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singularities of type $x_1^3 + x_2^3 + x_3^3 + x_4^3$**

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Abstract

We describe, by matrix factorizations, all the rank two maximal Cohen–Macaulay modules over singularities of type $x_1^3 + x_2^3 + x_3^3 + x_4^3$.¹

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Introduction

Let R be a hypersurface ring, that is $R = S/(f)$ for a regular local ring (S, \mathfrak{m}) and $0 \neq f \in \mathfrak{m}$. Accordingly to Eisenbud [Ei], any maximal Cohen–Macaulay (briefly MCM) module over R has a minimal free resolution of periodicity 2 which is completely given by a matrix factorization (φ, ψ) , φ, ψ being square matrices over S such that $\varphi\psi = \psi\varphi = f \text{Id}_n$, for a certain positive integer n . Therefore, in order to describe the MCM R -modules, it is enough to describe their matrix factorizations. In this paper we give the description, by matrix factorizations, of the graded, rank two, indecomposable, MCM modules over $K[x_1, x_2, x_3, x_4]/(x_1^3 + x_2^3 + x_3^3 + x_4^3)$. Part of this study was done with the help of the Computer Algebra System SINGULAR [GPS].

The MCM modules over the hypersurface $f_3 = x_1^3 + x_2^3 + x_3^3$ were described in [LPP] as 1-parameter families indexed by the points of the curve $Z = V(f_3) \subset \mathbf{P}^2$. This description is mainly based on the Atiyah’s theory of the vector bundles classification over elliptic curves, in particular over Z , and on difficult computations made with the Computer Algebra System SINGULAR. The description depends on two discrete invariants – the rank and the degree of the bundle – and on a continuous invariant – the points of the curve Z .

It is of high interest the classification of vector bundles, in particular of ACM bundles (i.e. those which corresponds to MCM modules) over the singularities of higher dimension. In the paper [EP] are described the matrix factorizations which define the graded MCM modules of rank one over $f_4 = x_1^3 + x_2^3 + x_3^3 + x_4^3$. There is a finite number of such modules which correspond to 27 lines, 27 pencils of quadrics and 72 nets of twisted cubic curves lying on the surface $Y = V(f_4) \subset \mathbf{P}^3$. From geometrical point of view the problem is easy, but the effective description of the matrix factorizations is difficult and SINGULAR has been intensively used.

In the present paper we continue this study for the graded MCM modules of rank two. We

obtain a general description of the MCM orientable modules of rank two. They are given by skew-symmetric matrix factorizations (see Theorem 2.2). The technique is based on the results of Herzog and Kühn (see [HK]) concerning the so called *Bourbaki exact sequences*. The matrix factorizations of the graded, orientable, rank two, 4-generated MCM modules are parameter families indexed by the points of the surface Y , that is two parameter families and some finite ones in bijection with rank one MCM modules described in [EP] (see Theorem 3.2, here an important fact is that two Gorenstein ideals of codimension 2 define the same MCM module via the associated Bourbaki sequence if and only if they belong to the same even linkage class). We also describe the non-orientable MCM modules of rank two over f_4 . There is a finite number of such modules which correspond somehow to the rank one modules described in [EP]. The graded MCM modules, non-orientable, of rank two are 2-syzygy over f_4 of some ideals of the form $J/(f_4)$, J being an ideal of the polynomial ring $S = K[x_1, x_2, x_3, x_4]$, (K is an algebraically closed field of characteristic zero), with $f_4 \in J$, $\dim S/J = 2$, $\text{depth } S/J = 1$, whose Betti numbers over S satisfy $\beta_1(J) = \beta_0(J) + 1$ and $\beta_2(J) = 1$ (see Lemma 4.4). This result has been essential in the description of the graded, non-orientable MCM modules. The paper highlights bijections between the classes of indecomposable, graded, non-orientable MCM modules of rank two, 4 and 5-generated and the classes of rank one, graded, MCM modules (see Theorem 4.7 and Theorem 5.2). Consequently, there exists a bijection between the classes of indecomposable, graded, non-orientable MCM modules of rank three, 5-generated and the classes of rank one, graded, MCM modules (see Corollary 5.3). These results remind us the theory of Atiyah and give a small hope that the non-orientable case behaves in the same way for higher rank. We also show that there are no indecomposable, graded, non-orientable MCM modules of rank two 6-generated. Consequently, there exist no indecomposable, graded, non-orientable MCM modules of rank four, 6-generated.

Until now the description of graded rank two MCM modules is not too far from the theory of Atiyah. But the description of graded, rank two, 6-generated MCM modules is different (see Section 6) as we expected since a part of them given by Gorenstein ideals defined by 5 general points on Y forms a 5-parameter family (see [Mi], [IK]). However we believe that behind these results there exists a nice theory of graded MCM modules over a cubic hypersurface in four variables which waits to be discovered.

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1 Preliminaries

Let $R_n := K[x_1, x_2, \dots, x_n]/(f_n)$, where $f_n = x_1^3 + x_2^3 + \dots + x_n^3$ and K is an algebraic closed field of characteristic 0. Using the classification of vector bundles over elliptic curves obtained by Atiyah [At], Laza, Pfister and Popescu [LPP] describe the matrix factorizations of the graded, indecomposable and reflexive modules over R_3 . They give canonical normal forms for the matrix factorizations of all graded reflexive R_3 -modules of rank one (see Section 3 in [LPP]) and show effectively how we can produce the indecomposable graded reflexive

R_3 -modules of rank ≥ 2 using SINGULAR (see Section 5 in [LPP]). We recall from [LPP] the description of the rank one, three-generated, non-free, graded MCM R_3 -modules since we shall use it in the last section of our paper. First we recall the notations. Let $P_0 = [-1 : 0 : 1] \in V(f_3)$. For each $\lambda = [\lambda_1 : \lambda_2 : 1] \in V(f_3)$, $\lambda \neq P_0$, we set

$$\alpha_\lambda = \begin{pmatrix} 0 & x_1 - \lambda_1 x_3 & x_2 - \lambda_2 x_3 \\ x_1 + x_3 & -x_2 - \lambda_2 x_3 & -w x_3 \\ x_2 & w x_3 & (1 - \lambda_1) x_3 - x_1 \end{pmatrix},$$

where $w = \frac{\lambda_2^2}{\lambda_1 + 1}$ and, if $\lambda = [\lambda_1 : 1 : 0] \in V(f_3)$, we set

$$\alpha_\lambda = \begin{pmatrix} 0 & x_1 - \lambda_1 x_2 & x_3 \\ x_1 + x_3 & -\lambda_1 x_1 & \lambda_1 x_1 + \lambda_1^2 x_2 \\ x_2 & x_3 - x_1 & -x_1 \end{pmatrix}.$$

Let β_λ the adjoint matrix of α_λ .

Theorem 1.1 ((3.7) in [LPP]). $(\alpha_\lambda, \beta_\lambda)$ is a matrix factorization for all $\lambda \in V(f_3)$, $\lambda \neq P_0$, and the set of three-generated MCM graded R_3 -modules,

$$\mathcal{M}_0 = \{\text{Coker } \alpha_\lambda \mid \lambda \in V(f_3), \lambda \neq P_0\}$$

has the following properties:

- (i) All the modules from \mathcal{M}_0 have rank one.
- (ii) Each two different modules from \mathcal{M}_0 are not isomorphic.
- (iii) Every three-generated, rank one, non-free, graded MCM R_3 -module is isomorphic with one module from \mathcal{M}_0 .

Now we consider the case $n = 4$. In this case we do not have the support of Atiyah classification. The complete description by matrix factorizations of the rank one, graded, indecomposable MCM modules over R_4 was given in [EP].

The aim of the present paper is to classify the rank two, graded, indecomposable MCM modules over R_4 . From now on, we shall denote $R = R_4$, $f = f_4$ and we preserve the hypothesis on K to be algebraically closed and of characteristic zero.

Let M be a rank two MCM module over R and let $\mu(M)$ be the minimal number of generators of M . By Corollary 1.3 of [HK], we obtain that $\mu(M) \in \{3, 4, 5, 6\}$.

First of all we consider **the three-generated case**. The description of the rank one MCM R -modules is given in [EP]. We recall the notations. For $a, b, c, d, \varepsilon \in K$ such that $a^3 = b^3 = c^3 = d^3 = -1$, $\varepsilon^3 = 1$, $\varepsilon \neq 1$, and $bcd = \varepsilon a$, we set

$$\alpha(b, c, d, \varepsilon) = \begin{pmatrix} 0 & x_1 - ax_4 & x_2 - bx_3 \\ x_1 - cx_2 & -b^2 x_3 - abc^2 \varepsilon^2 x_4 & b^2 c^2 x_3 - abc \varepsilon^2 x_4 \\ x_3 - dx_4 & c^2 x_2 + bc^2 x_3 + acx_4 & -x_1 - cx_2 - ax_4 \end{pmatrix}$$

and

$$\beta(b, c, d, \varepsilon) = \alpha(b, c, d, \varepsilon)^t,$$

that is, the transpose of $\alpha(b, c, d, \varepsilon)$. Then each of the matrices $\alpha(b, c, d, \varepsilon)$ and $\beta(b, c, d, \varepsilon)$ forms with its adjoint, $\alpha(b, c, d, \varepsilon)^*$, respectively $\beta(b, c, d, \varepsilon)^*$, a matrix factorization of f .

For $a, b, c \in K$, distinct roots of -1 , and ε as above, we set

$$\eta(a, b, c, \varepsilon) = \begin{pmatrix} 0 & x_1 + x_2 & x_3 - ax_4 \\ x_1 + \varepsilon x_2 & -x_3 + cx_4 & 0 \\ x_3 - bx_4 & 0 & -x_1 - \varepsilon^2 x_2 \end{pmatrix}$$

and

$$\vartheta(a, b, c) = \begin{pmatrix} 0 & x_1 + x_3 & x_2 - ax_4 \\ x_1 - a^2 bx_3 & -x_2 + cx_4 & 0 \\ x_2 - bx_4 & 0 & -x_1 + ab^2 x_3 \end{pmatrix}.$$

The matrices $\eta(a, b, c, \varepsilon)$ and $\vartheta(a, b, c)$ form with their adjoint, $\eta(a, b, c, \varepsilon)^*$, respectively $\vartheta(a, b, c)^*$, a matrix factorization of f .

Theorem 1.2 ((3.4) in [EP]). *Let*

$$\mathcal{M} = \{\text{Coker } \alpha(b, c, d, \varepsilon), \text{Coker } \beta(b, c, d, \varepsilon) \mid b, c, d, \varepsilon \in K, \\ b^3 = c^3 = d^3 = -1, bcd = \varepsilon a, \varepsilon^3 = 1, \varepsilon \neq 1\}$$

and

$$\mathcal{N} = \{\text{Coker } \eta(a, b, c, \varepsilon), \text{Coker } \vartheta(a, b, c) \mid \varepsilon^3 = 1, \varepsilon \neq 1 \\ \text{and } (a, b, c) \text{ is a permutation of the roots of } -1\}.$$

Then the sets \mathcal{M}, \mathcal{N} of rank one, three-generated, MCM graded R -modules have the following properties:

- (i) *Every three-generated, rank one, indecomposable, graded MCM R -module is isomorphic with one module from $\mathcal{M} \cup \mathcal{N}$.*
- (ii) *If $M = \text{Coker } \alpha(b, c, d, \varepsilon)$ (or $M = \text{Coker } \beta(b, c, d, \varepsilon)$) belongs to \mathcal{M} and $N \in \mathcal{M}$, then $N \simeq M$ if and only if $N = \text{Coker } \alpha(b\varepsilon, c\varepsilon, d\varepsilon, \varepsilon^2)$ (or $N = \text{Coker } \beta(b\varepsilon, c\varepsilon, d\varepsilon, \varepsilon^2)$).*
- (iii) *Any two different modules from \mathcal{N} are not isomorphic.*
- (iv) *Any module of \mathcal{N} is not isomorphic with some module of \mathcal{M} .*

The map $M \mapsto \Omega_R^1(M)$ is a bijection between the three-generated, indecomposable, graded, MCM R -modules of rank two and the three-generated, indecomposable, graded, MCM R -modules of rank one. Thus, from the above theorem we obtain the description of the rank two, three-generated, indecomposable, graded MCM R -modules.

Theorem 1.3. *Let*

$$\mathcal{M}^* = \{\text{Coker } \alpha(b, c, d, \varepsilon)^*, \text{Coker } \beta(b, c, d, \varepsilon)^* \mid b, c, d, \varepsilon \in K, \\ b^3 = c^3 = d^3 = -1, bcd = \varepsilon a, \varepsilon^3 = 1, \varepsilon \neq 1\}$$

and

$$\mathcal{N}^* = \{\text{Coker } \eta(a, b, c, \varepsilon)^*, \text{Coker } \vartheta(a, b, c)^* \mid \varepsilon^3 = 1, \varepsilon \neq 1 \\ \text{and } (a, b, c) \text{ is a permutation of the roots of } -1\}.$$

Then the sets $\mathcal{M}^*, \mathcal{N}^*$ of rank two, three-generated, MCM graded R -modules have the following properties:

- (i) *Every three generated, rank two, indecomposable, graded MCM R -module is isomorphic with one module from $\mathcal{M}^* \cup \mathcal{N}^*$.*
- (ii) *If $M = \text{Coker } \alpha(b, c, d, \varepsilon)^*$ (or $M = \text{Coker } \beta(b, c, d, \varepsilon)^*$) belongs to \mathcal{M}^* and $N \in \mathcal{M}^*$, then $N \simeq M$ if and only if $N = \text{Coker } \alpha(b\varepsilon, c\varepsilon, d\varepsilon, \varepsilon^2)^*$ (or $N = \text{Coker } \beta(b\varepsilon, c\varepsilon, d\varepsilon, \varepsilon^2)^*$).*
- (iii) *Any two different modules from \mathcal{N}^* are not isomorphic.*
- (iv) *Any module of \mathcal{N}^* is not isomorphic with some module of \mathcal{M}^* .*

Corollary 1.4. *There are 72 isomorphism classes of rank two, indecomposable, graded MCM modules over R with three generators.*

2 Skew symmetric matrices and rank 2 orientable MCM modules

Let $\varphi = (a_{ij})_{1 \leq i, j \leq 2s}$ be a generic skew symmetric matrix, that is

$$a_{ii} = 0, a_{ij} = -a_{ji}, \text{ for all } i, j = \overline{1, 2s}.$$

Then

$$\det(\varphi) = \text{pf}(\varphi)^2,$$

where $\text{pf}(\varphi)$ denotes the Pfaffian of φ (see [Bo1, §5, no. 2] or [BH, (3.4)]). Like determinants Pfaffians can be developed along a row. Set φ_{ij} the matrix obtained from φ by deleting the i^{th} and j^{th} rows and columns. Then, for all $i = 1, \dots, 2s$,

$$\text{pf}(\varphi) = \sum_{\substack{j=1 \\ j \neq i}}^{2s} (-1)^{i+j} \sigma(i, j) a_{ij} \text{pf}(\varphi_{ij}), \quad (2.1)$$

where $\sigma(i, j)$ denotes $\text{sign}(j - i)$. Multiplying (2.1) by $\text{pf}(\varphi)$, we get

$$\det(\varphi) = \sum_{j=1}^{2s} (-1)^{i+j} a_{ij} b_{ij}, \quad (2.2)$$

for $b_{ij} = \sigma(i, j) \text{pf}(\varphi_{ij}) \text{pf}(\varphi)$ when $i \neq j$ and $b_{ii} = 0$. Since φ is a generic matrix we see from (2.2) that b_{ij} is exactly the algebraic complement of a_{ij} and so the transpose matrix B of (b_{ij}) is the adjoint matrix of φ . Set

$$\psi = \frac{1}{\text{pf}(\varphi)} B.$$

Then

$$\varphi\psi = \psi\varphi = \text{pf}(\varphi) \text{Id}_{2s},$$

as it is stated also in [[JP], §3].

Proposition 2.1. *Let $f = x_1^3 + x_2^3 + x_3^3 + x_4^3$ and φ a skew symmetric matrix over $S = K[x_1, x_2, x_3, x_4]$ of order 4 or 6 such that $\det \varphi = f^2$, K being a field. Then Coker φ is a MCM module over $R := S/(f)$ of rank 2.*

Proof. Let ψ be given for φ as above, that is the (i, j) entry of ψ is $\sigma(i, j) \text{pf}(\varphi_{ij})$. As above we have

$$\varphi\psi = \psi\varphi = f \cdot \text{Id}_n, \quad n = 4 \text{ or } 6$$

because $\text{pf}(\varphi) = f$. Then (φ, ψ) is a matrix factorization which defines a MCM R -module of rank 2. \square

Theorem 2.2. *Preserving the hypothesis of Proposition 2.1, the cokernel of a homogeneous skew symmetric matrix over S of order 4 or 6 of determinant f^2 defines a graded orientable MCM R -module M of rank two. Conversely, each non-free graded orientable MCM R -module M of rank two is the cokernel of a map given by a skew symmetric homogeneous matrix φ over S of order 4 or 6, whose determinant is f^2 and φ together with ψ , defined above, form the matrix factorization of M .*

Proof. After Herzog and Kühn [HK], M must be 4 or 6 minimally generated. Suppose that M is 6-generated (the other case is similar). Then M is the second syzygy over R of a Gorenstein ideal $I \subset R$ of codimension 2 which is 5-generated by [HK]. Using Buchsbaum–Eisenbud Theorem (see e.g. [BH], (3,4)) there exists an exact sequence

$$0 \rightarrow S(-5) \xrightarrow{d_3} S^5(-3) \xrightarrow{d_2} S^5(-2) \xrightarrow{d_1} S \quad (2.3)$$

such that $J = \text{Im } d_1, I = J/(f), d_2$ is a skew symmetric homogeneous matrix, d_3 is the dual of $d_1, d_3 = d_1^t$, and

$$d_1 = \left(\text{pf}((d_2)_1), -\text{pf}((d_2)_2), \dots, \text{pf}((d_2)_5) \right),$$

where $(d_2)_i$ denotes the 4×4 skew symmetric matrix obtained by deleting the i^{th} row and column of d_2 . Since $f \in J$ there exists $v : S(-1) \rightarrow S^5$ such that $d_1 v = f$ (v is given by

linear forms). It is easy to see from (2.3) that $I = J/(f)$ has the following minimal resolution over S :

$$0 \rightarrow S(-5) \xrightarrow{\begin{pmatrix} d_3 \\ 0 \end{pmatrix}} S^6(-3) \xrightarrow{(d_2, v)} S^5(-2) \xrightarrow{\bar{d}_1} I \rightarrow 0.$$

Like in [Ei], since $fI = 0$, there exists a map $h : S^5(-5) \rightarrow S^6(-3)$ such that $(d_2, v)h = f \cdot \text{Id}_5$ and we get the following exact sequence

$$R^6(-5) \xrightarrow{\begin{pmatrix} \bar{h} \\ \bar{d}_3 \\ 0 \end{pmatrix}} R^6(-3) \xrightarrow{(\bar{d}_2, \bar{v})} R^5(-2) \xrightarrow{\bar{d}_1} I \rightarrow 0. \quad (2.4)$$

On the other hand, $\varphi = \begin{pmatrix} d_2 & v \\ -v^t & 0 \end{pmatrix}$ is a skew symmetric homogeneous matrix of order 6. Let

ψ given as above. By construction ψ has the form $\begin{pmatrix} C & -d_1^t \\ d_1 & 0 \end{pmatrix}$ and so $(d_2, v) \begin{pmatrix} C \\ d_1 \end{pmatrix} = f \cdot \text{Id}_5$.

