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Multiscale Solution for the Molodensky Problem on Regular Telluroidal Surfaces

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Abstract

Based on the well-known results of classical potential theory, viz. the limit and jump relations for layer integrals, a numerically viable and efficient multiscale method of approximating the disturbing potential from gravity anomalies is established on regular surfaces, i.e., on telluroids of ellipsoidal or even more structured geometric shape. The essential idea is to use scale dependent regularizations of the layer potentials occurring in the integral formulation of the linearized Molodensky problem to introduce scaling functions and wavelets on the telluroid.
As an application of our multiscale approach some numerical examples are presented on an ellipsoidal telluroid.

Key words. Molodensky problem, multiscale approximation on regular telluroidal surfaces, harmonic scaling functions and wavelets

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1 Introduction

Wavelets are known as mathematical means for breaking up a complicated function (signal) into many simple pieces at different scales and positions. Thus wavelets have become a powerful and flexible tool for scientific computation and data handling. Basically, wavelet analysis is done by convolving the function under consideration against 'dilated' and 'shifted' versions of one fixed function, viz. the 'mother wavelet'. Traditionally, applications of wavelets have been signal analysis, image processing, noise cancellation, etc, but there is also a growing interest in the numerical treatment of partial differential equations. However, wavelet methods are merely known for unfolding their computational economy and efficiency when applied to problems on Euclidean spaces, the sphere or the torus. The aim of this article is to present a new numerically viable and efficient wavelet approach to the linearized Molodensky problem of physical geodesy corresponding to a telluroid of geometrically complex structure, such as an ellipsoid or an approximation of the actual geoid (e.g. the EGM96-geoid). Our purpose is to develop a multiscale theory on telluroids which are understood as regular surfaces which at the same time establishes harmonic scaling functions and wavelets in the outer space. Our concept is essentially based on a connection of classical results of potential theory with new methods of modern analysis and scientific computing.

In more detail, this paper follows the standard procedure in potential theory by transforming the linearized Molodensky boundary–value problem corresponding to a telluroid understood to be a low-scale approximation of the geoid into a Fredholm integral equation of the second kind. We choose a single layer potential to be a solution of the Molodensky problem. However, instead of applying conventional wavelet constructions oriented on Euclidean or spherical theory for discretizing the integral equations in accordance with a collocational, Galerkin or least squares procedure we use the kernels of the layer potentials themselves to establish a new class of wavelets on (general) regular surfaces. In other words, a new wavelet theory is developed on regular surfaces that arises naturally as a result of scale discretization of the limit and jump relations of potential theory. The wavelet theory based on layer potentials provides well–promising efficient and fast approximation methods for the boundary–value problems of physical geodesy corresponding to geoscientifically relevant telluroids. In particular, the Molodensky problem of determining the geoid from gravity anomalies on a not-necessarily spherical or ellipsoidal telluroid becomes a numerically attackable procedure. As a matter of fact it may be expected that, in combination with an improved data situation in the near future, our wavelet approach opens new perspectives in modelling geopotential level surfaces with an accuracy unattainable until now.

The outline of this paper is as follows: First we introduce the notations and preliminaries that are needed for our wavelet approach to the solution of the Molodensky problem. We specify regular surfaces on which our theory is established. Then we introduce in standard way the boundary–value problem, i.e., the Molodensky problem, which we are concerned with. In the next section we introduce potential operators with respect to the normal vector field of the regular surface (telluroid) which are the main ingredients of this work. We develop the limit and jump relations of the involved potential operators formulated in the framework of the Hilbert space of square–integrable functions on the regular surface (telluroid). The setup of a multiresolution analysis (i.e., scaling functions, scale spaces, wavelets, detail spaces) is defined by interpreting the kernel functions of the limit and jump integral operators as scaling functions on regular surfaces. In this context, the normal distance to the parallel surfaces of the regular surface under consideration represents the scale level in the scaling function. At the end we deal with the already mentioned discretization of the occurring Fredholm integral equations in order to give a locally reflected multiscale representation of the solution of the (exterior) Molodensky boundary–value problems (EMP) in three dimensions corresponding

to geoscientifically relevant regular surfaces. The regular surface thereby represents a low-scale approximation of the geoid.
At the end of the paper we discuss some examples of local multiscale approximation within the global framework of the numerical solution of the Molodensky problem. In particular we are interested in the zoom–in property and the detection of a high frequency perturbation which are typical features within a wavelet framework.

