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by

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Abstract

The Earth's surface is an almost perfect sphere. Deviations from its spherical shape are less than 0.4% of its radius and essentially arise from its rotation. All equipotential surfaces are nearly spherical, too. In consequence, multiscale modelling of geoscientifically relevant data on the sphere involving rotational symmetry of the trial functions used for the approximation plays an important role. In this paper we deal with isotropic kernel functions showing local support and (one-dimensional) polynomial structure (briefly called isotropic finite elements) for reconstructing square–integrable functions on the sphere. Essential tool is the concept of multiresolution analysis by virtue of the spherical up function. The main result is a tree algorithm in terms of (low–order) isotropic finite elements.

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1   Preliminaries

A radial basis function on the unit sphere $\Omega \subset \mathbb{R}^3$ is introducable as follows: To a function $K \in L^1[-1,1]$ we are able to associate a kernel function $\tilde{K} : \Omega \times \Omega \to \mathbb{R}$ by letting $(\xi, \eta) \mapsto \tilde{K}(\xi, \eta) = K(\xi \cdot \eta)$, $(\xi, \eta) \in \Omega \times \Omega$. Then $\tilde{K}$ is called *radial basis function* on Ω (note that $\tilde{K}$ actually depends on the Euclidean distance $|\xi - \eta|$ because of the fact that $|\xi - \eta|^2 = 2 - 2\xi \cdot \eta$). $\tilde{K}$ is said to be in $C(\Omega \times \Omega)$ and $L^p(\Omega \times \Omega)$, $1 \le p < \infty$, if and only if K is in $C[-1,1]$ and $L^p[-1,1]$, respectively. Moreover, it should be mentioned that, for $\xi \in \Omega$ fixed, the (spherical) function $\eta \mapsto K(\xi, \eta)$, $\eta \in \Omega$, is in $C(\Omega)$ and $L^p(\Omega)$, $1 \le p < \infty$, if and only if K is in $C[-1,1]$ and $L^p[-1,1]$, respectively. The C–norm and the L^p–norm of $\eta \mapsto K(\xi \cdot \eta)$, $\eta \in \Omega$, respectively, do not depend on $\xi \in \Omega$. This result is obvious for a continuous functions, and in the case of L^p–functions it is implied by $\int_\Omega K(\xi \cdot \eta) \, d\omega(\eta) = 2\pi \int_{-1}^{+1} K(t) \, dt$ for $K \in L^1[-1,1]$ and $\xi \in \Omega$. By use of radial basis functions, a variety of different approximation procedures can be formulated on the sphere. For a review the reader is referred e.g. to [7].

In constructive approximation, certain interrelations of radial basis functions to the 'spherical polynomials' on the sphere, i.e. the *spherical harmonics*, play an important role, which should be recapitulated briefly: Let $\{Y_{n,k}\}$ be a maximal orthonormal system of spherical harmonics $Y_{n,k}$ of degree n and order k. Then $\{Y_{n,k}\}$ is known to be closed and complete in $L^2(\Omega)$. Any $K \in L^2[-1,1]$ admits an orthogonal expansion of the form

$$K(\xi \cdot \eta) = \sum_{n=0}^{\infty} \frac{2n+1}{4\pi} K^\wedge(n) P_n(\xi \cdot \eta), \quad (\xi, \eta) \in \Omega \times \Omega, \tag{1}$$

where $K^\wedge(n)$, $n \in \mathbb{N}_0$, is the n–th Legendre coefficient of K given by

$$K^\wedge(n) = 2\pi \int_{-1}^{+1} K(t) P_n(t) \, dt, \tag{2}$$

and P_n is the Legendre polynomial of degree n. By virtue of the addition theorem of spherical harmonics (see e.g. [17], [7]) the identity (1) can be rewritten as

$$K(\xi \cdot \eta) = \sum_{n=0}^{\infty} K^\wedge(n) \sum_{j=1}^{2n+1} Y_{n,j}(\xi) Y_{n,j}(\eta), \quad (\xi, \eta) \in \Omega \times \Omega. \tag{3}$$

An important result connecting spherical harmonics and radial basis functions is the *Funk–Hecke formula*: Suppose that K is of class $L^1[-1,1]$. Then for $\xi, \eta \in \Omega$,

$$\int_\Omega K(\xi \cdot \eta) P_n(\eta \cdot \zeta) \, d\omega(\eta) = K^\wedge(n) P_n(\xi \cdot \zeta). \tag{4}$$

Suppose that $F \in L^2(\Omega)$ and $K \in L^2[-1,1]$. Then the *spherical convolution* $K * F$ is well–defined by

$$(K * F)(\xi) = \int_\Omega K(\xi \cdot \eta) F(\eta) d\omega(\eta), \quad \xi \in \Omega, \tag{5}$$

and $K * F$ is of class $L^2(\Omega)$. More explicitly, for $F \in \mathcal{L}^2(\Omega)$ and $K \in \mathcal{L}^2[-1,1]$, $K * F$ can be expressed as an orthogonal expansion in terms of spherical harmonics

$$(K * F)(\xi) = \sum_{n=0}^{\infty} \sum_{j=1}^{2n+1} K^\wedge(n) F^\wedge(n,j) Y_{n,j}(\xi), \tag{6}$$

where $F^\wedge(n, j)$ are the orthogonal (Fourier) coefficients of F

$$F^\wedge(n, j) = \int_\Omega F(\eta) Y_{n,j}(\eta) d\omega(\eta).$$

Of particular importance is the convolution with a second zonal function. Assume that H is of class $L^2[-1, 1]$. An easy application of the Funk–Hecke formula shows that

$$(G * H)(\xi \cdot \zeta) = \int_\Omega G(\xi \cdot \eta) H(\eta \cdot \zeta) d\omega(\eta), \quad \xi, \zeta \in \Omega, \tag{7}$$

depends only on the inner product of ξ and ζ. Thus, $G * H$ is considered as another zonal function and can immediately be seen to be a continuous function defined on the interval $[-1, 1]$. It easily follows that for $G, \ H \in L^2[-1, 1]$

$$(G * H)^\wedge(n) = G^\wedge(n) H^\wedge(n), \quad n = 0, 1, \dots . \tag{8}$$

The convolution of a function $G \in L^2[-1, 1]$ with itself constitutes the so–called iterated function:

$$G^{(2)} = G * G, \quad G^{(k+1)} = G * G^{(k)}, \quad k = 2, 3, \dots . \tag{9}$$

Obviously, $(G^{(2)})^\wedge(n) = (G^\wedge(n))^2$, and $G^{(2)} \in C[-1, 1]$.

The concept of convolutions against scale dependent radial basis functions enables us to introduce singular integrals on the unit sphere: Let $\{K_h\}_{h\in(-1,1)}$ be a family of functions in $L^2[-1, 1]$ satisfying the condition $(K_h)^\wedge(0) = 1$ for all $h \in (-1, 1)$. Then the family of bounded linear operators $\{I_h\}_{h\in(-1,1)}$, $I_h : L^2(\Omega) \to L^2(\Omega)$, $F \mapsto I_h(F)$, given by

$$I_h(f)(\xi) = (K_h * F)(\xi) = \int_\Omega K_h(\xi \cdot \eta) F(y) d\omega(\eta), \quad \xi \in \Omega, \tag{10}$$

is called a *spherical singular integral*. The family $\{K_h\}_{h\in(-1,1)}$ is called the kernel of the singular integral. $\{I_h\}_{h\in(-1,1)}$ is said to be an *approximate identity* (in $L^2(\Omega)$) corresponding to the scaling function $\{K_h\}_{h\in(-1,1)}$, if the following limit relation holds true:

$$\lim_{\substack{h\to 1 \\ h<1}} \|F - I_h(F)\|_{\mathcal{L}^2(\Omega)} = 0$$

for all $F \in L^2(\Omega)$. Note that the assumption that a kernel $\{K_h\}_{h\in(-1,1)}$ is a *scaling function* implies that $\{I_h\}_{h\in(-1,1)}$ is an approximate identity. Conventionally, the scaling function is said to generate the approximate identity $\{I_h\}_{h\in(-1,1)}$. The kernel $\{K_h\}_{h\in(-1,1)}$ is called C–kernel, L^1–kernel, L^2–kernel, non–negative kernel, etc, if all members K_h have this property.

Singular integrals on the sphere have been studied by many authors (for example [3], [7], [8]). Of particular significance for our considerations is the following result: Assume that $\{K_h\}_{h\in(-1,1)}$ is a non–negative family of functions in $L^2[-1, 1]$, which satisfy $(K_h)^\wedge(0) = 1$. Then the following properties are equivalent (see e.g. [3], [8]):

(i) $\{K_h\}_{h\in(-1,1)}$ is a non–negative scaling function.

(ii) $\{I_h\}_{h\in(-1,1)}$ is an approximate identity.

(iii) $\lim_{h\to 1, h<1}(K_h)^\wedge(n) = 1$ for all $n \in \mathbb{N}_0$.

(iv) $\lim_{h\to 1, h<1}(K_h)^\wedge(1)$.

(v) $\{K_h\}_{h\in(-1,1)}$ satisfies the localization property $\lim_{h\to 1, h<1} \int_{-1}^\delta K_h(t) \, dt = 0$ for all $\delta \in (-1, 1)$.

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Moreover, it is worth mentioning that a non–negative scaling function satisfies the estimate

$$\|K_h * F\|_{L^2(\Omega)} \leq \|F\|_{L^2(\Omega)} \tag{11}$$

for all $h \in (-1, 1)$ and $F \in L^2(\Omega)$.

In multiscale approximation singular integrals, which form a semigroup of contraction operators (of class C_0) on $L^2(\Omega)$ play an outstanding role. An approximate identity $\{I_h\}_{h\in(-1,1)}$ is called a *semigroup of contraction operators (of class C_0)* on $L^2(\Omega)$ (cf. [3]), if the following properties are satisfied: (i) $I_{h_1+h_2} = I_{h_1} I_{h_2}$ for $h_1, h_2 \in (-1, 1)$ (ii) $\|I_h(F)\|_{L^2(\Omega)} \leq \|F\|_{L^2(\Omega)}$ for all $h \in (-1, 1)$ and $F \in L^2(\Omega)$. Examples are the Abel–Poisson singular integral and the Gauß–Weierstraß singular integral (see [3], [7]). Based on these examples, [7], [10] we are led to exponential wavelets, particularly to spherical reproducing wavelets (i.e., R–wavelets), which canonically allow a multiresolution analysis in $L^2(\Omega)$, hence, fast calculation of convolutions against scaling functions and wavelets (see [6], [7]).