Taking $h = \begin{pmatrix} C \\ d_1 \end{pmatrix}$ above, we get from (2.4), the following exact sequence:

$$R^6(-6) \xrightarrow{\varphi} R^6(-5) \xrightarrow{\psi} R^6(-3) \xrightarrow{(\bar{d}_2, \bar{v})} R^5(-2) \xrightarrow{\bar{d}_1} I \rightarrow 0,$$

which gives

$$\text{Coker } \varphi \cong \text{Im } \psi = \Omega_R^2(I).$$

We have $\varphi(h|_{-d_1^t}) = f \cdot \text{Id}_6$ and so $\det(\varphi)$ is a power of f . Since the entries of φ are linear forms, we get $\det(\varphi) = f^2$. \square

3 Orientable, rank 2, 4-generated MCM modules

Let K be an algebraically closed field of characteristic zero, $S = K[x_1, x_2, x_3, x_4]$, $f = x_1^3 + x_2^3 + x_3^3 + x_4^3$, and $R = S/(f)$. Let M be a graded, indecomposable, 4-generated MCM R -module of rank two. After Herzog and Kühl [HK], $M \cong \Omega_R^2(I)$, where I is a graded 3-generated Gorenstein ideal such that $\dim R/I = 1$. Then $I = J/(f)$, with $J \subset S$ a graded, 3-generated ideal containing f . Let $\alpha_1, \alpha_2, \alpha_3$ be a minimal system of homogeneous generators of J . Since $\dim S/J = 1$, it follows that $\alpha_1, \alpha_2, \alpha_3$ is a regular system of elements in S .

Let $u, a, b \in K$ with $a^3 = b^3 = -1, u^2 + u + 1 = 0$ and $\sigma = (i \ j \ s)$ be a permutation of the set $\{2, 3, 4\}$ with $i < j$. Set

$$\begin{aligned} w_{\sigma 1} &= x_1 - ax_s, & w_{\sigma 2} &= x_i - bx_j, \\ v_{\sigma 1} &= x_1^2 + ax_1x_s + a^2x_s, & v_{\sigma 2} &= x_i^2 + bx_ix_j + b^2x_j^2. \end{aligned}$$

then we have

$$f = w_{\sigma 1}v_{\sigma 1} + w_{\sigma 2}v_{\sigma 2}.$$

Let $\lambda = [\lambda_1 : \lambda_2 : \lambda_3 : 1]$ be a point of the surface $V(f) \subset \mathbb{P}^3$. We set

$$p_{i\lambda} = x_i - \lambda_i x_4, \text{ and } q_{i\lambda} = x_i^2 + \lambda_i x_i x_4 + \lambda_i^2 x_4^2, \text{ for } 1 \leq i \leq 3.$$

Let $\lambda = [\lambda_1 : \lambda_2 : 1 : 0]$ be a point of $V(f)$. We set

$$p_{i\lambda} = x_i - \lambda_i x_3, q_{i\lambda} = x_i^2 + \lambda_i x_i x_3 + \lambda_i^2 x_3^2, \text{ for } 1 \leq i \leq 2$$

and

$$p_{3\lambda} = x_4, q_{3\lambda} = x_4^2.$$

If $\lambda = [\lambda_1 : 1 : 0 : 0] \in V(f)$, we set

$$p_{1\lambda} = x_1 - \lambda_1 x_2, q_{1\lambda} = x_1^2 + \lambda_1 x_1 x_2 + \lambda_1^2 x_2^2$$

and

$$p_{2\lambda} = x_3, p_{3\lambda} = x_4, q_{2\lambda} = x_3^2, q_{3\lambda} = x_4^2.$$

In all cases we have

$$f = \sum_{i=1}^3 p_{i\lambda} q_{i\lambda}.$$

These are the only ways to write f as a linear combination of two or three forms of degrees 1, 2, provided that the 1-forms are linearly independent. Since $f \in (\alpha_1, \alpha_2, \alpha_3)$, we may suppose that either α_i is in the set $\{p_{i\lambda}, q_{i\lambda}\}$ for each $1 \leq i \leq 3$, or α_i is in the set $\{w_{\sigma i}, v_{\sigma i}\}$ for each $1 \leq i \leq 2$ and $\beta = \alpha_3$ is a regular element in $R/(\alpha_1, \alpha_2)$.

Lemma 3.1. *Let M be a graded, indecomposable, 4-generated MCM R -module of rank 2. Then M is one of the following modules:*

- (1) $\Omega_R^2(p_{1\lambda}, p_{2\lambda}, p_{3\lambda})$ or $\Omega_R^2(q_{1\lambda}, q_{2\lambda}, q_{3\lambda})$, for some $\lambda \in V(f)$,
- (2) $\Omega_R^2(w_{\sigma 1}, v_{\sigma 2}, \beta)$ or $\Omega_R^2(w_{\sigma 2}, v_{\sigma 1}, \beta)$ for some a, b, σ and β as above.

Proof. Set

$$I_\lambda = (p_{1\lambda}, p_{2\lambda}, p_{3\lambda})$$

and

$$\varphi_\lambda = \begin{pmatrix} 0 & p_{3\lambda} & -p_{2\lambda} & -q_{1\lambda} \\ -p_{3\lambda} & 0 & -p_{1\lambda} & q_{2\lambda} \\ p_{2\lambda} & p_{1\lambda} & 0 & q_{3\lambda} \\ q_{1\lambda} & -q_{2\lambda} & -q_{3\lambda} & 0 \end{pmatrix}, \psi_\lambda = \begin{pmatrix} 0 & -q_{3\lambda} & q_{2\lambda} & p_{1\lambda} \\ q_{3\lambda} & 0 & q_{1\lambda} & -p_{2\lambda} \\ -q_{2\lambda} & -q_{1\lambda} & 0 & -p_{3\lambda} \\ -p_{1\lambda} & p_{2\lambda} & p_{3\lambda} & 0 \end{pmatrix}.$$

We have the following exact sequence:

$$R^3(-5) \oplus R(-6) \xrightarrow{\varphi_\lambda} R^4(-4) \xrightarrow{\psi_\lambda} R^3(-2) \oplus R(-3) \xrightarrow{A} R^3(-1) \xrightarrow{\tau} I_\lambda \rightarrow 0,$$

where $\tau = (-p_{1\lambda}, p_{2\lambda}, p_{3\lambda})$ and A is given by the first three rows of φ_λ . Thus $\Omega^2(I_\lambda) \cong \text{Coker}(\varphi_\lambda)$ and $(\varphi_\lambda, \psi_\lambda)$ is a matrix factorization of $\Omega^2(I_\lambda)$. The ideals I_λ and $(q_{1\lambda}, q_{2\lambda}, p_{3\lambda})$ belong to the same even linkage class since

$$I_\lambda \sim (q_{1\lambda}, p_{2\lambda}, p_{3\lambda}) \sim (q_{1\lambda}, q_{2\lambda}, p_{3\lambda}).$$

For the first link we consider the regular sequence $\{p_{1\lambda}q_{1\lambda}, p_{2\lambda}, p_{3\lambda}\}$ and for the second one the sequence $\{q_{1\lambda}, p_{2\lambda}q_{2\lambda}, p_{3\lambda}\}$. Similarly one can see that I_λ is evenly linked with the ideals $(q_{1\lambda}, p_{2\lambda}, q_{3\lambda})$ and $(p_{1\lambda}, q_{2\lambda}, q_{3\lambda})$. By Theorem 2.1 [HK], we get that

$$\text{Coker}(\varphi_\lambda) \cong \Omega_R^2(I_\lambda) \cong \Omega_R^2(q_{1\lambda}, q_{2\lambda}, p_{3\lambda}) \cong \Omega_R^2(q_{1\lambda}, p_{2\lambda}, q_{3\lambda}) \cong \Omega_R^2(p_{1\lambda}, q_{2\lambda}, q_{3\lambda}).$$

Analogously we see that

$$\begin{aligned} \text{Coker}(\psi_\lambda) &\cong \Omega_R^2(q_{1\lambda}, q_{2\lambda}, q_{3\lambda}) \cong \Omega_R^2(p_{1\lambda}, p_{2\lambda}, q_{3\lambda}) \cong \Omega_R^2(p_{1\lambda}, q_{2\lambda}, p_{3\lambda}) \cong \\ &\cong \Omega_R^2(q_{1\lambda}, p_{2\lambda}, p_{3\lambda}). \end{aligned}$$

Thus the case when α_i is one of the forms $\{p_{i\lambda}, q_{i\lambda}\}$ gives (1).

Now let σ, a, b as above and $\beta \in S$ which is regular on $R/(w_{\sigma_1}, v_{\sigma_2})$. Set

$$I_{\sigma\beta}(a, b, u) = (w_{\sigma_1}, v_{\sigma_2}, \beta)$$

and

$$\begin{aligned} \varphi_{\sigma\beta}(a, b, u) &= \begin{pmatrix} 0 & w_{\sigma_1} & -v_{\sigma_2} & 0 \\ -w_{\sigma_1} & 0 & -\beta & w_{\sigma_2} \\ v_{\sigma_2} & \beta & 0 & v_{\sigma_1} \\ 0 & -w_{\sigma_2} & -v_{\sigma_1} & 0 \end{pmatrix}, \\ \psi_{\sigma\beta}(a, b, u) &= \begin{pmatrix} 0 & -v_{\sigma_1} & w_{\sigma_2} & \beta \\ v_{\sigma_1} & 0 & 0 & -v_{\sigma_2} \\ -w_{\sigma_2} & 0 & 0 & -w_{\sigma_1} \\ -\beta & v_{\sigma_2} & w_{\sigma_1} & 0 \end{pmatrix}. \end{aligned}$$

We have the following exact sequence:

$$R^4 \xrightarrow{\varphi_{\sigma\beta}(a, b, u)} R^4 \xrightarrow{\psi_{\sigma\beta}(a, b, u)} R^4 \xrightarrow{B} R^3 \xrightarrow{\tau'} I_{\sigma\beta}(a, b, u) \longrightarrow 0,$$

where $\tau' = (-\beta, v_{\sigma_2}, w_{\sigma_1})$ and B is the matrix given by the first three rows of $\varphi_{\sigma\beta}(a, b, u)$. Thus

$$\Omega_R^2(I_{\sigma\beta}(a, b, u)) \cong \text{Coker}(\varphi_{\sigma\beta}(a, b, u)).$$

As above we see that

$$\Omega_R^2(I_{\sigma\beta}(a, b, u)) \cong \Omega_R^2(w_{\sigma_2}, v_{\sigma_1}, \beta)$$

and

$$\Omega_R^2(w_{\sigma_1}, w_{\sigma_2}, \beta) \cong \Omega_R^2(v_{\sigma_1}, v_{\sigma_2}, \beta) \cong \text{Coker}(\psi_{\sigma\beta}(a, b, u)).$$

Thus the case when α_i is one of the forms $\{w_{\sigma_i}, v_{\sigma_i}\}$ for $i \leq 2$ gives (2). \square

Let

$$\mathcal{M} = \{\text{Coker}(\varphi_\lambda), \text{Coker}(\psi_\lambda) \mid \lambda \in V(f)\}.$$

For a, b, σ as above, set

$$\varphi_\sigma(a, b, u) = \varphi_{\sigma, x_j x_s}(a, b, u), \quad \psi_\sigma(a, b, u) = \psi_{\sigma, x_j x_s}(a, b, u),$$

that is $\beta = x_j x_s$. Let

$$\mathcal{P} = \{\text{Coker}(\varphi_\sigma(a, b, u)), \text{Coker}(\psi_\sigma(a, b, u)) \mid a, b, \sigma \text{ as above}\}.$$

Theorem 3.2. *The set $\mathcal{M} \cup \mathcal{P}$ contains only non-isomorphic, indecomposable, graded, orientable, 4-generated MCM R -modules of rank 2 and every indecomposable, graded, orientable, 4-generated MCM R -module of rank 2 is isomorphic with one module of $\mathcal{M} \cup \mathcal{P}$.*

Proof. Applying Lemma 3.1 we must show in the case (2) that β can be taken $x_j x_s$. Since $v_{\sigma_1} - w_{\sigma_1}(x_1 + 2ax_s) = 3a^2 x_s^2$, adding in $\varphi_{\sigma, \beta}(a, b, u)$ multiples of the last row to the second one and multiples of the first column to the third one, we may suppose the entry (2, 3) of the form $\gamma + x_s \delta$, with γ, δ depending only on x_j, x_i . These transformations modify the entries (2, 2), (3, 3) which are now possibly non-zero. Adding similar multiples of the last column to the second one and multiples of the first row to the third one, we get $\varphi_{\sigma, \beta}(a, b, u)$ of the same type as before but with $\beta = \gamma + x_s \delta$. Similarly, adding in $\varphi_{\sigma, \beta}(a, b, u)$ multiples of the first row to the second one and multiples of the last column to the third one we may suppose that the entry (2, 3) has the form $\varepsilon x_j x_s$ with $\varepsilon \in K$. These transformations modify the entries (2, 2), (3, 3). After similar transformations we get $\varphi_{\sigma, \beta}(a, b, u)$ of the same type as before but with $\beta = \varepsilon x_j x_s$. If $\varepsilon = 0$ we see that $\varphi_{\sigma, \beta}(a, b, u)$ is a direct sum of two 2×2 -matrices which contradicts the indecomposability of $M = \text{Coker}(\varphi_{\sigma, \beta}(a, b, u))$. So $\varepsilon \neq 0$. Divide the second and the third column of $\varphi_{\sigma, \beta}(a, b, u)$ with ε and multiply the first and the last row of $\varphi_{\sigma, \beta}(a, b, u)$ with ε . We reduce to the case $\varepsilon = 1$, that is $\beta = x_j x_s$.

Now we show that two different modules from $\mathcal{M} \cup \mathcal{P}$ are not isomorphic. Note that the Fitting ideals of φ_λ (respectively ψ_λ) modulo $(x_1, \dots, x_4)^2$ have the form $(p_{1\lambda}, p_{2\lambda}, p_{3\lambda})$ and the Fitting ideals of $\varphi_\sigma(a, b, u)$ (respectively $\psi_\sigma(a, b, u)$) modulo $(x_1, \dots, x_4)^2$ have the form $(w_{\sigma_1}, w_{\sigma_2})$ and these ideals are all different. Thus

$$\{\text{Coker}(\varphi_\lambda) \mid \lambda \in V(f)\} \cup \{\text{Coker}(\varphi_\sigma(a, b, u)) \mid \sigma, a, b \text{ as above}\}$$

contains only non-isomorphic modules (similarly for ψ -es). It follows that, if $N, P \in \mathcal{M} \cup \mathcal{P}$ are isomorphic and different, then $N \simeq \Omega_R^1(P)$.

If $N = \text{Coker}(\varphi_\lambda)$, for $\lambda \in V(f)$, then this is not possible since the ideals $(p_{1\lambda}, p_{2\lambda}, p_{3\lambda})$ and $(q_{1\lambda}, q_{2\lambda}, q_{3\lambda})$ are not in the same even linkage class. Indeed, by the proof of (i) in Lemma 3.1, $(p_{1\lambda}, p_{2\lambda}, p_{3\lambda})$ is evenly linked with $(q_{1\lambda}, q_{2\lambda}, p_{3\lambda})$ and this last ideal is obviously directly linked with $(q_{1\lambda}, q_{2\lambda}, q_{3\lambda})$. If $N = \text{Coker}(\varphi_\sigma(a, b, u))$ for some σ, a, b , and $N \simeq \Omega_R^1(N)$, then the ideals $(w_{\sigma_1}, v_{\sigma_2}, x_j x_s)$ and $(w_{\sigma_1}, w_{\sigma_2}, x_j x_s)$ are evenly linked. But these ideals are directly linked by the regular sequence $\{w_{\sigma_1}, v_{\sigma_2} w_{\sigma_2}, x_j x_s\}$, contradiction!

It remains to show that $\mathcal{M} \cup \mathcal{P}$ contains only indecomposable modules. If $N \in \mathcal{M}$, let us say $N = \text{Coker}(\varphi_\lambda)$ for $\lambda = [\lambda_1 : \lambda_2 : \lambda_3 : 1]$ we see that $N/x_4 N$ is exactly the module corresponding to the matrix

$$\begin{pmatrix} 0 & x_3 & -x_2 & -x_1^2 \\ -x_3 & 0 & -x_1 & x_2^2 \\ x_2 & x_1 & 0 & x_3^2 \\ x_1^2 & -x_2^2 & -x_3^2 & 0 \end{pmatrix}$$

whose cokernel is the special module M_2 (see [LPP] for the special module of rank two which corresponds to the special bundle from Atiyah classification). Thus $N/x_4 N$ is indecomposable and, by Nakayama's Lemma, N is indecomposable. Now let $N \in \mathcal{P}$, $N =$

$\text{Coker}(\psi_\sigma(a, b, u))$. By the permutation of the rows and the columns of $\psi_\sigma(a, b, u)$, we may suppose that it has the form:

$$\begin{pmatrix} w_{\sigma 1} & -v_{\sigma 2} & x_j x_s & 0 \\ w_{\sigma 2} & v_{\sigma 1} & 0 & x_j x_s \\ 0 & 0 & v_{\sigma 1} & v_{\sigma 2} \\ 0 & 0 & -w_{\sigma 2} & w_{\sigma 1} \end{pmatrix}.$$

Suppose N is decomposable. Then $\psi_\sigma(a, b, u)$ is a direct sum of two matrices of order two which we may suppose to be given by the submatrices of the above one given by the first two lines and columns respectively the last two lines and columns (this is obvious modulo x_j or x_s). Due to the particular form of $\psi_\sigma(a, b, u)$ this means that there exist two matrices A, B of order two such that

$$x_j x_s \cdot \text{Id}_2 = \begin{pmatrix} w_{\sigma 1} & -v_{\sigma 2} \\ w_{\sigma 2} & v_{\sigma 1} \end{pmatrix} A + B \begin{pmatrix} v_{\sigma 1} & v_{\sigma 2} \\ -w_{\sigma 2} & w_{\sigma 1} \end{pmatrix}$$

which is impossible. □

Remarks 3.3. (i) There exists a bijection between

$$\mathcal{P}_1 = \{\text{Coker}(\varphi_\sigma(a, b, u)) \mid \sigma, a, b\}$$

and the two-generated non-free MCM R -modules which remind us Atiyah's classification. Thus \mathcal{P}_1 contains 54 modules corresponding to 27 lines and 27 pencils of conics of $V(f)$. Similarly, $\mathcal{P}_2 = \{\text{Coker}(\psi_\sigma(a, b, u)) \mid \sigma, a, b\}$ contains 54 modules.

(ii) \mathcal{M} is a kind of "blowing up" of $M_2, \Omega_R^1(M_2)$ from [LPP] (see the proof of Theorem 3.2). Note also that \mathcal{M} consists of two classes of modules parameterized by the points of $V(f)$, which is also in Atiyah's idea.

(iii) The matrices φ defining the modules of $\mathcal{M} \cup \mathcal{P}$ are skew symmetric as our Theorem 2.2 predicted.

4 Non-orientable, rank 2, 4-generated MCM modules

Let M be a graded non-orientable, rank 2, MCM R -module, without free direct summands. We should like to express M as a 2-syzygy of an ideal I , $M \cong \Omega_R^2(I)$, with $\mu(M) = \mu(I) + 1$ (this is known in orientable case by [HK], see here Section 3). For this aim we shall modify a little Theorem 6 from [Bo2, Ch. VII, 4, no. 9], but first we need some preparations.

