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On the adaptive selection of the parameter in regularization of ill-posed problems

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On the adaptive selection of the parameter in regularization of ill-posed problems

by

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Abstract

We study a possibility to use the structure of the regularization error for a posteriori choice of the regularization parameter. As a result, a rather general form of a selection criterion is proposed, and its relation to the heuristical quasi-optimality principle of Tikhonov and Glasko (1964), and to an adaptation scheme propsed in a statistical context by Lepskii (1990), is discussed. The advantages of the proposed criterion are illustrated by using such examples as self-regularization of the trapezoidal rule for noisy Abel-type integral equations, Lavrentiev regularization for non-linear ill-posed problems and an inverse problem of the two-dimensional profile reconstruction.

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1. Introduction and the general theorems

How to choose implicitly (a posteriori) a suitable value for the regularization parameter in ill-posed problems without knowledge about the solution smoothness which may not be accessible? This question is discussed extensively in the regularization theory. A first a posteriori rule of choice is described in the paper by Phillips [21], which predates even Tikhonov's paper [26] recognized as a reference point of regularization theory.
We define an operator equation

$$Ax = y \tag{1.1}$$

with a linear operator $A \in \mathcal{L}(X,Y)$ between Banach spaces X and Y as essentially ill-posed if the range $R(A)$ of A is not closed in Y. This nonclosed range is associated with the discontinuity of the inverse operator A^{-1}, presumably it exists. In general, the best approximate solution $A^{+}y$, where A^{+} is the Moore-Penrose inverse of A, does not depend continuously on the right-hand side y. Since in practice data will almost never be available exactly, but distorted by some measurement error, one has to be aware of numerical instabilities when instead of y only a noisy observation $y_\delta \in Y$ with

$$\|y - y_\delta\|_Y \leq \delta \tag{1.2}$$

is known. Hence, in order to approximate $A^{+}y$ in a stable way, regularization methods should be applied. In general, regularization methods for the solution of (1.1) replace the generalized inverse A^{+} by a family of continuous operators R_α, which converge pointwise to A^{+}. If $Qy \in R(A)$, where Q is the orthogonal projector onto $\overline{R(A)}$, then the standard regularization methods have in common that the approximation error $\|A^{+}y - R_\alpha y\|_X$ is monotonically decreasing for decreasing α-values. In general, it is natural to assume that there exists an increasing continuous function $\varphi(\alpha) = \varphi(\alpha; A, y)$ such that $0 = \varphi(0) \leq \varphi(\alpha) \leq 1$, and

$$\left\|A^{+}y - R_\alpha y\right\|_X \leq \varphi(\alpha)\ . \tag{1.3}$$

This property is no longer true for the regularization error $\|A^{+}y - R_\alpha y_\delta\|_X$. The regularized solutions $R_\alpha y_\delta$ converge to $A^{+}y$ as $\delta \to 0$ only if the regularization parameter α is properly chosen dependent upon the noise level and possibly upon the data, i.e. $\alpha = \alpha(\delta, y_\delta)$.There are several methods that have been proposed and used for the a posteriori choice of the regularization parameter α as a function of the noise level and the data. These include the discrepancy principle (DP) originally proposed by Phillips [21] and later reinvented by Morozov [20] and Marti [18], a method developed by Gferer [9], Engl [8] and Raus [23], which is sometimes called the minimum-bound (MB) method [17], and the monotone error rule (ME) proposed recently by Tautenhahn and Hämarik [25]. The MB and ME methods have been designed for ill-posed problems in Hilbert spaces. The DP method is more universal, because, as it was shown by Plato (see e.g. [22]) it can be successfully applied to problems in Banach spaces. However, it is known that DP does not provide the best order of approximation for all problems, which could be, in principle, treated by a fixed regulariziaton method with optimal order of accuracy, see e.g. [11]. The MB and ME methods are free from this drawback of discrepancy principle, but a disadvantage of both methods is that they require knowledge of an additional approximate solution obtained within the framework of the regularization method of higher qualification. For example, in [25] to choose the regularization parameter for the ordinary Tikhonov regularization one should construct an additional approximate solution using iterated Tikhonov regulatization, i.e. another regularization method should be involved in the choice procedure, that is not always reasonable.

At the same time the structure of regularization error is very similar to the loss function of statistical estimation, where some parameter always controls the trade-off between the bias and the variance of the risk. It gives a hint that the statistical art of bias-variance balancing can be used for choosing the regularization parameter.

Indeed, the regularization error can be estimated by

$$\left\|A^{+}y - R_\alpha y_\delta\right\|_X \leq \left\|A^{+}y - R_\alpha y\right\|_X + \left\|R_\alpha y - R_\alpha y_\delta\right\|_X \ , \tag{1.4}$$

where the first term on the right-hand side is an approximation error, whereas the second term is stability bound on the regularizing operator R_α, If R_α possesses a locally uniformly bounded Fréchet derivative $R_\alpha^{'}$ in a ball of radius δ around the exact free term y then

$$\left\|R_\alpha y - R_\alpha y_\delta\right\|_X \leq \delta \left\|R_\alpha^{'}\right\|_{Y\to X} + o(\delta) \ .$$

For linear problems (1.1) R_α is usually linear, and $R'_\alpha = R_\alpha$. Keeping in mind that $\{R_\alpha\}$ approximates the unbounded Moore-Penrose inverse A^+, it is easy to realize that $\|R_\alpha\|$ (or $\|R'_\alpha\|$) should increase for $\alpha \to 0$. Thus, there exists an increasing continuous function $\lambda(\alpha)$ such that $\lambda(0) = 0$, and

$$\|R_\alpha y - R_\alpha y_\delta\| \leq \frac{\delta}{\lambda(\alpha)} \ . \tag{1.5}$$

For each regularization method $\lambda(\alpha)$ is known, or at least it can be estimated effectively. For the standard regularization methods $\lambda(\alpha) = \gamma\sqrt{\alpha}$, where γ is a known constant. Another forms of $\lambda(\alpha)$ will be discussed below.
Thus, from (1.3)-(1.5) it follows that

$$\left\|A^+ y - R_\alpha y_\delta\right\|_X \leq \varphi(\alpha) + \frac{\delta}{\lambda(\alpha)} \ . \tag{1.6}$$

