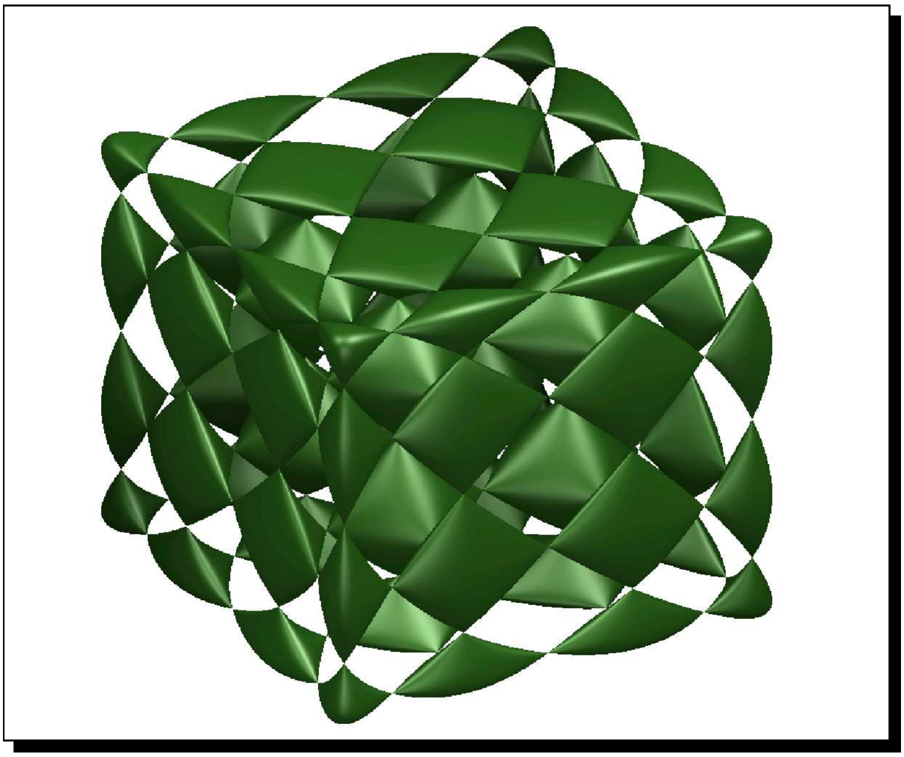
Families of hypersurfaces with many prescribed singularities

Eric Westenberger



Vom Fachbereich Mathematik der Universität Kaiserslautern
zur Verleihung des akademischen Grades Doktor der Naturwissenschaften
(Doctor rerum naturalium, Dr. rer. nat.) genehmigte Dissertation

1. Gutachter: Prof. Dr. G.-M. Greuel
2. Gutachter: Prof. Dr. E. Shustin

Vollzug der Promotion: 18.06.2004

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The image on the title page shows the surface of degree 8 defined by the equation

$$(x^4 - x^2 + 1)^2 + (y^4 - y^2 + 1)^2 + (z^4 - z^2 + 1)^2 = 1 \,.$$

It has 144 real ordinary double points. The equation is similar to Chmutov's construction [Ch92], and I would like to thank O. Labs for showing it to me.

The image was produced using the programme `Surf` written by S. Endrass.

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Introduction

Studying varieties with prescribed properties is one of the fundamental tasks in algebraic geometry. It was one of the main achievements in the last century to realize that it is not only necessary to study these varieties for themselves, but also to consider families of varieties. This means that a “good” space has to be found parametrizing all varieties with the fixed prescribed properties. Once such a *moduli space* has been constructed, then understanding its geometry allows us to answer a lot of interesting questions.

In this thesis, we consider reduced hypersurfaces W in a smooth, projective variety V defined over the complex numbers, and we assume that W has at most isolated singularities. The singular points are considered up to analytical or topological equivalence, and the equivalence classes are called singularity types. In the case of topological types we restrict ourselves to singularities, which are either quasihomogeneous or of corank less than 2.

Let $S_1, \ldots, S_r$ be types of isolated singularities, and let H be an ample divisor of V and $d \geq 0$. The space

$$V_d(S_1 + \cdots + S_r) \;:=\; \{\text{Reduced hypersurfaces } W \in |dH| \text{ with } r \text{ isolated singular points } z_1, \ldots, z_r \text{ of types } S_1, \ldots, S_r \text{ as its only singularities}\}$$

is a locally closed subspace of the linear system $|dH|$, and it is called the equisingular stratum. The fundamental questions concerning this space are:

- Is $V_d(S_1 + \cdots + S_r)$ non-empty, i.e. does there exist a hypersurface with singularities of the prescribed types and belonging to the given linear system?
- Is $V_d(S_1 + \cdots + S_r)$ smooth and what is its dimension? In particular, does $V_d(S_1 + \cdots + S_r)$ have the “expected dimension”, i.e. do the singular points impose independent conditions on hypersurfaces in the given linear system?
- Is $V_d(S_1 + \cdots + S_r)$ irreducible and what is its degree in the linear space $|dH|$?

These questions have attracted the continuous attention of algebraic geometers since the beginning of the 20th century, where the foundations were laid by Severi, Plücker, Segre, Zariski and others. It was realized quite early that all these questions appeared to be rather hard, the last one being the most difficult. In fact the only case for which a complete answer is known, is the classical case of plane, nodal curves solved essentially by Severi [Sev21] and completed by Harris [Har85]. Severi showed that $V_d^{irr}(r \cdot A_1)$ is non-empty if and only if

$$r \leq \frac{(d-1)(d-2)}{2} \,,$$

where $V_d^{irr}(r \cdot A_1)$ is the open subset of $V_d(r \cdot A_1)$ corresponding to irreducible curves. Furthermore, if $V_d^{irr}(r \cdot A_1)$ is non-empty, then it is smooth of the expected dimension $\frac{d(d+3)}{2} - k$ and also irreducible.

For other singularities and more general hypersurfaces, examples were found where the spaces $V_d(S_1 + \ldots + S_r)$ are reducible or non-reduced or singular or have dimension bigger than the expected one. Note that such pathological behaviour of singular hypersurfaces has already been observed for plane curves with only nodes and cusps. Since there are no more complete answers, the problem is to find *necessary* and *sufficient* conditions (in terms of numerical invariants of the singular points and the linear system) for the "good" properties to hold.

Our approach consists of reformulating the questions above in terms of H^1-vanishing conditions for ideal sheaves of zero-dimensional schemes associated to the singular points. Developing new techniques for deducing H^1-vanishing criteria is then one of the fundamental tasks in this area. This approach was applied very successfully, mainly by Greuel, Lossen and Shustin in the study of families of plane curves, and generalized to some extent by Keilen and Tyomkin to curves on more general surfaces. In this thesis we study the problem also for hypersurfaces in $\mathbb{P}^n$.

We are particularly interested in "asymptotically proper" conditions, i.e. criteria where the necessary and sufficient parts are asymptotically of the same order. Let us explain this more closely by means of the existence problem of hypersurfaces in $\mathbb{P}^n$ with many singularities of a certain fixed type S. A sufficient condition

$$r \cdot \sigma(S) \leq \alpha_1 \cdot d^m - R(d) \,, \quad \alpha_1 > 0 \,, \; R(d) \in O(d^{m-1}) \,, \tag{0.0.1}$$

for the existence of hypersurfaces in $V_d(rS)$ will be called *asymptotically proper* if there exists an absolute constant $\alpha_2 > \alpha_1$ and infinitely many pairs $(d, r) \in \mathbb{N}^2$ for which $V_d(rS) = \emptyset$ but $r \cdot \sigma(S) \leq \alpha_2 \cdot d^m$.
In fact, if $\sigma(S) = \tau(S)$ then (0.0.1) is asymptotically proper if and only if $m = n$, as we shall see later.

Main results

The main results of this thesis deal with the existence and the smoothness problem. We improve several conditions for the existence of hypersurfaces in $\mathbb{P}^n$ with prescribed singularities and develop the first asymptotically proper existence theorems for higher dimensional hypersurfaces with many singularities. On the other hand, some of the results are new even in the case of plane curves. Our main tool for constructing hypersurfaces with prescribed singularities is the so-called patchworking method (also called Viro's glueing method) combined with the theory of zero-dimensional schemes.

For obtaining new conditions for T-smoothness we study the Castelnuovo function of zero-dimensional schemes on surfaces in $\mathbb{P}^3$, which generalizes the theory in the plane case. This can be applied to derive H^1-vanishing theorems for these schemes.

Existence of hypersurfaces in $\mathbb{P}^n$ with prescribed singularities

We introduce an invariant of sets of singularity types, which is essentially the leading coefficient in a sufficient condition for the existence of hypersurfaces in $\mathbb{P}^n$ with these singularities.

Let $\mathcal{S}$ be a set of singularity types and denote by $\tau^s(S)$ the equianalytic or equisingular Tjurina number of $S \in \mathcal{S}$. Consider the set of all $\alpha \geq 0$ such that for all $\{S_1, \ldots, S_r\} \subset \mathcal{S}$, the condition

$$\sum_{i=1}^{r} k_i \tau^s(S_i) \leq \alpha \cdot d^n + O(d^{n-1}),$$

implies the existence of a non-empty T-smooth component of $V_d^n(k_1 S_1 + \ldots + k_r S_r)$. The T-smoothness requirement implies that

$$0 \leq \alpha \leq \frac{1}{n!},$$

and the existence result is asymptotically proper if $\alpha > 0$. The supremum of all α satisfying the property above is denoted by $\alpha_n^{reg}(\mathcal{S})$.

Let $n > 2$, and let $\mathcal{S} = \mathcal{S}_a \cup \mathcal{S}_t$ where $\mathcal{S}_a$ is a set of analytic singularity types of corank $< n$ and $\mathcal{S}_t$ is a set of topological singularity types of corank ≤ 2. If the (analytic, respectively topological) Tjurina number $\tau^s(S)$ is bounded as S varies in $\mathcal{S}$, then

$$\alpha_n^{reg}(\mathcal{S}) \geq \frac{\alpha_{n-1}^{reg}(\mathcal{S})}{n}. \qquad (0.0.2)$$

Hence, asymptotically proper existence results automatically carry over to higher dimensions, so this result can be seen as a kind of stabilization of the existence problem. Generalizing a result of [Sh01] we obtain as an immediate consequence that

$$\alpha_n^{reg}(\mathcal{S}) \geq \frac{2}{9 \cdot n!} \qquad (0.0.3)$$

for a set $\mathcal{S}$ of (analytical or topological) singularity types of corank ≤ 2 with bounded Tjurina number. Note that the lower bound (0.0.3) differs from the natural upper bound $\alpha_n^{reg}(\mathcal{S}) \leq 1/n!$ only by a constant factor.

For plane curves with only tacnodes and cusps, the sharpest known lower bound of $\frac{1}{6}$ was found by J. Roé in [Ro01]. Using our methods we are able to improve this result substantially and even determine the precise asymptotic factor for hypersurfaces of arbitrary dimension. If $\mathcal{S}$ is a finite set of simple singularity types, then

$$\alpha_n^{reg}(\mathcal{S}) = \frac{1}{n!}\,. \tag{0.0.4}$$

This is also an extension of the results of [Sh93], where it was shown for plane curves with only ordinary nodes and cusps.

We also prove an asymptotically proper existence result for hypersurfaces with quasi-homogeneous singularities. For fixed $\mathbf{a} = (a_2, \ldots, a_n) \in \mathbb{N}^{n-1}$, denote by $\mathcal{S}_{\mathbf{a}}$ the set of all analytical types of singularities defined by polynomials of the form

$$f(x_1, \ldots, x_n) = c_1 x_1^k + c_2 x_2^{a_2} + \ldots + c_n x_n^{a_n}\,,$$

with $k \in \mathbb{N}$ and $c_1, \ldots, c_n \in \mathbb{C} \setminus \{0\}$. Explicit constructions in combination with a local patchworking method allows us to deduce

$$\alpha_n^{reg}(\mathcal{S}_{\mathbf{a}}) \geq \frac{c}{2^n \cdot n \cdot l^n} > 0\,, \tag{0.0.5}$$

where $l = 1 + \sum_{i=2}^n a_i$ and $c = \prod_{i=2}^n (a_i - 1)$. Note that in $\mathcal{S}_{\mathbf{a}}$ the Tjurina number is not bounded. We can use this result to deduce an asymptotically proper existence result for hypersurfaces with singularities of modality ≤ 2.

Zero-dimensional schemes on surfaces in $\mathbb{P}^3$

We study the behaviour of the Castelnuovo function for zero-dimensional schemes on a general surface $S_n \subset \mathbb{P}^3$ of degree $n \geq 4$. Then $\mathrm{Pic}(S_n) = \mathbb{Z}$, and the Castelnuovo function of a zero-dimensional scheme $X \subset S_n$ is defined by

$$\mathcal{C}_X : \mathbb{N} \to \mathbb{N}\,, \quad d \mapsto h^1(\mathcal{I}_{X/S_n}(d-1)) - h^1(\mathcal{I}_{X/S_n}(d))\,.$$

We show that the graph of $\mathcal{C}_X$ has similar properties as for schemes in $\mathbb{P}^2$, in particular, we prove that if $H^0(\mathcal{I}_{X/S_n}(d)) \neq 0$ for some d then starting at least with $d+n-1$ the function $\mathcal{C}_X$ is descending. In analogy to the $\mathbb{P}^2$ case, we study fixed components of the linear systems $H^0(\mathcal{I}_X(d))$ and their influence onto the graph of $\mathcal{C}_X$. We prove that if $\mathcal{C}_X$ has a "long stair" at d_0, i.e.

$$\mathcal{C}_X(d_0 - 1) < \mathcal{C}_X(d_0) = \mathcal{C}_X(d_0 + 1) > 0\,, \tag{0.0.6}$$

and the linear system $H^0(\mathcal{I}_X(d_0))$ has a fixed component $D \in |\mathcal{O}_S(e)|$, then

$$\mathcal{C}_{X \cap D}(d) = \min\{\mathcal{C}_X(d), h^0(\mathcal{O}_C(d)) - h^0(\mathcal{O}_C(d-e))\}\,,$$

where $C \subset S$ is a generic hyperplane section of S. This is an analogue of the so-called Davis' Lemma for $\mathbb{P}^2$ [Da86].

In fact, we conjecture that (0.0.6) already implies the existence of the fixed curve D in $H^0(\mathcal{I}_X(d_0))$. If this conjecture holds, then for any zero-dimensional scheme on S, there exists a curve D such that the graph of $\mathcal{C}_{X\cap D}$ has no long stairs, and we can use this property to derive the same H^1-vanishing results as in [Ke03]. However, since this conjecture has not been proven, we derive a slighter weaker vanishing result. We show that if $X \subset S \subset \mathbb{P}^3$ is a zero-dimensional scheme contained in the equianalytic scheme associated to an irreducible curve $C_d \in |\mathcal{O}_S(d)|$, $d \geq 3$, then $h^1(\mathcal{I}_X(d)) = 0$ if

$$\gamma_1(X; C) < (d + n - 4)^2 \,,$$

where $\gamma_1(X; C)$ is a certain invariant introduced in [LoK03]. Furthermore, we compare the Castelnuovo function approach with the application of Bogomolov instability described in [CS97, Ke03].

Finally, we derive some properties of the Castelnuovo function of zero-dimensional schemes in $\mathbb{P}^3$, and calculate some examples.

Organization of the material

In Chapter 1 we introduce the main objects which we shall study in this thesis, i.e. hypersurfaces in smooth projective varieties with at most isolated singularities, and study their deformation theory. We formalize the concept of equisingular families and recall results concerning their geometry such as dimension and smoothness of these strata.
The second part of this preliminary chapter deals with the Newton polytope, which plays a major role in the patchworking method, which is our main tool for constructing hypersurfaces with prescribed singularities.

Chapters 2 and 3 are devoted to the existence problem. We review previously known existence results and theoretical restrictions. For hypersurfaces in $\mathbb{P}^n$ we formalize the general asymptotic existence problem by introducing the invariant $\alpha_n^{reg}(\mathcal{S})$. Furthermore, we describe the patchworking method in detail and discuss the connections between the existence problem and H^1-vanishing. In the third chapter we present our existence results.

In Chapter 4 we concentrate on H^1-vanishing theorems, and study in particular the Castelnuovo function of zero-dimensional schemes. After reviewing the theory for schemes in $\mathbb{P}^2$, we present our results concerning schemes on general surfaces in $\mathbb{P}^3$. We also review briefly the concept of Bogomolov instability and its relation to the H^1-vanishing problem, and compare this approach to the Castelnuovo function.

Publications of the results

It should be mentioned that many of the results are the product of the collaboration with Prof. E. Shustin from Tel Aviv University. Some of the results are published in [Wes03], some will appear in [SW03]. The results of Chapter 4 will be contained in a forthcoming paper.

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First of all, I should like to express my gratitude to Prof. Dr. G.-M. Greuel and Prof. Dr. E. Shustin. I am very glad that Gert-Martin has introduced me to this interesting subject, and I should like to thank him for his continuous support and encouragement. Furthermore I owe very much to many valuable discussions with Eugenii Shustin during my stay at Tel Aviv University in October 2003 and during the short mutual visits during recent years. I should also like to thank Prof. Dr. I. Luengo for fruitful discussions and suggestions and the hospitality during a stay at the UCM in Madrid in July 2002.

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Zum Schluss möchte ich gerne noch meiner Familie und meinen Freunden für ihre Unterstützung herzlich danken.

Contents

Chapter 1

Preliminaries

In this first chapter we introduce notation, define the main objects of interest and recall known results, which are necessary for the following chapters.

After some notation is fixed, we discuss classifications of isolated hypersurface singularities and their deformation theory. We study equisingular families of hypersurfaces and recall results concerning the geometry of these strata.

Then we introduce the Newton polytope, a combinatorial object associated to polynomials, containing information about the local and global behaviour of the corresponding hypersurfaces. In this context, the theory of toric varieties plays a major role, and we recall some facts which are of interest to us.

1.1 Notations and general remarks

Throughout this whole thesis we will use the following conventions. We study reduced hypersurfaces (i.e. reduced divisors) W in smooth projective varieties V defined over the complex numbers (unless otherwise stated). We assume in general that W has at most isolated singularities, and we denote the set of singular points by $\mathrm{Sing}(W)$. Furthermore we fix an ample divisor H, e.g. a hyperplane section if $V \subset \mathbb{P}^N$, and consider hypersurfaces belonging to a linear system $|dH|$ for some $d > 0$.

A smooth projective surface is usually denoted by S. If C, D are divisors in S then we write $C.D$ for their intersection number.

If X is any subscheme of V, then we denote by $\mathcal{I}_{X/V}$ (or just by $\mathcal{I}_X$) the ideal sheaf of X in V. The *support of* X is the set

$$\mathrm{supp}(X) = \{z \in V \mid \mathcal{I}_{X,z} \neq \mathcal{O}_{V,z}\}\,.$$

If $\mathcal{F}$ is a coherent sheaf on V and $W \subset V$ is a divisor, then we write $\mathcal{F}(W)$ for $\mathcal{F} \otimes_{\mathcal{O}_V} \mathcal{O}_V(W)$.

Let V be a smooth projective variety of dimension n, and let $W \subset V$ be a hypersurface. Let $z \in \operatorname{Sing}(W)$ be an isolated singular point and assume that $f \in \mathbb{C}\{\mathbf{x}\} := \mathbb{C}\{x_1, \dots, x_n\} \cong \mathcal{O}_{V,z}$ is a local equation for (W, z). We introduce the following invariants

$$\begin{aligned}
\tau^{ea}(W,z) &:= \tau(W,z) := \dim_{\mathbb{C}} \mathbb{C}\{\mathbf{x}\}/\langle f, f_{x_1}, \dots, f_{x_n}\rangle\,, \quad f_{x_i} := \tfrac{\partial f}{\partial x_i}\,,\\
\mu(W,z) &:= \dim_{\mathbb{C}} \mathbb{C}\{\mathbf{x}\}/\langle f_{x_1}, \dots, f_{x_n}\rangle\,,\\
\operatorname{mult}(W,z) &:= \max\{m \in \mathbb{Z} \mid f \in \langle \mathbf{x}\rangle^m\}\,,\\
\operatorname{corank}(W,z) &:= \operatorname{corank}(f) := \operatorname{corank}(H_f(\mathbf{x}))\,, \text{where } H_f \text{ is the Hessian of } f,\\
\delta(W,z) &:= \dim_{\mathbb{C}}(\overline{\mathcal{O}}_{W,z}/\mathcal{O}_{W,z})\,, \text{where } \overline{\mathcal{O}}_{W,z} \text{ is the normalization of } \mathcal{O}_{W,z}.
\end{aligned}$$

Note that all these numbers are well-defined, i.e. they are finite and independent of the choice of the local equation f. Furthermore, observe that $\delta(W, z) = 0$ for all isolated singularities (W, z) if $n > 2$.

We usually denote by $\mathbf{x} = (x_1, \dots, x_n)$ a coordinate vector, and by $\omega = (\omega_1, \dots, \omega_n)$ a vector in $\mathbb{Z}^n$, and write $\mathbf{x}^\omega$ for the (Laurent-)monomial $x_1^{\omega_1} \cdot \dots \cdot x_n^{\omega_n}$ of degree $|\omega| := \omega_1 + \ldots + \omega_n$. The set $(\mathbb{C}^*)^n$, $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$, is called the *($n$-dimensional) complex algebraic torus*. Obviously, any $\omega \in \mathbb{Z}^n$ defines a regular function

$$(\mathbb{C}^*)^n \to \mathbb{C}\,, \quad \mathbf{x} \mapsto \mathbf{x}^\omega\,.$$

If $\mathcal{A} \subset \mathbb{R}^n$ is a finite set, then $\operatorname{Conv}(\mathcal{A})$ denotes the convex hull of $\mathcal{A}$ and we write $\operatorname{vol}(\mathcal{A}) := \operatorname{vol}(\operatorname{Conv}(\mathcal{A}))$ for its euclidean volume.

If $f \in \mathbb{C}[\mathbf{x}]$ and $U \subset \mathbb{C}^n$ is open, then $V(f) \subseteq U$ denotes the hypersurface of U defined by f,

$$V(f) := \{\mathbf{x} \in U \mid f(\mathbf{x}) = 0\}\,.$$

On some occasions, we are also interested in hypersurfaces defined over the real numbers. We denote the real projective space by $\mathbb{RP}^n$ and call $(\mathbb{R}^*)^n$ the *(n-dimensional) real algebraic torus*. Furthermore, we denote by $\mathbb{R}^n_{\geq 0}$, respectively $\mathbb{R}^n_+$, the positive (respectively strictly positive) orthant.

Moreover, we would like to mention that even though we restrict ourselves to (subfields of) the complex numbers, the results of this thesis are also valid over any algebraically closed field of characteristic zero by the Lefschetz principle.

1.2 Hypersurfaces with isolated singularities

1.2.1 Singularity types and their local deformation theory

In this section, we study the local structure of isolated hypersurface singularities (W, z). We classify these singularities up to different equivalence relations, and recall invariants with respect to these classifications.
Moreover, we study equisingular deformations over complex space germs and recall results concerning infinitesimal equisingular deformations and obstructions for lifting these infinitesimal deformations to the next order.

Definition 1.2.1. Let (W_i, z_i), $i = 1, 2$, be isolated hypersurface singularities.

(i) The two germs are called *analytically equivalent*, $(W_1, z_1) \overset{a}{\sim} (W_2, z_2)$, if their local rings $\mathcal{O}_{W_1,z_1}$ and $\mathcal{O}_{W_2,z_2}$ are isomorphic. The equivalence classes are called *analytic types*.

(ii) The two germs are called *topologically equivalent*, $(W_1, z_1) \overset{t}{\sim} (W_2, z_2)$, if there exists a local homeomorphism $(V_1, z_1) \xrightarrow{\cong} (V_2, z_2)$ mapping (W_1, z_1) to (W_2, z_2). The equivalence classes are called *topological types*.

Notation 1.2.2. In the following, a (topological or analytic) singularity type is denoted by S. We use the convention that the singularity type carries implicitly the information whether this singularity is considered up to topological or analytical equivalence.

Remark 1.2.3. Obviously, if (W_1, z_1) and (W_2, z_2) are analytically equivalent then they are topologically equivalent. However, the opposite is not true. Consider for example the family of plane curves

$$F_t(x, y) = (xy) \cdot (x - y) \cdot (x + (1 - t)y) .$$

The elements of this family consist of four lines meeting in one point, and all these singular points are topologically equivalent. However, the analytic type depends on the cross-ratio of the four lines, so that the family consists of infinitely many analytically non-equivalent singularities.
Indeed, the topological equivalence coincides with the analytical equivalence if the singularity is simple. However, the converse is not true (cf. Remark 1.2.17).

Notation 1.2.4. If σ is an invariant of isolated hypersurface singularities with respect to some classification and S denotes an equivalence class, then we define $\sigma(S) := \sigma(W, z)$, where (W, z) is an arbitrary germ representing the type S.

In this way we introduce

(i) for topological or analytical types S

- the Milnor number $\mu(S)$ (cf. [Tei73]),
- the delta invariant $\delta(S)$,

(ii) and for analytic types S

- the Tjurina number $\tau^{ea}(S) = \tau(S)$,
- the multiplicity $\mathrm{mult}(S)$,
- the corank $\mathrm{corank}(S)$.

Remark 1.2.5. (a) The Tjurina number is not a topological invariant. Consider for example the plane curve germs defined by

$$f = x^5 + y^5 \text{ and } g = x^5 + y^5 + x^4y^2 + x^3y^3 .$$

They are topologically equivalent (cf. Remark 1.2.19 below) but

$$16 = \tau(f, 0) \neq \tau(g, 0) = 15 .$$

(b) Zariski has conjectured that the multiplicity is also a topological invariant (cf. [Za71a]). However, this conjecture is known to be true only in special cases (e.g. for plane curves (cf. below), and for quasihomogeneous singularities [Gr85]).

(c) If $n > 2$, then the delta invariant is trivial (i.e. $\delta(S) = 0$) since the singularity is isolated.

We introduce the concept of embedded deformations of reduced hypersurface singularities with special emphasis on those deformations which preserve the type of the singularity. We start by introducing the deformation category. For simplicity of notation we assume that $(W, 0) \subset (\mathbb{C}^n, 0)$.

Definition 1.2.6. Assume that $(W, 0) \subset (\mathbb{C}^n, 0)$ is a reduced hypersurface germ.

(1) An *embedded deformation of* $(W, 0)$ *over a complex germ* $(T, 0)$ is given by a commutative diagram

$$\begin{tikzcd} (\mathscr{W}, 0) \arrow[r, hook, "i"] \arrow[d, "\varphi"'] & (\mathbb{C}^n \times T, 0) \arrow[dl, "\pi"] \\ (T, 0) & \end{tikzcd}$$

where π denotes the natural projection, $(\mathscr{W}, 0) \subseteq (\mathbb{C}^n \times T, 0)$ is a hypersurface germ with $\varphi^{-1}(0) \cong (W, 0)$ and φ is a flat morphism of complex germs. If, in addition, a section $s : (T, 0) \longrightarrow (\mathscr{W}, 0)$ is given then we call (φ, s) a deformation with section.

(2) Two deformations $\varphi_1\colon (\mathscr{W}_1, 0) \to (T, 0)$ and $\varphi_2\colon (\mathscr{W}_2, 0) \to (T, 0)$ are called *isomorphic* if there exists an analytic isomorphism of hypersurface germs $\psi\colon (\mathscr{W}_1, 0) \xrightarrow{\cong} (\mathscr{W}_2, 0)$ such that the following diagram commutes

$$\begin{tikzcd}
(\mathscr{W}_1,0) \arrow[r, hook] \arrow[rr, bend left, "\psi"', "\cong"] \arrow[dr, "\varphi_1"'] & (\mathbb{C}^n \times T, 0) \arrow[d, "pr"] & (\mathscr{W}_2,0) \arrow[l, hook'] \arrow[dl, "\varphi_2"] \\
 & (T,0) &
\end{tikzcd}$$

(3) A deformation $\varphi\colon (\mathcal{W}, 0) \to (T, 0)$ is called *equianalytic* or *trivial* if it is isomorphic to the trivial deformation $(W \times T, 0) \to (T, 0)$.

The equianalytic deformations over $T_\varepsilon := \mathrm{Spec}(\mathbb{C}[\varepsilon]/\varepsilon^2)$ are of particular interest.

Proposition 1.2.7. *Let $(W, 0) \subset (\mathbb{C}^n, 0)$ be a hypersurface germ defined by $f \in \mathcal{O}_{\mathbb{C}^n,0}$. The set*

$$I^{ea}(W,0) := \Big\{ g \in \mathcal{O}_{\mathbb{C}^n,0} \mid F = f + \varepsilon \cdot g \text{ defines an equianalytic deformation of } (W,0) \text{ over } T_\varepsilon \Big\}$$

is equal to the Tjurina ideal generated by f and its partial derivatives. We call $I^{ea}(W, 0)$ the equianalytic ideal of $(W, 0)$.

Throughout this dissertation we shall primarily use the analytic equivalence. However, we discuss in the following some aspects of the theory for the topological classification because in some cases we can proceed analogously to the analytic case.

Remark 1.2.8. The local equianalytic stratum in the semi-universal deformation of any isolated singularity is always smooth (since it consists of just one reduced point). However, for the topological classification the situation is much more complicated in higher dimensions. Luengo gave an example of an isolated surface singularity, for which the μ-constant stratum in the semi-universal deformation is not smooth (cf. [Lue87]).

In the case of curves, we introduce a class of deformations, which do not change the topological type of the singularity.

Definition 1.2.9. Let $\varphi\colon (\mathscr{C}, 0) \to (T, 0)$ be a deformation of $(C, 0) \subset (\mathbb{C}^2, 0)$. Assume that $(C, 0)$ is defined by $f \in \mathcal{O}_{\mathbb{C}^2,0}$ and $(\mathscr{C}, 0)$ is given by $F \in \mathcal{O}_{\mathbb{C}^2\times T,0}$.

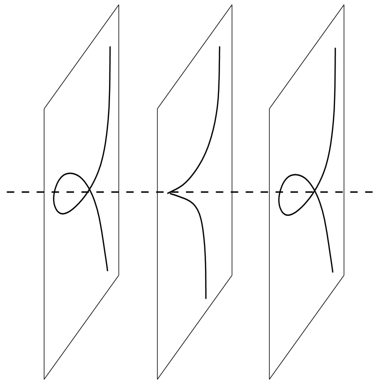


Figure 1.1: The deformation of the cusp given by $F_t(x,y) = x^3 - y^2 - t^2x^2$ is equimultiple but not equisingular along the trivial section.

(i) The deformation φ is called *equimultiple* along a section $s\colon (T,0) \to (\mathscr{C},0)$ if $F \in I_s^m$ where $m = \operatorname{mult}(C,0)$ and I_s denotes the ideal of $s(T,0) \subset (\mathbb{C}^2 \times T, 0)$.

(ii) The deformation φ is called *equisingular* along a section $s\colon (T,0) \to (\mathscr{C},0)$ if

- φ is equimultiple along s and
- there exists a sequence of morphisms

$$\pi\colon \Sigma_N \xrightarrow{\pi_N} \cdots \xrightarrow{\pi_2} \Sigma_1 \xrightarrow{\pi_1} \Sigma_0 = (\mathbb{C}^2 \times T)$$

and sections $s_k^{(i)}\colon (T,0) \to \widehat{\mathscr{C}_k}$, $i = 1,\dots,r_k$, where $\widehat{\mathscr{C}_k}$ is the reduction of $(\pi_1 \circ \cdots \circ \pi_k)^{-1}(\mathscr{C},0)$ satisfying the following properties:

(a) The mappings π_k simultaneously blow up the sections $s_k^{(i)}$, $i = 1,\dots,r_k$.

(b) $s_k^{(i)}(0)$, $i = 1,\dots,r_k$, are the singularities of

$$\widehat{C}_k = ((\pi_1 \circ \cdots \circ \pi_k)^{-1}(C,0))^{red}$$

of types different from A_1, i.e. π induces a minimal resolution of $(C,0)$ over $(\mathbb{C}^2 \times \{0\})$.

(c) The sections are compatible, that is, for all $k = 1,\dots,N$ and $j = 1,\dots,r_k$, there exists $1 \le i \le r_{k-1}$ with $\pi_k \circ s_k^{(j)} = s_{k-1}^{(i)}$.

(d) the deformations of $(\widehat{C}_k, s_k^{(i)}(0))$ are equimultiple along $s_k^{(i)}$ for $k = 1,\dots,N$, $i = 1,\dots,r_k$.

For arbitrary hypersurface singularities there is no notion of equisingular deformations over non-reduced spaces. However, for semi-quasihomogeneous singularities we can generalize this concept.

Definition 1.2.10. (i) A polynomial $f \in \mathbb{C}[\mathbf{x}]$ is *quasihomogeneous* if there exist $\alpha \in \mathbb{N}^n$ and $d \in \mathbb{Z}$ such that

$$f = \sum_{\alpha_1\omega_1+\ldots+\alpha_n\omega_n=d} a_\omega \mathbf{x}^\omega .$$

Then we say that f has type (α, d) and define

$$\deg_\alpha(f) := \langle \alpha, \omega \rangle := \alpha_1\omega_1 + \ldots + \alpha_n\omega_n = d .$$

(ii) A quasihomogeneous polynomial $f \in \mathbb{C}[\mathbf{x}]$ is called *non-degenerate* if it has an isolated singular point at the origin.

(iii) A polynomial f is called *semi-quasihomogeneous* of type (α, d), $\alpha \in \mathbb{N}^n$, $d \in \mathbb{Z}$, if f can be written as $f_0 + f'$ where f_0 is a non-degenerate quasihomogeneous polynomial of type (α, d), and all monomials $\mathbf{x}^\omega$ of f' satisfy $\langle \alpha, \omega \rangle > d$. The polynomial f_0 is called the *quasihomogeneous initial form of f*.

Example 1.2.11. Let $f \in \mathbb{C}[x, y]$ be homogeneous of degree m. Then f can be written in the form

$$f(x, y) = \prod_{k=1}^{m} (a_k x + b_k y) .$$

Then f is non-degenerate if and only if $(a_k : b_k) \in \mathbb{P}^1$ are pairwise distinct for $k = 1, \ldots, m$, i.e. if the curve defined by f consists of m distinct lines. If f is non-degenerate, and g is a polynomial of degree $\geq m + 1$, then $f + g$ is semi-quasihomogeneous.

We introduce a class of singularity types, which can be treated similarly to the analytic case.

Definition 1.2.12. Let $(W_1, z_1), (W_2, z_2)$ be isolated hypersurface singularities. Then we say that $(W_1, z_1), (W_2, z_2)$ have the same *admissible topological type S* if either

- there exist local equations f, g for $(W_1, z_1), (W_2, z_2)$ of the form

  $$f(\mathbf{x}) \overset{a}{\sim} f_0(x_1, x_2) + x_3^2 + \cdots + x_n^2 , \qquad g(\mathbf{x}) \overset{a}{\sim} g_0(x_1, x_2) + x_3^2 + \cdots + x_n^2 ,$$

  such that the plane curve singularities defined by f_0 and g_0 are topologically equivalent, or

- $(W_1, z_1), (W_2, z_2)$ are semi-quasihomogeneous of the same type, i.e. there exists a semi-quasihomogeneous local equations for $(W_1, z_1), (W_2, z_2)$ which have the same type in the sense of Definition 1.2.10.

If the initial form is homogeneous of degree m, then (W, z) is called a *semi-homogeneous singularity* or an *ordinary multiple point (of multiplicity m)* .

Note that if $(W_1, z_1), (W_2, z_2)$ have the same admissible topological type, then they have the same topological type by the following remark.

Remark 1.2.13. Let $f, g \in \mathbb{C}\{\mathbf{x}\}$ be germs of corank ≤ 2. If

$$f(\mathbf{x}) \overset{a}{\sim} f_0(x_1, x_2) + x_3^2 + \cdots + x_n^2\,, \qquad g(\mathbf{x}) \overset{a}{\sim} g_0(x_1, x_2) + x_3^2 + \cdots + x_n^2\,,$$

then $f \overset{t}{\sim} g$ if $f_0 \overset{t}{\sim} g_0$ because the links of f, g are suspensions of the links of f_0, g_0.

For admissible topological types we generalize the concept of equisingular deformations.

Definition 1.2.14. Let $\varphi\colon (\mathscr{W}, 0) \to (T, 0)$ be a deformation of $(W, 0) \subset (\mathbb{C}^n, 0)$ and assume that $(W, 0)$ is of admissible topological type. We call φ *equisingular* if it satisfies the corresponding condition below.

- If $\operatorname{corank}(W, 0) \leq 2$, then φ induces a deformation of a plane curve singularity, and φ is equisingular if the deformation of this curve singularity is equisingular in the sense of Definition 1.2.9.
- If $(W, 0)$ is semi-quasihomogeneous, defined in some local coordinates $x_1, \ldots, x_n$ by f of type (α, d). Then φ is equisingular if it is isomorphic to a deformation

$$\varphi_1\colon (\mathscr{W}_1, 0) \to (T, 0)$$

satisfying $\mathcal{O}_{T,0} \cong \mathbb{C}\{t_1, \ldots, t_r\}/I$ and $(\mathscr{W}_1, 0) \subset (\mathbb{C}^n \times T, 0)$ is defined by $F(\mathbf{x}, \mathbf{t})$ such that all monomials $\mathbf{x}^{\omega}\mathbf{t}^{\omega'}$ of F satisfy $\deg_{\alpha}(\mathbf{x}^{\omega}) \geq d$.

We shall use the latter definition from a pragmatic viewpoint, and do no deal with questions of functoriality here. Furthermore, note that equisingular deformations respect the topological type.