Let A be a Noetherian normal domain with $\dim A \geq 2$ and N be a finite torsion-free A -module. Let $\mathcal{P}(A)$ be the set of prime ideals of height one from A . We remind to the reader:

Lemma 4.1 ([Bo2, Ch. VII, 4, no. 9, Lemma 7]). *Let $x = (x_1, \dots, x_t)$ be a system of linearly independent elements of N and L the free sub- A -module generated by x in N . Then for all but finite $p \in \mathcal{P}(A)$ the elements $x(p) = (x_1(p), \dots, x_t(p))$ induced by x in $N \otimes_A k(p)$, $k(p) = A_p/pA_p$ are linearly independent over $k(p)$ and N/L is torsion-free, if and only if, for all $p \in \mathcal{P}(A)$, $x(p)$ are linearly independent over $k(p)$.*

The following lemma is a variation of Lemma 8 from [Bo2, Ch. VII, 4, no. 9].

Lemma 4.2. *Let (A, m) be a Noetherian normal local domain with $\dim A \geq 2$ and N be a finite A -module torsion-free of rank ≥ 2 . Then there exists $x \in N$ which is not in mN such that N/Ax is torsion-free.*

Proof after [Bo2]. Let $y \in N$ which is not in mN . By Lemma 4.1 the set $Y \subset \mathcal{P}(A)$ of all prime ideals p such that y induces 0 in N_p/pN_p , is finite. If $Y = \emptyset$ then, by Lemma 4.1, we may take $x = y$. Suppose $Y \neq \emptyset$. There exists $z \in N$ such that z induces a nonzero element z_p in N_p/pN_p for all $p \in Y$. Moreover as in the proof of Lemma 8 [Bo2] we may suppose $\{y, z\}$ linearly independent over A . Then the set $Z \subset \mathcal{P}(A)$ of all prime ideals $p \notin Y$ for which $\{y_p, z_p\}$ are linearly dependent over $k(p)$ is finite by Lemma 4.1. For each $p \in Z$ we have $z_p = \lambda_p y_p$ for a certain $\lambda_p \in k(p)$ ($p \notin Y!$). Choose $b \in m$ such that $b \notin p$ for all $p \in Y$ and $b_p \neq \lambda_p$ in $k(p)$ for all $p \in Z$. Set $x = y - bz$. Clearly $x \in y + mN$ and so $x \notin mN$. Note $x_p = -bz_p \neq 0$ in $N(p)$ for $p \in Y$ because $b_p \neq 0$, $z_p \neq 0$. If $p \in Z$ then $x_p \neq 0$ because $b_p \neq \lambda_p$. If $p \notin Y \cup Z$ then $\{y_p, z_p\}$ are linearly independent over $k(p)$ and so x_p must be nonzero. Thus $x_p \neq 0$ for all $p \in \mathcal{P}(A)$ and so N/Ax is torsion-free by Lemma 4.1. \square

The following proposition is a variation of Bourbaki theorem [Bo2, Ch. VII, 4, no. 9, Theorem 6].

Proposition 4.3. *Let (A, m) be a Noetherian normal local domain with $\dim A \geq 2$ and N a finite torsion-free A -module. Then there exists a finite free submodule $F \subset N$ such that N/F is isomorphic with an ideal of A and the canonical map $F/mF \rightarrow N/mN$ is injective.*

Proof. Apply induction on $r = \text{rank}_A N - 1$. If $r = 0$ then clearly N is isomorphic with an ideal of A . Suppose $r > 0$. By Lemma 4.2 there exists $x_r \in N$ which is not in mN such that $\bar{N} = N/Ax_r$ is torsion-free. Clearly $\text{rank } \bar{N} < r$ and by induction hypothesis there exists a finite free submodule $L \subset \bar{N}$ such that \bar{N}/L is isomorphic with an ideal of A and the canonical map $L/mL \rightarrow \bar{N}/m\bar{N}$ is injective. Let F be the submodule of N containing x_r corresponding to L . We have $N/F \cong \bar{N}/L$ is isomorphic with an ideal of A . Let $x_1, \dots, x_{r-1} \in F$ be elements inducing a basis in L . Then x_1, \dots, x_r is a basis in F inducing a linearly independent system of elements in N/mN , which is enough. \square

Back to our non-orientable MCM R -module M of rank 2, we get the following exact sequence from the above proposition:

$$0 \rightarrow R \rightarrow M \rightarrow I \rightarrow 0 \tag{4.1}$$

for an ideal $I \subset R$, which induces an exact sequence

$$0 \rightarrow K = R/m \rightarrow M/mM \rightarrow I/mI \rightarrow 0.$$

Thus $\mu(M) = \mu(I) + 1$.

As we know in the orientable case to get MCM R -modules of rank 2 we must choose I such that $\text{Ext}_R^1(I, R)$ is a cyclic R -module or more precisely such that R/I is Gorenstein. In the non-orientable case one can also show that $\text{Ext}_R^1(I, R)$ must be a cyclic R -module, but this is not very helpful since it is hard to check this condition for arbitrary I . Below we shall state an easier condition.

Let $J \subset S = K[X_1, \dots, X_4]$ be an ideal such that $f \in J$ and $I = J/(f)$.

Lemma 4.4. *Let*

$$0 \rightarrow S^{s_3} \xrightarrow{d_3} S^{s_2} \xrightarrow{d_2} S^{s_1} \xrightarrow{d_1} J \rightarrow 0$$

be a minimal free S -resolution of an ideal J with depth $S/J = 1$.

If $\text{rank } \Omega_R^2(J/(f)) = 2$ and $\mu(\Omega_R^2(J/(f))) = \mu(I) + 1$ then $s_1 = s_2 \leq 5$ and $s_3 = 1$.

Proof. As in the proof of Theorem 2.2 we get a minimal free resolution of $I = J/(f)$ over S in the following way:

Let $v : S \rightarrow S^{s_1}$ be an S -linear map such that $jd_1v = f\text{Id}_S$, where $j : J \rightarrow S$ is the inclusion. Let \tilde{d}_1 be the composite map $S^{s_1} \xrightarrow{d_1} J \rightarrow J/(f) = I$. Then the following sequence

$$0 \rightarrow S^{s_3} \xrightarrow{\begin{pmatrix} d_3 \\ 0 \end{pmatrix}} S^{s_2+1} \xrightarrow{(d_2, v)} S^{s_1} \xrightarrow{\tilde{d}_1} I \rightarrow 0$$

is exact and forms a minimal free S -resolution of I over S . Since

$$f \cdot S^{s_1} \subset \text{Im}(d_2, v),$$

there exists an S -linear map $h : S^{s_1} \rightarrow S^{s_2+1}$ such that

$$(d_2, v)h = f\text{Id}_{S^{s_1}}$$

and we get the following exact sequence

$$R^{s_3+s_2} \xrightarrow{\begin{pmatrix} \tilde{h} & \tilde{d}_3 \\ 0 \end{pmatrix}} R^{s_2+1} \xrightarrow{(\tilde{d}_2, \tilde{v})} R^{s_1} \xrightarrow{\tilde{d}_1} I \rightarrow 0,$$

which is part of a minimal free R -resolution of I . Thus $M = \Omega_R^2(I)$ is the image of the first map above and so $s_2 + s_3 = s_2 + 1 = s_1 + 1$ because $\mu(M) = \mu(\Omega_R^1(M)) = \mu(I) + 1$ by hypothesis. It follows $s_3 = 1$, $s_1 = s_2$. As $\mu(M) \leq 3\text{rank}_R M = 6$ we get $s_1 \leq 5$. \square

Let $\det N$ be the corresponding class of the bidual $(\wedge^n N)^{**}$, $n = \text{rank } N$, in $Cl(R)$ for a torsion free R -module N . Since \det is an additive function, we get $\det(M) = 0$ if and only if $\det(I) = 0$. Thus M is non-orientable if and only if I is non-orientable, that is $\text{codim}(I) \leq 1$, after [HK]. Since M has rank 2, we get $\text{codim}(I) = 1$. Thus $\dim R/I = 2$ and, from (4.1), we get $\text{depth } R/I = 1$, that is R/I is not Cohen-Macaulay. Also from (4.1) we get $\Omega_R^2(M) \simeq \Omega_R^2(I)$ and so $M \simeq \Omega_R^2(I)$.

Proposition 4.5. *Each graded, non-orientable, rank two, s -generated MCM R -module is the second syzygy $\Omega_R^2(I)$ of an $(s-1)$ -generated graded ideal $I \subset R$ with $\text{depth } R/I = 1$ and $\dim R/I = 2$.*

Proposition 4.6. *Let $I \subset R$ be a graded ideal with $\text{depth } R/I = 1$ and $\dim R/I = 2$. Suppose that $\Omega_R^1(I)$ is minimally generated by s elements and I is minimally generated by $s-1$ elements. Then $\Omega_R^2(I)$ is a graded non-orientable MCM module of rank two which is s -generated.*

The proof follows as above. □

As in Section 3, let $u, a, b \in K$, with

$$a^3 = b^3 = -1, \quad u^2 + u + 1 = 0,$$

$\sigma = (i \ j \ s)$ be a permutation of the set $\{2, 3, 4\}$ with $i < j$ and set

$$\begin{aligned} w_{\sigma 1} &= x_1 - ax_s, \quad w_{\sigma 2} = x_i - bx_j, \\ v_{\sigma 1} &= x_1^2 + ax_1x_s + a^2x_s^2, \quad v_{\sigma 2} = x_i^2 + bx_ix_j + b^2x_j^2. \end{aligned}$$

We have

$$v_{\sigma 1} = v'_{\sigma 1}v''_{\sigma 1}, \quad v_{\sigma 2} = v'_{\sigma 2}v''_{\sigma 2}$$

for

$$\begin{aligned} v'_{\sigma 1} &= x_1 - uax_s, \quad v''_{\sigma 1} = x_1 + (1+u)ax_s, \\ v'_{\sigma 2} &= x_i - ubx_j, \quad v''_{\sigma 2} = x_i + (1+u)bx_j. \end{aligned}$$

Set

$$\begin{aligned} I_{1\sigma}(a, b, u) &= (x_s v'_{\sigma 2}, v_{\sigma 2}, w_{\sigma 1}), \\ I_{2\sigma}(a, b, u) &= (x_j v''_{\sigma 1}, v_{\sigma 1}, w_{\sigma 2}), \\ I_{3\sigma}(a, b, u) &= (x_s v''_{\sigma 2}, v_{\sigma 2}, v_{\sigma 1}), \\ I_{4\sigma}(a, b, u) &= (x_j v'_{\sigma 1}, v_{\sigma 1}, v_{\sigma 2}). \end{aligned}$$

Set

$$\begin{aligned} \varphi_{1\sigma}(a, b, u) &= \begin{pmatrix} 0 & w_{\sigma 1} & -v''_{\sigma 2} & 0 \\ -w_{\sigma 1} & 0 & -x_s & w_{\sigma 2} \\ v_{\sigma 2} & x_s v'_{\sigma 2} & 0 & v_{\sigma 1} \\ 0 & -w_{\sigma 2} v'_{\sigma 2} & -v_{\sigma 1} & 0 \end{pmatrix}, \\ \psi_{1\sigma}(a, b, u) &= \begin{pmatrix} 0 & -v_{\sigma 1} & w_{\sigma 2} & x_s \\ v_{\sigma 1} & 0 & 0 & -v''_{\sigma 2} \\ -w_{\sigma 2} v'_{\sigma 2} & 0 & 0 & -w_{\sigma 1} \\ -x_s v'_{\sigma 2} & v_{\sigma 2} & w_{\sigma 1} & 0 \end{pmatrix}, \end{aligned}$$

$$\varphi_{2\sigma}(a, b, u) = \begin{pmatrix} 0 & w_{\sigma 2} & -v'_{\sigma 1} & 0 \\ -w_{\sigma 2} & 0 & -x_j & w_{\sigma 1} \\ v_{\sigma 1} & x_j v''_{\sigma 1} & 0 & v_{\sigma 2} \\ 0 & -w_{\sigma 1} v''_{\sigma 1} & -v_{\sigma 2} & 0 \end{pmatrix},$$

$$\psi_{2\sigma}(a, b, u) = \begin{pmatrix} 0 & -v_{\sigma 2} & w_{\sigma 1} & x_j \\ v_{\sigma 2} & 0 & 0 & -v'_{\sigma 1} \\ -w_{\sigma 1} v''_{\sigma 1} & 0 & 0 & -w_{\sigma 2} \\ -x_j v''_{\sigma 1} & v_{\sigma 1} & w_{\sigma 2} & 0 \end{pmatrix},$$

$$\varphi_{3\sigma}(a, b, u) = \begin{pmatrix} 0 & v_{\sigma 1} & -v'_{\sigma 2} & 0 \\ -v_{\sigma 1} & 0 & -x_s & w_{\sigma 2} \\ v_{\sigma 2} & x_s v''_{\sigma 2} & 0 & w_{\sigma 1} \\ 0 & -w_{\sigma 2} v''_{\sigma 2} & -w_{\sigma 1} & 0 \end{pmatrix},$$

$$\psi_{3\sigma}(a, b, u) = \begin{pmatrix} 0 & -w_{\sigma 1} & w_{\sigma 2} & x_s \\ w_{\sigma 1} & 0 & 0 & -v'_{\sigma 2} \\ -w_{\sigma 2} v''_{\sigma 2} & 0 & 0 & -v_{\sigma 1} \\ -x_s v''_{\sigma 2} & v_{\sigma 2} & v_{\sigma 1} & 0 \end{pmatrix},$$

$$\varphi_{4\sigma}(a, b, u) = \begin{pmatrix} 0 & v_{\sigma 2} & -v''_{\sigma 1} & 0 \\ -v_{\sigma 2} & 0 & -x_j & w_{\sigma 1} \\ v_{\sigma 1} & x_j v'_{\sigma 1} & 0 & w_{\sigma 2} \\ 0 & -w_{\sigma 1} v'_{\sigma 1} & -w_{\sigma 2} & 0 \end{pmatrix},$$

$$\psi_{4\sigma}(a, b, u) = \begin{pmatrix} 0 & -w_{\sigma 2} & w_{\sigma 1} & x_j \\ w_{\sigma 2} & 0 & 0 & -v''_{\sigma 1} \\ -w_{\sigma 1} v'_{\sigma 1} & 0 & 0 & -v_{\sigma 2} \\ -x_j v'_{\sigma 1} & v_{\sigma 1} & v_{\sigma 2} & 0 \end{pmatrix}.$$

Theorem 4.7. (i) For each $1 \leq t \leq 4$, the pair $(\varphi_{t\sigma}(a, b, u), \psi_{t\sigma}(a, b, u))$ forms a matrix factorization of $\Omega_R^2(I_{t\sigma}(a, b, u))$.

(ii) The set

$$\mathcal{N} = \{\text{Coker}(\varphi_{t\sigma}(a, b, u)), \text{Coker}(\psi_{t\sigma}(a, b, u)) \mid 1 \leq t \leq 4, \sigma, a, b, u\}$$

contains only graded, indecomposable, non-orientable, 4-generated MCM R -modules of rank 2.

(iii) Every indecomposable, graded, non-orientable, 4-generated MCM module over R of rank 2 is isomorphic with one module of \mathcal{N} .

Proof. (i). It is easy to check that

$$\varphi_{t\sigma}(a, b, u) \cdot \psi_{t\sigma}(a, b, u) = f \cdot \text{Id}_4$$

and the following sequence is exact:

$$\begin{aligned} R(-6)^4 &\xrightarrow{\varphi_{1\sigma}(a, b, u)} R(-5)^2 \oplus R(-4)^2 \xrightarrow{\psi_{1\sigma}(a, b, u)} R(-3)^4 \xrightarrow{A_1} \\ &\xrightarrow{A_1} R(-2)^2 \oplus R(-1) \longrightarrow I_{1\sigma}(a, b, u) \longrightarrow 0, \end{aligned}$$

where A_1 is the 3×4 -matrix formed by the first three rows of $\varphi_{1\sigma}(a, b, u)$. Thus (i) holds for $t = 1$, the other cases being similar.

(ii). Clearly $I_{1\sigma}(a, b, u) \subset (v'_{\sigma 2}, w_{\sigma 1})$ and so $\dim R/I_{1\sigma}(a, b, u) = 2$. As x_s is zero-divisor in $R/I_{1\sigma}(a, b, u)$ we see that $\text{depth } R/I_{1\sigma}(a, b, u) = 1$ and, by Proposition 4.6, $\Omega_R^2(I)$ is non-orientable, 4-generated of rank 2. Note that after some linear transformations $\varphi_{1\sigma}(a, b, u)$ becomes

$$\begin{pmatrix} w_{\sigma 2} & -w_{\sigma 1} & 0 & x_s \\ v_{\sigma 1} & v_{\sigma 2} & x_s v'_{\sigma 2} & 0 \\ 0 & 0 & w_{\sigma 1} & v''_{\sigma 2} \\ 0 & 0 & -w_{\sigma 2} v'_{\sigma 2} & v_{\sigma 1} \end{pmatrix},$$

and as in the last part of the proof of Theorem 3.2 we see that $\text{Coker}(\varphi_{1\sigma}(a, b, u))$ is indecomposable because there exist no two matrices A, B of order two such that

$$\begin{pmatrix} 0 & x_s \\ x_s v'_{\sigma 2} & 0 \end{pmatrix} = \begin{pmatrix} w_{\sigma 2} & -w_{\sigma 1} \\ v_{\sigma 1} & v_{\sigma 2} \end{pmatrix} A + B \begin{pmatrix} w_{\sigma 1} & v''_{\sigma 2} \\ -w_{\sigma 2} v'_{\sigma 2} & v_{\sigma 1} \end{pmatrix}.$$

Similarly follows the cases $t > 1$.

(iii). Now let M be an indecomposable, graded, non-orientable, 4-generated MCM R -module of rank 2. By Proposition 4.5 there exists a graded ideal $I \subset R$ with $\dim R/I = 2$, $\text{depth } R/I = 1$, which is 3-generated and such that $M \simeq \Omega_R^2(I)$. Then $I = J/(f)$ with $J \subset S = K[x_1, x_2, x_3, x_4]$ a three generated ideal containing f . Let $\alpha_1, \alpha_2, \alpha_3$ be a minimal system of homogeneous generators of J . If f does not belong to the ideal generated by two α_t , then, as in Section 3, $f = \sum_{t=1}^3 p_t q_t$ and, after a renumbering, we may suppose that α_t is necessarily either p_t or q_t , for all $1 \leq t \leq 3$. Then $\alpha_1, \alpha_2, \alpha_3$ is a regular system of elements in S and so $R/I = S/I$ is Cohen-Macaulay which is false.

Thus we may suppose $f \in (\alpha_1, \alpha_2)$. Then there exist $a, b \in K$ with $a^3 = b^3 = -1$ and $\sigma = (i \ j \ s)$ a permutation of the set $\sigma = \{2, 3, 4\}$, $i < j$, such that α_t is necessarily either $w_{\sigma t}$ or $v_{\sigma t}$, for $t = 1, 2$. If $\alpha_1 = w_{\sigma 1}, \alpha_2 = w_{\sigma 2}$, then $R/(\alpha_1, \alpha_2)$ is a domain and $\alpha_1, \alpha_2, \alpha_3$ must be a regular system of elements in S and so, again, $R/I = S/I$ is Cohen-Macaulay, contradiction!