All aforementioned examples of spherical singular integrals generating a multiresolution analysis of $L^2(\Omega)$, however, show a global support, i.e. supp$K_h = [-1, 1]$ for all $h \in (-1, 1)$. Seen from practical point of view, singular integrals with a local support would be the better choice. In this respect one is naturally led to start from the *piecewise polynomial function* $L_h^{(k)} : [-1, 1] \to \mathbb{R}$ (see [5], [7], [9], [23]) of the form

$$L_h^{(k)}(t) = \left(\left(B_h^{(k)} \right)^\wedge (0) \right)^{-1} B_h^{(k)}(t), \quad t \in [-1, 1] \tag{12}$$

with

$$B_h^{(k)}(t) = \begin{cases} 0, & \text{if } t \in [-1, h) \\ \frac{(t-h)^k}{(1-h)^k}, & \text{if } t \in [h, 1] \end{cases}. \tag{13}$$

Obviously, $L_h^{(k)}$ is non–negative, has the support $[h, 1]$, and satisfies $(L_h^{(k)})^\wedge(0) = 1$. Moreover, it is easy to see that for each $\xi \in \Omega$

$$\left| F(\xi) - \left(L_h^{(k)} * F \right)(\xi) \right| \leq \sup_{\substack{\eta \in \Omega \\ h \leq \xi \cdot \eta \leq 1}} |F(\xi) - F(\eta)|. \tag{14}$$

The iterated kernels $(L_h^{(k)})^{(2)}$, which have the support supp$(L_h^{(k)})^{(2)} = [2h^2 - 1, 1]$, similarly fulfill

$$\left| F(\xi) - \left((L_h^{(k)})^{(2)} * F \right)(\xi) \right| \leq \sup_{\substack{\eta \in \Omega \\ 2h^2 - 1 \leq \xi \cdot \eta \leq 1}} |F(\xi) - F(\eta)|. \tag{15}$$

Consequently, $\{L_h^{(k)}\}_{h\in(-1,1)}$ is a non–negative scaling function with supp$L_h^{(k)} = [h, 1]$. The family $\{L_h^{(k)}\}_{h\in(-1,+1)} \subset L^2[-1, +1]$ is called *Haar scaling function* of order k.

For fixed $h_1 \in (-1, +1)$ and $k \in \mathbb{N}_0$, let $\Gamma_1 \in L^2[-1, +1]$ be defined as the convolution integral

$$\Gamma_1(\xi \cdot \eta) = \left(L_{h_1}^{(k)} * L_{h_1}^{(k)} \right)(\xi \cdot \eta) = \int_\Omega L_{h_1}^{(k)}(\xi \cdot \zeta) L_{h_1}^{(k)}(\eta \cdot \zeta) \, d\omega(\zeta), \quad \xi, \eta \in \Omega. \tag{16}$$

Then, with $(\Gamma_1)^\wedge(n) = ((L_{h_1}^{(k)})^\wedge(n))^2$, $n \in \mathbb{N}_0$, we have

$$\Gamma_1 = \sum_{n=0}^\infty \frac{2n+1}{4\pi} (\Gamma_1)^\wedge(n) P_n \tag{17}$$

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and

$$\sum_{n=0}^{\infty} \frac{2n+1}{4\pi} \left((\Gamma_1)^{\wedge}(n) \right)^2 < \infty. \tag{18}$$

Moreover, it is easily seen that

$$\sum_{n=0}^{\infty} \frac{2n+1}{4\pi} \left((\Gamma_1)^{\wedge}(n) \right)^{2^j} < \infty \tag{19}$$

holds for all $j \in \mathbb{N}$. In consequence, we are able to introduce a family $\{\Gamma_j\} \subset L^2[-1,+1]$, $j \in \mathbb{N}$, by

$$\Gamma_j = \sum_{n=0}^{\infty} \frac{2n+1}{4\pi} \left(\left(L_{h_1}^{(k)} \right)^{\wedge}(n) \right)^{2^j} P_n, \quad j \in \mathbb{N}. \tag{20}$$

According to our construction it is clear that $\Gamma_j * \Gamma_j = \Gamma_{j+1}$ for $j \in \mathbb{N}$. The kernel Γ_j is the jth iteration of Γ_1 which means that $\mathrm{supp}\Gamma_j = [h_j, 1]$ with $h_j = \underbrace{h_1 \uplus \ldots \uplus h_1}_{2^j - \text{times}}$, where

$$h \uplus h' = \cos(\min\{\pi, \arccos h + \arccos h'\}), \quad h, h' \in (-1,+1)) \tag{21}$$

(note that $h_1 \uplus h_1 = -1 + 2h_1^2$). Altogether, our procedure leads to a family $\{\Gamma_j\}$ of locally supported kernels Γ_j with the "contraction property" $\Gamma_j * \Gamma_j = \Gamma_{j+1}$. In consequence, the results obtained for the operators $P_{\Gamma_j} : L^2(\Omega) \to L^2(\Omega)$ defined by

$$P_{\Gamma_j}(F)(\xi) = (\Gamma_j * F)(\xi), \quad \xi \in \Omega \tag{22}$$

can be summarized as follows: For each $j \in \mathbb{N}$, the operators $\{P_{\Gamma_j}\}_{j=1,\ldots,J}$ define a family of linear bounded mappings P_{Γ_j} from $L^2(\Omega)$ to $C^{(k-1)}(\Omega)$ with the following properties: (i) $P_{\Gamma_{j+1}} = P_{\Gamma_j} P_{\Gamma_j}$, (ii) $\|P_{\Gamma_j}(F)\|_{L^2(\Omega)} \leq \|F\|_{L^2(\Omega)}$, $F \in L^2(\Omega)$.

2 The Problem

In this paper our purpose is to establish a fast multiscale approximation by means of 'isotropic finite elements', i.e. the locally supported radial basis functions $L_h^{(k)}$. In doing so it must be emphasized that the structure of a semigroup of contraction operators (of class (C_0)) on $\mathrm{L}^2(\Omega)$ based on locally supported kernels of the representation $L_h^{(k)}$ is by no means obvious, if possible at all. Thinking of a multiresolution analysis based on the kernels $L_h^{(k)}$ as scaling functions with scale parameter h we loose the essential property that the scale spaces form a nested sequence of $L^2(\Omega)$. This observation will become clear from the behaviour of the Legendre coefficients $(L_h^{(k)})^{\wedge}(n)$, $n \in \mathbb{N}_0$ (see [7], [19]). Nevertheless, when we are interested in a multiscale approximation using a tree algorithm, the complete structure of a semigroup of contraction operators (of class (C_0)) on $L^2(\Omega)$ is not necessary. It suffices to have a *finite contraction process*. To be more specific, as pointed out above, Γ_j will be understood to be the convolution

$$\Gamma_j = \operatornamewithlimits{\text{\Large $*$}}_{l=1}^{2^j} L_{h_1}^{(k)} = \underbrace{L_{h_1}^{(k)} * \ldots * L_{h_1}^{(k)}}_{2^j - \text{times}}, \tag{23}$$

where the (fixed) value $h_1 \in (-1,+1)$ is chosen (as close to 1) such that

$$F(\xi) \simeq \int_{\Omega} \left(L_{h_1}^{(k)} * L_{h_1}^{(k)} \right) (\xi \cdot \eta) F(\eta) \, d\omega(\eta), \quad \xi \in \Omega, \tag{24}$$

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which is possible, since $\{I_h^{(k)}\}$ as defined above forms an approximate identity on $L^2(\Omega)$ (note that the symbol $\simeq$ always means that the error is negligible). In other words, seen in comparison with a semigroup of contraction operators (of class (C_0)) on $L^2(\Omega)$ we have developed a simulated procedure that stops after a finite number of steps for $j = 1$ that is for a fixed 'window size' h_1. It must be mentioned that the approximation cannot be performed with arbitrary accuracy. Actually we end up with a nested sequence V_{Γ_j} of scale spaces $V_{\Gamma_j} = \{\Gamma_j * F \mid F \in L^2(\Omega)\}$ (with index running from top to bottom). The finest scale is based on V_1, hence, the $L^2(\Omega)$–closure of the union of scale spaces is not $L^2(\Omega)$, but it may be assumed that any member $F \in L^2(\Omega)$ satisfies $F(\xi) \simeq (\Phi_1 * F)(\xi)$, $\xi \in \Omega$. Obviously, our approach is of practical importance. In applications only discrete data material of a function $F \in L^2(\Omega)$ is available. This is the reason why an infinite process for approximation is not necessary from practical point of view. Instead we are confronted with the problem of reaching in a finite number of steps a numerically relevant approximation (based on a parameter h_1 chosen in close adaptation to the data situation under consideration). For a discretization of the convolution integrals $\Gamma_j * F$ we need approximate integration rules. Many integration techniques are known from the literature (for a survey on approximate integration on the sphere see, for example, [7], [21] and the references therein). It should be mentioned that locally adapted integration formulae on the sphere involving space localized radial basis functions can be found e.g. in [2], [5]. Observing the fact that $\Gamma_j * \Gamma_j = \Gamma_{j+1}$ and basing numerical integration of $P_{\Gamma_{j+1}}(F) = \Gamma_{j+1} * F$ on approximate formulae associated to known (fixed) weights $w_i^{N_j}$ and knots $\eta_i^{N_j} \in \Omega$ we are able to deduce that

$$P_{\Gamma_{j+1}}(F)(\xi) = (\Gamma_{j+1} * F)(\xi) \simeq \sum_{i=1}^{N_j} a_i^{N_j} \Gamma_j \left(\xi \cdot \eta_i^{N_j} \right), \; j = 1, \dots, J,$$

where

$$a_i^{N_j} = w_i^{N_j} (\Gamma_j * F)(\eta_i^{N_j}), \; j = J, \dots, 1, \; i = 1, \dots, N_j \, .$$

Now it can be seen that

$$
\begin{aligned}
a_i^{N_{j+1}} &= w_i^{N_{j+1}} (\Gamma_{j+1} * F) \left(\eta_i^{N_{j+1}} \right) \\
&= w_i^{N_{j+1}} (\Gamma_j * \Gamma_j * F) \left(\eta_i^{N_{j+1}} \right) \\
&\simeq w_i^{N_{j+1}} \sum_{l=1}^{N_j} w_l^{N_j} (\Gamma_j * F) \left(\eta_l^{N_j} \right) \Gamma_j \left(\eta_l^{N_j}, \eta_i^{N_{j+1}} \right) \\
&= w_i^{N_{j+1}} \sum_{l=1}^{N_j} a_l^{N_j} \Gamma_j \left(\eta_l^{N_j}, \eta_i^{N_{j+1}} \right) \, ,
\end{aligned}
$$

$i = 1, \dots, N_{j+1}$, $j = 1, \dots, J$. In other words, the coefficients $a^{N_2} = (a_1^{N_2}, \dots, a_{N_2}^{N_2})^T$ can be calculated recursively starting from the initial level, a^{N_1}, $a^{N_3} = (a_1^{N_3}, \dots, a_{N_3}^{N_3})^T$ can be calculated recursively from $a^{N_2} = (a_1^{N_2}, \dots, a_{N_2}^{N_2})^T$, etc.

