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# Vorwort

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Darüberhinaus bietet die Reihe ein Forum für die Berichterstattung über die zahlreichen Kooperationsprojekte des Instituts mit Partnern aus Industrie und Wirtschaft.

Berichterstattung heißt hier Dokumentation darüber, wie aktuelle Ergebnisse aus mathematischer Forschungs- und Entwicklungsarbeit in industrielle Anwendungen und Softwareprodukte transferiert werden, und wie umgekehrt Probleme der Praxis neue interessante mathematische Fragestellungen generieren.

Prof. Dr. Dieter Prätzel-Wolters
Institutsleiter

Kaiserslautern, im Juni 2001

# Parameter influence on the zeros of network determinants

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**Abstract:** To a network $\mathcal{N}(q)$ with determinant $\Delta(s;q)$ depending on a parameter vector $q \in \mathbb{R}^{\varrho}$ via identification of some of its vertices, a network $\widehat{\mathcal{N}}(q)$ is assigned. The paper deals with procedures to find $\widehat{\mathcal{N}}(q)$, such that its determinant $\widehat{\Delta}(s;q)$ admits a factorization in the determinants of appropriate subnetworks, and with the estimation of the deviation of the zeros of $\widehat{\Delta}$ from the zeros of $\Delta$. To solve the estimation problem state space methods are applied.

*Keywords:* Networks, Equicofactor matrix polynomials, Realization theory, Matrix perturbation theory

## 1 Introduction

We consider determinants, which are generated through weighted undirected graphs consisting of the vertices $V$ and the edges $E$:

$$V := \{1, \ldots, n+1\}, \quad E := \{(i_1, j_1), \ldots, (i_m, j_m)\} \subseteq V^2.$$

The weight of every edge $(i_k, j_k)$ is given by a scalar polynomial. Let $w : E \to \mathbb{R}[s]$ be the corresponding weight function, which assigns every edge its polynomial weight. Then the triple $(V, E, w) =: \mathcal{N}$ is said to be a *network*. In the case where all weights are constants, $w$ is said to be a *length function*, $w(e)$ the *length* of the edge $e \in E$, and $\mathcal{N}$ turns out to be a network in the sense of [10]. The network $\mathcal{N}$ is parametrized via the coefficients of the polynomials $p_k := w(i_k, j_k)$ collected in the vector $q$:

$$q := [q_1^T, \ldots, q_m^T]^T \in \mathbb{R}^{\varrho}, \quad p_k(s) = [1, s, \ldots, s^{d_k}]q_k, \; q_k \in \mathbb{R}^{d_k+1}.$$

For the assignment of a scalar polynomial $\Delta(s;q)$ to $\mathcal{N}(q)$, a matrix polynomial $P(.;q) \in \mathbb{R}^{(n+1)\times(n+1)}[s]$ is generated according to

$$P(s;q) := \mathcal{A} \operatorname{diag}(p_1(s), \ldots, p_m(s))\, \mathcal{A}^T, \tag{1}$$

where $\mathcal{A} \in \mathbb{R}^{(n+1)\times m}$ denotes the *all vertex matrix* of $\mathcal{N}$. Now we set

$$\Delta(s;q) := |L(s;q)| := \det L(s;q), \quad L(s;q) := P^{\{i\}\{j\}}(s;q),$$

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and call $\Delta$ the *determinant* of $\mathcal{N}$. Here $P^{\{i\}\{j\}}$ is obtained by deleting row $i$ and column $j$ of $P$. The independence of $\Delta$ on $i, j \in V$ turns out to be a consequence of the *equicofactor* property of $P$, which is an implication of the *equivalence* of all elements of $\{P^{\{i\}\{j\}} | i, j \in V\}$.

The paper focuses on the influence of $q \in \mathbb{R}^\varrho$ on the zeros of $\Delta(.; q)$ taking into account the *structure* of $\mathcal{N}(q)$.

In the sequel we denote this zero set by $\sigma(\Delta(.; q))$. The elements of $\sigma(\Delta(.; q))$ are called the *finite eigenvalues* of $L(.; q)$. We further assume $\Delta(.; q) \not\equiv 0$ for at least one $q \in \mathbb{R}^\varrho$. Then $L$ is said to be *nonsingular*, and it is well known, for example see [[8], chapter 8], that the eigenvalues of $L$ determine the dynamics of the linear differential algebraic equation (DAE) system

$$L(d/dt)v(t) = f(t), \tag{2}$$

where $f$ is a $\mathbb{C}^n$-valued function of the real variable $t$. Due to equation (1) the coefficients of $L$ are symmetric matrices. Vice versa, for every matrix polynomial $Q \in \mathbb{R}^{n \times n}[s]$ with symmetrical coefficients there exists a matrix polynomial $P$ satisfying (1), such that $Q = P^{\{n+1\}\{n+1\}}$. Hence, the question for the parameter influence on the finite eigenvalues of a matrix polynomial with symmetrical coefficients can be transformed into the corresponding question for the zeros of a network determinant.

In the case where $Q$ is quadratic, and its coefficients are symmetric $Z$-matrices with positive main diagonal, the resulting network $\mathcal{N}$ is interpretable as an electrical circuit, and the independence of $\Delta$ on $i, j \in V$ reflects the arbitrariness of defining the zero potential. A large class of electrical circuits admits the description through DAE systems of the form (2). Therefore, our results are convenient to discuss the influence of electrical parameters on the oscillation behavior of electrical circuits. An excellent source for the theory of $Z$-matrices can be found in [[9], chapter 2.5].

Our approach to deal with the above stated influence problem consists in the identification of appropriate vertices of $\mathcal{N}$ with the intention to obtain an especially *structured* network $\widehat{\mathcal{N}}$, the determinant $\widehat{\Delta}$ of which falls into a product $\widehat{\Delta}_1\widehat{\Delta}_2$ of determinants of certain subnetworks $\widehat{\mathcal{N}}_\vartheta$. Then $\sigma(\widehat{\Delta}) = \sigma(\widehat{\Delta}_1) \cup \sigma(\widehat{\Delta}_2)$, and hence the dependency of $\sigma(\widehat{\Delta})$ on $q$ is given by the according dependencies of $\sigma(\Delta_1)$ and $\sigma(\Delta_2)$, which are much more easier to discuss as for $\sigma(\Delta)$. Because $\widehat{\mathcal{N}}$ does not deviate very strongly from $\mathcal{N}$, it can be expected, that $\sigma(\widehat{\Delta})$ does not deviate very strongly from $\sigma(\Delta)$. Via linearizations of $\Delta$ and $\widehat{\Delta}$ by the Theorem of Bauer - Fike [[2], Theorem 7.2.2, p.342] an estimation of this deviation is obtained.

With respect to this program, we examine factorization properties of $\Delta$, which are provided by the network structure. We say that $\mathcal{N}$ is *structured*, if it is generated by concatenation of similar or dual networks, or it is symmetric or cyclic. Of course combinations of such structures are allowed, too. For example, let $\mathcal{N}$ be given according to

$$\mathcal{N}: \quad \begin{array}{c} (\kappa_1) - \mathcal{N}_1 - (\kappa_2) - \mathcal{N}_2 - (\kappa_3) - \mathcal{N}_3 - (\kappa_4) - \mathcal{N}_4 - (\kappa_1) \\ (\kappa_2) \underline{\qquad\quad} \mathcal{N}_5 \underline{\qquad\quad} (\kappa_4) \end{array} . \tag{3}$$

Then $\mathcal{N}$ is structured in our sense, namely $\mathcal{N}$ can be seen as a connection of the subnetwork $\mathcal{N}_5$ with the cyclical network $\widetilde{\mathcal{N}} : (\kappa_1) - \mathcal{N}_1 - (\kappa_2) - \mathcal{N}_2 - (\kappa_3) - \mathcal{N}_3 - (\kappa_4) - \mathcal{N}_4 - (\kappa_1)$. Since $\mathcal{N}_1$ and $\mathcal{N}_4$ are connected in $\kappa_1$, it is natural to call $\widetilde{\mathcal{N}}$ cyclical. A different interpretation of $\mathcal{N}$ as structured network is given by the representation in the cyclical form

$$(\kappa_1) - \mathcal{N}_1 - (\kappa_2) - \mathcal{N}_6 - (\kappa_4) - \mathcal{N}_4 - (\kappa_1),$$

where $\mathcal{N}_6$ represents itself a cyclical network

$$(\kappa_2) - \mathcal{N}_2 - (\kappa_3) - \mathcal{N}_3 - (\kappa_4) - \mathcal{N}_5 - (\kappa_2).$$

It turns out, that the determinant $\Delta$ of a structured network can be represented as a sum of products consisting of determinants generated through its subnetwork $\mathcal{N}_\vartheta$. Under certain symmetry conditions, such a representation falls into a product, that means $\Delta$ is factorizable in the desired sense. Then we say that $\mathcal{N}$ possesses a *factorizable structure*. To illustrate the strength of our vertex identification approach, we apply our results for Darlington networks, that are networks of structure (3), where in addition some similarity and duality assumptions concerning the subnetworks $\mathcal{N}_\vartheta$ are made. Such networks play an important role in the realization of desired electrical voltage relations. For example, in the case where $\mathcal{N}_1 = \mathcal{N}_3$ and $\mathcal{N}_2 = \mathcal{N}_4$, the network (3) obtains a factorizable structure, namely

$$\Delta = \widehat{\Delta}_1(2\Delta_1\Delta_2\widehat{\Delta}_1 + \widehat{\Delta}_2\Delta_5), \tag{4}$$

where $\Delta_\vartheta$ denotes the determinant of $\mathcal{N}_\vartheta$, and $\widehat{\Delta}_\vartheta$ the determinant of a slight modified appropriate subnetwork of $\mathcal{N}$.

Let the distance $\epsilon$ between $\sigma(\widehat{\Delta}) =: \{\mu_1, \ldots, \mu_{\mathbf{d}}\}$ and $\sigma(\Delta) =: \{\lambda_1, \ldots, \lambda_{\mathbf{d}}\}$ be defined by $\epsilon = \max\{\epsilon_\lambda, \epsilon_\mu\}$, where

$$\begin{aligned} \epsilon_\lambda &= \max\{m_{\lambda_1}, \ldots, m_{\lambda_{\mathbf{d}}}\}, \quad m_{\lambda_\ell} = \min\{|\lambda_\ell - \mu_k| : k = 1, \ldots, \mathbf{d}\}, \\ \epsilon_\mu &= \max\{m_{\mu_1}, \ldots, m_{\mu_{\mathbf{d}}}\}, \quad m_{\mu_\ell} = \min\{|\mu_\ell - \lambda_k| : k = 1, \ldots, \mathbf{d}\}. \end{aligned} \tag{5}$$

That means for every $\lambda \in \sigma(\Delta)$ ($\mu \in \sigma(\widehat{\Delta})$) there exists at least one $\mu \in \sigma(\widehat{\Delta})$ ($\lambda \in \sigma(\Delta)$) with a distance of at most $\epsilon$. Then $\epsilon$ can be estimated by the Theorem of Bauer - Fike via two matrices $A$ and $\widehat{A}$, which are the main operators of minimal state space descriptions of the strictly proper parts of $L^{-1}$ and $\widehat{L}^{-1}$:

$$\epsilon_\lambda \le \|\widehat{T}^{-1}(A - \widehat{A})\widehat{T}\|_p, \quad \epsilon_\mu \le \|T^{-1}(A - \widehat{A})T\|_p.$$

Here, the matrix $T$ ($\widehat{T}$) is chosen to diagonalize $A$ ($\widehat{A}$), and $\|.\|_p$ denotes any $p$ - matrix norm. As main result we obtain, that by identification of two vertices, the difference $A - \widehat{A}$ admits the representation $A - \widehat{A} = UV^T$, where $U$ and $V$ are appropriately chosen vectors, and the equation $\deg \Delta = \deg \widehat{\Delta}$ is supposed. Consequently, for the 2-matrix norm the deviations $\epsilon_\lambda$ and $\epsilon_\mu$ are bounded through products of the Euclidean lengths of certain vectors, namely

$$\epsilon_\lambda \le \|\widehat{T}^{-1}U\|_2 \, \|\widehat{T}^T V\|_2, \quad \epsilon_\mu \le \|T^{-1}U\|_2 \, \|T^T V\|_2. \tag{6}$$

Since $U$ and $V$ explicitly contain the network parameters, the influence of $q$ on $\epsilon$ gets more transparent.

The paper is organized as follows. In section 2 the network determinant concept is introduced. Then formulas are provided, which express $\Delta$ through the determinants of appropriate subnetworks. In section 3 the results of the previous section are applied to structured networks. Subsequently the determinants of Darlington networks are treated. In section 4 the inequalities (6) are derived. The construction of $A$ and $\widehat{A}$ is done via classical realization theory exploiting natural given decomposition possibilities for the coefficients of the underlying matrix polynomials. The existence of two vectors $U$ and $V$ with $A - \widehat{A} = UV^T$ is guaranteed as a consequence of the Sherman - Morrison - Woodbury formula. Combination of this representation result with the Theorem of Bauer - Fike yields our main result summarized in Theorem 2. A concluding example illustrates both its feasibility and usefulness.

## 2 The determinant of $\mathcal{N}$

Definition 1 Let a network $\mathcal{N} := (\mathcal{G}, w)$ be given, where $\mathcal{G} := (V, E)$, and

$$V := \{1, \dots, n+1\},\ E := \{(i_1, j_1), \dots, (i_m, j_m)\} \subseteq V^2,\ w : E \to \mathbb{R}[s]. \tag{7}$$

The all vertex matrix $\mathcal{A}$ of $\mathcal{G}$ is generated via $E$ according to

$$\mathcal{A} := [e_{i_1} - e_{j_1}, \dots, e_{i_m} - e_{j_m}], \tag{8}$$

where $e_i$ denotes the $i$-th unit vector of length $n+1$. Depending on the vertex sets

$$\alpha := \{i_1, \dots, i_r\},\ \beta := \{j_1, \dots, j_r\} \subseteq V$$

the restricted matrix polynomial $P^{\alpha\beta}$ is defined by deleting the rows $i_1, \dots, i_r$ and the columns $j_1, \dots, j_r$ in

$$P := \mathcal{A}\, \mathrm{diag}(w(i_k, j_k))_{k=1}^m\, \mathcal{A}^T \ \in \mathbb{R}^{(n+1)\times(n+1)}[s]. \tag{9}$$

For abbreviation we set $\Delta^{\alpha\beta} := |P^{\alpha\beta}|$ and $P^\alpha := P^{\alpha\alpha}$, $\Delta^\alpha := \Delta^{\alpha\alpha}$. The determinant $(-1)^{\alpha+\beta}\Delta^{\alpha\beta}$ is said to be a *cofactor of order* $r$ of $P$, where $\alpha + \beta = \sum_{k=1}^r (i_k + j_k)$. If $\alpha = V$, then let $\Delta^\alpha = 1$.