We have the following cases:

Case I: $\alpha_1 = w_{\sigma_1}$

Then α_2 must be v_{σ_2} and we have

$$(\alpha_1, \alpha_2) = (v'_{\sigma_2}, w_{\sigma_1}) \cap (v''_{\sigma_2}, w_{\sigma_1}).$$

It follows that a zero-divisor of $R/(\alpha_1, \alpha_2)$ must be either in $(v'_{\sigma_2}, w_{\sigma_1})$ or in $(v''_{\sigma_2}, w_{\sigma_1})$. As we know α_3 is a zero-divisor in $R/(\alpha_1, \alpha_2)$ and so $\alpha_3 \in (v'_{\sigma_2}, w_{\sigma_1})$ or $\alpha_3 \in (v''_{\sigma_2}, w_{\sigma_1})$.

I (a). Suppose

$$\alpha_3 \in (v'_{\sigma_2}, w_{\sigma_1}).$$

Subtracting from α_3 a multiple of w_{σ_1} , we may take $\alpha_3 = v'_{\sigma_2}\beta$ for a form β of S . Note that the matrices

$$\varphi = \begin{pmatrix} 0 & w_{\sigma_1} & -v''_{\sigma_2} & 0 \\ -w_{\sigma_1} & 0 & -\beta & w_{\sigma_2} \\ v_{\sigma_2} & \beta v'_{\sigma_2} & 0 & v_{\sigma_1} \\ 0 & -w_{\sigma_2}v'_{\sigma_2} & -v_{\sigma_1} & 0 \end{pmatrix},$$

$$\psi = \begin{pmatrix} 0 & -v_{\sigma_1} & w_{\sigma_2} & \beta \\ v_{\sigma_1} & 0 & 0 & -v''_{\sigma_2} \\ -w_{\sigma_2}v'_{\sigma_2} & 0 & 0 & -w_{\sigma_1} \\ -\beta v'_{\sigma_2} & v_{\sigma_2} & w_{\sigma_1} & 0 \end{pmatrix}$$

give the following exact sequence:

$$\rightarrow R^4 \xrightarrow{\varphi} R^4 \xrightarrow{\psi} R^4 \xrightarrow{B_1} R^3 \rightarrow I \rightarrow 0,$$

where B_1 is given by the first three rows of φ . Thus (φ, ψ) is a matrix factorization of $\Omega_R^2(I) \simeq M$. Adding in φ multiples of the first row to the second one and adding multiples of the fourth column to the third one, we may suppose that the entry $(2, 3)$ of φ depends only on x_1, x_s . These transformations modify also the entries $(2, 2)$ and $(3, 3)$ which are now not zero. Adding similar multiples of first column to the second one and of the fourth row to the third one, we get φ of the same type as before but with β depending only on x_1, x_s . Since $v_{\sigma_1} - w_{\sigma_1}(x_1 + 2ax_s) = 3ax_s^2$, adding in φ multiples of the first column to the third one and multiples of the fourth row to the second row, we may suppose that the entry $(2, 3)$ has the form λx_s for some $\lambda \in K$. These transformations modify also the entries $(3, 3)$ and $(2, 2)$ which are now not zero. Adding similar multiples of the first row to the third one and of the fourth column to the second column, we get φ of the same type as before but with $\beta = \lambda x_s$. If $\lambda = 0$, then clearly φ is the direct sum of two 2-matrices which contradicts that M is indecomposable. So $\lambda \neq 0$. Now we divide the second and the third column of φ by λ and multiply the first and the fourth row by λ . The new φ is as before but with $\lambda = 1$, that is $\varphi = \varphi_{1\sigma}(a, b, u)$.

I (b). Suppose

$$\alpha_3 \in (v''_{\sigma_2}, w_{\sigma_1}).$$

Then we may take $\alpha_3 = v''_{\sigma_2}\beta$, for a form β . With a similar proof as above, we obtain $M \simeq \text{Coker}(\psi_{3\sigma}(a, b, u))$.

Case II: $\alpha_2 = w_{\sigma_2}$.

Then $\alpha_1 = v_{\sigma_1}$. It follows that $(\alpha_1, \alpha_2) = (v'_{\sigma_1}, w_{\sigma_2}) \cap (v''_{\sigma_1}, w_{\sigma_2})$. We have the following two subcases:

II (a). $\alpha_3 \in (v'_{\sigma_1}, w_{\sigma_2})$. We may suppose $\alpha_3 = v'_{\sigma_1}\beta$, for a form β and we obtain that $M \simeq \text{Coker}(\psi_{4\sigma}(a, b, u))$.

II (b). $\alpha_3 \in (v''_{\sigma_1}, w_{\sigma_2})$. In this subcase we may take $\alpha_3 = v''_{\sigma_1}\beta$, for a form β and we obtain that $M \simeq \text{Coker}(\varphi_{2\sigma}(a, b, u))$.

Case III: $\alpha_1 = v_{\sigma_1}, \alpha_2 = v_{\sigma_2}$.

Then $(\alpha_1, \alpha_2) = (v'_{\sigma_1}, v'_{\sigma_2}) \cap (v'_{\sigma_1}, v''_{\sigma_2}) \cap (v''_{\sigma_1}, v'_{\sigma_2}) \cap (v''_{\sigma_1}, v''_{\sigma_2})$. We proceed like in the above cases taking α_3 from one prime ideal of the above decomposition of (α_1, α_2) , let us say $\alpha_3 \in (v'_{\sigma_1}, v'_{\sigma_2})$, that is $\alpha_3 = v'_{\sigma_1}\beta + v'_{\sigma_2}\gamma$ for some $\beta, \gamma \in S$. Suppose that one cannot reduce the problem to the case $\beta = 0$ or $\gamma = 0$, this implies for example that v'_{σ_1} does not divide γ and v'_{σ_2} does not divide β . Then $\Omega_S^1((\alpha_1, \alpha_2, \alpha_3)) \subset S^3$ contains the columns of the following matrix

$$\begin{pmatrix} v_{\sigma_2} & \alpha_3 & 0 & v''_{\sigma_2}\beta \\ -v_{\sigma_1} & 0 & \alpha_3 & v''_{\sigma_1}\gamma \\ 0 & -v_{\sigma_1} & -v_{\sigma_2} & -v''_{\sigma_1}v''_{\sigma_2} \end{pmatrix}$$

and we can see that $\mu(\Omega_S^1((\alpha_1, \alpha_2, \alpha_3))) \geq 4$, which contradicts Lemma 4.4. Thus we may suppose, let us say $\alpha_3 = v'_{\sigma_1}\beta$ where β is not a multiple of v''_{σ_1} . Now we may proceed as in the above cases and we obtain, in order, $M \simeq \text{Coker}(\varphi_{4\sigma}(a, b, u))$, $M \simeq \text{Coker}(\varphi_{3\sigma}(a, b, u))$, $M \simeq \text{Coker}(\psi_{1\sigma}(a, b, u))$, and $M \simeq \text{Coker}(\psi_{2\sigma}(a, b, u))$. \square

Proposition 4.8. *The representative set for the isomorphism classes of indecomposable, graded, non-orientable MCM modules over R with 4 generators is:*

$$\begin{aligned} \mathcal{S} = & \{ \text{Coker}(\varphi_{1\sigma}(-1, -1, u)), \text{Coker}(\varphi_{1\sigma}(-1, -1, u^2)), \text{Coker}(\varphi_{1\sigma}(-1, -u, u^2)) \} \cup \\ & \cup \{ \text{Coker}(\varphi_{2\sigma}(-1, -1, u)), \text{Coker}(\varphi_{2\sigma}(-1, -1, u^2)), \text{Coker}(\varphi_{2\sigma}(-u, -1, u)) \}, \end{aligned}$$

where $u \neq 1$ is a fixed cubic root of unity in K . In particular, there are 18 such classes of isomorphism.

Proof. We already saw in the above proof that the matrix $\varphi_{1\sigma}(a, b, u)$ can be given in the form:

$$\varphi_{1\sigma}(a, b, u) = \begin{pmatrix} w_{\sigma_2} & -w_{\sigma_1} & 0 & x_s \\ v_{\sigma_1} & v_{\sigma_2} & x_s v'_{\sigma_2} & 0 \\ 0 & 0 & w_{\sigma_1} & v''_{\sigma_2} \\ 0 & 0 & -w_{\sigma_2} v'_{\sigma_2} & v_{\sigma_1} \end{pmatrix}.$$

Consequently, the $\psi_{1\sigma}(a, b, u)$ becomes:

$$\psi_{1\sigma}(a, b, u) = \begin{pmatrix} v_{\sigma 2} & w_{\sigma 1} & -x_s v'_{\sigma 2} & 0 \\ -v_{\sigma 1} & w_{\sigma 2} & 0 & x_s \\ 0 & 0 & v_{\sigma 1} & -v''_{\sigma 2} \\ 0 & 0 & w_{\sigma 2} v'_{\sigma 2} & w_{\sigma 1} \end{pmatrix}.$$

Acting by permutations on the rows and columns of $\varphi_{2\sigma}(a, b, u)$, we may take

$$\varphi_{2\sigma}(a, b, u) = \begin{pmatrix} -w_{\sigma 2} & w_{\sigma 1} & 0 & -x_j \\ v_{\sigma 1} & v_{\sigma 2} & x_j v''_{\sigma 1} & 0 \\ 0 & 0 & w_{\sigma 2} & -v'_{\sigma 1} \\ 0 & 0 & -w_{\sigma 1} v''_{\sigma 1} & -v_{\sigma 2} \end{pmatrix}$$

and so on.

First we show that the representative set for the isomorphism classes is contained in the set

$$\mathcal{S}' = \{\text{Coker}(\varphi_{1\sigma}(a, b, u)), \text{Coker}(\varphi_{2\sigma}(a, b, u)) \mid a, b, u, \sigma\}.$$

Using SINGULAR we can show that, for all $1 \leq t \leq 4$,

$$\text{Coker}(\varphi_{t\sigma}(a, b, u)) \simeq \text{Coker}(\psi_{t\sigma}(a, b, u)).$$

Take, for instance, $t = 1$ and apply the procedure `isomorph4` to the matrices $\varphi_{1\sigma}(n, p, v)$ and $\psi_{1\sigma}(a, b, u)$, where n, p, a, b are cubic roots of -1 and $u, v \neq 1$ are cubic roots of 1 .

```
LIB"matrix.lib";
option(redSB);
proc isomorph4(matrix X, matrix Y)
{
matrix U[4][4]=u(1..16);
matrix V[4][4]=v(1..16);
matrix C=U*X-Y*V;
ideal I=flatten(C);
ideal I1=transpose(coeffs(I,x(1)))[2];
ideal I2=transpose(coeffs(I,x(2)))[2];
ideal I3=transpose(coeffs(I,x(3)))[2];
ideal I4=transpose(coeffs(I,x(4)))[2];
ideal J=I1+I2+I3+I4+ideal(det(U)-1,det(V)-1);
ideal K=std(J);
return(K);
}

ring R=0,(u(1..16),v(1..16),x(1..4),a,b,n,p,u,v),lp;
ideal FI=a3+1,b3+1,p3+1,n3+1,u2+u+1,v2+v+1;
qring Q=std(FI);
```

```

matrix phi1[4][4]=x(2)-p*x(3),-x(1)+n*x(4),0,x(4),
x(1)^2+n*x(1)*x(4)+n2*x(4)^2,x(2)^2+p*x(2)*x(3)+p2*x(3)^2,x(4)*(x(2)-v*p*x(3)),0,
0,0,x(1)-n*x(4),x(2)+(1+v)*p*x(3),
0,0,-(x(2)-p*x(3))*(x(2)-v*p*x(3)),x(1)^2+n*x(1)*x(4)+n2*x(4)^2;

```

matrix

```

psi1[4][4]=x(2)^2+b*x(2)*x(3)+b2*x(3)^2,x(1)-a*x(4),-x(4)*(x(2)-u*b*x(3)),0,
-x(1)^2-a*x(1)*x(4)-a2*x(4)^2,x(2)-b*x(3),0,x(4),
0,0,x(1)^2+a*x(1)*x(4)+a2*x(4)^2,-x(2)-(1+u)*b*x(3),
0,0,(x(2)-b*x(3))*(x(2)-u*b*x(3)),x(1)-a*x(4);

```

```
isomorph4(phi1,psi1);
```

The first three generators of the ideal K are:

```
K[1]=u-v
```

```
K[2]=b-p
```

```
K[3]=a-n.
```

This implies that

$$\text{Coker}(\varphi_{1\sigma}(a, b, u)) \simeq \text{Coker}(\psi_{1\sigma}(a, b, u)).$$

With the same procedure we prove the following equalities of sets:

$$\{\text{Coker}(\varphi_{1\sigma}(a, b, u)) \mid \sigma, a, b, u\} = \{\text{Coker}(\varphi_{3\sigma}(a, b, u)) \mid \sigma, a, b, u\}$$

and

$$\{\text{Coker}(\varphi_{2\sigma}(a, b, u)) \mid \sigma, a, b, u\} = \{\text{Coker}(\varphi_{4\sigma}(a, b, u)) \mid \sigma, a, b, u\}.$$

Applying the procedure `isomorph4` to the matrices $\varphi_{1\sigma}(n, p, v)$ and $\varphi_{3\sigma}(a, b, u)$, we obtain the following first three generators for K :

```
K[1]=u+v+1
```

```
K[2]=b-p
```

```
K[3]=a-n.
```

It follows that

$$\text{Coker}(\varphi_{1\sigma}(a, b, u)) \simeq \text{Coker}(\varphi_{3\sigma}(a, b, u^2)).$$

Similarly we deduce the second equality of sets.

It remains to find the representatives for the set \mathcal{S}' . First we find the isomorphic modules of the form $\text{Coker}(\varphi_{1\sigma}(a, b, u))$, for a fixed permutation σ . Applying our procedure we obtain that

$$\text{Coker}(\varphi_{1\sigma}(n, p, v)) \simeq \text{Coker}(\varphi_{1\sigma}(a, b, u)),$$

if and only if,

$$b + pu v + pv = 0$$

and

$$(a - n)(u - v) = 0.$$

If $v = u$, then $b = p$ and if $v = u^2$, then $n = a$ and $p = bu^2$. This shows that, for a given σ , we have 3 non-isomorphic modules of type $\text{Coker}(\varphi_{1\sigma}(a, b, u))$, namely:

$$\text{Coker}(\varphi_{1\sigma}(-1, -1, u)), \text{Coker}(\varphi_{1\sigma}(-1, -1, u^2)), \text{Coker}(\varphi_{1\sigma}(-1, -u, u^2)),$$

where $u \neq 1$ is a fixed cubic root of unity. Similarly we get that we have 3 non-isomorphic modules of type $\text{Coker}(\varphi_{2\sigma}(a, b, u))$, namely:

$$\text{Coker}(\varphi_{2\sigma}(-1, -1, u)), \text{Coker}(\varphi_{2\sigma}(-1, -1, u^2)), \text{Coker}(\varphi_{2\sigma}(-u, -1, u)),$$

where $u \neq 1$ is a fixed cubic root of unity.

Using again the procedure `isomorph4`, we obtain that all the modules in the list \mathcal{S} are pairwise non-isomorphic. In fact we have to apply the procedure only for the pairs of matrices $(\varphi_{1\sigma}(n, p, v), (\varphi_{1\tau}(a, b, u)))$, where $\sigma = (2\ 3\ 4)$ and τ is an arbitrary permutation. \square

5 Non-orientable, rank 2, 5-generated MCM modules

As in Section 3, let $u, a, b \in K$, with

$$a^3 = b^3 = -1, \quad u^2 + u + 1 = 0,$$

$\sigma = (i\ j\ s)$ be a permutation of the set $\{2, 3, 4\}$ with $i < j$ and set

$$\begin{aligned} w_{\sigma 1} &= x_1 - ax_s, \quad w_{\sigma 2} = x_i - bx_j, \\ v_{\sigma 1} &= x_1^2 + ax_1x_s + a^2x_s^2, \quad v_{\sigma 2} = x_i^2 + bx_ix_j + b^2x_j^2. \end{aligned}$$

We have

$$v_{\sigma 1} = v'_{\sigma 1}v''_{\sigma 1}, \quad v_{\sigma 2} = v'_{\sigma 2}v''_{\sigma 2}$$

for

$$\begin{aligned} v'_{\sigma 1} &= x_1 - uax_s, \quad v''_{\sigma 1} = x_1 + (1+u)ax_s, \\ v'_{\sigma 2} &= x_i - ubx_j, \quad v''_{\sigma 2} = x_i + (1+u)bx_j. \end{aligned}$$

Consider the following ideals:

Set

$$\begin{aligned} J_{1\sigma}(a, b, u) &= (v_{\sigma 1}, v_{\sigma 2}, v'_{\sigma 1}v'_{\sigma 2}, v''_{\sigma 1}v''_{\sigma 2}), \\ J_{2\sigma}(a, b, u) &= (v_{\sigma 1}, v_{\sigma 2}, v'_{\sigma 1}v''_{\sigma 2}, v''_{\sigma 1}v'_{\sigma 2}). \end{aligned}$$

Denote by \mathcal{J} the union of the above families of ideals.

Set

$$\rho_{1\sigma}(a, b, u) = \begin{pmatrix} 0 & -v''_{\sigma 2} & -v'_{\sigma 2} & w_{\sigma 1} & 0 \\ v'_{\sigma 1} & 0 & 0 & w_{\sigma 2} & -v''_{\sigma 2}v''_{\sigma 1} \\ -v''_{\sigma 2} & 0 & v''_{\sigma 1} & 0 & 0 \\ 0 & v'_{\sigma 1} & 0 & 0 & v_{\sigma 2} \\ 0 & -w_{\sigma 2} & 0 & 0 & w_{\sigma 1}v''_{\sigma 1} \end{pmatrix}$$

and

$$\omega_{1\sigma}(a, b, u) = \begin{pmatrix} -w_{\sigma_2}v''_{\sigma_1} & w_{\sigma_1}v''_{\sigma_1} & -w_{\sigma_2}v'_{\sigma_2} & 0 & v''_{\sigma_1}v''_{\sigma_2} \\ 0 & 0 & 0 & w_{\sigma_1}v''_{\sigma_1} & -v_{\sigma_2} \\ -w_{\sigma_2}v''_{\sigma_2} & -w_{\sigma_1}v''_{\sigma_2} & w_{\sigma_1}v'_{\sigma_1} & 0 & v''_{\sigma_2}{}^2 \\ v_{\sigma_1} & v_{\sigma_2} & v'_{\sigma_1}v'_{\sigma_2} & v''_{\sigma_1}v''_{\sigma_2} & 0 \\ 0 & 0 & 0 & w_{\sigma_2} & v'_{\sigma_1} \end{pmatrix}.$$

Clearly the pair of the matrices above forms the matrix factorization of $\Omega_R^2(J_{1\sigma}(a, b, u)/(f))$. By permutations of $v'_{\sigma_1}, v''_{\sigma_1}, v'_{\sigma_2}, v''_{\sigma_2}$ one can find easily the matrix factorization of the module $\Omega_R^2(J_{2\sigma}(a, b, u)/(f))$.