Proposition 1.2.15. *Let $(W, 0) \subset (\mathbb{C}^n, 0)$ be a reduced hypersurface singularity and assume that $(W, 0)$ is of admissible topological type. Let $f \in \mathbb{C}\{\mathbf{x}\} \cong \mathcal{O}_{\mathbb{C}^n,0}$ be a local equation of $(W, 0)$.*

(a) If $f = f_0(x_1, x_2) + x_3^2 + \cdots + x_n^2$, let W_0 be the plane curve germ defined by f_0. Then the set $I^{es}(f, 0) := I^{es}(f_0) + \langle x_3, \ldots, x_n \rangle \cdot \mathcal{O}_{\mathbb{C}^n,0}$ is an ideal containing the equianalytic Tjurina ideal $I^{ea}(f, 0)$ where

$$I^{es}(f_0, 0) := \big\{ g \in \mathcal{O}_{\mathbb{C}^2,0} \mid F = f_0 + \varepsilon \cdot g \text{ defines an equisingular deformation of } (W_0, 0) \text{ over } T_\varepsilon \big\}\,.$$

(b) If f is semi-quasihomogeneous of type (α, d), then

$$I^{es}(f,0) = I^{ea}(f,0) + \langle \mathbf{x}^\omega \mid \deg_\alpha(\mathbf{x}^\omega) \geq d \rangle \, .$$

Proof. Cf. [Wa74] for singularities of corank ≤ 2 and [AGV88, GLS] for semi-quasihomogeneous singularities. □

Definition 1.2.16. The *equisingular Tjurina number of* (W, z) is defined by

$$\tau^{es}(W,z) := \dim_{\mathbb{C}}(\mathcal{O}_{\mathbb{C}^n,0}/I^{es}(W,z)) \, .$$

Remark 1.2.17. Let (C, z) be a plane curve germ. Then $\tau^{es}(C, z)$ equals the codimension of the μ-constant stratum in the semi-universal deformation of (C, z). Hence, if

$$\tau^{ea}(C,z) = \tau^{es}(C,z) \, , \tag{1.2.1}$$

then the μ-constant stratum consists of just one point. In particular, the topological classification coincides with the analytic classification for this singularity. Note that for example the (non-simple) curve germ defined by

$$f = (x^2 - y^3) \cdot (x^3 - y^2)$$

satisfies (1.2.1).

Notation 1.2.18. We use the notation $\tau^s(S)$ for either $\tau^{ea}(S)$ or $\tau^{es}(S)$ depending on whether S is an analytic or topological type. Similarly we write $I^s(W, z)$ for either $I^{ea}(W, z)$ or $I^{es}(W, z)$.

Concluding this section, we discuss some relations between the different classes of deformations introduced before.

Remark 1.2.19. (i) Topologically trivial deformations over a reduced complex space are always μ-constant (e.g. [Tei73]). The converse is also true if $n \neq 3$ [LeR76].

(ii) Assume that $f \in \mathbb{C}[\mathbf{x}]$ is quasihomogeneous of type (α, d) and non-degenerate. If $g = \sum_{\langle \alpha, \omega \rangle > d} a_\omega \mathbf{x}^\omega$. Then the 1-parameter deformation

$$F_t = f + t \cdot g$$

is μ-constant [AGV88].

1.2.2 Equisingular families of hypersurfaces

Passing on to the global situation, we introduce the concept of families of (reduced) hypersurfaces on a smooth projective variety $V \hookrightarrow \mathbb{P}^N$.

Definition 1.2.20. Let T be a complex space. A *family* $\mathfrak{W}$ *of reduced hypersurfaces on* V *over* T is given by a commutative diagram

$$\begin{tikzcd} \mathfrak{W} \arrow[r, hook, "i"] \arrow[dr, "\varphi"'] & V \times T \arrow[d, "\pi"] \\ & T \end{tikzcd}$$

where π is the natural projection onto T, i is a closed embedding and φ is a flat morphism such that the fibres $\mathfrak{W}_t = \varphi^{-1}(t) \subset V \times \{t\}$ are reduced hypersurfaces for all $t \in T$.

Let $\varphi\colon \mathfrak{W} \to T$ be a family of hypersurfaces in V, and let $s : T \to W$ be a section. Then φ induces, for any $t \in T$, a deformation of the germ $(\varphi^{-1}(t), s(t))$ over (T, t).

Definition 1.2.21. The *Hilbert functor* $\mathcal{H}_V$ is the functor

$$\begin{array}{ccc} (\text{Complex spaces}) & \longrightarrow & (\text{Sets}) \\ T & \longrightarrow & \{\mathfrak{W} \longrightarrow T \text{ family of reduced hypersurfaces}\}\,. \end{array}$$

Proposition 1.2.22. *The Hilbert functor is representable by a complex space* H_V, *i.e. there exists an isomorphism of functors*

$$\mathcal{H}_V \xrightarrow{\cong} Hom(\_, H_V)\,.$$

Proof. cf. [Bin80]. □

Let pt be the reduced point. Then we have a bijection

$$\mathcal{H}_V(pt) \xleftrightarrow{1:1} Hom(pt, H_V) = H_V\,.$$

Hence, we can identify the points of H_V with hypersurfaces W in V. The Zariski tangent space to H_V at W is given by $H^0(\mathcal{N}_{W/V})$, where $\mathcal{N}_{W/V} = \mathcal{O}_V(W) \otimes \mathcal{O}_W$ denotes the normal sheaf of W in V. Furthermore, there exists a unique preimage $\mathcal{U}$ of $id_{H_V} \in Hom(H_V, H_V)$, which is called the *universal family* of $\mathcal{H}_V$.

Remark 1.2.23. Since the Hilbert polynomial h_t defined by

$$h_t(n) = \chi(\mathcal{O}_{\mathfrak{W}_t}(n))$$

is constant on any connected component of T, the universal family $\mathcal{U}$ splits into components on which the Hilbert polynomial is constant. This implies that the Hilbert scheme decomposes in the form

$$H_V = \coprod_{h\in\mathbb{C}[z]} H_V^h ,$$

where the H_V^h are unions of connected components of H_V.

Proposition 1.2.24. *Let W be a reduced hypersurface in V, and let $U \subseteq |W|$ be the open subset of reduced hypersurfaces linearly equivalent to W. Denote by h the Hilbert polynomial of W. There exists a natural injective morphism $U \hookrightarrow H_V^h$. If $h^1(\mathcal{O}_V) = 0$ then we can identify U with H_V^h.*

Proof. Consider the incidence variety

$$\begin{array}{ccc} \{(W', z) \in U \times V \mid z \in W'\} & \subseteq & U \times V \\ \downarrow & & \\ U & & \end{array}$$

which is flat over U. Thus, there exists a natural morphism $U \to H_V^h$. The corresponding map on the level of tangent spaces at W is the map

$$H^0(\mathcal{O}_V(W))/H^0(\mathcal{O}_V) \hookrightarrow H^0(\mathcal{N}_{W/V}) . \tag{1.2.2}$$

The map (1.2.2) is coming from the exact sequence

$$0 \longrightarrow \mathcal{O}_V \longrightarrow \mathcal{O}_V(W) \longrightarrow \mathcal{O}_W(W) \longrightarrow 0 .$$

If $h^1(\mathcal{O}_V) = 0$ then the map $H^0(\mathcal{O}_V(W))/H^0(\mathcal{O}_V) \xrightarrow{\cong} H^0(\mathcal{N}_{W/V})$ is an isomorphism, which implies the claim (using e.g. the implicit function theorem). $\square$

Let $\varphi\colon \mathfrak{W} \to T$ be a family of reduced hypersurfaces, $t \in T$ and $z \in \mathfrak{W}$. Then there is an induced local deformation

$$\psi : (\mathfrak{W}, z) \to (T, t) .$$

This allows us to introduce the concept of equianalytic and equisingular families.

Definition 1.2.25. Let $\varphi\colon \mathfrak{W} \longrightarrow T$ be a family of hypersurfaces such that, for all $t \in T$, the fibre W_t has only isolated singularities.

(i) We call φ an *equianalytic* or *locally trivial family of hypersurfaces* if, for all $t \in T$, the induced deformation of each singular point of W_t is equianalytic.

(ii) Assume in addition that, for all $t \in T$, the topological types of the singular points of W_t are admissible. Then we call the family *equisingular* if, for all $t \in T$, and every singular point of W_t, the induced deformation is equisingular.

We define the equisingular stratum, i.e. the space of all hypersurfaces in a linear system with singularities of prescribed types. For the construction and properties of this space we refer to [GrK89, GrL96].

Definition 1.2.26. Let $S_1, \ldots, S_r$ be analytic or admissible topological singularity types. We write

$$V_D(S_1 + \ldots + S_r) \subset |D|$$

for the (locally closed) space of all reduced hypersurfaces linearly equivalent to D with precisely r singularities which are of types $S_1, \ldots, S_r$. We call $V_D(S_1 + \ldots + S_r)$ the *equisingular stratum*.

The open subset of all *irreducible* hypersurfaces is denoted by $V_D^{irr}(S_1 + \ldots + S_r)$.

Remark 1.2.27. If $V = \mathbb{P}^n$, $n > 2$, then the space $V_D^{irr}(S_1 + \ldots + S_r)$ coincides with $V_D(S_1 + \ldots + S_r)$, because hypersurfaces in $\mathbb{P}^n$ of dimension ≥ 2 having only isolated singularities are always irreducible.

Notation 1.2.28. Let $S_1, \ldots, S_r$ be analytic or admissible topological types. If an ample divisor H is fixed, then to simplify notation we write $V_d(S_1 + \ldots + S_r)$ for $V_{dH}(S_1 + \ldots + S_r)$, respectively $V_d^{irr}(S_1 + \ldots + S_r)$ for $V_{dH}^{irr}(S_1 + \ldots + S_r)$. Furthermore, we abbreviate $V_D(S + \ldots + S)$ by $V_D(rS)$.

1.2.3 Zero-dimensional schemes and H^1-vanishing

Again, $V \subset \mathbb{P}^N$ denotes a smooth, projective variety.

Definition 1.2.29. Let $S_1, \ldots, S_r$ be analytic or admissible topological singularity types, and let D be an ample divisor. Assume that $V_D(S_1 + \ldots + S_r) \neq \emptyset$. Then we call the germ of $V_D(S_1 + \ldots + S_r)$ at a hypersurface W *T-smooth* if it is smooth and has the "expected" codimension $\sum_{i=1}^{r} \tau^s(S_i)$ in $|D|$.

Given a hypersurface having only isolated singularities, we associate to it certain zero-dimensional schemes concentrated in the singular points. The cohomology groups of the ideal sheaves of these schemes contain fundamental geometric information.

Definition 1.2.30. Let $W \subset V$ be a hypersurface having r isolated points $z_1, \ldots, z_r$ as its only singularities. Then we introduce the following schemes (cf. also Proposition 1.2.15):

(1) $X^{ea}(W) := \bigcup_{i=1}^r X^{ea}(W, z_i)$ with $\mathcal{I}_{X^{ea}(W),z_i} = I^{ea}(W, z_i)$.

(2) $X^{es}(W) := \bigcup_{i=1}^r X^{es}(W, z_i)$ with $\mathcal{I}_{X^{es}(W),z_i} = I^{es}(W, z_i)$.

We write $X^s(W)$ for either $X^{ea}(W)$ or $X^{es}(W)$.

The following theorem shows that the first cohomology group of the ideal sheaf of $X^s(W)$ obstructs the T-smoothness of the equianalytic stratum.

Theorem 1.2.31. *Let $W \in |D|$ be a reduced hypersurface in V having r singular points $z_1, \ldots, z_r$ of analytic or admissible topological types $S_1, \ldots, S_r$, and let $X^s = X^s(W)$.*

(i) The Zariski tangent space of $V_D(S_1+\ldots+S_r)$ at W is $H^0(\mathcal{I}_{X^s/V}(W))/H^0(\mathcal{O}_V)$ and

$$h^0(\mathcal{I}_{X^s/V}(W)) - h^1(\mathcal{I}_{X^s/V}(W)) \leq dim(V_D(S_1+\ldots+S_r), W) + 1 \leq h^0(\mathcal{I}_{X^s/V}(W)).$$

In particular, if $h^1(\mathcal{I}_{X^s/V}(W)) = 0$ then $V_D(S_1 + \ldots + S_r)$ is T-smooth at W.

(ii) If $h^1(\mathcal{I}_{X^s/V}(W)) = 0$ then all possible local deformations of the singular points of W are realizable when varying W in $|D|$.

Proof. See [GrK89] for the equianalytic case and [GrL96] for equisingular case. □

Remark 1.2.32. If $h^1(\mathcal{I}_{X^{ea}/V}(W)) = 0$, then the natural map of germs

$$(H^0(\mathcal{O}_V(D)), W) \longrightarrow \coprod_{z_i \in \mathrm{Sing}(W)} Def(W, z_i)$$

is surjective, where $Def(W, z_i)$ denotes the semi-universal deformation of (W, z_i). In other words, any local deformation of the singular points of W can be realized simultaneously when varying W in $|D|$.

Remark 1.2.33. Let $W \in |D|$ be a reduced hypersurface with isolated singularities $z_1, \ldots, z_r$ of (analytic or admissible topological) types $S_1, \ldots, S_r$, and some additional singularities $w_1, \ldots, w_s$ of arbitrary types. We denote by

$$V_D(S_1 + \ldots + S_r; W) \subset |D|$$

the germ at W of the variety of reduced hypersurfaces with singularities $z'_1, \ldots, z'_r$ in neighbourhoods of $z_1, \ldots, z_r$ having the same types $S_1, \ldots, S_r$. Note that the hypersurfaces are allowed to have also singularities other than the r prescribed ones. Then it can be shown that $V_D(S_1 + \ldots + S_r; W)$ is T-smooth if $h^1(\mathcal{I}_{X/V}(W)) = 0$, where $X = \bigcup_{i=1}^r X^s(W, z_i)$ (see [GrL96]).

The schemes defined by powers of the maximal ideal are of particular interest.

Definition 1.2.34. Let $\underline{m} = (m_1, \ldots, m_r) \in \mathbb{N}^r$. We introduce the zero-dimensional scheme $X(\underline{m}; \underline{z}) \subset V$ of r multiple points z_i of multiplicities m_i, which is defined by the ideal sheaf $\mathcal{I}_{X(\underline{m};\underline{z})/V}$ with

$$(\mathcal{I}_{X(\underline{m};\underline{z})/V})_z \;=\; \begin{cases} \mathfrak{m}_{V,z_i}^{m_i}, & z = z_i,\ i = 1, \ldots, r \\ \mathcal{O}_{V,z} & else\,. \end{cases}$$

We call $X(\underline{m}; \underline{z})$ a *fat point scheme.* If the points $z_1, \ldots, z_r$ are in general position then $X(\underline{m}; \underline{z})$ is a *generic fat point scheme*, and we denote it by $X(\underline{m})$.

The degree of a fat point scheme $X(\underline{m}; \underline{z}) \subset \mathbb{P}^n$ can be determined as follows:

$$\deg(X(\underline{m}; \underline{z})) = \sum_{i=1}^{r} dim_{\mathbb{C}} \mathcal{O}_{\mathbb{P}^n, z_i} / \mathfrak{m}_{\mathbb{P}^n, z_i}^{m_i} = \sum_{i=1}^{r} \binom{m_i + n - 1}{n}.$$

The Horace Method. The Horace method introduced by Hirschowitz ([Hir85]) is an inductive method, which can be used to prove H^1-vanishing theorems for zero-dimensional schemes concentrated in generic points. It is based on the *residual exact sequence* (cf. below) and on specializing the position of points.

Definition 1.2.35. Let W be a divisor in a smooth variety V and X a closed subscheme of V. Then we define

(i) the *trace of* X *on* W to be the scheme theoretic intersection $X \cap W$ defined by the ideal sheaf

$$\mathcal{I}_{(X \cap W)/W} = (\mathcal{I}_{X/V} + \mathcal{I}_{W/V}) \cdot \mathcal{O}_W \subset \mathcal{O}_W\,,$$

(ii) and the *residual of* X *with respect to* W, $X : W$, given by the ideal sheaf

$$\mathcal{I}_{(X:W)/V} = \mathcal{I}_{X/V} : \mathcal{I}_{W/V} \subset \mathcal{O}_V\,.$$

Note that $X : W$ is a closed subscheme of V.

The following lemma is easily verified.

Lemma 1.2.36. *Let V be a smooth projective variety and W a divisor in V. For all divisors D we have an exact sequence*

$$0 \longrightarrow \mathcal{I}_{(X:W)/V}(D - W) \longrightarrow \mathcal{I}_{X/V}(D) \longrightarrow \mathcal{I}_{(X \cap W)/W}(D) \longrightarrow 0\,. \qquad (1.2.3)$$

We call the exact sequence (1.2.3) the *reduction sequence (of X with respect to W)*.

Now consider the case $V = \mathbb{P}^n$. Let X be a zero-dimensional scheme concentrated in points $z_1, \dots, z_r \in \mathbb{P}^n$, and let H a hyperplane in $\mathbb{P}^n$. For any $d \in \mathbb{Z}$, we have an exact sequence

$$\dots \longrightarrow H^1(\mathcal{I}_{(X:H)/\mathbb{P}^n}(d-1)) \longrightarrow H^1(\mathcal{I}_{X/\mathbb{P}^n}(d)) \longrightarrow H^1(\mathcal{I}_{(X\cap H)/H}(d)) \longrightarrow \dots .$$

In order to show $h^1(\mathcal{I}_{X/\mathbb{P}^n}(d)) = 0$ it is sufficient to verify that

$$h^1(\mathcal{I}_{(X:H)/\mathbb{P}^n}(d-1)) = 0 \text{ and } h^1(\mathcal{I}_{(X\cap H)/H}(d)) = 0\,.$$

Note that the first condition is a condition in *lower degree* while the latter condition is a condition in *lower dimension*, which makes this procedure well suited for inductive proofs.

Let us discuss two simple applications of the Horace method. The following lemma is useful in combination with Bertini's theorem (cf. Appendix A).

Lemma 1.2.37. *Let $X \subset \mathbb{P}^n$ be a zero-dimensional scheme and assume that*

$$h^1(\mathcal{I}_X(d-1)) = 0\,.$$

Then the linear system $|H^0(\mathcal{I}_X(d))|$ has no base points outside the support of X.

Proof. Assume that $w \notin \operatorname{supp}(X)$ is a base point of $|H^0(\mathcal{I}_X(d))|$. Hence

$$H^0(\mathcal{I}_X(d)) = H^0(\mathcal{I}_{X\cup\{w\}}(d))\,. \tag{1.2.4}$$

Let $H \subset \mathbb{P}^n$ be a hyperplane with $w \in H$ and $X \cap H = \emptyset$. The reduction sequence for X with respect to H together with the assumption implies that

$$H^1(\mathcal{I}_{X\cup\{w\}}(d)) = 0\,.$$

But this implies that we obtain an exact sequence

$$0 \longrightarrow H^0(\mathcal{I}_{X\cup\{w\}}(d)) \longrightarrow H^0(\mathcal{I}_X(d)) \longrightarrow \underbrace{H^0(\mathcal{I}_{\{w\}}(d))}_{\neq\{0\}} \longrightarrow 0\,,$$

which contradicts (1.2.4). □

The following lemma is a simple H^1-vanishing condition.

Lemma 1.2.38. *Assume that $H_1 \subset \ldots \subset H_{n-1} \subset \mathbb{P}^n$ is a chain of linear subspaces with $H_i \cong \mathbb{P}^i$, $i = 1, \dots, n-1$. Let $X_1, \dots, X_n \subset \mathbb{P}^n$ be zero-dimensional schemes satisfying* *(i)* $X_i \subset H_i$ *for* $i = 1, \ldots, n-1$,

(ii) $X_i \cap H_{i-1} \subseteq X_{i-1}$ *for* $i = 2, \ldots, n$,

(iii) $X_i \subset H_{i-1}^{s_i} := s_i H_{i-1}$ *for* $i = 2, \ldots, n$, *and*

(iv) $\deg(X_1) \leq s_1$.

Then

$$h^1(\mathcal{I}_{X_n/\mathbb{P}^n}(d)) = 0 \textit{ if } d \geq s_1 + \ldots + s_n \,.$$

Before starting the proof, we would like to mention an elementary property, which we shall use several times. Assume that $X \subset Y$ are zero-dimensional schemes in V. Then

$$h^1(\mathcal{I}_{Y/V}(W)) = 0 \implies h^1(\mathcal{I}_{X/V}(W)) = 0\,,$$

since the support of $\mathcal{I}_X/\mathcal{I}_Y$ is zero-dimensional and hence $h^1(\mathcal{I}_X/\mathcal{I}_Y) = 0$.

Proof. We use induction over the dimension n.
For $n = 1$ the assumptions imply $h^1(\mathcal{I}_{X_1}(s_1)) = 0$ by the Riemann-Roch Theorem for curves because $deg(X_1) \leq s_1 \leq d < d+1$.

Now assume the claim is true for n, and let $H_1, \ldots, H_n$ and $X_1, \ldots, X_{n+1}$ satisfy the assumptions of the lemma. Hence $h^1(\mathcal{I}_{X_n}(d)) = 0$ whenever $d \geq s_1 + \ldots + s_n$.
We have to show that $h^1(\mathcal{I}_{X_{n+1}}(d)) = 0$ if $d \geq s_1 + \ldots + s_{n+1}$. Note that it is enough to verify

(i) $h^1(\mathcal{I}_{X_{n+1}:H_{n+1}^j \cap H_{n+1}}(d-j)) = 0$ for $j = 0, \ldots, s_{n+1}-1$, and

(ii) $h^1(\mathcal{I}_{X_{n+1}:H_{n+1}^{s_{n+1}}}(d - s_{n+1})) = 0$.

The first assertion is true since by assumption $X_{n+1} : H_{n+1}^j \cap H_{n+1} \subseteq X_n$ and by induction hypothesis $h^1(\mathcal{I}_{X_n}(d-j)) = 0$ as $d - j \geq s_1 + \ldots + s_n$ for $j \leq s_{n+1}$.
The second claim follows since $X_{n+1} : H_{n+1}^{s_{n+1}} = \emptyset$. □

1.3 Newton polytopes, toric geometry and isolated singular points

1.3.1 Newton polytope and Newton diagram

We recall the definition of the Newton polytope and Newton diagram and introduce some notation related to them.

Definition 1.3.1. (i) A *convex polytope* Δ in $\mathbb{R}^n$ is the convex hull of a discrete set $\mathcal{A} \subset \mathbb{R}^n$. A convex polytope is called *non-degenerate* if $\dim(\Delta) = n$.

(ii) Let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be an affine linear function with $\varphi(\Delta) \subset \mathbb{R}_{\geq 0}$. Then the intersection $\sigma := \{\varphi(x) = 0\} \cap \Delta$ is called a *face* of Δ. A face of dimension $dim(\Delta) - 1$ is called a *facet*. A zero dimensional face is a *vertex* of Δ.

(iii) If all vertices of Δ are integral, i.e. $\mathcal{A}$ can be chosen as $\mathcal{A} \subset \mathbb{Z}^n$, then Δ is called a *Newton polytope*.

(iv) For a Newton polytope Δ we denote by $\deg(\Delta) = n! \cdot \mathrm{vol}(\Delta)$ the *affine volume* or *degree* of Δ. Here $\mathrm{vol}(\Delta)$ denotes the usual euclidean volume of Δ.

Let $f = \sum_{\omega \in \mathbb{Z}^n} a_\omega \cdot \mathbf{x}^\omega \in \mathbb{C}[x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}]$ be a Laurent polynomial. The finite set $\mathrm{supp}(f) = \{\omega \in \mathbb{Z}^n \mid a_\omega \neq 0\}$ is called the *support* of f. The convex hull of $supp(f)$ is the *Newton polytope* of f and is denoted by $\Delta(f)$, or just by Δ if no confusion arises. The combinatorial structure of the set $\Delta(f)$ contains interesting information about the hypersurface defined by f.
We are only considering convex polytopes arising in this way. So when we speak about convex polytopes, we shall always mean convex polytopes having integral vertices. Furthermore we usually assume $\Delta \subset \mathbb{R}^n_{\geq 0}$.

Definition 1.3.2. (i) Let Δ be a convex polytope. We denote by $\mathcal{P}(\Delta)$ the space of (Laurent-)polynomials with monomials from Δ, i.e.

$$\mathcal{P}(\Delta) = \Big\{ f = \sum_{\omega \in \Delta \cap \mathbb{Z}^n} a_\omega \mathbf{x}^\omega \Big\}.$$

(ii) Let $f = \sum_{\omega \in \Delta \cap \mathbb{Z}^n} a_\omega x^\omega \in \mathcal{P}(\Delta)$ and let $\sigma \subset \partial\Delta$ be a face of Δ. The polynomial

$$f^\sigma = \sum_{\omega \in \sigma \cap \mathbb{Z}^n} a_\omega \mathbf{x}^\omega$$

is called the *σ-truncation of f*.

If Δ is contained in $\mathbb{R}^n_{\geq 0}$ then we consider $\mathcal{P}(\Delta)$ also as a subspace of $H^0(\mathcal{O}_{\mathbb{P}^n}(d))$ where $d \geq \max\{|\omega| \mid \omega \in \Delta\}$. We introduce the notation Δ^n_d for the simplex

$$\Big\{ (x_1, \ldots, x_n) \in \mathbb{R}^n | x_i \geq 0, \sum_{i=1}^n x_i \leq d \Big\}.$$

such that $\mathcal{P}(\Delta^n_d)$ corresponds to the complete linear system $|\mathcal{O}_{\mathbb{P}^n}(d)|$. We abbreviate $\mathcal{P}(\Delta^n_d)$ by $\mathcal{P}(d)$ if no confusion arises. The polytope Δ^2_d is shown in Figure 1.2.

Let us denote by $\mathrm{Aff}_n(\mathbb{Z})$ the group of affine lattice isomorphisms, i.e.

$$\mathrm{Aff}_n(\mathbb{Z}) = \{A : \mathbb{Z}^n \to \mathbb{Z}^n \mid A(\omega) = B\omega + b, \text{ with } B \in Gl_n(\mathbb{Z}),\ b \in \mathbb{Z}^n\}.$$

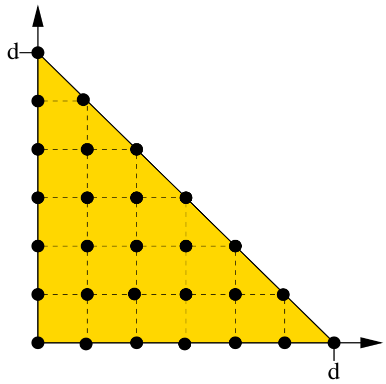


Figure 1.2: The polytope Δ_d^2: The circles correspond to points $(i,j) \in \Delta_d^2 \cap \mathbb{Z}^2$, and hence to monomials $x^i y^j \in \mathbb{C}[x,y]$ of degree less than d.

Definition 1.3.3. The group $\mathrm{Aff}_n(\mathbb{Z})$ acts naturally on $\mathbb{C}[x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}]$ by sending

$$f(\mathbf{x}) = \sum_{\omega \in \mathbb{Z}^n} a_\omega \mathbf{x}^\omega \quad \text{to} \quad f^A(\mathbf{x}) = \sum_{\omega \in \mathbb{Z}^n} a_\omega \mathbf{x}^{A(\omega)}\,, \qquad A \in \mathrm{Aff}_n(\mathbb{Z}).$$

Note that $\Delta(f^A) = A(\Delta)$.

Remark 1.3.4. Any $A \in \mathrm{Aff}_n(\mathbb{Z})$ induces an automorphism of $(\mathbb{C}^*)^n$ by

$$\begin{array}{ccc} (\mathbb{C}^*)^n & \longrightarrow & (\mathbb{C}^*)^n \\ (x_1, \ldots, x_n) & \mapsto & (\mathbf{x}^{A \cdot e_1}, \ldots, \mathbf{x}^{A \cdot e_n}), \end{array}$$

where e_i denotes the i-th unit vector. Thus A induces an isomorphism between $V(f) \cap (\mathbb{C}^*)^n$ and $V(f^A) \cap (\mathbb{C}^*)^n$.

While the Newton polytope contains global information about the hypersurface

$$V(f) = \{\mathbf{x} \in \mathbb{C}^n \mid f(\mathbf{x}) = 0\}\,,$$

the *Newton diagram* contains information about the hypersurface locally at the origin.

Definition 1.3.5. Let $f \in \mathbb{C}[\mathbf{x}]$ be a polynomial with Newton polytope Δ. Let $K_0(f)$ be the closure of the set $\mathrm{Conv}(\{0\} \cup \Delta) \setminus \Delta$. The *Newton diagram* $\Gamma(f, 0)$ of f at the origin is defined by

$$\Gamma(f, 0) := K_0(f) \cap \Delta\,.$$

If $\mathbf{a} \in \mathbb{C}^n$, then we define $\Gamma(f, \mathbf{a}) := \Gamma(f(\mathbf{x} - \mathbf{a}), 0)$.

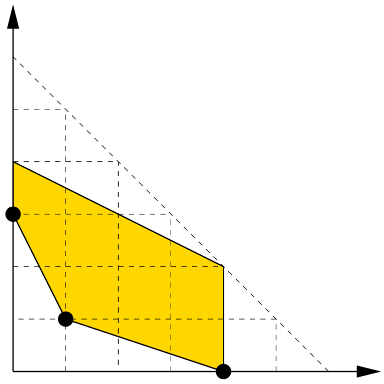


Figure 1.3: Example: The Newton polytope of $f = x^4 + xy + y^3 + y^4 + x^4y^2$. The circles correspond to the integral points of the Newton diagram at the origin.

Remark 1.3.6. From the Newton polytope we can read off information concerning the behaviour of the corresponding hypersurfaces at the coordinate hyperplanes, in particular at the fundamental points. Note that fixing the Newton polytope is equivalent to fixing the degree and the local Newton diagrams at the $n+1$ fundamental points $(0 : ... : 1 : ... : 0)$.

For example the polynomial $f = x^4 + xy + y^3 + y^4 + x^4y^2$ has A_1 singularities at $(0 : 0 : 1)$ and $(1 : 0 : 0)$ and an A_3 singularity at $(0 : 1 : 0)$, which can be read off the Newton polytope shown in Figure 1.3.

1.3.2 Toric varieties from convex polytopes

Definition 1.3.7. A *complex toric variety* is an irreducible, complex algebraic variety X equipped with an action of the algebraic torus $(\mathbb{C}^*)^n$ having an open dense orbit.

We associate to any convex lattice polytope a toric variety in the following way.

Definition 1.3.8. Let $\Delta \subset \mathbb{R}^n$ be a lattice polytope of dimension n. Assume that $\Delta \cap \mathbb{Z}^n = \{\omega_0, \ldots, \omega_N\}$ and consider the map

$$\varphi : (\mathbb{C}^*)^n \longrightarrow \mathbb{P}^N, \quad \varphi(\mathbf{x}) = (\mathbf{x}^{\omega_0} : \ldots : \mathbf{x}^{\omega_N}).$$

Then we define $\mathrm{Tor}(\Delta)$ to be the closure in the Zariski topology of $\varphi((\mathbb{C}^*)^n) \subset \mathbb{P}^N$. We denote the coordinates of $\mathbb{P}^N$ by $(z_{\omega_0} : \ldots : z_{\omega_N})$.

Theorem 1.3.9. $\mathrm{Tor}(\Delta)$ *is a projective, normal, toric variety of degree*

$$\deg(\Delta) = n! \cdot \mathrm{vol}(\Delta).$$

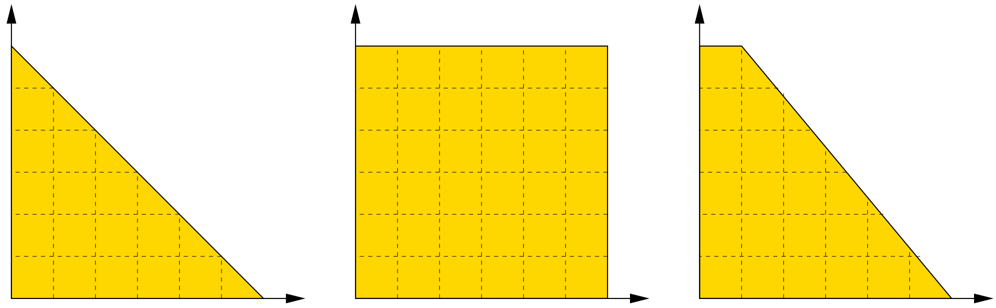


Figure 1.4: Polytopes defining $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{F}_3$.

Furthermore, there is a $1:1$ *correspondence between faces of* Δ *and* $(\mathbb{C}^*)^n$ *orbits, and for any face* σ *of* Δ *of dimension* k*, there is a* k*-dimensional orbit* $O_\sigma \cong (\mathbb{C}^*)^k$ *given by*

$$O_\sigma = \mathrm{Tor}(\Delta) \cap \{(z_\omega)_{\omega \in \Delta \cap \mathbb{Z}^n} \mid z_\omega = 0 \textit{ for } \omega \notin \sigma \textit{ and } z_\omega \neq 0 \textit{ for } \omega \in \sigma\}\,.$$

Proof. See [GKZ94]. □

Remark 1.3.10. (i) The same toric variety may arise from different polytopes. For example $\mathrm{Tor}(\Delta) \cong \mathrm{Tor}(A(\Delta))$ for any $A \in \mathrm{Aff}_n(\mathbb{Z})$.

(ii) The embedding $\mathrm{Tor}(\Delta) \subset \mathbb{P}^N$ corresponds to a very ample sheaf $\mathcal{O}_\Delta$ on $\mathrm{Tor}(\Delta)$ satisfying $H^0(\mathcal{O}_\Delta) \cong \mathcal{P}(\Delta)$, and we denote the global section of $\mathcal{O}_\Delta$ corresponding to a lattice point ω by χ^ω.

Example 1.3.11. Let us discuss some examples (cf. Figure 1.4).

(1) Let $n, d \geq 1$, and let $\Delta = \Delta^n_d$. Then $\mathrm{Tor}(\Delta)$ is the image of $\mathbb{P}^n$ under its d-uple embedding and $\mathcal{O}_\Delta \cong \mathcal{O}_{\mathbb{P}^n}(d)$.

(2) Let $a, b \geq 1$, and let $\Delta = \mathrm{Conv}\{(0,0), (a,0), (0,b), (a,b)\}$. Then $\mathrm{Tor}(\Delta) \cong \mathbb{P}^1 \times \mathbb{P}^1$ and $\mathcal{O}_\Delta \cong \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(a,b)$.

(3) Let $a \geq 1$, and let Δ_a be the rectangle $\mathrm{Conv}\{(0,0), (2a,0), (0,2a), (1,2a)\}$. Then $\mathrm{Tor}(\Delta) \cong \mathbb{F}_a$, the Hirzebruch surface $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(a))$.

Real part of complex toric varieties, the moment map and charts of polynomials.

Inside a complex toric variety $\mathrm{Tor}(\Delta)$ we have its real part $\mathrm{Tor}_{\mathbb{R}}(\Delta)$, which is the intersection of $\mathrm{Tor}(\Delta) \subset \mathbb{P}^N$ with $\mathbb{RP}^n \subset \mathbb{P}^N$. The real algebraic torus $(\mathbb{R}^*)^n$ is the union of 2^n orthants

$$\mathbb{R}^n_\varepsilon = \{(x_1, \ldots, x_n) \mid \varepsilon_i x_i > 0\}$$

where $\varepsilon \in \{\pm 1\}^n$. We define the subspaces $\mathrm{Tor}_\varepsilon(\Delta) \subseteq \mathrm{Tor}_{\mathbb{R}}(\Delta)$ to be the closure (in the real topology) of the subsets

$$\mathbb{R}^n_\varepsilon \subseteq (\mathbb{R}^*)^n \hookrightarrow \mathrm{Tor}_{\mathbb{R}}(\Delta).$$

Thus,

$$\mathrm{Tor}_{\mathbb{R}}(\Delta) = \bigcup_{\varepsilon \in \{\pm 1\}^n} \mathrm{Tor}_\varepsilon(\Delta).$$

We abbreviate the sign vector $\varepsilon = (1, \ldots, 1)$ by $+$. Using the same philosophy, we define the real parts $(O_\sigma)_{\mathbb{R}}$ of the orbits and decompose them in the form

$$(O_\sigma)_{\mathbb{R}} = \bigcup_{\varepsilon \in \{\pm 1\}^n} (O_\sigma)_\varepsilon.$$

Definition 1.3.12. The *moment map* is the map $\mu := \mu_\Delta : \mathrm{Tor}(\Delta) \to \Delta$ defined by

$$\mu(x) = \frac{1}{\sum_{\omega \in \Delta \cap \mathbb{Z}^n} |\chi^\omega(x)|} \cdot \sum_{\omega \in \Delta \cap \mathbb{Z}^n} |\chi^\omega(x)| \cdot \omega, \tag{1.3.5}$$

where we consider the sections χ^ω as functions $\mathrm{Tor}(\Delta) \to \mathbb{C}$.

In general, moment maps arise in the context of Hamiltonian actions of Lie groups on symplectic manifolds. For details about relations between general moment maps and the map defined above we refer to [Fu93] or [Ri92].

The following theorem is the key to a purely combinatorial description of the real toric variety $\mathrm{Tor}_{\mathbb{R}}(\Delta)$.

Theorem 1.3.13. *The moment map μ induces a real analytic homeomorphism from $\mathrm{Tor}_+(\Delta)$ onto Δ respecting the stratification of Δ by its faces.*
More precisely: for any face $\sigma \subseteq \Delta$, the corresponding orbit $(O_\sigma)_+$ is mapped by μ diffeomorphically onto the interior of σ.

Proof. See [Fu93]. □

Instead of $\mathbb{R}^n_+$ we can consider any other orthant $\mathbb{R}^n_\varepsilon$. Then completely analogous results are true for $\mathrm{Tor}_\varepsilon(\Delta)$. In particular, Theorem 1.3.13 implies that, from the topological point of view, $\mathrm{Tor}_{\mathbb{R}}(\Delta)$ can be thought of as 2^n copies of Δ glued together appropriately.

For every vector $\varepsilon \in \{\pm 1\}^n$ take a copy Δ_ε of Δ and for every face σ of Δ denote the corresponding face in Δ_ε by σ_ε. We say that two vectors $\varepsilon, \delta \in \{\pm 1\}^n$ *agree on a face* $\sigma \subseteq \Delta$ if either $\varepsilon^\omega = \delta^\omega$ for all $\omega \in \sigma \cap \mathbb{Z}^n$ or $\varepsilon^\omega \neq \delta^\omega$ for all $\omega \in \sigma \cap \mathbb{Z}^n$.