Introducing the operator $R_{\Gamma_j} : L^2(\Omega) \to L^2(\Omega)$, $F \mapsto R_{\Gamma_j}(F) = (\Gamma_{j-1} - \Gamma_j) * F$, $j = 2, \dots, J$, we finally end up with a tree algorithm for the decomposition of a function $F \in L^2(\Omega)$ from discretely given data as follows:

$$
\begin{array}{ccccccccc}
F & \to & a^{N_1} & \to & & a_{N_2} & \to \cdots \to & a_{N_{J-1}} & \to & & a^{N_J} \\
& & \downarrow & & & \downarrow & & \downarrow & & & \downarrow \\
& & P_{\Gamma_1}(F) & \searrow & \nearrow & P_{\Gamma_2}(F) & & P_{\Gamma_{J-1}}(F) & \searrow & \nearrow & P_{\Gamma_J}(F) \; . \\
& & & R_{\Gamma_2}(F) & & & & & R_{\Gamma_{J(F)}} & &
\end{array}
$$

Obviously, a tree algorithm using locally supported radial basis function, i.e. *isotropic finite elements* on the sphere has been established. The kernels Γ_j, as defined by (23), are known in spectral form as orthogonal expansions in terms of Legendre functions, where the coefficients $(L_h^{(k)})^\wedge(n) = 2\pi \int_{-1}^{+1} L_h^{(k)}(t) P_n(t)\, dt$, $n = 0, 1, \ldots$, are available by recursion (see the considerations given in Chapter 3). However, until now the explicit representation of Γ_j (in space domain) is unknown. Moreover, the increase of the iteration order 2^j for the members of the constituting system $\{\Gamma_j\}_{j=1,\ldots,J}$ from level to level seems to be a non–canonical procedure seen from numerical point of view. Both features together reduce the the numerical acceptance of the tree algorithm presented above.

Our interest in this paper is twofold: first we make the attempt to develop explicit representations (in space domain) of the iterated kernel functions $(L_h^{(k)})^{(2)} = L_h^{(k)} * L_h^{(k)}$, $h \in (-1, +1)$ for different choices of k. Second we use the construction process of the so–called spherical up function (see [7], [11])

$$ Up = \left(L_{h_1}^{(k)}\right)^{(2)} * \left(L_{h_2}^{(k)}\right)^{(2)} * \left(L_{h_3}^{(k)}\right)^{(2)} * \ldots $$

with suitably chosen numbers $h_i \in (-1, 1)$, $i = 1, 2, \ldots$, (for example, $h_i = \cos 2^{-i} \arccos h$) to establish (smooth) rotation invariant finite elements of the type

$$ \Phi^j = \underset{i=j}{\overset{\infty}{\text{\Large $*$}}} \left(L_{h_i}^{(k)}\right)^{(2)}, \quad j = 1, 2, \ldots . $$

Accordingly we introduce (smooth isotropic finite element) wavelets by setting

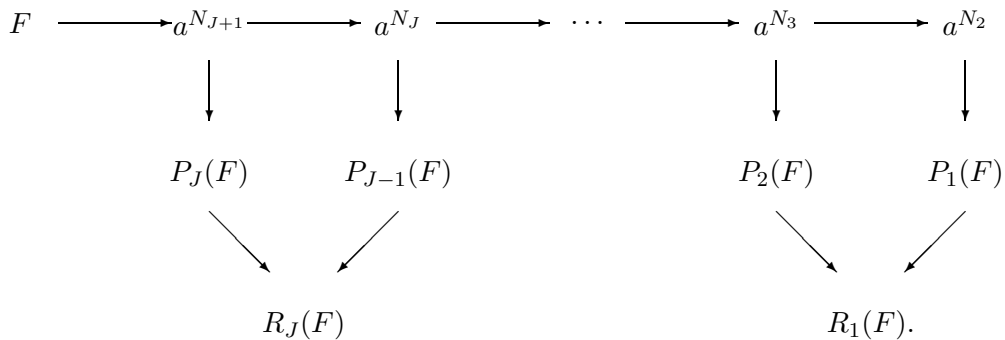$$ \Psi^j = \Phi^{j+1} - \Phi^j, \quad j = 1, 2, \ldots . $$

As usual, we let

$$ P_j(F)(\xi) = \left(\Phi^j * F\right)(\xi) = \int_\Omega \Phi^j(\xi \cdot \zeta) F(\zeta)\, d\omega(\zeta), \quad \xi \in \Omega, $$

$$ R_j(F)(\xi) = \left(\Psi^j * F\right)(\xi) = \int_\Omega \Psi^j(\xi \cdot \zeta) F(\zeta)\, d\omega(\zeta), \quad \xi \in \Omega, $$

and

$$ P_j^{(2)}(F)(\xi) = \left(\Phi^j * \Phi^j * F\right)(\xi) = \int_\Omega \left(\Phi^j\right)^{(2)}(\xi \cdot \zeta) F(\zeta)\, d\omega(\zeta), \quad \xi \in \Omega, $$

$$ R_j^{(2)}(F)(\xi) = \left(\Psi^j * \Psi^j * F\right)(\xi) = \int_\Omega \left(\Psi^j\right)^{(2)}(\xi \cdot \zeta) F(\zeta)\, d\omega(\zeta), \quad \xi \in \Omega . $$

Our aim is to establish a tree algorithm by exclusively using the kernels $(L_{h_j}^{(k)})^{(2)}$ with *fixed* order k: In fact, we end up with the following tree algorithm for the decomposition of an $L^2(\Omega)$–function F as follows (see Chapter 8):

The essential tool for the establishment of the tree algorithm is the scaling equation yielding Φ^j as convolution of Φ^{j+1} against $(L_{h_j}^{(k)})^{(2)}$.

The outline of the remaining part of the paper is as follows: Chapter 3 presents spectral and spatial properties of the locally supported kernels $(L_h^{(k)})^{(2)}$. The problem of finding elementary representations of $(L_h^{(k)})^{(2)}$ is discussed for $k = 0, \ldots, 3$. The spherical up function is defined in Chapter 4. Then, in Chapter 5, the multiresolution analysis of the space $L^2(\Omega)$ using the up function is discussed in detail (cf. [11]). Chapter 6 deals with locally supported wavelets. Decomposition and reconstruction schemes by use of the up function are illustrated in Chapter 7. Finally, the results of Chapter 7 will be used in Chapter 8 to develop the announced tree algorithm involving the system $\{(L_{h_j}^{(k)})^{(2)}\}$ with fixed order k.

3 Spectral and Spatial Properties of the Kernels $L_h^{(k)}$

In this section, we develop new results concerning the Legendre transform of $L_h^{(k)}$ and find an explicit representation of the iterated kernels $(L_h^{(k)})^{(2)}$ for $k = 0, \ldots, 3$ (in space domain).

We start by repeating some formulas for $L_h^{(k)}$ known from the investigations in e.g. [7], [20], [22], [23].

THEOREM 3.1. *Let $h \in (-1, 1)$, $k = 0, 1, \ldots$. Then the following statements are valid:*

(i) $\mathrm{supp} L_h^{(k)}(\eta \cdot) = \{\xi \in \Omega | h \leq \xi \cdot \eta \leq 1\}$,

(ii) *The Legendre transform fulfills the following recursion formulas:*
$$(L_h^{(k)})^\wedge(0) = 1,$$
$$(L_h^{(k)})^\wedge(1) = (L_h^{(k)})^\wedge(0)\left(1 - \frac{1-h}{k+2}\right),$$
$$(L_h^{(k)})^\wedge(n+1) = \frac{2n+1}{n+k+2}h(L_h^{(k)})^\wedge(n) + \frac{k+1-n}{n+k+2}(L_h^{(k)})^\wedge(n-1),$$

(iii) $(L_h^{(k)})^\wedge(0) = 1$, $|(L_h^{(k)})^\wedge(n)| < 1$, $n \geq 1$,

(iv) $\displaystyle\lim_{\substack{h \to 1 \\ h < 1}}(L_h^{(k)})^\wedge(n) = 1$,

(v) $|(L_h^{(k)})^\wedge(n)| = O(n^{-3/2-k})$, $n \to \infty$.

Next we develop an explicit representation for the Legendre transform $(L_h^{(k)})^\wedge(n)$ in terms of the hypergeometric function (see e.g. [1], [14]). We will strongly rely on results stated in[1]. For the sake of simple notation we also introduce a rescaled version of the Legendre function by letting
$$p_\nu^\mu(t) = (t^2 - 1)^{-\nu/2}P_\nu^\mu(t), \quad t \in [-1, +1],$$
which fulfills for its derivative
$$\begin{aligned}
\frac{\partial p_\nu^\mu}{\partial z}(z) &= \frac{\partial}{\partial z}((z^2-1)^{-\mu/2}P_\nu^\mu(z)) \\
&= (-\mu/2)(z^2-1)^{-\mu/2-1}2zP_\nu^\mu(z) \\
&\quad + (z^2-1)^{-\mu/2-1}(z\nu P_\nu^\mu(z) - (\nu+\mu)P_{\nu-1}^\mu(z)) \\
&= (z^2-1)^{-\mu/2-1}(z(\nu-\mu)P_\nu^\mu(z) - (\nu+\mu)P_{\nu-1}^\mu(z)) \\
&= (z^2-1)^{-\mu/2-1/2}P_\nu^{\mu+1}(z) \\
&= p_\nu^{\mu+1}(z) .
\end{aligned}$$

It is a well-known fact that the hypergeometric function and the associated Legendre functions are related as follows:

$$F\left(-n, n+1, 2+k, \frac{1-x}{2}\right) = c_k(x)P_n^{-k-1}(x) = e_k(x)p_n^{-k-1}(x), \qquad (25)$$

where

$$P_n^{-k-1}(x) = d_k(x)p_n^{-k-1}, \qquad (26)$$

c_k, d_k are given by

$$c_k(x) = (k+1)! \left(\frac{x+1}{x-1}\right)^{(k+1)/2}, \qquad (27)$$

$$d_k(x) = \left(x^2 - 1\right)^{-k/2-1/2}, \qquad (28)$$

and

$$e_k(x) = c_k(x) \; d_k(x) = \frac{(k+1)!}{(x-1)^{k+1}} . \qquad (29)$$

The associated Legendre function fulfills the recursion relation

$$P_{n+1}^{-k-1}(x) = \frac{2n+1}{n+k+2}xP_n^{-k-1}(x) + \frac{-n+k+1}{n+k+2}P_{n-1}^{-k-1}(x) .$$

Observing these results we are therefore led to the following theorem.

THEOREM 3.2. *The Fourier coefficients of $L_h^{(k)}$ possess the explicit representation*

$$\begin{aligned}
(L_h^{(k)})^\wedge(n) &= F\left(-n, n+1, 2+k, \frac{1-h}{2}\right) \\
&= c_k(h)P_n^{-k-1}(h) \\
&= e_k(h)p_n^{-k-1}(h),
\end{aligned}$$

where the multiplicative terms $c_k(h)$ and $e_k(h)$ are given by (27) and (29), respectively.