Almost all existing results about the accuracy of regularization methods are asymptotic results as $\delta \to 0$. These results indicate that a choice of

$$\alpha = \alpha_{opt} = (\varphi\lambda)^{-1}(\delta) \ , \tag{1.7}$$

that balances $\varphi(\alpha)$ with $\frac{\delta}{\lambda(\alpha)}$, will lead to an error estimate

$$\left\|A^+ y - R_{\alpha_{opt}} y_\delta\right\|_X \leq 2\varphi\left((\varphi\lambda)^{-1}(\delta)\right) \ , \tag{1.8}$$

which has at least optimal order with respect to δ. Unfortunately, an a priori parameter choice (1.7) can seldom be used in practice since the smoothness properties of the unknown solution A^+y reflected in function φ from (1.3) are generally unknown.
In practical applications different regularization parameters α_i are often selected from some finite set

$$\Delta_N = \{\alpha_i : 0 < \alpha_0 < \alpha_1 < \cdots < \alpha_N\}$$

and the corresponding regularization solutions

$$x^{\delta}_{\alpha_i} = R_{\alpha_i} y_\delta \ , \ i = 1, 2, ..., N \ ,$$

are studied on-line. In view of the representation

$$\alpha_{opt} = \max\left\{\alpha : \varphi(\alpha) \le \frac{\delta}{\lambda(\alpha)}\right\}$$

the optimal choice of α_i from Δ_N would be

$$\alpha_* = \alpha_\ell = \max\left\{\alpha_i : \alpha_i \in M(\Delta_N)\right\} \ ,$$

where

$$M(\Delta_N) := \left\{\alpha_i : \alpha_i \in \Delta_N, \ \varphi(\alpha_i) \le \frac{\delta}{\lambda(\alpha_i)}\right\}.$$

But if φ is unknown such a choice is also not feasible. At the same time, for any $\alpha_i, \alpha_j, \ \alpha_i \ge \alpha_j$, from the set $M(\Delta_N)$, containing α_* as an upper bound, the estimation of the norm $\left\|x^{\delta}_{\alpha_i} - x^{\delta}_{\alpha_j}\right\|$ does not require knowledge of φ. Indeed, due to the monotonicity of $\varphi(\alpha), \ \lambda(\alpha)$ from (1.6) it follows that

$$\begin{aligned}
\left\|x^{\delta}_{\alpha_i} - x^{\delta}_{\alpha_j}\right\| &\le \|A^+y - R_{\alpha_i} y_\delta\| + \left\|A^+y - R_{\alpha_j} y_\delta\right\| \\
&\le \varphi(\alpha_i) + \varphi(\alpha_j) + \tfrac{\delta}{\lambda(\alpha_i)} + \tfrac{\delta}{\lambda(\alpha_j)} \\
&\le 2\varphi(\alpha_i) + \tfrac{\delta}{\lambda(\alpha_i)} + \tfrac{\delta}{\lambda(\alpha_j)} \\
&\le \tfrac{4\delta}{\lambda(\alpha_j)}
\end{aligned}$$

It gives a hint that upper bound of the subset

$$M^+(\Delta_N) := \left\{ \alpha_i \in \Delta_N : \left\| x^\delta_{\alpha_i} - x^\delta_{\alpha_j} \right\| \le \frac{4\delta}{\lambda(\alpha_j)} \ , \ j = 0, 1, 2, ..., i \right\} \qquad (1.9)$$

should be sufficiently close to a desirable value α_*. The following proposition shows that it is really the case.

Theorem 1.1.
Let $\Delta_N = \Delta_N^{\lambda,q}$ *be such that* $M(\Delta_N) \neq \varnothing$, $\Delta_N \backslash M(\Delta_N) \neq \varnothing$, *and for any* $\alpha_i \in \Delta_N$, $i = 1, 2, N$

$$\lambda(\alpha_i) \le q\lambda(\alpha_{i-1}) \ , \qquad (1.10)$$

where q *is some fixed constant. Then under the assumptions (1.2), (1.3), (1.5) for* $\alpha_+ = \alpha_k \in \Delta_N$ chosen as

$$\alpha_+ = \max \left\{ \alpha_i : \alpha_i \in M^+(\Delta_N) \right\} \qquad (1.11)$$

the following estimate holds:

$$\left\| A^+ y - x^\delta_{\alpha_+} \right\| \le 6q\varphi \left((\varphi\lambda)^{-1}(\delta) \right) \ . \qquad (1.12)$$

Proof:
From the definition of $\alpha_* = \alpha_\ell$ it follows that for $\alpha_{\ell+1} > \alpha_\ell$

$$\varphi(\alpha_{\ell+1})\lambda(\alpha_{\ell+1}) > \delta = \varphi(\alpha_{opt})\lambda(\alpha_{opt}) \ ,$$

and using monotonicity of $\varphi(\alpha)$, $\lambda(\alpha)$ we deduce $\alpha_{\ell+1} > \alpha_{opt}$. Then under the assumption of our proposition

$$\lambda(\alpha_{opt}) < \lambda(\alpha_{\ell+1}) \le q\lambda(\alpha_\ell) = q\lambda(\alpha_*) \ .$$

Hence

$$\frac{\delta}{\lambda(\alpha_*)} \le q\frac{\delta}{\lambda(\alpha_{opt})} \ . \tag{1.13}$$

As already shown above, $M(\Delta_N) \subset M^+(\Delta_N)$, and therefore

$$\alpha_* = \alpha_\ell = \max\{\alpha_i \in M(\Delta_N)\} \le \alpha_+ = \alpha_k = \max\{\alpha_i \in M^+(\Delta_N)\} \ .$$

From the definition of $M^+(\Delta_N)$ and (1.13) we conclude

$$\begin{aligned} \left\|A^+y - x^\delta_{\alpha_+}\right\| &= \left\|A^+y - x^\delta_{\alpha_k}\right\| \le \left\|A^+y - x^\delta_{\alpha_\ell}\right\| + \left\|x^\delta_{\alpha_\ell} - x^\delta_{\alpha_k}\right\| \\ &\le \varphi(\alpha_\ell) + \tfrac{\delta}{\lambda(\alpha_\ell)} + \tfrac{4\delta}{\lambda(\alpha_\ell)} \le \tfrac{6\delta}{\lambda(\alpha_*)} \le 6q\tfrac{\delta}{\lambda(\alpha_{opt})} \\ &= 6q\varphi((\varphi\lambda)^{-1}(\delta)). \end{aligned}$$

The theorem is proved.

If we would know in advance the function $\varphi(\alpha)$ reflecting the smoothness properties of the unknown solution A^+y, we could achieve the accuracy of the optimal order given in (1.8). Comparing (1.8) with (1.12) we can conclude that the choice of the regularization parameter $\alpha = \alpha_+$ is also order optimal in the sense of accuracy. We would like to stress, however, that the selection criterion (1.9), (1.11) producing α_+ is adaptive to the unknown smoothness, because φ is not involved in its construction. Observe, that α_+ depends only on the noisy data y_δ, on the noise level δ, and on the discrete set $\Delta_N = \Delta_N^{\lambda;q}$ which should meet the conditions of Theorem 1.1. The conditions $M(\Delta_N) \ne \varnothing$, $\Delta_N \backslash M(\Delta_N) \ne \varnothing$ are rather natural. It is satisfied if, for example, $\alpha_0 = \lambda^{-1}(\delta) \in \Delta_N, \alpha_N = \lambda^{-1}(1) \in \Delta_N$. The condition (1.10) is also not so restrictive. Recall, that for the standard regularization methods $\lambda(\alpha) = \gamma\sqrt{\alpha}$. Then to meet (1.10) one can take Δ_N in the form of a geometric sequence