Proposition 1.3.14. *The real toric variety* $\mathrm{Tor}_{\mathbb{R}}(\Delta)$ *is homeomorphic to* 2^n *copies of* Δ *glued together by the following recipe: for any* $\varepsilon, \delta \in \{\pm 1\}^n$ *and any face* $\sigma \subseteq \Delta$ *identify* σ_ε *with* σ_δ *if* ε *and* δ *agree on* σ*.*

Proof. By the description of the orbits given in Theorem 1.3.9, we have $(O_\sigma)_\varepsilon = (O_\sigma)_\delta$ if and only if either $\varepsilon^\omega = \delta^\omega$ or $\varepsilon^\omega \neq \delta^\omega$ for all $\omega \in \sigma$. □

Let f be a real (Laurent) polynomial

$$f(\mathbf{x}) = \sum_{\omega \in \mathbb{Z}^n} a_\omega \cdot \mathbf{x}^\omega$$

and Δ its Newton polytope. We usually assume that Δ is *nondegenerate*, i.e. $\dim(\Delta) = n$.

Let us consider the zero set of f in the toric variety $\mathrm{Tor}_{\mathbb{R}}(\Delta)$, by which we mean the closure of $V(f) \subset (\mathbb{R}^*)^n$ under the embedding $(\mathbb{R}^*)^n \hookrightarrow \mathrm{Tor}_{\mathbb{R}}(\Delta)$, and which we also denote by $V(f)$ by abuse of notation. We map the intersection of this toric hypersurface with each orthant into a separate copy of Δ by means of the moment map. This leads to the definition of the chart of a polynomial.

Definition 1.3.15. Let $f(\mathbf{x}) \in \mathbb{R}[\mathbf{x}]$ and let Δ be its Newton polytope. For every $\varepsilon \in \{\pm 1\}^n$ let Δ_ε be a copy of Δ in the orthant $\mathbb{R}^n_\varepsilon$ (i.e. the image of Δ under the map $\mathbb{R}^n \to \mathbb{R}^n$, $(x_1, \ldots, x_n) \mapsto (\varepsilon_1 x_1, \ldots, \varepsilon_n x_n)$). Define

$$\mathrm{Ch}_\varepsilon(f) = \mu(Z(f)) \cap \mathrm{Tor}_\varepsilon(\Delta) \subset \Delta_\varepsilon \,.$$

The *chart of* f is the union

$$\mathrm{Ch}(f) = \bigcup_{\varepsilon \in \{\pm 1\}^n} \mathrm{Ch}_\varepsilon(f).$$

Remark 1.3.16. Clearly $\mathrm{Ch}_\varepsilon(f) \cong \mathrm{Ch}_+(f^\varepsilon)$ where $f^\varepsilon(x_1, \ldots, x_n) = f(\varepsilon_1 x_1, \ldots, \varepsilon_n x_n)$. This implies that it is often enough just to consider the positive orthant.

Example 1.3.17. Consider the space of affine linear polynomials, i.e. polynomials of the form

$$f(x) = a_0 + \sum_{i=1}^{n} a_i x_i \,.$$

Assume $a_i \neq 0$ for all i, so that the zero locus of f is a hyperplane in $\mathbb{RP}^n$ transversal to all the coordinate hyperplanes and the Newton polytope $\Delta(f)$ is the simplex

$$\left\{ (t_1, \ldots, t_n) \,\middle|\, t_i \geq 0,\ \sum t_i \leq 1 \right\} \,.$$

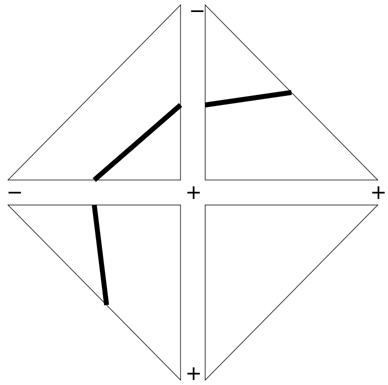


Figure 1.5: Example: The 4 charts of an affine linear polynomial f. The signs correspond to the signs of the coefficients of f^ε, $\varepsilon \in \{\pm 1\}^2$.

The corresponding real toric variety is $\mathbb{RP}^n$ and $x_1, \ldots, x_n$ are coordinates in an affine chart. In this chart the moment map is given by

$$\mu(x) = \left(\tfrac{|x_1|}{1+\sum |x_i|}, \ldots, \tfrac{|x_n|}{1+\sum |x_i|} \right).$$

Consider the intersection of $Z(f)$ with the positive part of $\mathrm{Tor}_{\mathbb{R}}(\Delta) = \mathbb{RP}^n$, i.e. the set of zeros of $f(x_1, \ldots, x_n) = 0$ with all $x_i > 0$. The image of this set under the moment map is

$$\left\{ (t_1, \ldots, t_n) \in \Delta \,\middle|\, a_0 \Big(1 - \sum_{i=1}^{n} t_i \Big) + \sum_{i=1}^{n} a_i t_i = 0 \right\},$$

which is a hyperplane section in Δ separating the vertices of Δ with $a_i < 0$ from those with $a_i > 0$. In particular, the chart of linear polynomials only depends on the signs of the coefficients (cf. Figure 1.5).

1.3.3 Singular points and combinatorics of the Newton polytope

In this section we discuss some connections between Newton polytopes and isolated singular points. Assume that Δ is a Newton polytope and $f \in \mathcal{P}(\Delta)$. For the patchworking method, which shall be described in Chapter 2, we need to understand the intersections of the zero locus $Z(f) \subset \mathrm{Tor}(\Delta)$ with $\mathrm{Tor}(\sigma)$ for faces σ of Δ.

The following lemma is obvious.

Lemma 1.3.18. *Let σ be a face of Δ and consider $\mathrm{Tor}(\sigma) \subseteq \mathrm{Tor}(\Delta)$. Then*

$$Z(f) \cap \mathrm{Tor}(\sigma) = Z(f^\sigma) \subseteq \mathrm{Tor}(\sigma).$$

Definition 1.3.19. Let $\Delta \subset \mathbb{R}^n$ be a Newton polytope and $f \in \mathcal{P}(\Delta)$.

(i) Let $\sigma \subset \partial\Delta$ be a face. Then f is called *non-singular along* σ (respectively *non-critical along* σ) if the truncation f^σ has no singular point (respectively critical point) in $(\mathbb{C}^*)^n$.

(ii) f is called *peripherally non-singular (PNS)* if f is non-singular on all proper faces σ of Δ, and has only isolated singular points in $(\mathbb{C}^*)^n$.

(iii) f is called *peripherally non-critical (PNC)* if f is non-critical on all proper faces σ of Δ, and has only isolated critical points in $(\mathbb{C}^*)^n$.

Remark 1.3.20. (i) If $f \in \mathcal{P}(\Delta)$ is non-singular along a face $\sigma \subset \Delta$. Then the intersection $Z(f) \cap \mathrm{Tor}(\sigma) \subset \mathrm{Tor}(\Delta)$ is transversal.

(ii) Let $\Delta = \Delta_d^n$ and $f \in \mathcal{P}(\Delta)$ with only isolated singular points. Then there exist a linear coordinate change such that f is PNS. Just choose $n+1$ hyperplanes H_i as coordinate hyperplanes such that $V(f)$ is non-singular along all the H_i.

Since the hyperplane sections of $\mathrm{Tor}(\Delta)$ are compactifications of zero loci of polynomials from $\mathcal{P}(\Delta)$ and $\deg(\mathrm{Tor}(\Delta)) = \deg(\Delta)$ by Theorem 1.3.9 we expect that n generic (Laurent) polynomials from $\mathcal{P}(\Delta)$ have $\deg(\Delta)$ common solutions in $(\mathbb{C}^*)^n$.

Kouchnirenko proved the following theorem [Kou76].

Theorem 1.3.21. *Let $\Delta \subset \mathbb{Z}^n$ be an integral, convex, nondegenerate polytope. If $f_1, \dots, f_n \in \mathcal{P}(\Delta)$ are PNC polynomials, then the number of common solutions of $f_1, \dots, f_n$ in $(\mathbb{C}^*)^n$ (counted with multiplicity) is equal to* $\deg(\Delta) = n! \cdot \mathrm{vol}(\Delta)$.

Proof. [Kou76]. □

Concluding this section, we introduce a special class of local deformations, which play an important role in the local patchworking method. They are called *lower deformations*, because we deform with monomials which correspond to lattice points *below* the Newton diagram.

Definition 1.3.22. Let $(W, z) \subset (\mathbb{C}^n, z)$ be a semi-quasihomogeneous singularity whose initial form is of type (α, d) such that $\alpha_i | d$ for all i, i.e. the hyperplane

$$\{\omega \in \mathbb{R}^n \mid \langle \alpha, \omega \rangle = d\}$$

meets the coordinate axes in integral points. Let

$$\Delta := \mathrm{Conv}\{\omega \in \mathbb{N}^n \mid \langle \alpha, \omega \rangle < d\}.$$

A *pattern for a lower deformation of* (W, z) is a hypersurface W_z defined by a polynomial $g(\mathbf{x}) \in \mathcal{P}(\Delta)$ such that the quasihomogeneous leading form of g is non-degenerate.

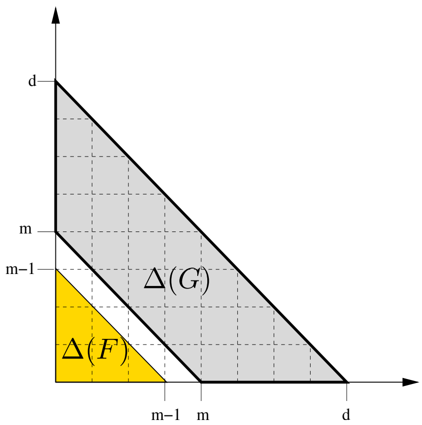


Figure 1.6: Example: Lower deformation deformation of an ordinary singular point of multiplicity m.

Example 1.3.23. Let $C \subset \mathbb{P}^2$ be defined by a polynomial F of degree d. Assume that C has an ordinary multiple point of multiplicity $m < d$ at the origin $(0:0:1)$, and that C does not pass through $(1:0:0)$ and $(0:1:0)$. Then the Newton diagram of F looks as in Figure 1.6, and a lower deformation of $(C, 0)$ is given by any polynomial G of degree $\leq m-1$ whose leading form is non-degenerate. Note that the latter property can be achieved by a generic linear coordinate change.

The local patchworking method, which we shall present in Chapter 2, is based on the idea that we can find a 1-parameter family F_t such that $F_0 = F$ and for $t > 0$ small enough, F_t has in a neighbourhood of the origin the same number and types of singularities as G in $\mathbb{C}^n$.

Chapter 2

Hypersurfaces with prescribed singularities

In this chapter, we give a detailed introduction to the asymptotic existence problem in order to distinguish it from the existence problem in small degrees. After that we review classical construction methods and previous existence results as well as restrictions for the existence of singular hypersurfaces.

The rest of the chapter is devoted to the patchworking method, which is our main tool for constructing hypersurfaces with prescribed singularities. The idea is essentially to construct a new hypersurface out of old ones, such that the new hypersurface inherits all the (isolated) singularities of the initial hypersurfaces. We present two different versions of this patchworking method (which we call local and global patchworking) and discuss some examples.

2.1 The asymptotic existence problem

Let again $V \subset \mathbb{P}^N$ be a fixed smooth variety. In the following we shall refer to the following problem as *the existence problem*:

> Given an ample linear system D and singularity types $S_1, \ldots, S_r$, decide whether $V_D(S_1 + \ldots + S_r)$ is non-empty.

Obviously, it is impossible to find complete solutions to this problem in this generality. In fact, the only case where a complete answer is known is the classical case of nodal curves in $\mathbb{P}^2$. It was shown by Severi [Sev21] that there is an irreducible curve of degree d having r nodes if and only if $r \leq \frac{(d-1)(d-2)}{2}$, i.e.

$$V_d^{irr}(rA_1) \neq \emptyset \Longleftrightarrow r \leq \frac{(d-1)(d-2)}{2}\,.$$

The main step of the proof was to show that nodes on plane curves can be "smoothed" independently.

In the general case, we may hardly expect that sufficient and necessary conditions for the existence agree. In fact, already for cuspidal plane curves, a complete answer is only known up to degree 11 (cf. below). Hence, the general existence problem consists of two parts:

(1) Find criteria in terms of certain invariants which prohibit the existence of hypersurfaces with a given collection of isolated singularities and belonging to the prescribed linear system. We shall refer to these criteria as *restrictions for the existence* or *upper bounds* (since we are primarily interested in the *maximal* number of singularities of certain types, and these criteria bound these numbers from above).

(2) Construct reduced hypersurfaces with prescribed singularities in a given linear system. As mentioned above, the most interesting case is to realize hypersurfaces with as many singularities of the prescribed types as possible. We shall refer to an existence result (expressed in terms of certain invariants) as a *lower bound* for the existence.

Example 2.1.1. Consider irreducible, plane curves of degree d having r nodes as only singularities. Then the genus formula implies the restriction

$$r \leq \frac{(d-1)(d-2)}{2}\,. \tag{2.1.1}$$

On the other hand, Severi showed that the singularities of a nodal, plane curve, irreducible or not, can be smoothed independently [Sev21]. Hence, by taking the union of d generic lines and smoothing out suitable intersection points, one can construct irreducible curves with any number of nodes allowed by (2.1.1).

Essentially there are two diametrical approaches to the constructive part of the existence problem.

Complete results for small degrees and very special singularities.
The idea of this approach is to obtain concrete maximal numbers of singularities of some type, which can be realized on curves or surfaces of some small fixed degree.

This approach was used in the following situations.

(1) Curves with ordinary cusps in $\mathbb{P}^2$.
The following table displays the maximal number of cusps of a (complex) cuspidal curve of degree less than 10. In all cases there exist curves corresponding to T-smooth germs, which implies that any cusp can be smoothed independently, and curves with lower number of cusps can be obtained in the same degree.

degree	2	3	4	5	6	7	8	9	10
number of cusps	0	1	3	5	9	10	15	20	26

For $d \leq 6$ the results are classical. For a detailed construction of a curve of degree 5 having 5 (real) cusps see [Gu82]. The cases $d = 7, 8, 9, 10$ can be found in [IS96, Sh98].
Note that for real curves the maximal number of singularities may be smaller than in the complex case. For example Shustin and Itenberg [IS96] showed that a real curve of degree 6 cannot have more than 7 cusps, which is less than in the complex case.

(2) Nodal surfaces in $\mathbb{P}^3$.
The following table displays upper and lower bounds for the number of ordinary nodes on a (complex) nodal surface of degree less than 10.

degree	2	3	4	5	6	7	8	9	10
upper bound	1	4	16	31	65	104	174	246	360
lower bound	1	4	16	31	65	93	168	216	345

For $d \leq 5$, the results are classical. The cases $d = 6, 10$ can be found in [Bar96], $d = 7, 9$ in [Ch92] and for $d = 8$ see [End97]. A nice overview can be found on [URL:Mainz].

(3) As an example of non-simple singularities, consider surfaces in $\mathbb{P}^3$ with ordinary triple points. They were studied in [EPS00, Ste03] and the following bounds were found.

degree	3	4	5	6	7
upper bound	1	1	5	10	17
lower bound	1	1	5	10	16

For other classes of singularities, or more general varieties V, almost nothing is known. Moreover, the construction methods are very specific to the singularity types and to the degrees of the curves or surfaces. Hence, for every new degree or other singularity type a new idea is needed.

The asymptotic approach.
In contrast to the approach described above, we try to find conditions for the existence in linear systems $|dH|$ for d sufficiently large. More precisely, this means the following:
Fix an ample divisor H, for example a hyperplane section, and a set of singularity types $\mathcal{S}$. We try to find sufficient conditions for the existence of hypersurfaces $W \in V_d(S_1 + \cdots + S_r)$, $S_1, \ldots, S_r \in \mathcal{S}$, of the form

$$\sum_{i=1}^{r} \sigma(S_i) < f(d)\,, \tag{2.1.2}$$

where $\sigma(S)$ is an invariant of the singularities $S \in \mathcal{S}$, and $f(d)$ is some function. Note that this implies in particular that if there exists $W \in V_d(S_1 + \cdots + S_r)$ satisfying (2.1.2), then there is also $W' \in V_d(S_1, \ldots, S_{r'})$ with $r' \leq r$. If $h^1(\mathcal{O}_V(d)) = 0$, then the latter property always holds for example for T-smooth hypersurfaces since we can smooth any of the singular points independently of the others by Remark 1.2.32.

In order to understand the asymptotic quality of a condition of the form (2.1.2) we have to find necessary conditions with the same left hand side. More precisely, this means that we try to find a function $g(d)$ such that there exist infinitely many $d \in \mathbb{N}$ and infinitely many $S_1, \ldots, S_r \in \mathcal{S}$ with

$$\sum_{i=1}^{r} \sigma(S_i) \leq g(d) \text{ and } V_d(S_1 + \cdots + S_r) = \emptyset\,. \tag{2.1.3}$$

Assume we have a sufficient condition (2.1.2) and a necessary condition (2.1.3). If the functions $f(d)$ and $g(d)$ satisfy $O(f) = O(g)$, then we call this an *asymptotically proper* solution to the given existence problem. In case f and g are polynomials of the same degree and their leading coefficients agree, we speak of an *asymptotically optimal* existence result.

The existence problem for $V = \mathbb{P}^n$.
In the most interesting case $V = \mathbb{P}^n$ we express the asymptotic existence problem in terms of certain invariants.

Definition 2.1.2. Fix $n \geq 2$, and let $\mathcal{S}$ be a set of some (analytic or topological) singularity types.

(i) We define $\mathcal{A}_n^{max}(\mathcal{S})$ to be the set of all $\alpha \in \mathbb{R}$ such that there exists a function $R(d) \leq \beta d^{n-1}$ with $\beta > 0$, depending only on n, α, and $\mathcal{S}$, with the property:

For all $\{S_1, \ldots, S_r\} \subset \mathcal{S}$ and infinitely many $k_1, k_2 \ldots$ and k_r there exists a $d \geq 0$ such that $V_d(k_1S_1 + \ldots + k_rS_r)$ is non-empty and

$$\sum_{i=1}^{r} k_i \tau^s(S_i) \geq \alpha \cdot d^n - R(d)\,.$$

We put $\alpha_n^{max}(\mathcal{S}) = \sup \mathcal{A}_n^{max}(\mathcal{S})$.

(ii) We define $\mathcal{A}_n^{reg}(\mathcal{S})$ to be the set of all $\alpha \in \mathbb{R}$ such that there exists a function $R(d) \leq \beta d^{n-1}$ with $\beta > 0$, depending only on n, α, and $\mathcal{S}$, with the property:

If for some subset $\{S_1, \ldots, S_r\} \subset \mathcal{S}$ and positive integers $k_1, \ldots, k_r$, the relation

$$\sum_{i=1}^{r} k_i \tau^s(S_i) \leq \alpha \cdot d^n - R(d)$$

holds true, then there is a hypersurface $W \in V_d(k_1S_1+\ldots+k_rS_r)$ corresponding to a T-smooth germ.

We put $\alpha_n^{reg}(\mathcal{S}) = \sup \mathcal{A}_n^{reg}(\mathcal{S})$.

Remark 2.1.3. Let us make some easy remarks concerning these invariants.

(a) If $\alpha_n^{max}(\mathcal{S}) \in \mathcal{A}_n^{max}(\mathcal{S})$, respectively $\alpha_n^{reg}(\mathcal{S}) \in \mathcal{A}_n^{reg}(\mathcal{S})$, then the coefficient β in Definition 2.1.2 can be chosen independently of α.

(b) Obviously $\alpha_n^{reg}(\mathcal{S}) \leq \alpha_n^{max}(\mathcal{S})$. Furthermore, we know

$$0 \leq \alpha_n^{reg}(\mathcal{S}) \leq \frac{1}{n!}\,,$$

since for a T-smooth hypersurface the total Tjurina number cannot exceed the dimension of the space of polynomials of degree d, which equals $\frac{d^n}{n!} + O(d^{n-1})$. Moreover,

$$0 \leq \alpha_n^{max}(\mathcal{S}) \leq 1\,,$$

since $\sum_{i=1}^{r} \mu(S_i) > (d-1)^n$ implies that $V_d(S_1 + \cdots + S_r) = \emptyset$ by Bézout's Theorem.

(c) However, it is not known whether $\alpha_n^{max}(\mathcal{S}) \neq 0$ for an arbitrary set $\mathcal{S}$. In case we know $\alpha_n^{max}(\mathcal{S}) > 0$ we have an *asymptotically proper* existence result for $\mathcal{S}$.

The main conjecture concerning the asymptotic existence problem for $\mathbb{P}^n$ states that there is a general sufficient condition ensuring existence with a function of order d^n on the right-hand side. More precisely,

Conjecture. *Let $\mathcal{S}$ be an arbitrary set of isolated singularity types of corank $\leq n$. Then*

$$\alpha_n^{reg}(\mathcal{S}) > 0\,.$$

Evidence for this conjecture can be found in Section 2.2 and in Chapter 3. In particular we will see that $\alpha_n^{reg}(\mathcal{S}) > 0$ if

- the Tjurina number in $\mathcal{S}$ is bounded, i.e. $\sup\{S \in \mathcal{S}\} < \infty$,
- $\mathcal{S}$ is a set of curve singularities, i.e. $\mathrm{corank}(S) \leq 2$ for $S \in \mathcal{S}$,
- $\mathcal{S}$ is a set of ordinary singularities,
- $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2$ with $\alpha_n^{reg}(\mathcal{S}_i) > 0$.

The case of hypersurfaces of minimal degree having only one prescribed singularity is of its own interest but, as we shall see in Section 2.3, plays also a major role in the application of the local patchworking method.

Definition 2.1.4. Let S be an (analytic or topological) singularity type.

(i) We denote by $d_n(S)$ the smallest natural number d for which there exists a hypersurface $W \subset \mathbb{P}^n$ of degree d having one singular point of type S as its only singularity.

(ii) We define $d_n^{reg}(S)$ as above but require furthermore that the hypersurface corresponds to a T-smooth germ.

Remark 2.1.5. Obviously $d_n(S) \leq d_n^{reg}(S)$. Furthermore,

$$\sqrt[n]{\mu(S)} + 1 \leq d_n(S) \leq \tau^{ea}(S) + 1$$

by Bézout and the finite determinacy theorem.

There is also a conjecture concerning this number $d_n(S)$, which is equivalent to the conjecture mentioned before.

Conjecture. *There exists an absolute constant $\alpha > 0$ such that*

$$d_n^{reg}(S) \leq \alpha \sqrt[n]{\tau^s(S)}$$

for any singularity type S of corank $\leq n$.

2.2 Overview of known necessary and sufficient conditions for the existence

2.2.1 Restrictions for the existence

The only effective way to proof non-existence of a hypersurface with certain singularity collection and belonging to a prescribed linear system, is to show that it would violate certain constraints on invariants. A common approach is consider some invariant which behaves semi-continuously under deformations, and derive properties for a general singular hypersurface from a special one (e.g. one having many ordinary singular points).

We review general restrictions for the existence and group them according to the surrounding variety V.

Singular hypersurfaces in $\mathbb{P}^n$.

(i) For every hypersurface $W \subset \mathbb{P}^n$ of degree d with $\mathrm{Sing}(W) = \{z_1, \ldots, z_r\}$ we have
$$\sum_{i=1}^{r} \mu(W, z_i) \leq (d-1)^n.$$
This is an immediate consequence of Bézout's Theorem applied to the intersection of n generic polars.

(ii) (Bruce's bound [Bru81]). Let $W \subset \mathbb{P}^n$ be a hypersurface of degree d with r isolated singularities. Then
$$r \leq \tfrac{1}{2} d^n + O(d^{n-1}).$$

(iii) (Spectrum bound [Var83]). Let $W \subset \mathbb{P}^n$ be a hypersurface of degree d with r isolated singularities having spectra $D_1, \ldots, D_r \in \mathrm{Div}(\mathbb{Q})$. Moreover, let $D \in \mathrm{Div}(\mathbb{Q})$ be the spectrum of an ordinary singular point of multiplicity d. Then for any open interval I of length 1 the following semi-continuity property holds
$$\sum_{i=1}^{r} |D_i \cap I| \;\leq\; |D \cap I|.$$
While this inequality often gives the best bounds in concrete examples, the asymptotic quality is only known in the case of nodal hypersurfaces.

Curves on general smooth, projective surfaces. Let $C \subset S$ be a curve having r singular points $z_1, \ldots, z_r$. Then the genus formula implies that
$$\sum_{i=1}^{r} \delta(C, z_i) \leq \frac{C^2 + C.K_S + 2}{2}.$$

Plane curves $C \subset \mathbb{P}^2$.

Let $C \subset \mathbb{P}^2$ be a plane curve of degree d having r singular points $z_1, \ldots, z_r$.

(1) The first Plücker formula implies
$$\sum_{i=1}^{r} (\mu(C, z_i) + \mathrm{mult}(C, z_i) - 1) \leq d^2 - d - 2.$$
In particular, if C has r_1 nodes and r_2 cusps, $r_1 + r_2 = r$, then
$$2r_1 + 3r_2 \leq d^2 - d - 2.$$

(2) The genus formula implies that $\sum_{i=1}^{r} \delta(C, z_i) \leq \frac{(d-1)(d-2)}{2}$.

(3) Applying the log-Miyaoka inequality F. Sakai [Sak93] proved that

$$\sum_{i=1}^{r} \mu(C, z_i) \leq \frac{2\nu}{2\nu+1}\Big(d^2 - \frac{3}{2}d\Big),$$

where ν denotes the maximum of the multiplicities $\mathrm{mult}(C, z_i)$, $i = 1, \ldots, r$.

(4) The best known upper bound for plane curves of degree d having r nodes and s cusps was given by Ivinskis [Iv85],

$$\frac{9}{8}r + 2s \leq \frac{5}{8}d^2 - \frac{3}{4}d.$$

Singular surfaces $S \subset \mathbb{P}^3$.

Assume that $S \subset \mathbb{P}^3$ is a normal surface of degree d having only log-canonical singularities. Then Wahl's generalization of Miyaoka's bound (cf. [Mi84, Wa94]) implies that

$$\sum_{z \in \mathrm{Sing}(S)} \mu(S, z) \leq \frac{4}{9} d(d-1)^2.$$

2.2.2 Asymptotic proper existence results

In this section we review previously known asymptotically proper existence results. Only in the case of curves, there are results dealing with general singularity types. In the higher-dimensional case, there are only asymptotically proper results in very few cases.

Existence of singular curves on surfaces

(i) **Plane curves.**

(a) General results. The first general asymptotically proper existence result for plane curves with singularities prescribed up to topological equivalence was proven by Greuel, Lossen and Shustin in [GLS98]. They showed that there exists a plane curve of degree with r singular points of prescribed topological types $S_1, \ldots, S_r$ and corresponding to a T-smooth germ if

$$\sum_{i=1}^{r} \mu(S_i) \leq \frac{1}{392}(d+2)^2.$$

This result has been improved successively ([GLS98, Los99]), and the best general sufficient condition up to now was found by Shustin [Sh01]. He proved that for arbitrary analytic or topological types $S_1, \ldots, S_r$, the condition

$$\sum_{i=1}^{r} \tau^s(S_i) \leq \frac{1}{9}d^2 + O(d)$$

implies the existence of a non-empty T-smooth component of $V_d(S_1+\cdots+S_r)$, i.e. in our notation

$$\alpha_2^{reg}(\mathcal{S}) \geq \frac{1}{9}\,.$$

(b) Cuspidal curves. The best general existence result for cuspidal plane curves was found by Hirano [Hi92] and reads $\alpha_2^{max}(A_2) \geq \frac{9}{16}$.
In particular, this implies that these curves cannot correspond to T-smooth germs. Hence already for plane curves with only ordinary cusps, the picture is much more complicated than for nodal curves, where the nodes always imposed independent conditions.

(c) Curves with ordinary nodes and cusps. Shustin proved the first asymptotically optimal existence result for curves with nodes and cusps [Sh93]. Using the patchworking method he proved that $\alpha_2^{reg}(A_1, A_2) = \frac{1}{2}$. Note that this was the only case where an asymptotically optimal result is known. This will be extended in Chapter 3.

(d) Curves with general tacnodes and cusps. Using the theory of Enriques diagrams, Roé [Ro01] showed that $\alpha_2^{reg}(\{A_k | k \geq 1\}) \geq \frac{1}{6}$. Our results presented in Chapter 3 improve this bound substantially.

(e) Curves with one A_k singularity. Lossen [Los99] generalized a construction of Gusein-Zade and Nekhoroshev [GZN83] to produce a series of polynomials f_k of degree d_k with one A_k singularity and satisfying

$$\lim_{k\to\infty} \frac{k}{d^2} = \frac{1}{2}\,.$$

Gusein-Zade and Nekhoroshev [GZN00] improved the coefficient to $\frac{15}{28}$.

(ii) **Curves on general surfaces.**

The first general asymptotically proper sufficient conditions for the existence for curves on more general surfaces S were found by Keilen and Tyomkin [KeT02] (cf. also [Sh01]). Let L be an ample divisor and D a divisor such that $D - K_S$ is nef. Then there exists an $\alpha > 0$ such that

$$\sum_{i=1}^{r} \delta(S_i) \leq \alpha(D - K_S - L)^2\,, \text{ respectively } \sum_{i=1}^{r} \mu(S_i) \leq \alpha(D - K_S - L)^2\,,$$

together with some additional conditions (cf. [KeT02]) implies the existence of a curve $C \in |D|$ with topological (respectively analytic) singularities of types $S_1, \ldots, S_r$.

The higher-dimensional case

We only mention the classical case of nodal surfaces in $\mathbb{P}^3$ since it has been studied by many people, and moreover, hypersurfaces in $\mathbb{P}^n$ with only ordinary singular points.

(i) **Nodal surfaces in $\mathbb{P}^3$.** The best asymptotic bound was given by Chmutov [Ch92] and reads $\alpha_3^{max}(A_1) \geq \frac{5}{12}$. Again, this already implies that these surfaces do not correspond to T-smooth germs of the equisingular stratum, and we cannot expect that the maximal number of nodes on surfaces in $\mathbb{P}^3$ has a regular behaviour.

(ii) **Hypersurfaces in $\mathbb{P}^n$ with ordinary multiple points.** Using the Horace method outlined in Section 1.2.3, Shustin derived the following existence theorem for hypersurfaces with ordinary multiple points (cf. [Sh00]). If

$$\sum_{i=1}^{r} \binom{m_i+n-1}{n} \leq 2 \cdot \binom{\lfloor \frac{d-1}{2} \rfloor + n}{n} = \frac{1}{2^{n-1}n!} d^n + O(d^{n-1}),$$

then there exists a hypersurface $W \subset \mathbb{P}^n$ having r ordinary multiple points of multiplicity $m_1, \ldots, m_r$ as its only singular points. Note that this result is asymptotically proper since the terms on the left hand side are asymptotically of order m_i^n, just as the Milnor number of the multiple points.

2.2.3 Obstructed and reducible families of hypersurfaces

In order to understand the (asymptotic) quality of a sufficient condition for some property (e.g. T-smoothness or irreducibility of an equisingular stratum), it is indispensable to construct (series of) examples, where this property fails. In this section we review some classical and new examples.

Families of plane curves which are smooth but not T-smooth

Already Segre [Seg29] constructed a series of plane curves, which correspond to smooth germs of the equisingular stratum of dimension bigger than the expected one. Let $m \geq 3$ and consider two generic homogeneous polynomials $F_{2m}(x,y,z)$ and $G_{3m}(x,y,z)$ of degrees $2m$, respectively $3m$. The curve $C_{6m} \subset \mathbb{P}^2$ defined by

$$\Big(F_{2m}(x,y,z)\Big)^3 + \Big(G_{3m}(x,y,z)\Big)^2 = 0,$$

has $6m^2$ ordinary cusps and the dimension of the family of such curves has dimension

$$\frac{2m(2m+3)}{2} + \frac{3m(3m+3)}{2} + 1 = \frac{d(d+3)}{2} - 6m^2 + \frac{(m-1)(m-2)}{2},$$

which is bigger than the expected dimension since $m \geq 3$. On the other hand, a simple calculation of the Zariski tangent space of $V_{6m}^{irr}(6m^2 \cdot A_2)$ at C_{6m} yields that this space is in fact smooth.

A slight modification of this example can be found in [GLS]. They show that the space $V^{irr}_{7m-3}(6m^2 \cdot A_2)$ has a non-T-smooth component. Note that the number r of cusps satisfies

$$r = \frac{6}{49}d^2 + O(d)\,.$$

Non-smooth families of curves

The following series of irreducible curves where the T-property fails is due to Lossen ([Los02]). It generalizes the example of Luengo [Lue87].

Theorem 2.2.1. *Let $l \geq 2$, $0 \leq s \leq l-2$, $k > 2l - s$, $q \geq 3/(l-s-1)$ be integers. Then there exist irreducible curves C_d such that*

(i) the degree of C_d is $d = qk + qs$,

(ii) C_d has precisely q^2 singularities of type A_{lk+s-1}, and

(iii) $V^{irr}_d(q^2 \cdot A_{\ell k+s-1})$ is is non-T-smooth at C_d.

Moreover, if $k > 4l - s$ then $V^{irr}_d(q^2 \cdot A_{lk+s-1})$ has a component of the expected dimension but is singular at C_d.

Proof. [Los02]. □

Remark 2.2.2. (i) There exists also examples of non-smooth families of curves with many D_μ singularities, or, more generally, quasihomogeneous singularities (cf. [Los02]).

(ii) Furthermore, this series can be used to construct examples of obstructed singular hypersurfaces in $\mathbb{P}^n$. For example consider the surface in $\mathbb{P}^3$ defined by

$$F_d(x,y,z,w) := C_d(x,y,z) + w^j \cdot G_{d-j}(x,y,z,w)\,,$$

with G_{d-j} a generic polynomial, $j \geq 2$. Then F has q^2 singularities of types A_{lk+s-1} and corresponds to a non-T-smooth germ because the singularities of F_d are contained in the plane $\{w = 0\}$.

Reducible families

The main example of a reducible stratum is due to Zariski [Za71], and was generalized by Shustin [Sh94].

Example 2.2.3. For all $p \geq 1$, the space $V^{irr}_{6p}(6p^2 \cdot A_2)$ is reducible. For $p = 1, 2$ the stratum consists of two different T-smooth components and for $p \geq 3$ there exist components with different dimensions.

Obstructed families of hypersurfaces

The following lemma allows us to generalize of examples of obstructed families of plane curve to arbitrary dimensions.

Lemma 2.2.4. *Let C be a reduced plane curve of degree $d > 2$. Then, for any $n > 2$, there exists a hypersurface $W \subset \mathbb{P}^n$ of degree d having only isolated singular points, such that $\tau(W) = \tau(C)$ and*

$$h^1\big(\mathcal{I}_{X^{ea}(C)/\mathbb{P}^2}(d)\big) = h^1\big(\mathcal{I}_{X^{ea}(W)/\mathbb{P}^n}(d)\big)\,.$$

Proof. [GLS]. □

The following theorem implies that the $4d - 4$ criterion (cf. Section 4.1) for T-smoothness is sharp.

Theorem 2.2.5. *Let $n \geq 2$ and $d \geq 10$. Then there exists a reduced hypersurface $W \subset \mathbb{P}^n$ of degree d having an isolated singular point z with Tjurina number $\tau(W, z) = 4d - 4$ and being smooth outside z such that the corresponding germ of the analytic equisingular stratum is non-smooth.*

Proof. For $n = 2$ the result is due to du Plessis and Wall [dPW00]. The higher dimensional case can be derived using Lemma 2.2.4. □

2.3 Patchworking of projective varieties

The patchworking method, sometimes also called Viro's glueing method, was introduced by Oleg Viro in order to construct non-singular real algebraic curves in $\mathbb{RP}^2$ with prescribed isotopy type [Vir, Vir84], which is a part of Hilbert's 16th problem. The isotopy type of a real, non-singular, plane curve is determined by the mutual disposition of its components, which, from the topological viewpoint, are either so-called "ovals" (they divide $\mathbb{RP}^2$ into two parts) or so-called "one-sided curves" (their complement is connected). Viro's idea was to construct a new curve out of old ones such that the isotopy type of the new curve was some kind of patchwork of the components of the original curves.

It was realized, mainly by Shustin, that the patchworking method can in fact be applied in many contexts to construct objects defined by polynomials with certain prescribed properties, as long as one is able to control these properties in deformations. Except for the construction of hypersurfaces with prescribed singularities, which we shall describe in detail in this section, there exist also versions of Viro's method in the following situations:

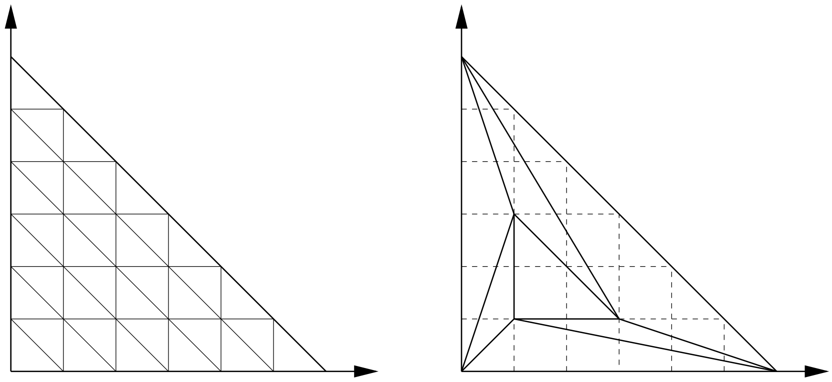


Figure 2.1: A regular and a non regular triangulation.

- non-singular complete intersections with prescribed isotopy type [St94],
- real polynomial vector fields with prescribed singular points [IS94],
- real polynomials with prescribed critical points [Sh96],
- construction of ample divisors out of old ones [Bir99].

Moreover, we should mention that recent new observations about the patchworking method provided valuable new insight and led to a generalization of the construction presented below in the case of curves (see [Sh02, ShT03]). This revealed interesting connections between enumerative geometry and the new field of tropical geometry.

2.3.1 Preliminaries

Viro's original method and all of its modifications are based on subdivisions of Newton polytopes corresponding to the defining polynomials. We start by defining regular polyhedral subdivisions of Newton polytopes, and formulate the general setup of patchworking, which is similar in all variants.

Furthermore, we introduce the notion of transversality, which is the key condition to control equisingular deformations of polynomials with fixed Newton polytope.

In the following we consider the singularities either up to analytical or up to topological equivalence, and *singularity type* always means the equivalence class with respect to the fixed classification.