The proof of Theorem 3.2 is an obvious consequence of the fact that

$$F\left(0, 1, 2, +k, \frac{1-h}{2}\right) = 1.$$

As mentioned in the last chapter, our tree algorithm strongly relies on the iterated kernels $(L_h^{(k)})^{(2)}$. Therefore we are interested in the explicit representation of these iterated kernels. The starting point are formulae for the rescaled Legendre functions, which are immediate reformulations of identities to be found in [13], [25].

LEMMA 3.3. *The following identities hold true:*

$$\sum_{n=0}^{\infty} \frac{2n+1}{4\pi} p_n^0(h)p_n^0(h)P_n(t)$$

$$= \begin{cases} \dfrac{1}{2\pi^2\sqrt{t-2h^2+1}\sqrt{1-t}} & if \quad t \in [2h^2-1, 1] \\ 0 & if \quad t \in [0, 2h^2-1), \end{cases}$$

$$\sum_{n=0}^{\infty} \frac{2n+1}{4\pi} p_n^1(h) p_n^{-1}(h) P_n(t)$$

$$= \begin{cases} \dfrac{t-h^2}{2\pi^2(h^2-1)\sqrt{t-2h^2+1}\sqrt{1-t}} & if \quad t \in [2h^2-1,1] \\ 0 & if \quad t \in [0, 2h^2-1), \end{cases}$$

$$\sum_{n=0}^{\infty} \frac{2n+1}{4\pi} p_n^2(h) p_n^{-2}(h) P_n(t)$$

$$= \begin{cases} \dfrac{-1+h^4+h^2(2-4t)+2t^2}{2\pi^2(h^2-1)^2\sqrt{t-2h^2+1}\sqrt{1-t}} & if \quad t \in [2h^2-1,1] \\ 0 & if \quad t \in [0, 2h^2-1), \end{cases}$$

$$\sum_{n=0}^{\infty} \frac{2n+1}{4\pi} p_n^3(h) p_n^{-3}(h) P_n(t)$$

$$= \begin{cases} \dfrac{(-h^2+t)(-3+h^4+h^2(6-8t)+4t^2)}{2\pi^2(h^2-1)^3\sqrt{t-2h^2+1}\sqrt{1-t}} & if \quad t \in [2h^2-1,1] \\ 0 & if \quad t \in [0, 2h^2-1), \end{cases}$$

$$\sum_{n=0}^{\infty} \frac{2n+1}{4\pi} p_n^4(h) p_n^{-4}(h) P_n(t) =$$

$$= \begin{cases} \begin{aligned} & (2(-1+h^2)^4\pi^2\sqrt{1-t}\sqrt{1-2h^2+t})^{-1} \\ & \quad \left(1+h^8-8t^2+8t^4-4h^6(-3+4t) \right. \\ & \quad - 4h^2(1+4(-1+t)t(1+2t)) \\ & \quad \left. + h^4(-2+8t(-4+5t)) \right) \end{aligned} & if \quad t \in [2h^2-1,1] \\ 0 & if \quad t \in [0, 2h^2-1[\,. \end{cases}$$

Note that we are not concerned with formulations of a sum of the type

$$\sum_{n=0}^{\infty} \frac{2n+1}{4\pi} p_n^k(h) p_n^{-k}(h) P_n(t). \tag{30}$$

Instead we are interested in discussing sums of the form

$$\sum_{n=0}^{\infty} \frac{2n+1}{4\pi} p_n^{-k}(h) p_n^{-k}(h) P_n(t) \,. \tag{31}$$

For our calculations we will use the fact that $\partial_h p_n^k(h) = p_n^{k+1}(h)$ which we have verified beforehand. In other words, we will not compute our kernels directly, but their derivative with respect to h.

LEMMA 3.4. *For $k \in \{0,1,2,3\}$:*

$$\frac{\partial^{2k+2}}{\partial h^{2k+2}} \left(\sum_{n=0}^{\infty} \frac{2n+1}{4\pi} (p_n^{-k-1}(h))^2 P_n(t) \right) =$$

$$\begin{cases} \dfrac{2^k(1-t)^{k+(1/2)}}{\pi^2(1-h^2)^{k+1}\sqrt{1-2h^2+t}} & if \quad t \in [2h^2-1,1] \\ 0 & if \quad t \in [0, 2h^2-1). \end{cases}$$

Proof. The result follows by straightforward computation using the product rule, the previous lemma, and the fact that we may interchange the sum and the derivative. $\qquad\square$

Obviously, our above strategy can be applied to kernels with any degree k. For simplicity, however, we restrict ourselves to the degrees $k = 0, 1, 2, 3$, which are particularly important in our finite element applications. In order to keep our notation transparent it is advisable to introduce the following abbreviations:

$$\alpha_h(t) = \begin{cases} \sqrt{1-t}\sqrt{1-2h^2+t} & \text{if} \quad t \in [2h^2-1, 1] \\ 0 & \text{if} \quad t \in [0, 2h^2-1), \end{cases}$$

$$\beta_h(t) = \begin{cases} \arctan(\dfrac{\sqrt{1-t}\sqrt{1-2h^2+t}}{1+t+2h}) & \text{if} \quad t \in [2h^2-1, 1] \\ 0 & \text{if} \quad t \in [0, 2h^2-1), \end{cases}$$

and

$$\gamma_h(t) = \begin{cases} \arctan(\dfrac{\sqrt{1-t}\sqrt{1-2h^2+t}}{-1-t+2h}) + \pi & \text{if} \quad t \in [2h-1, 1] \\ \arctan(\dfrac{\sqrt{1-t}\sqrt{1-2h^2+t}}{-1-t+2h}) & \text{if} \quad t \in [2h^2-1, 2h-1) \\ 0 & \text{if} \quad t \in [0, 2h^2-1). \end{cases}$$

The following illustrations give a graphical impression of the functions defined above:
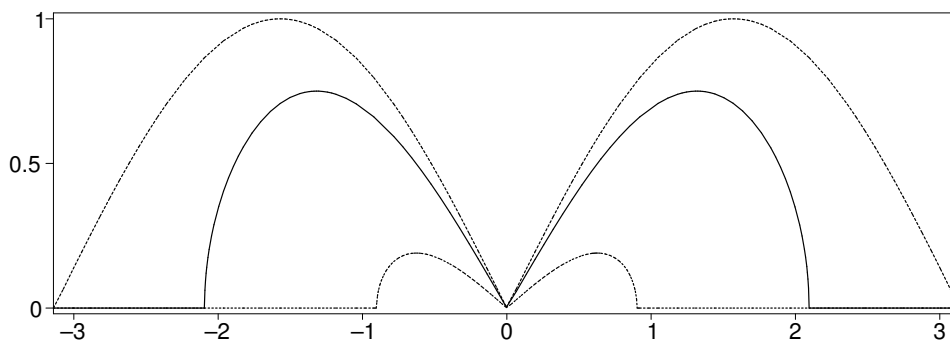

Figure 1: $\vartheta \mapsto \alpha_h(\cos\vartheta)$ for different values of h, viz. $h = 0.01$, $h = 0.5$, and $h = 0.9$.
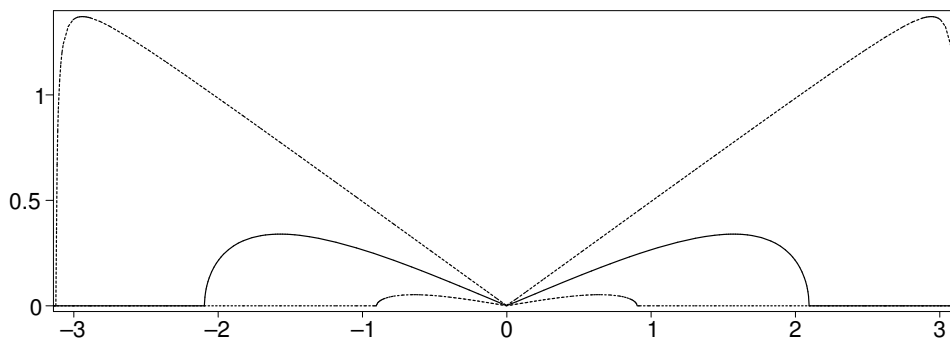

Figure 2: $\vartheta \mapsto \beta_h(\cos\vartheta)$ for different values of h, viz. $h = 0.01$, $h = 0.5$, and $h = 0.9$.
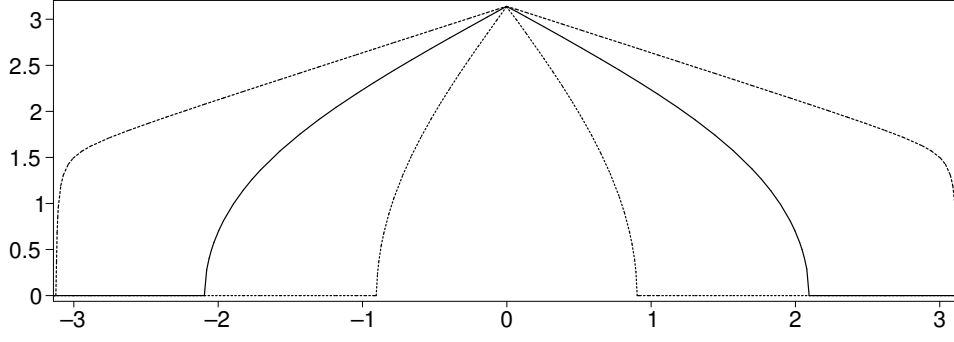
Figure 3: $\vartheta \mapsto \gamma_h(\cos\vartheta)$ for different values of h, viz. $h = 0.01$, $h = 0.5$, and $h = 0.9$.