$$\Delta_N = \{\alpha_i : \alpha_i = \mu^i \alpha_0 \ , \ i = 0, 1, N \quad \} \tag{1.14}$$

with $\mu = q^2 > 1$.
It is interesting to note that the first time a geometric sequence was used as a set of regularization parameters in the papers by Tikhonov and Glasko [27], [28], where a method of choosing a paramter $\alpha_T = \alpha_m = \mu^m \alpha_0$ from such a sequence, termed quasi-optimality criterion, was suggested for which

$$\sigma(\alpha_i) := \left\| x^\delta_{\alpha_i} - x^\delta_{\alpha_{i-1}} \right\| \tag{1.15}$$

has the minimum value $\sigma(\alpha_m)$ in the chosen net (1.14). It is worth to mention that this quasi-optimality criterion is chronologically the first in the class of the heuristically motivated regulariziaton parameter choice rules that seek to avoid any a-priori knowledge of the noise level δ. There is, however, a negative result [1] which tells us that no convergence theory and error estimates as above can exist for noise level-free rules, and for the quasi-optimality criterion in particular.
At the same time the quasi-optimality criterion gives a hint that the quantities (1.15) can be used as indicators for the order optimal regularization parameter choice. Indeed, if α_{i-1}, $\alpha_i = \mu\alpha_{i-1}$ belong to the set $M(\Delta_N)$ containing the optimal parameter value $\alpha = \alpha_*$ then the quantitiy (1.15) can be estimated as

$$\left\| x^\delta_{\alpha_i} - x^\delta_{\alpha_{i-1}} \right\| \leq \frac{4\delta}{\lambda(\alpha_{i-1})}. \tag{1.16}$$

The right-hand side of (1.16) is a decreasing function of α. Therefore, the largest $\alpha_i \in \Delta_N$ satisfying (1.16) can not be far from α_T minimizing (1.15). This observation leads to the following noise level-dependent analog of the quasi-optimality criterion:

$$\overline{\alpha} = \max \left\{ \alpha_j \in \Delta_N : \left\| x^\delta_{\alpha_i} - x^\delta_{\alpha_{i-1}} \right\| \leq \frac{4\delta}{\lambda(\alpha_{i-1})} \ , \ i = 0, 1, 2, ..., j \right\} \tag{1.17}$$

Theorem 1.2.
Assume (1.2), (1.3), (1.5) to hol

$$\tfrac{\delta}{\alpha}\| \leq c\varphi(\varphi\lambda)^{-1}(\delta)), \tag{1.18}$$

where the constant c depends only on q, κ, κ_1, μ

Proof.
Let $\overline{\alpha} = \alpha_m \in \Delta_N$. From (1.9), (1.11) and (1.17) it follows that $\overline{\alpha} \geq \alpha_+$. Then, as in the proof of Theorem 1.1, one can deduce $\overline{\alpha} = \alpha_m \geq \alpha_+ = \alpha_k \geq \alpha_* = \alpha_\ell$, and using the triangle inequality successively, we arrive at

$$\begin{aligned}
\left\|A^+y - x^\delta_{\overline{\alpha}}\right\| &\leq \left\|A^+y - x^\delta_{\alpha_*}\right\| + \sum_{i=\ell+1}^{m} \left\|x^\delta_{\alpha_{i-1}} - x^\delta_{\alpha_i}\right\| \\
&\leq \left\|A^+y - x^\delta_{\alpha_*}\right\| + \sum_{i=\ell+1}^{m} \frac{4\delta}{\lambda(\alpha_{i-1})} \\
&\leq \left\|A^+y - x^\delta_{\alpha_*}\right\| + \sum_{\nu=0}^{m-\ell-1} \frac{4\delta}{\lambda(\alpha_*\mu^\nu)} \ .
\end{aligned}$$

On the other hand, for any $\mu > 1$, $b \quad 1$ and integers n, j such that $2^n \leq \mu \leq 2^{n+1}$, $2^j \leq b \leq 2^{j+1}$ iterating the strong Δ_2-condition, if necessary, one obtains

$$\frac{1}{\lambda(b\alpha_*)} \leq \frac{1}{\lambda(2^j\alpha_*)} \leq \frac{1}{\kappa^j\lambda(\alpha_*)} \leq \frac{\kappa}{\kappa^{\log_2 b}\lambda(\alpha_*)} \ ;$$
$$\lambda(\alpha_i) = \lambda(\alpha_{i-1}\mu) \leq \kappa_1^{n+1}\lambda(\alpha_{i-1}) \leq \kappa_1^{\log_2 2\mu}\lambda(\alpha_{i-1}).$$

It means that (1.10) is satisfied with $q = \kappa_1^{\log_2 2\mu}$. Using these observations and (1.13) we conclude

$$
\begin{aligned}
\left\|A^+y - x_{\bar{\alpha}}^{\delta}\right\| &\leq \varphi(\alpha_*) + \tfrac{\delta}{\lambda(\alpha_*)} + \tfrac{4\kappa\delta}{\lambda(\alpha_*)} \sum_{\nu=0}^{m-\ell-1} \left(\tfrac{1}{\kappa^{\log_2 \mu}}\right)^{\nu} \\
&\leq \tfrac{\delta}{\lambda(\alpha_*)} \left[2 + \tfrac{4\kappa^{\log_2 2\mu}}{\kappa^{\log_2 \mu} - 1}\right] = \tfrac{c_1\delta}{\lambda(\alpha_*)} \\
&\leq \tfrac{c_1\kappa_1^{\log_2 2\mu}}{\lambda(\alpha_{opt})}\, \delta = c\varphi((\varphi\lambda)^{-1}(\delta)) \ .
\end{aligned}
$$

The theorem ist proved.

At first glance the rule (1.17) looks like a simplified version of (1.9), (1.11), because it requires to compare the regularized solutions $x_{\alpha_i}^{\delta}$ corresponding to parameters with adjacent numbers only. But as it has been mentioned above, there are two different ideas behind these rules. The rule (1.17) is related to the heuristical quasi-optimality criterion. Up to a certain extent it supports heuristic theoretically. Moreover, numerical tests from [12] show that in some important particular cases both these criteria give the same value of regularization parameter. At the same time the rule (1.9), (1.11) has a statistical root. It was first studied in the paper [15] by Lepskii, devoted to statistical estimation from direct white noise observations that corresponds to (1.1) with identity operator A, but with random noisy data y_δ. Since then many authors have adopted this approach towards various statistical applications, we mention only [10], [5], [29], where the same idea has been realized in the context of ill-posed problems of the form (1.1) with compact operator A, but still with random noise. Deterministic noise model (1.2) allows to improve the order of accuracy of regularized solution, as it has been shown in [12], [4], [19] for the Hilbert space setting. Theorems 1.1 and 1.2 provide an uniform approach to such results. The rest of the present paper will be actually devoted to the illustration of general Theorems 1.1 and 1.2 on several new examples including a discussion about the advantages of the rules (1.9), (1.11) and (1.17).