Also set

$$\begin{aligned} T_{1\sigma}(a, b, u) &= (v_{\sigma_1}, v_{\sigma_2}, v'_{\sigma_1}v''_{\sigma_2}, v''_{\sigma_2}{}^2), \\ T_{2\sigma}(a, b, u) &= (v_{\sigma_1}, v_{\sigma_2}, v''_{\sigma_1}v''_{\sigma_2}, v''_{\sigma_2}{}^2), \\ T_{3\sigma}(a, b, u) &= (v_{\sigma_1}, v_{\sigma_2}, v''_{\sigma_1}v'_{\sigma_2}, v'_{\sigma_2}{}^2), \\ T_{4\sigma}(a, b, u) &= (v_{\sigma_1}, v_{\sigma_2}, v'_{\sigma_1}v'_{\sigma_2}, v'_{\sigma_2}{}^2), \\ T_{5\sigma}(a, b, u) &= (v_{\sigma_1}, v_{\sigma_2}, v'_{\sigma_1}v''_{\sigma_2}, v'_{\sigma_1}{}^2), \\ T_{6\sigma}(a, b, u) &= (v_{\sigma_1}, v_{\sigma_2}, v'_{\sigma_1}v'_{\sigma_2}, v'_{\sigma_1}{}^2), \\ T_{7\sigma}(a, b, u) &= (v_{\sigma_1}, v_{\sigma_2}, v''_{\sigma_1}v''_{\sigma_2}, v''_{\sigma_1}{}^2), \\ T_{8\sigma}(a, b, u) &= (v_{\sigma_1}, v_{\sigma_2}, v''_{\sigma_1}v'_{\sigma_2}, v''_{\sigma_1}{}^2), \end{aligned}$$

and denote by \mathcal{T} the set of all these ideals.

Set

$$\mu_{1\sigma}(a, b, u) = \begin{pmatrix} v''_{\sigma_2} & 0 & 0 & 0 & w_{\sigma_1} \\ 0 & v''_{\sigma_2} & -v'_{\sigma_1} & 0 & w_{\sigma_2} \\ -v''_{\sigma_1} & 0 & v'_{\sigma_2} & v''_{\sigma_2} & 0 \\ 0 & -v'_{\sigma_2} & 0 & -v'_{\sigma_1} & 0 \\ 0 & -v''_{\sigma_1}w_{\sigma_1} & 0 & w_{\sigma_2}v''_{\sigma_2} & 0 \end{pmatrix}$$

and

$$\nu_{1\sigma}(a, b, u) = \begin{pmatrix} v'_{\sigma_2}w_{\sigma_2} & -v'_{\sigma_2}w_{\sigma_1} & -v'_{\sigma_1}w_{\sigma_1} & -v''_{\sigma_2}w_{\sigma_1} & 0 \\ 0 & 0 & 0 & -v''_{\sigma_2}w_{\sigma_2} & -v'_{\sigma_1} \\ v''_{\sigma_1}w_{\sigma_2} & -v''_{\sigma_1}w_{\sigma_1} & w_{\sigma_2}v''_{\sigma_2} & 0 & -v''_{\sigma_2} \\ 0 & 0 & 0 & -v''_{\sigma_1}w_{\sigma_1} & v'_{\sigma_2} \\ v_{\sigma_1} & v_{\sigma_2} & v''_{\sigma_2}v'_{\sigma_1} & v''_{\sigma_2}{}^2 & 0 \end{pmatrix}.$$

The pair of the matrices above forms the matrix factorization of $\Omega_R^2(T_{1\sigma}(a, b, u)/(f))$. By permutations of $v'_{\sigma_1}, v''_{\sigma_1}, v'_{\sigma_2}, v''_{\sigma_2}$ one can find easily the matrix factorization for the 2-syzygy of the other ideals of \mathcal{T} .

Lemma 5.1. *Let M be a graded non-orientable, rank two, 5-generated MCM R -module, without free direct summands. Then there exists an ideal $J \in \mathcal{J} \cup \mathcal{T}$ such that $f \in J$ and $M \cong \Omega^2(J/(f))$. Conversely, for every $J \in \mathcal{J} \cup \mathcal{T}$, the module $\Omega^2(J/(f))$ is a non-orientable, rank two, 5-generated MCM R -module without free direct summands.*

Proof. The second statement follows easily since we have already the matrix factorizations above of those ideals. Let M be as above. As in the beginning of Section 4 we see that $M \cong \Omega^2(J/(f))$, for J an ideal of S containing f , with $\mu(J) = 4$, $\dim S/J = 2$, $\text{depth } S/J=1$ and $\mu(\Omega_S^1(J)) = 5$. We may also suppose $J = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ with $f \in (\alpha_1, \alpha_2)$, where α_t is necessarily either $w_{\sigma t}$ or $v_{\sigma t}$ for $t = 1, 2$ for some a, b and a certain permutation σ as above. Clearly we cannot have simultaneously $\alpha_t = w_{\sigma t}$ because then (α_1, α_2) is a prime ideal and one cannot find α_3, α_4 zero divisors as we need. We treat the following cases:

Case I $\alpha_1 = w_{\sigma 1}$

Then we have $\alpha_2 = v_{\sigma 2}$ and (α_1, α_2) is the intersection of the prime ideals $(v'_{\sigma 2}, w_{\sigma 1})$, $(v''_{\sigma 2}, w_{\sigma 1})$. Since α_3, α_4 must be zero divisors in $S/(\alpha_3, \alpha_4)$ we have the following possibilities:

(I1) $\alpha_3 = v'_{\sigma 2}\beta$, $\alpha_4 = v'_{\sigma 2}\gamma$, (I2) $\alpha_3 = v''_{\sigma 2}\beta$, $\alpha_4 = v''_{\sigma 2}\gamma$, (I3) $\alpha_3 = v'_{\sigma 2}\beta$, $\alpha_4 = v''_{\sigma 2}\gamma$, (I4) $\alpha_3 = v''_{\sigma 2}\beta$, $\alpha_4 = v'_{\sigma 2}\gamma$

for some homogeneous β, γ from $\mathfrak{m} = (x_1, x_2, x_3, x_4)$. In the first case we see that the relations given by the columns of the following matrix:

$$\begin{pmatrix} v_{\sigma 2} & \alpha_3 & \alpha_4 & 0 & 0 \\ -w_{\sigma 1} & 0 & 0 & \gamma & \beta \\ 0 & -w_{\sigma 1} & 0 & 0 & -v''_{\sigma 2} \\ 0 & 0 & -w_{\sigma 1} & -v''_{\sigma 2} & 0 \end{pmatrix},$$

are elements in $\Omega_S^1(J) \subset S^4$. Clearly these columns are part in the minimal system of generators of $\Omega_S^1(J)$ because $w_{\sigma 1}, v''_{\sigma 2}$ form a regular system in S . The subcase (I2) is similar, this contradicts Lemma 4.4.

Suppose now (I3) holds. Then the relations given by the columns of the following matrix

$$\begin{pmatrix} v_{\sigma 2} & v'_{\sigma 2}\beta & v''_{\sigma 2}\beta & 0 & 0 \\ -w_{\sigma 1} & 0 & 0 & \beta & \gamma \\ 0 & -w_{\sigma 1} & 0 & -v''_{\sigma 2} & 0 \\ 0 & 0 & -w_{\sigma 1} & 0 & -v'_{\sigma 2} \end{pmatrix},$$

are part of a minimal set of generators of $\Omega_S^1(J)$ (note that $w_{\sigma 1}, v''_{\sigma 2}, v'_{\sigma 2}$ form a regular system in S). Contradiction! Case (I4) is similar.

Case II $\alpha_1 = v_{\sigma 1}$, $\alpha_2 = v_{\sigma 2}$

Since $(\alpha_1, \alpha_2) = (v'_{\sigma 1}, v'_{\sigma 2}) \cap (v'_{\sigma 1}, v''_{\sigma 2}) \cap (v''_{\sigma 1}, v''_{\sigma 2}) \cap (v''_{\sigma 1}, v'_{\sigma 2})$ we see that the zero divisors of $S/(\alpha_1, \alpha_2)$ must be in one of the prime ideals of the above decomposition. Suppose $\alpha_3 \in (v'_{\sigma 1}, v'_{\sigma 2})$. If $\alpha_3 = \beta_1 v'_{\sigma 1} + \beta_2 v'_{\sigma 2}$ then as in the proof of Case III of Proposition 4.7 we see that there are at least 4 minimal relations between first three α . Then all α have at least 5 minimal relations. Contradiction! Thus α_3 as well α_4 are multiples of one $v'_{\sigma t}, v''_{\sigma t}$. So we have the following possibilities:

(II1) $\alpha_3 = v'_{\sigma 1}\beta$, $\alpha_4 = v'_{\sigma 1}\gamma$, (II2) $\alpha_3 = v''_{\sigma 1}\beta$, $\alpha_4 = v''_{\sigma 1}\gamma$, (II3) $\alpha_3 = v'_{\sigma 2}\beta$, $\alpha_4 = v'_{\sigma 2}\gamma$, (II4) $\alpha_3 = v''_{\sigma 2}\beta$, $\alpha_4 = v''_{\sigma 2}\gamma$ (II5) $\alpha_3 = v'_{\sigma 1}\beta$, $\alpha_4 = v''_{\sigma 1}\gamma$ (II6) $\alpha_3 = v'_{\sigma 2}\beta$, $\alpha_4 = v''_{\sigma 2}\gamma$ (II7)

$\alpha_3 = v'_{\sigma_2}\beta, \alpha_4 = v''_{\sigma_1}\gamma$ (II8) $\alpha_3 = v'_{\sigma_2}\beta, \alpha_4 = v''_{\sigma_2}\gamma$ (II9) $\alpha_3 = v'_{\sigma_1}\beta, \alpha_4 = v'_{\sigma_2}\gamma$ (II10) $\alpha_3 = v''_{\sigma_1}\beta, \alpha_4 = v''_{\sigma_2}\gamma$.

Subcase $\alpha_3 = v'_{\sigma_1}\beta, \alpha_4 = v'_{\sigma_1}\gamma, (v_{\sigma_2}v''_{\sigma_1}, \gamma) \cong 1, (v_{\sigma_2}v''_{\sigma_1}, \beta) \cong 1$

We see that the relations given by the columns of the following matrix

$$\begin{pmatrix} v_{\sigma_2} & \beta & \gamma & 0 & 0 \\ -v_{\sigma_1} & 0 & 0 & \alpha_3 & \alpha_4 \\ 0 & -v''_{\sigma_1} & 0 & -v_{\sigma_2} & 0 \\ 0 & 0 & -v''_{\sigma_1} & 0 & -v_{\sigma_2} \end{pmatrix},$$

are part from a minimal system of generators of $\Omega_S^1(J)$ which must be false. Indeed, it is easy to see that the last 4 columns are part in a minimal system of generators of $\Omega_S^1(J)$. If the first column belongs to the module generated by the last four then there exist $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \in S$ such that:

$$v_{\sigma_2} = \lambda_1\beta + \lambda_2\gamma,$$

$$-v_{\sigma_1} = \lambda_3v'_{\sigma_1}\beta + \lambda_4v''_{\sigma_1}\gamma,$$

$$0 = \lambda_1v''_{\sigma_1} + \lambda_3v_{\sigma_2},$$

$$0 = \lambda_2v''_{\sigma_1} + \lambda_4v_{\sigma_2}.$$

It follows that $v_{\sigma_2}|\lambda_1$ and $v_{\sigma_2}|\lambda_2$ and so we get $1 \in (\beta, \gamma)$. Contradiction! If $(v_{\sigma_2}v''_{\sigma_1}, \beta) \not\cong 1$ then we are in the subcase (II5), (II6), ... In the same way we treat (II2), (II3), (II4).

Subcase $\alpha_3 = v'_{\sigma_1}\beta, \alpha_4 = v''_{\sigma_1}\gamma$

We see that the relations given by the columns of the following matrix

$$\begin{pmatrix} v_{\sigma_2} & \beta & \gamma & 0 & 0 \\ -v_{\sigma_1} & 0 & 0 & \alpha_3 & \alpha_4 \\ 0 & -v''_{\sigma_1} & 0 & -v_{\sigma_2} & 0 \\ 0 & 0 & -v'_{\sigma_1} & 0 & -v_{\sigma_2} \end{pmatrix},$$

are elements in $\Omega_S^1(J)$. The columns 2,3 together with the last two columns divided by (β, v_{σ_2}) , respectively (γ, v_{σ_2}) are part of a minimal system of generators. Since $\mu(\Omega_S^1(J)) = 4$ we see that the first column is a linear combination of the others as above. Thus there exist $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \in S$ such that:

$$v_{\sigma_2} = \lambda_1\beta + \lambda_2\gamma,$$

$$-v_{\sigma_1} = \lambda_3v'_{\sigma_1}\beta/(\beta, v_{\sigma_2}) + \lambda_4v''_{\sigma_1}\gamma/(\gamma, v_{\sigma_2}),$$

$$0 = \lambda_1v''_{\sigma_1} + \lambda_3v_{\sigma_2}/(\beta, v_{\sigma_2}),$$

$$0 = \lambda_2v'_{\sigma_1} + \lambda_4v_{\sigma_2}/(\gamma, v_{\sigma_2}).$$

It follows that $v_{\sigma_2}/(\beta, v_{\sigma_2})|\lambda_1$ and $v_{\sigma_2}/(\gamma, v_{\sigma_2})|\lambda_2$ and so we get $1 \in (\beta, \gamma)$ which is false as above if $(\beta, v_{\sigma_2}) \cong 1, (\gamma, v_{\sigma_2}) \cong 1$. Clearly β, γ cannot be multiples of v_{σ_2} because otherwise J is only 3 generated. Thus we may suppose for example $\beta = v'_{\sigma_2}$.

Then $J = (v_{\sigma_1}, v_{\sigma_2}, v'_{\sigma_1}v'_{\sigma_2}, v''_{\sigma_1}\gamma)$ and the matrix factorizations of $\Omega_R^2(J/(f))$ is given by the following matrices A, B :

$$A = \begin{pmatrix} 0 & -\gamma & -v'_{\sigma_2} & w_{\sigma_1} & 0 \\ v'_{\sigma_1} & 0 & 0 & w_{\sigma_2} & -\gamma v''_{\sigma_1} \\ -v''_{\sigma_2} & 0 & v''_{\sigma_1} & 0 & 0 \\ 0 & v'_{\sigma_1} & 0 & 0 & v_{\sigma_2} \\ 0 & -w_{\sigma_2} & 0 & 0 & w_{\sigma_1}v''_{\sigma_1} \end{pmatrix},$$

$$B = \begin{pmatrix} -w_{\sigma_2}v''_{\sigma_1} & w_{\sigma_1}v''_{\sigma_1} & -w_{\sigma_2}v'_{\sigma_2} & 0 & v''_{\sigma_1}\gamma \\ 0 & 0 & 0 & w_{\sigma_1}v''_{\sigma_1} & -v_{\sigma_2} \\ -w_{\sigma_2}v''_{\sigma_2} & w_{\sigma_1}v''_{\sigma_2} & w_{\sigma_1}v'_{\sigma_1} & 0 & \gamma v''_{\sigma_2} \\ v_{\sigma_1} & v_{\sigma_2} & v'_{\sigma_1}v'_{\sigma_2} & \gamma v''_{\sigma_1} & 0 \\ 0 & 0 & 0 & w_{\sigma_2} & v'_{\sigma_1} \end{pmatrix}.$$

We may add to γ multiples of v'_{σ_1} because this means to add to α_4 multiples of v_{σ_1} . Also adding multiples of the column 4 of A to the column 2 and then adding multiples of the row 5 to the row 2 we see that the result is just the addition of some multiples of w_{σ_1} to γ . On the other hand adding some multiples of the row 5 to the row 1 and then adding some multiples of the column 4 to the column 5 we see that the result is just the addition of some multiples of w_{σ_2} to γ .

So, after some elementary transformations on A , we may suppose γ to be a polynomial in v''_{σ_2} and it is enough to see that $\deg(\gamma) = 1$. However adding to α_4 multiples of α_2 we may add to γ multiples of v_{σ_2} . Since $v''_{\sigma_2} \in (v_{\sigma_2}, w_{\sigma_2})$ we may suppose $\deg(\gamma) = 1$, that is $\gamma = qv''_{\sigma_2}$ for a certain nonzero constant q . Now we multiply the row 1 of A with a q^{-1} then the columns 3,4 with q , then rows 2,3 with q^{-1} and finally the column 1 with q . So we reduce to the case $q = 1$.

Thus $J = J_{1\sigma}(a, b, u)$. If we take $\beta = v''_{\sigma_2}$ then similarly we get $J = J_{2\sigma}(a, b, u)$. If $(\gamma, v_{\sigma_2}) \not\cong 1$ we get similarly $J = J_{2\sigma}(a, b, u)$, $J = J_{1\sigma}(a, b, u)$.

Subcase $\alpha_3 = v'_{\sigma_1}\beta$, $\alpha_4 = v''_{\sigma_2}\gamma$

We see that the relations given by the columns of the following matrix

$$\begin{pmatrix} v_{\sigma_2} & \beta & v''_{\sigma_2}\gamma & 0 & 0 \\ -v_{\sigma_1} & 0 & 0 & \alpha_3 & \gamma \\ 0 & -v''_{\sigma_1} & 0 & -v_{\sigma_2} & 0 \\ 0 & 0 & -v_{\sigma_1} & 0 & -v'_{\sigma_2} \end{pmatrix},$$

are elements in $\Omega_S^1(J)$. As in the above subcase the columns 2,5 together with the columns 3,4 divided by (γ, v_{σ_2}) , respectively (β, v_{σ_2}) , form a minimal system of generators in $\Omega_S^1(J)$. Thus the first column is a linear combinations of the others and there exist $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \in S$ such that:

$$v_{\sigma_2} = \lambda_1\beta + \lambda_2v''_{\sigma_1}\gamma/(\gamma, v_{\sigma_1}),$$

$$-v_{\sigma_1} = \lambda_3v'_{\sigma_1}\beta/(\beta, v_{\sigma_2}) + \lambda_4\gamma,$$

$$0 = \lambda_1 v''_{\sigma_1} + \lambda_3 v_{\sigma_2} / (\beta, v_{\sigma_2}),$$

$$0 = \lambda_2 v_{\sigma_1} / (\gamma, v_{\sigma_1}) + \lambda_4 v'_{\sigma_2}.$$

It follows $v'_{\sigma_2} | \lambda_2$ and from the first identity we see that we get $v_{\sigma_2} | \lambda_1 \beta$. If $\lambda_1 = 0$ then $\lambda_3 = 0$ and so $\gamma | v_{\sigma_1}$. If γ is a multiple of v''_{σ_1} then we are in the preceding subcase. If γ is a multiple of v'_{σ_1} we change α_3 with α_4 and so we may suppose the new β to be a multiple of v''_{σ_2} . This is exactly one possibility which follows from $\lambda_1 \neq 0$ which we treat now. From first identity we see that $\deg(\lambda_1 \beta) = 2$ and so either $\beta = v'_{\sigma_2}$ or $\beta = v''_{\sigma_2}$. The first situation lead us to the proceeding subcase, that is $J \in \mathcal{J}$. So we may suppose $\beta = v''_{\sigma_2}$.

Thus we reduce to the case $J = (v_{\sigma_1}, v_{\sigma_2}, v'_{\sigma_1} v''_{\sigma_2}, v''_{\sigma_2} \gamma)$. Adding to γ multiples of $v'_{\sigma_1}, v'_{\sigma_2}$ this means to add some multiples of α_3, α_2 we may suppose that $\gamma \in (v''_{\sigma_1}, v''_{\sigma_2})$. As before $\deg(\alpha_4) = 1$ because for instance (α_2, α_3) contains $v''_{\sigma_2} (v'_{\sigma_1}, v'_{\sigma_2})^2$. Then $\gamma = \tau_1 v''_{\sigma_1} + \tau_2 v''_{\sigma_2}$ for some $\tau_1, \tau_2 \in K$. Then another relation of $\Omega_S^1(J)$ is the transpose of $(\tau_1 v''_{\sigma_2}, 0, \tau_2 v''_{\sigma_2}, -v'_{\sigma_1})$. If $\tau_1 \neq 0$ then this relation together with the last 4 columns of the previous matrix (some of them divided by something) form 5 elements from a minimal system of generators of $\Omega_S^1(J)$. Contradiction! Thus $\tau_1 = 0$ and $J \in \mathcal{T}$.