In terms of the abbreviations introduced above we obtain from Lemma 3.4

LEMMA 3.5. *For* $t \in [-1, 1]$

$$\sum_{n=0}^{\infty} \frac{2n+1}{4\pi} p_n^{-1}(h) p_n^{-1}(h) P_n(t) = \frac{1}{2\pi^2}\Big((1-h)\gamma_h(t) - (h+1)\beta_h(t)\Big)$$

$$\sum_{n=0}^{\infty} \frac{2n+1}{4\pi} p_n^{-2}(h) p_n^{-2}(h) P_n(t) = \frac{1}{12\pi^2}\Big((h-1)^2(th - 3h + 2t)\gamma_h(t)$$
$$+ (h+1)^2(th - 3h - 2t)\beta_h(t)$$
$$+ 2(h^2+1)\alpha_h(t)\Big),$$

$$\sum_{n=0}^{\infty} \frac{2n+1}{4\pi} p_n^{-3}(h) p_n^{-3}(h) P_n(t)$$
$$= \frac{1}{480\pi^2}\Big((1-h)^3(3t^2h^2 - 10th^2 + 19h^2 - 3h + 9ht^2 - 30th + 4 + 8t^2)\gamma_h(t)$$
$$- (h+1)^3(3t^2h^2 - 10th^2 + 19h^2 + 3h - 9ht^2 + 30th + 8t^2 + 4)\beta_h(t)$$
$$- 6(th^4 - 2t - 3th^2 - 9h^2 - 3h^4)\alpha_h(t)\Big),$$

and

$$\sum_{n=0}^{\infty} \frac{2n+1}{4\pi} p_n^{-4}(h) p_n^{-4}(h) P_n(t)$$
$$= \frac{1}{60480\pi^2}\Big(3(h-1)^4(-63h^3 + 5t^3h^3 - 21h^3t^2$$
$$+ 39th^3 + 28h^2 + 20t^3h^2 - 84h^2t^2 + 156th^2$$
$$+ 29t^3h - 105ht^2 - 9ht - 35h + 16t^3 + 24t)\gamma_h(t)$$
$$+ 3(h+1)^4(-63h^3 + 5t^3h^3 - 21h^3t^2$$
$$+ 39th^3 - 28h^2 - 20t^3h^2 + 84h^2t^2 - 156th^2$$
$$+ 29t^3h - 105ht^2 - 9ht - 35h - 16t^3 - 24t)\beta_h(t)$$
$$+ 2(44t^2 + 15h^6t^2 - 58h^4t^2 + 87h^2t^2 - 58h^6t + 276h^4t$$
$$+ 566th^2 + 16 + 682h^4 + 103h^6 + 247h^2)\alpha_h(t)\Big).$$

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Proof. Obviously, each of these results just is true up to an additive polynomial $a_k(h,t)$ in h with coefficients which are depending on t. Now, assume that $a_k(h,t)$ is non-zero. Thus there exists a pair (h_0, t_0), where $a_k(h_0, t_0) \neq 0$. Of course, a_k is a polynomial in h, hence, $a_k(h, t_0) \neq 0$ almost everywhere. On the other hand we already know that our result has to have a compact support on $[1 - 2h^2, 1]$. In fact, our solution proposed above fulfills this property. Hence, $a_k(h,t)$ also has to have compact support on $[1 - 2h^2, 1]$. But this contradicts $a_k(h, t_0) \neq 0$ almost everywhere because we may choose h such that $t_0 < 1 - 2h^2$. This proves that our assertion is actually correct. $\qquad\square$

In order to describe our kernels $(L_h^{(k)})^{(2)} = L_h^{(k)} * L_h^{(k)}$ we just need to include the missing factor $(e_k(h))^2$.

THEOREM 3.6. *For $t \in [-1, +1]$*

$$\left(L_h^{(0)} * L_h^{(0)}\right)(t) = \frac{1}{2\pi^2(1-h)^2}\Big((1-h)\gamma_h(t) - (h+1)\beta_h(t)\Big),$$

$$
\begin{aligned}
\left(L_h^{(1)} * L_h^{(1)}\right)(t) = \ & \frac{1}{3\pi^2(1-h)^4}\Big((h-1)^2(th - 3h + 2t)\gamma_h(t) \\
& + (h+1)^2(th - 3h - 2t)\beta_h(t) \\
& + 2(h^2+1)\alpha_h(t)\Big),
\end{aligned}
$$

$$
\begin{aligned}
& \left(L_h^{(2)} * L_h^{(2)}\right)(t) \\
=\ & \frac{3}{40\pi^2(1-h)^6}\Big((1-h)^3(3t^2h^2 - 10th^2 + 19h^2 - 3h + 9ht^2 - 30th + 4 + 8t^2)\gamma_h(t) \\
& - (h+1)^3(3t^2h^2 - 10th^2 + 19h^2 + 3h - 9ht^2 + 30th + 8t^2 + 4)\beta_h(t) \\
& - 6(th^4 - 2t - 3th^2 - 9h^2 - 3h^4)\alpha_h(t)\Big),
\end{aligned}
$$

and

$$
\begin{aligned}
\left(L_h^{(3)} * L_h^{(3)}\right)(t) = \ & \frac{1}{105\pi^2(1-h)^8}\Big(3(h-1)^4(-63h^3 + 5t^3h^3 - 21h^3t^2 \\
& + 39th^3 + 28h^2 + 20t^3h^2 - 84h^2t^2 + 156th^2 \\
& + 29t^3h - 105ht^2 - 9ht - 35h + 16t^3 + 24t)\gamma_h(t) \\
& + 3(h+1)^4(-63h^3 + 5t^3h^3 - 21h^3t^2 \\
& + 39th^3 - 28h^2 - 20t^3h^2 + 84h^2t^2 - 156th^2 \\
& + 29t^3h - 105ht^2 - 9ht - 35h - 16t^3 - 24t)\beta_h(t) \\
& + 2(44t^2 + 15h^6t^2 - 58h^4t^2 + 87h^2t^2 - 58h^6t + 276h^4t \\
& + 566th^2 + 16 + 682h^4 + 103h^6 + 247h^2)\alpha_h(t)\Big)
\end{aligned}
$$

Proof. The assertion of Theorem 3.6 follows by straightforward calculations using the fact that $(L_h^{(k)})^\wedge(n) = \frac{(k+1)!}{(h-1)^{k+1}}p_n^{-k-1}(h)$. $\qquad\square$

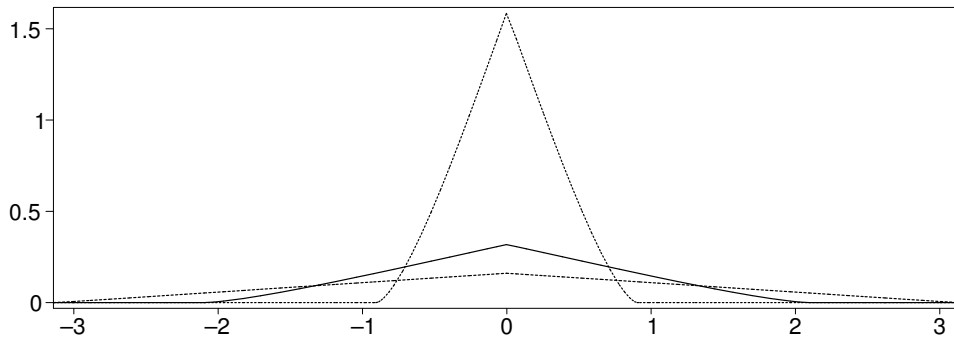Graphical impressions of the iterated kernels are given in the Figures 4.7.

Figure 4: Kernel $\vartheta \mapsto (L_h^{(0)} * L_h^{(0)})(\cos \vartheta)$ for different values of h, viz. $h = 0.01$, $h = 0.5$ and $h = 0.9$ ($t = \cos \vartheta$).
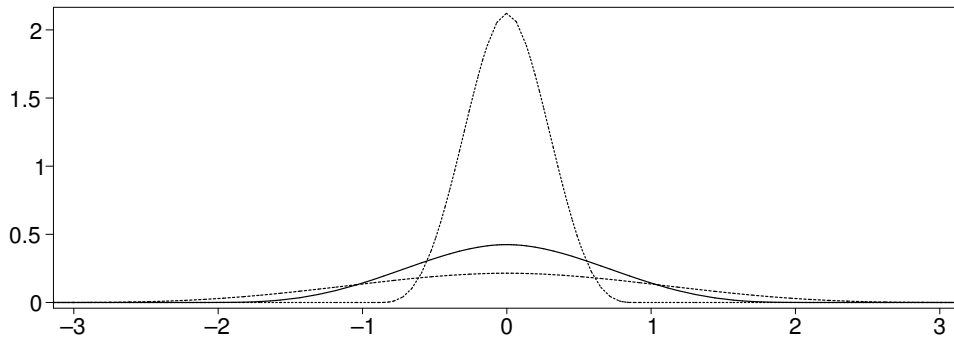

Figure 5: Kernel $\vartheta \mapsto (L_h^{(1)} * L_h^{(1)})(\cos \vartheta)$ for different values of h, viz. $h = 0.01$, $h = 0.5$ and $h = 0.9$ ($t = \cos \vartheta$).
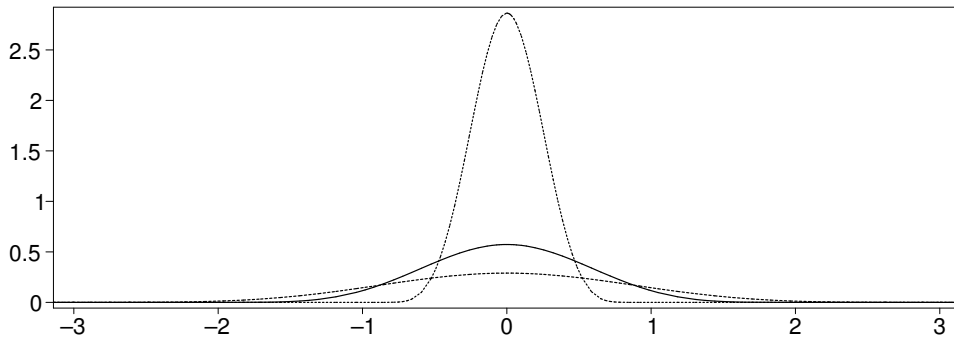

Figure 6: Kernel $\vartheta \mapsto (L_h^{(2)} * L_h^{(2)})(\cos \vartheta)$ for different values of h, viz. $h = 0.01$, $h = 0.5$ and $h = 0.9$ ($t = \cos \vartheta$).
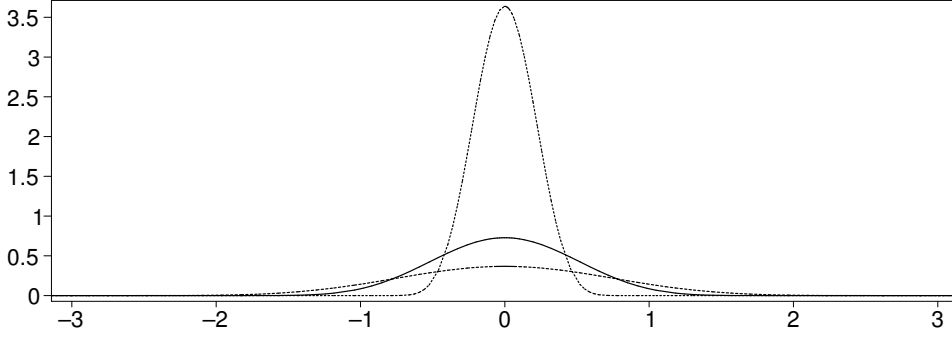
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Figure 7: Kernel $\vartheta \mapsto (L_h^{(3)} * L_h^{(3)})(\cos\vartheta)$ for different values of h, viz. $h = 0.01$, $h = 0.5$, and $h = 0.9$ $(t = \cos\vartheta)$.