Definition 2.3.1. Let $\Delta \subset \mathbb{R}^n$ be a non-degenerate Newton polytope.

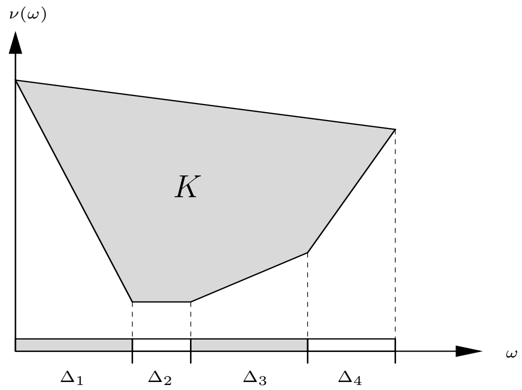


Figure 2.2: Graph of a function ν inducing a subdivision of $\Delta \subset \mathbb{R}$.

(i) A *polyhedral subdivision* of Δ is a subdivision

$$\Delta = \Delta_1 \cup \cdots \cup \Delta_N$$

into non-degenerate Newton polytopes Δ_i, $i = 1, \dots, N$, such that $\Delta_i \cap \Delta_j$ is either empty or a proper, common face of Δ_i and Δ_j.

(ii) A polyhedral subdivision $\Delta = \Delta_1 \cup \cdots \cup \Delta_N$ is called *regular* if there exists a convex, piecewise linear function $\nu \colon \Delta \to \mathbb{R}$ with $\nu(\Delta \cap \mathbb{Z}^n) \subset \mathbb{Z}$ and whose linearity domains are exactly the polytopes $\Delta_1, \dots, \Delta_N$.

Example 2.3.2. (i) The triangulation shown on the left in Figure 2.1 is an example of a regular subdivision. For let ν_1, ν_2, ν_3 be convex functions whose linearity domains are the strips $k \leq x \leq k+1$ (for ν_1), $k \leq y \leq k+1$ (for ν_2) and $k \leq x+y \leq k+1$ (for ν_3), where $k = 0, \dots, d-1$, and put $\nu = \nu_1 + \nu_2 + \nu_3$. This obviously is a convex, piecewise linear function and its linearity domains are exactly as shown in Figure 2.1.

(ii) Not every subdivision is regular. For example the triangulation shown on the right in Figure 2.1 is not regular ([GKZ94]).

(iii) A regular decomposition of Δ can be constructed by the following procedure. Let $\mathcal{A} \subseteq \mathbb{Z}^n$ be a finite set with $\mathrm{Conv}(\mathcal{A}) = \Delta$ and let $\nu : \mathcal{A} \to \mathbb{Z}$ be any function. Consider the convex hull $K \subset \mathbb{R}^{n+1}$ of the graph of ν. The *lower boundary* of K projects bijectively onto Δ and the projections of the faces define a regular subdivision of Δ. Figure 2.2 illustrates this construction for $n = 1$.

Obviously, every regular decomposition can be obtained via this procedure. Furthermore, a generic function ν induces a triangulation of Δ (cf. [GKZ94]).

Let us formulate the setup of the patchworking method.

Input data of the patchworking method

Assume we are given

- a non-degenerate Newton polytope $\Delta \subset \mathbb{R}^n$,
- a regular polyhedral subdivision $\Delta = \Delta_1 \cup \cdots \cup \Delta_N$ of Δ,
- complex numbers $a_\omega \in \mathbb{C}$ with $\omega \in \Delta \cap \mathbb{Z}^n$ and $a_\omega \neq 0$ whenever ω is a vertex of some Δ_i.

Using this data we define the polynomials

$$F_k(\mathbf{x}) = \sum_{\omega \in \Delta_k} a_\omega \mathbf{x}^\omega \in \mathcal{P}(\Delta_k)\,, \quad k = 1, \ldots, N\,,$$

which obviously satisfy $\Delta(F_k) = \Delta_k$.
Furthermore, they are *compatible* in the sense that $F_i^\sigma = F_j^\sigma$ for $\sigma = \Delta_i \cap \Delta_j$.

Given initial data as above, consider the polynomial

$$F^{(t)}(\mathbf{x}) \;=\; \sum_{\omega \in \Delta} a_\omega \mathbf{x}^\omega \cdot t^{\nu(\omega)}\,,$$

where t is a parameter and ν is a convex, piecewise linear function inducing the given subdivision of Δ. This polynomial is sometimes also called the *Viro polynomial* since it was used in Viro's original construction and will reappear in slightly modified form in the patchworking theorem for singular hypersurfaces. Let us discuss some elementary properties of this polynomial.

(i) Let $l\colon \mathbb{R}^n \to \mathbb{R}$ be an affine linear function defined by

$$l(\omega_1, \ldots, \omega_n) = \alpha_0 + \alpha_1\omega_1 + \ldots + \alpha_n\omega_n$$

with $\alpha_0, \ldots, \alpha_n \in \mathbb{Z}$. Put $\widetilde{\nu} := \nu - l$ and define $\widetilde{F}^{(t)}$ in the same way as $F^{(t)}$ with ν replaced by $\widetilde{\nu}$. Note that $\widetilde{\nu}$ is also convex and piecewise linear.

Let T be the linear transformation $\mathbb{R}^n \to \mathbb{R}^n$ given by $T(\mathbf{x}) = (t^{\alpha_1}x_1, \ldots, t^{\alpha_n}x_n)$. Then

$$t^{\alpha_0} \cdot \left(\widetilde{f}^{(t)}\left(T(x)\right)\right) = f^{(t)}(x) \text{ and thus } V\left(\widetilde{f}^{(t)}\right) = T\left(V\left(f^{(t)}\right)\right).$$

(ii) Now let l_k be the linear function equal to ν on Δ_k and put $\nu_k = \nu - l_k$. So $\nu_k|_{\Delta_k} = 0$ and hence,

$$F_k^{(t)}(x) \;:=\; \sum_{\omega \in \Delta} a_\omega \mathbf{x}^\omega \cdot t^{\nu_k(\omega)} \;=\; F_k(x) \;+\; G_k^{(t)}(x),$$

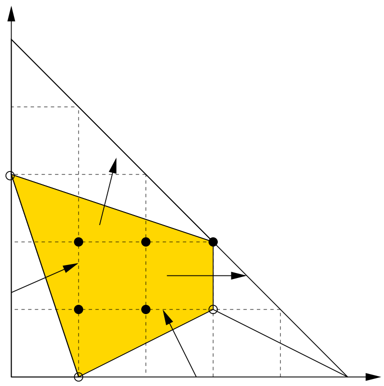


Figure 2.3: Example of a subdivision together with an admissible directed graph.

where every monomial of $G_k^{(t)}$ contains t to a positive power due to the convexity of ν. This implies that $F_k^{(t)}$ is a small deformation of the polynomial F_k, and indicates the necessity for restricting to regular subdivisions.

The rest of this section is devoted to the notion of transversality.

Notation 2.3.3. Let Δ be a Newton polytope and $F \in \mathcal{P}(\Delta)$. If $\Delta_+ \subset \partial\Delta$ is a union of faces of Δ then we denote by $\mathcal{P}(\Delta, \Delta_+, F)$ the space of polynomials from $\mathcal{P}(\Delta)$ with fixed coefficients on Δ_+, i.e.

$$\mathcal{P}(\Delta, \Delta_+, F) := \Big\{ G \in \mathcal{P}(\Delta) \mid F^\sigma = G^\sigma, \text{ for all faces } \sigma \subset \Delta_+ \Big\}.$$

Again we consider $\mathcal{P}(\Delta, \Delta_+, F)$ also as a subspace of $H^0(\mathcal{O}_{\mathbb{P}^n}(d))$, $d \geq \deg(F)$, by homogenizing the polynomials.

Definition 2.3.4. Let Δ be a Newton polytope and assume that $\Delta = \Delta_1 \cup \cdots \cup \Delta_N$ is a polyhedral subdivision. Let $\mathcal{G} = (V, E)$ be the dual graph of the subdivision, i.e. $V = \{1, \ldots, N\}$ and $(i, j) \in E$ if $\Delta_i \cap \Delta_j$ is a common facet of Δ_i and Δ_j. Denote by $\vec{\mathcal{G}} \neq \emptyset$ the set of directed graphs with support $\mathcal{G}$ and without oriented cycles. We call these graphs *admissible*, and for any such graph $\Gamma \in \vec{\mathcal{G}}$ we define

$$\Delta_{k,\Gamma} := \bigcup_{(j,k)\in\Gamma} \Delta_k \cap \Delta_j \subseteq \partial\Delta_k \,,$$

i.e. the union of those facets of Δ_k which "go into" Δ_k with respect to the directed graph Γ.

Example 2.3.5. Consider the subdivision of $\Delta(5)$ shown in Figure 2.3 together with the indicated directed graph Γ. Let $\Delta_1 = \mathrm{Conv}\{(1, 0), (3, 1), (3, 2), (0, 3)\}$ be the shaded polytope. The non-filled circles correspond to $\Delta_{1,\Gamma} \cap \mathbb{Z}^2$. The idea is that the coefficients corresponding to these lattice points are already prescribed by conditions on the polynomials defined on neighbouring polytopes, i.e. these coefficients are no free parameters.

Definition 2.3.6. Let $F \in \mathbb{C}[x_1, \ldots, x_n]$ and assume that

- $\Delta = \Delta(F)$ has dimension n and
- $\mathrm{Sing}(F) \cap (\mathbb{C}^*)^n = \{z_1, \ldots, z_r\}$.

If Δ_+ is a subset of $\partial\Delta$, then we call the triple (Δ, Δ_+, F) *transversal* if the natural map

$$\mathcal{P}(\Delta, \Delta_+, F) \longrightarrow \bigoplus_{i=1}^{r} \mathcal{O}_{\mathbb{P}^n, z_i}/I^s(W, z_i)$$

is surjective, where $W \subset \mathbb{P}^n$ is the hypersurface defined by F.

The following remark justifies the name of "transversality".

Remark 2.3.7. Let $F \in \mathbb{C}[\mathbf{x}]$ be as in Definition 2.3.6 and let $d \geq \deg(F)$. Denote by $\mathcal{M}_d(F)$ the germ at F of the space of all polynomials G of degree less than d, having r singularities $w_1, \ldots, w_r$ in neighbourhoods of $z_1, \ldots, z_r$ and such that $(F, z_i) \sim (G, w_i)$, $i = 1, \ldots, r$. In other words $\mathcal{M}_d(F)$ denotes the germ of the equisingular stratum corresponding to the singularities in $(\mathbb{C}^*)^n$ (embedded via multiplying F with $x_0^{d-\deg(F)}$).

Then a triple (Δ, Δ_+, F) as in Definition 2.3.6 is transversal if and only if for $d \gg 0$ the intersection

$$\mathcal{M}_d(F) \cap \mathcal{P}(\Delta, \Delta_+, F) \subset \mathbb{C}[x_1, \ldots, x_n]_{\leq d}$$

is transversal. This follows since for d sufficiently large the sequence

$$0 \longrightarrow H^0(\mathcal{I}_X(d)) \longrightarrow H^0(\mathcal{O}_{\mathbb{P}^n}(d)) \longrightarrow H^0(\mathcal{O}_X) \longrightarrow 0$$

is exact, where $X = \bigcup_{i=1}^r X^s(F, z_i)$. Hence (Δ, Δ_+, F) is transversal if and only if the sequence

$$0 \longrightarrow H^0(\mathcal{I}_X(d)) \cap \mathcal{P}(\Delta, \Delta_+, F) \longrightarrow \mathcal{P}(\Delta, \Delta_+, F) \longrightarrow H^0(\mathcal{O}_X) \longrightarrow 0$$

is exact, which is equivalent to $\mathcal{P}(\Delta, \Delta_+, F)$ intersecting transversally with $\mathcal{M}_d(F)$.

Let us give a condition for transversality in terms of H^1-vanishing.

Proposition 2.3.8. *Let us be given a triple (Δ, Δ_+, F) and assume that Δ meets all the coordinate axes. Let Y be the subscheme of $\mathbb{P}^n$ defined by the homogeneous ideal*

$$I = \langle x_0^{d-\omega_1-\ldots-\omega_n} x_1^{\omega_1} \ldots x_n^{\omega_n} \mid \omega \in \Delta \setminus \Delta_+ \rangle ,$$

where $d = \deg(F)$. Furthermore, we introduce the scheme

$$X = \bigcup_{z \in \mathrm{Sing}(F) \setminus Y} X^s(F, z) \text{ and put } Z = X \cup Y .$$

Then (Δ, Δ_+, F) is transversal if and only if $h^1(\mathcal{I}_{Z/\mathbb{P}^n}(d)) = 0$.

Proof. If $h^1(\mathcal{I}_{Z/\mathbb{P}^n}(d)) = 0$ then we have exact sequences

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
 & & & H^0(\mathcal{I}_X(d)) \arrow[d] & \\
 & & & H^0(\mathcal{O}_{\mathbb{P}^n}(d)) \arrow[d] & \\
0 \arrow[r] & H^0(\mathcal{I}_Z(d)) \arrow[r] & H^0(\mathcal{I}_Y(d)) \arrow[r] \arrow[ur, hook] & H^0(\mathcal{O}_X(d)) \arrow[r] & 0
\end{tikzcd}$$

This implies that $h^1(\mathcal{I}_X(d)) = 0$ and that the intersection $H^0(\mathcal{I}_X(d)) \cap H^0(\mathcal{I}_Y(d))$ has the expected dimension. Since

$$H^0(\mathcal{I}_Y(d)) = \Big\{ \sum_{\omega \in \Delta \setminus \Delta_+} a_\omega x_0^{d-\omega_1-\ldots-\omega_n} x_1^{\omega_1} \ldots x_n^{\omega_n} \Big\}$$

the claim follows. □

Transversality also implies that we can prescribe generic coefficients on Δ_+. This is very helpful to make given polynomials compatible on common faces of their Newton polytopes.

Lemma 2.3.9. *Assume that (Δ, Δ_+, F) is transversal, then for a generic polynomial*

$$f(\mathbf{x}) = \sum_{\omega \in \Delta_+ \cap \mathbb{Z}^n} a_\omega \mathbf{x}^\omega ,$$

there exists a polynomial $F' \in \mathcal{P}(\Delta)$ such that $\mathcal{S}(F) = \mathcal{S}(F')$, $(F')^\sigma = f$ and (Δ, Δ_+, F') is transversal.

Proof. Easy application of the implicit function theorem. □

2.3.2 The patchworking theorems

Again we assume that the classification of singular points is fixed. In order to simplify the formulation of the patchworking theorem, we introduce the following notation.

Notation 2.3.10. (i) Let F be a polynomial with $\mathrm{Sing}(F)\cap(\mathbb{C}^*)^n = \{z_1, \ldots, z_r\}$. If (F, z_i) has type S_i, we write $\mathcal{S}(F)$ for the formal sum $S_1 + \ldots + S_r$.

(ii) Analogously, if W is a hypersurface contained in a toric variety $\mathrm{Tor}(\Delta)$ with $\mathrm{Sing}(W) \cap (\mathbb{C}^*)^n = \{z_1, \ldots, z_r\}$ and (W, z_i) has type S_i, then we write $\mathcal{S}(W)$ for the formal sum $S_1 + \ldots + S_r$.

Theorem 2.3.11 (Patchworking of singular hypersurfaces). *Assume we are given a regular subdivision $\Delta = \Delta_1 \cup \cdots \cup \Delta_N$ and let $F_1, \ldots, F_N$ be compatible PNS polynomials with $\Delta(F_k) = \Delta_k$ and having only isolated singular points in $(\mathbb{C}^*)^n$.*

If there exists an admissible graph $\Gamma \in \vec{\mathcal{G}}$, such that the triples $(\Delta_k, \Delta_{k,\Gamma}, F_k)$ are transversal for $k = 1, \ldots, N$, then there exists a polynomial $F \in \mathcal{P}(\Delta)$ with

$$\mathcal{S}(F) = \mathcal{S}(F_1) + \ldots + \mathcal{S}(F_N)\,.$$

Furthermore, $(\Delta, \emptyset, F)$ is transversal.

Proof. [Sh98]. □

The following corollary reformulates the theorem above in the language of hypersurfaces in smooth toric varieties.

Corollary 2.3.12. *Assume that* $\mathrm{Tor}(\Delta)$ *is smooth. Then under the assumptions of the patchworking theorem, there exists a toric hypersurface $W \subset \mathrm{Tor}(\Delta)$ such that*

$$\mathcal{S}(W) = \mathcal{S}(F_1) + \ldots + \mathcal{S}(F_N)\,,$$

and W is smooth outside the torus $(\mathbb{C}^*)^n$, and corresponds to a T-smooth germ.

Remark 2.3.13. The patchworking theorem works completely analogously in the real case since it is based on the implicit function theorem (cf. [Sh98] for details). In fact, using the charts of polynomials introduced in Section 1.3.2 we can give a nice topological picture of the resulting hypersurface, at least in the case of curves.

Since the initial polynomials F_k are assumed to be PNS, Remark 1.3.20 implies that all charts $\mathrm{Ch}_\varepsilon(F_k)$ intersect transversally with the boundaries $\partial\Delta_k$. Viro's original theorem (e.g. [Vir84]) states that the chart of the constructed polynomial F satisfies

$$\mathrm{Ch}_\varepsilon(F) = \bigcup_{k=1}^{N} \mathrm{Ch}_\varepsilon(F_k)\,,$$

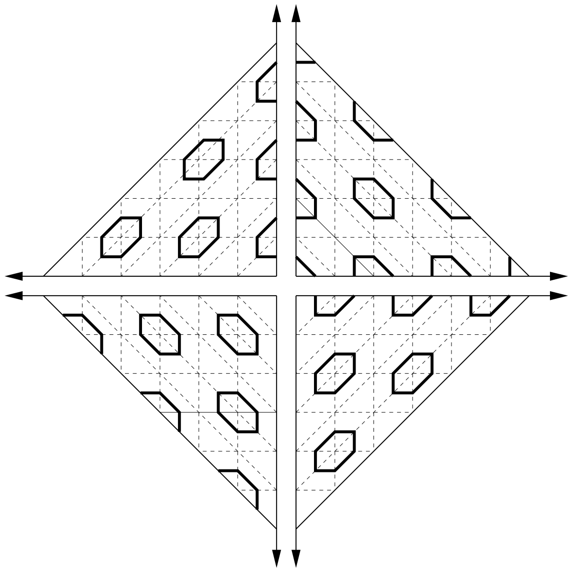


Figure 2.4: A patchwork of a non-singular real algebraic curve of degree 6.

for all $\varepsilon \in \{\pm 1\}^n$. Hence the charts are "glued" together, which explains the name "patchworking" (respectively "Viro's glueing method"). Note that if the initial polynomials F_k have no singular points in $(\mathbb{C}^*)^n$, then the triples $(\Delta, \Delta_{k,+}, F_k)$ are always transversal.

Figure 2.4 shows an example of a patchwork of a non-singular real curve of degree 6. The Newton polytopes of the initial polynomials are all triangles of area $\frac{1}{2}$ and their charts look like the chart of a linear function, which were computed in Example 1.3.17. In particular, their shape only depends on the signs of the coefficients. In the displayed example we chose the following coefficients $a_{i,j}$ of the monomials $x^i y^j$, $i + j \leq 6$:

$$a_{i,j} = \begin{cases} +1 & \text{if } i \text{ and } j \text{ are even,} \\ -1 & \text{otherwise.} \end{cases}$$

The resulting curve is a so-called Harnack curve; it has 11 connected components, which is the maximal number for curves of degree 6.

The following variant of the patchworking theorem allows a simpler application, and we refer to it as *local patchworking.* We formulate and apply this theorem only for hypersurfaces in $\mathbb{P}^n$, but it is possible to generalize it to hypersurfaces in smooth varieties.

Theorem 2.3.14 (Local patchworking). *Let $S_1, \ldots, S_r$ be isolated singularity types and assume that*

- *there exists an irreducible hypersurface in $\mathbb{P}^n$ of degree d having $r + r'$ ordinary singular points $z_1, \ldots, z_{r+r'}$ of multiplicities $m_1, \ldots, m_{r+r'}$ as its only* *singularities, and*

$$h^1(\mathcal{I}_{X(\underline{m};\underline{z})/\mathbb{P}^n}(d)) = 0\,, \tag{2.3.4}$$

- *for any $i = 1, \ldots, r$ there exists a hypersurface in $V_{m_i-1}(S_i)$ corresponding to a T-smooth germ.*

Then there exists an irreducible hypersurface $W \subset \mathbb{P}^n$ of degree d with r singular points of types $S_1, \ldots, S_r$ and r' ordinary singular points of multiplicities $m_{r+1}, \ldots, m_{r+r'}$ as its only singularities.

The following proof is a generalization of the proof from [GLS98] to arbitrary smooth varieties. For a different proof, dealing also with lower deformations of Newton non-degenerate points, see [Sh00].

Proof. We denote the hypersurface having $r + r'$ ordinary singular points by F and let $G_i \in V_{s_i}(S_i)$, $s_i = m_i - 1$, be T-smooth. The idea of the proof is to deform F by "glueing" a local equation of G_i into the Newton diagram of a local equation of F. We shall proceed in several steps.

Step 1. Choose $\mathbb{A}^n \subset \mathbb{P}^n$ containing $z_1, \ldots, z_{r+r'}$. We introduce local coordinates $\mathbf{x_i} = (x_{i1}, \ldots, x_{in})$ around z_i inducing isomorphisms

$$\mathcal{O}_{\mathbb{C}^n,z_i} \cong \mathbb{C}\{x_{i1}, \ldots, x_{in}\}.$$

Assumption (2.3.4) implies that the natural morphism

$$H^0(\mathcal{O}_{\mathbb{P}^n}(d)) \longrightarrow H^0(\mathcal{O}_{X(\underline{m};\underline{z})}(d)) \cong \bigoplus_{i=1}^{r+r'} \mathbb{C}\{\mathbf{x_i}\}/\mathfrak{m}^{m_i}\,,$$

is surjective, which means that we can realize any variation of the $(m_i - 1)$-jets at z_i in $\mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^n}(d)))$.
We consider the following system of coordinates for $H^0(\mathcal{O}_{\mathbb{P}^n}(d))$ in a sufficiently small neighbourhood U of F:

- choose preimages $A_\omega^{(i)}$ of the monomials $\mathbf{x_i}^\omega$, $0 \le |\omega| \le m_i - 1,\ i = 1, \ldots, r + r'$, and
- some additional parameters B_j, $j = 1, \ldots, N = \binom{d+n}{n} - \sum_{i=1}^{r+r'} \binom{m_i+n-1}{n}$

such that F corresponds to the parameter values

- $A_\omega^{(i)} = 0$, $0 \le |\omega| \le m_i - 1,\ i = 1, \ldots, r + r'$,
- $B_j = 0$, $j = 1, \ldots, N$.

Step 2. Assume that

$$g_i(\mathbf{x_i}) = \sum_{0\le|\omega|\le m_i-1} a_\omega^{(i)} \mathbf{x_i}^\omega$$

is an affine equation for G_i at z_i such that the leading form of g_i is nondegenerate. We define the hypersurface G corresponding to the parameter values

- $A_\omega^{(i)} = \begin{cases} a_\omega^{(i)} & \text{if } |\omega| = s_i = m_i - 1 \text{ and } i \le r \\ 0 & \text{otherwise} \end{cases}$
- $B_j = b_j$ for $j = 1, \ldots, N$ with $b_j \in \mathbb{C}$ arbitrary.

Thus, the hypersurface G has semi-quasihomogeneous singularities at $z_1, \ldots, z_r$ and m_i-fold points at $z_{r+1}, \ldots, z_{r+r'}$. Furthermore, we may assume that G is a small deformation of F by choosing the coefficients $a_\omega^{(i)}, b_j$ sufficiently small such that

(i) the m_i-fold points $z_{r+1}, \ldots, z_{r+r'}$ are ordinary,

(ii) G is irreducible and

(iii) G has no singular points other than $z_1, \ldots, z_{r+r'}$.

Step 3. We consider the family W_t of hypersurfaces close to G corresponding to the parameter values

- $A_\omega^{(i)} = \begin{cases} a_\omega^{(i)}(t) \cdot t^{s_i - |\omega|} & \text{if } |\omega| \le s_i = m_i - 1 \text{ and } i \le r\,, \\ 0 & \text{otherwise} \end{cases}$
- $B_j = b_j(t)$ for $j = 1, \ldots, N$.

where $a_\omega^{(i)}(t)$ and $b_j(t)$ are smooth functions in a neighbourhood of zero with

$$a_\omega^{(i)}(0) = a_\omega^{(i)}, \quad b_j(0) = b_j.$$

Then the affine equation of W_t at z_i is given by

$$f_t^{(i)}(\mathbf{x_i}) = \sum_{0\le|\omega|\le s_i} a_\omega^{(i)}(t) \cdot t^{s_i-|\omega|} \mathbf{x_i}^\omega + \sum_{|\omega|>s_i} a_\omega^{(i)}(t) \mathbf{x_i}^\omega$$

where the coefficients $a_\omega^{(i)}(t)$, $|\omega| > s_i$, are affine functions in $A_\omega^{(m)}$, $m \ne i$.

Step 4. Now we have to show that we can define the functions $a_\omega^{(i)}(t)$ and $b_j(t)$ as smooth functions of t such that the hypersurface belongs to the intersection of the local equisingular strata corresponding to the singular types S_i and has ordinary singular points at $z_{r+1}, \ldots, z_{r+r'}$. For any $i \in \{1, \ldots, r\}$ we first apply the (local) coordinate change

$$T_i \colon (x_{i1}, \ldots, x_{in}) \mapsto (tx_{i1}, \ldots, tx_{in}),$$

which induces an isomorphism for $t \neq 0$. This coordinate change turns W_t into a hypersurface with affine equation

$$\widetilde{f}_t^{(i)}(\mathbf{x_i}) = \sum_{0 \le |\omega| \le s_i} a_\omega^{(i)}(t) \mathbf{x_i}^\omega + \sum_{|\omega| > s_i} a_\omega^{(i)}(t) \cdot t^{|\omega| - s_i} \mathbf{x_i}^\omega$$

satisfying $\widetilde{f}_0^{(i)}(\mathbf{x_i}) = g_i(\mathbf{x_i})$.
Since G_i is T-smooth the germ of the local equianalytic stratum at $G_i \cdot H_\infty^{d-s_i}$ in $H^0(\mathcal{O}_{\mathbb{P}^n}(d))$ can be described by $c_i = \tau(S_i)$ equations

$$\varphi^{(i)} \colon U \longrightarrow \mathbb{A}^{c_i}$$

depending only on $A_\omega^{(i)}$, $|\omega| \le d$, and with the property that there exists a subset

$$\Lambda_i \subseteq \{\omega \in \mathbb{N}^n \mid |\omega| \le s_i\}$$

with cardinality c_i and satisfying

$$det \left(\frac{\partial \varphi_u^{(i)}}{\partial A_\omega^{(i)}} \right)_{\substack{\omega \in \Lambda_i \\ u = 1, \ldots, c_i}} \neq 0$$

at the point $A_\omega^{(i)} = a_\omega^{(i)}$, $\omega \le s_i$, $A_\omega^{(i)} = 0$, $\omega > s_i$, i.e. at the parameter values corresponding to G_i. Now consider the function

$$\Phi = (\Phi^{(1)}, \ldots, \Phi^{(r)}) \colon \mathbb{A}^1 \times U \longrightarrow \prod_{i=1}^r \mathbb{A}^{c_i}$$

given by

$$\begin{aligned} &\Phi_u^{(i)} \left(t, \left\{ A_\omega^{(i)} \mid i = 1, \ldots, r, |\omega| \le s_i \right\}, \left\{ B_j \mid j = 1, \ldots, N \right\} \right) \\ &\qquad = \varphi_u^{(i)} \left(\left\{ A_\omega^{(i)} \mid |\omega| \le s_i \right\}, \left\{ t^{|\omega| - s_i} \cdot A_\omega^{(i)} \mid s_i < |\omega| \right\} \right). \end{aligned}$$

By applying the chain rule we obtain

$$\left. \frac{\partial \Phi_u^{(i)}}{\partial A_\omega^{(m)}} \right|_{\substack{A_\omega^{(j)} = a_\omega^{(j)} \\ t=0,\ B_j = b_j}} = \begin{cases} \frac{\partial \varphi_u^{(i)}}{\partial A_\omega^{(m)}} \left(\{ a_\omega^{(i)} \mid |\omega| \le s_i \}, 0, \ldots, 0 \right) & \text{if } i = m \\ \sum\limits_{|\lambda| \le s_i} \frac{\partial \varphi_u^{(i)}}{\partial A_\lambda^{(i)}} \left(\{ a_\omega^{(i)} \mid |\omega| \le s_i \}, 0, \ldots, 0 \right) \cdot \underbrace{\frac{\partial A_\lambda^{(i)}}{\partial A_\omega^{(m)}}}_{=0} & \text{if } i \neq m. \end{cases}$$

This implies that at the point $t=0$, $A^{(j)}_\omega = a^{(j)}_\omega$, $B_j = b_j$, the determinant

$$det\left(\frac{\partial \Phi^{(i)}_u}{\partial A^{(m)}_\omega}\right)_{\substack{\omega\in\Lambda_m,\ u=1,\dots,c_i\\ i,m=1,\dots,r}}$$

decomposes into the product

$$\prod_{i=1}^{r} det\left(\frac{\partial \varphi^{(i)}_u}{\partial A^{(i)}_\omega}\left(\left\{a^{(i)}_\omega \mid |\omega|\le s_i\right\},0,\dots,0\right)\right)_{\substack{\omega\in\Lambda_i\\ u=1,\dots,c_i}} \neq 0\,.$$

Hence we can apply the implicit function theorem and obtain the existence of (smooth) functions

$$\begin{aligned} t &\ \mapsto\ A^{(i)}_\omega = \begin{cases} a^{(i)}_\omega(t) & \text{if } \omega\in\Lambda_i,\ i\in\{1,\dots,r\}\\ a^{(i)}_\omega & \text{if } \omega\notin\Lambda_i,\ i\in\{1,\dots,r\}\\ 0 & \text{if } i\in\{r+1,\dots,r+r'\}\end{cases}\\ t &\ \mapsto\ B_j = b_j, \quad j=1,\dots,N, \end{aligned}$$

which parametrizes locally at G a 1-parameter subfamily of the solution set of $\Phi=0$. The hypersurface W_t (for t sufficiently small) corresponding to those parameter values has then the desired singularities.

Step 5. It remains to verify that the hypersurface $W=W_t$ (for t sufficiently small) has no other than the $r+r'$ prescribed singularities.
Recall that the hypersurface W is a small deformation of the hypersurface G having semi-quasihomogeneous singular points at $z_1,\dots,z_{r+r'}$ as its only singularities. Thus, there exist open neighbourhoods $U_i(0)\subset\mathbb{A}^n$ and $V(0)\subset\mathbb{C}$ such that for $t\in V(0)$ the singular points of $f^{(i)}_t$ in $U_i(0)$ come from the singularity of $f^{(i)}_0$ at the origin. But the singularities of $f^{(i)}_t$ in $U_i(0)$ coincide with the singularities of $\widetilde{f}^{(i)}_t$ in $T_i^{-1}(U_i(0))$ and since $\widetilde{f}^{(i)}_t$ is a small deformation of the polynomials g_i, which have exactly one singular point z, we know that there exists a neighbourhood $U(z)$ such that

$$\mathrm{Sing}\left(\widetilde{f}^{(i)}_t\right)\cap T_i^{-1}(U_i(0)) = \{z'\}\subset U(z).$$

This implies that $W=W_t$ (t small) cannot have singularities other than the $r+r'$ prescribed ones. □

Remark 2.3.15. The local patchworking theorem can be seen as a special case of global patchworking. Let us explain this in the case $r=1$, i.e. we want to specify a lower deformation of one ordinary multiple point z of multiplicity m while keeping r' ordinary singular points.

Denote the polynomial having $r'+1$ ordinary singular points by F and let G be a polynomial of degree $m-1$ having exactly one singular point situated at z such that the hypersurface defined by G corresponds to a T-smooth germ of $V_{m-1}(S)$.

Assume without loss of generality that z is the affine origin and that

(i) $\Delta(F) = \{\omega \in \mathbb{R}^n_{\geq 0} \mid m \leq \omega_i \leq d\,, i = 1, \ldots, n\}$,

(ii) $\Delta(G) = \{\omega \in \mathbb{R}^n_{\geq 0} \mid 0 \leq \omega_i \leq m - 1\,, i = 1, \ldots, n\}$,

(iii) F and G are PNS polynomials.

As in the proof of Theorem 2.3.14 we can deform F into a PNS polynomial F' such that

(i) $\Delta(F') = \{\omega \in \mathbb{R}^n_{\geq 0} \mid m - 1 \leq \omega_i \leq d\,, i = 1, \ldots, n\}$,

(ii) The terms of degree $m - 1$ of the polynomials F' and G agree.

Consider the subdivision of $\Delta(F') \cup \Delta(G)$ of Δ^n_d. Then the assumptions of the local patchworking theorem imply that the triples

$$(\Delta(G), \emptyset, G) \text{ and } (\Delta(F'), \Delta(F') \cap \Delta(G), F')$$

are transversal, and hence the global patchworking theorem applies.

2.4 Existence and H^1-vanishing

In this section we give conditions for the existence of hypersurfaces with prescribed singularities, purely in terms of H^1-vanishing. We start by expressing the existence problem in terms of a certain zero-dimensional scheme.

Definition 2.4.1. Let $(W, z) \subset (V, z)$ be a hypersurfaces germ defined by $f \in \mathbb{C}\{\mathbf{x}\} \cong \mathcal{O}_{V,z}$.

(i) We define

$$I^a(W, z) := \Big\{ g \in \mathbb{C}\{\mathbf{x}\} \mid \langle g, g_{x_1}, \ldots, g_{x_n} \rangle \subset \langle f, f_{x_1}, \ldots, f_{x_n} \rangle \Big\}\,,$$

and let $Z^a(W, z)$ be the zero-dimensional scheme defined by $I^a(W, z)$.

(ii) Let $I^a_1(W, z) := \mathfrak{m}_z \cdot I^a(W, z)$ and denote by $Z^a_1(W, z)$ the zero-dimensional scheme defined by $I^a_1(W, z)$.

The following lemma is an immediate consequence of the Mather-Yau theorem.

Lemma 2.4.2. *(i) If $g \in I^a(W,z)$, then for almost all $t \in \mathbb{C}$, the germs $\{f=0\}$ and $\{f+tg=0\}$ are analytically equivalent.*

(ii) If $g \in I_1^a(W,z)$, then for all $t \in \mathbb{C}$, the germs $\{f=0\}$ and $\{f+tg=0\}$ are analytically equivalent.

This yields the following existence criterion for hypersurfaces with one singularity.

Proposition 2.4.3. *Given a zero-dimensional scheme $X \subset V$, a point $z \in V \setminus \operatorname{supp}(X)$ and a reduced hypersurface germ $(W,z) \subset (V,z)$. If*

$$H^1(\mathcal{I}_{X \cup Z_1^a(W,z)}(d)) = 0\,, \tag{2.4.5}$$

then there exists a hypersurface $W' \in |H^0(\mathcal{I}_{X\cup Z^a(W,z)}(d))|$ such that (W',z) is analytically equivalent to (W,z). Moreover, these hypersurfaces form an open, dense subset in $|H^0(\mathcal{I}_{X\cup Z^a(W,z)}(d))|$.

Proof. [Sh01]. □

Remark 2.4.4. It is enough to require (2.4.5) for $X \cup Z$, where Z is a generic scheme isomorphic to $Z_1^a(W,z)$ and satisfying $X \cap Z = \emptyset$.

This allows to deduce the following existence theorem for hypersurfaces in $\mathbb{P}^n$ with many singularities.

Proposition 2.4.5. *Let $(W_1,z_1),\dots,(W_r,z_r)$ be hypersurface germs and let*

$$Z = Z_1^a(W_1,z_1) \cup \dots \cup Z_1^a(W_r,z_r)\,.$$

If $H^1(\mathcal{I}_Z(d-1)) = 0$, then there exists a reduced hypersurface W of degree d, such that (W,z_i) is analytically equivalent to (W_i,z_i) and W has no further singular points.

Proof. We introduce the zero-dimensional schemes

$$Z_z := Z \cup \{z\} \text{ for } z \in \mathbb{P}^n \setminus \{z_1,\dots,z_r\}\,.$$

Let H be a generic hyperplane through z, then $Z_z : H = Z$ and hence

$$H^1(\mathcal{I}_{Z_z:H/\mathbb{P}^n}(d-1)) = 0\,.$$

Using the reduction sequence and $H^1(\mathcal{I}_{Z_z\cap H/H}(d)) = 0$ we obtain $H^1(\mathcal{I}_{Z_z}(d)) = 0$.

By Proposition 2.4.3 there exist hypersurfaces $D_1,\dots,D_r \in |H^0(\mathcal{I}_Z(d))|$ such that (D_i,z_i) is analytically equivalent to (W_i,z_i) for $i=1,\dots,r$. By Lemma 2.4.2 a hypersurface

$$D = \lambda_1 D_1 + \ldots + \lambda_r D_r\,,$$

with $\lambda_1, \ldots, \lambda_r \in \mathbb{C}$ generic, satisfies $(D, z_i) \sim (W_i, z_i)$ and is furthermore reduced.

Let $w_1, \ldots, w_s$ be the singular points of D outside $z_1, \ldots, z_r$. Since $H^1(\mathcal{I}_{Z\cup\{w_j\}}(d)) = 0$ for $j = 1, \ldots, s$, we obtain the existence of hypersurfaces $D'_j \in |H^0(\mathcal{I}_Z(d))|$ not passing through w_j. Hence a generic hypersurface W in the linear system

$$\lambda'_0 D + \lambda'_1 D'_1 + \ldots + \lambda'_s D'_s\,,$$

$\lambda'_0, \ldots, \lambda'_s \in \mathbb{C}$, has singularities at $z_1, \ldots, z_r$ equivalent to $(W_1, z_1), \ldots, (W_r, z_r)$ and no further singular points by Bertini's theorem (cf. Theorem A.1.1). $\square$

Chapter 3

Existence results

In this chapter we present our main results concerning the existence problem. First, we present a general existence theorem, which roughly states that from the asymptotic viewpoint it suffices to study the existence problem in the minimal dimension where the singularities appear.
After that, we study the existence problem in special situations. We derive new results for hypersurfaces with simple singularities, which are even asymptotically optimal. Furthermore, we give explicit equations of certain hypersurfaces with one singularity and glue them by the local patchworking method. Finally we apply our results also to the construction of real hypersurfaces.