With similar procedures we treat the other cases. \square

Set

$$\rho_{\sigma}(a, b, u) = \begin{pmatrix} 0 & w_{\sigma_1} & -v'_{\sigma_2} & -x_j & 0 \\ v'_{\sigma_1} & w_{\sigma_2} & 0 & 0 & -x_j v''_{\sigma_1} \\ -v''_{\sigma_2} & 0 & v''_{\sigma_1} & 0 & 0 \\ 0 & 0 & 0 & v'_{\sigma_1} & v_{\sigma_2} \\ 0 & 0 & 0 & -w_{\sigma_2} & w_{\sigma_1} v''_{\sigma_1} \end{pmatrix},$$

$$\omega_{\sigma}(a, b, u) = \begin{pmatrix} -w_{\sigma_2} v''_{\sigma_1} & w_{\sigma_1} v''_{\sigma_1} & -w_{\sigma_2} v'_{\sigma_2} & 0 & x_j v''_{\sigma_1} \\ v_{\sigma_1} & v_{\sigma_2} & v'_{\sigma_1} v'_{\sigma_2} & x_j v''_{\sigma_1} & 0 \\ -w_{\sigma_2} v''_{\sigma_2} & -w_{\sigma_1} v''_{\sigma_2} & w_{\sigma_1} v'_{\sigma_1} & 0 & x_j v''_{\sigma_2} \\ 0 & 0 & 0 & w_{\sigma_1} v''_{\sigma_1} & -v_{\sigma_2} \\ 0 & 0 & 0 & w_{\sigma_2} & v'_{\sigma_1} \end{pmatrix},$$

and

$$\mu_{\sigma}(a, b, u) = \begin{pmatrix} 0 & w_{\sigma_1} & v''_{\sigma_2} & 0 & 0 \\ -v'_{\sigma_1} & w_{\sigma_2} & 0 & 0 & x_j \\ v'_{\sigma_2} & 0 & -v''_{\sigma_1} & x_j & 0 \\ 0 & 0 & 0 & -v'_{\sigma_1} & -v'_{\sigma_2} \\ 0 & 0 & 0 & w_{\sigma_2} v''_{\sigma_2} & -v''_{\sigma_1} w_{\sigma_1} \end{pmatrix},$$

$$\nu_{\sigma}(a, b, u) = \begin{pmatrix} v''_{\sigma_1} w_{\sigma_2} & -v''_{\sigma_1} w_{\sigma_1} & v''_{\sigma_2} w_{\sigma_2} & 0 & -x_j \\ v_{\sigma_1} & v_{\sigma_2} & v''_{\sigma_2} v'_{\sigma_1} & x_j v''_{\sigma_2} & 0 \\ v'_{\sigma_2} w_{\sigma_2} & -v'_{\sigma_2} w_{\sigma_1} & -v'_{\sigma_1} w_{\sigma_1} & -x_j w_{\sigma_1} & 0 \\ 0 & 0 & 0 & -v''_{\sigma_1} w_{\sigma_1} & v'_{\sigma_2} \\ 0 & 0 & 0 & -v''_{\sigma_2} w_{\sigma_2} & -v'_{\sigma_1} \end{pmatrix}.$$

Theorem 5.2. *Let*

$$\mathcal{E} = \{\text{Coker}(\rho_\sigma(a, b, u)), \text{Coker}(\mu_\sigma(a, b, u)) \mid \sigma, a, b, u\}$$

- (i) *The set \mathcal{E} contains only indecomposable, graded, non-orientable, 5-generated MCM R -modules of rank 2.*
- (ii) *Every indecomposable, graded, non-orientable, 5-generated MCM module over R of rank 2 is isomorphic with one module of \mathcal{E} .*
- (iii) *All the modules of the set \mathcal{E} are non-isomorphic. In particular, there are 108 isomorphism classes of indecomposable, graded, non-orientable MCM modules over R with 5 generators.*

Proof. (i). The pairs of matrices $(\rho_\sigma(a, b, u), \omega_\sigma(a, b, u))$ and $(\mu_\sigma(a, b, u), \nu_\sigma(a, b, u))$ have been obtained from the pairs $(\rho_{1\sigma}(a, b, u), \omega_{1\sigma}(a, b, u))$, respectively $(\mu_{1\sigma}(a, b, u), \nu_{1\sigma}(a, b, u))$ by elementary operations on rows and columns. By the above Lemma, the set \mathcal{E} satisfies the part (i) of the theorem. For the proof of indecomposability we may proceed as in the last part of the proof of Theorem 3.2.

(ii). Preserving the notations of Lemma 5.1, set

$$\mathcal{J}_i = \{J_{i\sigma}(a, b, u) \mid \sigma, a, b, u\}, i = 1, 2$$

and

$$\mathcal{T}_i = \{T_{i\sigma}(a, b, u) \mid \sigma, a, b, u\},$$

for $1 \leq i \leq 8$. We claim that

$$\mathcal{J}_1 = \mathcal{J}_2$$

and

$$\mathcal{T}_1 = \mathcal{T}_2 = \dots = \mathcal{T}_8.$$

Indeed, take, for instance, $J_{2\sigma}(a, b, u) = (v_{\sigma_1}, v_{\sigma_2}, v'_{\sigma_1}v''_{\sigma_2}, v''_{\sigma_1}v'_{\sigma_2}) \in \mathcal{J}_2$. Then we may find $\bar{v}'_{\sigma_1}, \bar{v}'_{\sigma_2}, \bar{v}''_{\sigma_1}, \bar{v}''_{\sigma_2}$, depending on some other cubic roots of -1 , let us say n, p , and v , a cubic root of unity different from 1, such that

$$J_{2\sigma}(a, b, u) = J_{1\sigma}(n, p, v) = (\bar{v}_{\sigma_1}, \bar{v}_{\sigma_2}, \bar{v}'_{\sigma_1}\bar{v}'_{\sigma_2}, \bar{v}''_{\sigma_1}\bar{v}''_{\sigma_2}).$$

(iii). In order to check that the modules of the list are pairwise non-isomorphic, we use the following procedure in SINGULAR.

```
LIB"matrix.lib";
option(redSB);
proc isomorph5(matrix X, matrix Y)
{
matrix U[5][5]=u(1..25);
matrix V[5][5]=v(1..25);
matrix C=U*X-Y*V;
```

```

ideal I=flatten(C);
ideal I1=transpose(coeffs(I,x(1)))[2];
ideal I2=transpose(coeffs(I,x(2)))[2];
ideal I3=transpose(coeffs(I,x(3)))[2];
ideal I4=transpose(coeffs(I,x(4)))[2];
ideal J=I1+I2+I3+I4+ideal(det(U)-1,det(V)-1);
ideal K=std(J);
return(K);
}

```

We give some examples of such tests. Choose $a, b, n, p \in K$, cubic roots of -1 , $u, v \in K$, cubic roots of unity, different from 1, and $\sigma = (2\ 3\ 4)$ a permutation of the set $\{2, 3, 4\}$. Take the matrices $\rho_\sigma(n, p, v)$ and $\rho_\sigma(a, b, u)$ and apply the above procedure:

```

ring R=0,(u(1..25),v(1..25),x(1..4),a,b,n,p,u,v),lp;

ideal FI=a3+1,b3+1,p3+1,n3+1,u2+u+1,v2+v+1; qring Q=std(FI);

matrix J01[5][5]=0,x(1)-n*x(4),-x(2)+v*p*x(3),-x(3),0,
x(1)-v*n*x(4),x(2)-p*x(3),0,0,-x(3)*x(1)-(1+v)*n*x(3)*x(4),
-x(2)-(1+v)*p*x(3),0,x(1)+(1+v)*n*x(4),0,0,
0,0,0,x(1)-v*n*x(4),x(2)^2+p*x(2)*x(3)+p2*x(3)^2,
0,0,0,-x(2)+p*x(3),x(1)^2+v*n*x(1)*x(4)+v2*n2*x(4)^2;

matrix J1[5][5]=0,x(1)-a*x(4),-x(2)+u*b*x(3),-x(3),0,
x(1)-u*a*x(4),x(2)-b*x(3),0,0,-x(3)*x(1)-(1+u)*a*x(3)*x(4),
-x(2)-(1+u)*b*x(3),0,x(1)+(1+u)*a*x(4),0,0,
0,0,0,x(1)-u*a*x(4),x(2)^2+b*x(2)*x(3)+b2*x(3)^2,0,0,
0,-x(2)+b*x(3),x(1)^2+u*a*x(1)*x(4)+u2*a2*x(4)^2;
isomorph5(J01,J1)

```

The first three generators of the returned ideal K are:

```

K[1]=u-v
K[2]=b-p
K[3]=a-n

```

This shows that

$$\text{Coker}(\rho_\sigma(n, p, v)) \simeq \text{Coker}(\rho_\sigma(a, b, u))$$

if and only if

$$u = v, \quad a = n \text{ and } b = p.$$

Now let $\sigma = (2\ 3\ 4)$, and $\tau = (2\ 4\ 3)$ be permutations of the set $\{2, 3, 4\}$. We check that $\text{Coker}(\rho_\sigma(n, p, v)) \not\simeq \text{Coker}(\rho_\tau(a, b, u))$.

```

matrix L01 [5] [5]=0,x(1)-n*x(4),-x(2)+v*p*x(3),-x(3),0,
x(1)-v*n*x(4),x(2)-p*x(3),0,0,-x(3)*x(1)-(1+v)*n*x(3)*x(4),
-x(2)-(1+v)*p*x(3),0,x(1)+(1+v)*n*x(4),0,0,
0,0,0,x(1)-v*n*x(4),x(2)^2+p*x(2)*x(3)+p^2*x(3)^2,
0,0,0,-x(2)+p*x(3),x(1)^2+v*n*x(1)*x(4)+v^2*n^2*x(4)^2;

```

```

matrix L1 [5] [5]=0,x(1)-a*x(3),-x(2)+u*b*x(4),-x(4),0,
x(1)-u*a*x(3),x(2)-b*x(4),0,0,-x(4)*x(1)-(1+u)*a*x(3)*x(4),
-x(2)-(1+u)*b*x(4),0,x(1)+(1+u)*a*x(3),0,0,
0,0,0,x(1)-u*a*x(3),x(2)^2+b*x(2)*x(4)+b^2*x(4)^2,
0,0,0,-x(2)+b*x(4),x(1)^2+u*a*x(1)*x(3)+u^2*a^2*x(3)^2;
isomorph5(L01,L1);

```

The returned ideal K is

```
K[1]=1
```

Finally we give an example to check that no module of type $\text{Coker}(\rho_\sigma(a, b, u))$ is isomorphic with some module of type $\text{Coker}(\mu_\sigma(a, b, u))$. In fact we need to prove the non-equivalence between the matrices of the pairs $(\mu_\tau(n, p, v), \rho_\tau(a, b, u))$, where $\sigma = (2\ 3\ 4)$ and τ is an arbitrary permutation. We give now the test for $\tau = (2\ 3\ 4)$.

```

ring R=0,(u(1..25),v(1..25),x(1..4),a,b,n,p,u,v),lp;
ideal FI=a^3+1,b^3+1,p^3+1,n^3+1,u^2+u+1,v^2+v+1;
qring Q=std(FI);

```

```

matrix T1 [5] [5]=0,x(1)-n*x(4),x(2)+(1+v)*p*x(3),0,0,
-x(1)+v*n*x(4),x(2)-p*x(3),0,0,x(3),
x(2)-v*p*x(3),0,-x(1)-(1+v)*n*x(4),x(3),0,
0,0,0,-x(1)+v*n*x(4),-x(2)+v*p*x(3),
0,0,0,x(2)^2+v*p*x(2)*x(3)+v^2*p^2*x(3)^2,-x(1)^2-v*n*x(1)*x(4)-v^2*n^2*x(4)^2;

```

```

matrix J1 [5] [5]=0,x(1)-a*x(4),-x(2)+u*b*x(3),-x(3),0,
x(1)-u*a*x(4),x(2)-b*x(3),0,0,-x(3)*x(1)-(1+u)*a*x(3)*x(4),
-x(2)-(1+u)*b*x(3),0,x(1)+(1+u)*a*x(4),0,0,
0,0,0,x(1)-u*a*x(4),x(2)^2+b*x(2)*x(3)+b^2*x(3)^2,
0,0,0,-x(2)+b*x(3),x(1)^2+u*a*x(1)*x(4)+u^2*a^2*x(4)^2;

```

```
isomorph5(T1,J1);
```

```
K[1]=1
```

□

Corollary 5.3. *Let*

$$\mathcal{F} = \{\text{Coker}(\omega_\sigma(a, b, u)), \text{Coker}(\nu_\sigma(a, b, u)) \mid \sigma, a, b, u\}$$

- (i) *The set \mathcal{F} contains only indecomposable, graded, non-orientable, 5-generated MCM R -modules of rank 3.*
- (ii) *Every indecomposable, graded, non-orientable, 5-generated MCM module over R of rank 3 is isomorphic with one module of \mathcal{F} .*
- (iii) *All the modules of the set \mathcal{F} are non-isomorphic.*

Proof. The map $M \mapsto \Omega_R^1(M)$ is a bijection between the 5-generated, indecomposable, graded, MCM R -modules of rank 2 and the 5-generated, indecomposable, graded, MCM R -modules of rank 3. \square

Lemma 5.4. *There exist no graded, indecomposable, non-orientable, rank two, six-generated MCM modules.*

Proof. Suppose there exist such MCM module M . Then $M \cong \Omega_R^2(J/(f))$ for a certain 5-generated ideal $J = (\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5)$ of S as is hinted by first part of Section 4. Then any 4 elements from the α_t must generate an ideal J'' in $\mathcal{J} \cup \mathcal{T}$ because otherwise $\mu(\Omega_S^1(J''/(f))) > 4$ and so obvious $\mu(\Omega_S^1(J/(f))) > 5$. So we may suppose $\alpha_t = v_{\sigma t}$ for $t = 1, 2$ and after some permutations $\alpha_3 = v'_{\sigma 1} v''_{\sigma 2}$. Set $J' = (\alpha_1, \alpha_2, \alpha_3)$. If $(J', \alpha_4) \in \mathcal{J}$ then $(J', \alpha_5) \notin \mathcal{J}$ because otherwise we get $\alpha_4 = \alpha_5$. Thus $(J', \alpha_5) \in \mathcal{T}$ and so either $\alpha_5 = v''_{\sigma 2}$ or $\alpha_5 = v'_{\sigma 1}$. But then $(\alpha_1, \alpha_2, \alpha_4, \alpha_5) \notin \mathcal{J} \cup \mathcal{T}$. If $(J', \alpha_4) \notin \mathcal{J}$ and $(J', \alpha_5) \notin \mathcal{J}$ then $(J', \alpha_4), (J', \alpha_5) \in \mathcal{T}$ and so $\alpha_4 = v''_{\sigma 2}$ and $\alpha_5 = v'_{\sigma 1}$ or conversely. But then $(\alpha_1, \alpha_2, \alpha_4, \alpha_5) \notin \mathcal{J} \cup \mathcal{T}$. \square

Corollary 5.5. *There exist no indecomposable, graded, non-orientable, rank 4, 6-generated MCM modules.*

6 Orientable, rank 2, 6-generated MCM modules

Consider $S = K[x_1, x_2, x_3, x_4]$, K an algebraically closed field of characteristic zero and $R = S/(f)$, $f = x_1^3 + x_2^3 + x_3^3 + x_4^3$.

We have proved that a non-free graded orientable six-generated MCM R -module corresponds to a skew symmetric homogeneous matrix over S of order 6, whose determinant is f^2 .

Let Λ be such a matrix. Notice that Λ has linear entries and the matrix $\underline{\Lambda} := \Lambda|_{x_4=0}$, obtained from Λ by restricting the entries to $x_4 = 0$, is a homogeneous matrix over $S_3 = K[x_1, x_2, x_3]$, whose determinant is f_3^2 , where $f_3 = x_1^3 + x_2^3 + x_3^3$. Therefore, $\text{Coker} \underline{\Lambda}$ defines a graded rank two, six-generated MCM over $R_3 = S_3/(f_3)$. These modules were explicitly described in [LPP].

Lemma 6.1. *Let M be a non-free graded orientable six-generated MCM module over R . Then there exists $\lambda \in V(f_3) \setminus \{P_0\}$ and a skew symmetric matrix $\Gamma \in \mathcal{M}_{6 \times 6}(K)$, such that M is the cokernel of a map given by the matrix $\Lambda = x_4 \cdot \Gamma + \begin{pmatrix} 0 & -\alpha_\lambda^t \\ \alpha_\lambda & 0 \end{pmatrix}$.*

(The same notations as in [LPP] and in "Preliminaries".)

Proof. Let Λ_1 be a skew symmetric homogeneous matrix over S , corresponding to M , and denote $\underline{\Lambda}_1 = \Lambda_1|_{x_4=0}$. Suppose that the MCM S_3 -module corresponding to $\underline{\Lambda}_1$ is indecomposable. Then we can generate it using Theorem 4.2 and Lemma 5.4 from [LPP]. Denote with D the matrix which we obtain by this means.

Since $D \sim \underline{\Lambda}_1$, and $\underline{\Lambda}_1$ is skew symmetric, there exist two invertible matrices $U, V \in \mathcal{M}_{6 \times 6}(K)$ such that $U \cdot D \cdot V + (U \cdot D \cdot V)^t = 0$. Therefore, there exists $T \in \mathcal{M}_{6 \times 6}(K)$ an invertible matrix such that $T \cdot D + (TD)^t = 0$. (Take $T = (V^t)^{-1} \cdot U$.)

With the help of SINGULAR, we find that, in fact, there is no invertible matrix T such that $T \cdot D$ is skew symmetric. Therefore, the module corresponding to $\underline{\Lambda}_1$ should decompose.

//First, we generate the matrix D

```
LIB"matrix.lib"; option(redSB); proc reflexivHull(matrix M) {
  module N=mres(transpose(M),3)[3];
  N=prune(transpose(N));
  return(matrix(N));
}
```

```
proc tensorCM(matrix Phi, matrix Psi) {
  int s=nrows(Phi);
  int q=nrows(Psi);
  matrix A=tensor(unitmat(s),Psi);
  matrix B=tensor(Phi,unitmat(q));
  matrix R=concat(A,B,U);
  return(reflexivHull(R));
}
```

```
proc M2(ideal I) {
  matrix A=syz(transpose(mres(I,3)[3]));
  return(transpose(A));
}
```

```
ring R=0,(x(1..3)),(c,dp); qring S=std(x(1)^3+x(2)^3+x(3)^3);
ideal I=maxideal(1); matrix C=M2(I);
```

```
ring R1=(0,a),(x(1..3),e,b),lp; ideal I=x(1)^3+x(2)^3+x(3)^3,
(a-1)^3+b3+1,e*b+a2-3*a+3,e*a-b2;
qring S1=std(I);
```

```

matrix B[3][3]=      0,  x(1)-(a-1)*x(3),  x(2)-b*x(3),
                   x(1)+x(3),  -x(2)-x(3)*b,  -x(3)*e,
                   x(2),      x(3)*e,  -x(1)+(-a+2)*x(3);
matrix C=imap(S,C); matrix D=tensorCM(C,B);

//We check the existence of the invertible matrix T

ring R2=0,(x(1..3),a,e,b,t(1..36)),dp; ideal
I=x(1)^3+x(2)^3+x(3)^3,(a-1)^3+b3+1,e*b+a2-3*a+3,e*a-b2; qring
S2=std(I); matrix D=imap(S1,D); matrix T[6][6]=t(1..36); matrix
A=T*D+transpose(T*D); ideal I=flatten(A); ideal
I1=transpose(coeffs(I,x(1)))[2]; ideal
I2=transpose(coeffs(I,x(2)))[2]; ideal
I3=transpose(coeffs(I,x(3)))[2]; ideal J=I1+I2+I3+ideal(det(T)-1);
ideal L=std(J);
L;

L[1]=1

```

//Therefore, does not exists T an invertible matrix such that T:D skew
//symmetric.