2. Example 1: Self-regularization of the trapezoidal rule for noisy Abel-type integral equations.

Consider an equation of the form (1.1) with the Abel-type integral operator

$$
Ax(t) = A_\beta x(t) := \int\limits_0^t \frac{a(t,\tau)}{(t-\tau)^\beta} x(\tau)d\tau, \quad t \in [0,1] \ , \tag{2.1}
$$

in Banach spaces $X = Y = C = C_{[0,1]}$, where $a(t,\tau)$ is at least Lipschitz continuous on $0 \le \tau \le t \le 1$, and

$$
|a(t,t)| \ge a_0 > 0 \ . \tag{2.2}
$$

The parameter β satisfies $0 < \not{\beta} \quad 1$.
The trapezoidal-discretization method for (1.1), (2.1) has been intensively studied in [2], [30], [7]. It consists of replacing (1.1), (2.1) by a set of linear equations

$$
\int\limits_0^{\frac{i}{n}} \frac{a(\frac{i}{n},\tau)}{(\frac{i}{n}-\tau)^\beta} x(\tau)d\tau = y(\frac{i}{n}), \quad i = 1, 2, ..., n \ .
$$

Then one replaces each of them by means of discretizing the integral on the left as follows:

$$
n \sum_{j=1}^{i} \int\limits_{\frac{j-1}{n}}^{\frac{j}{n}} \frac{(\tau - \frac{j-1}{n}) a_{ij}^n x_{n,j} + (\frac{j}{n} - \tau) a_{ij-1}^n x_{n,j-1}}{(\frac{i}{n} - \tau)^\beta} d\tau = y(\frac{i}{n}) \ , \ i = 1, 2, n \tag{2.3}
$$

where $a_{ij}^n = a(\frac{i}{n}, \frac{j}{n})$, and $x_{n,j}$ denotes the numerical approximation to $x(\frac{j}{n})$, $j = 0, 1, 2, ..., n$. Thus, (2.3) is a system of n equations in $n+1$ unknown. For starting value $x_{n,0}$ one can take

$$
x(0) = \lim_{t \to 0} \frac{(1-\beta)}{a(0,0)} \frac{y(t)}{t^{1-\beta}} < 0 \ ,
$$

which exists, whenever (1.1), (2.1) has a continuous solution, or, as in [7],

$$x_{n,0} = \frac{(1-\beta)}{a(0,0)} \left\{ 3g(\frac{1}{n}) - 3g(\frac{2}{n}) + g(\frac{3}{n}) \right\} \ , \tag{2.4}$$

with $g(t) = t^{\beta-1}y(t)$. This yields the following triangular system for the approximations $\bar{x}_n = (x_{n,1}, x_{n,2}, ..., x_{n,n})^T$

$$\frac{n^{\beta-1}}{(1-\beta)(2-\beta)} \bar{A}_n \bar{x}_n = \bar{y}_n - \frac{n^{\beta-1}}{(1-\beta)(2-\beta)} \bar{b}_n \ , \tag{2.5}$$

where $\bar{y}_n = (y(\frac{1}{n}), y(\frac{2}{n}), y \quad (1))^T \ , \ \bar{b}_n = (b_{n,1}, b_{n,2}, ..., b_{n,n})^T$

$$b_{n,i} = a(\tfrac{i}{n}, 0) x_{n,0}, \ i = 1, 2, n$$

$$(\bar{A}_n)_{i,j} = \begin{cases} a^n_{i,j} \kappa_{i-j} & , \quad 1 \le j \le i \le n \ , \\ & \\ 0 & , \quad \text{otherwise} \ , \end{cases}$$

$$\begin{aligned} \kappa_\ell &= (\ell+1)^{2-\beta} - 2\ell^{2-\beta} + (\ell-1)^{2-\beta} \ , \ \ell \ge 1 \ . \\ \kappa_0 &= 1 \ . \end{aligned}$$

The question of the existence and uniqueness of a solution (2.5) is summarized in the following

Proposition 2.1. [7].
If $a(t,\tau)$ is Lipschitz continuous on $0 \le \tau \le t \le 1$, then there is a constant $\tilde{c}_{\beta,a}$ depending on β and such that $\left\|(\bar{A}_n)^{-1}\right\|_\infty \le \tilde{c}_{\beta,a}$, i.e. for any $\bar{f}_n = (f_1, f_2, ..., f_n)^T$, $\left\|\bar{f}_n\right\|_\infty := \max_i |f_i|$,

$$\left\|(\bar{A}_n)^{-1}\bar{f}_n\right\|_\infty \le \tilde{c}_{\beta,a} \left\|\bar{f}_n\right\|_\infty \ .$$

Moreover, in [2] (see also [31]) the convergence of the trapezoidal-discretization method has been shown to hold when the solution $x(t)$ of (1.1), (2.1) has only Lipschitz continuity and the same conditions on $a(t,\tau)$ apply. It means that there exists an increasing continuous function $\psi_{a,\beta}(x;t)$ such that $\psi_{a,\beta}(x;0) = 0$ and

$$\max_{0 \le i \le n} \left| x(\frac{i}{n}) - x_{n,i} \right| \le \psi_{a,\beta}(x, \frac{1}{n}) \ . \tag{2.6}$$

Let us turn to the case of the noisy equation

$$Ax(t) = y_\delta(t) \ , \tag{2.7}$$

where A has the form (2.1), and y_δ can be only element from $Y = C_{[0,1]}$ such that (1.2) holds.

The trapezoidal-discretization method can be applied directly to (2.7) if in (2.4), (2.5) $y(\frac{i}{n})$ will be replaced by $y_\delta(\frac{i}{n})$, $i = 1, 2, ..., n$. Then from the Proposition 2.1 it follows that there is always a unique solution $\bar{x}_n^\delta$ of the system

$$\frac{n^{\beta-1}}{(1-\beta)(2-\beta)} \bar{A}_n \bar{x}_n^\delta = \bar{y}_n^\delta - \frac{n^{\beta-1}}{(1-\beta)(2-\beta)} \bar{b}_n^\delta \ .$$

It is easy to see that $\left\| \bar{y}_n - \bar{y}_n^\delta \right\|_\infty \leq \delta$,

$$\left| x_{n,0} - x_{n,0}^\delta \right| \leq \frac{(1-\beta)}{|a(0,0)|} n^{1-\beta} \delta \left(3 + 3 \cdot 2^{\beta-1} + 3^{\beta-1} \right) = c_{\beta_{\mathcal{A}},\, 1} n^{1-\beta} \delta \ , \tag{2.8}$$

and

$$\left\| \bar{b}_n - \bar{b}_n^\delta \right\|_\infty \leq \| a(\cdot, 0) \|_C \left| x_{n,0} - x_{n,0}^\delta \right| \leq c_{\beta_{\mathcal{A}},\, 2} n^{1-\beta} \delta \ .$$