The coefficient of the weight polynomials $p_k := w(i_k, j_k)$ play the role of the parameters for $\mathcal{N}$ and $P$, where $q := [q_1^T, \dots, q_m^T]^T \in \mathbb{R}^\varrho$, $p_k(s) = [s^0, \dots, s^{d_k}]q_k$, $q_k \in \mathbb{R}^{d_k+1}$.

A matrix polynomial $P \in \mathbb{R}^{(n+1)\times(n+1)}[s]$ is referred to as *equicofactor*, if all its first order cofactors are equal [[6], chapter 26], more precisely if for all $i, j, k, \ell \in V$ the equation $(-1)^{i+j}\Delta^{\{i\}\{j\}} = (-1)^{k+\ell}\Delta^{\{k\}\{\ell\}}$ holds true. Two matrix polynomials $P_1$ and $P_2$ of the same size are called *equivalent*, $P_1 \sim P_2$, if $P_1 = U_1 P_2 U_2$ for some matrix polynomials $U_\vartheta$ with constant nonzero determinant.

To justify the network determinant concept we need Proposition 1.

PROPOSITION 1 *Let $P$ be of the form $P = \mathcal{A}_L \operatorname{diag}(p_1, \dots, p_m)\, \mathcal{A}_R^T \in \mathbb{R}^{(n+1)\times(n+1)}[s]$, where $\mathcal{A}_L$ and $\mathcal{A}_R$ are the all vertex matrices of two graphs $\mathcal{G}_L$ and $\mathcal{G}_R$. Then for all $i, j, k, \ell \in V$ the restricted matrix polynomials $P^{\{i\}\{j\}}$ and $P^{\{k\}\{\ell\}}$ are equivalent, that means in particular $P$ is equicofactor.*

*Proof.* Suppose that a matrix $M$ fulfils the equations $Mv = 0$, $v^T M = 0$, where $v = [1, \dots, 1]^T$. Then $M^{\{i\}\{j\}}$ is related to $M^{\{n+1\}}$ according to $M^{\{i\}\{j\}} = U_i^T M^{\{n+1\}} U_j$, where

$$U_j := \begin{cases} [e_1, \dots, e_{j-1}, e_{j+1}, \dots, e_n, -e_1 - \dots - e_n], & j < n+1 \\ I_n, & j = n+1 \end{cases},$$

$e_k$ denotes the $k$-th unit vector of length $n$, and $|U_j| = (-1)^{n+1-j}$. Now, we have $v^T[\mathcal{A}_L, \mathcal{A}_R] = 0$, hence $P$ fulfils the assumptions made for $M$. □

Consequently, the following definition is natural.

DEFINITION 2 The first order cofactor $\Delta := |L|$, $L := P^{\{1\}}$ is said to be the *determinant* of $\mathcal{N}$, and $(-1)^{\alpha+\beta}\Delta^{\alpha\beta}$ a *cofactor of $\mathcal{N}$ of order $\#\alpha$*. For fixed $q \in \mathbb{R}^\varrho$ we set $\sigma(\mathcal{N}(q)) = \{\lambda \in \mathbb{C} : \Delta(\lambda; q) = 0\}$.

Two networks $\mathcal{N}_\vartheta$ are said to be *similar*, $\mathcal{N}_1 \sim_{\mathcal{P}} \mathcal{N}_2$, if for a permutation matrix $\mathcal{P}$ the equation $P_1 = \mathcal{P}^T P_2 \mathcal{P}$ is fulfiled, that means one network is generated by the other only by a new enumeration of the vertices. $\mathcal{N}_1$ and $\mathcal{N}_2$ are said to be *dual* with respect to $\alpha := \{i_1, i_2\}$ and $\beta := \{j_1, j_2\}$, if for some $c \in \mathbb{C} \setminus \{0\}$ the equation $c\Delta_1^\alpha \Delta_2^\beta = \Delta_1 \Delta_2$ is satisfied.

In the framework of realization theory dual networks admit the following interpretation, supposed the realization problem is introduced as follows: for a given rational function $h$ find a network $\mathcal{N}$ and a vertex pair $\alpha$ with $h = \Delta^\alpha/\Delta$. In the case where $P$ is quadratic, $\mathcal{N}$ can be interpreted as an electrical circuit and $h$ as the driving point impedance between the terminal pair $\alpha$. The following equivalence is evident: $\mathcal{N}_1$ and $\mathcal{N}_2$ are dual with respect to $\alpha$ and $\beta$, if and only if $\mathcal{N}_1$ realizes $h$ in $\alpha$ and $\mathcal{N}_2$ realizes $(hc)^{-1}$ in $\beta$.

For similar networks the coincidence of all its cofactors can be stated.

PROPOSITION 2 *Let $\mathcal{N}_1 \sim_{\mathcal{P}} \mathcal{N}_2$ and $[\pi_1, \dots, \pi_{n+1}] = [1, \dots, n+1]\mathcal{P}$. Then for all vertex sets $\alpha$, $\beta$, $\widehat{\alpha} = \{\pi_{i_1}, \dots, \pi_{i_r}\}$, $\widehat{\beta} = \{\pi_{j_1}, \dots, \pi_{j_r}\}$ we have $\Delta_1^{\alpha\beta} = \Delta_2^{\widehat{\alpha}\widehat{\beta}}$.*

*Proof.* For $\overline{\alpha} := V \setminus \alpha$, and $I_\alpha := [e_{i_1}, \dots, e_{i_r}] \in \mathbb{R}^{(n+1)\times r}$ one gets

$$P_1^{\alpha\beta} = I_{\overline{\alpha}}^T P_1 I_{\overline{\beta}} = I_{\overline{\alpha}}^T \mathcal{P}^T P_2 \mathcal{P} I_{\overline{\beta}} = I_{\overline{\widehat{\alpha}}}^T P_2 I_{\overline{\widehat{\beta}}} = P_2^{\widehat{\alpha}\widehat{\beta}}.$$

□

In order to express the determinant of $\mathcal{N}$ through the determinants of its subnetworks $\mathcal{N}_\vartheta$, cofactors of the form $\Delta_\vartheta^\alpha$ are suitable.

Proposition 3 *Suppose that $\mathcal{N}$ is generated by identification of the ordered vertex sets $\alpha$ of $\mathcal{N}_1$ and $\beta$ of $\mathcal{N}_2$, and let the identified vertices $(i_\ell, j_\ell)$ be denoted by $\kappa_\ell$:*

$$
\begin{array}{ccccc}
 & \overline{\phantom{xxxx}} & (\kappa_1) & \overline{\phantom{xxxx}} & \\
 & | & \vdots & | & \\
\mathcal{N}: & \mathcal{N}_1 \,\_\_\_\_ & (\kappa_\ell) & \_\_\_\_\, \mathcal{N}_2 & . \\
 & | & \vdots & | & \\
 & \overline{\phantom{xxxx}} & (\kappa_r) & \overline{\phantom{xxxx}} &
\end{array}
$$

*Then for $r = 1$, the equation $\Delta = \Delta_1\Delta_2$, and for $r = 2$ the equation $\Delta = \Delta_1\Delta_2^\beta + \Delta_1^\alpha\Delta_2$ holds true.*

*Proof.* Due to Proposition 2 w.l.o.g. $\{i_\ell, j_\ell\} = \{n_1 - r + \ell, \ell\}$, $\ell = 1, \ldots, r$, can be assumed, where $n_\vartheta$ denotes the total vertex number of $\mathcal{N}_\vartheta$. Then with respect to (9) the corresponding matrix polynomials $P, P_1, P_2$ are related to each other according to

$$P = \mathrm{diag}(P_1, O_{n_2-r}) + \mathrm{diag}(O_{n_1-r}, P_2),$$

that means $P$ has the form of a quasi block diagonal matrix consisting of two blocks, which are overlapping in a $(r \times r)$ area. Immediately the equation

$$P^{\{n_1\}} = \mathrm{diag}(P_1^{\{n_1\}}, O_{n_2-r}) + \mathrm{diag}(O_{n_1-r}, P_2^{\{r\}}),$$

follows, that means for $r = 1$, $P^{\{n_1\}}$ is block diagonal and hence $|P^{\{n_1\}}| = |P_1^{\{n_1\}}||P_2^{\{r\}}|$. With the equicofactor property of all involved matrix polynomials for $r = 1$ the statement follows. To prove it for $r = 2$, one develops $|P^{\{n_1\}}|$ into a sum of products, such that every factor depends only on $\mathcal{N}_1$ or only on $\mathcal{N}_2$:

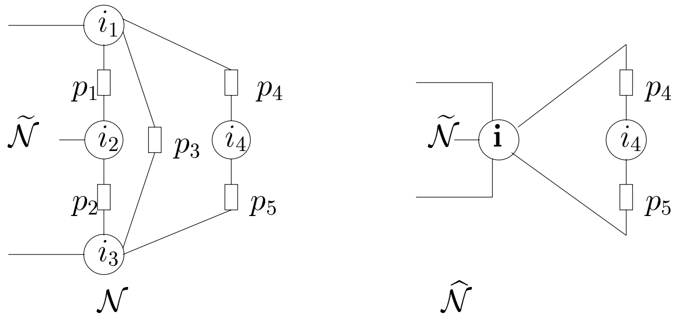
$$|P^{\{n_1\}}| = \begin{vmatrix} A & v & \\ z & a+b & x \\ & w & B \end{vmatrix} = \begin{vmatrix} A & v \\ z & a \end{vmatrix} |B| + |A| \begin{vmatrix} b & x \\ w & B \end{vmatrix}.$$

With the denotations introduced in Definition 1, the equation

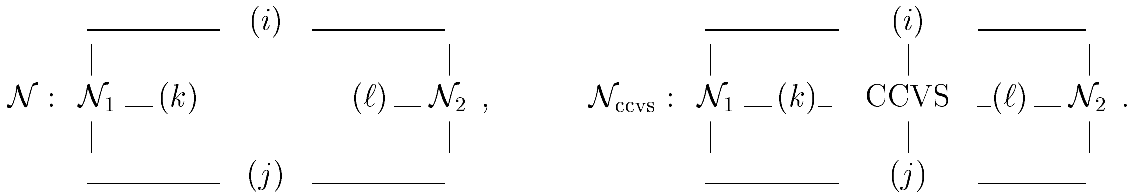
$$|P^{\{n_1\}}| = |P_1^{\{n_1\}}||P_2^{\{1,2\}}| + |P_1^{\{n_1-1,n_1\}}||P_2^{\{2\}}|$$

follows. Referring to Proposition 1, the proof of the first statements becomes complete. □

Remark 1 Higher order cofactors $\Delta^\alpha$ of $\mathcal{N}$ admit the interpretation as the determinant of a restricted version of $\mathcal{N}$: if $\widehat{\mathcal{N}}$ is generated by identifying all vertices of $\alpha$ with the artificial vertex $\mathbf{i}$ and by deleting all edges $(i_\mu, i_\eta) \in \alpha^2$ arising in the edge set $E$ of $\mathcal{N}$, then $\Delta^\alpha = \widehat{\Delta}$. For example, for $\alpha = \{i_1, i_2, i_3\}$ one obtains




To illustrate the strength of the network determinant concept, we like to represent the determinant $\Delta_{\text{ccvs}}$ of a network $\mathcal{N}_{\text{ccvs}}$, which one obtains through connection of a structured network $\mathcal{N}$ with a *current controlled voltage source* (CCVS) in the vertex quadruple $\alpha := \{i, j\}, \beta := \{k, \ell\}$:




A mathematical model of such a circuit is given by

$$P_{\text{ext}}(s) \begin{bmatrix} \text{v}(s) \\ \text{i}(s) \end{bmatrix} = \begin{bmatrix} e_\eta - e_\mu \\ 0 \end{bmatrix} \text{I}(s), \quad P_{\text{ext}}(s) = \left[ \begin{array}{c|c} P(s) & e_k - e_\ell \\ \hline e_i^T - e_j^T & r \end{array} \right]. \tag{10}$$

Here, $P(s)$ denotes the matrix polynomial generated through $\mathcal{N}$, $\text{v}(s)$ the node potentials, $\text{i}(s)$ the current flowing through the CCVS from node $k$ to node $\ell$, $\text{I}(s)$ the current, which is injected from outside into the node $\mu$, and is extracted from the node $\eta$, and $r \in \mathbb{R}$ the resistance of the CCVS. By consideration of the last line of (10), obviously the potential difference $v_i(s) - v_j(s)$ is equal to the product $r\text{i}(s)$. Now we are able to represent the eigenfrequencies of such a circuit as the zeros of a linear combination of the determinant of $\mathcal{N}$ and one of its second order cofactors.

LEMMA 1 *Let $P$ and $P_{\text{ext}}$ be defined as in* (9) *and* (10). *Then $\Delta_{\text{ccvs}} = r\Delta - d$ holds true, where $\Delta$ denotes the determinant of $\mathcal{N}$, and $d$ its second order cofactor $(-1)^{\alpha+\beta}\Delta^{\alpha\beta}$.*

*Proof.* Using appropriate column and row permutations, $\alpha = \beta = \{1, n+1\}$ can be achieved. Then $P$ is of the form as in Proposition 1. If one defines for $* \in \{L, R\}$ the all vertex matrix $A_*$ according to

$$A_* = \left[ \begin{array}{c|c|c|c} \mathcal{A}_* & e_1 - e_{n+1} & e_1 - e_{n+2} & e_{n+1} - e_{n+2} \\ o^T & & & \end{array} \right] \in \mathbb{R}^{(n+2)\times(m+3)},$$

then $P_{\text{ext}} = U_{n+2} Q U_{n+2}^T$, where $Q = A_L \text{diag}(p_1, \ldots, p_m, -1, r-1, 1) A_R^T$, $U_n = I_n - e_1 e_n^T$.

Hence, $P_{\text{ext}}^{\{\kappa\}} = U_{n+2,\{\kappa\}} Q U_{n+2,\{\kappa\}}^T$, where $U_{\{\kappa\}}$ is obtained by deleting of row $\kappa$ in $U$. Now, for $(\kappa, *) \in \{2, \ldots, n+1\} \times \{L, R\}$, the equation $U_{n+1,\{\kappa\}} A_* = U_{n+1} A_{*,\{\kappa\}}$ is evident, consequently $P_{\text{ext}}^{\{\kappa\}} = U_{n+1} Q^{\{\kappa\}} U_{n+1}^T$, that means $P_{\text{ext}}^{\{\kappa\}} \sim Q^{\{\kappa\}}$.

$A_L$ and $A_R$ are all vertex matrices, therefore $Q$ fulfils the assumption of Proposition 1. Their application yields $Q^{\{\kappa\}} \sim Q^{\{\xi\}}$ for $\kappa, \xi \in V \setminus \{1\}$, what implies $P_{\text{ext}}^{\{\kappa\}} \sim P_{\text{ext}}^{\{\xi\}}$ for $\kappa, \xi \in V \setminus \{1\}$.