3.1 Stabilization of the Existence Problem

The goal of this section is to prove a general existence theorem, which allows us to carry over existence results to higher dimensions.

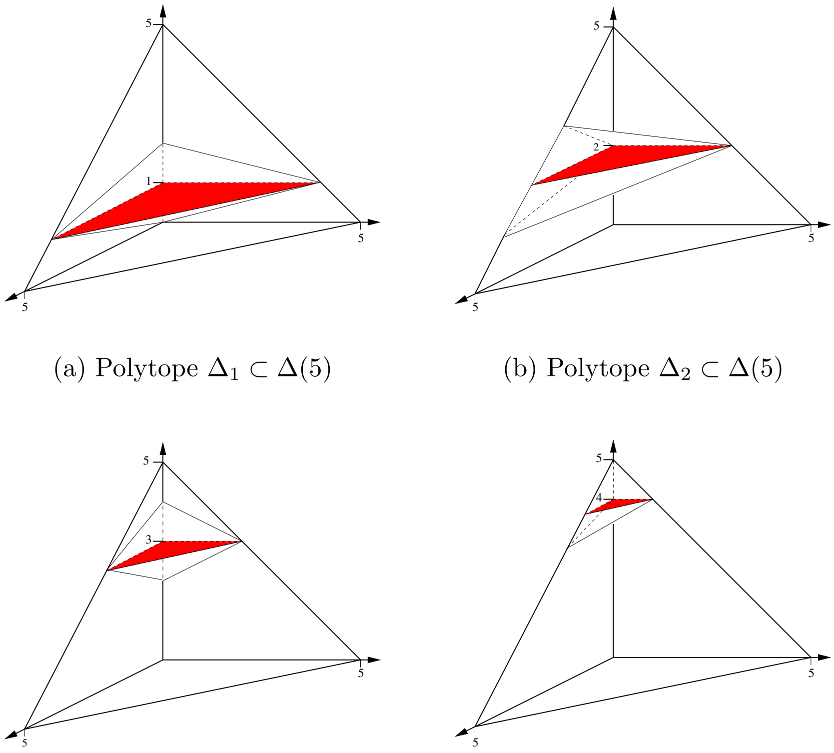
Given a set of singularity types $\mathcal{S}$ of corank $n \geq 1$, let us assume we have an asymptotic proper existence result, e.g. assume we know that

$$\alpha_n^{reg}(\mathcal{S}) > 0\,. \tag{3.1.1}$$

It is natural to ask whether this implies that we also have asymptotically proper existence results for hypersurfaces with singularities of types $\mathcal{S}$ in higher dimensions, i.e. does (3.1.1) imply

$$\alpha_m^{reg}(\mathcal{S}) > 0\,, \quad \forall m \geq n\,?$$

Example 3.1.1. Let $\mu \geq 1$. Clearly, $d_1(A_\mu) = \mu + 1$ and a defining polynomial is $x^{\mu+1}$. However, taking the polynomial $x^{\mu+1} + y^2$ we only obtain $d_2(A_\mu) \leq \mu + 1$,



(a) Polytope $\Delta_1 \subset \Delta(5)$ (b) Polytope $\Delta_2 \subset \Delta(5)$

(c) Polytope $\Delta_3 \subset \Delta(5)$ (d) Polytope $\Delta_4 \subset \Delta(5)$

Figure 3.1: The subdivision for $n = 3, d = 5$.

which is *not* asymptotically proper, i.e. just adding squares is not enough.
On the other hand consider

$$f_k(x, y) = (y - x^k)^2 + y^{2k},$$

which has an A_{2k^2-1} singularity at the origin. Hence for $\mu = 2k^2 - 1$

$$d_2(A_\mu) \leq \tfrac{1}{\sqrt{2}} \cdot \sqrt{\mu + 1} \text{ for } \mu = 2k^2 - 1,$$

which is asymptotically proper.

We start by introducing a subdivision of Δ^n_d containing Δ^{n-1}_p as "slices" for $p = 1, \ldots, d-1$. Consider the following convex, integral sub-polytopes $\Delta_1, \ldots, \Delta_{d-1}$ of Δ^n_d:

- If $1 \leq i \leq d-1$, i is odd, we define Δ_i to be the convex hull of the union of the simplex $\Delta(d) \cap \{\omega_n = i\}$ with the points $(0, \ldots, 0, i-1)$ and $(0, \ldots, 0, i+1)$;

- If $1 \le i \le d-1$, i is even, we define Δ_i to be the convex hull of the union of the simplex $\Delta(d) \cap \{\omega_n = i\}$ with the points $(0,\ldots,0,d-i+1,i-1)$ and $(0,\ldots,0,d-i-1,i+1)$.

The closure of $\Delta(d) \setminus (\Delta_1 \cup \ldots \cup \Delta_{d-1})$ is the union of $d-1$ polytopes $\Delta'_1,\ldots,\Delta'_{d-1}$ yielding altogether a convex subdivision of $\Delta(d)$. Note that every lattice point of the Δ'_i, $i = 1,\ldots,d-1$ is contained in at least one of the Δ_j, $j = 1,\ldots,d-1$. Hence if for $i = 1,\ldots,d-1$ PNS polynomials $F_i \in \mathcal{P}(\Delta_i)$ are given, then the resulting polynomials $G_i \in \mathcal{P}(\Delta'_i)$ are also PNS. In fact if the coefficients of the F_i on the boundary are generic, then the G_i themselves have no singular point in $(\mathbb{C}^*)^n$.

Figure 3.1 shows an example for $n = 3$, and Figure 3.2 shows the projection of $\Delta_1,\ldots,\Delta_{d-1}$ onto the plane (ω_{n-1},ω_n).

Notice that

$$\Big(\Delta \setminus (\Delta_1 \cup \ldots \cup \Delta_{d-1})\Big) \cap \mathbb{Z}^n = \Big\{(\omega',0) \mid \omega' \in \mathbb{Z}^{n-1} \setminus \{0\}\Big\},$$

and furthermore, $\Delta_i \cap \Delta_j = \emptyset$ if $|i-j| > 1$ and

$$\Big(\Delta_{2i-1} \cap \Delta_{2i}\Big) \cap \mathbb{Z}^n = \mathrm{Conv}\{(0,\ldots,0,2i),(0,\ldots,0,d-2i+1,2i-1)\}\ ,$$

$$\Big(\Delta_{2i} \cap \Delta_{2i+1}\Big) \cap \mathbb{Z}^n = \mathrm{Conv}\{(0,\ldots,0,d-2i-1,2i+1),(0,\ldots,0,2i)\}\ .$$

Hence, $\mathrm{codim}_{\mathbb{R}^n}(\Delta_i \cap \Delta_j) \ge 2$ for all i,j.

Let us define the polynomials which we want to glue:
Assume $f(x_1,\ldots,x_{n-1})$ is a PNS polynomial of degree $d-1$ having only isolated singular points. Then we consider

$$F(x_1,\ldots,x_n) := x_n^2 - 2x_n(f(x_1,\ldots,x_{n-1})+1)+1\,.$$

Obviously, $\Delta(F) = \Delta_1$ and F is PNS.

Lemma 3.1.2. *If 2 is not a critical value of f, then* $\mathrm{Sing}(F) \cap (\mathbb{C}^*)^n = \{(\mathbf{x}',1) \mid \mathbf{x}' \in \mathrm{Sing}(f) \cap (\mathbb{C}^*)^n\}$ *and for all $z \in \mathrm{Sing}(F)$,*

$$I^{ea}(F,z) = \langle f, f_{x_1},\ldots,f_{x_{n-1}}, x_n - 1\rangle\,.$$

Proof. Firstly, note that

$$F(x_1,\ldots,x_n) = (x_n - f - 1)^2 - f^2 - 2f = 0\,.$$

Let $\mathbf{x} = (x_1,\ldots,x_n) \in \mathrm{Sing}(F) \cap (\mathbb{C}^*)^n$. Calculating the partial derivatives yields

$$\begin{aligned} F_{x_1} &= -2x_n \cdot f_{x_1} \\ \vdots & \qquad \vdots \\ F_{x_{n-1}} &= -2x_n \cdot f_{x_{n-1}} \\ F_{x_n} &= 2(x_n - f - 1)\,. \end{aligned}$$

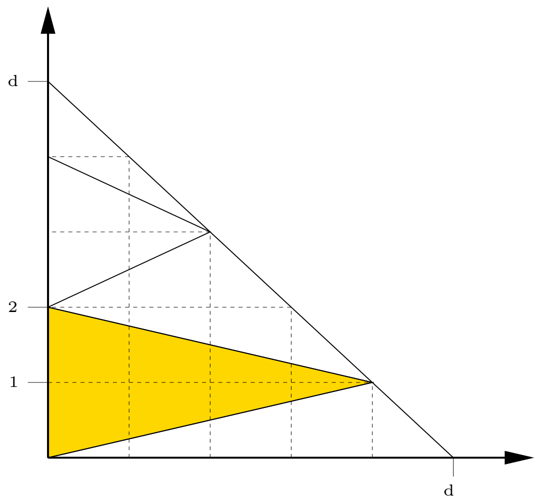


Figure 3.2: Projection of the subdivision to the (ω_{n-1}, ω_n) plane (the shaded part is Δ_1).

Since $x_n \neq 0$, we obtain $f_{x_i} = 0$, $i = 1, \ldots, n-1$. Furthermore, $F = F_{x_n} = 0$ implies that $f^2 + 2f = 0$, hence $f(\mathbf{x}) = 0$ (since 2 is not a critical value of f) and thus $x_n = 1$. $\square$

Lemma 3.1.3. *Let f be a PNS polynomial as above, and assume that f defines a T-smooth hypersurface in $\mathbb{P}^{n-1}$. Then the triad $(\Delta_1, \emptyset, F)$ is transversal.*

Proof. This follows since the map

$$\mathcal{P}(\Delta_1) \supseteq \langle (\mathbf{x}')^{\omega'} x_n \mid |\omega'| \leq d-1 \rangle \longrightarrow \bigoplus_{z_i \in \mathrm{Sing}(F) \cap (\mathbb{C}^*)^n} \mathcal{O}_{\mathbb{P}^n, z_i} / I^s(F, z_i)\,,$$

is surjective. $\square$

We obtain the following general existence result:

Proposition 3.1.4. *Fix $n \geq 3$, and let $f_i(x_1, \ldots, x_{n-1})$ be PNS polynomials of degree i, $i = 1, \ldots, d-1$, defining T-smooth hypersurfaces $W_i \in V_i(\mathcal{S}_i)$, $\mathcal{S}_i = S_1^i + \ldots + S_{r_i}^i$.*
Then there exists a T-smooth hypersurface W in $\mathbb{P}^n$ whose singular points correspond precisely to the singularities of $W_1, \ldots, W_{d-1}$, i.e.

$$W \in V_d(\mathcal{S}_1 + \ldots + \mathcal{S}_{d-1})\,.$$

Proof. Let $\Delta_i' = \mathrm{Conv}\{(0,..,0), (0,..,0,2), (d-i,0,..,0,1), \ldots, (0,..,0,d-i,1)\}$, and let $\phi_i : \mathbb{Z}^n \to \mathbb{Z}^n$ be an affine linear lattice isomorphism taking Δ_k' to Δ_k.

Hence ϕ_i induces isomorphisms $\mathcal{P}(\Delta_i') \xrightarrow{\cong} \mathcal{P}(\Delta_i)$ and $(\mathbb{C}^*)^n \xrightarrow{\cong} (\mathbb{C}^*)^n$, which, for simplicity of notation, we denote also by ϕ_i. We define again

$$F_i'(x_1,\dots,x_n) := x_n^2 - 2x_n(f_i(x_1,\dots,x_{n-1})+1)+1\,,$$

and put $F_i = \phi_i(F_i')$. Then F_i has the same singularities in the torus as F_i and moreover $\Delta(F_i) = \Delta_i$.

By the preceding lemmas the assumptions of the patchworking theorem are satisfied and the claim follows. □

Using these steps we obtain the following

Theorem 3.1.5. *Let $n > 2$, and let $\mathcal{S} = \mathcal{S}_a \cup \mathcal{S}_t$ where $\mathcal{S}_a$ is a set of analytic singularity types of corank $< n$ and $\mathcal{S}_t$ is a set of admissible topological singularity types. If $\sup_{S\in\mathcal{S}} \tau^s(S) < \infty$, then*

$$\alpha_n^{reg}(\mathcal{S}) \geq \frac{\alpha_{n-1}^{reg}(\mathcal{S})}{n}\,.$$

Furthermore, if $\alpha_{n-1}^{reg}(\mathcal{S}) \in \mathcal{A}_{n-1}^{reg}(\mathcal{S})$, then $\alpha_{n-1}^{reg}(\mathcal{S})/n \in \mathcal{A}_n^{reg}(\mathcal{S})$.

Proof. For all $p = 1,\dots,d-1$, we have the following condition for putting singularities on the polynomials situated on the "slice of height p" in the subdivision:

$$\sum_{i=1}^{r} k_i^{(p)}\tau^s(S_i) \leq \alpha p^{n-1} + R_{n-1}(p), \quad \alpha \in \mathcal{A}_{n-1}^{reg}(\mathcal{S})\,,$$

where $R_{n-1}(p) \in O(p^{n-2})$. Summing over all p yields the condition

$$\sum_{i=1}^{r} k_i\tau^s(S_i) = \sum_{i=1}^{r}\sum_{p=1}^{d-1} k_i^{(p)}\tau^s(S_i) \leq \alpha\sum_{p=1}^{d-1} p^{n-1} + \sum_{p=1}^{d-1} R_{n-1}(p)\,.$$

The right hand side of the inequality satisfies

$$\alpha\sum_{p=1}^{d-1} p^{n-1} + \sum_{p=1}^{d-1} R_{n-1}(p) \geq \frac{\alpha}{n}d^n + R_n(d)\,,$$

where $R_n(d) \in O(d^{n-1})$, which implies the claim. □

Let us discuss two immediate consequences of Theorem 3.1.5. Firstly, it allows us to deduce a general existence theorem for hypersurfaces in $\mathbb{P}^n$ with singularities of corank less than 2.

Corollary 3.1.6 (Hypersurfaces with singularities of corank ≤ 2). *Let $n \geq 2$, and let $\mathcal{S}$ be a set of (analytical or topological) singularity types of corank ≤ 2. If $\sup_{S\in\mathcal{S}} \tau^s(S) < \infty$, then*

$$\frac{2}{9 \cdot n!} \in \mathcal{A}_n^{reg}(\mathcal{S}) .$$

Proof. Immediately by Theorem 3.1.5 since $\frac{1}{9} \in \mathcal{A}_2^{reg}(\mathcal{S})$ by [Sh01]. □

Furthermore, let us briefly discuss the case of nodal surfaces in $\mathbb{P}^3$.

Corollary 3.1.7. *If*

$$k \leq \frac{1}{6}d^3 - d^2 + \frac{11}{6}d - 1$$

there is an irreducible, reduced surface $S \subset \mathbb{P}^3$ of degree d having exactly k nodes and no other singular points. Moreover, the equisingular stratum is T-smooth at S.

Proof. We apply Theorem 3.1.5 to irreducible plane curves with $\frac{(p-1)(p-2)}{2}$ nodes, $p = 2, \ldots, d-1$. By Severi's classical result [Sev21], these curves correspond to T-smooth germs. The result follows by calculating the sum $\sum_{p=2}^{d-1} \frac{(p-1)(p-2)}{2}$. □

3.2 Hypersurfaces in $\mathbb{P}^n$ with ordinary multiple points

The local patchworking theorem requires the existence of hypersurfaces with many ordinary multiple points. Hence, if we want to use local patchworking for obtaining general asymptotically proper existence results, it is first necessary to have an asymptotically proper existence result for hypersurfaces with multiple points.

The following proposition reduces this existence problem to an H^1-vanishing condition.

Proposition 3.2.1. *Given integers $\underline{m} = (m_1, \ldots, m_r)$ and $d > 0$ with $m_i \leq d$ for all i, and generic points $z_1, \ldots, z_r \in \mathbb{P}^n$ such that*

$$h^1(\mathcal{I}_{X(\underline{m},\underline{z})/\mathbb{P}^n}(d-1)) = 0 ,$$

then there exists a reduced, irreducible hypersurface of degree d in $\mathbb{P}^n$ having ordinary singular points $z_1, \ldots, z_r$ of multiplicities $m_1, \ldots, m_r$ as its only singularities.

Proof. The proof is a generalization of the proof for $n = 2$ in [GLS98].
For fixed $j \in \{1, \ldots, r\}$ let H_j be a generic hyperplane through z_j (in particular not containing any z_i, $i \neq j$). Then we consider the zero-dimensional schemes $X := X(\underline{m}, \underline{z})$ and X^j concentrated in $\underline{z} = (z_1, \ldots, z_r)$ and given by the ideals

$$\mathcal{I}_{X,z_i} = \mathfrak{m}_{z_i}^{m_i}, \ i = 1, \ldots, r, \text{ respectively } \mathcal{I}_{X^j,z_i} = \begin{cases} \mathfrak{m}_{z_i}^{m_i} & i \neq j \\ \mathfrak{m}_{z_j}^{m_j+1} & i = j \end{cases}.$$

Since $X^j : H_j = X$ we have $h^1(\mathbb{P}^n, \mathcal{I}_{X^j:H_j}(d-1)) = 0$ and moreover $h^1(H_j, \mathcal{I}_{X^j \cap H_j}(d)) = 0$ as $m_j + 1 \leq d + 1$. Hence we can conclude that

$$h^1(\mathcal{I}_{X^j}(d)) = 0\,.$$

This implies that for any $j \in \{1, \ldots, r\}$ we can find a hypersurface

$$W_j \in H^0(\mathcal{I}_X(d)) \setminus H^0(\mathcal{I}_{X^j}(d))\,,$$

which has an m_j-fold point at z_j and at least multiplicity m_i at z_i, $i \neq j$. Then by Bertini's Theorem A.1.1 and Lemma 1.2.37 a generic hypersurface in the linear system

$$\{\lambda_1 W_1 + \ldots + \lambda_r W_r\} \subset |H^0(\mathcal{I}_{X/\mathbb{P}^n}(d))|$$

has ordinary singularities of multiplicity m_i at z_i, $i = 1, \ldots, r$, and no other singularities.

If $n > 2$, then the hypersurface is clearly irreducible as it has only isolated singular points. For $n = 2$ irreducibility follows by Bertini's Theorem (cf. [GLS98]). □

Using the Horace method, Shustin generalized an H^1-vanishing theorem by Hirschowitz for zero-dimensional schemes in $\mathbb{P}^2$ to higher dimensions.

Let $n \geq 1$ and $d \geq 1$. We define

- $M(n, d) = 2 \cdot \binom{k+n}{n}$ for $d = 2k$ and
- $M(n, d) = \binom{k+n}{n} + \binom{k+n-1}{n}$ for $d = 2k - 1$.

Note that

$$M(n, d-1) \leq 2 \cdot \binom{\lfloor \frac{d-1}{2} \rfloor + n}{n} = \frac{1}{2^{n-1} n!} d^n + O(d^{n-1})\,. \tag{3.2.2}$$

Theorem 3.2.2. *Let $\underline{m} = (m_1, \ldots, m_r)$ and $d > 0$. If*

$$\sum_{i=1}^{r} \binom{m_i + n - 1}{n} \ < \ M(n, d)\,, \tag{3.2.3}$$

then $h^1(\mathcal{I}_{X(\underline{m})}(d)) = 0$.

Proof. For $n = 2$ see [Hir85], the generalization to higher dimensions is contained in [Sh00]. □

Using the local patchworking theorem and the theorem above, we obtain

Corollary 3.2.3. *Let $n \geq 2$, and for $i = 1, \dots, r$ let $W_i \in V_{d_i}(S_i)$ be a T-smooth hypersurface. Then there exists a T-smooth hypersurface $W \in V_d(S_1 + \ldots + S_r)$ if*

$$\sum_{i=1}^{r} \binom{d_i + n}{n} \leq M(n, d-1)\,.$$

Remark 3.2.4. Let $\mathcal{S}$ be a set of singularity types and assume we know that there exists a constant $\alpha > 0$ such that

$$d_n^{reg}(S) \leq \alpha \cdot \sqrt[n]{\tau^s(S)} \tag{3.2.4}$$

for all $S \in \mathcal{S}$. This implies

$$\binom{d_n^{reg}(S) + n}{n} < \frac{2}{(n-1)!} \cdot (d_n^{reg}(S))^n \leq \frac{2 \cdot \alpha^n}{(n-1)!} \tau^s(S)\,.$$

Hence, using estimate (3.2.2) we obtain

$$\alpha_n^{reg}(\mathcal{S}) \geq \frac{1}{n \cdot 2^n \cdot \alpha^n}\,.$$

Note that if the Tjurina number in $\mathcal{S}$ is bounded (e.g. if $\mathcal{S}$ is finite), then there always exists $\alpha > 0$ satisfying (3.2.4) (e.g. $\alpha = M + 1$ where $M = \max_{S \in \mathcal{S}}\{\tau(S)\}$). In particular, for any set $\mathcal{S}$ of singularity types with bounded Tjurina number we know

$$\alpha_n^{reg}(\mathcal{S}) > 0\,.$$

3.3 Curves with simple singularities

We apply again the patchworking procedure to construct curves with many simple singularities. The idea for obtaining an asymptotically optimal existence result is to look for polynomials f such that the sum of the Tjurina numbers of its singular points in $(\mathbb{C}^*)^2$ equals the euclidian volume of the Newton polytope of f.

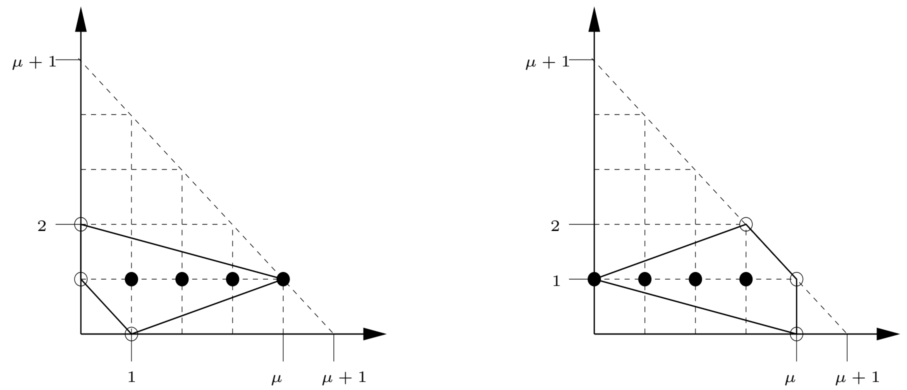


Figure 3.3: The polytopes $\Delta_{A_\mu}, \Delta'_{A_\mu}$ for the A_μ case. The non-filled circles correspond to $\Delta_+ \cap \mathbb{Z}^2$, the filled ones to $(\Delta \setminus \Delta_+) \cap \mathbb{Z}^2$, $\Delta = \Delta_{A_\mu}$ (respectively $\Delta = \Delta'_{A_\mu}$).

3.3.1 The A_μ case

We shall use the following polytopes, which are shown in Figure 3.3:

$\Delta_{A_\mu} := \mathrm{Conv}\{(1,0),(0,1),(0,2),(\mu,1)\}, \quad \Delta'_{A_\mu} = \mathrm{Conv}\{(0,1),(\mu{-}1,2),(\mu,1),(\mu,0)\}\,.$

Lemma 3.3.1. *Let $\mu \geq 1$, $\Delta = \Delta_{A_\mu}$ or Δ'_{A_μ}. Let Δ_+ be the union of any two edges of Δ with a common vertex. There exists a PNS polynomial $f \in \mathcal{P}(\Delta)$ such that*

(i) f has an A_μ singularity as its only singular point in $(\mathbb{C}^*)^2$,

(ii) the coefficients of f along Δ_+ are prescribed non-zero numbers,

(iii) the triad (Δ, Δ_+, f) is transversal.

Proof. Assume that $\Delta = \Delta_{A_\mu}$. We look for the desired polynomial in the form

$$f(x,y) = ay^2 - 2y \cdot Q(x) + cx, \quad ac \neq 0, \quad \deg Q = \mu, \quad Q(0) \neq 0\ .$$

A singular point $(\alpha,\beta) \in (\mathbb{C}^*)^2$ of f must be isolated and of multiplicity 2, that is, of type A_k. The condition that (α,β) is of type A_μ means that

$$f(\alpha,\beta) = 0, \quad f_x(\alpha,\beta) = 0, \quad f_y(\alpha,\beta) = 0\ ,$$

or, equivalently

$$\beta = \frac{Q(\alpha)}{a}, \quad (Q^2(x) - acx)\big|_{x=\alpha} = 0, \quad (2Q'(x)Q(x) - ac)\big|_{x=\alpha} = 0\ ,$$

where $x = \alpha$ is a root of multiplicity μ, respectively $\mu-1$, of the second, respectively third, equation. To satisfy the requirement, we choose a branch of $\sqrt{acx}$ in a neighborhood of $x = \alpha$, and demand that $Q(x) - \sqrt{acx} = (x-\alpha)^{\mu+1}\varphi(x)$ with a function

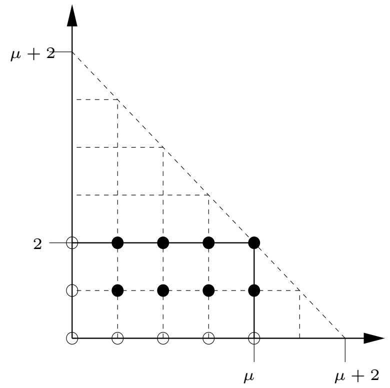


Figure 3.4: The polytope Δ_{D_μ} for the D_μ case. The non-filled circles correspond to $\Delta_+ \cap \mathbb{Z}^2$, the filled ones to $(\Delta \setminus \Delta_+) \cap \mathbb{Z}^2$, $\Delta = \Delta_{D_\mu}$.

$\varphi(x)$ holomorphic in a neighborhood of $x = \alpha$. This yields that $Q(x)$ is the μ-jet of the Taylor series of $\sqrt{acx}$ at $x = \alpha$. It is an easy exercise to verify that, chosen any three of the coefficients a, $b = 2Q(0)$, c, or $d = \frac{2}{\mu!}Q^{(\mu)}(0)$ of f at the vertices of Δ, the rest of the coefficients of f depend (locally) smoothly on the chosen parameters, and this means the transversality of the triad (Δ, Δ_+, f). Furthermore, by acting as $f(x,y) \mapsto \lambda_1 f(\lambda_2 x, \lambda_3 y)$, $\lambda_1, \lambda_2, \lambda_3 \neq 0$, we can prescribe the values of the chosen coefficients.

It remains to notice that the required polynomial with the Newton polytope Δ'_{A_μ} can be chosen in the form $x^\mu y^2 f(x^{-1}, y^{-1})$, with f being constructed above. $\square$

3.3.2 The D_μ case

We use the polytopes shown in Figure 3.4:

$$\Delta_{D_\mu} := \mathrm{Conv}\{(0,0), (\mu,0), (\mu,2), (0,2)\}\,.$$

Let Δ_+ be the union of the left and bottom parts of $\partial\Delta_{D_\mu}$, i.e. $\Delta_+ = [(0,0),(0,2)] \cup [(0,0),(\mu,0)]$.

Lemma 3.3.2. *Let $\mu \geq 3$, $\Delta = \Delta_{D_\mu}$. There exists a PNS polynomial $f \in \mathcal{P}(\Delta)$ such that*

(i) f has two D_μ singularities as its only singular points in $(\mathbb{C}^*)^2$,

(ii) the coefficients of f along Δ_+ are prescribed generic non-zero numbers,

(iii) the triad (Δ, Δ_+, f) is transversal.

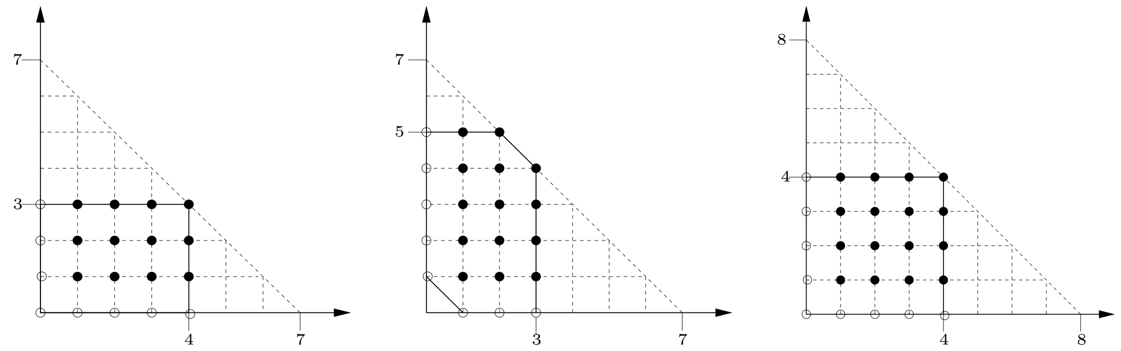


Figure 3.5: The polytopes $\Delta_{E_6}, \Delta_{E_7}, \Delta_{E_8}$ for the E_6, E_7, E_8 singularities. The non-filled circles correspond to $\Delta_+ \cap \mathbb{Z}^2$, the filled ones to $(\Delta \setminus \Delta_+) \cap \mathbb{Z}^2$, $\Delta = \Delta_{E_k}$, $k = 6, 7, 8$.

Proof. Assume that

$$f(x,y) = (x-\alpha_1)(x-\alpha_2)\Big(y^2 \cdot P(x) - 2y \cdot Q(x) + R(x)\Big) = (x-\alpha_1)(x-\alpha_2) \cdot g(x,y)\,.$$

Let $\alpha = \alpha_i$, $i = 1, 2$. The curve $\{f = 0\}$ has a D_μ singularity at $(\alpha, \beta) \in (\mathbb{C}^*)^2$ if and only if the curve $\{g = 0\}$ has an $A_{\mu-3}$ singularity at (α, β). The latter property is equivalent to

$$\beta = \frac{Q(\alpha)}{P(\alpha)}\,, \quad R(\alpha) = \frac{Q^2(\alpha)}{P(\alpha)}\,, \quad R'(\alpha) = \Big(\frac{Q^2}{P}\Big)'(\alpha)\,.$$

Since the polars $\{g_x = 0\}$, $\{g_y = 0\}$ of g intersect with multiplicity $\mu-3$ at (α, β), we obtain that $PR - Q^2 = c \cdot (x-\alpha_1)^{\mu-2}(x-\alpha_2)^{\mu-2}$. The latter equation can be solved for P and Q by interpolation: the degree of Q is $\mu-2$ and we know the value of Q^2 at the $\mu - 1$ points $0, x_1, \ldots, x_{\mu-2}$ where x_i are the (distinct) roots of $R(x)$. Note that in particular the value of c is uniquely determined by $P(0), Q(0), R(0)$ and α_1, α_2. Hence we obtain the existence of f and the transversality of (Δ, Δ_+, f), since the coefficients of P, Q depend smoothly on the fixed data $R(x), P(0), Q(0), \alpha_1, \alpha_2$. $\square$

3.3.3 The exceptional cases

For the three exceptional types E_6, E_7, E_8 we proceed slightly differently. We use the polytopes shown in Figure 3.5

$$\begin{aligned}
\Delta_{E_6} &= \mathrm{Conv}\{(0,0),(4,0),(0,3),(4,3)\}\,,\\
\Delta_{E_7} &= \mathrm{Conv}\{(1,0),(3,0),(3,4),(2,5),(0,5),(0,1)\}\,,\\
\Delta_{E_8} &= \mathrm{Conv}\{(0,0),(4,0),(0,4),(4,4)\}\,,
\end{aligned}$$

and let $\Delta_{E_k,+} \subset \partial\Delta_{E_k}$ be the union of the left and lower part of the boundary.

Then we proceed as follows:

(1) Find explicit polynomials $F_k \in \mathbb{Q}[x,y]$ with prescribed Newton polytopes Δ_{E_k} and two E_k singularities in $(\mathbb{Q}^*)^2$.

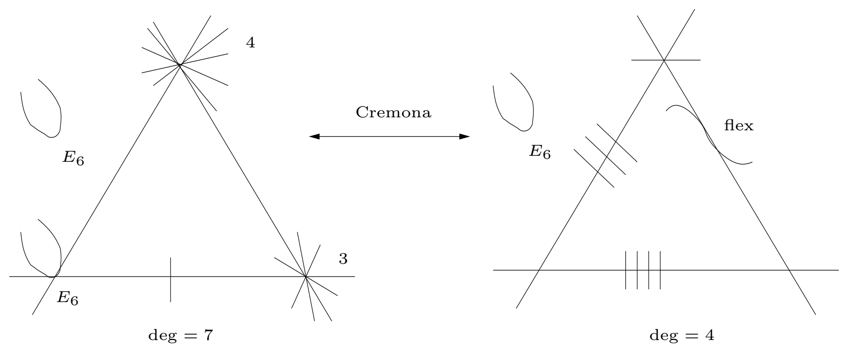
(2) Compute the tangent space to the equisingular stratum and check that it intersects transversally with the respective space of polynomials. Since everything is defined over the rational numbers, this can be done using the computer algebra system SINGULAR [GPS01]. For details cf. Appendix B.

Lemma 3.3.3. *For $k = 6, 7, 8$ there exists a polynomial $F_k \in \mathbb{Q}[x,y]$ with $\Delta(F_k) = \Delta_{E_k}$ and having two E_k singularities in $(\mathbb{Q}^*)^2$. Furthermore, the triad $(\Delta_{E_k}, \Delta_{E_k,+}, F_k)$ is transversal.*

In the proof we shall use elementary properties of the Cremona transform, cf. Appendix A.

Proof. $S = E_6$**.** We show the existence of a curve of degree 7 with an ordinary 4-fold point at $(0:1:0)$, an ordinary 3-fold point at $(1:0:0)$, an E_6 singularity at $(0:0:1)$ and another E_6 singularity somewhere outside the coordinate triangle. The Cremona transform of such a curve has degree 4 and clearly can be realized (just take a generic polynomial with Newton polytope $\Delta = \text{Conv}\{(0,3),(4,0),(3,1)\}$ and apply a linear coordinate change).

The picture below shows the essential behaviour of these curves.



An explicit example looks as follows:

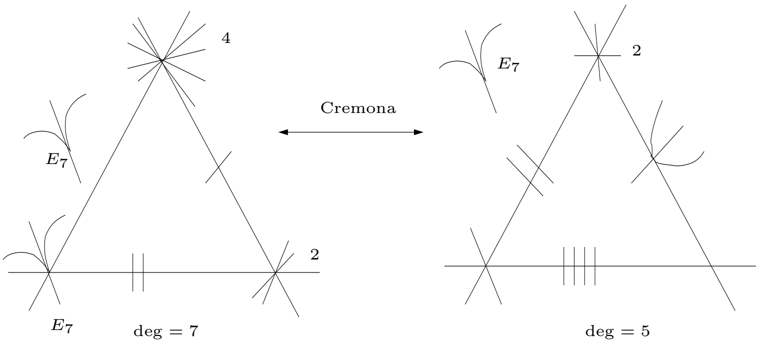
$$f_0 := y^3 + x^4 + x^3 y\,, \quad f_1 := f_0(x+y+2, x-y-4)\,, \quad f_2 := \text{cremona}(f_1)\,,$$

where cremona(g) denotes the (strict) Cremona transform of g. Then the polynomial $F_6 := f_2(x-4, y+3)$ has the desired properties. Using SINGULAR it has been checked that

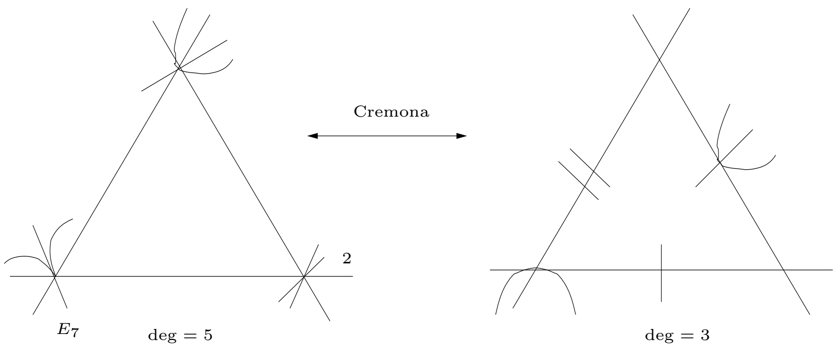
$$\dim_{\mathbb{Q}}(I_d \cap \mathcal{P}(\Delta_{E_6}, \Delta_{E_6,+}, F_6)) = 0\,, \quad I_d = \langle F_6, \tfrac{\partial F_6}{\partial x}, \tfrac{\partial F_6}{\partial y}\rangle \cap \mathbb{Q}[x,y]_{\le d}\,.$$

Hence $(\Delta_{E_6}, \Delta_{E_6,+}, F_6)$ is transversal.

$S = E_7$**.** We show the existence of a curve of degree 7 with an ordinary 4-fold point at $(0:1:0)$, an ordinary 2-fold point at $(1:0:0)$, an E_7 singularity at $(0:0:1)$ and another E_7 singularity somewhere outside the coordinate triangle. The Cremona transform of such a curve has degree 5 and is displayed below.



Then we apply a linear coordinate change such that the curve looks as in the left of the next picture below. Another Cremona transform yields a curve of degree 3 as shown on the right. Such a curve can easily be realized by taking a generic polynomial with Newton polytope $\Delta = \mathrm{Conv}\{(3,0),(2,0),(1,1),(0,1)\}$ and applying a linear coordinate change.