Thus, under the condition of Proposition 2.1 the following bound holds

$$\begin{aligned} \max_{0 \leq i \leq n} \left| x_{n,i} - x_{n,i}^\delta \right| &\leq n^{1-\beta} \left\| (\bar{A}_n)^{-1} \right\|_\infty ((1-\beta)(2-\beta) + c_{\beta_{\mathcal{A}},\, 2}) \delta \\ &\leq c_{\beta_{\mathcal{A}},\, 3} n^{1-\beta} \delta \end{aligned} \tag{2.9}$$

Within the framework of trapezoidal-discretization method the approximate solution of (1.1), (2.1) can be taken as piecewise linear interpolation spline $x_n(t)$ with uniform interpolation knots such that $x_n(\frac{i}{n}) = x_{n,i}$, $i = 0, 1, 2, ..., n$. If only noisy right-hand side $y_\delta(t)$ is available then such a spline will interpolate $x_{n,i}^\delta$ and have the form

$$x_n^\delta(t) = \sum_{i=0}^{n} x_{n,i}^\delta \ell_{n,i}(t) \ ,$$

where $\ell_{n,i}(t)$ are so-called fundamental linear splines with knots $\{\frac{i}{n}\}_{i=0}^{n}$ such that $\ell_{n,i}(t) \geq 0$ for $t \in [0,1]$, and $\ell_{n,i}(\frac{j}{n}) = \delta_{ij}$. From (2.8), (2.9) it follows that

$$\begin{aligned}\left|x_n(t) - x_n^\delta(t)\right| &\leq \sum_{i=0}^{n} \left|x_{n,i} - x_{n,i}^\delta\right| \ell_{n,i}(t) &\leq \max_{0\leq i\leq n} \left|x_{n,i} - x_{n,i}^\delta\right| \\ &\leq n^{1-\beta}\delta \max\{c_{\beta a,\,1}, c_{\beta a,\,3}\} &= c_{\beta a}\, n^{1-\beta}\delta\end{aligned} \tag{2.10}$$

Let now $s_n(x;t)$ be a piecewise linear spline with knots $\{\frac{i}{n}\}_{i=0}^{n}$ interpolating the values $x(\frac{i}{n}),\ i = 0,1,...,n,$ of the solution (1.1), (2.1). It is well known that

$$|x(t) - s_n(x;t)| \leq c\omega_2(x; \tfrac{1}{n}) \ ,$$

where $\omega_2(x;h)$ is the second-order modulus of smoothness, $\omega_2(x;h) \to 0$, and c is some absolute constant. Using (2.6) this yields

$$\begin{aligned}|x(t) - x_n(t)| &\leq |x(t) - s_n(x;t)| + |s_n(x;t) - x_n(t)| \\ &\leq c\omega_2(x;\tfrac{1}{n}) + \sum_{i=0}^{n} \left|x(\tfrac{i}{n}) - x_{n,i}\right| \ell_{n,i}(t) \\ &\leq c\omega_2(x;\tfrac{1}{n}) + \psi_{a,\beta}(x;\tfrac{1}{n}) = \varphi(\tfrac{1}{n}) \ .\end{aligned}$$

Combining it with (2.10) we obtain

$$\left|x_n(t) - x_n^\delta(t)\right| \leq \varphi(\frac{1}{n}) + c_{\beta a}\, n^{1-\beta}\delta \ . \tag{2.11}$$

Here the function φ depends on the smoothness of the solution (1.1), (2.1) and usually is unknown. But (2.11) has the same form as (1.6), where $\alpha = \frac{1}{n},\ \lambda(\alpha) = c_{\beta a}^{-1}\alpha^{1-\beta}$. For such α and $\lambda(\alpha)$ we have $\Delta_N = \left\{\alpha_i = \frac{1}{N-i+1}\right\}_{i=0}^{N}$ and

$$\begin{aligned}M^+(\Delta_N) &= \{\alpha_i = \tfrac{1}{N-i+1} : \left\|x_{N-i+1}^\delta - x_{N-j+1}^\delta\right\|_C \leq 4\delta c_{\beta a}(N-j+1)^{1-\beta}, \\ &\quad j = 0,1,2,i \quad \} \\ &= \left\{n : \left\|x_n^\delta - x_m^\delta\right\|_C \leq 4\delta c_{\beta a}\, m^{1-\beta},\ m = N+1, N, ..., n\right\} \ .\end{aligned}$$

Hence the selection criterion (1.9), (1.11) can be written as

$$n_+ := \min\left\{n : \left\|x_n^\delta - x_m^\delta\right\|_C \le 4\delta c_{\beta,a} m^{1-\beta},\ m = N+1, N, ..., n\right\} \qquad (2.12)$$

and the conditions of Theorem 1.1 are satisfied with $q = 2^{1-\beta}, \succcurlyeq N \quad (c_{\beta,a}\delta)^{\frac{1}{\beta-1}}$. Thus, we have

Theorem 2.1.
If $a(t,\tau)$ and the solution $x(t)$ of the equation (1.1), (2.1) are Lipschitz continuous on $0 \le \tau \le t \le 1$, then for $N > (c_{\beta,a}\delta)^{\frac{1}{\beta-1}}$ and n_+ chosen as (2.12)

$$\left|x(t) - x_{n+}^\delta(t)\right| \le 6 \cdot 2^{1-\beta}\varphi((\varphi\lambda)^{-1}(\delta)) \ ,$$

where $\lambda(\alpha) = c_{\beta,a}^{-1}\alpha^{1-\beta}$, $\varphi(\alpha) = c\omega_2(x;\alpha) + \psi_{a,\beta}(x;\alpha)$, and $\psi_{a,\beta}$ is the function from (2.6).

Remark 2.1.
We can indicate only one case when the order of the function φ is known. Namely, in [7] it has been shown that $\psi_{a,\beta}(x, \frac{1}{n}) = c_1 n^{-2}$ for all $\beta \in (0,1)$, and for $x(t)$, $a(t,\tau)$ having Lipschitz continuous second derivatives. For such $x(t)$ $\omega_2(x; \frac{1}{n})$ has the best possible order $\omega_2(x; \frac{1}{n}) = c_2 n^{-2}$. Thus, in the considered case $\varphi(\frac{1}{n}) = c_3 n^{-2}$, and to balance both terms in (2.11) one should take $n = n_{opt} \asymp \delta^{\frac{1}{\beta-3}}$ that gives an accuracy of order $O(\delta^{\frac{2}{3-\beta}})$. Note that Theorem 2.1 gives the same order of accuracy automatically without knowledge of φ.