So, after some linear transformations, $\underline{\Lambda}_1$ decomposes into two matrices of order three and rank one with determinant $f_3 = x_1^3 + x_2^3 + x_3^3$, which correspond to two points λ_1, λ_2 in $V(f_3) \setminus \{P_0\}$, $P_0 = [-1 : 0 : 1]$. Let us denote them by A and B . We can consider $A = \alpha_{\lambda_1}$, $B = \alpha_{\lambda_2}$.

Since $\underline{\Lambda}_1$ is skew symmetric, there exists an invertible matrix $U \in \mathcal{M}_{6 \times 6}(K)$ such that $U \cdot \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ is skew symmetric. Therefore, if we consider $U = \begin{pmatrix} U_1 & U_2 \\ U_3 & U_4 \end{pmatrix}$, we have the following equalities:

$$\begin{cases} U_1 \cdot A + (U_1 \cdot A)^t = 0 \\ U_4 \cdot B + (U_4 \cdot B)^t = 0 \\ U_2 \cdot B + A^t \cdot U_3^t = 0 \\ U_3 \cdot A + B^t \cdot U_2^t = 0 \end{cases}$$

So $U_1 \cdot \alpha_{\lambda_1}$ and $U_4 \cdot \alpha_{\lambda_2}$ are skew symmetric, so they have only zeros on the main diagonal. Since the entries of the second and third line and column of α_{λ_1} and α_{λ_2} are linearly independent, we easily obtain that $U_1 = U_4 = 0$. Therefore, U_2 and U_3 are invertible matrices and $B = -U_2^{-1} \cdot A^t \cdot U_3^t$.

We have obtained $\underline{\Lambda}_1 \sim \begin{pmatrix} \alpha_{\lambda_1} & 0 \\ 0 & \alpha_{\lambda_1}^t \end{pmatrix} \sim \begin{pmatrix} 0 & -\alpha_{\lambda_1}^t \\ \alpha_{\lambda_1} & 0 \end{pmatrix}$.

Therefore, there exists $\Gamma \in \mathcal{M}_{6 \times 6}(K)$ skew symmetric and $\lambda \in V(f_3) \setminus \{P_0\}$ such that $\Lambda_1 \sim \Lambda = x_4 \cdot \Gamma + \begin{pmatrix} 0 & -\alpha_{\lambda}^t \\ \alpha_{\lambda} & 0 \end{pmatrix}$. We can write $\Gamma = \begin{pmatrix} \Gamma_1 & -\Gamma_2^t \\ \Gamma_2 & \Gamma_3 \end{pmatrix}$, $\Gamma_i \in \mathcal{M}_{3 \times 3}(K)$, $i = 1, 2, 3$, Γ_1 and Γ_3 skew symmetric. \square

Remark 6.2 (Notation). For any $\lambda = [a : b : c] \in V(f_3) \setminus \{P_0\}$ there exists a unique point in $V(f_3) \setminus \{P_0\}$ which we denote as λ^t , such that $\alpha_\lambda^t \sim \alpha_{\lambda^t}$. We find $\lambda^t = [c : b : a]$.

For $\lambda = [a : b : 1]$ we denote with U_λ and V_λ two invertible matrices such that $U_\lambda \cdot \alpha_\lambda^t = \alpha_{\lambda^t} \cdot V_\lambda$.

If $a \neq 0$, then we can take $U_\lambda = \begin{pmatrix} b^2 & b(a+1) & -(a+1)^2 \\ -(a+1)^2 & b^2 & -b(a+1) \\ b(a+1) & (a+1)^2 & b^2 \end{pmatrix}$ and $V_\lambda = U_\lambda^t$.

If $a = 0$, then we can take $U_\lambda = \begin{pmatrix} -b^2 & -b & 1 \\ -2b & 1 & b^2 \\ 2b^2 & 2b & 1 \end{pmatrix}$ and $V_\lambda = \begin{pmatrix} 1 & -2b & 2b^2 \\ -b & -b^2 & -1 \\ -b & -b^2 & 2 \end{pmatrix}$.

Notice that for $\lambda = [1 : b : 1] \in V(f_3)$, $\lambda^t = \lambda$ and for all other $\lambda \in V(f_3) \setminus \{P_0\}$, $\lambda \neq \lambda^t$. \square

Remark 6.3. For any $\lambda = [a : 1 : 0] \in V(f_3) \setminus \{P_0\}$ and any $\Lambda = x_4 \cdot \Gamma + \begin{pmatrix} 0 & -\alpha_\lambda^t \\ \alpha_\lambda & 0 \end{pmatrix}$ skew symmetric with $\det \Lambda = f^2$, there exist $\lambda^t = [0 : -a^2 : 1] \in V(f_3) \setminus \{P_0\}$ and $\Lambda' = x_4 \Gamma' + \begin{pmatrix} 0 & -\alpha_{\lambda^t}^t \\ \alpha_{\lambda^t} & 0 \end{pmatrix}$ skew symmetric with $\det \Lambda' = f^2$ such that $\Lambda \sim \Lambda'$.

Indeed, take $\Lambda' = U \cdot \Lambda \cdot U^t$ where $U = \begin{pmatrix} 0 & T_1 \\ T_2 & 0 \end{pmatrix}$, $T_1 = \frac{1}{\sqrt{3}} \begin{pmatrix} -a^2 & -a & 0 \\ 0 & 1 & a \\ 2 & 0 & 1 \end{pmatrix}$ and $T_2 = \frac{1}{\sqrt{3}} \begin{pmatrix} -1 & -a^2 & -a^2 \\ -2a^2 & -a & -a \\ 2a & 1 & -2 \end{pmatrix}$.

Therefore, Coker Λ and Coker Λ' define two isomorphic MCM modules. This is the reason why, from now on, we shall consider only the case $\lambda = [a : b : 1] \in V(f_3) \setminus \{P_0\}$. \square

Remark 6.4. Consider $\lambda = [a : b : 1] \in V(f_3) \setminus \{P_0\}$ and $\Lambda = x_4 \cdot \Gamma + \begin{pmatrix} 0 & -\alpha_\lambda^t \\ \alpha_\lambda & 0 \end{pmatrix}$ as in Lemma 6.1. Then there exists $\bar{\Lambda} = x_4 \cdot \bar{\Gamma} + \begin{pmatrix} \alpha_\lambda & 0 \\ 0 & \alpha_{\lambda^t} \end{pmatrix}$ with $\det \bar{\Lambda} = f^2$ such that $\bar{\Lambda} \sim \Lambda$.

Indeed, consider $\bar{\Lambda} = \begin{pmatrix} 0 & \text{Id} \\ -U_\lambda & 0 \end{pmatrix} \cdot \Lambda \cdot \begin{pmatrix} \text{Id} & 0 \\ 0 & V_\lambda^{-1} \end{pmatrix}$.

We obtain $\bar{\Gamma} = \begin{pmatrix} \Gamma_2 & \Gamma_3 \cdot V_\lambda^{-1} \\ -U_\lambda \cdot \Gamma_1 & U_\lambda \cdot \Gamma_2^t \cdot V_\lambda^{-1} \end{pmatrix}$. \square

Lemma 6.5. Consider $\Lambda = x_4 \cdot \Gamma + \begin{pmatrix} 0 & -\alpha_\lambda^t \\ \alpha_\lambda & 0 \end{pmatrix}$ as above. Then the MCM module M corresponding to Λ is indecomposable if and only if $\Gamma_1 \neq 0$ or $\Gamma_3 \neq 0$.

Proof. Suppose M is indecomposable. If $\Gamma_1 = \Gamma_3 = 0$, then $\begin{pmatrix} 0 & \text{Id} \\ \text{Id} & 0 \end{pmatrix} \cdot \Lambda = \begin{pmatrix} x_4 \cdot \Gamma_2 + \alpha_\lambda & 0 \\ 0 & -x_4 \cdot \Gamma_2^t - \alpha_\lambda^t \end{pmatrix}$, so Λ decomposes after some linear transformation.

This contradicts the indecomposability of $M = \text{Coker } \Lambda$, so we must have $\Gamma_1 \neq 0$ or $\Gamma_3 \neq 0$.

Now, let us suppose $\Gamma_1 \neq 0$ or $\Gamma_3 \neq 0$ and prove that M is indecomposable.

Suppose M decomposes. Then there exists a matrix $\begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix}$ equivalent to Λ with T_1, T_2 two matrices of order three and rank one, with $\det T_1 = \det T_2 = f$ and $T_1|_{x_4=0} = \alpha_{\lambda_1}$, $T_2|_{x_4=0} = \alpha_{\lambda_2}$, where $\lambda_1, \lambda_2 \in V(f_3) \setminus \{P_0\}$.

Since Λ is skew symmetric, after some linear transformations, $\begin{pmatrix} \alpha_{\lambda_1} & 0 \\ 0 & \alpha_{\lambda_2} \end{pmatrix}$ should also become skew symmetric. As we saw in the proof of Lemma 6.1, this gives $\alpha_{\lambda_2} \sim \alpha_{\lambda_1}^t$, so $\lambda_2 = \lambda_1^t$.

Using Remark 6.3, there exist $U, V \in \mathcal{M}_{6 \times 6}(K)$ invertible matrices such that $U \cdot \bar{\Lambda} \cdot V = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix} = x_4 \cdot \begin{pmatrix} N_1 & 0 \\ 0 & N_2 \end{pmatrix} + \begin{pmatrix} \alpha_{\lambda_1} & 0 \\ 0 & \alpha_{\lambda_1^t} \end{pmatrix}$.

Therefore,

$$\begin{cases} U \cdot \begin{pmatrix} \alpha_\lambda & 0 \\ 0 & \alpha_{\lambda^t} \end{pmatrix} \\ U \cdot \begin{pmatrix} \Gamma_2 & \Gamma_3 \cdot V_\lambda^{-1} \\ -U_\lambda \cdot \Gamma_1 & U_\lambda \cdot \Gamma_2^t \cdot V_\lambda^{-1} \end{pmatrix} \end{cases} = \begin{cases} \begin{pmatrix} \alpha_{\lambda_1} & 0 \\ 0 & \alpha_{\lambda_1^t} \end{pmatrix} \\ \begin{pmatrix} N_1 & 0 \\ 0 & N_2 \end{pmatrix} \end{cases} \cdot V^{-1} \quad (1)$$

Let us consider $U = \begin{pmatrix} U_1 & U_2 \\ U_3 & U_4 \end{pmatrix}$ and $V^{-1} = \begin{pmatrix} V_1 & V_2 \\ V_3 & V_4 \end{pmatrix}$ with $U_i, V_i \in \mathcal{M}_{3 \times 3}(K)$, $i = \overline{1, 4}$.

The first system of equations gives:

$$\begin{cases} U_1 \cdot \alpha_\lambda = \alpha_{\lambda_1} \cdot V_1 \\ U_2 \cdot \alpha_{\lambda^t} = \alpha_{\lambda_1} \cdot V_2 \\ U_3 \cdot \alpha_\lambda = \alpha_{\lambda_1^t} \cdot V_3 \\ U_4 \cdot \alpha_{\lambda^t} = \alpha_{\lambda_1^t} \cdot V_4 \end{cases}$$

By comparing the coefficients of x_1, x_2, x_3 on the left-hand side and right-hand side of the above equalities, we obtain easily:

$$U_i = V_i = K_i \cdot \text{Id}_3 \text{ with } K_i \in K, i = \overline{1, 4}.$$

Moreover, if $\lambda \neq \lambda_1$, then $K_1 = K_4 = 0$ and if $\lambda \neq \lambda_1^t$, then $K_2 = K_3 = 0$. Since U is invertible, we have $\lambda = \lambda_1$ or $\lambda = \lambda_1^t$.

We know that $\alpha_{\lambda_1} = T_1|_{x_4=0}$ where T_1 is a matrix of order three over $S = K[x_1, x_2, x_3, x_4]$ of rank one and with determinant f . So Coker T_1 is a graded three-generated rank one MCM R -module. In [EP], all the isomorphism classes of such modules are given explicitly. We obtain $\alpha_{\lambda_1} \sim \alpha|_{x_4=0}$ or $\alpha_{\lambda_1} \sim \alpha^t|_{x_4=0}$ or $\alpha_{\lambda_1} \sim \eta|_{x_4=0}$ or $\alpha_{\lambda_1} \sim \nu|_{x_4=0}$.

With the help of computers, we obtain that none of the above matrices is equivalent to $\alpha_{[1:\ell:1]}$, therefore, $\lambda_1 \neq \lambda_1^t$.

```
LIB"matrix.lib"; option(redSB);
```

```
ring r=0,(x(1..3),1,a,b,c,d,e,v(1..9),u(1..9)),dp; ideal
I=x(1)^3+x(2)^3+x(3)^3,
  1^3+2,
  a3+1,b3+1,c3+1,d3+1,e2+e+1,bcd-e*a;
qring s=std(I);
```

```
proc isomorf(matrix X,matrix Y)
matrix U[3][3]=u(1..9);
matrix V[3][3]=v(1..9);
matrix C=U*X-Y*V;
ideal I=flatten(C);
ideal I1=transpose(coeffs(I,x(1)))[2];
ideal I2=transpose(coeffs(I,x(2)))[2];
```

```

ideal I3=transpose(coeffs(I,x(3)))[2];
ideal J=I1+I2+I3+ideal(det(U)-1,det(V)-1);
ideal L=std(J);
L;

matrix A[3][3]=0,    x(1)-x(3),  x(2)-1*x(3),
                 x(1)+x(3), -x(2)-1*x(3), -1/2*1^2*x(3),
                 x(2),  1/2*1^2*x(3),          -x(1);

//This is the matrix corresponding to the point (1:1:1)

//We now write the matrices corresponding to the rank one three-
generated MCM modules, restricted to x(4)=0

matrix alpha[3][3]=0,          x(1), -x(3)*b+x(2),
                            -x(2)*c+x(1),    -x(3)*b^2, x(3)*b^2*c^2,
                            x(3), x(3)*b*c^2+x(2)*c^2, -x(2)*c-x(1);

matrix alphas=transpose(alpha);

matrix eta[3][3]=0,x(1)+x(2),    x(3),
                  x(1)+e*x(2),  -x(3),    0,
                  x(3),        0,-x(1)-e^2*x(2);

matrix nu[3][3]=0,x(1)+x(3),    x(2),
                  x(1)-a^2*b*x(3), -x(2),    0,
                  x(2),        0,-x(1)+a*b^2*x(3);

isomorf(alpha,A); L[1]=1 isomorf(alphas,A); L[1]=1 isomorf(eta,A);
L[1]=1 isomorf(teta,A); L[1]=1

// Therefore none is isomorphic to  $\alpha_{[1:\ell:1]}$  and this means  $\lambda_1 \neq \lambda_1^t$ .

If  $\lambda = \lambda_1 \neq \lambda_1^t$  as a solution of the system (1), we obtain:  $U = V = \begin{pmatrix} K_1 \cdot \text{Id} & 0 \\ 0 & K_4 \cdot \text{Id} \end{pmatrix}$ ,  $K_1 \cdot K_4 \neq 0$ .

Replacing  $U$  and  $V$  in (2), we obtain: 
$$\begin{cases} K_1 \cdot \Gamma_3 \cdot V_\lambda = 0 \\ K_4 \cdot U_\lambda \cdot \Gamma_1 = 0. \end{cases}$$


Since  $K_1 \neq 0$ ,  $K_4 \neq 0$  and  $U_\lambda, V_\lambda$  are invertible matrices, we obtain  $\Gamma_1 = \Gamma_3 = 0$ , which is a contradiction to our hypothesis.

```

If $\lambda = \lambda_1^t \neq \lambda$, we obtain as a solution of (1): $U = V = \begin{pmatrix} 0 & K_2 \cdot \text{Id} \\ K_3 \text{Id} & 0 \end{pmatrix}$, $K_2 \cdot K_3 \neq 0$.

Replacing U and V in (2), we obtain: $\begin{cases} K_2 \cdot U_\lambda \cdot \Gamma_1 = 0 \\ K_3 \cdot \Gamma_3 \cdot V_\lambda = 0. \end{cases}$ Therefore, we must have again $\Gamma_1 = \Gamma_3 = 0$. \square

For each $\lambda = [a : b : 1] \in V(f_3) \setminus \{P_0\}$, we define a family of skew symmetric homogeneous indecomposable matrices of order six over $S = K[x_1, x_2, x_3, x_4]$ with determinant f^2 :

$$\mathcal{M}_\lambda := \left\{ \Lambda_{(\lambda, \Gamma)} = x_4 \cdot \Gamma + \begin{pmatrix} 0 & -\alpha_\lambda^t \\ \alpha_\lambda & 0 \end{pmatrix}, \det \Lambda_{(\lambda, \Gamma)} = f^2, \right. \\ \left. \Gamma = \begin{pmatrix} \Gamma_1 & -\Gamma_2^t \\ \Gamma_2 & \Gamma_3 \end{pmatrix}, \Gamma_1, \Gamma_3 \text{ skew symmetric}, \Gamma_1 \neq 0 \text{ or } \Gamma_3 \neq 0 \right\}.$$

Notice that, as in the proof of Lemma 6.5, if $\Lambda_{(\lambda, \Gamma)} \sim \Lambda_{(\lambda', \Gamma')}$, then $\lambda' = \lambda$ or $\lambda' = \lambda^t$.

Lemma 6.6. *Let $\lambda = [a : b : 1] \in V(f_3) \setminus \{P_0\}$ with $a \neq 0$, $a \neq 1$.*

1. *Inside the family \mathcal{M}_λ , two matrices, Λ and Λ' , are equivalent if and only if there exists $k \in K^*$ such that $\Lambda' = U_k \cdot \Lambda \cdot U_k^t$, $U_k = \begin{pmatrix} k \text{Id} & 0 \\ 0 & \frac{1}{k} \text{Id} \end{pmatrix}$. This condition means:*

$$\begin{cases} \Gamma'_2 = \Gamma_2 \\ \Gamma'_1 = k^2 \cdot \Gamma_1 \\ \Gamma'_3 = \frac{1}{k^2} \cdot \Gamma_3. \end{cases}$$

2. *A matrix Λ from \mathcal{M}_λ is equivalent to a matrix Λ' from $\mathcal{M}_{\lambda'}$, $\lambda' \neq \lambda$ if and only if $\lambda' = [\frac{1}{a} : \frac{b}{a} : 1]$ and $\Lambda' = U_k \cdot \Lambda \cdot U_k^t$, where $k \in K^*$ and $U_k = \begin{pmatrix} 0 & k \cdot U_\lambda^{-1} \\ -\frac{1}{k} U_\lambda & 0 \end{pmatrix}$.*

Proof. Two matrices, $\Lambda = \Lambda_{(\lambda, \Gamma)}$ and $\Lambda' = \Lambda_{(\lambda', \Gamma')}$, are equivalent if and only if $\bar{\Lambda}$ and $\bar{\Lambda}'$ are equivalent (see Remark 6.4).