In particular $|P_{\text{ext}}^{\{n+1\}}| = r\Delta - |P^{\{1,n+1\}}|$ is valid. Finally, for $\kappa = 1$ one obtains

$$|P_{\text{ext}}^{\{1\}}| = \det \left[ \begin{array}{c|c} P^{\{1\}} & -e_{n+1} \\ \hline -e_{n+1}^T & r \end{array} \right] = r\Delta - |P^{\{1\}}| e_{n+1}^T (P^{\{1\}})^{-1} e_{n+1} = r\Delta - |P^{\{1,n+1\}}|.$$

□

Now we exploit the structure of $\mathcal{N}$ to express $\Delta_{\text{ccvs}}$ through appropriate cofactors of $\mathcal{N}_\vartheta$. To do this we assume that $\mathcal{N}_\vartheta$ consists of $n_\vartheta$ vertices, $\alpha$ is formed through identification of the vertex pair $\{n_1 - 1, n_1\}$ of $\mathcal{N}_1$ with the vertex pair $\{1, 2\}$ of $\mathcal{N}_2$, and $\beta$ coincide with vertex 1 of $\mathcal{N}_1$ and vertex $n_2$ of $\mathcal{N}_2$:

$$\mathcal{N}: \begin{array}{ccccc} \underline{\qquad\quad} & & (n_1-1, 1) & & \underline{\qquad\quad} \\ | & & & & | \\ \mathcal{N}_1 \,\_\,(1) & & & & (n_2)\,\_\,\mathcal{N}_2 \\ | & & & & | \\ \underline{\qquad\quad} & & (n_1, \qquad 2) & & \underline{\qquad\quad} \end{array},$$

| | $\mathcal{N}_1$ | $\mathcal{N}_2$ | $\mathcal{N}$ |
|---|---|---|---|
| $k$ | 1 | | 1 |
| $\ell$ | | $n_2$ | $n$ |
| $i$ | $n_1 - 1$ | 1 | $n_1 - 1$ |
| $j$ | $n_1$ | 2 | $n_1$ |

(11)

Obviously, the total vertex number $n$ of $\mathcal{N}$ amounts $n_1 + n_2 - 2$. Table (11) shows the enumeration of $k, \ell, i, j$ with respect to $\mathcal{N}_1$, $\mathcal{N}_2$, and $\mathcal{N}$. Then combination of Proposition 3 and Lemma 1 yields with $\beta_1 := \{k, j\}$, $\beta_2 := \{\ell, j\}$ for $\Delta_{\text{ccvs}}$ the representation

$$\Delta_{\text{ccvs}} = (r\Delta_1 - (-1)^{n_1} \Delta_1^{\alpha\beta_1}) \Delta_2^\alpha + (r\Delta_2 + (-1)^{n_2} \Delta_2^{\alpha\beta_2}) \Delta_1^\alpha. \tag{12}$$

*Proof.* Due to our enumeration assumption the matrix polynomials $P, P_\vartheta$ generated through $\mathcal{N}$, $\mathcal{N}_\vartheta$, satisfy $P = \text{diag}(P_1, O_{n_2-2}) + \text{diag}(O_{n_1-2}, P_2)$. According to the last column of table (11) the cofactor $d$ arising in Lemma 1 fulfils $d = (-1)^n \Delta^{\{1,n\},\{n_1-1,n_1\}}$. Now, the quasi block diagonal structure of $P$ yields

$$d = (-1)^n (\Delta_1^{\{1,n_1\}\{n_1-1,n_1\}} \Delta_2^{\{1,n_2\}\{1,2\}} - \Delta_1^{\{1,n_1-1\}\{n_1-1,n_1\}} \Delta_2^{\{2,n_2\}\{1,2\}}). \tag{13}$$

In general, the equations

$$\begin{aligned} \Delta_1^{\{1,n_1-1\},\{n_1-1,n_1\}} &= (-1)^{n_1} \Delta_1^{\{n_1-1,n_1\}} - \Delta_1^{\{1,n_1\}\{n_1-1,n_1\}}, \\ \Delta_2^{\{1,n_2\}\{1,2\}} &= (-1)^{n_2} \Delta_2^{\{1,2\}} - \Delta_2^{\{2,n_2\}\{1,2\}} \end{aligned} \tag{14}$$

hold true. Replacement of $\Delta_1^{\{1,n_1-1\},\{n_1-1,n_1\}}$ and $\Delta_2^{\{1,n_2\}\{1,2\}}$ in (13) through the right hand sides of (14) yields

$$d = (-1)^{n_1} \Delta_1^{\{1,n_1\}\{n_1-1,n_1\}} \Delta_2^{\{1,2\}} - (-1)^{n_2} \Delta_2^{\{2,n_2\}\{1,2\}} \Delta_1^{\{n_1-1,n_1\}}.$$

Finally with Proposition 3 we get $\Delta_{\text{ccvs}} = r\Delta_1 \Delta_2^{\{1,2\}} + r\Delta_2 \Delta_1^{\{n_1-1,n_1\}} - d$, and therefore (12). □

## 3 Determinants of structured networks

We say that $\mathcal{N}$ possesses a structure, if it is generated by connection of similar or dual networks or if it is symmetric or cyclic. In the following we show, how such structural properties of the network lead to corresponding factorization properties of its determinantal polynomial $\Delta$. For symmetrical networks a surprising result is obtained, namely $\sigma(\mathcal{N})$ contains elements, which are independent on the parameters located on the symmetry axis.

DEFINITION 3 Let $\mathcal{N}$ be generated by identification of the ordered vertex sets $\alpha$ and $\beta$ of $\mathcal{N}_1$ and $\mathcal{N}_2$. Suppose that $\mathcal{N}_1 \sim_{\mathcal{P}} \mathcal{N}_2$, and let $\widehat{\beta} = \{\pi_{j_1}, \ldots, \pi_{j_r}\}$, where $[\pi_1, \ldots, \pi_{n+1}] = [1, \ldots, n+1]\mathcal{P}$. If $\alpha = \widehat{\beta}$, then $\alpha$ is said to be a *symmetry axis* of $\mathcal{N}$, and $\mathcal{N}$ is said to be *symmetric*.

LEMMA 2 *Let $\mathcal{N}$ be generated by identification of the ordered vertex sets $\alpha$ and $\beta$ of $\mathcal{N}_1$ and $\mathcal{N}_2$, and suppose that $\mathcal{N}_1 \sim_{\mathcal{P}} \mathcal{N}_2$. Then the determinant $\Delta$ of $\mathcal{N}$ admits for $r = 1$ the representation $\Delta = \Delta_1^2$, and for $r = 2$ the representation $\Delta = \Delta_1(\Delta_1^\alpha + \Delta_1^{\widehat{\beta}})$, where $\widehat{\beta}$ is defined as above. In the case, that $\mathcal{N}$ is generated by reflection of $\mathcal{N}_1$ along its vertex set $\alpha$, we have $\Delta = 2^{r-1}\Delta_1\Delta_1^\alpha$.*

*Proof.* Due to $\mathcal{N}_1 \sim_{\mathcal{P}} \mathcal{N}_2$, with respect to Proposition 2, the equations $\Delta_1 = \Delta_2$ and $\Delta_1^{\widehat{\beta}} = \Delta_2^\beta$ hold. Hence,

$$\Delta = \Delta_1\Delta_2^\beta + \Delta_1^\alpha\Delta_2 = \Delta_1(\Delta_1^\alpha + \Delta_2^\beta) = \Delta_1(\Delta_1^\alpha + \Delta_1^{\widehat{\beta}}).$$

If $\mathcal{N}$ is generated by reflection of $\mathcal{N}_1$ along its vertex set $\alpha$, then $\alpha = \widehat{\beta}$ and for $r = 2$ the last equations imply $\Delta = 2\Delta_1\Delta_1^\alpha$, that means for $r \leq 2$ the second statement is true. To prove it in the general case, the following fact from linear algebra can be applied: let $A \in \mathbb{C}^{s\times s}$, $B \in \mathbb{C}^{s\times r}$, $C \in \mathbb{C}^{r\times r}$, $D \in \mathbb{C}^{r\times r}$. Then

$$\begin{vmatrix} A & B & \\ C & 2D & C \\ & B & A \end{vmatrix} = 2^r \begin{vmatrix} A & B \\ C & D \end{vmatrix} |A|. \tag{15}$$

Now, let $\alpha = \{n-(r-1), \ldots, n\}$, where $n$ denotes the total vertex number of $\mathcal{N}_1$. Then the matrix polynomials $P$ and $Q$ generated through $\mathcal{N}$ and $\mathcal{N}_1$ admit the partitions

$$P = \begin{bmatrix} Q^\alpha & W & \\ W^T & 2Q^{\overline{\alpha}} & W^T \\ & W & Q^\alpha \end{bmatrix}, \quad Q = \begin{bmatrix} Q^\alpha & W \\ W^T & Q^{\overline{\alpha}} \end{bmatrix}, \quad \overline{\alpha} = \{1, \ldots, s\}, \quad W = [w_1, \ldots, w_r].$$

Using formula (15) and the equation $\widetilde{W} = [w_1, \ldots, w_{r-1}]$ one obtains

$$|P^{\{n\}}| = \begin{vmatrix} Q^\alpha & \widetilde{W} & \\ \widetilde{W}^T & 2Q^{\overline{\alpha}\cup\{n\}} & \widetilde{W}^T \\ & \widetilde{W} & Q^\alpha \end{vmatrix} = 2^{r-1} \begin{vmatrix} Q^\alpha & \widetilde{W} \\ \widetilde{W}^T & Q^{\overline{\alpha}\cup\{n\}} \end{vmatrix} |Q^\alpha| = 2^{r-1}\,|Q^{\{n\}}|\,|Q^\alpha|.$$

Together with Proposition 1, the second statement follows. □

Remark 2 Since the determinant of a symmetrical network $\mathcal{N}$ can be factorized according to $\Delta = 2^{\#\alpha-1}\Delta_1\Delta_1^\alpha$, the zero set $\sigma(\mathcal{N})$ can be decomposed as $\sigma(\mathcal{N}) = \sigma(\Delta_1)\cup\sigma(\Delta_1^\alpha)$. Therefore with respect to Remark 1, the existence of elements in $\sigma(\mathcal{N})$, which are independent on the weights distributed on the symmetry axis $\alpha$, follows.

Now we consider cyclical networks.

Definition 4 The network $\mathcal{N}$ is said to be *cyclic*, if it is of the form

$$(\kappa_1) - \mathcal{N}_1 - (\kappa_2) - \mathcal{N}_2 - (\kappa_3) - \ldots - (\kappa_{n-1}) - \mathcal{N}_{n-1} - (\kappa_n) - \mathcal{N}_n - (\kappa_1), \tag{16}$$

where $\mathcal{N}_\vartheta$ represents a network in the sense of Definition 1.

Lemma 3 *Let $\mathcal{N}$ be cyclic. Then* $\Delta = \sum_{i=1}^n \Delta_i^{\{\kappa_i,\kappa_{i+1}\}} \prod_{j=1,j\neq i}^n \Delta_j$, *where* $\kappa_{n+1} := \kappa_1$.

*Proof.* Since $\mathcal{N}$ is cyclic, its determinant admits the representation

$$\Delta = \Delta_n^{\{\kappa_n,\kappa_1\}} \prod_{i=1}^{n-1} \Delta_i + \widetilde{\Delta}\Delta_n, \tag{17}$$

where $\widetilde{\Delta}$ denotes the determinant of the shortened network

$$\widetilde{\mathcal{N}}:\ (\kappa_1) - \mathcal{N}_1 - (\kappa_2) - \mathcal{N}_2 - (\kappa_3) - \mathcal{N}_3 - (\kappa_4) - \ldots - (\kappa_{n-1}) - \mathcal{N}_{n-1} - (\kappa_1),$$

and the vertex $\kappa_n$ of $\mathcal{N}_{n-1}$ is identified with the vertex $\kappa_1$ of $\mathcal{N}_1$. Namely, if one sets

$$\widehat{\mathcal{N}}:\ \mathcal{N}_1 - (\kappa_2) - \mathcal{N}_2 - (\kappa_3) - \mathcal{N}_3 - (\kappa_4) \quad \ldots \quad (\kappa_{n-1}) - \mathcal{N}_{n-1},$$

then $\mathcal{N}$ is generated by connection of $\mathcal{N}_n$ with $\widehat{\mathcal{N}}$ in the vertex $\kappa_1$ of $\mathcal{N}_1$ and the vertex $\kappa_n$ of $\mathcal{N}_{n-1}$, and Proposition 3 implies $\Delta = \Delta_n^{\{\kappa_1,\kappa_n\}}\widehat{\Delta} + \widehat{\Delta}^{\{\kappa_1,\kappa_n\}}\Delta_n$.

Since $\widehat{\mathcal{N}}$ is generated by single vertex connections of $\mathcal{N}_1,\ldots,\mathcal{N}_{n-1}$, its determinant fulfils $\widehat{\Delta} = \prod_{i=1}^{n-1}\Delta_i$. With respect to Remark 1 and to the construction of $\widetilde{\mathcal{N}}$, we have $\widehat{\Delta}^{\{\kappa_1,\kappa_n\}} = \widetilde{\Delta}$. By induction starting from (17), the statement of the lemma follows. □

To illustrate the use of our representation formulas, we apply our results to *Darlington networks*, which are obtained connecting three networks $\mathcal{N}_\vartheta$ in the following manner

$$\mathcal{N}:\quad \begin{array}{c}(\kappa_1) - \mathcal{N}_1 - (\kappa_2) - \mathcal{N}_2 - (\kappa_3) - \mathcal{N}_1 - (\kappa_4) - \mathcal{N}_2 - (\kappa_1)\\ (\kappa_2) \,\rule{1.5cm}{0.4pt}\, \mathcal{N}_3 \,\rule{1.5cm}{0.4pt}\, (\kappa_4)\end{array}. \tag{18}$$

Obviously, $\mathcal{N}$ is a periodic cyclic network with a secant between $\kappa_2$ and $\kappa_4$. Note however that figure (18) does not uniquely define $\mathcal{N}$. Additionally we have to specify which vertices of $\mathcal{N}_\vartheta$ are used to generate $\kappa_\xi$. To fill this gap, let $\kappa_\xi$ be formed by identification of $\alpha := \{i_1, i_2\}$, $\beta := \{j_1, j_2\}$, $\gamma := \{k_1, k_2\}$ according to the table

| | $\mathcal{N}_1$ | $\mathcal{N}_2$ | $\mathcal{N}_1$ | $\mathcal{N}_2$ | $\mathcal{N}_3$ |
|---|---|---|---|---|---|
| $\kappa_1$ | $i_1$ | | | $j_1$ | |
| $\kappa_2$ | $i_2$ | $j_1$ | | | $k_1$ |
| $\kappa_3$ | | $j_2$ | $i_1$ | | |
| $\kappa_4$ | | | $i_2$ | $j_2$ | $k_2$ |

.