Theorem 2.1 shows that the regularization of ill-posed problem (1.1), (2.1) with noisy right-hand side y_δ can be achieved by just choosing the number of knots in the trapezoidal rule properly. This is called self-regularization. Self-regularization adapted to unknown smoothness in a Hilbert space has been discussed recently in [13], [12], [4] To the best of our knowledge, Theorem 2.1 is the first example of adaptive self-regularization in Banach space.

3. Example 2: Lavrentiev regularization for nonlinear ill-posed problems with monotone operators.

Throughout this section we assume that $A : D(A) \to X$ is a nonlinear monotone operator with domain $D(A)$ in a real Hilbert space X. Monotonicity means that for all $x_1, x_2 \in D(A)$

$$\langle A(x_1) - A(x_2),\ x_1 - x_2 \rangle \geq 0 \ ,$$

where $\langle \cdot, \cdot \rangle$ is the inner product associated with norm $\|\cdot\| = \|\cdot\|_X$.
We further assume throughout that a nonlinear equation

$$A(x) = y$$

has a solution x^+, but only a noisy data y_δ with a known noise level δ is available, i.e. $\|y - y_\delta\| \leq \delta$. we do not assume that x^+ depends continuously on the data. It means that the stable reconstruction of x^+ from the noisy equation

$$A(x) = y_\delta \tag{3.1}$$

requires the application of special regularization methods. In the well-known Tikhonov regularization method a regularized approximation x_α^δ is obtained by minimizing the functional

$$J_\alpha(x) = \|A(x) - y_\delta\|^2 + \alpha \|x - \bar{x}\|^2 \ ,$$

with some initial guess $\bar{x} \in X$ and some properly chosen regularization parameter $\alpha > 0$. If A is Fréchet differentiable in some ball $B_\rho(x^+)$ of radius ρ around x^+, and x_α^δ is an interior point of $D(A)$ then it can be found from the (nonlinear) normal equation of Tikhonov's functional $J_\alpha(x)$

$$[A'(x)]^* \left[A(x) - y^\delta\right] + \alpha(x - \bar{x}) = 0 \ ,$$

where $[A'(x)]^*$ is the adjoint of the Fréchet-derivative $A'(x)$. As it has been indicated in [16], [24], for the problems with monotone operators the least squares minimization (and hence the use of the Fréchet-derivaties) can be avoided and one can use the simpler regularized equation

$$A(x) + \alpha(x - \bar{x}) = y_\delta \tag{3.2}$$

known as Lavrentiev regularization.
If $D(A) = X$ and $A(x)$ is a continuous operator, then as it has been shown in [6], pp. 97, 100, the monotonicity implies that for $\alpha > 0$ the operator

$F(x) = \alpha x + A(x),\ x \in X$, is strongly monotone, and $F^{-1}(x) = (\alpha I + A)^{-1}(x)$ is Lipschitz with constant $\frac{1}{\alpha}$, i.e. for any $u, v \in X$

$$\left\|(\alpha I + A)^{-1}(u - v)\right\| \leq \frac{1}{\alpha} \left\|u - v\right\| \ . \tag{3.3}$$

Applying Lavrentiev regularization one usually assumes that for pure data $y = A(x^+)$ it produces an approximate solution $x_\alpha = (\alpha I + A)^{-1}(y + \alpha\bar{x})$ converging to x^+ as $\alpha \to 0$. It means that there exists an increasing continuous function $\varphi(\alpha) = \varphi(x^+; \alpha)$ such that $\varphi(0) = 0$ and

$$\left\|x^+ - x_\alpha\right\| \leq \varphi(\alpha) \ . \tag{3.4}$$

Theorem 3.1.
Let $A(x)$ be a continuous monotone operator in a real Hilbert space X. Consider $\Delta_N = \{\alpha_i = q^i\delta,\ i = 0, 1, ..., N\}$, $q > 1$, $\alpha_N \simeq 1$, and $\bar{\alpha} = \max\left\{\alpha_j \in \Delta_N : \left\|x^\delta_{\alpha_i} - x^\delta_{\alpha_{i-1}}\right\| \leq 4q^{1-i},\ i = 1, 2, ..., j\right\}$, where $x^\delta_{\alpha_i}$ is the unique solution of (3.2) for $\alpha = \alpha_i$. Then under the assumption (3.4)

$$\left\|x^+ - x^\delta_{\bar{\alpha}}\right\| \leq \frac{(6q-2)q}{q-1}\varphi(x^+; \theta^{-1}_\varphi(\delta)) \ , \tag{3.5}$$

where $\theta_\varphi(t) = \varphi(t)t$.

Proof.
From (3.3) and (3.4) it follows that for any $\alpha > 0$

$$\begin{aligned}
\left\|x^+ - x^\delta_\alpha\right\| &\leq \left\|x^+ - x_\alpha\right\| + \left\|x_\alpha - x^\delta_\alpha\right\| \\
&\leq \varphi(\alpha) + \left\|(\alpha I + A)^{-1}(y - y_\delta)\right\| \\
&\leq \varphi(\alpha) + \tfrac{\delta}{\alpha} \ ,
\end{aligned}$$

Hence, the error bound has the form (1.6) with $\lambda(\alpha) = \alpha$. It is easy to see that for such $\lambda(\alpha)$ all conditions of Theorem 1.2 are satisfied. Moreover, for $\lambda(\alpha) = \alpha$ the arguments from the proof can be simplified, and it gives an explicit form of the constant c near the optimal order.

Remark 3.1.
Lavrentiev regularization is usually studied under the assumption (3.4) with $\varphi(\alpha) = c\alpha^p$, $p \in (0, 1]$. For example, the case of unknown p has been discussed

recently in [24], where it has been shown that for α chosen as the solution of the nonlinear equation

$$\left\|\alpha(\alpha I + A'(x_\alpha^\delta))^{-1}(A(x_\alpha^\delta) - y_\delta)\right\| = c_1\delta,\ c_1 > 1\ ,$$

one has

$$\left\|x^+ - x_\alpha^\delta\right\| \le c_p\delta^{\frac{p}{p+1}}\ .$$

The disadvantage of this a posteriori rule is that its combination with Lavrentiev regularization (3.2) does not allow to avoid the use of the Fréchet-derivatives. At the same time, an a posteriori rule presented in Theorem 3.1 is free from the above mentioned drawback and gives the same order of accuracy $0(\delta^{\frac{p}{p+1}})$ for $\varphi(\alpha) = c\alpha^p$.
Moreover, to our knowledge, the rule from Theorem 3.1 is the only one which allows to reach the best possible order of accuracy of Lavrentiev regularization automatically, and does not involve another regularization methods.