4 The Spherical Up Function

Now, we deal with a spherical counterpart of the so–called up function (see [11]) which is, for one dimensional problems, described e.g. in [18]. The main idea is to build an infinite convolution of locally supported functions, where the support of each of the building blocks is chosen carefully to ensure that the resulting convolution is additionally locally supported. Even more, the infinite convolution turns out to be infinitely often differentiable. The reason is that the symbol of the up function decays for increasing n faster than any rational function (in n).

DEFINITION 4.1 *Suppose that $h \in (-1,1)$ and $k = 0,1,\ldots$. We let $\varphi_0 = \arccos h$ and $\varphi_i = 2^{-i}\varphi_0$, , $h_i = \cos(\varphi_i/2)$, $i = 1,2,\ldots$. Then $Up_h^{(k)}$ defined by*

$$Up_h^{(k)} = (L_{h_1}^{(k)})^{(2)} * (L_{h_2}^{(k)})^{(2)} * \ldots = \mathop{\Huge *}_{i=1}^{\infty} (L_{h_i}^{(k)})^{(2)} \tag{32}$$

is called up function (more precisely: (h,k)–up function).

Each $\vartheta \mapsto L_{h_i}^{(k)}(\cos\vartheta)$ possesses the support $[0,\varphi_i/2]$, so that $\vartheta \mapsto (L_{h_i}^{(k)})^{(2)}(\cos\vartheta)$ has the support $[0,\varphi_i]$. Thus, the function $\vartheta \mapsto Up_h^{(k)}(\cos\vartheta)$ has the support $[0,\sum_{i=1}^{\infty}\varphi_i] = [0,\varphi_0]$, so that $\operatorname{supp} Up_h^{(k)}(t) = [h,1]$ (what justifies our way of writing). We know that, for each i, we have

$$0 \le ((L_{h_i}^{(k)})^{(2)})^{\wedge}(n) \le ((L_{h_i}^{(k)})^{(2)})^{\wedge}(0) = 1, \quad n = 1,2,\ldots$$

so that the infinite convolution (32) is well–defined, and we have

$$(Up_h^{(k)})^{\wedge}(n) = \prod_{i=1}^{\infty}((L_{h_i}^{(k)})^{(2)})^{\wedge}(n). \tag{33}$$

In particular,

$$0 \le (Up_h^{(k)})^{\wedge}(n) \le (Up_h^{(k)})^{\wedge}(0) = 1, \quad n = 1,2,\ldots \tag{34}$$

From Theorem 3.1 it follows that

$$\lim_{h\to 1}(Up_h^{(k)})^{\wedge}(n) = 1, \quad n = 1,2,\ldots$$

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Furthermore, as a consequence of the asymptotic behavior of the Legendre transform $(L_h^{(k)})^\wedge(n)$ for $n \to \infty$, we have for every $k \in \mathbb{N}$

$$(Up_h^{(k)})^\wedge(n) = O(n^{-k}), \quad n \to \infty. \tag{35}$$

Hence we are able to deduce from the Sobolev Lemma (cf. [7]) that $Up_h^{(k)}(\eta \cdot\,) \in C^{(\infty)}(\Omega)$ for every $\eta \in \Omega$.

Summarizing the properties of the (h, k)– spherical up function we obtain

THEOREM 4.2. *Let, for $h \in (-1, 1)$ and $k = 0, 1, \ldots$, the (h, k)–up function $Up_h^{(k)} : [-1, 1] \to \mathbb{R}$ be defined as in (32). Then the following statements are valid:*

(i) *$Up_h^{(k)}$ is locally supported with $\operatorname{supp} Up_h^{(k)} = [h, 1]$.*

(ii) *For every $\eta \in \Omega$: $Up_h^{(k)}(\eta \cdot\,)$ is of class $C^{(\infty)}(\Omega)$.*

(iii) *$Up_h^{(k)} : [-1, 1] \to \mathbb{R}$ admits the uniformly convergent orthogonal expansion in terms of Legendre polynomials*

$$Up_h^{(k)} = \sum_{n=0}^{\infty} \frac{2n+1}{4\pi} (Up_h^{(k)})^\wedge(n) P_n, \tag{36}$$

where $(Up_h^{(k)})^\wedge(0) = 1$ and

$$0 \leq (Up_h^{(k)})^\wedge(n) = \prod_{i=1}^{\infty} \left((L_{h_i}^{(k)})^\wedge(n) \right)^2 \leq 1, \; n = 0, 1, 2, \ldots. \tag{37}$$

(iv) *For $n = 1, 2, \ldots$*

$$\lim_{h \to 1} (Up_h^{(k)})^\wedge(n) = 1. \tag{38}$$

(v) *For all $t \in [-1, 1]$*

$$0 \leq Up_h^{(k)}(t) \leq Up_h^{(k)}(1) = \sum_{n=0}^{\infty} \frac{2n+1}{4\pi} (Up_h^{(k)})^\wedge(n). \tag{39}$$

(vi) *For any $l \in \mathbb{N}$,*

$$(Up_h^{(k)})^\wedge(n) = O(n^{-l}), \quad n \to \infty. \tag{40}$$

Proof. In the light of the previous considerations, we only have to prove the statement *(v)*: Since $Up_h^{(k)}$ is built by a convolution in terms of positive functions $L_{h_i}^{(k)}$, it is clear that $Up_h^{(k)}(t) \geq 0$ for $t \in [-1, 1]$. Since $(Up_h^{(k)})^\wedge(n) \geq 0$ for all $n = 0, 1, \ldots$, and $|P_n(t)| \leq P_n(1) = 1$ for all $t \in [-1, 1]$, the series expansion (36) leads us to the inequality

$$0 \leq Up_h^{(k)}(t) \leq Up_h^{(k)}(1) = \sum_{n=0}^{\infty} \frac{2n+1}{4\pi} (Up_h^{(k)})^\wedge(n),$$

which completes the proof. $\qquad \square$

5 Multiresolution Analysis Using the Up Function

Next we come to the characterization of a multiresolution analysis within the space $L^2(\Omega)$ involving the spherical up function. Starting point is the fact, that the up functions define a spherical singular integral. From [11] we know that

THEOREM 5.1. *Suppose that $k = 0, 1, \ldots$. For all $F \in L^2(\Omega)$,*

$$\lim_{h \to 1} \left\| Up_h^{(k)} * F - F \right\|_{L^2(\Omega)} = 0. \tag{41}$$

Furthermore, from $|(Up_h^{(k)})^\wedge(n)| \leq 1$, it follows that

$$\left\| Up_h^{(k)} * F \right\|_{L^2(\Omega)} \leq \|F\|_{L^2(\Omega)}, \tag{42}$$

which motivates the terminology of a multiresolution analysis $L^2(\Omega)$. However, when $h_1 \geq h_2$, we are *not* able to guarantee that

$$\left\| Up_{h_2}^{(k)} * F \right\|_{L^2(\Omega)} \leq \left\| Up_{h_1}^{(k)} * F \right\|_{L^2(\Omega)} \tag{43}$$

holds for all $F \in L^2(\Omega)$. This is due to the fact that the Legendre transform $(Up_h^{(k)})^\wedge(n)$ is not monotone with respect to h. As an counterexample for (43) take values h_1 and h_2 so that $(Up_{h_1}^{(k)})^\wedge(n) = 0$ for an $n \in \mathbb{N}$, but $(Up_{h_2}^{(k)})^\wedge(n) \neq 0$. Then it can be easily seen that (43) is *not* true for a spherical harmonic of order n, $F = Y_n \in \text{Harm}_n$. We can overcome this calamity by restricting ourselves to discrete values of h, i.e. we are looking for a *scale discrete multiresolution analysis* of $L^2(\Omega)$.

We assume from now on, that $h \in (-1, 1)$ and $k = 0, 1, \ldots$ are fixed. For this h, the numbers h_i, $i = 1, 2, \ldots$ are defined as in Definition 4.1. The scaling function $\Phi_{h,k}^j : [-1, 1] \to \mathbb{R}$ is introduced by

$$\Phi_{h,k}^j = (Up_h^{(k)})^{j,\ldots,\infty} = \overset{\infty}{\underset{i=j}{\circledast}} \left(L_{h_i}^{(k)} \right)^{(2)}, \quad j = 1, 2, \ldots. \tag{44}$$

By construction, $\text{supp}\Phi_{h,k}^j = [h_{j-1}, 1]$, and we have the *refinement equation*

$$\Phi_{h,k}^{j+1} * (L_{h_j}^{(k)})^{(2)} = \Phi_{h,k}^j, \quad j \geq 1. \tag{45}$$

Using the previous results we, therefore, obtain for every $F \in L^2(\Omega)$

$$\lim_{j \to \infty} \left\| \Phi_{h,k}^j * F - F \right\|_{L^2(\Omega)} = 0.$$

Moreover, for every $F \in L^2(\Omega)$, we get

$$\left\| \Phi_{h,k}^j * F \right\|_{L^2(\Omega)} \leq \left\| \Phi_{h,k}^{j+1} * F \right\|_{L^2(\Omega)} \leq \|F\|_{L^2(\Omega)}.$$

These facts give rise to interpret the convolution with $\Phi_{h,k}^j$ as low–pass filter. Obviously, we define, for $j = 1, 2, \ldots$, the projection operators $P_j : L^2(\Omega) \to \mathcal{C}^{(\infty)}(\Omega) \subset L^2(\Omega)$ by

$$P_j(F) = \Phi_{h,k}^j * F = \int_\Omega \Phi_{h,k}^j(\eta \cdot \,)F(\eta)d\omega(\eta). \tag{46}$$

Correspondingly, the scale spaces are introduced as follows:

$$V_j = \{P_j(F) | F \in L^2(\Omega)\}. \tag{47}$$

Altogether we find the following results.

THEOREM 5.2. *For $h \in (-1, 1)$, $k = 0, 1, \ldots$, the scale spaces $V_j = \{\Phi_{h,k}^j * F | F \in L^2(\Omega)\}$ define a multiresolution analysis of $L^2(\Omega)$ in the following sense:*

(i) $V_j \subset L^2(\Omega)$ is a linear subspace with $V_j \subset C^{(\infty)}$,

(ii) $V_1 \subset V_2 \subset V_3 \subset \ldots$,

(iii) $\bigcap\limits_{j=1}^{\infty} V_j = V_1$,

(iv) $\overline{\bigcup\limits_{j=1}^{\infty} V_j} = L^2(\Omega)$.