For example, $\kappa_4$ is formed by identification of $i_2, j_2, k_2$ of $\mathcal{N}_1$, $\mathcal{N}_2$, $\mathcal{N}_3$. A detailed description of the meaning of Darlington networks can be found in [[5], section 7]. Now, suppose that $\mathcal{N}_1$ and $\mathcal{N}_3$ realize the rational functions $f$ and $g$ in $\alpha$ and $\gamma$, respectively. In the case that $\mathcal{N}_2$ realizes $g^2/f$ in $\beta$, with the preceding results the equation

$$\Delta = (\Delta_1\Delta_2)^2\Delta_3(f^2 + g^2)(f + g)^2 f^{-2} \tag{19}$$

for the determinant $\Delta$ of $\mathcal{N}$ can be proved. Obviously, if $g$ is a constant, then $\mathcal{N}_1$ and $\mathcal{N}_2$ are dual to each other.

*Proof* of formula (19). Let $\mathcal{N}_{\text{cycl}}$ be defined by the cyclical component of $\mathcal{N}$:

$$\mathcal{N}_{\text{cycl}}: \ (\kappa_1) - \mathcal{N}_1 - (\kappa_2) - \mathcal{N}_2 - (\kappa_3) - \mathcal{N}_1 - (\kappa_4) - \mathcal{N}_2 - (\kappa_1).$$

Then with respect to Proposition 3 the equation

$$\Delta = \Delta_{\text{cycl}}\Delta_3^{\{\kappa_2,\kappa_4\}} + \Delta_{\text{cycl}}^{\{\kappa_2,\kappa_4\}}\Delta_3 \tag{20}$$

holds true. Corresponding to Lemma 3, the determinant $\Delta_{\text{cycl}}$ satisfies

$$\Delta_{\text{cycl}} = \Delta_1\Delta_2(\Delta_1^{\{\kappa_1,\kappa_2\}}\Delta_2 + \Delta_1\Delta_2^{\{\kappa_2,\kappa_3\}} + \Delta_2\Delta_1^{\{\kappa_3,\kappa_4\}} + \Delta_1\Delta_2^{\{\kappa_4,\kappa_1\}}).$$

Replacement of $\kappa_i$ according to the identification table yields

$$\Delta_{\text{cycl}} = 2\Delta_1\Delta_2\widetilde{\Delta}, \quad \widetilde{\Delta} := \Delta_1^\alpha\Delta_2 + \Delta_1\Delta_2^\beta. \tag{21}$$

The use of Proposition 3 backwards admits the interpretation of $\widetilde{\Delta}$ as the determinant of

$$\widetilde{\mathcal{N}}: \quad \begin{array}{ccc} \underline{\qquad\quad} & (i_1, j_1) & \underline{\qquad\quad} \\ | & & | \\ \mathcal{N}_1 & & \mathcal{N}_2 \\ | & & | \\ \underline{\qquad\quad} & (i_2, j_2) & \underline{\qquad\quad} \end{array}$$

By identification of $\kappa_2$ with $\kappa_4$, and referring to Remark 1, the coincidence of $\Delta_{\text{cycl}}^{\{\kappa_2,\kappa_4\}}$ with the determinant $\widehat{\Delta}$ of

$$\widehat{\mathcal{N}}: \quad \begin{array}{l} (\kappa_1) - \mathcal{N}_1 - (\kappa_2) - \mathcal{N}_2 - (\kappa_3) \\ (\kappa_1) - \mathcal{N}_2 - (\kappa_2) - \mathcal{N}_1 - (\kappa_3) \end{array}$$

is obtained. Obviously, $\widehat{\mathcal{N}}$ falls into two networks

$$\begin{array}{c}(\kappa_1) - \mathcal{N}_1 - (\kappa_2) \\ (\kappa_1) - \mathcal{N}_2 - (\kappa_2)\end{array}, \qquad \begin{array}{c}(\kappa_2) - \mathcal{N}_2 - (\kappa_3) \\ (\kappa_2) - \mathcal{N}_1 - (\kappa_3)\end{array},$$

which are connected in the single vertex $\kappa_2$. Again with Proposition 3 we get

$$\widehat{\Delta} = (\Delta_1^{\{\kappa_1,\kappa_2\}}\Delta_2 + \Delta_1\Delta_2^{\{\kappa_1,\kappa_2\}})(\Delta_1^{\{\kappa_2,\kappa_3\}}\Delta_2 + \Delta_1\Delta_2^{\{\kappa_2,\kappa_3\}}).$$

Replacement of $\kappa_i$ according to the identification table yields

$$\Delta_{\text{cycl}}^{\{\kappa_2,\kappa_4\}} = \widehat{\Delta} = \widetilde{\Delta}^2. \tag{22}$$

If one replaces in (20) the quantities $\Delta_{\text{cycl}}$ and $\Delta_{\text{cycl}}^{\{\kappa_2,\kappa_4\}}$ through the right hand sides of (21) and (22), and $\{\kappa_2, \kappa_4\}$ according to the identification table through $\gamma$, then one gets

$$\Delta = 2\Delta_1\Delta_2\widetilde{\Delta}\Delta_3^\gamma + \widetilde{\Delta}^2\Delta_3 = \widetilde{\Delta}(2\Delta_1\Delta_2\Delta_3^\gamma + \widetilde{\Delta}\Delta_3).$$

(If one replaces $\Delta_3$ by $\Delta_5$ and defines $\widehat{\Delta}_1 := \widetilde{\Delta}$, $\widehat{\Delta}_2 := \Delta_3^\gamma$, then formula (4) arising in the introduction is obtained. With respect to Remark 1, $\widehat{\Delta}_2$ can be interpreted as the determinant of a network.)

Our realization assumption concerning $\mathcal{N}_3$ means $g = \Delta_3^\gamma/\Delta_3$. Therefore, $\Delta$ admits the factorization

$$\Delta = \widetilde{\Delta}(2\Delta_1\Delta_2 g + \widetilde{\Delta})\Delta_3. \tag{23}$$

Our realization assumptions concerning $\mathcal{N}_1$ and $\mathcal{N}_2$ yields $\widetilde{\Delta} = \Delta_1\Delta_2(f^2 + g^2)f^{-1}$, that means

$$2\Delta_1\Delta_2 g + \widetilde{\Delta} = 2\Delta_1\Delta_2 g + \Delta_1\Delta_2(f^2 + g^2)f^{-1} = \Delta_1\Delta_2(f + g)^2 f^{-1}.$$

If one inserts these equations in (23), finally formula (19) is obtained.

# 4 Disturbance of $\sigma(\mathcal{N})$ due to vertex identification

The question arises, what is the distance $\epsilon$ between $\sigma(\mathcal{N})$ and $\sigma(\widehat{\mathcal{N}})$, if $\widehat{\mathcal{N}}$ is generated by identification of two vertices of $\mathcal{N}$, and $\epsilon$ is defined through (5). Let $P$ be the matrix polynomial generated by $\mathcal{N}$ consisting of $n + 1$ vertices. Due to Proposition 1 we know that for two different vertices $i$ and $j$ the matrix polynomials $P^{\{i\}}$ and $P^{\{j\}}$ are equivalent. Hence we suppress the superscript $\{*\}$ and abbreviate $P^{\{*\}}$ by $L$. Now, if $L^{-1}$ is represented as

$$L^{-1}(s) = Q(s) + R(s),\ R(s) = C(sI - A)^{-1}B, \quad Q \in \mathbb{R}^{n\times n}[s],\ A \in \mathbb{R}^{\mathbf{d}\times\mathbf{d}},\ B^T, C \in \mathbb{R}^{n\times\mathbf{d}},$$

with minimal state space dimension $\mathbf{d}$, then $\sigma(\mathcal{N}) = \sigma(A)$. The arising matrix triple $\Sigma := (A, B, C)$ is said to be a *minimal realization* of $R$. Note, that $\Sigma$ is not uniquely determined by $R$, but for two different minimal realizations $\Sigma_\vartheta$ of the same rational matrix function there exists a regular matrix $T$ with $A_1 = T^{-1}A_2T,\ B_1 = T^{-1}B_2,\ C_1 = C_2T$. For abbreviation we set

$$\mathcal{M}(\mathcal{N}) := \mathcal{M}(L) := \{A \in \mathbb{R}^{\mathbf{d}\times\mathbf{d}} | \exists C, B^T \in \mathbb{R}^{n\times\mathbf{d}} : R(s) = C(sI_{\mathbf{d}} - A)^{-1}B\}.$$

As a consequence of Proposition 1 we get the similarity of all elements of $\mathcal{M}(\mathcal{N})$. In [1] for arbitrary nonsingular matrix polynomials $L$ an iterative algorithm is provided to compute a representative of $\mathcal{M}(L)$. Now let $A \in \mathcal{M}(\mathcal{N})$ and $\widehat{A} \in \mathcal{M}(\widehat{\mathcal{N}})$ be of the same size. Then the deviations $\epsilon_\lambda$ and $\epsilon_\mu$ defined by (5) can be estimated by the Theorem of Bauer-Fike [[2], Theorem 7.2.2, p.342].

Theorem 1 [Bauer-Fike] *If $\mu$ is an eigenvalue of $\widehat{A}$ and $A = T\mathrm{diag}(\lambda_k)_{k=1}^{\mathbf{d}}T^{-1}$, then*

$$\min\{|\mu - \lambda_k| \,|\, k = 1, \ldots, \mathbf{d}\} \le \|T^{-1}(A - \widehat{A})T\|_p,$$

*where $\|.\|_p$ denotes any of the p-norms.*

Corollary 1 *The deviations $\epsilon_\lambda$ and $\epsilon_\mu$ satisfy the inequalities $\epsilon_\lambda \le \|\widehat{T}^{-1}(A - \widehat{A})\widehat{T}\|_p$, and $\epsilon_\mu \le \|T^{-1}(A - \widehat{A})T\|_p$, where $A = T\mathrm{diag}(\lambda_k)_{k=1}^{\mathbf{d}}T^{-1}$, and $\widehat{A} = \widehat{T}\mathrm{diag}(\mu_k)_{k=1}^{\mathbf{d}}\widehat{T}^{-1}$.*

Proposition 3 gives a hint, which vertices of $\mathcal{N}$ are convenient for identification to get a network $\widehat{\mathcal{N}}$ with factorizable structure. In the next lemma the identification process is realized by border crossing.

Lemma 4 *Let $\mathcal{N}$ be of the form*

$$\mathcal{N}(\delta):\ \begin{array}{ccccc} \overline{\quad\quad} & & (i_1,i_2) & & \overline{\quad\quad} \\ | & & | & & | \\ \mathcal{N}_1 & & p\delta & & \mathcal{N}_2 \\ | & & | & & | \\ \overline{\quad\quad} & & (j_1,j_2) & & \overline{\quad\quad} \end{array}\ ,\ \delta \in \mathbb{R},\ p \in \mathbb{R}[s],$$

*and $\widehat{\mathcal{N}}_\vartheta$ be generated from $\mathcal{N}_\vartheta$ by identification of $i_\vartheta$ with $j_\vartheta$ generating the artificial vertex* $\mathbf{i}$. *Then* $\lim_{\delta\to\infty}\sigma(\mathcal{N}(\delta)) = \sigma(\widehat{\mathcal{N}}_1) \cup \sigma(\widehat{\mathcal{N}}_2) \cup \sigma(p)$. *In addition, if one sets*

$$\widehat{\mathcal{N}}:\quad \begin{array}{c} \widehat{\mathcal{N}}_1 - (\mathbf{i}) - \widehat{\mathcal{N}}_2 \\ | \\ p - (\mathbf{j}) \end{array} \tag{24}$$

*where $\mathbf{j}$ denotes a new vertex, then* $\lim_{\delta\to\infty}\sigma(\mathcal{N}(\delta)) = \sigma(\widehat{\mathcal{N}})$.

*Proof.* Application of Proposition 3 yields $\Delta = \widetilde{\Delta} + \widetilde{\Delta}^{\{\kappa_1,\kappa_2\}} p\delta$, where $\widetilde{\Delta}$ denotes the determinant of

$$\widetilde{\mathcal{N}}:\ \begin{array}{ccc} \overline{\quad\quad} & (\kappa_1) & \overline{\quad\quad} \\ | & & | \\ \mathcal{N}_1 & & \mathcal{N}_2 \\ | & & | \\ \overline{\quad\quad} & (\kappa_2) & \overline{\quad\quad} \end{array},\ \kappa_1 = (i_1, i_2),\ \kappa_2 = (j_1, j_2).$$

Corresponding to Remark 1, $\widetilde{\Delta}^{\{\kappa_1,\kappa_2\}}$ coincides with the determinant of $\widehat{\mathcal{N}}_1 - (\mathbf{i}) - \widehat{\mathcal{N}}_2$. Again with Proposition 3 we get $\widetilde{\Delta}^{\{\kappa_1,\kappa_2\}} = \widehat{\Delta}_1\widehat{\Delta}_2$, and hence $\Delta = (\widetilde{\Delta}\delta^{-1} + \widehat{\Delta}_1\widehat{\Delta}_2 p)\delta$, that means the first statement holds true. The use of Proposition 3 backwards, yields the interpretation of $\widehat{\Delta}_1\widehat{\Delta}_2 p$ as the determinant of network (24). $\square$

To get an upper bound for the distance $\epsilon$ between $\sigma(\mathcal{N})$ and $\sigma(\widehat{\mathcal{N}})$, now $A \in \mathcal{M}(\mathcal{N})$ is constructed exploiting the all vertex matrix $\mathcal{A}$ of $\mathcal{G}$, where $\mathcal{G} := (V, E)$ represents the underlying graph of $\mathcal{N}$. An element $\widehat{A}$ of $\mathcal{M}(\widehat{\mathcal{N}})$ is obtained by border crossing within $A$. To do this, we introduce the *rank defect numbers* of a matrix polynomial $Q(s) =: \sum_{\ell=0}^{\nu} Q_\ell s^{\nu-\ell} \in \mathbb{R}^{n\times n}[s]$ according to

$$r_0 = \text{rank}\ Q_0,\ r_\ell = \text{rank}\ [Q_0, \ldots, Q_\ell] - \text{rank}\ [Q_0, \ldots, Q_{\ell-1}],\ \ell = 1, \ldots, \nu.$$

In particular, for nonsingular $Q$, the equation $r_0 + \ldots + r_\nu = n$ holds true.