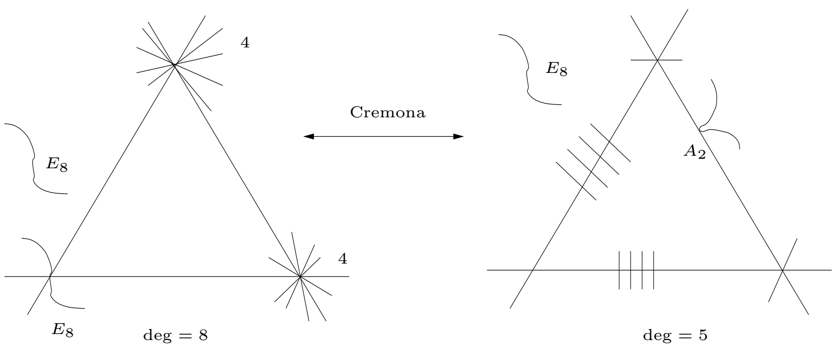


An explicit example looks as follows:

$$f_0 := 3x^3 - x^2 - 2xy + y\,, \quad f_1 := f_0(x+y,y)\,, \quad f_2 := \mathrm{cremona}(f_1)\,,$$
$$f_3 := f_2(x+y-1, y+\tfrac{1}{2})\,, \quad f_4 := \mathrm{cremona}(f_3)\,.$$

Then $F_7 := f_4(x+\frac{28}{9}, y+4)$ is the desired polynomial. The transversality of the triad $(\Delta_{E_7}, \Delta_{E_7,+}, F_7)$ has been checked as in the E_6 case.

$S = E_8$**.** We show the existence of a curve of degree 8 with ordinary 4-fold points at $(0:1:0)$ and $(1:0:0)$, an E_8 singularity at $(0:0:1)$ and another E_8 singularity somewhere outside the coordinate triangle (cf. Figure 3.4). The Cremona transform of such a curve has degree 5. It has a cusp on the line at infinity and an E_8 singularity outside the coordinate triangle and such a curve clearly exists.



A concrete example looks as follows:

$$f_0 := y^3 + x^5 - y^2x^3\,, \quad f_1 := f_0(x+y+2, x-y-3)\,, \quad f_2 := \text{cremona}(f_1)\,.$$

Then $F_8 := f_2(x-4, y+3)$ is the desired polynomial. The transversality of the triple $(\Delta_{E_8}, \Delta_{E_8,+}, F_8)$ has been checked as in the previous cases. □

3.3.4 Result

Combining the results of this section yields

Corollary 3.3.4. *Let $\mathcal{S}$ be a finite set of simple singularities. Then $\alpha_n^{reg}(\mathcal{S}) = \frac{1}{n!}$, and, moreover, $\frac{1}{n!} \in \mathcal{A}_n^{reg}(\mathcal{S})$.*

Proof. By Theorem 3.1.5 we only have to consider the case $n = 2$. Assume that $\mathcal{S} = \{S_1, \ldots, S_r\}$. If $S_i = D_\mu$ or $S_i = E_\mu$, we put Δ_i to be Δ_{S_i}. If $S_i = A_\mu$, we put Δ_i to be the hexagon $\text{Conv}\{(1,0),(0,1),(0,2),(\mu-1,3),(\mu,2),(\mu,1)\}$, the union of Δ_{A_μ} and a suitable translate of Δ'_{A_μ}.

There exists a linear function $R(d)$ such that if

$$\sum_{i=1}^{r} k_i \,\text{vol}(\Delta_i) \leq \text{vol}(\Delta(d)) - R(d) \tag{3.3.5}$$

we can pack k_i translate copies of Δ_i simultaneously into $\Delta(d)$, and so that these copies will be the linearity domains of some convex piece-wise linear function $\nu : \Delta(d) \to \mathbb{R}$. If we orient all common edges upward and to the right, Lemmas 3.3.1, 3.3.2 and 3.3.3 guarantee that, first, the coefficients of the polynomials in these lemmas can be made compatible, and, second, the transversality assumptions of the patchworking theorem are satisfied.
Since $\text{vol}(\Delta_i) = \tau(S_i)$ and $\text{vol}(\Delta(d)) = \frac{d^2}{2}$, inequality (3.3.5) implies the claim. □

Remark 3.3.5. Let $m = \max_{S \in \mathcal{S}}\{\tau(S)\}$. Then $R(d) = md + 3$ satisfies (3.3.5) and we obtain the sufficient condition

$$\sum_{i=1}^{r} k_i \tau(S_i) \leq \frac{d^2}{2} - md - 3$$

for the existence of a plane curve in $V_d(k_1 S_1 + \ldots + k_r S_r)$ with only simple singularities.

3.4 Hypersurfaces with one singularity

In this section we study the existence problem for hypersurfaces having only one singularity and corresponding to a T-smooth germ, i.e. we want to estimate $d_n^{reg}(S)$ for given types S.

Quasihomogeneous singularities

We construct hypersurfaces in $\mathbb{P}^n$ with one quasihomogeneous singularity and corresponding to T-smooth germs. These hypersurfaces can be glued using Corollary 3.2.3.

Let $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{N}^n$, $a_i \geq 2$. Consider the polynomials

$$f_k(x_1, \ldots, x_n) = c_2 \cdot (x_2 - x_1^k)^{a_2} + \ldots + c_n \cdot (x_n - x_{n-1}^k)^{a_n} + c_1 \cdot x_n^{a_1 k}\,,$$

where $(c_1, \ldots, c_n) \in (\mathbb{C}^*)^n$. Then $deg(f_k) = \max\{a_1, \ldots, a_n\} \cdot k$.

Lemma 3.4.1. *If $k \geq \max\{a_1, \ldots, a_n\}$, then the germ of f_k at the origin is contact equivalent to*

$$f(\mathbf{x}) := c_1 \cdot x_1^{a_1 k^n} + c_2 \cdot x_2^{a_2} + \ldots + c_n \cdot x_n^{a_n}\,.$$

Proof. By applying successively the *Jung automorphisms*

$$\begin{array}{rcl} \widetilde{x}_n & = & x_n - x_{n-1}^k \\ \vdots & & \vdots \\ \widetilde{x}_2 & = & x_2 - x_1^k\,, \end{array}$$

we deduce that f_k is at the origin locally equivalent to $f(\mathbf{x}) + h_k(\widetilde{\mathbf{x}})$, where $h_k(\widetilde{\mathbf{x}})$ is a polynomial with

$$h_k(\widetilde{\mathbf{x}}) \in \langle \widetilde{x}_2^{a_2}, \ldots, \widetilde{x}_n^{a_n} \rangle\,,$$

since $k \geq \max\{a_1, \ldots, a_n\}$. Hence $h_k \in \mathfrak{m} \cdot \tau^{ea}(f)$ and thus, by the theorem of Mather-Yau, f_k is contact equivalent to f at the origin. $\square$

Example 3.4.2. Let $\mathbf{a} = (2, \ldots, 2)$. Then f_k has an A_{2k^n-1} singularity, and we can deduce in particular that for $\mu = 2k^n - 1$

$$d_2(A_\mu) \leq 2 \sqrt[n]{\tfrac{\mu+1}{2}}\,,$$

which is asymptotically proper. However, f_k cannot correspond to a T-smooth germ of $V_{2k}(A_{2k^n-1})$ since $2k^n - 1$ exceeds the dimension of the space of hypersurfaces of degree $2k$, which equals

$$h^0(\mathcal{O}_{\mathbb{P}^n}(2k)) - 1 = \frac{(2k)^n}{n!} + O(k^{n-1})\,.$$

We can modify f_k slightly to construct singularities of type D_μ. Consider the polynomials of degree $2k+1$

$$g_k(\mathbf{x}) := x_1(x_2 - x_1^k)^2 + (x_3 - x_2^k)^2 + \ldots + (x_n - x_{n-1}^k)^2 + x_n^{2k+1}\,.$$

Then using the same coordinate changes as above we deduce that g_k has a D_μ-singularity, $\mu = (2k+1) \cdot k^{n-1} + 1$. Hence we obtain

$$d_2(D_\mu) \leq 2 \sqrt[n]{\tfrac{\mu-1}{2}} + 1\,,$$

which is again asymptotically proper.

The following lemma is a simple application of the Horace method, which can be used to deduce T-smoothness.

Lemma 3.4.3. *Consider the ideal $I \subset \mathbb{C}\{\mathbf{x}\}$ given by*

$$I := \langle (x_2 - x_1^k)^{a_1}, \ldots, (x_n - x_{n-1}^k)^{a_{n-1}}, x_n^{ka_n} \rangle\,,$$

where $a_i \geq 2$ and $k \geq 1$. Denote by X the zero-dimensional scheme in $\mathbb{P}^n$ defined by I. Then $h^1(\mathcal{I}_X(d)) = 0$ if $d \geq (a_1 + \ldots + a_n) \cdot k$.

Proof. Using Lemma 1.2.38 we only have to note that for all $i = 1, \ldots, n$

$$x_i^{ka_i} \in I_i := I + \langle x_{i+1}, \ldots, x_n \rangle / \langle x_{i+1}, \ldots, x_n \rangle\,.$$

□

The previous lemma indicates how much we have to increase the degree in order to obtain a T-smooth hypersurface.

Lemma 3.4.4. *Let $m := \max\{a_1, \ldots, a_n\}$ and let l be an integer satisfying $l \geq \frac{\sum_{i=1}^n a_i}{m}$. Then for $k \geq m$, the germ at the origin of*

$$F_k(\mathbf{x}) = f_k(\mathbf{x}) + (x_2 - x_1^k)^{m \cdot l} + \ldots + (x_n - x_{n-1}^k)^{m \cdot l} + x_n^{m \cdot l \cdot k} =: f_k(\mathbf{x}) + g_k(\mathbf{x}) .$$

is analytically equivalent to the germ of f_k. Furthermore, the hypersurface in $\mathbb{P}^n$ defined by the homogenization of F_k is smooth outside the origin and corresponds to a T-smooth germ.

Proof. The first part follows by the theorem of Mather-Yau since

$$g_k(\mathbf{x}) \in \mathfrak{m} \cdot \tau(f_k) .$$

The equisingular stratum is T-smooth at F_k by Lemma 3.4.3 since

$$\langle (x_2 - x_1^k)^{a_1}, \ldots, (x_n - x_{n-1}^k)^{a_{n-1}}, x_n^{k a_n} \rangle \subset I^{ea}(F_k, 0) .$$

The fact that F_k is smooth outside the origin can be checked by calculating the partial derivatives. □

Corollary 3.4.5. *Let $n \geq 2$ and fix $\mathbf{a} = (1, a_2, \ldots, a_n) \in \mathbb{N}^n$. Denote by $\mathcal{S}_{\mathbf{a}}$ the set of all analytical types of singularities defined by polynomials of the form*

$$f(\mathbf{x}) = c_1 \cdot x_1^k + c_2 \cdot x_2^{a_2} + \ldots + c_n \cdot x_n^{a_n} ,$$

with $k \in \mathbb{N}$, $(c_1, \ldots, c_n) \in (\mathbb{C}^*)^n$. Then

$$\alpha_n^{reg}(\mathcal{S}_{\mathbf{a}}) \geq \frac{c}{2^n \cdot n \cdot l^n} > 0 , \tag{3.4.6}$$

where $l = \sum_{i=1}^n a_i$ and $c = \prod_{i=2}^n (a_i - 1)$.

Proof. Let $S \in \mathcal{S}_{\mathbf{a}}$, then by Lemma 3.4.4 the condition $\tau^{ea}(S) \leq \frac{c \cdot d^n}{l^n}$, implies the existence of a T-smooth hypersurface $W \in V_d(S)$. Applying Corollary 3.2.3 and Remark 3.2.4

$$\alpha_n^{reg}(\mathcal{S}_{\mathbf{a}}) \geq \frac{c}{2^n \cdot n \cdot l^n} .$$

□

Remark 3.4.6. Corollary 3.4.5 implies that $\alpha_n(\mathcal{S}) > 0$ for any finite union of sets $\mathcal{S}_{\mathbf{a}}$. However, we do not obtain a uniform bound, which is independent of $\mathbf{a}$.

Simple singularities

We apply these constructions also to obtain hypersurfaces with D_μ-singularities. Furthermore, we improve the result from Lemma 3.4.4 in the case of A_μ-singularities.

Proposition 3.4.7. *Let $n \geq 2$. Then*

(a) Let $\mu \geq 1$. Then $d_n^{reg}(A_\mu) \leq (2n-2) \cdot \lceil \sqrt[n]{\mu+1} \rceil$.

(b) Let $\mu \geq 4$. Then $d_n^{reg}(D_\mu) \leq (2n-1) \cdot \lceil \sqrt[n]{\mu-1} \rceil + 1$.

Proof. We define the polynomials

$$f_{k,l}(\mathbf{x}) = (x_2 - x_1^k)^l + \ldots + (x_n - x_{n-1}^k)^l + x^{lk} .$$

(a) Let $l = 2n-2$ and define

$$F_k(\mathbf{x}) := f_{k,2}(\mathbf{x}) + f_{k,l}(\mathbf{x}) + x_n^k .$$

Then $\deg(F_k) = (2n-2) \cdot k$ and F_k has an A_{k^n-1}-singularity at the origin and no further singular points. We show T-smoothness again using Lemma 1.2.38. Choose $H_i = \{x_i = 0\}$ and let $X_i \subset \mathbb{P}^i$, $i = 1, \ldots, n$, be the zero-dimensional scheme $X^{ea}(F_k(x_1, \ldots, x_i, 0, \ldots, 0), 0)$. Then $s_n = k$ and $s_2, \ldots, s_{n-1} = 2k$ satisfy the conditions of the lemma.
Furthermore, if we denote for simplicity $f_k(x_1, x_2, 0, \ldots, 0)$ by f then

$$deg(X_2 \cap \{x_2 = 0\}) = dim_{\mathbb{C}} \left(\mathbb{C}\{x_1, x_2\} / \langle x_2, \alpha \cdot f_{x_1} + \beta \cdot f_{x_2} \rangle \right) = k$$

and thus $s_1 = k$. This implies $s_1 + \ldots + s_n = k + (n-2) \cdot 2k + k = (2n-2) \cdot k$ and hence by Lemma 1.2.38

$$h^1(\mathbb{P}^n, \mathcal{I}_{X_n}((2n-2) \cdot k)) = 0 .$$

(b) Let $l = 2n-2$ and define

$$g_k = x_1(x_2 - x_1^k)^2 + (x_3 - x_2^k)^2 + \ldots + (x_n - x_{n-1}^k)^2 + f_{k,l} + x_n^k$$

Then $d := deg(g_k) = (2n-1)k+1$ and g_k has an D_{k^n+1}-singularity at the origin and no further singular points. Then $s_n = k$, $s_2, \ldots, s_{n-1} = 2k+1$ satisfy the conditions of Lemma 1.2.38 Furthermore, denote by g the polynomial $g_k(x_1, x_2, 0, \ldots, 0)$ then

$$deg(X_2 \cap \{x_2 = 0\}) = dim_{\mathbb{C}} \left(\mathbb{C}\{x_1, x_2\} / \langle x_2, \alpha \cdot g_{x_1} + \beta \cdot g_{x_2} \rangle \right) = k+1$$

and hence $s_1 = k+1$. This implies

$$\begin{aligned} s_1 + \ldots + s_n = k + (n-2)(2k+1) + k + 1 &= (2n-2)k + n - 1 \\ &\leq (2n-1)k + 1 = d \end{aligned}$$

if $k \geq n-2$. Hence $h^1(\mathcal{I}_{X_n}(d) = 0$ by Lemma 1.2.38.

□

Example 3.4.8. (i) The hypersurface in $\mathbb{P}^4$ of degree 8 defined by the (affine) polynomial

$$f = (w - z^4)^2 + (z - y^4)^2 + (y - x^4)^2 + w^8$$

has an A_{511}-singularity at the origin and no other singular points. Note that $511 > 495 = h^0(\mathcal{O}_{\mathbb{P}^4}(8))$ and hence the hypersurface can not be T-smooth.

(ii) The hypersurface defined by the polynomial

$$F = (w-z^4)^2+(w-z^4)^6+(z-y^4)^2+(z-y^4)^6+(y-x^4)^2+(y-x^4)^6+w^8+w^{24}+w^4$$

of degree $24 = 6k$ has an $A_{255} = A_{k^4-1}$ singularity as its only singular point and corresponds to a T-smooth germ.

Hence, we obtain an asymptotically proper existence result for the set of *all* simple singularities (not just finite subsets). Proposition 3.4.7 implies in particular the weaker estimate

$$d_n^{reg}(S) \leq 2n \sqrt[n]{\mu(S)}$$

for any simple singularity type[1] S. By Remark 3.2.4 we obtain that

$$\alpha_n^{reg}(\mathcal{S}) \geq \frac{1}{4^n \cdot n^{n+1}} > 0\,,$$

for the set of all simple singularity types.

Unimodal singularities

The methods of this section can also be applied to construct hypersurfaces with singularities which are not quasihomogeneous.

The hypersurface W_k defined by

$$F_k(x,y,z) = (y-x^k)^{a_2} + (z-y^k)^{a_3} + a \cdot x \cdot (y-x^k) \cdot (z-y^k) + z^{a_1 \cdot k}\,,$$

$a \in \mathbb{C}$, $a_1, a_2, a_3 \in \mathbb{N}$, has a $T_{a_1k^3,a_2,a_3}$ singularity at the origin. If X_k is the zero dimensional scheme concentrated at the origin and defined by the Tjurina ideal, we obtain that $h^1(\mathcal{I}_{X_k}(d)) = 0$ if $d \geq (a_1 + a_2 + a_3) \cdot k$ using Lemma 3.4.3 because

$$\langle (y-x^k)^{a_2}, (z-y^k)^{a_3}, z^{a_1 \cdot k} \rangle \subset I^{ea}(F_k)\,.$$

By increasing the degree of F_k without changing the singularity as before we obtain the following existence result.

[1] Note that $d_n^{reg}(E_6) = d_n^{reg}(E_7) = 4$, $d_n^{reg}(E_8) = 5$.

Corollary 3.4.9. *Let $\mathcal{S}_{a,b}$ be the set of all singularities of type $T_{k,a,b}$. Then*

$$\alpha_3^{reg}(\mathcal{S}_{a,b}) \geq \frac{1}{24 \cdot (1+a+b)^3} > 0 \, .$$

Note that together with the result for simple singularities, this gives an asymptotic proper existence result for hypersurfaces with singularities of modality < 2 since the Tjurina number of the other unimodal singularities (3 parabolic and the 14 exceptional families, cf. [AGV88]) is bounded (cf. Remark 3.2.4).

3.5 Existence on other projective surfaces

In this section we present applications of our constructions to curves in other smooth, projective surfaces.

Toric surfaces

Assume that $V = \mathrm{Tor}(\Delta)$ is a smooth, projective, toric surface. We denote the very ample divisor corresponding to Δ by H. Let $d\Delta \subset \mathbb{R}^2$ be the polytope obtained by multiplying all points in Δ by d. Then

$$H^0(\mathcal{O}_{\mathrm{Tor}(\Delta)}(dH)) \cong \mathcal{P}(d\Delta) \, .$$

We can use the methods of Section 3.3 to construct curves in the linear system $|H^0(\mathcal{O}_\Delta(d))|$, which has dimension

$$h^0(\mathcal{O}_\Delta(d)) - 1 = \#\{d\Delta \cap \mathbb{Z}^2\} - 1 = \mathrm{vol}(\Delta) + O(d) \, .$$

This yields

Theorem 3.5.1. *Let $\mathcal{S} = \{S_1, \ldots, S_r\}$ be a finite set of simple singularity types. There exists a linear function $R(d)$ such that for all $k_1, \ldots, k_r$ the inequality*

$$\sum_{i=1}^{r} k_i \tau^s(S_i) \leq \mathrm{vol}(\Delta) - R(d) \qquad (3.5.7)$$

implies that $V_{|dH|}(k_1 S_1 + \ldots + k_r S_r)$ has a non-empty T-smooth component.

Proof. Analogous to the proof of Corollary 3.3.4. □

Example 3.5.2. Let $\Delta = \mathrm{Conv}\{(0,0),(1,0),(0,1),(1,1)\}$. Then $\mathrm{Tor}(\Delta) \cong \mathbb{P}^1 \times \mathbb{P}^1$ and H is of type $(1,1)$ and $h^0(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(d,d)) = (d+1)^2$. Hence condition (3.5.7) reads

$$\sum_{i=1}^{r} k_i \tau^s(S_i) \leq d^2 - R(d) \, .$$

Surfaces in $\mathbb{P}^3$

The following application of the techniques and results of Sections 3.1 and 3.3 is due to Shustin and Tyomkin [ShT03].

Let k be a positive integer. Define two sequences $a_k(n)$ and $b_k(n)$ recursively as follows $a_k(1) = -k$, $b_k(1) = -3$, and

$$\begin{aligned} a_k(n+1) &= a_k(n) + a_k(1) - (n+1)\,, \\ b_k(n+1) &= b_k(n) - k + \frac{(n+1)^2}{2} - (n+1)a_k(1) + b_k(1)\,. \end{aligned}$$

Theorem 3.5.3. *Consider a generic surface $S \subset \mathbb{P}^3$ of degree $n \geq 1$ and let $\mathcal{S} = \{S_1, \ldots, S_r\}$ be a finite set of simple singularity types with $k = \max_{i=1..r}\{\mu(S_i)\}$. If for $k_1, \ldots, k_r \geq 0$ and $d \geq 1$ the inequality*

$$\sum_{i=1}^{r} k_i \mu(S_i) \leq \frac{nd^2}{2} + a_k(n)d + b_k(n) \tag{3.5.8}$$

holds true, then there exists a curve $C \in |\mathcal{O}_S(d)|$ satisfying $C \in V_d(k_1S_1 + \ldots + k_rS_r)$ and corresponding to a T-smooth germ.

Proof. For $n = 1$ the theorem coincides with Corollary 3.3.4. For $n > 1$, see [ShT03]. □

Note that since $h^0(\mathcal{O}_S(d)) = \frac{nd^2}{2} + O(d)$, Theorem 3.5.3 is also asymptotically optimal.

3.6 Real curves with many singularities

In this section we study plane curves defined over the real numbers and consider the singular points up to topological equivalence.

Throughout this section, σ denotes the action on $\mathbb{P}^2 = \mathbb{CP}^2$ induced by complex conjugation.

Definition 3.6.1. If z, w are points in $\mathbb{RP}^2$ (respectively pairs of conjugate imaginary points), and $(C, z) \subset (C, z)$, $(D, w) \subset (\mathbb{C}^2, w)$ are (multi-)germs, then (C, z) and (D, w) are called *topologically equivalent over* $\mathbb{R}$, if there exists a local equivariant homeomorphism mapping (C, z) to (D, w). The equivalence classes are called *real topological types.*

We start by making a few general remarks concerning real singularity types:

(i) The real equisingular stratum is locally the real part of the complex equisingular stratum. This implies in particular that if the complex ES-stratum is T-smooth, then the real stratum is also T-smooth.

(ii) The real topological type S of a real, curve singularity is determined by the $(\mu(S)+1)$-jet of a defining equation (cf. [AGV88]).

(iii) There are three different real types corresponding to a complex node: Real nodes with real tangents (e.g. $x^2 - y^2 = 0$), single points (e.g. $x^2 + y^2 = 0$) and pairs of conjugated imaginary nodes.

(iv) For unibranched singularities, the real topological type is uniquely determined by the complex type since their resolution tree is a chain. For other singularities the real type depends not only on the multiplicity sequence, but also on the positions of the infinitely near points (cf. [Los99]).

(v) A real singularity $(C, z) \subset (\mathbb{RP}^2, z)$ is simple if and only if its real topological type belongs to the following list:

Real type		**Equation**	**Real picture**
A_{2k}	$k \geq 1$	$x^{2k+1} + y^2$	unibranched
A^+_{2k-1}	$k \geq 1$	$x^{2k} + y^2$	isolated point
A^-_{2k-1}	$k \geq 1$	$x^{2k} - y^2$	2 branches
D_{2k+1}	$k \geq 2$	$x^{2k} + xy^2$	2 branches
D^+_{2k}	$k \geq 2$	$x^{2k-1} + xy^2$	line
D^-_{2k}	$k \geq 2$	$x^{2k-1} - xy^2$	3 branches
E_6		$x^3 + y^4$	unibranched
E_7		$x^3 + xy^3$	2 branches
E_8		$x^3 + y^5$	unibranched

Reviewing the constructions from Section 3.3 we obtain

Proposition 3.6.2. *Let $\{S_1, \ldots, S_r\}$ be a finite set of real, simple singularity types, and let $k = \sup_{i=1..r}\{\mu(S_i)\}$. If*

$$\sum_{i=1}^{r} k_i \mu(S_i) \leq \frac{1}{2}d^2 - kd - 3$$

then there exists a real curve $C \subset \mathbb{RP}^2$ of degree d with k_i singularities of type S_i for $i = 1, \ldots, r$, and no further singular points. Moreover, the real equisingular stratum is T-smooth at C.

Proof. Using the remarks above we have to verify that in the constructions described in Lemmas 3.3.1, 3.3.2 and 3.3.3 we find representatives defined over the real numbers. Moreover, whenever a complex type S splits into two real types S^+ and S^-, we have to check that both types can be realized.

If $S = A_\mu$ we consider the polynomials

$$f(x,y) = ay^2 - 2y \cdot Q(x) + cx, \quad ac \neq 0, \quad \deg Q = \mu, \quad Q(0) \neq 0\,,$$

as in the proof of Lemma 3.3.1. It is easy to see that we find a polynomial defined over the real numbers. Furthermore, the sign of Q_0 determines the choice of the branch of $\sqrt{acx}$ and hence, if $\mu = 2k-1$, the corresponding real type A^+_{2k-1} or A^-_{2k-1}.

For $S = D_\mu$ consider again the polynomials

$$f(x,y) = (x-\alpha_1)(x-\alpha_2)\Big(y^2 \cdot P(x) - 2y \cdot Q(x) + R(x)\Big)\,.$$

If we assume that all roots of R are real, then we can solve the system appearing in the proof of Lemma 3.3.2 over the real numbers. Since we require that

$$PR - Q^2 = c \cdot (x-\alpha_1)^{\mu-2}(x-\alpha_2)^{\mu-2}\,,$$

we see that the type depends on the sign of c if $\mu = 2k$. But in this case the sign of c is determined by the sign of $P(0) \cdot R(0) - Q^2(0)$, which implies that both types can be constructed.

For $S = E_k$, $k = 6, 7, 8$, the result follows immediately by Lemma 3.3.3 since the polynomials are defined even over the rational numbers. □

Remark 3.6.3. We should mention that also the results from Section 3.4 can be transfered to the real case by simply choosing real values for the parameters in the constructed polynomials. This implies that we obtain precisely the same bounds as in the complex case. For more details concerning real curves with one prescribed singularity cf. [Sh93, Los99].

Chapter 4

H^1-vanishing

As we have seen in the previous chapters, H^1-vanishing of certain ideal sheaves plays a major role for the geometry of equisingular strata. In this chapter we study methods for deriving H^1-vanishing criteria.

We start by reviewing classical and modern approaches, and give an overview of vanishing theorems, most of which concern zero-dimensional schemes in $\mathbb{P}^2$.

Then we recall properties of the Castelnuovo function of zero-dimensional schemes in $\mathbb{P}^2$, and show how to generalize this approach to zero-dimensional schemes on general surfaces in $\mathbb{P}^3$. The graph of the Castelnuovo function of these schemes is to some extent similar to the $\mathbb{P}^2$ case, and we show how to apply this theory in order to obtain H^1-vanishing theorems. Furthermore, we review the approach based on Bogomolov instability, and discuss relations with the Castelnuovo function.

Finally, we present some examples and properties of the Castelnuovo function of zero-dimensional schemes in $\mathbb{P}^3$, which might serve as a motivation for future studies.

4.1 Introduction to the problem

Assume that $X \subset V \subset \mathbb{P}^N$ is a zero-dimensional scheme, and let H be a fixed ample divisor. Then for $d > 0$ sufficiently large

$$H^1(\mathcal{I}_X(dH)) = 0 \tag{4.1.1}$$

since H is ample (cf. Appendix A). The goal is to find the smallest number d such that (4.1.1) is valid.

As for the existence problem discussed in Chapters 2 and 3 we are in particular interested in the *asymptotic* behaviour. Let σ be an invariant of zero-dimensional schemes, and assume that the inequality

$$\sigma(X) \leq f(d) \tag{4.1.2}$$

implies that $H^1(\mathcal{I}_X(dH)) = 0$, where $f : \mathbb{N} \to \mathbb{N}$ is some function. Assume furthermore that there exists a function $g : \mathbb{N} \to \mathbb{N}$ such that, for infinitely many d, there exists a zero-dimensional scheme X with $h^1(\mathcal{I}_X(d)) > 0$ and

$$\sigma(X) \leq g(d)\,. \tag{4.1.3}$$

As before, if $O(f) = O(g)$, then we call (4.1.2) an *asymptotically proper* H^1-vanishing criterion, and in case f and g are polynomials whose leading terms agree, then (4.1.2) is called *asymptotically optimal*.

Remark 4.1.1. There exists a series of zero-dimensional schemes $X_k \subset \mathbb{P}^n$ with

$$d_k := \deg(X_k) = \alpha k^2 + O(k)\,,$$

and $h^1(\mathcal{I}_{X_k}(d_k)) > 0$ (cf. [Sh00]). This shows that a general H^1-vanishing result with $\sigma(X) = \deg(X)$ can be at most quadratic in d on the right-hand side.

We distinguish between several classes of zero-dimensional schemes, which require or allow special techniques for dealing with them:

- General zero-dimensional schemes $X \subset V$,
- Zero-dimensional schemes contained in a hypersurfaces $X \subset W \subset V$,
- Zero-dimensional schemes associated to singular points of a hypersurface W and satisfying $X \subset X^{ea}(W)$,
- (Generic) fat point schemes $X(\underline{m}, \underline{z})$.

Let us briefly review some approaches and results.

Classical approaches based on the Riemann-Roch Theorem.

This approach was used already by Severi, Segre and Zariski, and it is based on the following idea: If X is a zero-dimensional scheme on a surface S contained in a curve C (for example $X = X^{ea}(C)$), then consider $\mathcal{I}_{X/C} = \mathcal{I}_{X/S} \otimes \mathcal{O}_C$ instead of $\mathcal{I}_{X/S}$, and apply vanishing theorems for coherent $\mathcal{O}_C$-modules using that X is a divisor on C.

Applying this method Greuel and Karras proved a general H^1-vanishing theorem for zero-dimensional schemes $X \subset C \subset S$, where S is a smooth, projective surface and C a curve. If $C_1, \ldots, C_r$ are the irreducible components of C, they proved that $H^1(\mathcal{I}_{X/C}(D)) = 0$ if

$$\deg(X \cap C_i) - \mathrm{isod}_{C_i}(X, C) < (D - K_S - C).C_i$$

for $i = 1, \ldots, r$ (cf. [GrK89]). The so-called *isomorphism defect* $\mathrm{isod}_{C_i}(X, C)$ is a certain positive number introduced in [GrK89], which we shall not define here.

If $S = \mathbb{P}^2$, $X \subset X^{ea}(C)$ and C is not a union of $d \geq 3$ lines, $d = \deg(C)$, then the above condition was used to produce the famous $4d - 4$ criterion

$$\deg(X) - \mathrm{isod}(X, C') < 4d - 4 \implies H^1(\mathcal{I}_{X/C}(d)) = 0\,,$$

where C' denotes a generic polar of C. It can be shown that for a curve C having r nodes as only singular points, we have $\mathrm{isod}(X^{ea}(C), C') = \deg(X^{ea}(C)) = r$. Hence, this criterion implies in particular the classical result that nodal curves are always T-smooth.

Application of Kodaira vanishing.

This approach was introduced by Xu [Xu95], and it was used to prove vanishing theorems for generic fat point schemes $X(\underline{m}) \subset \mathbb{P}^2$. It is based on a criterion for showing ampleness of divisors $dH - m_1E_1 - \ldots - m_rE_r$ on $\mathbb{P}^2_r$, the projective plane blown up in the r generic points, and then applying the Kodaira vanishing theorem.

Using this idea he proved the following vanishing criterion for generic fat point schemes:
Let $\underline{m} = (m_1, \ldots, m_r)$, $m_1 \geq m_2 \geq \cdots \geq m_r$, and d be positive integers satisfying $m_1 + m_2 \leq d$, $m_1 + \ldots + m_5 \leq 2d$, and

$$\sum_{i=1}^{r} \frac{(m_i + 1)^2}{2} < \frac{9(d + 3)^2}{20}\,.$$

Then $H^1(\mathcal{I}_{X(\underline{m})}(d)) = 0$.

Keilen and Tyomkin [KeT02] generalized this approach to more general projective surfaces S, and showed that $H^1(\mathcal{I}_{X(\underline{m})/S}(d)) = 0$ if

$$2 \cdot \sum_{i=1}^{r} (m_i + 1)^2 < (D - K_S)^2\,,$$

and, in addition, for any irreducible curve B with $B^2 = 0$ and $\dim |B|_a > 0$ (the system of curves algebraically equivalent to B)

$$\max\{m_i \mid i = 1, \ldots, r\} < (D - K_S).B\,.$$

Bogomolov instability of rank 2 vector bundles.

Chiantini and Sernesi [CS97] used Bogomolov's theory of unstable rank 2 vector bundles (cf. [Bog79]) to produce H^1-vanishing theorems for some zero-dimensional schemes on smooth projective surfaces. We will discuss this approach in more detail in Section 4.3 below.

In the $\mathbb{P}^2$ case they obtained the following result:
Let $C \subset \mathbb{P}^2$ be an irreducible curve of degree $d \geq 3$, and let $X \subset X^{ea}(C)$ be a zero-dimensional scheme with connected components X_i, $i = 1, \ldots, r$. For an arbitrary zero-dimensional scheme Z define

$$c(Z) = \max\{\deg(Z') \mid Z' \subset Z \text{ locally a complete intersection}\}\,.$$

Then $H^1(\mathcal{I}_X(d)) = 0$ if

$$\sum_{i=1}^{r}(c(X_i)+1)^2 < d^2 + 6d\,.$$

Keilen and Tyomkin used this approach in combination with ideas of [GLS00] to produce a new vanishing result in terms of new invariants $\gamma_\alpha(S)$ discussed in detail in [LoK03] (cf. also Section 4.4). Let S be a surface with Néron-Severi group equal to $\mathbb{Z}$ and denote by L an ample generator. Assume that $C \subset S$ is an irreducible curve, and let $X = X_1 \cup \cdots \cup X_r \subset X^{ea}(C)$ be a zero-dimensional scheme.
Then $H^1(\mathcal{I}_{X/S}(dL)) = 0$ if

$$\sum_{i=1}^{r} \gamma_\alpha(C; X_i) < \alpha(d-\kappa)^2 \cdot L^2\,, \tag{4.1.4}$$

where $\alpha = \frac{1}{\max\{1, 1+\kappa\}}$ and $K_S = \kappa \cdot L$.

The Horace method.

Hirschowitz [Hir85] initiated a new approach for finding H^1-vanishing criteria, which has been used in many variants. We have already discussed the idea of the basic Horace method in Section 1.2.3.

The following vanishing theorem for generic fat point schemes in $\mathbb{P}^n$ was proven by Hirschowitz [Hir89] for $n = 2$, and generalized to higher dimensions by Shustin in [Sh00].

If $\underline{m} = (m_1, \ldots, m_r)$ and $d > 0$ are positive integers such that

$$\sum_{i=1}^{r} \binom{m_i + n - 1}{n} \; < \; M(n,d)\,,$$

then $h^1(\mathcal{I}_{X(\underline{m})/\mathbb{P}^n}(d)) = 0$, where $M(n,d)$ is the number introduced in 3.2. Note that this result is asymptotically proper.

The $4d-4$ criterion (cf. above) can be generalized to higher dimensions. If $X \subset \mathbb{P}^n$ is an arbitrary zero-dimensional scheme, then $H^1(\mathcal{I}_X(d)) = 0$ if

$$\deg(X) < \begin{cases} 8 & \text{if } d = 2, \\ 16 & \text{if } d = 3, \\ 18 & \text{if } d = 4, \\ 4d-4 & \text{if } d \geq 5. \end{cases}$$

(cf. [dPW98]). Shustin and Tyomkin proved this theorem by induction on the degree and the dimension by applying the Horace method [ShT99]. This is in fact the only general H^1-vanishing result in the higher dimensional case.

The Castelnuovo function.

Greuel, Lossen and Shustin used the Castelnuovo function to prove new H^1-vanishing results for zero-dimensional scheme in the plane [GLS00]. This approach is based on the works of Davis [Da86] and Barkats [Ba93], and shall be discussed in detail in the next section. The main result was the following theorem [GLS00]:

Let $C \subset \mathbb{P}^2$ be an irreducible curve of degree $d \geq 6$, with r singular points $z_1, \ldots, z_r$. If a zero-dimensional scheme of the form

$$X = X_1 \cup \cdots \cup X_r\,, \quad X_i \subset X^{ea}(C, z_i)\,,$$

satisfies

$$\sum_{i=1}^{r} \gamma_1(X_i, C) < (d+3)^2\,,$$

then $h^1(\mathcal{I}_X(d)) = 0$.

4.2 The Castelnuovo function

In this section we discuss the Castelnuovo function of zero-dimensional schemes in detail. We start by recalling the definition and some elementary properties, which are valid in general. In Section 4.2.1 the case of zero-dimensional schemes in the plane is discussed and the results of [GLS00] are recalled. Then in Section 4.2.2 we study the behaviour of the Castelnuovo function for schemes on general surfaces in $\mathbb{P}^3$, which has not been studied before.

Let us define the Castelnuovo function of zero-dimensional schemes. Assume that $V \subset \mathbb{P}^N$ is a smooth, projective variety and that $H \subset V$ is a fixed ample divisor. If no confusion arises we write $\mathcal{F}(d)$ for $\mathcal{F}(dH)$ if $\mathcal{F}$ is a coherent sheaf on V.

Definition 4.2.1. Let $\emptyset \neq X \subset V$ be a zero-dimensional scheme defined by the ideal sheaf $\mathcal{I}_X \subset \mathcal{O}_V$. Then the *Castelnuovo function of* X is given by

$$\mathcal{C}_X : \mathbb{Z} \to \mathbb{Z}\,, \quad d \mapsto h^1(\mathcal{I}_{X/V}((d-1)H)) + h^1(\mathcal{I}_{X/V}(dH))\,.$$

We associate to X the numbers

$$\begin{aligned} a(X) &= \min\{d \,|\, H^0(\mathcal{I}_X(d)) \neq 0\}\,, \\ b(X) &= \min\{d \,|\, |H^0(\mathcal{I}_X(d))| \neq \emptyset \text{ has no fixed component}\} \text{ and} \\ t(X) &= \min\{d \,|\, h^1(\mathcal{I}_X(d)) = 0\}\,. \end{aligned}$$

Here, a fixed component is a divisor D such that every element of the linear system $|H^0(\mathcal{I}_X(dH))|$ contains D as a component. We call the maximal divisor satisfying this property *the fixed component of* $|H^0(\mathcal{I}_X(dH))|$.