6 Locally Supported Wavelets

In what follows, we assume that h and k are fixed, and that the corresponding h_i are given as in Definition 4.1. In doing so, we obtain with $\Phi^j = \Phi_{h,k}^j$ the family $\{\Phi^j \mid j = 1, 2, \ldots\}$, which we interpret as scale discrete scaling function. This scaling function allows us to introduce scale discrete locally supported wavelets on the sphere. In consequence, we are able to represent an $L^2(\Omega)$–function F by a two parameter family $(j; \eta)$, $j \in \mathbb{N}$, $\eta \in \Omega$, breaking up the function F into "pieces" at different locations and different levels of resolution. The refinement equation corresponding to the scaling function $\{\Phi_j | j = 1, 2, \ldots\}$ reads as follows:

$$\Psi^j = \Phi^{j+1} - \Phi^j, \quad j = 1, 2, \ldots, \tag{48}$$

Clearly, Ψ^j is a locally supported infinitely often differentiable function with $\mathrm{supp}\Psi^j = [h_j, 1]$. We use Ψ^j to introduce the spherical wavelet at level j and point $\eta \in \Omega$ by setting $\Psi_{j;\eta}(\xi) = \Psi^j(\eta \cdot \xi)$, $(\xi, \eta) \in \Omega \times \Omega$. For the scaling function we analogously write $\Phi_{j;\eta}(\xi) = \Phi^j(\eta \cdot \xi)$.
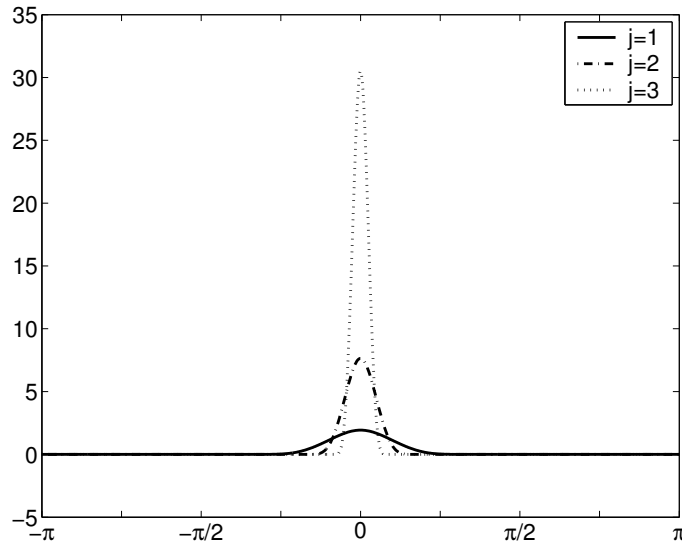

Figure 8: The scaling function $\vartheta \mapsto \Phi^j(\cos\vartheta)$ for $j = 1, 2, 3$, $k = 0$, and $h = -0.9$.

From the definition of the wavelets it is obvious that $(\xi, \eta) \mapsto \Psi_{j;\eta}(\xi) = \Psi^j(\xi \cdot \eta)$ is a radial basis function on the sphere. It is easily seen that

$$(\Psi^j)^\wedge(n) = (\Phi^{j+1})^\wedge(n) - (\Phi^j)^\wedge(n) = (\Phi^{j+1})^\wedge(n)\left[1 - ((L_{h_j}^{(k)})^{(2)})^\wedge(n)\right].$$
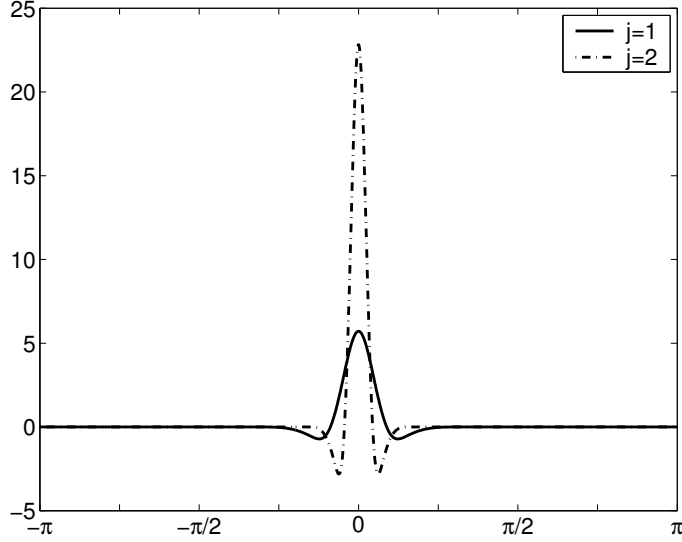
Figure 9: The wavelets $\vartheta \mapsto \Psi^j(\cos\vartheta)$ for $j = 1, 2$, $k = 0$, and $h = -0.9$.

In particular,

$$(\Psi^j)^\wedge(0) = 0 \tag{49}$$

which is nothing else than the zero–mean property known from Euclidean wavelet theory.
Given a function $F \in L^2(\Omega)$, we define its wavelet transform by

$$(\text{WT})(F)(j; \eta) = (\Psi_{j;\eta}, F), \quad j = 1, 2, \ldots, \eta \in \Omega, \tag{50}$$

which allows to break up F into "pieces" at different locations and different scales. This statement is made rigorous in the following theorem, which is a reconstruction formula for linear wavelets:

THEOREM 6.1. *Let F be of class $L^2(\Omega)$. Then, in the $\|\cdot\|_{L^2(\Omega)}$–sense,*

$$F(\eta) = (\Phi_{1;\eta}, F) + \sum_{j=1}^{\infty} (\text{WT})(F)(j; \eta). \tag{51}$$

Proof. The statement is a reformulation of Theorem 5.1. $\qquad\qquad\square$

In (46) and (47) the projection operators P_j and the corresponding scale spaces V_j are introduced by

$$P_j(F) = (\Phi_{j;.}, F), \quad V_j = \left\{ P_j(F) \mid F \in L^2(\Omega) \right\}.$$

Analogously, we let the operator R_j and the detail spaces to be given by

$$R_j(F) = (\Psi_{j;.}, F), \quad W_j = \left\{ R_j(F) \mid F \in L^2(\Omega) \right\}. \tag{52}$$

It follows from the zero–mean property (49) that $F^\wedge(0,0) = 0$ for all $F \in W_j$. Thus the wavelet transform can be seen as a band–pass filter. By construction, we have

$$V_{J+1} = V_J + W_J = V_1 + \sum_{j=1}^{J} W_j . \tag{53}$$

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It is worth mentioning that the decomposition (53) is neither direct nor orthogonal.

The described wavelet analysis, which may be seen as a linear wavelet theory, can be extended to a bilinear reconstruction scheme. We introduce a second family of wavelets $\tilde{\Psi}_{j;\eta}$ (dual wavelets) and understand the reconstruction process by a convolution of the wavelet transform against the dual wavelets. The dual wavelets are given by

$$\tilde{\Psi}^j = \Phi^{j+1} + \Phi^j, \quad j = 1, 2, \dots . \tag{54}$$

As usual, we let

$$\tilde{\Psi}_{j;\eta}(\xi) = \tilde{\Psi}^j(\eta \cdot \xi), \ (\xi, \eta) \in \Omega \times \Omega. \tag{55}$$

The dual wavelet $\tilde{\Psi}^j$ has the local support $\mathrm{supp}\tilde{\Psi}^j = [h_j, 1]$, and its Legendre transform reads as follows:

$$(\tilde{\Psi}^j)^\wedge(n) = (\Phi^{j+1})^\wedge(n) + (\Phi^j)^\wedge(n) = (\Phi^{j+1})^\wedge(n) \left[1 + ((L_{h_j}^{(k)})^{(2)})^\wedge(n) \right].$$

A reconstruction scheme involving the dual wavelets can be formulated as follows:

THEOREM 6.2. *Suppose that $F \in L^2(\Omega)$. Then, again in the $\|\cdot\|_{L^2(\Omega)}-sense$,*

$$F = \int_\Omega (\Phi_{1;\eta}, F)\Phi_{1;\eta}(\cdot)d\omega(\eta) + \sum_{j=1}^\infty \int_\Omega (\mathrm{WT})(F)(j;\eta)\tilde{\Psi}_{j;\eta}(\cdot)d\omega(\eta). \tag{56}$$

Proof. From the completeness of the spherical harmonics in $L^2(\Omega)$ we are able to deduce, that convergence in $L^2(\Omega)$ is equivalent to the convergence of the Fourier transform. For the first summand in (56) we get

$$\left(\int_\Omega (\Phi_{1;\eta}, F)\Phi_{1;\eta}(.)d\omega(\eta) \right)^\wedge (n, m) = F^\wedge(n, m) \left[(\Phi^1)^\wedge(n) \right]^2 .$$

Furthermore, we have

$$\begin{aligned}
\left(\int_\Omega (\mathrm{WT})(F)(j;\eta)\tilde{\Psi}_{j;\eta}(.)d\omega(\eta) \right)^\wedge (n, m) &= F^\wedge(n, m)(\Psi^j)^\wedge(n)(\tilde{\Psi}^j)^\wedge(n) \\
&= F^\wedge(n, m) \left[(\Phi^{j+1})^\wedge(n) - (\Phi^j)^\wedge(n) \right] \\
&\quad \left[(\Phi^{j+1})^\wedge(n) + (\Phi^j)^\wedge(n) \right] \\
&= F^\wedge(n, m) \left[((\Phi^{j+1})^\wedge(n))^2 - ((\Phi^j)^\wedge(n))^2 \right] .
\end{aligned}$$

In conclusion,

$$\begin{aligned}
\left(\int_\Omega (\Phi_{1;\eta}, F)\Phi_{1;\eta}(.)d\omega(\eta) + \sum_{j=1}^J \int_\Omega (\mathrm{WT})(F)(j;\eta)\tilde{\Psi}_{j;\eta}(.)d\omega(\eta) \right)^\wedge (n, m) \\
= F^\wedge(n, m) \left((\Phi^{J+1})^\wedge(n) \right)^2 .
\end{aligned}$$

Observing the fact that

$$\lim_{J \to \infty} \left((\Phi^J)^\wedge(n) \right)^2 = 1,$$

the proof of Theorem 6.2 follows in a similar way as the proof of Theorem 5.1. $\qquad \square$

7 Decomposition and Reconstruction Schemes Involving the Up Functions

For numerical purposes it is important to know, how the wavelet decomposition and reconstruction can be organized in an efficient way, so that information is transported from level to level, which characterizes the essence of a tree algorithm or a pyramid scheme. The decompositions are based an the refinement equation (45)

$$\Phi^{j+1} * (L_{h_j}^{(k)})^{(2)} = \Phi^j, \quad j = 1, 2, \dots. \tag{57}$$

In the following, we present schemes for the decomposition and reconstruction of a function $F \in L^2(\Omega)$. We assume, that we start from a finest level $J \in \mathbb{N}$. To be specific, the wavelet decomposition of a signal $F \in L^2(\Omega)$ looks as follows:

Wavelet Decomposition
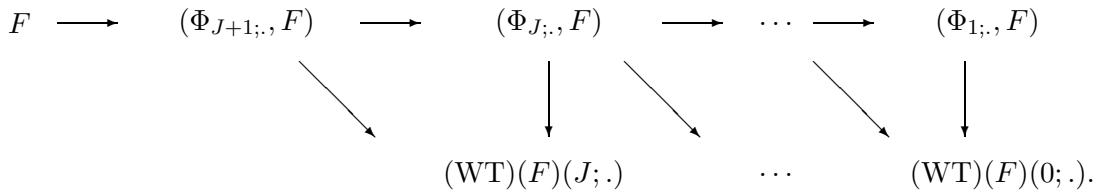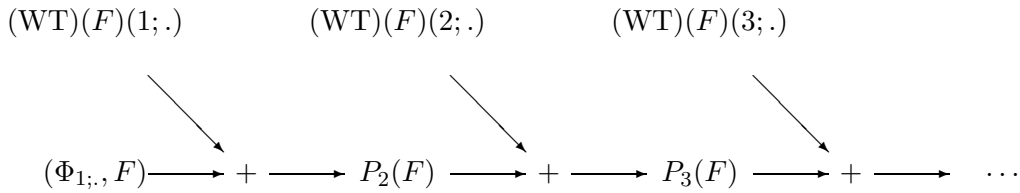

The scheme works because we have from (57) that

$$(\Phi_{j;.}, F) = \Phi^j * F = \Phi^{j+1} * (L_{h_j}^{(k)})^{(2)} * F = (L_{h_j}^{(k)})^{(2)} * (\Phi_{j+1;.}, F), \; j = 1, 2, \dots.$$

and since we can deduce from (48) that

$$(WT)(F)(j;.) = (\Phi_{j+1;.}, F)_{\mathcal{L}^2(\Omega)} - (\Phi_{j;.}, F)_{\mathcal{L}^2(\Omega)}.$$

The reconstruction in the linear case (Theorem 6.1) can be organized as follows:
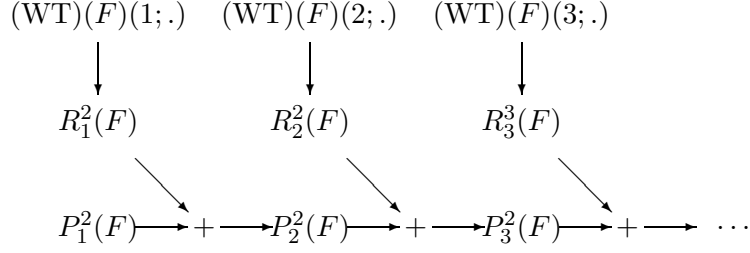
Wavelet Reconstruction (Linear Case)

In order to formulate the reconstruction for the bilinear case, we introduce the following variants of the projection operators P_j and R_j:

$$P_j^2(F) = \int_\Omega (\Phi_{j;\eta}, F)\Phi_{j;\eta}(.)d\omega(\eta) = \Phi^j * \Phi^j * F,$$

$$R_j^2(F) = \int_\Omega (WT)(F)(j, \eta)\tilde{\Psi}_{j;\eta}(.)d\omega(\eta) = \tilde{\Psi}^j * \Psi^j * F.$$

Consequently, we obtain the following scheme from Theorem 6.2:

$$\begin{array}{ccccc}
(\text{WT})(F)(1;.) & (\text{WT})(F)(2;.) & (\text{WT})(F)(3;.) \\
\downarrow & \downarrow & \downarrow \\
R_1^2(F) & R_2^2(F) & R_3^3(F) \\
& & \\
P_1^2(F) \longrightarrow + \longrightarrow & P_2^2(F) \longrightarrow + \longrightarrow & P_3^2(F) \longrightarrow + \longrightarrow & \cdots
\end{array}$$

8 Tree Algorithms Involving the System $\{(L_{h_j}^{(k)})^{(2)}\}$

In what follows we base numerical integration on the approximate formulas associated to known (fixed) weights $w_i^{N_j}$ and known (fixed) knots $\eta_i^{N_j} \in \Omega$ such that, in connection with (57),

$$\begin{aligned}
P_j(F)(\eta) &= (\Phi^j * F)(\eta) \\
&\simeq \sum_{i=1}^{N_{j+1}} w_i^{N_{j+1}} (L_{h_j}^{(k)})^{(2)}(\eta \cdot \eta_i^{N_{j+1}})(\Phi^{j+1} * F)(\eta_i^{N_{j+1}})
\end{aligned}$$

$j = 1, \ldots, J$ (once again, the symbol $\simeq$ means that the error is assumed to be negligible). What we are going to realize is a tree algorithm by use of the "isotropic finite element system" $\{(L_{h_j}^{(k)})\}$ with the following ingredients: Starting from a sufficiently large J such that

$$P_J(F)(\eta) = (\Phi^J * F)(\eta) \simeq \sum_{i=1}^{N_{J+1}} a_i^{N_{J+1}} (L_{h_J}^{(k)})^{(2)}(\eta \cdot \eta_i^{N_{J+1}})$$

with

$$a_i^{N_{J+1}} = w_i^{N_{J+1}}(\Phi^{J+1} * F)(\eta_i^{N_{J+1}}), \quad i = 1, \ldots, N_{J+1}, \tag{58}$$

we want to show that the coefficients vectors $a^{N_j} = (a_1^{N_j}, \ldots, a_{N_j}^{N_j})^T \in \mathbb{R}^{N_j}$, $j = 2, \ldots, J+1$ (being, of course, dependent on the function F under consideration) can be determined such that the following statements are valid:

(i) The vectors $a^{N_{j+1}}$, $J-1, \ldots, 1$, can be calculated recursively from $a^{N_{J+1}}$.

(ii) For $j = 1, \ldots, J$ and $\eta \in \Omega$,

$$P_j(F)(\eta) = (\Phi^j * F)(\eta) \simeq \sum_{i=1}^{N_{j+1}} a_i^{N_{j+1}} (L_{h_j}^{(k)})^{(2)}(\eta \cdot \eta_i^{N_{j+1}}).$$

Our considerations are divided into two parts, i.e. the initial step and the pyramid step establishing the recursion procedure:

The initial step. For sufficiently large J the kernel Φ^J replaces the Dirac kernel in the formal sense that $\Phi^J * F \simeq F = \delta * F$. Therefore, we have

$$(\Phi^J * F)(\eta_i^{N_{J+1}}) \simeq F(\eta_i^{N_{J+1}}), \quad i = 1, \ldots, N_{J+1}.$$

It follows that the initial vector $a^{N_{J+1}} = (a_1^{N_{J+1}}, \ldots, a_{N_{J+1}}^{N_{J+1}})^T \in \mathbb{R}^{N_{J+1}}$ is given in the form

$$a_i^{N_{J+1}} = w_i^{N_{J+1}} F(\eta_i^{N_{J+1}}), \quad i = 1, \ldots, N_{J+1}.$$

The pyramid step. In accordance with our construction we obtain for $j = 1, \ldots, J-1$

$$
\begin{aligned}
a_i^{N_{j+1}} &= w_i^{N_{j+1}} \left(\Phi^{j+1} * F \right) \left(\eta_i^{N_{j+1}} \right) \\
&\simeq w_i^{N_{j+1}} \sum_{l=1}^{N_{j+2}} w_l^{N_{j+2}} (L_{h_{j+1}}^{(k)})^{(2)} \left(\eta_i^{N_{j+1}} \cdot \eta_l^{N_{j+2}} \right) \left(\Phi^{j+2} * F \right) \left(\eta_l^{N_{j+2}} \right) \\
&= w_i^{N_{j+1}} \sum_{l=1}^{N_{j+2}} a_l^{N_{j+2}} (L_{h_{j+1}}^{(k)})^{(2)} \left(\eta_i^{N_{j+1}} \cdot \eta_l^{N_{j+2}} \right),
\end{aligned}
$$

$i = 1, \ldots, N_{j+1}$. In other words, the coefficients a^{N_J} can be calculated recursively starting from the data $a_{N_{J+1}}$ for the initial level J, $a^{N_{J-1}}$ can be calculated recursively from a^{N_J}, etc. This leads to a tree algorithm as shown at the end of Chapter 2.

From the reconstruction theorem (Theorem 6.1) we, therefore, obtain the fully discretized variant.

THEOREM 8.1. *For $F \in L^2(\Omega)$ and $\eta \in \Omega$*

$$
\begin{aligned}
F(\eta) \simeq{}& \sum_{i=1}^{N_1} (L_{h_1}^{(k)})^{(2)} (\eta \cdot \eta_i^2) \\
&+ \sum_{j=1}^{J-1} \left(\sum_{i=1}^{N_{j+2}} a_i^{N_{j+2}} (L_{h_{j+1}}^{(k)})^{(2)} (\eta \cdot \eta_i^{N_{j+2}}) - \sum_{i=1}^{N_{j+1}} a_i^{N_{j+1}} (L_{h_j}^{(k)})^{(2)} (\eta \cdot \eta_i^{N_{j+1}}) \right),
\end{aligned}
$$

where

$$
a_i^{N_{J+1}} \simeq w_i^{N_{J+1}} F(\eta_i^{N_{J+1}}), \quad i = 1, \ldots, N_{J+1},
$$

and

$$
a_i^{N_{j+1}} \simeq w_i^{N_{j+1}} \sum_{l=1}^{N_{j+1}} a_l^{N_{j+2}} (L_{h_{j+1}}^{(k)})^{(2)} (\eta_i^{N_{j+1}} \cdot \eta_l^{N_{j+2}}), \quad i = 1, \ldots, N_{j+1}, \quad j = 1, \ldots, J-1.
$$

In case of hierarchical knots, i.e. $\eta_i^{N_{j+1}} = \eta_i^{N_j}$, $i = 1, \ldots, N_j$, $j = 2, \ldots, J-1$, we get

$$
\begin{aligned}
F(\eta) \simeq{}& \sum_{i=1}^{N_1} (L_{h_1}^{(k)})^{(2)} (\eta \cdot \eta_i^2) \\
&+ \sum_{j=1}^{J-1} \left(\sum_{i=1}^{N_{j+1}} (a_i^{N_{j+2}} - a_i^{N_{j+1}}) (L_{h_{j+1}}^{(k)})^{(2)} (\eta \cdot \eta_i^{N_{j+1}}) \right. \\
&+ \sum_{i+1}^{N_{j+1}} a_i^{N_{j+1}} ((L_{h_{j+1}}^{(k)})^{(2)} (\eta \cdot \eta_i^{N_{j+1}}) - (L_{h_j}^{(k)})^{(2)} (\eta \cdot \eta_i^{N_{j+1}})) \\
&+ \left. \sum_{i=N_{j+1}+1}^{N_{j+2}} a_i^{N_{j+2}} (L_{h_{j+1}}^{(k)})^{(2)} (\eta \cdot \eta_i^{N_{j+2}}) \right).
\end{aligned}
$$

Altogether, a fully discretized multiscale approximation involving locally supported radial basis functions (i.e. isotropic finite elements) is established only by use of numerical integration rules. The generalization of the tree algorithm to the bilinear theory is obvious and will be omitted here.

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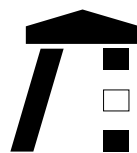
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