Since the matrix polynomial $L(s) =: \sum_{\ell=0}^{\nu} L_\ell s^{\nu-\ell}$ is generated through the restriction of a matrix polynomial (9), its coefficients admit factorizations of the form

$$L_\ell = \mathcal{A}_\ell\, \text{diag}(\hat{q}_\ell)\, \mathcal{A}_\ell^T, \quad \mathcal{A}_\ell = [v_1^\ell, \ldots, v_{m_\ell}^\ell],\ q = [\hat{q}_0, \ldots, \hat{q}_\nu]^T \in \mathbb{R}^\varrho,$$

where $\mathcal{A}_\ell$ represents an appropriate section of the restricted all vertex matrix $\mathcal{A}_{\{*\}}$. Due to the definition of $r_\ell$, w.o.l.g. the rank equations

$$\text{rank}\,[\mathcal{A}_0, \ldots, \mathcal{A}_\ell] = \text{rank}\,[v_1^0, \ldots, v_{r_0}^0|\ldots|v_1^\ell, \ldots, v_{r_\ell}^\ell],\ \ell = 0, \ldots, \nu,$$

can be assumed. The $(n \times n)$ matrix $\mathcal{A}_{\text{ext}} := [v_1^0, \ldots, v_{r_0}^0|\ldots|v_1^\nu, \ldots, v_{r_\nu}^\nu]$ is said to be a *canonical extension* of $\mathcal{A}_0$. In particular, $|L| \not\equiv 0$ implies $|\mathcal{A}_{\text{ext}}| \neq 0$.

In order to construct $A \in \mathcal{M}(\mathcal{N})$ we need the concept of a column reduced matrix polynomial $Q$, and its *Schur complement* $S_r(Q)$, which is defined as usual for $r = 0$ by $S_0(Q) = Q$, and for $r > 0$, and $|Q_{22}| \not\equiv 0$, where $Q = [Q_{ij}]_{ij=1}^2 \in \mathbb{R}^{n\times n}[s]$, $Q_{22} \in \mathbb{R}^{r\times r}[s]$, by the rational matrix function $S_r(Q) = Q_{11} - Q_{12}Q_{22}^{-1}Q_{21} \in \mathbb{R}^{(n-r)\times(n-r)}(s)$.

Definition 5 Let the nonsingular matrix polynomial $Q \in \mathbb{R}^{n\times n}[s]$ be represented in the form $Q(s) = Q_{\text{hc}}\,\text{diag}(s^{k_1}, \ldots, s^{k_n}) + [v_1(s), \ldots, v_n(s)]$, $Q_{\text{hc}} \in \mathbb{R}^{n\times n}$, $v_i \in \mathbb{R}^n[s]$, $\deg v_i(s) < k_i$. Then the integers $k_i$ are said to be the *column degrees*, and $Q_{\text{hc}}$ the *highest column degree coefficient matrix* of $Q$. The matrix polynomial $Q$ is said to be *column reduced* if $|Q_{\text{hc}}| \neq 0$. Since $|Q(s)| = |Q_{\text{hc}}|s^{k_1+\ldots+k_n} + \ldots + |Q(0)|$, $Q$ is column reduced if and only if $\deg|Q(s)| = k_1 + \ldots + k_n$. To our knowledge the first introduction of this concept is due to Wedderburn, [[7], chapter 4].

PROPOSITION 4 *Let* $L(s) = \sum_{\ell=0}^{\nu} \mathcal{A}_\ell \operatorname{diag}(\hat{q}_\ell)\, \mathcal{A}_\ell^T s^{\nu-\ell} \in \mathbb{R}^{n\times n}[s]$, $q = [\hat{q}_0^T, \dots, \hat{q}_\nu^T]^T \in \mathbb{R}^\varrho$, *be generated through a network* $\mathcal{N}$, *and* $\mathcal{A}_{\text{ext}}$ *be a canonical extension of* $\mathcal{A}_0$. *Then for almost every* $q$ *the matrix polynomial* $\widetilde{P} := \mathcal{A}_{\text{ext}}^{-1} L \mathcal{A}_{\text{ext}}^{-T}$ *is column reduced, its Schur complement* $\check{P} := S_{r_\nu}(\widetilde{P})$ *exists, and turns out to be a column reduced matrix polynomial as well.*

*Proof.* Since $\mathcal{A}_{\text{ext}}$ is constructed by appropriate chosen columns of the restricted all vertex matrix $\mathcal{A}_{\{*\}}$, the matrix polynomial $L$ admits the representation

$$L = \mathcal{A}_{\text{ext}} \operatorname{diag}(p_1^0, \dots, p_{r_0}^0, \dots, p_1^\nu, \dots, p_{r_\nu}^\nu) \mathcal{A}_{\text{ext}}^T + \dots,$$

where $\deg p_k^\ell = \nu - \ell$, $p_k^\ell \in w(E) \subset \mathbb{R}[s]$, and $r_0, \dots, r_\nu$ are the rank defect numbers of $L$. Hence, the inequality $\sum_{\ell=0}^{\nu-1} r_\ell(\nu-\ell) \le \deg|L| = \deg|\widetilde{P}|$ is obtained for almost every coefficient vector $q$. Due to the construction of $\mathcal{A}_{\text{ext}}$ we have

$$\widetilde{P}(s) = \operatorname{diag}(\mathcal{P}_0, O_{n-r_0}) s^\nu + \operatorname{diag}(\mathcal{P}_1, O_{n-r_0-r_1}) s^{\nu-1} + \dots + \operatorname{diag}(\mathcal{P}_{\nu-1}, O_{r_\nu}) s + \mathcal{P}_\nu, \quad (25)$$

for appropriately chosen matrices $\mathcal{P}_\ell \in \mathbb{R}^{(r_0+\dots+r_\ell)\times(r_0+\dots+r_\ell)}$, that means the column degrees $k_i$ of $\widetilde{P}$ satisfy

$$[k_1, \dots, k_n] = [\underbrace{\nu, \dots, \nu}_{r_0}, \underbrace{\nu-1, \dots, \nu-1}_{r_1}, \dots, \underbrace{1, \dots, 1}_{r_{\nu-1}}, \underbrace{0, \dots, 0}_{r_\nu}].$$

Therefore, $\deg|\widetilde{P}| \le \mathbf{d} := k_1 + \dots + k_n = \sum_{\ell=0}^{\nu-1} r_\ell(\nu-\ell)$. Combination with the first inequality yields $\deg|\widetilde{P}| = \mathbf{d}$ for almost every $q$, that means $\widetilde{P}$ is column reduced for almost every $q$.

In order to prove the existence of the Schur complement $\check{P}$, we consider the triangular structure of $\widetilde{P}_{\text{hc}}$:

$$\widetilde{P}_{\text{hc}} = \begin{bmatrix} P_{00} & P_{01} & \dots & P_{0\nu} \\ & P_{11} & \dots & P_{1\nu} \\ & & \ddots & \vdots \\ O & & & P_{\nu\nu} \end{bmatrix}, \quad P_{\ell\ell} \in \mathbb{R}^{r_\ell \times r_\ell}.$$

Since $\widetilde{P}$ is column reduced, $\widetilde{P}_{\text{hc}}$ is regular, that means the regularity of all main diagonal blocks $P_{\ell\ell}$. Since the right lower $(r_\nu \times r_\nu)$ corner of $\widetilde{P}$ coincides with $P_{\nu\nu}$, the existence of $\check{P}$ is proved. Explicitly, from (25) we get

$$\check{P}(s) = \operatorname{diag}(\mathcal{P}_0, O_{n-r_0-r_\nu}) s^\nu + \operatorname{diag}(\mathcal{P}_1, O_{n-r_0-r_1-r_\nu}) s^{\nu-1} + \dots + \mathcal{P}_{\nu-1} s + S_{r_\nu}(\mathcal{P}_\nu).$$

Therefore, $\check{P}$ is a matrix polynomial, and immediately the coincidence of its column degrees with the column degrees of $\widetilde{P}$ follows. Finally, with $|\widetilde{P}| = |P_{\nu\nu}||\check{P}|$ the column reducedness of $\check{P}$ is obtained. $\square$

Now, all quantities are available to construct $A \in \mathcal{M}(\mathcal{N})$:

1. Let $n+1$ be the vertex number of $\mathcal{N}$, and $\eta := n - r_\nu$.

2. Let $\check{P}$ be defined as in Proposition 4, and the vector polynomials $V_k \in \mathbb{R}^\eta[s]$ by its columns:

$$\check{P} =: [V_1, \ldots, V_\eta],\ V_\ell(s) =: V_\ell^{k_\ell} s^{k_\ell} + \ldots + V_\ell^1 s + V_\ell^0, \quad V_\ell^i \in \mathbb{R}^\eta,\ V_\ell^{k_\ell} \neq o.$$

3. Let the matrices $\Phi$, $\Theta$, $J$, and $\Psi$ be defined by

$$\begin{aligned} &\Phi = \check{P}_{\mathrm{hc}}^{-1} \in \mathbb{R}^{\eta\times\eta},\ \Theta = [V_1^{k_1-1}, \ldots, V_1^0|\ldots|V_\eta^{k_\eta-1}, \ldots, V_\eta^0] \in \mathbb{R}^{\eta\times\mathbf{d}}, \\ &J = \mathrm{diag}(J_{k_1}, \ldots, J_{k_\eta}) \in \mathbb{R}^{\mathbf{d}\times\mathbf{d}}, \Psi = \mathrm{diag}(e^{k_1}, \ldots, e^{k_\eta}) \in \mathbb{R}^{\mathbf{d}\times\eta}, \end{aligned} \tag{26}$$

where $\mathbf{d} = k_1 + \ldots + k_\eta$, $J_k = \begin{bmatrix} 0 & & & \\ 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{bmatrix} \in \mathbb{R}^{k\times k}$, $e^k = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \in \mathbb{R}^k$.

4. Let $A$ be defined by $A := J - \Psi\Phi\Theta \in \mathbb{R}^{\mathbf{d}\times\mathbf{d}}$.

Then $A \in \mathcal{M}(\mathcal{N})$.

This construction receipt is an adapted version of the general receipt to obtain a minimal state space description for the inverse of a column reduced matrix polynomial. A detailed description can be found in [[4], section 6.4].

In order to find vectors $U, V \in \mathbb{R}^{\mathbf{d}}$ for $(A, \widehat{A}) \in \mathcal{M}(\mathcal{N}) \times \mathcal{M}(\widehat{\mathcal{N}})$, such that $\widehat{A} - A = UV^T$, we return to the representation of $L$ in the form

$$L(s;q) = \sum_{\ell=0}^{\nu} \mathcal{A}_\ell \,\mathrm{diag}\,(\hat{q}_\ell)\, \mathcal{A}_\ell^T s^{\nu-\ell},\ q = [\hat{q}_0^T, \ldots, \hat{q}_\nu^T]^T \in \mathbb{R}^\varrho.$$

Consequently, a disturbance $\delta$ of an arbitrary component of $q$ can be represented according to

$$L(s; q + e_\kappa\delta) = L(s;q) + v(\delta s^k)v^T,\ v \in \mathbb{R}^n. \tag{27}$$

Here, $e_\kappa$ denotes the $\kappa$-th unit vector of length $\varrho$. That means such a disturbance can be described simply through the addition of a rank 1 matrix. Now it turns out that this option stays valid, if one passes over from $L \in \mathbb{R}^{n\times n}[s]$ to $A \in \mathcal{M}(\mathcal{N})$. The proof is essentially based on the Sherman - Morrison - Woodbury formula [[2], p.51]: Let $A \in \mathbb{C}^{\mathbf{d}\times\mathbf{d}}, U, V \in \mathbb{C}^{\mathbf{d}\times r}$, and suppose the existence of $A^{-1}$ and $H := (I_r + V^T A^{-1} U)^{-1}$. Then $A + UV^T$ is invertible and satisfies

$$(A + UV^T)^{-1} = A^{-1} - A^{-1}UHV^TA^{-1}.$$

Proposition 5 *Let $L$ be generated through a network $\mathcal{N}$, and let $r_\nu$ be the last rank defect number of $L$. Then with respect to every parameter $q_\kappa$ of $L$ there exists a canonical* *extension* $\mathcal{A}_{\text{ext}}$ *of* $\mathcal{A}_0$*, such that for almost every* $q \in \mathbb{R}^\varrho$ *the Schur complement* $\check{P} := S_{r_\nu}(\widetilde{P})$*,* $\widetilde{P} := \mathcal{A}_{\text{ext}}^{-1} L \mathcal{A}_{\text{ext}}^{-T}$*, exists and satisfies*

$$\check{P}(s; q + e_\kappa \delta) = \check{P}(s; q) + u(q) h(q, \delta) s^k u(q)^T, \ u(q) \in \mathbb{R}^{n - r_\nu},$$

*where* $h$ *is of the form* $h(q, \delta) = \begin{cases} \delta/(1 + c_1(q)\delta), & k = 0, \\ \delta, & k > 0. \end{cases}$

*Proof.* Because $L$ is generated by a network the disturbance of an parameter $q_\kappa$ can be described through (27). If the power $k$ of $s$ is positive, then for all canonical extensions $\mathcal{A}_{\text{ext}}$ of $\mathcal{A}_0$ we have

$$\widetilde{P}(s; q + e_\kappa \delta) = \mathcal{A}_{\text{ext}}^{-1} L(s; q + e_\kappa \delta) \mathcal{A}_{\text{ext}}^{-T} = \begin{bmatrix} P_{11}(s; q) + u \delta s^k u^T & P_{12}(q) \\ P_{21}(q) & P_{22}(q) \end{bmatrix}, \begin{bmatrix} u \\ o \end{bmatrix} := \mathcal{A}_{\text{ext}}^{-1} v,$$

where $u \in \mathbb{R}^{(n - r_\nu) \times (n - r_\nu)}$. With respect to Proposition 4 we know that $\widetilde{P}$ is column reduced for almost every $q$, that means in particular the matrix $P_{22}$ is invertible for almost every $q$, and hence

$$\check{P}(s; q + e_\kappa \delta) = P_{11}(s; q) + P_{12}(q) P_{22}(q)^{-1} P_{21}(q) + u \delta s^k u^T = \check{P}(s; q) + u \delta s^k u^T.$$

Now, for $k = 0$ we have to distinguish between two cases: either $v$ is a linear combination of the first $n - r_\nu$ columns of $\mathcal{A}_{\text{ext}}$, or $v$ can be used to generate the last $r_\nu$ columns of a canonical extension $\mathcal{A}_{\text{ext}}$ of $\mathcal{A}_0$. In the first case one can argue as in in the case $k > 0$. For the second case we assume that $v$ arises as one of the last $r_\nu$ columns of $\mathcal{A}_{\text{ext}}$. Then we have

$$\widetilde{P}(s; q + e_\kappa \delta) = \begin{bmatrix} P_{11}(s; q) & P_{12}(q) \\ P_{21}(q) & P_{22}(q) + e_j \delta e_j^T \end{bmatrix}.$$

With the Sherman - Morrison - Woodbury formula one obtains

$$\check{P}(s; q + e_\kappa \delta) = P_{11}(s; q) - P_{12}(q)(P_{22}(q) + e_j \delta e_j^T)^{-1} P_{21}(q) = \check{P}(s; q) + u(q) h(q, \delta) u(q)^T,$$

where $u := P_{12} P_{22}^{-1} e_j$, $h := \delta/(1 + c_1 \delta)$, and $c_1 := e_j^T P_{22}^{-1} e_j$. □

Moreover, the statement of Proposition 5 transfers to an analog statement for $A \in \mathcal{M}(\mathcal{N})$.

