi_\xi \longrightarrow 0$$

2) By lemma 4.4, Coker α_ξ correspond to the line bundle $\mathcal{B}(0, 1, -1)$. From the proof of the Theorem 3.1 we obtain that deg($\widetilde{\text{Coker } \psi_\xi}$)=1, so deg($(\Omega^1_R(M_2) \otimes \text{Coker } \varphi_\xi)^{\vee\vee} \otimes R(l)$)=1. From the exact sequence (1) we see that deg($\Omega^1_R(M_2)$)=-9.
Because deg($(\Omega^1_R(M_2) \otimes \text{Coker } \varphi_\xi)^{\vee\vee} \otimes R(l)$)=$-9 - 2 + 6l$, we find $l = 2$.

Proposition 4.8. *Let be* $\widetilde{\text{Coker } \psi_{\lambda_0}} = \mathcal{B}(1, 1, -\mu_0)$, *where* $\lambda_0 = (0 : 1 : 0)$. *Then the module corresponding to* $\mathcal{B}((0, 1), 1, \mu_0)$ *is* $(\Omega^1_R(M_2)\otimes \text{Coker } \varphi_{\lambda_0})^{\vee\vee}\otimes R(2)$.

Proof:

We know that exists the graded exact sequence:

$$0 \longrightarrow \text{Coker } \alpha_\xi \longrightarrow (\Omega^1_R(M_2) \otimes \text{Coker } \varphi_\xi)^{\vee\vee} \otimes R(2) \longrightarrow \text{Coker } \psi_\xi \longrightarrow 0$$

Therefore, there exists the exact sequence of bundles:

$$0 \longrightarrow \mathcal{B}(0,1,-1) \longrightarrow \mathcal{B} \longrightarrow \mathcal{B}(1,1,\mu_0) \longrightarrow 0$$

where $\mathcal{B} = \widetilde{(\Omega^1_R(M_2) \otimes \text{Coker } \varphi_\xi)^{\vee\vee} \otimes R(2)}$ and $\mathcal{B}(1,1,\mu_0) = \widetilde{\text{Coker } \psi_\xi}$ (see lemma 4.4). By tensorization with the locally free sheaf $\mathcal{B}(0,1,-1)$ we get:

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{B} \otimes \mathcal{B}(0,1,-1) \longrightarrow \mathcal{B}(1,1,-\mu_0) \longrightarrow 0$$

Therefore the bundle $\mathcal{B} \otimes \mathcal{B}(0,1,-1)$ is isomorphic to the vector bundle $\mathcal{B}((0,1),1,\mu_0)$.
From lemma 4.4 we get that the module corresponding to this bundle is $(\Omega^1_R(M_2) \otimes \text{Coker } \varphi_{\lambda_0})^{\vee\vee} \otimes R(2)$.

Lemma 4.9. *Let be* $\xi = (1:0:1)$ *and* $\lambda_0 = (0:1:0)$. *With the notations from lemma* 4.4 *we have* $\widetilde{\Omega^1_R(M_2)} = \mathcal{B}((-4,-5),1,\mu_0^{-9})$.

Proof:

From the Proposition 4.8, we see that $\widetilde{\Omega^1_R(M_2)}$ has the form $\mathcal{B}((a,b),1,\zeta)$ with $a+b=-9$. We know from lemma 4.4 that $\mathcal{O}_X(1) = \mathcal{B}(3,1,-\mu_0^3)$. Therefore $\widetilde{\Omega^1_R(M_2)}(2) = \mathcal{B}((a+6,b+6),1,\zeta\mu_0^{12})$.
Thus $\mathcal{B}((0,1),1,\mu_0) = \mathcal{B}((a+6,b+6),1,\zeta\mu_0^{12}) \otimes \mathcal{B}(-1,1,-\mu_0^{-1})$. We obtain $\widetilde{\Omega^1_R(M_2)} = \mathcal{B}((-4,-5),1,\mu_0^{-9})$.

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