Let us assume that $h^1(\mathcal{O}_V(d)) = 0$ for all d, which is the case in particular for all smooth hypersurfaces $V \subset \mathbb{P}^n$, $n \geq 3$. Then

$$h^1(\mathcal{I}_X(d)) = h^0(\mathcal{I}_X(d)) - h^0(\mathcal{O}_V(d)) + \deg(X)\,,$$

and hence

$$\begin{aligned} \mathcal{C}_X(d) &= h^1(\mathcal{I}_X(d-1)) - h^1(\mathcal{I}_X(d)) \\ &= h^0(\mathcal{O}_V(d)) - h^0(\mathcal{O}_V(d-1)) - \big(h^0(\mathcal{I}_X(d)) - h^0(\mathcal{I}_X(d-1))\big)\,. \end{aligned}$$

If W is a generic hyperplane section of $V \subset \mathbb{P}^N$, then we have the *reduction sequence*

$$0 \longrightarrow \mathcal{I}_X(d-1) \longrightarrow \mathcal{I}_X(d) \longrightarrow \mathcal{O}_W(d) \longrightarrow 0\,.$$

Since $h^1(\mathcal{O}_V(d)) = 0$ we obtain

$$\begin{aligned} \mathcal{C}_X(d) &= h^0(\mathcal{O}_V(d)) - h^0(\mathcal{O}_V(d-1)) - \big(h^0(\mathcal{I}_X(d)) - h^0(\mathcal{I}_X(d-1))\big) \\ &= h^0(\mathcal{O}_W(d)) - \big(h^0(\mathcal{I}_X(d)) - h^0(\mathcal{I}_X(d-1))\big) \\ &= \dim \operatorname{coker}(\pi)\,. \end{aligned} \tag{4.2.5}$$

where $\pi : H^0(\mathcal{I}_X(d)) \to H^0(\mathcal{O}_W(d))$ is the natural restriction map.

The following lemma contains some elementary properties of the Castelnuovo function.

Lemma 4.2.2. *Let V be a smooth, projective variety with $h^1(\mathcal{O}_V(d)) = 0$ for all $d \in \mathbb{Z}$. Let $X \subset V$ be a zero-dimensional scheme, and asssume that W is a generic hyperplane section of V, hence in particular $X \cap W = \emptyset$. Then*

(a) $\mathcal{C}_X(d) \geq 0$ for all d, and $\mathcal{C}_X(d) = 0$ for $d \gg 0$.

(b) $\mathcal{C}_X(d) \le h^0(\mathcal{O}_W(d))$*, with equality if and only if* $d < a(X)$*.*

(c) $a(X) \le b(X) \le t(X) + 1$*.*

(d) $\mathcal{C}_X(d) = 0 \Longleftrightarrow d \ge t(X) + 1$*.*

(e) If $Y \subseteq X$ *then* $\mathcal{C}_Y(d) \le \mathcal{C}_X(d)$*.*

(f) $\sum_{s=0}^{d} \mathcal{C}_X(s) = \deg(X) - h^1(\mathcal{I}_X(d))$ *for all* $d \ge 0$*.*

Proof. (a) The first part is obvious, the second one follows from Proposition A.4.1.

(b) Follows from (4.2.5).

(c) The first inequality follows by definition. For the second one we have to show that $\mathcal{C}_X(d) = 0$ implies $d \ge b(X)$. Assume that $d < b(X)$. Then there is fixed component D of $H^0(\mathcal{I}_X(d))$, and its restriction to W is a fixed component of $\pi(H^0(\mathcal{I}_X(d)) \subset H^0(\mathcal{O}_W(d))$. But this implies that $\mathcal{C}_X(d) \ne 0$ by (4.2.5).

(d) If $d \ge t(X) + 1$, then $h^1(\mathcal{I}_X(s)) = 0$ for all $s \ge d - 1$, and hence in particular $\mathcal{C}_X(d) = 0$. Conversely if $\mathcal{C}_X(d) = 0$, then by (4.2.5) $\mathcal{C}_X(s) = 0$ for all $s \ge d$. Hence $h^1(\mathcal{I}_X(d-1)) = h^1(\mathcal{I}_X(s))$ for all $s \ge d-1$, and the claim follows since $h^1(\mathcal{I}_X(s)) = 0$ for $s \gg 0$ by Proposition A.4.1.

(e) If $Y \subseteq X$, then $H^0(\mathcal{I}_X(d)) \subseteq H^0(\mathcal{I}_Y(d))$ and the claim follows from (4.2.5).

(f) Follows since $\deg(X) = h^1(\mathcal{I}_X(-1))$.

□

In the following sections we will refer to *the graph of Castelnuovo function*, where we consider the Castelnuovo function as a function on $\mathbb{R}_{\ge 0}$ given by

$$\mathcal{C}_X : \mathbb{R}_{\ge 0} \longrightarrow \mathbb{R}_{\ge 0}\,, \quad d \mapsto \mathcal{C}_X([d])\,.$$

4.2.1 The Castelnuovo function of zero-dimensional schemes in $\mathbb{P}^2$

In this section, we review the behaviour of the Castelnuovo function for zero-dimensional schemes in $\mathbb{P}^2$. In the following L denotes a generic line in $\mathbb{P}^2$.

Lemma 4.2.3. *Let* $X \subset \mathbb{P}^2$ *be a zero-dimensional scheme.*

(a) If $d < a(X)$*, then* $\mathcal{C}_X(d) = d + 1$*.*

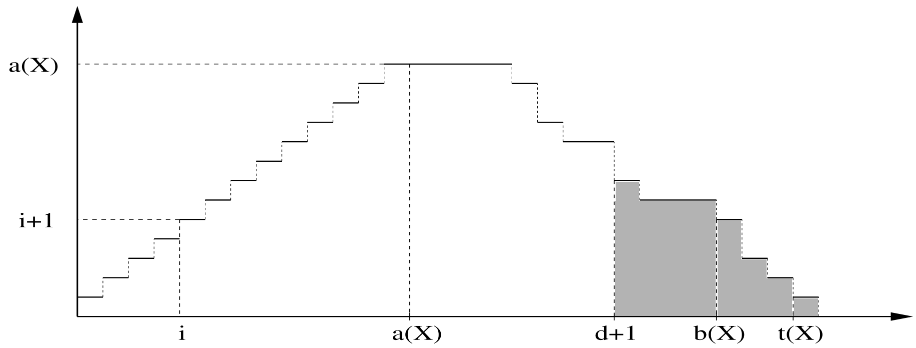


Figure 4.1: The graph of the Castelnuovo function of a scheme $X \subset \mathbb{P}^2$. The content of the shaded region is $h^1(\mathcal{I}_X(d))$.

(b) If $d \geq a(X)$, then $\mathcal{C}_X(d) \leq \mathcal{C}_X(d-1)$. In particular, $\mathcal{C}_X(d) \leq a(X)$ for all d.

(c) If $b(X) \leq d \leq t(X)+1$, then $\mathcal{C}_X(d) < \mathcal{C}_X(d-1)$.

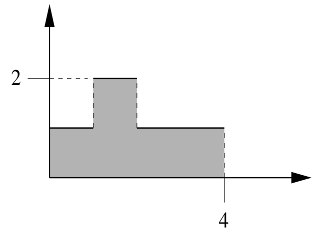
(d) Let $X \subset \mathbb{P}^2$ be the intersection of two curves C_a, C_b of degrees a, b with $a \leq b$. Then

$$\mathcal{C}_X(d) = \begin{cases} d+1 & \text{for } d < a\,, \\ a & \text{for } a \leq d < b\,, \\ a+b-d-1 & \text{for } b \leq d < a+b-1\,, \\ 0 & \text{for } d \geq a+b-1\,. \end{cases}$$

Proof. [Da86]. □

From these properties we can derive the typical form of the graph of $\mathcal{C}_X(d)$, which is displayed in Figure 4.1. For complete intersection schemes X the picture is symmetric, in particular there are no "long stairs" in the graph, i.e. $\mathcal{C}_X(d) \neq \mathcal{C}_X(d+1)$ if $a(X) > \mathcal{C}_X(d) \neq 0$. For more general schemes, there can be long stairs in the graph, and it is necessary to understand the reason for their appearance.

Example 4.2.4. Let $X \subset \mathbb{P}^2$ be a zero-dimensional scheme supported in $z_1, \ldots, z_5$, where $z_1, \ldots, z_4$ lie on a line L and z_5 is generic. The Castelnuovo function of X is shown below. Note that $\mathcal{C}_X(3) = \mathcal{C}_X(4) = 1$, which equals the degree of the fixed component L. Furthermore, $\mathcal{C}_{X \cap L}(d) = 1$ for $d = 0, \ldots, 4$.



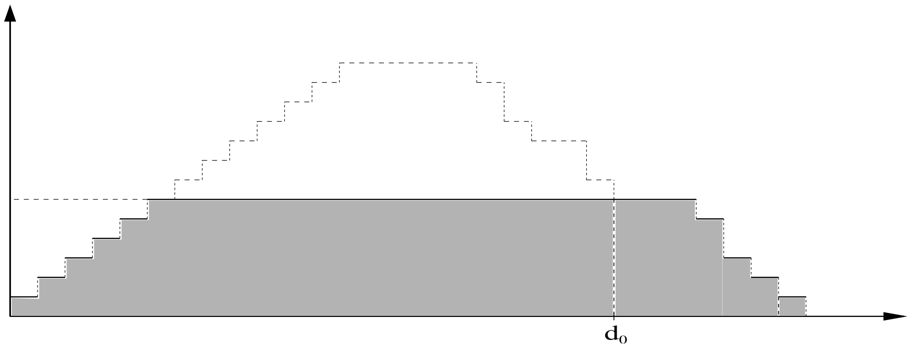


Figure 4.2: The Davis Lemma for $X \subset \mathbb{P}^2$. The shaded region is the graph of $\mathcal{C}_{X\cap D}$, where D is the fixed component in $H^0(\mathcal{I}_X(d_0))$ of degree $\mathcal{C}_X(d_0)$.

The example above is a special case of Davis' Lemma (cf. [Da86]).

Lemma 4.2.5. *Let $X \subset \mathbb{P}^2$ be a zero-dimensional scheme, and $d_0 \geq a(X)$ such that $\mathcal{C}_X(d_0) = \mathcal{C}_Z(d_0+1)$. Then the fixed curve D in $|H^0(\mathcal{I}_X(d_0))|$ is of degree $\mathcal{C}_X(d_0)$ and satisfies for $d \geq 0$*

$$\mathcal{C}_{X\cap D}(d) = \min\{\mathcal{C}_X(d), \mathcal{C}_X(d_0)\}\,. \tag{4.2.6}$$

Definition 4.2.6. We call a zero-dimensional scheme $X \subset \mathbb{P}^2$ *decomposable* if there exists a positive integer d_0 such that

$$\mathcal{C}_X(d_0-1) > \mathcal{C}_Z(d_0) = \mathcal{C}_X(d_0+1) > 0\,.$$

The following lemma is one step towards an H^1-vanishing theorem, and was derived by Barkats [Ba93].

Lemma 4.2.7. *Let $C_d \subset \mathbb{P}^2$ be an irreducible curve of degree $d > 0$, and let $X \subset C_d$ be a zero-dimensional scheme such that $h^1(\mathcal{I}_X(d)) > 0$. Suppose, moreover, $d > a(X)$. Then there exists a curve C_k of degree $k \geq 3$ such that $X \cap C_k \subset C_d \cap C_k$ is non-decomposable and satisfies*

(a) $h^1(\mathcal{I}_{X\cap C_k}(d)) = h^1(\mathcal{I}_X(d))$,

(b) $\deg(X\cap C_k) \geq k\cdot(d+3-k)$, *if* $k \leq \frac{d+3}{2}$, *and* $\deg(X\cap C_k) > \frac{(d+3)^2}{4}$, *if* $k > \frac{d+3}{2}$.

Note that, if X is contained in an irreducible curve C of degree d, then necessarily $d \geq b(X)$. This follows since $C \in |H^0(\mathcal{I}_X(d))|$ and any fixed component of this linear system would be a component of C, which is impossible since C is irreducible.

4.2.2 The Castelnuovo function of schemes $X \subset S \subset \mathbb{P}^3$

In this section, we study the Castelnuovo function for zero-dimensional schemes on a general surface $S \subset \mathbb{P}^3$ of degree n. Then $h^1(\mathcal{O}_S(d)) = 0$ for all d, and $\mathrm{Pic}(S) = \mathbb{Z}$ if $n \geq 4$ (cf. Proposition A.4.2). For simplicity of notation, we write $|\mathcal{O}_S(d)|$ for the complete linear system $|H^0(\mathcal{O}_S(d))|$.

We also consider X as a subscheme of $\mathbb{P}^3$, and the cohomology groups are connected by the exact sequence

$$0 \longrightarrow \mathcal{O}_S(d-n) \longrightarrow \mathcal{I}_{X/\mathbb{P}^3}(d) \longrightarrow \mathcal{I}_{X/S}(d) \longrightarrow 0\,.$$

Since $H^1(\mathcal{O}_S(d-n)) = 0$ we know that

$$H^1(\mathcal{I}_{X/S}(d)) \cong H^1(\mathcal{I}_{X/\mathbb{P}^3}(d))$$

if $H^2(\mathcal{O}_S(d-n)) = 0$. But the exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^3}(d-2n) \longrightarrow \mathcal{O}_{\mathbb{P}^3}(d-n) \longrightarrow \mathcal{O}_S(d-n) \longrightarrow 0$$

implies that $h^2(\mathcal{O}_S(d-n)) = h^3(\mathcal{O}_{\mathbb{P}^3}(d-2n)) - h^3(\mathcal{O}_{\mathbb{P}^3}(d-n))$, and hence we obtain $H^2(\mathcal{O}_S(d-n)) = 0$ if $d - 2n + 4 > 0$. In particular,

$$\mathcal{C}_{X/S}(d) = \mathcal{C}_{X/\mathbb{P}^3}(d) \qquad \text{if } d \geq 2n-2\,.$$

The Castelnuovo function of a complete intersection scheme shows a regular behaviour as expected. Its graph is depicted in Figure 4.3.

Lemma 4.2.8. *Let $X = C_a \cap C_b$ be the intersection of two curves $C_a \in |\mathcal{O}_S(a)|$, $C_b \in |\mathcal{O}_S(b)|$ without common component, and assume that $n \leq a \leq a+n \leq b$. Then:*

$$\mathcal{C}_X(d) = \begin{cases} \binom{d+2}{2} & \text{for } d < n-2 \\ n \cdot d - \frac{n(n-3)}{2} & \text{for } n-2 \leq d < a \\ n \cdot d - \frac{n(n-3)}{2} - \binom{d-a+2}{2} & \text{for } a \leq d < a+n-2 \\ n \cdot a & \text{for } a+n-2 \leq d < b \\ \mathcal{C}_X(a+b+n-d) & \text{for } b \leq d < a+b+n-2 \\ 0 & \text{for } d \geq a+b+n-2\,. \end{cases}$$

Proof. Since

$$\begin{aligned} h^0(\mathcal{I}_{X/S}(d)) &= h^0(\mathcal{O}_S(d-a)) + h^0(\mathcal{I}_{X/C_a}(d)) \\ &= h^0(\mathcal{O}_S(d-a)) + h^0(\mathcal{O}_S(d-b)) - h^0(\mathcal{O}_S(d-a-b))\,, \end{aligned}$$

we obtain that $\mathcal{C}_X(d) = H(d) - H(d-1)$, where

$$H(d) := h^0(\mathcal{O}_S(d)) - h^0(\mathcal{O}_S(d-a)) - h^0(\mathcal{O}_S(d-b)) + h^0(\mathcal{O}_S(d-a-b))\,.$$

□

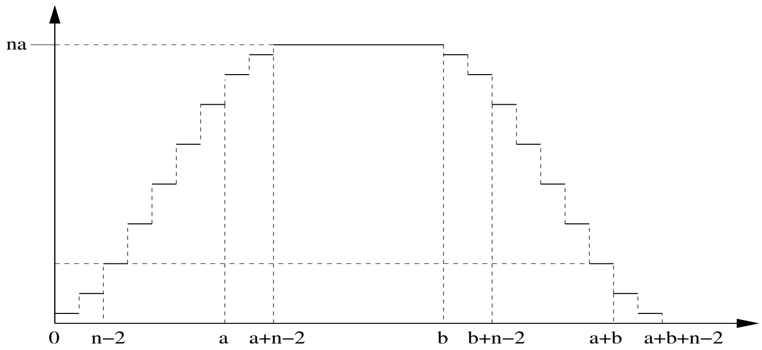


Figure 4.3: The Castelnuovo function of a complete intersection $X = C_a \cap C_b$ on a surface $S_n \subset \mathbb{P}^3$, $n \leq a \leq a+n \leq b$.

Example 4.2.9. Let $S \subset \mathbb{P}^3$ be a generic surface of degree $n \geq 4$. Let $X \subset S$ be the fat point supported in $z \in S$ and defined by the ideal $\mathfrak{m}_z^2$. Since S is locally isomorphic to $\mathbb{P}^2$, $X \subset S$ is isomorphic to a fat point $Y \subset \mathbb{P}^2$ of multiplicity 2. However, the Castelnuovo function of X and Y do *not* coincide since

$$a(X) \;=\; 1 \;<\; 2 \;=\; a(Y)\,,$$

because the curve $C \in |\mathcal{O}_S(1)|$, given by the intersection of the tangent plane to S at z with S, has multiplicity ≥ 2 at z.

Again we can estimate the maximal value of $\mathcal{C}_X(d)$.

Lemma 4.2.10. *Let $X \subset S$ be a zero-dimensional scheme. Then*

(i) $\mathcal{C}_X(d) \leq n \cdot a(X)$,

(ii) $t(X) \leq a(X) + b(X) + n - 2$.

Proof. Let $C_a \in |H^0(\mathcal{I}_X(a(X)))| \neq \emptyset$ and let $C_b \in |H^0(\mathcal{I}_X(b(X)))|$ be a curve without common component with C_a. Define $Y := C_a \cap C_b$. Then $X \subseteq Y$, and the assertions follow from Lemma 4.2.8 and Lemma 4.2.2(e). $\square$

The crucial step is to know when the Castelnuovo starts descending. While for schemes in $\mathbb{P}^2$ this was the case as soon as $d \geq a(X)$, the situation on a general surface in $\mathbb{P}^3$ is slightly different.

Lemma 4.2.11. *Let $S \subset \mathbb{P}^3$ be a smooth surface of degree n, and let $X \subset S$ be a zero-dimensional scheme. Then for all $p \geq 0$,*

$$\mathcal{C}_X(a(X)+p) \leq \mathcal{C}_X(a(X)+p-1) + \max\{0, n-p-1\}\,.$$

In particular, if $p \geq n-1$, then the Castelnuovo function is descending.

Proof. Let C be a generic hyperplane section of S, i.e. C is a plane curve of degree n. In generic affine coordinates, C is given by an equation $F(x, y)$ of degree n, and we denote by R the affine coordinate ring $\mathbb{C}[x, y]/F$ of C. The subspaces of $R_{\le d}$ corresponding to $\pi(H^0(\mathcal{I}_X(d))) \subset H^0(\mathcal{O}_C(d))$ are denoted by I_d. We represent the elements of $R_{\le d}$ in terms of monomials $x^i y^j$, $i + j \le d$, $j \le n - 1$.

Let $f \in I_{a(X)}$, and we may assume that $\deg(f) = a(X)$ since we chose generic affine coordinates. Let f_a be the leading form of f. Then the elements $f_a \cdot x^i y^j$, $i + j = d - a$, $j \le n - 1$, are linearly independent in $R_{\le d}$. Hence, if B is a basis of I_{d-1}, then the set

$$B \cup \{f_a \cdot x^i y^j \mid i + j = d - a,\ j \le n - 1\}$$

is linearly independent in $R_{\le d}$. □

Let us study the influence of fixed curves on the shape of the Castelnuovo function.

Lemma 4.2.12. *Let $D \in |H^0(\mathcal{O}_S(e))|$ be a fixed curve of $|H^0(\mathcal{I}_X(d))|$, and assume that $C \in |\mathcal{O}_S(1)|$ is a generic hyperplane section. Then*

(a) $e \le a(X)$, and if $e = a(X)$ then $X \cap D = X$.

(b) If $e < a(X)$, then

$$\mathcal{C}_X(d) = \mathcal{C}_{X:D}(d - e) + h^0(\mathcal{O}_C(d)) - h^0(\mathcal{O}_C(d - e))\,.$$

In particular, $\mathcal{C}_X(d) \le h^0(\mathcal{O}_C(d)) - h^0(\mathcal{O}_C(d - e))$ with equality if and only if $t(X : D) < d - e$.

Proof. The first assertion is clear. If $e < a(X)$, then $X : D \ne \emptyset$, and

$$H^0(\mathcal{I}_X(d)) = D \cdot H^0(\mathcal{I}_{X:D}(d - e))\,.$$

For $t \ge 0$, denote by π_t the restriction map

$$\pi_t \colon H^0(\mathcal{I}_X(t)) \longrightarrow H^0(\mathcal{O}_C(t))\,.$$

Then result follows from the equations

$$\begin{aligned} \mathcal{C}_X(d) &= h^0(\mathcal{O}_C(d)) - \pi_d(H^0(\mathcal{I}_X(d)) \\ \mathcal{C}_{X:D}(d - e) &= h^0(\mathcal{O}_C(d - e)) - \pi_{d-e}(H^0(\mathcal{I}_{X:D}(d - e)), \end{aligned}$$

noting that $\pi_d(H^0(\mathcal{I}_X(d)) = \pi_e(D) \cdot \pi_{d-e}(H^0(\mathcal{I}_{X:D}(d - e))$. □

Remark 4.2.13. If $e \leq d - n + 2$, then $h^0(\mathcal{O}_C(d)) - h^0(\mathcal{O}_C(d-e)) = n \cdot e$. Hence under this condition the previous lemma implies

$$e \leq \frac{\mathcal{C}_X(d)}{n}$$

with equality iff $t(X : D) < d - e$.

In order to proceed similarly to the $\mathbb{P}^2$ case, it is necessary to understand the appearance of fixed components in greater details. In fact, we believe that the following property holds for all zero-dimensional schemes X on a general surface in $\mathbb{P}^3$ of degree ≥ 4.

Conjecture 4.2.14. *If $d \geq \max\{b(X), a(X) + n - 1\}$, then*

$$\mathcal{C}_X(d) < \mathcal{C}_X(d-1)\,.$$

In particular, this implies that if $\mathcal{C}_X(d_0) = \mathcal{C}_X(d_0 + 1) > 0$, then the linear system $H^0(\mathcal{I}_X(d_0))$ has a fixed component. Conjecture 4.2.14 holds if X is a complete intersection scheme (cf. above), or if $X = \pi^* Y$ where $\pi : S \to \mathbb{P}^2$ is a branched covering (cf. Section 4.2.3 below). In the following, we will draw several conclusions of this conjecture, indicating always the steps where it is necessary.

The next lemma is the analogue of the Davis Lemma 4.2.5 for zero-dimensional schemes $X \subset S \subset \mathbb{P}^3$.

Lemma 4.2.15. *Assume that Conjecture 4.2.14 holds. Let $X \subset S$ be a zero-dimensional scheme, and let C be a generic hyperplane section. Assume that*

$$\mathcal{C}_X(d) \;=\; \mathcal{C}_X(d+1)\,,$$

and let $D \in |\mathcal{O}_S(e)|$ be the fixed curve of $|H^0(\mathcal{I}_X(d))|$.

(a) $\mathcal{C}_X(d) = h^0(\mathcal{O}_C(d)) - h^0(\mathcal{O}_C(d-e))$.

(b) For all $u \in \mathbb{Z}$, $\mathcal{C}_X(u) = \mathcal{C}_{X:D}(u-e) + \min\{\mathcal{C}_X(u), h^0(\mathcal{O}_C(u) - h^0(\mathcal{O}_C(u-e))\}$.

(c) $\deg(X \cap D) = \sum_{u \geq 0} \min\{\mathcal{C}_X(u), h^0(\mathcal{O}_C(u) - h^0(\mathcal{O}_C(u-e))\}$.

(d) $\mathcal{C}_{X \cap D}(u) = \min\{\mathcal{C}_X(u), h^0(\mathcal{O}_C(u) - h^0(\mathcal{O}_C(u-e))\}$ *for $u \in \mathbb{Z}$.*

Proof. (a) If $e = a(X)$, then the claim follows easily since $X : D = \emptyset$. Hence assume that $e < a(X)$. By Lemma 4.2.12 we have to verify that $t(X : D) < d - e$. The assumptions together with Lemma 4.2.12 imply that

$$\mathcal{C}_{X:D}(d-e) = \mathcal{C}_{X:D}(d-e+1)\,. \tag{4.2.7}$$

Furthermore, since D is the fixed curve of $H^0(\mathcal{I}_X(d))$, it follows that the linear system $H^0(\mathcal{I}_{X:D}(d-e))$ has no fixed curve, that is, $b(X : D) \leq d - e$. Hence, (4.2.7) together with Conjecture 4.2.14 implies that $\mathcal{C}_{X:D}(d-e) = 0$, that is, $t(X : D) < d - e$.

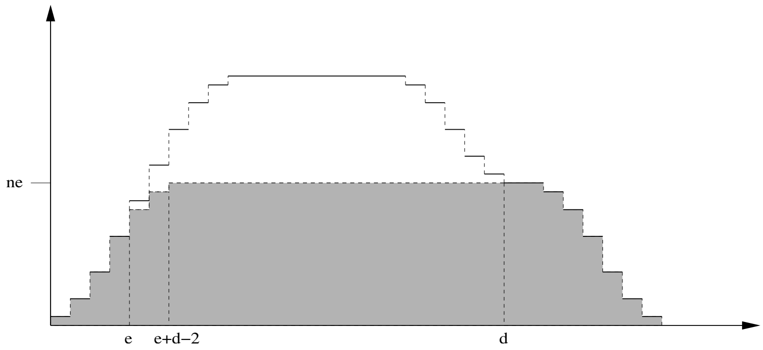


Figure 4.4: The Davis Lemma for $X \subset S \subset \mathbb{P}^3$.

(b) The formula is true for all $u \leq d$ by Lemma 4.2.12.
If $u > d$, then $\mathcal{C}_{X:D}(u-e) = 0$ since $u - e > d - e > t(X : D)$ by the proof of part (a).

(c) Follows by summing (b) over all $u \in \mathbb{Z}$ since

$$\deg(X) = \deg(X \cap D) + \deg(X : D)\,.$$

(d) Using (b) and (c) the claim follows since for all $u \in \mathbb{Z}$

$$\mathcal{C}_{X\cap D}(u) \leq h^0(\mathcal{O}_C(u)) - h^0(\mathcal{O}_C(u-e))\,.$$

□

As in the plane case we call a zero-dimensional scheme $X \subset S$ *decomposable* if there exists $d > 0$ with $\mathcal{C}_X(d-1) > \mathcal{C}_X(d) = \mathcal{C}_X(d+1) > 0$.

Corollary 4.2.16. *Assume that Conjecture 4.2.14 holds. Let $C_d \in |\mathcal{O}_S(d)|$ be an irreducible curve, and $X \subset C_d$ a zero-dimensional scheme such that $h^1(\mathcal{I}_{X/S}(d)) > 0$ and $d > a(X)$. Then there exists a curve $C_k \in |\mathcal{O}_S(k)|$ such that the scheme $Y = X \cap C_k$ is non-decomposable and satisfies*

$$h^1(\mathcal{I}_{Y/S}(d)) = h^1(\mathcal{I}_{X/S}(d))\,.$$

Proof. Without loss of generality we may assume that X is decomposable. Otherwise choose $Y = X$ and $k = a(X) < d$.

Let d_0 be maximal with the property $\mathcal{C}_X(d_0) = \mathcal{C}_X(d_0+1) > 0$. Since C_d is irreducible, we know $d_0 < d$. Let C_k be the fixed curve of $H^0(\mathcal{I}_X(d_0))$. Then $Y = X \cap C_k$ is non-decomposable by Lemma 4.2.15 and satisfies

$$h^1(\mathcal{I}_Y(d)) = \sum_{i=d+1}^{\infty} \mathcal{C}_Y(i) = \sum_{i=d+1}^{\infty} \mathcal{C}_X(i) = h^1(\mathcal{I}_X(d))\,.$$

□

4.2.3 The Castelnuovo function and branched coverings

In this section, we relate the Castelnuovo function of a scheme on a surface in $\mathbb{P}^3$ to the Castelnuovo function of a scheme in $\mathbb{P}^2$.

Assume that $S \subset \mathbb{P}^N$ is a smooth surface, and let $\pi : S \to H$ be a branched covering of degree n, where $\mathbb{P}^2 \cong H \subset \mathbb{P}^N$. Let $X \subset \mathbb{P}^2$ be a zero-dimensional scheme disjoint from the branch locus. Consider the scheme $\pi^* X \subset S$ of degree $n \cdot \deg(X)$. Then

$$\mathcal{I}_{X/\mathbb{P}^2}(d) \cong \pi_*\Big(\mathcal{I}_{\pi^* X/S}(d)\Big) . \tag{4.2.8}$$

Now assume that $S \subset \mathbb{P}^3$ is a smooth surface of degree $n \geq 1$, and let $\pi : S \to H$ be a generic projection to a plane H. Denote by C the plane curve of degree n given by the intersection of S with H, and let X be a zero-dimensional scheme with $C \cap X = \emptyset$. By (4.2.8)

$$\pi_1(H^0(\mathcal{I}_{X/\mathbb{P}^2}(d)) = \pi_2(H^0(\mathcal{I}_{\pi^* X/S}(d))) \subset H^0(\mathcal{O}_C(d)) ,$$

where π_1 is induced by $\mathcal{I}_{X/\mathbb{P}^2}(d) \to \mathcal{O}_C(d)$, and π_2 is induced by $\mathcal{I}_{\pi^* X/S}(d) \to \mathcal{O}_C(d)$. In particular, $a(\pi^* X) = a(X)$, $b(\pi^* X) = b(X)$ and

$$h^1(\mathcal{O}_X(d-n)) - h^1(\mathcal{O}_X(d)) = h^1(\mathcal{O}_{\pi^* X}(d-1)) - h^0(\mathcal{O}_{\pi^* X}(d)) = \mathcal{C}_{\pi^* X}(d) ,$$

which implies

$$\mathcal{C}_{\pi^* X}(d) = \sum_{i=0}^{n-1} \mathcal{C}_X(d-i) .$$

Let us make some observations:

(i) Let $a(X) \leq d \leq t(X)$, then $\mathcal{C}_{\pi^* X}(d) = \mathcal{C}_{\pi^* X}(d-1)$ if and only if $\mathcal{C}_X(d-n) = \mathcal{C}_X(d)$, i.e. if the graph of $\mathcal{C}_X$ has a long stair of length n.

(ii) $\mathcal{C}_{\pi^* X}(d) = 0$ if $\mathcal{C}_X(d-n+1) = 0$. This implies that $h^1(\mathcal{I}_{\pi^* X}(d)) = 0$ if

$$d \geq t(X) + n - 1 . \tag{4.2.9}$$

We shall apply these observations in Section 4.4.

4.3 Bogomolov instability of vector bundles

In this section, we explain how to apply Bogomolov instability of rank 2 vector-bundles (cf. Appendix A.3) to the study of zero-dimensional schemes on a surface in $\mathbb{P}^3$ (cf. also [CS97, Ke03]).

Lemma 4.3.1. *Let $S \subset \mathbb{P}^3$ be a smooth surface of degree $n \geq 1$ with* $\mathrm{Pic}(S) = \mathbb{Z}$. *Let $X \subset S$ be a zero-dimensional scheme with $h^1(\mathcal{I}_X(d)) > 0$. Then there exists a subscheme $X_0 \subseteq X$ and a rank 2 vector bundle E, fitting into an exact sequence*

$$0 \longrightarrow \mathcal{O}_S \longrightarrow E \longrightarrow \mathcal{I}_{X_0}(d-n+4) \longrightarrow 0\,.$$

In particular, X_0 is a complete intersection scheme.

Moreover, if $4 \cdot \deg(X_0) < (d-n+4)^2$, then E is Bogomolov unstable.

Proof. Choose $X_0 \subseteq X$ minimal such that still $h^1(\mathcal{I}_{X_0}(d)) > 0$. Since $h^1(\mathcal{O}_S(d)) = 0$ we know that $X_0 \neq \emptyset$. By Serre-Grothendieck duality (cf. Theorem A.2.3),

$$H^1(\mathcal{I}_{X_0}(d)) \cong \mathrm{Ext}^1\big(\mathcal{I}_{X_0}(d), \mathcal{O}_S(n-4)\big) \neq 0\,.$$

Hence, a general element $\xi \in \mathrm{Ext}^1\big(\mathcal{I}_{X_0}(d), \mathcal{O}_S(n-4)\big)$ defines an extension

$$0 \longrightarrow \mathcal{O}_S \longrightarrow E \longrightarrow \mathcal{I}_{X_0}(d-n+4) \longrightarrow 0\,,$$

where E is a coherent sheaf of rank 2. By [Laz97], Proposition 3.9, the sheaf E fails to be locally free if and only if there exists a proper subscheme $X' \subset X_0$ such that ξ is contained in the image of the natural map

$$\mathrm{Ext}^1\big((\mathcal{I}_{X'}(d), \mathcal{O}_S(n-4)\big) \longrightarrow \mathrm{Ext}^1\big((\mathcal{I}_{X_0}(d), \mathcal{O}_S(n-4)\big)\,.$$

But the group on the left hand side vanishes by the minimality of X_0, because it is isomorphic to $H^1(\mathcal{I}_{X'}(d))$ by Theorem A.2.3. Hence, E is locally free, and X_0 is a local complete intersection scheme.

If $4\deg(X_0) < (d-n+4)^2$, then by Remark A.3.2 (b),

$$c_1(E)^2 - 4c_2(E) = (d-n+4)^2 - 4\deg(X_0) > 0\,,$$

and E is Bogomolov unstable by Theorem A.3.3. □

Proposition 4.3.2. *Let S be a surface of degree n in $\mathbb{P}^3$ with* $\mathrm{Pic}(S) = \mathbb{Z}$, *and let $X \subset S$ be a zero-dimensional scheme. Assume that $h^1(\mathcal{I}_{X/S}(d)) > 0$, and let X_0 be the local complete intersection scheme from Lemma 4.3.1.*

If $4 \cdot \deg(X_0) < (d-n+4)^2$, then there exists a curve $C_k \in |H^0(\mathcal{O}_S(k))|$, $k \geq 1$, satisfying the following properties:

(i) $C_k \cap X_0 = X_0$, *i.e.* $C_k \in |H^0(\mathcal{I}_{X_0/S}(k))|$.

(ii) $\deg(X_0) \geq nk \cdot (d-n+4-k)$,

Proof. By Lemma 4.3.1, there exists an exact sequence

$$0 \longrightarrow \mathcal{O}_S \longrightarrow E \longrightarrow \mathcal{I}_{X_0}(d-n+4) \longrightarrow 0\,, \tag{4.3.10}$$

where E is a Bogomolov unstable rank 2 vector bundle with $c_2(E) = \deg(X_0)$. Thus, there is an exact sequence

$$0 \longrightarrow \mathcal{O}_S(d_0) \longrightarrow E \longrightarrow \mathcal{I}_Z(d-n+4-d_0) \longrightarrow 0\,, \tag{4.3.11}$$

where Z is some zero-dimensional scheme and d_0 is some positive number.

By twisting (4.3.11) with $-d_0$, we see that $E(-d_0)$ has a global section. Thus, twisting (4.3.10) with $-d_0$ and taking global sections, we obtain

$$0 \longrightarrow \underbrace{H^0(\mathcal{O}_S(-d_0))}_{=0} \longrightarrow \underbrace{H^0(E(-d_0))}_{\neq 0} \longrightarrow H^0(\mathcal{I}_{X_0}(d-n+4-d_0))\,.$$

Hence, we can deduce the existence of a curve C_k in $|H^0(\mathcal{I}_{X_0}(k))|$, where $k = d-n+4-d_0 \geq 1$. Furthermore, Remark A.3.2 applied to (4.3.11) implies

$$\begin{aligned} \deg(X_0) = c_2(E) &= n \cdot d_0 \cdot (d-n+4-d_0) \ + \ \deg(Z) \\ &= nk \cdot (d-n+4-k) \ + \ \deg(Z) \\ &\geq nk \cdot (d-n+4-k)\,. \end{aligned}$$

□

Remark 4.3.3. Under the same conditions as in Proposition 4.3.2, assume furthermore that $X_0 \subset C_d$, where $C_d \in |H^0(\mathcal{O}_S(d))|$ is an irreducible curve. Then by Bézout's Theorem

$$\begin{aligned} n \cdot k \cdot d &= \sum_{z \in C_d \cap C_k} i(C_d, C_k; z) \\ &\geq \deg(X_0) + \sum_{z \in \operatorname{supp}(X_0)} \min\{\deg(X_z), i(C_d, C_k; z) - \deg(X_z)\}\,, \end{aligned}$$

where X_z, $z \in \operatorname{supp}(X_0)$, denote the connected components of X_0.

Remark 4.3.4. Let $X \subset \mathbb{P}^2$ be a zero-dimensional scheme with $h^1(\mathcal{I}_X(d)) > 0$. Lemma 4.3.1 allows us to restrict ourselves to any minimal subscheme X_0 of X with $h^1(\mathcal{I}_{X_0}(d)) > 0$. These schemes X_0 are necessarily non-decomposable, that is, $\mathcal{C}_{X_0}$ has no long stairs, because otherwise there would exist a curve D, with $X_0 \cap D \subsetneq X$ and

$$h^1(\mathcal{I}_{X_0 \cap D}(d)) \ = \ h^1(\mathcal{I}_{X_0}(d)) \ > \ 0\,,$$

contradicting the minimality of X_0. Furthermore, notice that Proposition 4.3.2 for $S = \mathbb{P}^2$ coincides with Lemma 4.2.7.