PROPOSITION 6 *Let* $A \in \mathcal{M}(\mathcal{N})$*. Then* $A(q + e_\kappa \delta) = A(q) + U(q) H(q, \delta) V(q)^T$*, where* $U, V \in \mathbb{R}^{\mathbf{d}}$*, and* $H$ *is of the form* $H = h/(1 + c_2 h)$*, where* $h$ *is defined as in Proposition 5.*

*Proof.* Let $L$ be generated through $\mathcal{N}$, and $\mathcal{A}_{\text{ext}}$ be a canonical extension of $\mathcal{A}_0$, such that Proposition 5 can be applied, and let via the Schur complement $\check{P}$ the matrices $\Phi$, $\Theta$, $J$, $\Psi$ be constructed as in (26). Then Proposition 5 provides the representations

$$\Phi(q + e_\kappa \delta)^{-1} = \Phi(q)^{-1} + u(q) h(q, \delta) v(q)^T, \ \Theta(q + e_\kappa \delta) = \Theta(q) + u(q) h(q, \delta) w(q)^T.$$

Suppose $c_2 h \neq -1$, where $c_2(q) = v(q)^T \Phi(q) u(q)$, and set $H = h/(1 + c_2 h)$. Then with the Sherman - Morrison - Woodbury formula one obtains

$$\Phi(q + e_\kappa \delta) = \Phi(q) - \Phi(q) u(q) H(q, \delta) v(q)^T \Phi(q).$$

Due to the similarity of all elements of $\mathcal{M}(\mathcal{N})$, $A = J - \Psi\Phi\Theta$ can be assumed. Consequently, omitting the arguments $q$ and $\delta$, one obtains

$$\begin{aligned} A(q + e_\kappa \delta) &= J - \Psi\Phi(q + e_\kappa \delta)\Theta(q + e_\kappa \delta) = J - \Psi(\Phi - \Phi u H v^T \Phi)(\Theta + u h w^T) \\ &= A - \Psi\Phi u((1 - Hc_2)hw^T - Hv^T \Phi\Theta). \end{aligned}$$

The definition of $H$ yields $(1 - Hc_2)h = H$, what implies

$$A(q + e_\kappa \delta) = A(q) + \Psi\Phi(q) u(q) H(q, \delta)(v(q)^T \Phi(q)\Theta(q) - w(q)^T).$$

Hence, setting

$$U(q) := \Psi\Phi(q) u(q), \quad V(q) := \Theta(q)^T \Phi(q)^T v(q) - w(q) \in \mathbb{R}^{\mathbf{d}}, \tag{28}$$

the statement of the proposition follows. □

Corollary 2 *The matrix* $\widehat{A}(q) := \lim_{\delta\to\infty} A(q + e_\kappa \delta)$ *exists if and only if* $\mathcal{H}(q) := \lim_{\delta\to\infty} H(q, \delta)$ *exists. Let* $\mathcal{N}$ *and* $\widehat{\mathcal{N}}$ *be as in Lemma* 4*, and suppose the existence of* $\widehat{A}$*. If* $\#\sigma(\widehat{A}) = \mathbf{d}$*, then* $\widehat{A} \in \mathcal{M}(\widehat{\mathcal{N}})$.

Combination of the Theorem of Bauer - Fike with Lemma 4, Proposition 6, and Corollary 2 yields our main result.

Theorem 2 (Main Theorem) *Let* $\mathcal{N}$ *and* $\widehat{\mathcal{N}}_\vartheta$ *be as in Lemma* 4*,* $\sigma(\mathcal{N}) = \{\lambda_1, \ldots, \lambda_{\mathbf{d}}\}$*, and* $\mu \in \sigma(\widehat{\mathcal{N}}_1) \cup \sigma(\widehat{\mathcal{N}}_2) \cup \{0\}$*. Let* $A \in \mathcal{M}(\mathcal{N})$*, and suppose the existence of the limit* $\mathcal{H}$*. If* $A = T \operatorname{diag}(\lambda_k)_{k=1}^{\mathbf{d}} T^{-1}$*, and* $U, V$ *are defined as in* (28)*, then*

$$\min\{|\lambda_1 - \mu|, \ldots, |\lambda_{\mathbf{d}} - \mu|\} \leq \|T^{-1}U\|_2 \, |\mathcal{H}| \, \|T^T V\|_2.$$

*Proof.* By construction we have $\sigma(A(q + e_\kappa \delta)) = \sigma(\Delta(.; q + e_\kappa \delta))$. Hence, with respect to Lemma 4 the equation $\sigma(\widehat{A}) = \sigma(\widehat{\Delta}_1) \cup \sigma(\widehat{\Delta}_2) \cup \{0\}$ follows. If $\mu \in \sigma(\widehat{A})$, then with Theorem 1 one obtains $\min\{|\lambda_1 - \mu|, \ldots, |\lambda_{\mathbf{d}} - \mu|\} \leq \|T^{-1}(\widehat{A} - A)T\|_2$, where $T$ diagonalizes $A$. Because of Proposition 6 and Corollary 2 we have

$$T^{-1}(\widehat{A} - A)T = T^{-1} U \mathcal{H} V^T T.$$

Finally, since $\|M\|_2$ coincides with the maximal singular value of the matrix $M$, the estimation holds true. □

Coming to the end, we like to illustrate Theorem 2 by an example. Let the network $\mathcal{N}$ be given by

$$\mathcal{N}: \quad \begin{array}{c} (\kappa_1) - p_1(s) - (\kappa_2) - p_2(s) - (\kappa_3) - p_3(s) - (\kappa_4) - p_2(s) - (\kappa_1) \\ (\kappa_2) \underline{\qquad\qquad} \delta s \underline{\qquad\qquad} (\kappa_4) \end{array},$$

with the edge weights $p_1(s) = s^2 - q_1$, $p_2(s) = s - q_2$, $p_3(s) = s^2 - q_3$, and the parameter vector $q = [\hat{q}^T, \delta]^T$, $\hat{q} = [q_1, q_2, q_3]^T$. Definition 1 yields

$$P(s) = \mathcal{A}\,\mathrm{diag}(p_1, p_2, p_3, p_2, \delta s)\,\mathcal{A}^T,\ \mathcal{A} = [e_1 - e_2, e_2 - e_3, e_3 - e_4, e_1 - e_4, e_2 - e_4],$$

and for the determinant $\Delta$ of $\mathcal{N}$ the scalar polynomial

$$\begin{aligned}\Delta(s;q) = \ & (\delta+2)s^5 \\ & +2(\delta+1-q_2)s^4 \\ & +(\delta(1-q_3-q_1-2q_2)-2(2q_2+q_1+q_3))s^3 \\ & +(2(q_1q_2+q_2q_3+q_2^2)-q_1-q_3-\delta(q_3+2q_2+q_1))s^2 \\ & +(\delta(q_1q_2+q_1q_3+q_2^2+q_2q_3)+2(q_1q_3+q_1q_2+q_2q_3))s \\ & -q_2^2q_3-q_1q_2^2-2q_1q_2q_3.\end{aligned}$$

According to Definition 2 we have $\sigma(\mathcal{N}) = \sigma(\Delta)$. Obviously it is quite difficult to exploit this representation of $\Delta$ for further investigations of the influence of $q$ on $\sigma(\mathcal{N})$. Regarding the structure of $\mathcal{N}$ this influence can be discussed using the adjoined networks

$$\widehat{\mathcal{N}}_1 :\ (\kappa_1) - p_1(s) - (\kappa_2) - p_2(s) - (\kappa_3) - p_3(s) - (\kappa_4) - p_2(s) - (\kappa_1)$$

$$\widehat{\mathcal{N}}_2 :\ \begin{array}{c}(\kappa_1) - (p_1(s)+p_2(s)) - (\mathbf{i}) - (p_2(s)+p_3(s)) - (\kappa_3) \\ | \\ s - (\mathbf{j})\end{array}$$

Here, $\widehat{\mathcal{N}}_1$ corresponds with $\delta = 0$, and $\widehat{\mathcal{N}}_2$ with $\delta = \infty$. The construction receipt after the proof of Proposition 4 makes $A \in \mathcal{M}(\mathcal{N})$ available:

$$A = \frac{1}{2}\begin{bmatrix} -1 & 2q_1+q_2 & 0 & -1 & q_2 \\ 2 & 0 & 0 & 0 & 0 \\ -1 & q_2 & 2q_2 & -1 & q_2 \\ -1 & q_2 & 0 & -1 & q_2+2q_3 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \underbrace{\frac{\delta}{4+2\delta}}_{H(\delta)} [1, -q_2, -2q_2, -1, -q_2].$$

Because of the minimality of $A$, for all $q \in \mathbb{R}^4$, excepted the case $\delta = -2$, the equation $\sigma(\mathcal{N}) = \sigma(A)$ holds true. Due to $\mathcal{H} = \lim_{\delta\to\infty} H(\delta) = 1/2$, the matrix $\widehat{A} := A([\hat{q}, \infty])$ exists. From Lemma 3 and 4 we know, that

$$\begin{aligned}\sigma(\widehat{\mathcal{N}}_1) = \ & \sigma(A([\hat{q},0])) && = \sigma(p_1p_2p_3 + p_1p_2p_2 + p_1p_3p_2 + p_2p_3p_2), \\ \sigma(\widehat{\mathcal{N}}_2) = \ & \sigma(\widehat{A}) && = \sigma(p_1+p_2) \cup \sigma(p_2+p_3) \cup \{0\}.\end{aligned}$$

The dependency of $\sigma(\widehat{\mathcal{N}}_2)$ on $\hat{q}$ can be described easily, namely

$$\sigma(p_1+p_2) = \left\{(-1 \pm \sqrt{1+4(q_1+q_2)})/2\right\},\ \sigma(p_2+p_3) = \left\{(-1 \pm \sqrt{1+4(q_2+q_3)})/2\right\}.$$

To get a control over $\sigma(\widehat{\mathcal{N}}_1)$, we observe that $\sigma(\widehat{\mathcal{N}}_1) = \sigma(p_2) \cup \sigma(\hat{p})$, where

$$\begin{aligned}\hat{p}(s;\hat{q}) & = 2p_1(s)p_3(s) + (p_1(s)+p_3(s))p_2(s) \\ & = 2(s^4+s^3) - 2(q_1+q_2+q_3)s^2 - (q_1+q_3)s + (q_1+q_3)q_2 + 2q_1q_3.\end{aligned}$$

Simple calculation shows that

$$2((s+1/4)^2-q_1)((s+1/4)^2-q_3)=$$
$$2(s^4+s^3)+\left(\frac{3}{4}-2(q_1+q_3)\right)s^2+\left(\frac{1}{8}-(q_1+q_3)\right)s+\left(\frac{1}{128}-(q_1+q_3)q_2+2q_1q_3\right).$$

Under the condition $|q_2| \ll |q_1|, |q_3|$, comparison with the coefficients of $\hat{p}$ yields $\sigma(\hat{p}) \approx \left\{-1/4 \pm \sqrt{q_1}\right\} \cup \left\{-1/4 \pm \sqrt{q_3}\right\}$. Summing up the following approximations

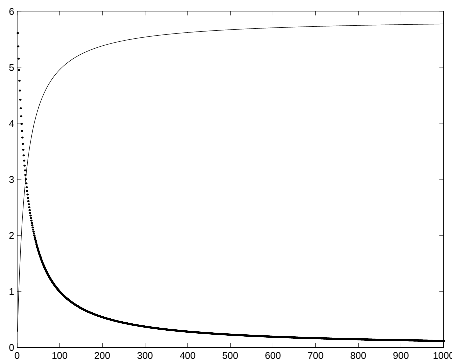
$$\begin{aligned}\sigma(\widehat{\mathcal{N}}_1(\hat{q})) &\approx \ \{q_2\} \cup \left\{-1/4 \pm \sqrt{q_1}\right\} \cup \left\{-1/4 \pm \sqrt{q_3}\right\},\\ \sigma(\widehat{\mathcal{N}}_2(\hat{q})) &\approx \ \{\,0\,\} \cup \left\{-1/2 \pm \sqrt{q_1}\right\} \cup \left\{-1/2 \pm \sqrt{q_3}\right\}\end{aligned}$$

can be stated. The right hand side defines the 5 intervals $[q_2, 0]$, and

$$[-1/2+\sqrt{q_1}, -1/4+\sqrt{q_1}\,], [-1/2-\sqrt{q_1}, -1/4-\sqrt{q_1}\,],$$

$$[-1/2+\sqrt{q_3}, -1/4+\sqrt{q_3}\,], [-1/2-\sqrt{q_3}, -1/4-\sqrt{q_3}\,].$$

Consequently it is to expect, that by the made assumption $|q_2| \ll |q_1|, |q_3|$, and by variation of $\delta$ in $[0, \infty]$, the 5 zeros of $\mathcal{N}$ are moving close to these 5 intervals. Concretely, for $q_1 = -100$, $q_2 = -4$, $q_3 = -200$, and $\delta_k = k/10$, $k = 1, \ldots, 1000$, the following figure visualizes the error estimates obtained by Theorem 2 for the deviation of $\sigma(\widehat{\mathcal{N}}_\vartheta)$ from $\sigma(\mathcal{N})$:




The solid (dotted) line represents the estimate for the deviation of $\sigma(\widehat{\mathcal{N}}_1)$ $(\sigma(\widehat{\mathcal{N}}_2))$ from $\sigma(\mathcal{N})$.