4. Example 3: Inverse problem of profile reconstruction in diffractive optics.

The statement of the problem discussed in this section is borrowed from [3]. Let the profile of two-dimensional diffraction grating is described by the curve

$$\Lambda_f := \{(x_1, f(x_1)) : x_1 \in \mathbb{R}\}$$

with 2π-periodic function f. Let

$$\Omega_f := \{x = (x_1, x_2) : x_2 > f(x_1),\ x_1 \in \mathbb{R}\}$$

be filled with a material whose index of refraction k is some positive constant. Suppose that a plane wave given by

$$u^{in}(x) = \exp(i\alpha x_1 - i\beta x_2)$$

is insident on Λ_f from the top, where $\alpha = k\sin\theta$, $\beta = k\cos\theta$, and $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ is the incident angle. Then the scattering of this wave by Λ_f is modelled by the Dirichlet problem for the Helmholz equation

$$\Delta u + k^2 u = 0 \ \text{ in } \ \Omega_f \ \ ,\ \ u = -u^{in} \ \text{ on } \ \Lambda_f \tag{4.1}$$

where the scattered field u is assumed to satisfy a radiation condition, i.e. u is composed of bounded outgoing plane waves:

$$u(x_1, x_2) = \sum_{n \in \mathbb{Z}} A_n \exp[i(n+\alpha)x_1 + i\beta_n x_2] \ , \tag{4.2}$$

with $\beta_n = \sqrt{k^2 - (n+\alpha)^2} \in \mathbb{C}$, and the Rayleigh coefficients $A_n \in \mathbb{C}$. To exclude resonances one assumes that $\beta_n \neq 0, \ n \in \mathbb{Z}$.
The inverse problem of profile reconstruction is to recover the profile function f from the trace $u_b(x) = u(x, b)$ of the scattered field $u(x_1, x_2)$ on the line $x_2 = b$ for a given incident wave u^{in}. Without loss of generality we can assume that the unknown profile Λ_f lies avbove the line $x_2 = b_0$ and below $x_2 = b$, i.e.

$$b_0 < f(x_1) < b, \quad x_1 \in \mathbb{R} \ . \tag{4.3}$$

Representing the scattered field as a single layer potential

$$u(x_1, x_2) = \int_0^{2\pi} z(t) G(x_1, x_2, t, 0) dt$$

with unknown density function $z \in L_2(0, 2\pi)$ and the free space quasiperiodic Green function

$$G(x_1, x_2, y_1, y_2) = \frac{i}{2\pi} \sum_{n \in \mathbb{Z}} \frac{1}{\beta_n} \exp[i(n+\alpha)(x_1 - y_1) + i\beta_n (x_2 - y_2)],$$

one can reduce the inverse problem of profile reconstruction to the following system of integral equations

$$\begin{array}{rclcl} Tz(x_1) & := & \int_0^{2\pi} z(t) G(x_1, b, t, 0) dt & = & u_b(x_1) \\ S_f z(x_1) & := & \int_0^{2\pi} z(t) G(x_1, f(x_1), t, 0) dt & = & -u^{in} \circ f(x_1) \end{array} \tag{4.4}$$

which is nonlinear with respect to f. Here $u^{in} \circ f(x_1) = \exp(i\alpha x_1 - i\beta f(x_1))$. Applying the arguments from the proof of Lemma 4.1 and Theorem 4.2 [3] one can obtain

Proposition 4.1.
Let $u_b(x_1)$ be the exact pattern of the scattered field $u(x_1, x_2)$ on $x_2 = b$ which corresponds to some 2π-periodic profile functions $f \in C^2(\mathbb{R})$ meeting (4.3). Then there exists a solution (z_0, f_0) of the system (4.4). If in addition the inverse problem of profile reconstruction is uniquely solvable then $f = f_0$.

Note that in problem (4.4) the knowledge of all Rayleigh coefficients A_n of the scattered waves is required. At the same time, the Fourier coefficients of $u_b(x_1) = u(x_1, b)$ with respect to orthonormal basis $\{\exp[i(n+\alpha)x_1]\}_{n\in\mathbb{Z}}$ of the complex Hilbert space $L_2(0, 2\pi)$ have the form $A_n e^{i\beta_n b}$, $n \in \mathbb{Z}$, and decay exponentially. Therefore, in practice one is able to measure only a finite number of A_n, $n \in U$, corresponding to outgoing plane waves (modes) of the scattered field (4.2) that can be observed on the line $x_2 = b$. Here U is some finite index set. Moreover, even these coefficients will not be given exactly but perturbed by measurement errors. To be more precise, we have only a vector $(A_n^\delta)_{n\in U}$ determining the "noisy trace"

$$u_b^\delta(x_1) = \sum_{n\in U} A_n^\delta \exp[i(n+\alpha)x_1 + i\beta_n b]$$

such that

$$\left\| u_b - u_b^\delta \right\| \leq \delta \ , \tag{4.5}$$

where $\|\cdot\|$ denotes the norm in the complex Hilbert space $L_2(0, 2\pi)$.
Thus, replacing the scattered field u_b by u_b^δ one obtains the system (4.4) containing the noisy equation $Tz = u_b^\delta$, and for a stable profile reconstruction its regularized version should be considered. Such an approach was first proposed by Kirsch and Kress [14] for acoustic obstacle scattering. For profile reconstruction problem it has been developed recently in [3]. These authors have observed that the structure of the system (4.4) allows to decompose the inverse problem of profile reconstruction into the severely ill-posed linear problem of estimating the scattered field potential density $z(t)$, and into the well-posed nonlinear problem of determining the unknown profile function as the location of the zeros of the total field; the later problem can be then replaced by the finite dimensional nonlinear least squares problem.
If $z(t)$ is given as a Fourier series

$$z(t) = \sum_{n\in\mathbb{Z}} z_n \exp[i(n+\alpha)t], \quad z_n \in \mathbb{C} \ ,$$

then the operators from the system (4.4) can be represented in the following form

$$Tz(x_1) = i \sum_{n\in\mathbb{Z}} z_n \beta_n^{-1} \exp[i(n+\alpha)x_1 + i\beta_n b], \tag{4.6}$$

$$S_f z(x_1) = i \sum_{n\in\mathbb{Z}} z_n \beta_n^{-1} \exp[i(n+\alpha)x_1 + i\beta_n f(x_1)]\ . \tag{4.7}$$