We would like to add that the same observations also hold for schemes on $S \subset \mathbb{P}^3$ provided that Conjecture 4.2.14 is true.

4.4 A vanishing theorem for $X \subset S \subset \mathbb{P}^3$

In this section, we apply the theory of the Castelnuovo function and Bogomolov instability to the H^1-vanishing problem of zero-dimensional schemes contained in the equianalytic scheme associated to an irreducible curve $C \subset S$.

Unless specified, S denotes a smooth, projective surface. We can estimate the degree of schemes contained in $X^{ea}(C, z)$ by using the following lemma.

Lemma 4.4.1. *Let $(C, z) \subset (S, z)$ be a reduced curve singularity, and let $I \subset \mathfrak{m}_z \subset \mathcal{O}_{S,z}$ be an ideal containing $I^{ea}(C, z)$. Then for any $g \in I$*

$$\dim_{\mathbb{C}} \mathcal{O}_{S,z}/I < \dim_{\mathbb{C}} \mathcal{O}_{S,z}/\langle g, C\rangle = (g, C)_z \,.$$

Proof. [Sh97], Lemma 4.1, since the estimate is a purely local computation of intersection numbers. □

The following invariants were introduced and studied in [LoK03] and applied in [Ke03]. They are an extension of the γ-invariant introduced in [GLS00] (more precisely, the γ invariant equals the γ_α invariant for $\alpha = 1$).

Definition 4.4.2. Let $f \in \mathcal{O}_{S,z}$ define a reduced curve singularity (C, z). If $I \subset \mathcal{O}_{S,z} =: R$ is an ideal containing $I^{ea}(f, 0)$, then we define for any real number $\alpha \geq 0$

$$\gamma_\alpha(f; I) := \max\bigl\{(1+\alpha)^2 \cdot \dim_{\mathbb{C}}(R/I), \lambda_\alpha(f; I, g) \mid g \in I\,,\ i(f,g) \leq 2 \cdot \dim_{\mathbb{C}}(R/I)\bigr\}\,,$$

where for $g \in I$

$$\lambda_\alpha(f; I, g) := \frac{\bigl(\alpha \cdot (i(f,g) - (1-\alpha) \cdot \dim_{\mathbb{C}}(R/I)\bigr)^2}{i(f,g) - \dim_{\mathbb{C}}(R/I)}\,.$$

Note that $i(f, g) - \dim_{\mathbb{C}}(R/I) > 0$ by Lemma 4.4.1.

Using this definition we define the following invariants of curve singularities.

Definition 4.4.3. Let (C, z) be a reduced curve singularity defined by $f \in \mathcal{O}_{S,z}$ and let $X \subset X^{ea}(C, z)$ be a zero-dimensional scheme supported in z and defined by an ideal $J \subset \mathcal{O}_{S,z}$. Then we define

$$\gamma_\alpha(X; C) := \max\{\gamma_\alpha(f, I) \mid J \subset I \text{ a complete intersection}\}\,.$$

In particular:

(a) If (C, z) has topological type S then we define

$$\gamma^{es}_\alpha(S) := \gamma^{es}_\alpha(C, z) := \max\{\gamma(f, I) \mid I^{es}(f, z) \subset I \text{ a complete intersection}\}\,.$$

(b) If (C, z) has analytic type S then we define

$$\gamma_\alpha^{ea}(S) := \gamma_\alpha^{ea}(C, z) := \max\{\gamma(f, I) \mid I^{ea}(f, z) \subset I \text{ a complete intersection}\}.$$

Again we write $\gamma_\alpha^s(S)$ for either $\gamma_\alpha^{es}(S)$ or $\gamma_\alpha^{ea}(S)$.

The following table contains the values of $\gamma_\alpha^{es}(S) = \gamma_\alpha^{ea}(S)$ for all simple singularity types S. In [LoK03] this invariant was calculated also for other singularity types, but we should point out that there is no general algorithm for computing $\gamma_\alpha^s(S)$.

Singularity type S	$\gamma_\alpha^s(S)$
$A_k, \quad k \geq 1$	$(k+\alpha)^2$
$D_k, \quad 4 \leq k < 4 + \sqrt{2}(2+\alpha)$	$\frac{(k+2\alpha)^2}{2}$
$D_k, \quad k \geq 4 + \sqrt{2}(2+\alpha)$	$(k-2+\alpha)^2$
$E_k, \quad k = 6, 7, 8$	$\frac{(k+2\alpha)^2}{2}$

In the general case, we only have the rough estimate:

Lemma 4.4.4. *Let $X \subset X^{ea}(C, z)$ be a zero-dimensional scheme, and assume that $\gamma_\alpha(X; C) = \gamma(f, I)$ for a complete intersection ideal I with $\mathcal{I}_{X,z} \subset I \subset \mathcal{O}_{S,z} =: R$. Then*

$$(1+\alpha)^2 \dim_{\mathbb{C}}(R/I) \leq \gamma_\alpha(C; X) \leq (\dim_{\mathbb{C}}(R/I) + \alpha)^2.$$

Proof. See [LoK03]. □

The strategy for obtaining an H^1-vanishing theorem is the same for the approaches based on the Castelnuovo function and on Bogomolov instability. Let us explain the main steps:

Assume that S is either $\mathbb{P}^2$ or a general surface in $\mathbb{P}^3$ of degree $n \geq 4$.

- Start with a zero-dimensional scheme $X \subset X^{ea}(C_d)$, $C_d \in |\mathcal{O}_S(d)|$ an irreducible curve, and assume that $h^1(\mathcal{I}_X(d)) > 0$.
- Consider a minimal subscheme X_0 of X with $h^1(\mathcal{I}_{X_0}(d)) > 0$. We denote the connected components of X_0 by X_z, $z \in \operatorname{supp}(X_0) = \{z_1, \ldots, z_r\}$.
- Both approaches[1] provide the existence of a curve $C_k \in |H^0(\mathcal{I}_{X_0}(k))|$ with

$$\begin{aligned} \deg(X_0) &\geq nk \cdot (d - n + 4 - k), \\ nkd &\geq \deg(X_0) + \sum_{z \in \operatorname{supp}(X_0)} \min\{\deg(X_z), i(C_d, C_k; z) - \deg(X_z)\}. \end{aligned}$$

[1] We assume again that Conjecture 4.2.14 holds.

- Deduce from these inequalities that

$$\gamma_\alpha(X;C) := \sum_{z\in\operatorname{supp}(X)} \gamma_\alpha(X_z;C) \;\geq\; n\cdot\alpha\cdot(d-n+4)^2\,,$$

  for an appropriate value of $\alpha > 0$.

In other words, by reversing the steps above, one can prove

Theorem 4.4.5. *Let $S \subset \mathbb{P}^3$ be a general surface of degree n, $n = 1$ or $n \geq 4$, and let $X \subset X^{ea}(C)$ be a zero-dimensional scheme, where $C \in |\mathcal{O}_S(d)|$ is an irreducible curve, $d \geq \max\{n-3, 3\}$.*

If the inequality

$$\gamma_\alpha(X;C) < n\cdot\alpha\cdot(d-n+4)^2$$

holds true, where $\alpha = \frac{1}{\max\{1,n-3\}}$, then $h^1(\mathcal{I}_X(d)) = 0$.

Proof. For $S = \mathbb{P}^2$, the theorem was shown in [GLS00] (in slightly weaker form) using the Castelnuovo function, and the general case was proven in [Ke03] using Bogomolov instability. □

Using generic projections to a plane, we obtain the following result which is, for general $n \geq 4$ and arbitrary schemes X, asymptotically slightly weaker than Theorem 4.4.5, but stronger in some special cases.

Corollary 4.4.6. *Let $S \subset \mathbb{P}^3$ be a smooth surface of degree $n > 1$. Let $C \subset S$ be an irreducible curve in $|H^0(\mathcal{O}_S(d))|$, $d \geq 3$, with r singular points $z_1, \ldots, z_r$. If a zero-dimensional scheme of the form*

$$X = X_1 \cup \cdots \cup X_r\,, \quad X_i \subset X^{ea}(C, z_i)$$

satisfies

$$\sum_{i=1}^{r} \gamma_1(X_i, C) < (d+4-n)^2\,,$$

then $h^1(\mathcal{I}_X(d)) = 0$.

Proof. Let $\pi : S \to H$ be a generic projection to a plane H such that

$$X \cap D = \emptyset\,, \quad \text{where } D = S \cap H\,.$$

Using (4.2.9) the result follows by Theorem 4.4.5 since $X \subset \pi^*(\pi(X))$. □

As an immediate corollary we obtain the following smoothness result.

Corollary 4.4.7. *Let $C \subset S$ be an irreducible curve in $|\mathcal{O}_S(d)|$, $d \geq 3$, having r singular points $z_1, \ldots, z_r$ of topological (respectively analytic) types $S_1, \ldots, S_r$ as its only singularities. Then $V_d^{irr}(S_1 + \ldots + S_r)$ is T-smooth at C if*

$$\sum_{i=1}^{r} \gamma_1^s(C, z_i) < (d - n + 4)^2 .$$

4.5 Some remarks about the Castelnuovo function of schemes in $\mathbb{P}^3$

The techniques for dealing with zero-dimensional schemes in $\mathbb{P}^n$, $n \geq 3$, are quite rare. In fact, all general H^1-vanishing results have been obtained via the Horace method. Since $\mathrm{Pic}(\mathbb{P}^n) = \mathbb{Z}$ and $h^1(\mathcal{O}_{\mathbb{P}^n}(d)) = 0$ for all d, the Castelnuovo function might provide also a useful tool for dealing with zero-dimensional schemes in higher dimensions.

In this final section, we make some observations about the Castelnuovo function of zero-dimensional schemes in $\mathbb{P}^3$.

Example 4.5.1. (i) Again, we consider first the case where $X \subset \mathbb{P}^3$ is a complete intersection of three surfaces of degrees $a, b, c \geq 1$,

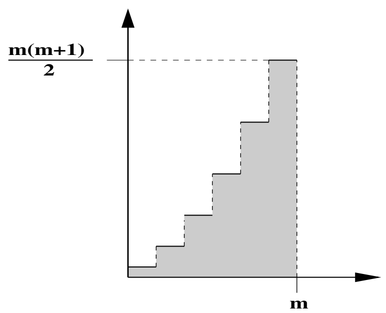
$$X = S_a \cap S_b \cap S_c\,, \quad a \leq b \leq a + b \leq c\,,$$

The Castelnuovo function of X can be calculated by the formula

$$\begin{aligned} \mathcal{C}_X(d) \;&=\; \binom{d+2}{2} - \binom{d-a+2}{2} - \binom{d-b+2}{2} + \binom{d-a-b+2}{2} \\ &-\; \binom{d-c+2}{2} + \binom{d-a-c+2}{2} + \binom{d-b-c+2}{2} \\ &-\; \binom{d-a-b-c+2}{2}, \end{aligned}$$

and its graph is shown in Figure 4.5. Obviously, $a(X) = a$ and $b(X) = b$.

(ii) Let $X = X(m, z)$ be a fat point of multiplicity m and supported in $z \in \mathbb{P}^3$. Then $a(X) = b(X) = m$, and the Castelnuovo function looks as follows:



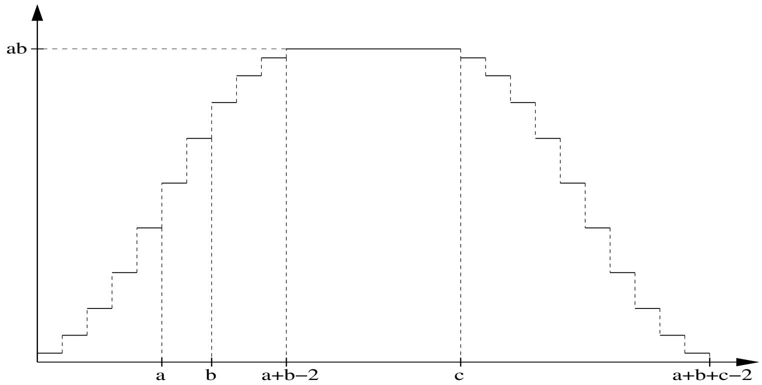


Figure 4.5: Example: The Castelnuovo function of a complete intersection scheme $X = C_a \cap C_b \cap C_c \subset \mathbb{P}^3$, $a \leq b \leq a+b \leq c$.

Remark 4.5.2. Recall that for zero-dimensional schemes on surfaces in $\mathbb{P}^3$, it is possible to estimate $t(X)$ by the numbers $a(X)$ and $b(X)$.

However, the previous example shows that this is not true for zero-dimensional schemes in $\mathbb{P}^3$. The reason is that the linear systems $H^0(\mathcal{I}_X(d))$, $d \geq b(X)$, may now still have fixed parts of positive dimension.

Definition 4.5.3. Let $X \subset \mathbb{P}^3$ be a zero-dimensional scheme. We introduce the number

$$c(X) := \min\{d \,|\, |H^0(\mathcal{I}_X(d))| \neq \emptyset \text{ has no fixed part of positive dimension}\}\,.$$

Obviously, $b(X) \leq c(X) \leq t(X) + 1$.

In particular, we can estimate $t(X)$ by

$$t(X) \leq a(X) + b(X) + c(X) - 2\,.$$

Let $X \subset \mathbb{P}^3$ be a zero-dimensional scheme, and assume that $b(X) < c(X)$. Let $X_0 \supset X$ be the scheme defined by the elements of $H^0(\mathcal{I}_X(c(X)-1)) \neq 0$, and put $Y = X_0 \cap H$, where H is a generic hyperplane. Hence, $Y \neq \emptyset$ is a zero-dimensional scheme in $H \cong \mathbb{P}^2$, and the reduction sequence implies

$$\mathcal{C}'_X(d) := \mathcal{C}_X(d) - \mathcal{C}_X(d-1) = \mathcal{C}_Y(d)\,.$$

Hence, we obtain

Corollary 4.5.4. *Let $X \subset \mathbb{P}^3$ be a zero-dimensional scheme. Then*

(i) If $d < a(X)$, then $\mathcal{C}_X(d) = h^0(\mathcal{O}_{\mathbb{P}^2}(d)) = \binom{d+2}{2}$.

(ii) $\mathcal{C}_X(d) \leq a(X) \cdot b(X)$.

(iii) *If* $d \geq a(X)$, *then* $\mathcal{C}'_X(d) \leq \mathcal{C}'_X(d-1)$.

(iv) *If* $b(X) \leq d < c(X)$, *then* $\mathcal{C}'_X(d) < \mathcal{C}'_X(d-1)$, *or* $\mathcal{C}'_X(d-1) = 0$.

We conjecture that the Castelnuovo is strictly descending as soon as the linear systems $H^0(\mathcal{I}_X(d))$ have only isolated base points.

Conjecture 4.5.5. *Let* $X \subset \mathbb{P}^3$ *be a zero-dimensional scheme. If* $d \geq c(X)$, *then*

$$\mathcal{C}_X(d) < \mathcal{C}_X(d-1)\,.$$

Remark 4.5.6. If we consider the Castelnuovo function of zero-dimensional schemes in $\mathbb{P}^n$, $n > 3$, then even higher derivatives of the Castelnuovo function may have to be taken into account. They were already studied to some extent in a different context in [DGM84].

Appendix A

Some general facts

In this appendix we review some classical results from algebraic geometry, and reformulate them, if necessary, according to the context in which we shall apply them. This includes Bertini's theorem, Cremona transformations, sheaf cohomology and Bogomolov instability.

Furthermore, we review some general facts about surfaces in $\mathbb{P}^3$.

A.1 Bertini's theorem and Cremona transformations

In our constructions we apply several times Bertini's theorem and Cremona transformations. Cremona transformations are used to construct curves of small degree with given Newton polytope and singularities. Bertini's theorem is used in order to show that a generic members of certain linear system are reduced and smooth outside the prescribed singularities. In this section we briefly review both theorems.

Theorem A.1.1. *Let V be a smooth projective variety, and let $\mathcal{L} \subset |D|$ be a linear system, where D is an effective divisor.*

(i) A generic member of $\mathcal{L}$ is reduced and non-singular outside the basepoints of $\mathcal{L}$.

(ii) Let $\varphi : V \dashrightarrow \mathbb{P}^m$ be the rational map defined by $\mathcal{L}$. If $\dim(\varphi(V)) \geq 2$, then a generic member is also irreducible.

Proof. E.g. [Ha77, III,10] or [vdW73]. $\square$

Cremona transformations are a useful tool for constructing curves with a few prescribed singularities and in addition certain multiple points at the fundamental points. This is of particular interest if we want to construct curves such that the defining equation has a specified Newton polytope.

Proposition A.1.2. *Let C be a reduced curve of degree d in $\mathbb{P}^2$ and assume that $m_1, m_2, m_3 \geq 0$ are the multiplicities of C at the fundamental points $P_1 = (1:0:0)$, $P_2 = (0:1:0)$, $P_3 = (0:0:1)$. Let C^* be the strict transform of C under the Cremona transformation*

$$\begin{array}{ccc} \mathbb{P}^2 & \longrightarrow & \mathbb{P}^2 \\ (x_0 : x_1 : x_2) & \mapsto & (x_1x_2 : x_0x_2 : x_0x_1)\,. \end{array}$$

Then

(i) $(C^*)^* = C$,

(ii) $\deg(C^*) = 2d - m_1 - m_2 - m_3$,

(iii) the tangents of C at P_i correspond precisely (including multiplicities) to intersection points of C^* with the line $x_i = 0$ outside the fundamental points,

(iv) the intersection points of C with the line $x_i = 0$ outside the fundamental points correspond precisely (including multiplicities) to tangents of C^* at P_i.

Proof. These statements follow immediately since the Cremona transformation corresponds to the blow up of $\mathbb{P}^2$ in the three fundamental points. For details we refer to [Wal50] □

A.2 Cohomology of coherent sheaves

We recall a few general theorems about cohomology of coherent sheaves.

Theorem A.2.1 (Cohomology and ample sheaves). *Let V be a smooth projective variety, and let $\mathscr{L}$ be an ample sheaf on V. If $\mathcal{F}$ is a coherent sheaf on V, then for $d \gg 0$*

$$H^i(V, \mathcal{F} \otimes \mathscr{L}^{\otimes d}) = 0 \text{ for all } i > 0\,.$$

Proof. [Ha77]. □

Hence in particular, if $X \subset V$ is a zero-dimensional scheme, then for d sufficiently large, the group $H^1(\mathcal{I}_X(d))$ vanishes.

Furthermore we would like to remind the fact that the dimensions of the cohomology groups behaves semi-continuously under deformations.

Theorem A.2.2 (Semicontinuity of cohomology). *Let $f : X \to Y$ be a proper morphism of complex spaces, and let $\mathcal{F}$ be a coherent sheaf on X, flat over Y. Then, for each $i \geq 0$, the function $h^i_{\mathcal{F}} : Y \to \mathbb{Z}_{\geq 0}$,*

$$h^i_{\mathcal{F}}(z) := \dim_{\mathbb{C}} H^i\big(f^{-1}(z), f_*\mathcal{F}(z)\big)$$

is an upper semicontinuous function on Y.

Proof. [Ha77, III.12.8]. □

Theorem A.2.3 (Serre-Grothendieck Duality). *Let $\mathcal{F}$ be a coherent sheaf on a smooth projective surface S. Then there is a natural isomorphism*

$$\mathrm{Ext}^1\big(\mathcal{F}, \mathcal{O}_S(K_S)\big) \xrightarrow{\cong} H^1\big(\Sigma, \mathcal{F}\big)^*$$

Proof. [Ha77]. □

A.3 Bogomolov instability

In this section, we briefly recall elements from Bogomolov's theory of unstable vector bundles. For details we refer to [Bog79] and [Laz97].

Throughout this section we assume that S is a smooth projective surface.

Definition A.3.1. A rank two vector bundle E on S is called *Bogomolov unstable* if there exist divisors A, B and a (possibly empty) zero-dimensional scheme Z in S, fitting into an exact sequence

$$0 \longrightarrow \mathcal{O}_S(A) \longrightarrow E \longrightarrow \mathcal{I}_{Z/S}(B) \longrightarrow 0\,, \tag{A.3.1}$$

and satisfying

(i) $(A - B)^2 > 0$ and

(ii) $(A - B).H > 0$ for all ample divisors H.

Remark A.3.2. (a) If $\mathscr{L}$ is an invertible sheaf, then a rank 2 vector bundle E is Bogomolov unstable if and only if $E \otimes \mathscr{L}$ is Bogomolov unstable.

(b) If E is a rank 2 vector bundle sitting in an exact sequence (A.3.1), then the Chern numbers of E are

$$c_1(E) = c_1(A) + c_1(B)\,, \quad c_2(E) = c_1(A) \cdot c_1(B) + \deg(Z)\,.$$

Bogomolov's famous theorem expresses this property in terms of Chern numbers.

Theorem A.3.3. *Let E be a rank two vector bundle on a smooth projective surface S. If the Chern numbers of E satisfy*

$$c_1(E)^2 - 4c_2(E) \; > \; 0\,,$$

then E is Bogomolov unstable.

Proof. [Bog79, Laz97]. □

A.4 Some facts about surfaces in $\mathbb{P}^3$

Let us review some invariants of surfaces in $\mathbb{P}^3$.

Proposition A.4.1. *Let $S \subset \mathbb{P}^3$ be a smooth surface of degree $n \geq 4$. Then*

(i) $K_S = \mathcal{O}_S(d - n + 4)$,

(ii) $h^0(\mathcal{O}_S(d)) = \binom{d+3}{3} - \binom{d-n+3}{3}$ *and* $h^1(\mathcal{O}_S(d)) = 0$ *for all* d.

(iii) Let $C \in |H^0(\mathcal{O}_S(1))|$ *be a general hyperplane section. Then*

$$\begin{aligned} h^0(\mathcal{O}_C(d)) = h^0(\mathcal{O}_S(d)) - h^0(\mathcal{O}_S(d-1)) \;&=\; h^0(\mathcal{O}_{\mathbb{P}^2}(d)) - h^0(\mathcal{O}_{\mathbb{P}^2}(d-n)) \\ &=\; \begin{cases} \binom{d+2}{2} & \textit{if } d < n-2 \\ n \cdot d - \frac{n(n-3)}{2} & \textit{if } d \geq n-2. \end{cases} \end{aligned}$$

In general the Picard number of a surface $S \subset \mathbb{P}^3$ may be arbitrarily large. However, there is always a distinguished element of the Picard group, namely the class of a hyperplane section. The following proposition shows that if $\deg(S) \geq 4$ and S is sufficiently general, then this class actually generates $\mathrm{Pic}(S)$, hence in particular $\mathrm{Pic}(S) \cong \mathbb{Z}$ for sufficiently general S.

Proposition A.4.2. *Let $n \geq 4$. Then the set U consisting of all smooth surfaces $S \subset \mathbb{P}^3$ of degree n with $\mathrm{Pic}(S)$ generated by the restriction of $\mathcal{O}_{\mathbb{P}^3}(1)$ is a very general subset of $|\mathcal{O}_{\mathbb{P}^3}(n)|$, i.e. the complement of U is an at most countable union of lower-dimensional subvarieties.*

Proof. [Ha75], Corollary 3.5. □

Appendix B

Some algorithms

In this appendix we show how to check certain properties and calculate invariants appearing in this thesis using the computer algebra system SINGULAR.

We show how to check T-smoothness of the equisingular stratum locally at a hypersurface $W \subset \mathbb{P}^n$, more precisely we compute $H^1(\mathcal{I}_{X/\mathbb{P}^n}(d))$ for any zero-dimensional scheme $X \subset \mathbb{P}^n$. We also discuss how to check the transversality property appearing in the patchworking method.

Finally we give an algorithm for computing the Castelnuovo function for zero-dimensional schemes in $\mathbb{P}^n$.

In this appendix we discuss some of the methods used throughout this thesis from the computational point of view. We shall present some algorithms and show how to implement them in the computer algebra system SINGULAR. We shall not explain the syntax extensively but refer to [GPS01, GP02] instead.

We start by giving a brief overview of some objects which can be computed.

- Let $X \subset \mathbb{P}^n$ be a zero-dimensional scheme defined over the rational numbers. Then we can compute the cohomology groups of $\mathcal{I}_X(d)$ (cf. Section B.1). In particular this allows to check whether a hypersurface in $\mathbb{P}^n$ defined over $\mathbb{Q}$ corresponds to a T-smooth germ and we can compute the Castelnuovo function of a zero-dimensional scheme (cf. Section B.3).

- The equisingular Tjurina ideal $I^{es}(f)$ can be computed for $f \in K[[x,y]]$ reduced, K algebraically closed of characteristic 0, hence in particular the equisingular Tjurina number (cf. [Los03]).

- We can check whether a given triad (Δ, Δ_+, F), $F \in \mathbb{Q}[\mathbf{x}]$, is transversal by computing the intersection of $H^0(\mathcal{I}_{X^s(F)/\mathbb{P}^n}(d))$ with the affine vector-space $\mathcal{P}(\Delta, \Delta_+, F)$ (cf. Section B.3).

- The equisingular stratum of any deformation of a reduced plane curve singularity over a complete, local K-algebra (cf. [Los03]). The algorithms are implemented in the SINGULAR library `equising.lib`.
- The ideal $I^a(W,z)$ introduced in Section 3.4 ([GLS00]).

Unfortunately, there are no algorithms for calculating the following objects.

- There is no general algorithm for computing the invariants $\gamma_\alpha(S)$ introduced in Chapter 4. For details how to calculate this invariant in special cases by hand we refer to [LoK03].
- It is not possible to compute explicit equations of the hypersurfaces constructed via the patchworking method.

B.1 Computing $h^0(\mathcal{I}_{X/\mathbb{P}^n}(d))$ and $h^1(\mathcal{I}_{X/\mathbb{P}^n}(d))$

In this section we show how to compute the dimensions of the cohomology groups of $\mathcal{I}_{X/\mathbb{P}^n}(d)$ with SINGULAR.

Let $X \subset \mathbb{Q}^n \subset \mathbb{P}^n$ be a zero-dimensional scheme defined by an ideal $I \subset \mathbb{Q}[x_1,\dots,x_n]$. Denote by $I^h \subset \mathbb{Q}[x_0,\dots,x_n]$ the homogenization of I with respect to x_0 and let

$$I^h_d := I^h \cap \mathbb{Q}[\mathbf{x}]_d\,.$$

The SINGULAR function `kbase` computes

$$\mathtt{kbase(I^h,d)} = \{m \in M \mid \deg(m) = d\}\,,$$

where M is a monomial basis of $\mathbb{Q}[\mathbf{x}]/I^h$.

Lemma B.1.1. *The set* $\mathtt{kbase(I^h,d)}$ *is a vector-space basis of* $\mathbb{Q}[\mathbf{x}]_d/I^h_d$.

Proof. The monomials are clearly linear independent since if $\sum_i a_i m_i \in I^h_d \subset I^h$, then $a_i = 0$ for all i.
Hence we have to show that they generate $K[\mathbf{x}]_d/I^h_d$. For that let $f \in K[\mathbf{x}]_d \subset K[\mathbf{x}]$. Then f can be written in the form

$$\begin{aligned} f &= \sum_{i\in I} a_i m_i + h\,, \quad h \in I^h \\ &= \sum_{\deg(m_i)=d} a_i m_i + h_d + \sum_{j\neq d}\Big(\sum_{\deg(m_i)=j} a_i m_i + h_j\Big)\,, \end{aligned}$$

where $h = h_m + h_{m+1} + \dots$ and h_i is homogeneous of degree i. Since $\sum_{\deg(m_i)=j} a_i m_i + h_j$ is homogeneous of degree $j \neq d$, it follows that it must be zero. Since I^h is a homogeneous ideal, $h_j \in I^h$ and hence $\sum_{\deg(m_i)=j} a_i m_i \in I^h$. But the m_i are linear independent, hence $a_i = 0$ if $\deg(m_i) \neq d$. □

Using the previous lemma we obtain

$$h^0(\mathcal{I}_{X/\mathbb{P}^n}(d)) = \binom{d+n}{n} \; - \; \texttt{size}(\texttt{kbase}(\texttt{I}^\texttt{h}, \texttt{d}))$$

and hence

$$h^1(\mathcal{I}_{X/\mathbb{P}^n}(d)) = \dim(\mathbb{Q}[x_1, \ldots, x_n]/I) - \texttt{size}(\texttt{kbase}(\texttt{I}^\texttt{h}, \texttt{d}))\,.$$

The Singular function `size()` computes the number of elements in a list of polynomials.

The following Singular procedure implements this computation. Note that generators of I^h can be obtained by homogenizing a standard basis of I with respect to a degree ordering (e.g. [CLS96, 8.4]). The function `binom(,)` computes the binomial coefficient of two integers, and its definition has been omitted.

```
proc cohom(ideal I, int d)
"USAGE:  cohom(I,d,k); I ideal, d integer
 ASSUME: I is an zero-dimensional ideal,
         the monomial ordering is a global degree ordering
 RETURN: intvec containing h^0(I(d)),h^1(I(d))
"
{
  def oldring=basering;
  int n=nvars(oldring);
  ideal j=std(i); // ordering has to be degree ordering
  int dim1=vdim(j);
  if (dim1==-1){
    "Error: ideal not zero-dimensional!} return (0);
  }

  // compute homogenization
  ring dummy=32003,(x(1..n+1)),dp;
  ideal i=fetch(r,i);
  ideal ih=std(homog(i,x(n+1)));

  // compute h^i(I(d))
  int dim2=size(kbase(ih,d));
  invec v=binom(d+n,n)-dim2,dim1-dim2;
  return (v);
}
```

B.2 Computing a vector-space basis of $I \cap K[\mathbf{x}]_{\leq d}$

In this section we how to compute a vector-space basis of $I \cap K[\mathbf{x}]_{\leq d}$, where $I \subset K[\mathbf{x}]$ is an ideal. If $f \in K[\mathbf{x}]$ and I is a Groebner basis of I with respect to some monomial ordering, then the Singular function `reduce(f, I)` returns the normal form of f with respect to this Groebner basis.

Lemma B.2.1. *Let K be any field, $I \subset K[x_1, \ldots, x_n]$ an ideal, and let $d \geq 0$. Then we can compute a vector space basis B of $I_d = I \cap K[\mathbf{x}]_{\leq d}$ as follows:*

(Step 0) *$B := \emptyset$, M = list of all monomials of degree $\leq d$.*

(Step 1) *G := leading terms of a Groebner basis of I with respect to a degree ordering.*

(Step 2) *For all $\mathbf{x}^\omega \in M$: If reduce$(\mathbf{x}^\omega, G) = 0$, then*

$$B := B \cup \{\mathbf{x}^\omega - \mathit{reduce}(x^\omega, I)\}\,.$$

Proof. If $\mathit{reduce}(\mathbf{x}^\omega, G) = 0$, then there is an element $g_\omega \in I$ with leading monomial $\mathbf{x}^\omega$ and B is the set of all these g_ω. This set is clearly linearly independent. Furthermore, since the chosen monomial ordering is a degree ordering we know that $\deg(g_\omega) \leq d$ for all ω and it is easy to see that B also spans I_d. $\square$

The Singular code looks essentially as follows:

```
proc zerohom(ideal I, int d)
"USAGE:  zerohom(I,d); I ideal, d integer
 ASSUME:  I is a standard basis of an affine ideal,
    the monomial ordering is a global degree ordering
 RETURN: a list consisting of h^0(I(d))
    and a vector space basis of H^0(I(d)) in terms of polynomials
    of degree less than d.
"
{
  ideal J=lead(I);
  int i;

  ideal M=1;
  for (i=1; i<=d; i++) { M=M,maxideal(i); }
  ideal R=reduce(M,J);

  ideal res;
```

```
  for (i=1; i<=size(M); i++){
    if (R[i]==0){ res=res+(M[i]-reduce(M[i],I)); }
  }
  list l=size(res),res;
  return(l);
}
```

B.3 Applications

Let us show how to apply the procedures from the previous sections to some problems.

Checking T-smoothness

Assume that $W \subset \mathbb{P}^n$ is a hypersurface defined by an (affine) polynomial $f \in \mathbb{Q}[\mathbf{x}]$ and having only isolated singularities. The following procedure checks if W is obstructed in degree $d \geq \deg(f)$, i.e. if $h^1(\mathcal{I}_{X^{ea}(W)}(d)) = 0$.

```
proc is_Tsmooth(poly f, int d)
"USAGE:  is_Tsmooth(f,d); f polynomial , d integer
 ASSUME: - ground field Q and singularities of f are contained in Q^n
 RETURN: -1: if f has non-isolated singularities
          0: if h^1(I_X(d))=0, X the zero-dimensional scheme
             associated to the singular points of f
          1: otherwise
"
{
  ideal i=std(f+jacob(f));
  int tau=vdim(i);
  if (tau==-1){
    dbprint (printlevel-voice+1,"Error: Non-isolated singularity!");
    return(-1);
  }
  else
  {
    intvec hi=cohom(i,d);
    if (hi[2]==0) {return(0);} else { return(1);}
  }
}
```

Computing the Castelnuovo function

Assume that $X \subset \mathbb{Q}^n \subset \mathbb{P}^n$ is a zero-dimensional scheme. Using Lemma B.1.1 we can compute the Castelnuovo function of X by calculating

$$\mathcal{C}_X(d) = \mathtt{size}(\mathtt{kbase}(\mathtt{I}^{\mathtt{h}}, \mathtt{d})) - \mathtt{size}(\mathtt{kbase}(\mathtt{I}^{\mathtt{h}}, \mathtt{d} - 1))\,.$$

The Singular code looks as follows:

```
proc CNFunc(ideal i, int max)
"USAGE:  CNFunc(i,max); i ideal, max integer
 ASSUME:  i is a projective zero-dimensional ideal
          (in particular homogeneous)
 RETURN:  the values of the Castelnuovo function of i
"
{
  ideal j=std(i);
  intvec cas;
  for (int k=0; k<=max; k++){
    cas[k]=size(kbase(j,k))-size(kbase(j,k-1));
  }
  return(cas);
}

proc affCNFunc(ideal i, int max)
"USAGE:  affCNFunc(i,max); i ideal, max integer
ASSUME:  i is an affine zero-dimensional ideal,
         monomial ordering is global degree ordering
RETURN:  the values of the Castelnuovo function of i
{
  def oldring=basering;
  int n=nvars(oldring);
  ideal j=std(i); // ordering has to be degree ordering
  ring dummy=32003,(x(1..n+1)),dp;
  ideal i=fetch(r,i);
  ideal ih=homog(i,x(n+1));
  intvec cas=CNFunc(ih,max);
  setring oldring;
}
```

Checking transversality of triads (Δ, Δ_+, F)

Let $W \subset \mathbb{P}^n$ be a hypersurface defined by the homogenization of a polynomial $f(x_1, \ldots, x_n) \in \mathbb{Q}[\mathbf{x}]_{\leq d}$. Let Δ be the Newton polytope of f, and let $\Delta_+ \subset \partial\Delta$. Let us assume that

(i) f is a PNS polynomial.

(ii) All singularities of W are defined over $\mathbb{Q}$.

(iii) W has only isolated singularities in $\mathbb{Q}^n$, all of them lying in $(\mathbb{Q}^*)^n$. Denote the types of the singularities by $S_1, \ldots, S_r$, $r = \#(\mathrm{Sing}(W) \cap (\mathbb{Q}^*)^n)$.

(iv) The equianalytic stratum corresponding to the singular points in $(\mathbb{Q}^*)^n$ is smooth of the expected dimension.

Let us show how to check whether the triad (Δ, Δ_+, f) is transversal. The tangent space to the germ $V_d(S_1 + \cdots + S_r; W)$ is given by

$$I_d := \langle f, f_{x_1}, \ldots, f_{x_n} \rangle \cap \mathbb{Q}[\mathbf{x}]_{\leq d} \subset \mathbb{Q}[\mathbf{x}]_{\leq d}\,,$$

which can be computed exactly (i.e. no approximations appear). For checking transversality of (Δ, Δ_+, f) we have to check whether I_d intersects transversally with

$$\Big\{ g \in \mathbb{Q}[\mathbf{x}] \;\mid\; g(\mathbf{x}) = \sum_{\omega \in (\Delta \setminus \Delta_+) \cap \mathbb{Z}^n} a_\omega \mathbf{x}^\omega \Big\}\,.$$

We omit the definition of the following two procedures:

- If $f \in k[\mathbf{x}]_{\leq d}$, then `poly2vec(f,d)` computes the coordinate representation of f with respect to some (monomial) basis of $k[\mathbf{x}]_{\leq d}$.
- Let U, V be matrices with entries in the ground field k having the same number of rows N. Then the procedure `isTransInt(U,V)` returns 1 if the column spaces of U and V intersect transversally in k^N, and 0 otherwise.

The Singular code for checking the transversality of a triad looks as follows:

```
proc isTriadTrans(poly f, ideal D, list #)
"USAGE:   isTriadTrans(f,D[,d]); f polynomial, D ideal
          (list of monomials), d integer
PURPOSE: checks whether the triad (Del(f),D_+,f) is transversal
          where Del(f) is the Newton polytope of f and D=Del(f)\D_+
RETURN:   -1: if f has non-isolated singularities
           0: if (Del(f),D_+,f) is not transversal
           1: otherwise
ASSUME:   - ground field is Q
          - f has only isolated singular points inside the (rational)
```

```
          torus and no singular points on the coordinate hyperplanes
"
{
  int d=deg(f);
  int n=nvars(basering);
  if ((size(#)>1) and (#[1]>d)) { d=#[1]; }
  int i;

  // compute basis of H^0
  ideal I=std(f+jacob(f));
  if (vdim(I)==-1){
    dbprint (printlevel-voice+1,"Error: Non-isolated singularity!");
    return(-1);
  }
  ideal J=lead(I);
  res=zerohom(J,d,1);

  // compute basis representations of the two vector-spaces
  int dimension=binom(d+n,n);
  matrix U[dimension][size(res)];
  for (i=1; i<=size(res); i++){
    U[1..dimension,i]=poly2vec(res[i],d);
  }
  matrix V[dimension][size(D)];
  for (i=1; i<=size(D); i++){
    V[1..dimension,i]=poly2vec(D[i],d);
  }

  // check for transversal intersection
  if (isTransInt(U,V)) { return (1); } else { return(0);}
}
```

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