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***A Finite - Volume Particle Method for Compressible Flows***

We derive a new class of particle methods for conservation laws, which are based on numerical flux functions to model the interactions between moving particles. The derivation is similar to that of classical Finite-Volume methods; except that the fixed grid structure in the Finite-Volume method is substituted by so-called mass packets of particles. We give some numerical results on a shock wave solution for Burgers equation as well as the well-known one-dimensional shock tube problem.
(19 S., 1998)

2. M. Feldmann, S. Seibold

***Damage Diagnosis of Rotors: Application of Hilbert Transform and Multi-Hypothesis Testing***

In this paper, a combined approach to damage diagnosis of rotors is proposed. The intention is to employ signal-based as well as model-based procedures for an improved detection of size and location of the damage. In a first step, Hilbert transform signal processing techniques allow for a computation of the signal envelope and the instantaneous frequency, so that various types of non-linearities due to a damage may be identified and classified based on measured response data. In a second step, a multi-hypothesis bank of Kalman Filters is employed for the detection of the size and location of the damage based on the information of the type of damage provided by the results of the Hilbert transform.
*Keywords:*
Hilbert transform, damage diagnosis, Kalman filtering, non-linear dynamics
(23 S., 1998)

3. Y. Ben-Haim, S. Seibold

***Robust Reliability of Diagnostic Multi-Hypothesis Algorithms: Application to Rotating Machinery***

Damage diagnosis based on a bank of Kalman filters, each one conditioned on a specific hypothesized system condition, is a well recognized and powerful diagnostic tool. This multi-hypothesis approach can be applied to a wide range of damage conditions. In this paper, we will focus on the diagnosis of cracks in rotating machinery. The question we address is: how to optimize the multi-hypothesis algorithm with respect to the uncertainty of the spatial form and location of cracks and their resulting dynamic effects. First, we formulate a measure of the reliability of the diagnostic algorithm, and then we discuss modifications of the diagnostic algorithm for the maximization of the reliability. The reliability of a diagnostic algorithm is measured by the amount of uncertainty consistent with no-failure of the diagnosis. Uncertainty is quantitatively represented with convex models.
*Keywords:*
Robust reliability, convex models, Kalman filtering, multi-hypothesis diagnosis, rotating machinery, crack diagnosis
(24 S., 1998)

4. F.-Th. Lentes, N. Siedow

***Three-dimensional Radiative Heat Transfer in Glass Cooling Processes***

For the numerical simulation of 3D radiative heat transfer in glasses and glass melts, practically applicable mathematical methods are needed to handle such problems optimal using workstation class computers. Since the exact solution would require super-computer capabilities we concentrate on approximate solutions with a high degree of accuracy. The following approaches are studied: 3D diffusion approximations and 3D ray-tracing methods.
(23 S., 1998)

5. A. Klar, R. Wegener

***A hierarchy of models for multilane vehicular traffic***
***Part I: Modeling***

In the present paper multilane models for vehicular traffic are considered. A microscopic multilane model based on reaction thresholds is developed. Based on this model an Enskog like kinetic model is developed. In particular, care is taken to incorporate the correlations between the vehicles. From the kinetic model a fluid dynamic model is derived. The macroscopic coefficients are deduced from the underlying kinetic model. Numerical simulations are presented for all three levels of description in [10]. Moreover, a comparison of the results is given there.
(23 S., 1998)

***Part II: Numerical and stochastic investigations***

In this paper the work presented in [6] is continued. The present paper contains detailed numerical investigations of the models developed there. A numerical method to treat the kinetic equations obtained in [6] are presented and results of the simulations are shown. Moreover, the stochastic correlation model used in [6] is described and investigated in more detail.
(17 S., 1998)

6. A. Klar, N. Siedow

***Boundary Layers and Domain Decomposition for Radiative Heat Transfer and Diffusion Equations: Applications to Glass Manufacturing Processes***

In this paper domain decomposition methods for radiative transfer problems including conductive heat transfer are treated. The paper focuses on semi-transparent materials, like glass, and the associated conditions at the interface between the materials. Using asymptotic analysis we derive conditions for the coupling of the radiative transfer equations and a diffusion approximation. Several test cases are treated and a problem appearing in glass manufacturing processes is computed. The results clearly show the advantages of a domain decomposition approach. Accuracy equivalent to the solution of the global radiative transfer solution is achieved, whereas computation time is strongly reduced.
(24 S., 1998)

7. I. Choquet

***Heterogeneous catalysis modelling and numerical simulation in rarified gas flows Part I: Coverage locally at equilibrium***

A new approach is proposed to model and simulate numerically heterogeneous catalysis in rarefied gas flows. It is developed to satisfy all together the following points:
1) describe the gas phase at the microscopic scale, as required in rarefied flows,
2) describe the wall at the macroscopic scale, to avoid prohibitive computational costs and consider not only crystalline but also amorphous surfaces,
3) reproduce on average macroscopic laws correlated with experimental results and
4) derive analytic models in a systematic and exact way.
The problem is stated in the general framework of a non static flow in the vicinity of a catalytic and non porous surface (without aging). It is shown that the exact and systematic resolution method based on the Laplace transform, introduced previously by the author to model collisions in the gas phase, can be extended to the present problem. The proposed approach is applied to the modelling of the Eley-Rideal and Langmuir-Hinshelwood recombinations, assuming that the coverage is locally at equilibrium. The models are developed considering one atomic species and extended to the general case of several atomic species. Numerical calculations show that the models derived in this way reproduce with accuracy behaviors observed experimentally.
(24 S., 1998)

8. J. Ohser, B. Steinbach, C. Lang

***Efficient Texture Analysis of Binary Images***

A new method of determining some characteristics of binary images is proposed based on a special linear filtering. This technique enables the estimation of the area fraction, the specific line length, and the specific integral of curvature. Furthermore, the specific length of the total projection is obtained, which gives detailed information about the texture of the image. The influence of lateral and directional resolution depending on the size of the applied filter mask is discussed in detail. The technique includes a method of increasing directional resolution for texture analysis while keeping lateral resolution as high as possible.
(17 S., 1998)

9. J. Orlik

***Homogenization for viscoelasticity of the integral type with aging and shrinkage***

A multi-phase composite with periodic distributed inclusions with a smooth boundary is considered in this contribution. The composite component materials are supposed to be linear viscoelastic and aging (of the non-convolution integral type, for which the Laplace transform with respect to time is not effectively applicable) and are subjected to isotropic shrinkage. The free shrinkage deformation can be considered as a fictitious temperature deformation in the behavior law. The procedure presented in this paper proposes a way to determine average (effective homogenized) viscoelastic and shrinkage (temperature) composite properties and the homogenized stress-field from known properties of the

components. This is done by the extension of the asymptotic homogenization technique known for pure elastic non-homogeneous bodies to the non-homogeneous thermo-viscoelasticity of the integral non-convolution type. Up to now, the homogenization theory has not covered viscoelasticity of the integral type.
Sanchez-Palencia (1980), Francfort & Suquet (1987) (see [2], [9]) have considered homogenization for viscoelasticity of the differential form and only up to the first derivative order. The integral-modeled viscoelasticity is more general then the differential one and includes almost all known differential models. The homogenization procedure is based on the construction of an asymptotic solution with respect to a period of the composite structure. This reduces the original problem to some auxiliary boundary value problems of elasticity and viscoelasticity on the unit periodic cell, of the same type as the original non-homogeneous problem. The existence and uniqueness results for such problems were obtained for kernels satisfying some constrain conditions. This is done by the extension of the Volterra integral operator theory to the Volterra operators with respect to the time, whose 1 kernels are space linear operators for any fixed time variables. Some ideas of such approach were proposed in [11] and [12], where the Volterra operators with kernels depending additionally on parameter were considered. This manuscript delivers results of the same nature for the case of the space-operator kernels.
(20 S., 1998)

10. J. Mohring

***Helmholtz Resonators with Large Aperture***

The lowest resonant frequency of a cavity resonator is usually approximated by the classical Helmholtz formula. However, if the opening is rather large and the front wall is narrow this formula is no longer valid. Here we present a correction which is of third order in the ratio of the diameters of aperture and cavity. In addition to the high accuracy it allows to estimate the damping due to radiation. The result is found by applying the method of matched asymptotic expansions. The correction contains form factors describing the shapes of opening and cavity. They are computed for a number of standard geometries. Results are compared with numerical computations.
(21 S., 1998)

11. H. W. Hamacher, A. Schöbel

***On Center Cycles in Grid Graphs***

Finding "good" cycles in graphs is a problem of great interest in graph theory as well as in locational analysis. We show that the center and median problems are NP hard in general graphs. This result holds both for the variable cardinality case (i.e. all cycles of the graph are considered) and the fixed cardinality case (i.e. only cycles with a given cardinality p are feasible). Hence it is of interest to investigate special cases where the problem is solvable in polynomial time.
In grid graphs, the variable cardinality case is, for instance, trivially solvable if the shape of the cycle can be chosen freely.
If the shape is fixed to be a rectangle one can analyze rectangles in grid graphs with, in sequence, fixed dimension, fixed cardinality, and variable cardinality. In all cases a complete characterization of the optimal cycles and closed form expressions of the optimal objective values are given, yielding polynomial time algorithms for all cases of center rectangle problems.
Finally, it is shown that center cycles can be chosen as rectangles for small cardinalities such that the center cycle problem in grid graphs is in these cases completely solved.
(15 S., 1998)

12. H. W. Hamacher, K.-H. Küfer

***Inverse radiation therapy planning - a multiple objective optimisation approach***

For some decades radiation therapy has been proved successful in cancer treatment. It is the major task of clinical radiation treatment planning to realize on the one hand a high level dose of radiation in the cancer tissue in order to obtain maximum tumor control. On the other hand it is obvious that it is absolutely necessary to keep in the tissue outside the tumor, particularly in organs at risk, the unavoidable radiation as low as possible.
No doubt, these two objectives of treatment planning - high level dose in the tumor, low radiation outside the tumor - have a basically contradictory nature. Therefore, it is no surprise that inverse mathematical models with dose distribution bounds tend to be infeasible in most cases. Thus, there is need for approximations compromising between overdosing the organs at risk and underdosing the target volume.
Differing from the currently used time consuming iterative approach, which measures deviation from an ideal (non-achievable) treatment plan using recursively trial-and-error weights for the organs of interest, we go a new way trying to avoid a priori weight choices and consider the treatment planning problem as a multiple objective linear programming problem: with each organ of interest, target tissue as well as organs at risk, we associate an objective function measuring the maximal deviation from the prescribed doses.
We build up a data base of relatively few efficient solutions representing and approximating the variety of Pareto solutions of the multiple objective linear programming problem. This data base can be easily scanned by physicians looking for an adequate treatment plan with the aid of an appropriate online tool.
(14 S., 1999)

13. C. Lang, J. Ohser, R. Hilfer

***On the Analysis of Spatial Binary Images***

This paper deals with the characterization of microscopically heterogeneous, but macroscopically homogeneous spatial structures. A new method is presented which is strictly based on integral-geometric formulae such as Crofton's intersection formulae and Hadwiger's recursive definition of the Euler number. The corresponding algorithms have clear advantages over other techniques. As an example of application we consider the analysis of spatial digital images produced by means of Computer Assisted Tomography.
(20 S., 1999)

14. M. Junk

***On the Construction of Discrete Equilibrium Distributions for Kinetic Schemes***

A general approach to the construction of discrete equilibrium distributions is presented. Such distribution functions can be used to set up Kinetic Schemes as well as Lattice Boltzmann methods. The general principles are also applied to the construction of Chapman Enskog distributions which are used in Kinetic Schemes for compressible Navier-Stokes equations.
(24 S., 1999)

15. M. Junk, S. V. Raghurame Rao

***A new discrete velocity method for Navier-Stokes equations***

The relation between the Lattice Boltzmann Method, which has recently become popular, and the Kinetic Schemes, which are routinely used in Computational Fluid Dynamics, is explored. A new discrete velocity model for the numerical solution of Navier-Stokes equations for incompressible fluid flow is presented by combining both the approaches. The new scheme can be interpreted as a pseudo-compressibility method and, for a particular choice of parameters, this interpretation carries over to the Lattice Boltzmann Method.
(20 S., 1999)

16. H. Neunzert

***Mathematics as a Key to Key Technologies***

The main part of this paper will consist of examples, how mathematics really helps to solve industrial problems; these examples are taken from our Institute for Industrial Mathematics, from research in the Technomathematics group at my university, but also from ECMI groups and a company called TecMath, which originated 10 years ago from my university group and has already a very successful history.
(39 S. (vier PDF-Files), 1999)

17. J. Ohser, K. Sandau

***Considerations about the Estimation of the Size Distribution in Wicksell's Corpuscle Problem***

Wicksell's corpuscle problem deals with the estimation of the size distribution of a population of particles, all having the same shape, using a lower dimensional sampling probe. This problem was originary formulated for particle systems occurring in life sciences but its solution is of actual and increasing interest in materials science. From a mathematical point of view, Wicksell's problem is an inverse problem where the interesting size distribution is the unknown part of a Volterra equation. The problem is often regarded ill-posed, because the structure of the integrand implies unstable numerical solutions. The accuracy of the numerical solutions is considered here using the condition number, which allows to compare different numerical methods with different (equidistant) class sizes and which indicates, as one result, that a finite section thickness of the probe reduces the numerical problems. Furthermore, the relative error of estimation is computed which can be split into two parts. One part consists of the relative discretization error that increases for increasing class size, and the second part is related to the relative statistical error which increases with decreasing class size. For both parts, upper bounds can be given and the sum of them indicates an optimal class width depending on some specific constants.
(18 S., 1999)

18. E. Carrizosa, H. W. Hamacher, R. Klein, S. Nickel

***Solving nonconvex planar location problems by finite dominating sets***

It is well-known that some of the classical location problems with polyhedral gauges can be solved in polynomial time by finding a finite dominating set, i. e. a finite set of candidates guaranteed to contain at least one optimal location.
In this paper it is first established that this result holds for a much larger class of problems than currently considered in the literature. The model for which this result can be proven includes, for instance, location problems with attraction and repulsion, and location-allocation problems.
Next, it is shown that the approximation of general gauges by polyhedral ones in the objective function of our general model can be analyzed with regard to the subsequent error in the optimal objective value. For the approximation problem two different approaches are described, the sandwich procedure and the greedy algorithm. Both of these approaches lead - for fixed epsilon - to polynomial approximation algorithms with accuracy epsilon for solving the general model considered in this paper.
*Keywords:*
Continuous Location, Polyhedral Gauges, Finite Dominating Sets, Approximation, Sandwich Algorithm, Greedy Algorithm
(19 S., 2000)

19. A. Becker

***A Review on Image Distortion Measures***

Within this paper we review image distortion measures. A distortion measure is a criterion that assigns a "quality number" to an image. We distinguish between mathematical distortion measures and those distortion measures in-cooperating a priori knowledge about the imaging devices ( e. g. satellite images), image processing algorithms or the human physiology. We will consider representative examples of different kinds of distortion measures and are going to discuss them.
*Keywords:*
Distortion measure, human visual system
(26 S., 2000)

20. H. W. Hamacher, M. Labbé, S. Nickel, T. Sonneborn

**Polyhedral Properties of the Uncapacit*ated Multiple Allocation Hub Location Problem***

We examine the feasibility polyhedron of the uncapacitated hub location problem (UHL) with multiple allocation, which has applications in the fields of air passenger and cargo transportation, telecommunication and postal delivery services. In particular we determine the dimension and derive some classes of facets of this polyhedron. We develop some general rules about lifting facets from the uncapacitated facility location (UFL) for UHL and projecting facets from UHL to UFL. By applying these rules we get a new class of facets for UHL which dominates the inequalities in the original formulation. Thus we get a new formulation of UHL whose constraints are all facet–defining. We show its superior computational performance by benchmarking it on a well known data set.
*Keywords:*
integer programming, hub location, facility location, valid inequalities, facets, branch and cut
(21 S., 2000)

21. H. W. Hamacher, A. Schöbel

***Design of Zone Tariff Systems in Public Transportation***

Given a public transportation system represented by its stops and direct connections between stops, we consider two problems dealing with the prices for the customers: The fare problem in which subsets of stops are already aggregated to zones and "good" tariffs have to be found in the existing zone system. Closed form solutions for the fare problem are presented for three objective functions. In the zone problem the design of the zones is part of the problem. This problem is NP hard and we therefore propose three heuristics which prove to be very successful in the redesign of one of Germany's transportation systems.
(30 S., 2001)