If U and V are two invertible matrices such that $U \cdot \bar{\Lambda} = \bar{\Lambda}' \cdot V$, as in the proof of Lemma 6.5, we obtain

$$U = V = \begin{pmatrix} K_1 \text{Id} & K_2 \text{Id} \\ K_3 \text{Id} & K_4 \text{Id} \end{pmatrix} \text{ with } \begin{aligned} K_1 = K_4 = 0 & \text{ if } \lambda \neq \lambda' \text{ and} \\ K_2 = K_3 = 0 & \text{ if } \lambda' \neq \lambda^t. \end{aligned}$$

Since $U \cdot \bar{\Lambda} = \bar{\Lambda}' \cdot V$, we have:

$$\begin{pmatrix} 0 & \text{Id} \\ -U_{\lambda'} & 0 \end{pmatrix}^{-1} \cdot U \cdot \begin{pmatrix} 0 & \text{Id} \\ -U_\lambda & 0 \end{pmatrix} \cdot \Lambda \cdot \begin{pmatrix} \text{Id} & 0 \\ 0 & V_\lambda^{-1} \end{pmatrix} \cdot U^{-1} \cdot \begin{pmatrix} \text{Id} & 0 \\ 0 & V_{\lambda'}^{-1} \end{pmatrix}^{-1} = \Lambda'. \quad (*)$$

1. If $\lambda = \lambda'$ then $\lambda' \neq \lambda^t$, so $U = \begin{pmatrix} K_1 \text{Id} & 0 \\ 0 & K_4 \text{Id} \end{pmatrix}$ with $K_1 \neq 0$, $K_4 \neq 0$. So (*) implies: $\begin{pmatrix} K_4 \cdot \text{Id} & 0 \\ 0 & K_1 \text{Id} \end{pmatrix} \cdot \Lambda \cdot \begin{pmatrix} \frac{1}{K_1} \text{Id} & 0 \\ 0 & \frac{1}{K_4} \text{Id} \end{pmatrix} = \Lambda'$. For $k = \sqrt{\frac{K_4}{K_1}}$ and $U_k = \begin{pmatrix} k \text{Id} & 0 \\ 0 & \frac{1}{k} \text{Id} \end{pmatrix}$ we have $\Lambda' = U_k \cdot \Lambda \cdot U_k^t$.

2. If $\lambda' = \lambda^t$ then $\lambda' \neq \lambda$, so $U = \begin{pmatrix} 0 & K_2 \text{Id} \\ K_3 \text{Id} & 0 \end{pmatrix}$, $K_2 \neq 0$, $K_3 \neq 0$. Replacing U in (*) we obtain:

$$\Lambda' = \begin{pmatrix} 0 & -K_3 U_{\lambda^t}^{-1} \\ -K_2 U_{\lambda} & 0 \end{pmatrix} \cdot \Lambda \cdot \begin{pmatrix} 0 & \frac{1}{K_3} V_{\lambda^t} \\ \frac{1}{K_2} V_{\lambda}^{-1} & 0 \end{pmatrix}.$$

Since $a \neq 0$ and $a \neq 1$, $V_{\lambda} = U_{\lambda}^t$, $\lambda^t = [\frac{1}{a} : \frac{1}{b} : 1]$, $U_{\lambda^t} = \frac{1}{a^2} \cdot U_{\lambda}$, $V_{\lambda^t} = \frac{1}{a^2} U_{\lambda}^t$ (see Remark 6.2).

So $\Lambda' = \begin{pmatrix} 0 & -K_3 a^2 U_{\lambda}^{-1} \\ -K_2 U_{\lambda} & 0 \end{pmatrix} \cdot \Lambda \cdot \begin{pmatrix} 0 & \frac{1}{K_3} \cdot \frac{1}{a^2} \cdot U_{\lambda}^t \\ \frac{1}{K_2} (U_{\lambda}^{-1})^t & 0 \end{pmatrix} = \begin{pmatrix} 0 & k U_{\lambda}^{-1} \\ -\frac{1}{k} U_{\lambda} & 0 \end{pmatrix} \cdot \Lambda \cdot \begin{pmatrix} 0 & k U_{\lambda}^{-1} \\ -\frac{1}{k} U_{\lambda} & 0 \end{pmatrix}^t$,
where $k^2 = -a^2 \cdot \frac{K_3}{K_2}$.

□

In a similar way, we can prove the following two lemmas:

Lemma 6.7. *Let $\lambda = [0 : b : 1] \in V(f_3) \setminus \{P_0\}$.*

1. *Inside the family \mathcal{M}_{λ} , two matrices Λ and Λ' are equivalent if and only if there exists*

$$k \in K^* \text{ such that } \Lambda' = U_k \cdot \Lambda \cdot U_k^t, \quad U_k = \begin{pmatrix} k \text{Id} & 0 \\ 0 & \frac{1}{k} \text{Id} \end{pmatrix}. \text{ This means } \begin{cases} \Gamma'_2 = \Gamma_2 \\ \Gamma'_1 = k^2 \cdot \Gamma_1 \\ \Gamma'_3 = \frac{1}{k^2} \cdot \Gamma_3 \end{cases}$$

2. *No $\lambda \in V(f_3) \setminus \{P_0, [0 : b : 1]\}$ exists, such that a matrix from \mathcal{M}_{λ} is equivalent to a matrix from $\mathcal{M}_{[0:b:1]}$.*

Lemma 6.8. *Let $\lambda = [1 : b : 1] \in V(f_3) \setminus \{P_0\}$.*

1. *Inside the family \mathcal{M}_{λ} , two matrices Λ and Λ' are equivalent if and only if $\Lambda' = T \cdot \Lambda \cdot T^t$, where*

$$T = \begin{pmatrix} K_4 \cdot \text{Id} & -K_3 \cdot U_{\lambda}^{-1} \\ -K_2 \cdot U_{\lambda} & K_1 \cdot \text{Id} \end{pmatrix}, \quad K_1, K_2, K_3, K_4 \in K \text{ such that } K_1 K_4 - K_2 K_3 = 1.$$

2. *No $\lambda \in V(f_3) \setminus \{P_0, [1 : b : 1]\}$ exists, such that a matrix from \mathcal{M}_{λ} is equivalent to a matrix from $\mathcal{M}_{[1:b:1]}$.*

Remark 6.9. For $K = \mathbb{C}$ we can restrict to the following families with no equivalents outside the families:

1. \mathcal{M}_{λ} , $\lambda = [a : b : 1] \in V(f_3) \setminus \{P_0\}$, $0 \leq |a| < 1$ or $|a| = 1$ and $\text{Im } a > 0$.

2. $\mathcal{M}_{[1:b:1]}$, $b^3 + 2 = 0$.

□

Now let us see “how large” the family \mathcal{M}_λ is for a given $\lambda = [a : b : 1]$ in $V(f_3) \setminus \{P_0\}$.

For $\Lambda = \Lambda_{(\lambda, \Gamma)}$ in \mathcal{M}_λ , we denote:

$$\Gamma_1 = \begin{pmatrix} 0 & a_1 & a_2 \\ -a_1 & 0 & a_3 \\ -a_2 & -a_3 & 0 \end{pmatrix}, \quad \Gamma_2 = \begin{pmatrix} a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} \end{pmatrix}, \quad \Gamma_3 = \begin{pmatrix} 0 & a_4 & a_5 \\ -a_4 & 0 & a_6 \\ -a_5 & -a_6 & 0 \end{pmatrix}.$$

The condition $\det \Lambda = f^2$ provides 10 equations in the above 15 parameters. Six of these equations are linear in the entries of Γ_2 and form a linear system of dimension three.

1. If $b = 0$ the solution of this system is:

$$\begin{cases} a_7 = -a_{12} \cdot (a^2 + 1) \\ a_8 = a_{10} = a_{15} = 0 \\ a_9 = a_{11} - a_{13} \\ a_{14} = a^2 \cdot a_{12}. \end{cases}$$

2. If $b \neq 0$, the system has the following solution:

$$\begin{cases} a_8 = -\frac{b}{a+1}a_7 + a_{15} \\ a_9 = -\frac{a-1}{b(a+1)}a_7 - \frac{a^2}{b^2} \cdot a_{15} \\ a_{10} = \frac{b}{a+1} \cdot a_7 \\ a_{12} = -\frac{a^2+3}{(a+1)^2} \cdot a_7 + \frac{b}{a+1}a_{11} + \frac{1-a}{b(a+1)}a_{15} \\ a_{13} = \frac{a-1}{b(a+1)} \cdot a_7 + a_{11} + \frac{a^2}{b^2} \cdot a_{15} \\ a_{14} = \frac{2(1-a)}{(a+1)^2} \cdot a_7 - \frac{b}{a+1} \cdot a_{11} + \frac{a-1}{b(a+1)} \cdot a_{15}. \end{cases}$$

The other four equations are linear in the entries of Γ_1 with coefficients in $K[a_4, \dots, a_{15}]$ and have dimension five:

```
LIB"matrix.lib"; option(redSB);
```

```
ring r=0,(x(4),x(1),x(2),x(3),e,a,b,a(1..15)),dp; ideal
ii=a3+b3+1,e*b+a2-a+1,e*a+e-b2; qring s=std(ii);
```

```
matrix B[10][1]; B[1,1]=x(4)*a(1); B[2,1]=x(4)*a(2);
B[3,1]=-x(4)*a(7); B[4,1]=-x(4)*a(10)-(x(1)+x(3));
B[5,1]=x(4)*a(3); B[6,1]=-x(4)*a(8)-(x(1)-a*x(3));
B[7,1]=-x(4)*a(11)+x(2)+b*x(3); B[8,1]=-x(4)*a(9)-x(2)+b*x(3);
B[9,1]=-x(4)*a(12)+e*x(3); B[10,1]=x(4)*a(4);
```



```

matrix V[1][5]; V[1,1]=-x(4)*a(13)-x(2);
V[1,2]=-x(4)*a(14)-e*x(3); V[1,3]=-x(4)*a(15)+x(1)+(a-1)*x(3);
V[1,4]=x(4)*a(5); V[1,5]=x(4)*a(6);

poly p1=B[5,1]*B[10,1]-B[6,1]*B[9,1]+B[7,1]*B[8,1]; poly
p2=B[2,1]*B[10,1]-B[3,1]*B[9,1]+B[4,1]*B[8,1]; poly
p3=B[1,1]*B[10,1]-B[3,1]*B[7,1]+B[4,1]*B[6,1]; poly
p4=B[1,1]*B[9,1]-B[2,1]*B[7,1]+B[4,1]*B[5,1]; poly
p5=B[1,1]*B[8,1]-B[2,1]*B[6,1]+B[3,1]*B[5,1];

poly g=V[1,1]*p1-V[1,2]*p2+V[1,3]*p3-V[1,4]*p4+V[1,5]*p5; poly
f=x(4)^3+x(1)^3+x(2)^3+x(3)^3; g=g-f;

//For our skew symmetric matrix the condition g=f is equivalent to
//det  $\Lambda = f^2$ .

matrix H=coef(g,x(4)*x(1)*x(2)*x(3)); for(int j=1;j<=13;j++)
H[1,j]=0;

ideal I=H; I=interred(I);

I[1]=a(9)-a(11)+a(13) I[2]=a(8)+a(10)-a(15) I[3]=a(7)+a(12)+a(14)
I[4]=a*a(10)-e*a(11)+b*a(12)+2*e*a(13)+2*b*a(14)-2*a*a(15)+a(10)+a(15)
I[5]=2*e*a(10)+2*b*a(11)-2*a*a(12)-b*a(13)-a*a(14)-e*a(15)+a(12)+2*a(14)
I[6]=a(3)*a(4)-a(2)*a(5)+a(1)*a(6)+a(11)^2+a(10)*a(12)-a(11)*a(13)+a(13)^2
-a(10)*a(14)-2*a(12)*a(15)-a(14)*a(15)
I[7]=a(1)*a(4)+a(3)*a(5)+a(2)*a(6)-a(10)^2+a(11)*a(12)+a(12)*a(13)+2*a(11)
*a(14)-a(13)*a(14)+a(10)*a(15)-a(15)^2
I[8]=2*e^2*a(12)+2*a*b*a(12)-3*b^2*a(13)+2*e^2*a(14)-a*b*a(14)-3*e*b*a(15)
-6*e*a(11)-b*a(12)+12*e*a(13)+2*b*a(14)-6*a*a(15)
I[9]=a(3)*a(5)*a(10)-a(2)*a(6)*a(10)-a(2)*a(5)*a(11)-a(1)*a(6)*a(11)+a(1)
*a(5)*a(12)+a(3)*a(6)*a(12)-a(2)*a(5)*a(13)+2*a(1)*a(6)*a(13)+a(13)^3
+a(2)*a(4)*a(14)+a(3)*a(6)*a(14)+a(10)*a(11)*a(14)+a(12)^2*a(14)-2
*a(10)*a(13)*a(14)+a(12)*a(14)^2+a(3)*a(5)*a(15)+2*a(2)*a(6)*a(15)
+a(11)*a(14)*a(15)-2*a(13)*a(14)*a(15)-a(15)^3-1
I[10]=2*e*a(2)*a(4)-2*e*a(1)*a(5)+2*b*a(2)*a(5)-2*a*a(3)*a(5)+2*b*a(1)*a(6)
-4*a*a(2)*a(6)-2*b*a(11)^2+2*a*a(11)*a(12)+2*e*a(12)^2+5*b*a(11)*a(13)
-4*a*a(12)*a(13)-2*b*a(13)^2-2*b*a(10)*a(14)-a*a(11)*a(14)+2*a*a(13)
*a(14)-2*e*a(14)^2+3*e*a(11)*a(15)-6*e*a(13)*a(15)-2*b*a(14)*a(15)
+6*a*a(15)^2+4*a(3)*a(5)+2*a(2)*a(6)-a(11)*a(12)+2*a(12)*a(13)+2*a(11)
*a(14)-4*a(13)*a(14)-6*a(15)^2

ideal J=I[1],I[2],I[3],I[4],I[5],I[8];

```

//This is the ideal generated by the linear equations in the entries
//of Γ_2 .

ideal JJ=std(J); dim(JJ); 14

ideal J1=I[6],I[7],I[9],I[10];

//This is the ideal generated by the other four equations.

ideal JJ1=std(J1); dim(JJ1); 16

Remark 6.10. For $a_4, a_5, a_6 \in K$ such that $\begin{vmatrix} a_6 & -a_5 & a_4 \\ a_4 & a_6 & a_5 \\ g_1 & g_2 & 0 \end{vmatrix} \neq 0$, where $g_1 = (a - 2)a_4 - e \cdot a_5 + ba_6$ and $g_2 = e \cdot a_4 + ba_5 - (a + 1)a_6$ ($e = \frac{b^2}{a+1}$), solving the linear system in a_1, a_2, a_3 , we have a homogeneous polynomial of degree three in a_4, a_5, a_6 .

Let $P_1 = [1 : b : 1] \in V(f_3)$, that is $b^3 + 2 = 0$, $Z := V(f_3) - \{P_0, P_1\}$, ω the automorphism of Z given by $[a : b : c] \rightarrow [c : b : a]$, $\bar{Z} := Z/\omega$ the factor set given by the equivalence relation $z \sim \omega(z)$. Choose a section $\tau : \bar{Z} \rightarrow Z$ of the canonical surjection, that is choose a representative for each class.

Theorem 6.11. *The set of isomorphism classes of indecomposable, graded, rank two, 6-generated MCM modules consists in a 5-parameter family \mathcal{G} and a 2-parameter family \mathcal{H} given by*

$$\mathcal{G} = \cup_{\bar{z} \in \bar{Z}} \mathcal{M}_{\tau(\bar{z})} / \sim_{K^*}, \mathcal{H} = \mathcal{M}_{P_1} / \sim_{P_1},$$

where \sim_{K^*} is the equivalence relation defined on \mathcal{M}_λ by Lemmas 6.6, 6.7, namely $\Lambda \cong U_k \Lambda U_k$ and \sim_{P_1} is defined in Lemma 6.8.

Proof. As above $\mathcal{M}_{\tau(\bar{z})}$ is a 5-parameter family and so $\mathcal{M}_{\tau(\bar{z})} / \sim_{K^*}$ is a 4-parameter family. Thus \mathcal{G} is again a 5-parameter family. It is not hard to see that \mathcal{H} is a 2-parameter family. \square

Remark 6.12. It is well known that the ideal defining 5 general points in \mathbf{P}_K^3 is Gorenstein (this means any four from them are not on a hyperplane). Restricting to the 5 general points on the surface $V(f)$ we get a family of Gorenstein ideals whose isomorphism classes of 2-syzygies over R (they are indecomposable, graded, rank two, 6-generated MCM modules) form a 5-parameter family (see [Mi], [IK]). Here we give an example. Let $[1 : 0 : 0 : -1], [1 : 0 : -1 : 0], [1 : -1 : 0 : 0], [1 : -u : 0 : 0], [1 : -u : 1 : -u]$, $u^2 + u + 1 = 0$ be 5 general points on $V(f)$ and I the ideal defined by these points in R . I is generated by the following quadratic forms: $x_2x_4 + ux_3x_4$, $-ux_2x_3 + ux_3x_4$, $x_1x_4 + x_4^2 - (1-u)x_3x_4$, $u(x_1 + x_3)x_3 + 2x_3x_4$, $-x_3x_4 - x_1^2 + ux_1x_2 - u^2x_2^2 + x_3^2 + x_4^2$. Then the second syzygy of I over R is the cokernel of a skew symmetric matrix A defined by

$$A[1, 1] = A[2, 2] = A[3, 3] = A[4, 4] = A[5, 5] = A[6, 6] = 0, A[1, 2] = (-3u - 2)x_3 + (2u - 1)x_4 = -A[2, 1], A[1, 3] = (-u)x_1 + (-2u + 1)x_2 + (u + 1)x_3 + (u)x_4 = -A[3, 1],$$

$$\begin{aligned}
A[1, 4] &= (u-2)x_1 - x_2 + (-3u-4)x_3 + (2u-1)x_4 = -A[4, 1], \quad A[1, 5] = (u+1)x_3 + (-u)x_4 = \\
&= -A[5, 1], \quad A[1, 6] = (-u)x_1 + (u+1)x_2 + (1/7u + 3/7)x_3 + (-3/7u - 2/7)x_4 = -A[6, 1], \\
A[2, 3] &= (u-2)x_1 - x_2 + x_3 + (-u+2)x_4 = -A[3, 2], \quad A[2, 4] = (3u+2)x_1 + (2u+3)x_2 + \\
&+ (4u)x_3 + x_4 = -A[4, 2], \quad A[2, 5] = (-3u-1)x_3 + (u-2)x_4 = -A[5, 2], \quad A[2, 6] = (-u-2)x_1 + \\
&+ (-u+1)x_2 + (-u-1)x_3 + (u)x_4 = -A[6, 2], \quad A[3, 4] = -3x_3 = -A[4, 3], \quad A[3, 5] = (u+1)x_3 = \\
&= -A[5, 3], \quad A[3, 6] = (-6/7u - 4/7)x_3 + x_4 = -A[6, 3], \quad A[4, 5] = (-3u-1)x_3 = -A[5, 4], \\
A[4, 6] &= (-u)x_3 + (u)x_4 = -A[6, 4], \quad A[5, 6] = -x_1 + (-u)x_2 = -A[6, 5].
\end{aligned}$$

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