Now it can be easily checked that T is an injective linear operator, whose inverse T^{-1} acts continuously from $L_2 = L_2(0, 2\pi)$ to the Hilbert space of generalized functions

$$L^{-b}_{2,\exp} := \left\{ z : \|z\|^2_{L^{-b}_{2,\exp}} := \sum_{n\in\mathbb{Z}} |z_n|^2 \left|e^{2i\beta_n b}\right| |\beta_n|^{-2} < \infty \right\} ,$$

where z_n is the value of the functional $\left\langle z, e^{\,i(n+\alpha)} \right\rangle_{L_2(0,2\pi)}$, $n \in \mathbb{Z}$. Thus, if the problem was to find the solution of equation $Tz = u_b^\delta$ in the space $L^{-b}_{2,\exp}$, it would be well-posed. But the second equation of (4.4) presumes $S_f z \in L_2(0, 2\pi)$ for all admissible function meeting (4.3). One can guarantee it if $z \in L^{-b_0+h}_{2,\exp}$ for some $0 < h < b_0$. Indeed, $|\beta_m| \sim m$ and

$$\begin{aligned} \|S_f z\|^2 &= \int_0^{2\pi} \left| \sum_{n\in\mathbb{Z}} z_n \beta_n^{-1} e^{i(\alpha+n)x_1} e^{i\beta_n f(x_1)} \right|^2 dx_1 \\ &\le c \left(\sum_{n\in\mathbb{Z}} \frac{|z_n|^2}{|\beta_n|^2} \left| e^{2i\beta_n(b_0-h)} \right| \right) \int_0^{2\pi} \sum_{n\in\mathbb{Z}} e^{-2|\beta_n|(f(x_1)-b_0+h)} dx_1 \\ &\le \tfrac{c}{1-e^{-2h}} \|z\|^2_{L^{-b_0+h}_{2,\exp}} = c_h \|z\|^2_{L^{-b_0+h}_{2,\exp}} \ , \end{aligned} \tag{4.8}$$

where the constant c_h depends only on h. Thus, it is reasonable to seek for solution of $Tz = u_b^\delta$ in the space $L^{-b_0+h}_{2,\exp}$.

Remark 4.1.
In [3] it has been proposed to regularize the first equation of (4.4) in the space L_2. Keeping in mind that $L_2 \hookrightarrow L^{-b_0+h}_{2,\exp}$ it is easy to realize that for the pair (L_2, L_2) the problem $Tz = u_b^\delta$ is more ill-posed than for $(L^{-b_0+h}_{2,\exp}, L_2)$. Moreover, for any regularized solution z_δ of equation $Tz = u_b$ one has

$$\|z_0 - z_\delta\|_{L^{-b_0+h}_{2,\exp}} \le \|z_0 - z_\delta\|_{L_2} \ ,$$

where $z_0 = T^{-1}u_b$. At the same time, from (4.8) it follows that the perturbation of the left-hand side of the second equation of (4.4) caused by the replacement z_0 for z_δ can be estimated as

$$\|S_f z_0 - S_f z_\delta\| \le c_h \|z_0 - z_\delta\|_{L^{-b_0+h}_{2,\exp}} \ .$$

It supports the use of $L^{-b_0+h}_{2,\exp}$ as a more suitable space for the problem under consideration.
Singular value expansion (4.6) of the operator T allows to apply the spectral cut off scheme for the regularization of the equation $Tz = u_b^\delta$. It gives the following sequence of regularized solutions:

$$z_{m,\delta}(x_1) = -i \sum_{|n|<m} A_n^\delta \beta_n \exp[i(\alpha + n)x_1], \ m = 1, 2, ..., M+1 \ , \tag{4.9}$$

where $M = \max\{m : (-m, -m+1, ..., m-1, m) \subset U\}$. Replacing in (4.9) A_n^δ with A_n, one obtains the partial sum $z_{m,0}$ of the Fourier series

$$z_0(x_1) = T^{-1}u_b(x_1) = -i \sum_{n\in\mathbb{Z}} A_n \beta_n \exp[i(\alpha + n)x_1] \ .$$

Keeping in mind that $\|z_{m,0} - z_0\| \to 0$ as $m \to \infty$, and $\|z_{m,0} - z_0\|_{L^{-b_0+h}_{2,\exp}} \le \|z_{m,0} - z_0\|$, we deduce that there exists an increasing continuous function $\varphi(\lambda)$ such that $\varphi(0) = 0$ and

$$\|z_0 - z_{m,0}\|_{L^{-b_0+h}_{2,\exp}} \le \varphi(\tfrac{1}{m}) \ . \tag{4.10}$$

Moreover, from (4.5) it follows that

$$\begin{aligned}
\|z_{m,0} - z_{m,\delta}\|^2_{L^{-b_0+h}_{2,\exp}} &= \sum_{|n|<m} \left|A_n - A_n^\delta\right|^2 \left|e^{2i\beta_n(b_0-h)}\right| \\
&= \sum_{|n|<m} \left|A_n - A_n^\delta\right|^2 \left|e^{2i\beta_n b}\right| \left|e^{2i\beta_n(b_0-h-b)}\right| \\
&\le e^{2|\beta_m|(b+h-b_0)} \left\|u_b - u_b^\delta\right\|^2 \\
&\le \delta^2 e^{2|\beta_m|(b+h-b_0)} .
\end{aligned}$$

Then

$$\|z_0 - z_{m,\delta}\|_{L^{-b_0+h}_{2,\exp}} \le \varphi(\tfrac{1}{m}) + \delta e^{|\beta_m|(b+h-b_0)} .$$

This estimation has the form (1.6) with $\alpha = \frac{1}{m}$ and

$$\lambda(\alpha) = \exp\left[-\sqrt{|k^2 - (\alpha^{-1} + k\sin\theta)^2|}(b+h-b_0)\right] . \tag{4.11}$$

As in Section 2 we consider $\Delta_M = \left\{\alpha_i = \frac{1}{M-i+1}\right\}_{i=0}^{M}$. Keeping in mind that

$$c_1 e^{-\frac{a}{\alpha}} \le \lambda(\alpha) \le c_1 e^{-\frac{a}{\alpha}}$$

with $a = (b+h-b_0)$ and some constants c_1, c_2 depending only on k and θ, it is easy to check that in considered case the condition (1.10) is satisfied with $q = \frac{c_2 e^a}{c_1}$. Then, as in Section 2, the straightforward application of Theorem 1.1 gives

Theorem 4.1
Assume that the inverse problem of profile reconstruction is uniquely solvable. If M is sufficiently large such that $M \sim (b+h-b_0)^{-1}\ln\frac{1}{\delta}$ then for m_+ chosen as

$$m_+ = \min\left\{m : \|z_{m,\delta} - z_{n,\delta}\|^2_{L^{-b_0+h}_{2,\exp}} \le 4\delta e^{|\beta_n|(b+h-b_0)} ,\ n = M+1, M\!\!\!/m \qquad\right\}$$

one has

$$\left\|z_0 - z_{m_+,\delta}\right\|_{L^{-b_0+h}_{2,\exp}} \le c\varphi((\varphi\lambda)^{-1}(\delta)) ,$$

where φ, λ are the functions from (4.10), (4.11), and c depends only on b, b_0, h, θ

Remark 4.2.
In the case under consideration the spectral cut of scheme (4.9) can be combined with the discrepancy principle. Then the regularization parameter m would be chosen as

$$m_d = \min \left\{ m : \left\| T z_{m,\delta} - u_b^\delta \right\|_{L_2} \leq d\delta;\ m = 1, 2, M + 1 \right\} \tag{4.12}$$

It is easy to observe that the combination of (4.9) with (4.12) does not take into account our wish to regularize a problem in such an "exotic" space as $L_{2,\exp}^{-b_0+h}$. In this respect the parameter choice rule discussed above is much more flexible, and it is one more advantage of it.

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