22. D. Hietel, M. Junk, R. Keck, D. Teleaga:

***The Finite-Volume-Particle Method for Conservation Laws***

In the Finite-Volume-Particle Method (FVPM), the weak formulation of a hyperbolic conservation law is discretized by restricting it to a discrete set of test functions. In contrast to the usual Finite-Volume approach, the test functions are not taken as characteristic functions of the control volumes in a spatial grid, but are chosen from a partition of unity with smooth and overlapping partition functions (the particles), which can even move along prescribed velocity fields. The information exchange between particles is based on standard numerical flux functions. Geometrical information, similar to the surface area of the cell faces in the Finite-Volume Method and the corresponding normal directions are given as integral quantities of the partition functions.
After a brief derivation of the Finite-Volume-Particle Method, this work focuses on the role of the geometric coefficients in the scheme.
(16 S., 2001)

23. T. Bender, H. Hennes, J. Kalcsics, M. T. Melo, S. Nickel

***Location Software and Interface with GIS and Supply Chain Management***

The objective of this paper is to bridge the gap between location theory and practice. To meet this objective focus is given to the development of software capable of addressing the different needs of a wide group of users. There is a very active community on location theory encompassing many research fields such as operations research, computer science, mathematics, engineering, geography, economics and marketing. As a result, people working on facility location problems have a very diverse background and also different needs regarding the software to solve these problems. For those interested in non-commercial applications (e. g. students and researchers), the library of location algorithms (LoLA can be of considerable assistance. LoLA contains a collection of efficient algorithms for solving planar, network and discrete facility location problems. In this paper, a detailed description of the functionality of LoLA is presented. In the fields of geography and marketing, for instance, solving facility location problems requires using large amounts of demographic data. Hence, members of these groups (e. g. urban planners and sales managers) often work with geographical information too s. To address the specific needs of these users, LoLA was inked to a geographical information system (GIS) and the details of the combined functionality are described in the paper. Finally, there is a wide group of practitioners who need to solve large problems and require special purpose software with a good data interface. Many of such users can be found, for example, in the area of supply chain management (SCM). Logistics activities involved in strategic SCM include, among others, facility location planning. In this paper, the development of a commercial location software tool is also described. The too is embedded in the Advanced Planner and Optimizer SCM software developed by SAP AG, Walldorf, Germany. The paper ends with some conclusions and an outlook to future activities.
*Keywords:*
facility location, software development, geographical information systems, supply chain management.
(48 S., 2001)

24. H. W. Hamacher, S. A. Tjandra

***Mathematical Modelling of Evacuation Problems: A State of Art***

This paper details models and algorithms which can be applied to evacuation problems. While it concentrates on building evacuation many of the results are applicable also to regional evacuation. All models consider the time as main parameter, where the travel time between components of the building is part of the input and the overall evacuation time is the output. The paper distinguishes between macroscopic and microscopic evacuation models both of which are able to capture the evacuees' movement over time.
Macroscopic models are mainly used to produce good lower bounds for the evacuation time and do not consider any individual behavior during the emergency situation. These bounds can be used to analyze existing buildings or help in the design phase of planning a building. Macroscopic approaches which are based on dynamic network flow models (minimum cost dynamic flow, maximum dynamic flow, universal maximum flow, quickest path and quickest flow) are described. A special feature of the presented approach is the fact, that travel times of evacuees are not restricted to be constant, but may be density dependent. Using multicriteria optimization priority regions and blockage due to fire or smoke may be considered. It is shown how the modelling can be done using time parameter either as discrete or continuous parameter.
Microscopic models are able to model the individual evacuee's characteristics and the interaction among evacuees which influence their movement. Due to the corresponding huge amount of data one uses simulation approaches. Some probabilistic laws for individual evacuee's movement are presented. Moreover ideas to model the evacuee's movement using cellular automata (CA) and resulting software are presented.
In this paper we will focus on macroscopic models and only summarize some of the results of the microscopic approach. While most of the results are applicable to general evacuation situations, we concentrate on building evacuation.
(44 S., 2001)

25. J. Kuhnert, S. Tiwari
***Grid free method for solving the Poisson equation***

A Grid free method for solving the Poisson equation is presented. This is an iterative method. The method is based on the weighted least squares approximation in which the Poisson equation is enforced to be satisfied in every iterations. The boundary conditions can also be enforced in the iteration process. This is a local approximation procedure. The Dirichlet, Neumann and mixed boundary value problems on a unit square are presented and the analytical solutions are compared with the exact solutions. Both solutions matched perfectly.
*Keywords:*
Poisson equation, Least squares method,
Grid free method
(19 S., 2001)

26. T. Götz, H. Rave, D. Reinel-Bitzer,
K. Steiner, H. Tiemeier
***Simulation of the fiber spinning process***

To simulate the influence of process parameters to the melt spinning process a fiber model is used and coupled with CFD calculations of the quench air flow. In the fiber model energy, momentum and mass balance are solved for the polymer mass flow. To calculate the quench air the Lattice Boltzmann method is used. Simulations and experiments for different process parameters and hole configurations are compared and show a good agreement.
*Keywords:*
Melt spinning, fiber model, Lattice Boltzmann, CFD
(19 S., 2001)

27. A. Zemitis
***On interaction of a liquid film with an obstacle***

In this paper mathematical models for liquid films generated by impinging jets are discussed. Attention is stressed to the interaction of the liquid film with some obstacle. S. G. Taylor [Proc. R. Soc. London Ser. A 253, 313 (1959)] found that the liquid film generated by impinging jets is very sensitive to properties of the wire which was used as an obstacle. The aim of this presentation is to propose a modification of the Taylor's model, which allows to simulate the film shape in cases, when the angle between jets is different from 180°. Numerical results obtained by discussed models give two different shapes of the liquid film similar as in Taylors experiments. These two shapes depend on the regime: either droplets are produced close to the obstacle or not. The difference between two regimes becomes larger if the angle between jets decreases. Existence of such two regimes can be very essential for some applications of impinging jets, if the generated liquid film can have a contact with obstacles.
*Keywords:*
impinging jets, liquid film, models, numerical solution, shape
(22 S., 2001)

28. I. Ginzburg, K. Steiner
***Free surface lattice-Boltzmann method to model the filling of expanding cavities by Bingham Fluids***

The filling process of viscoplastic metal alloys and plastics in expanding cavities is modelled using the lattice Boltzmann method in two and three dimensions. These models combine the regularized Bingham model for viscoplastic with a free-interface algorithm. The latter is based on a modified immiscible lattice Boltzmann model in which one species is the fluid and the other one is considered as vacuum. The boundary conditions at the curved liquid-vacuum interface are met without any geometrical front reconstruction from a first-order Chapman-Enskog expansion. The numerical results obtained with these models are found in good agreement with available theoretical and numerical analysis.
*Keywords:*
Generalized LBE, free-surface phenomena, interface boundary conditions, filling processes, Bingham viscoplastic model, regularized models
(22 S., 2001)

29. H. Neunzert
***»Denn nichts ist für den Menschen als Menschen etwas wert, was er nicht mit Leidenschaft tun kann«***
Vortrag anlässlich der Verleihung des Akademiepreises des Landes Rheinland-Pfalz am 21.11.2001

Was macht einen guten Hochschullehrer aus? Auf diese Frage gibt es sicher viele verschiedene, fachbezogene Antworten, aber auch ein paar allgemeine Gesichtspunkte: es bedarf der »Leidenschaft« für die Forschung (Max Weber), aus der dann auch die Begeisterung für die Lehre erwächst. Forschung und Lehre gehören zusammen, um die Wissenschaft als lebendiges Tun vermitteln zu können. Der Vortrag gibt Beispiele dafür, wie in angewandter Mathematik Forschungsaufgaben aus praktischen Alltagsproblemstellungen erwachsen, die in die Lehre auf verschiedenen Stufen (Gymnasium bis Graduiertenkolleg) einfließen; er leitet damit auch zu einem aktuellen Forschungsgebiet, der Mehrskalenanalyse mit ihren vielfältigen Anwendungen in Bildverarbeitung, Materialentwicklung und Strömungsmechanik über, was aber nur kurz gestreift wird. Mathematik erscheint hier als eine moderne Schlüsseltechnologie, die aber auch enge Beziehungen zu den Geistes- und Sozialwissenschaften hat.
*Keywords:*
Lehre, Forschung, angewandte Mathematik, Mehrskalenanalyse, Strömungsmechanik
(18 S., 2001)

30. J. Kuhnert, S. Tiwari
***Finite pointset method based on the projection method for simulations of the incompressible Navier-Stokes equations***

A Lagrangian particle scheme is applied to the projection method for the incompressible Navier-Stokes equations. The approximation of spatial derivatives is obtained by the weighted least squares method. The pressure Poisson equation is solved by a local iterative procedure with the help of the least squares method. Numerical tests are performed for two dimensional cases. The Couette flow, Poiseuelle flow, decaying shear flow and the driven cavity flow are presented. The numerical solutions are obtained for stationary as well as instationary cases and are compared with the analytical solutions for channel flows. Finally, the driven cavity in a unit square is considered and the stationary solution obtained from this scheme is compared with that from the finite element method.
*Keywords:*
Incompressible Navier-Stokes equations, Meshfree method, Projection method, Particle scheme, Least squares approximation
*AMS subject classification:*
76D05, 76M28
(25 S., 2001)

31. R. Korn, M. Krekel
***Optimal Portfolios with Fixed Consumption or Income Streams***

We consider some portfolio optimisation problems where either the investor has a desire for an a priori specified consumption stream or/and follows a deterministic pay in scheme while also trying to maximize expected utility from final wealth. We derive explicit closed form solutions for continuous and discrete monetary streams. The mathematical method used is classical stochastic control theory.
*Keywords:*
Portfolio optimisation, stochastic control, HJB equation, discretisation of control problems.
(23 S., 2002)

32. M. Krekel
***Optimal portfolios with a loan dependent credit spread***

If an investor borrows money he generally has to pay higher interest rates than he would have received, if he had put his funds on a savings account. The classical model of continuous time portfolio optimisation ignores this effect. Since there is obviously a connection between the default probability and the total percentage of wealth, which the investor is in debt, we study portfolio optimisation with a control dependent interest rate. Assuming a logarithmic and a power utility function, respectively, we prove explicit formulae of the optimal control.
*Keywords:*
Portfolio optimisation, stochastic control, HJB equation, credit spread, log utility, power utility, non-linear wealth dynamics
(25 S., 2002)

33. J. Ohser, W. Nagel, K. Schladitz
***The Euler number of discretized sets - on the choice of adjacency in homogeneous lattices***

Two approaches for determining the Euler-Poincaré characteristic of a set observed on lattice points are considered in the context of image analysis { the integral geometric and the polyhedral approach. Information about the set is assumed to be available on lattice points only. In order to retain properties of the Euler number and to provide a good approximation of the true Euler number of the original set in the Euclidean space, the appropriate choice of adjacency in the lattice for the set and its background is crucial. Adjacencies are defined using tessellations of the whole space into polyhedrons. In R 3 , two new 14 adjacencies are introduced additionally to the

well known 6 and 26 adjacencies. For the Euler number of a set and its complement, a consistency relation holds. Each of the pairs of adjacencies (14:1; 14:1), (14:2; 14:2), (6; 26), and (26; 6) is shown to be a pair of complementary adjacencies with respect to this relation. That is, the approximations of the Euler numbers are consistent if the set and its background (complement) are equipped with this pair of adjacencies. Furthermore, sufficient conditions for the correctness of the approximations of the Euler number are given. The analysis of selected microstructures and a simulation study illustrate how the estimated Euler number depends on the chosen adjacency. It also shows that there is not a uniquely best pair of adjacencies with respect to the estimation of the Euler number of a set in Euclidean space.
*Keywords: image analysis, Euler number, neighborhod relationships, cuboidal lattice*
(32 S., 2002)

34. I. Ginzburg, K. Steiner
***Lattice Boltzmann Model for Free-Surface flow and Its Application to Filling Process in Casting***

A generalized lattice Boltzmann model to simulate free-surface is constructed in both two and three dimensions. The proposed model satisfies the interfacial boundary conditions accurately. A distinctive feature of the model is that the collision processes is carried out only on the points occupied partially or fully by the fluid. To maintain a sharp interfacial front, the method includes an anti-diffusion algorithm. The unknown distribution functions at the interfacial region are constructed according to the first order Chapman-Enskog analysis. The interfacial boundary conditions are satisfied exactly by the coefficients in the Chapman-Enskog expansion. The distribution functions are naturally expressed in the local interfacial coordinates. The macroscopic quantities at the interface are extracted from the least-square solutions of a locally linearized system obtained from the known distribution functions. The proposed method does not require any geometric front construction and is robust for any interfacial topology. Simulation results of realistic filling process are presented: rectangular cavity in two dimensions and Hammer box, Campbell box, Sheffield box, and Motorblock in three dimensions. To enhance the stability at high Reynolds numbers, various upwind-type schemes are developed. Free-slip and no-slip boundary conditions are also discussed.
*Keywords: Lattice Boltzmann models; free-surface phenomena; interface boundary conditions; filling processes; injection molding; volume of fluid method; interface boundary conditions; advection-schemes; upwind-schemes*
(54 S., 2002)

35. M. Günther, A. Klar, T. Materne, R. Wegener
***Multivalued fundamental diagrams and stop and go waves for continuum traffic equations***

In the present paper a kinetic model for vehicular traffic leading to multivalued fundamental diagrams is developed and investigated in detail. For this model phase transitions can appear depending on the local density and velocity of the flow. A derivation of associated macroscopic traffic equations from the kinetic equation is given. Moreover, numerical experiments show the appearance of stop and go waves for highway traffic with a bottleneck.
*Keywords: traffic flow, macroscopic equations, kinetic derivation, multivalued fundamental diagram, stop and go waves, phase transitions*
(25 S., 2002)

36. S. Feldmann, P. Lang, D. Prätzel-Wolters
***Parameter influence on the zeros of network determinants***

To a network $\mathcal{N}(q)$ with determinant $\Delta(s;q)$ depending on a parameter vector $q \in \mathbb{R}^p$ via identification of some of its vertices, a network $\hat{\mathcal{N}}(q)$ is assigned. The paper deals with procedures to find $\hat{\mathcal{N}}(q)$, such that its determinant $\hat{\Delta}(s;q)$ admits a factorization in the determinants of appropriate subnetworks, and with the estimation of the deviation of the zeros of $\hat{\Delta}$ from the zeros of $\Delta$. To solve the estimation problem state space methods are applied.
*Keywords: Networks, Equicofactor matrix polynomials, Realization theory, Matrix perturbation theory*
(30 S., 2002)

Stand: Mai 2002