2 Preliminaries and Notation

We start with introducing some basic notation and the nomenclature which is used in our considerations.

A sphere of radius R centered around the origin is denoted by $\Omega_R = \{x \in \mathbb{R}^3 \mid |x| = R\}$. In particular, $\Omega = \Omega_1$ is the *unit sphere* in $\mathbb{R}^3$. We set Ω_R^{int} for the 'inner space' of Ω_R, $\Omega_R^{int} = \{x \in \mathbb{R}^3 \mid |x| < R\}$ while $\Omega_R^{ext} = \mathbb{R}^3 \setminus \overline{\Omega_R^{int}}$ is the 'outer space' of Ω_R. Clearly, $\Omega_R^{ext} = \{x \in \mathbb{R}^3 \mid |x| > R\}$. By $\Omega_{(R_1,R_2)}$ we denote the open spherical shell with inner radius R_1 and outer radius R_2 given by $\Omega_{(R_1,R_2)} = \{x \in \mathbb{R}^3 \mid R_1 < |x| < R_2\}$.
A major role in our considerations play regular surfaces which are introduced next.

Definition 2.1. A surface $\Sigma \subset \mathbb{R}^3$ is called *regular*, if it satisfies the following properties:

(i) Σ divides the three-dimensional Euclidean space $\mathbb{R}^3$ into the bounded region Σ_{int} (*inner space*) and the unbounded region Σ_{ext} (*outer space*) defined by $\Sigma_{\text{ext}} = \mathbb{R}^3 \backslash \overline{\Sigma_{\text{int}}}$, $\overline{\Sigma_{\text{int}}} = \Sigma_{\text{int}} \cup \Sigma$,

(ii) Σ_{int} contains the origin,

(iii) Σ is a closed and compact surface free of double points,

(iv) Σ has a continuously differentiable unit normal field ν pointing into the outer space Σ_{ext}.

Geoscientifically regular surfaces Σ are, for example, sphere, ellipsoid, spheroid, geoid, (regular) Earth's surface. In our approach to the Molodensky problem the telluroid will be understood as a regular surface approximating closely the (actual) geoid.
Given a regular surface, then there exist positive constants α, β such that

$$\alpha < \sigma^{\text{inf}} = \inf_{x \in \Sigma} |x| \leq \sup_{x \in \Sigma} |x| = \sigma^{\text{sup}} < \beta. \tag{1}$$

As usual, A_{int}, B_{int} (resp. A_{ext}, B_{ext}) denote the inner (resp. outer) space of the sphere A (resp. B) around the origin with radius α (resp. β). $\Sigma_{\text{int}}^{\text{inf}}$, $\Sigma_{\text{int}}^{\text{sup}}$ (resp. $\Sigma_{\text{ext}}^{\text{inf}}$, $\Sigma_{\text{ext}}^{\text{sup}}$) denote the inner (resp. outer) space of the sphere Σ^{inf} (resp. Σ^{sup}) around the origin with radius σ^{inf} (resp. σ^{sup}).

The set $\Sigma(\tau) = \{x \in \mathbb{R}^3 | x = y + \tau\nu(y), y \in \Sigma\}$ generates a *parallel surface* which is exterior to Σ for $\tau > 0$ and interior for $\tau < 0$. It is well known from differential geometry (see e.g. [30]) that if $|\tau|$ is sufficiently small, then the surface $\Sigma(\tau)$ is regular, and the normal to one parallel surface is a normal to the other.

In what follows we discuss function spaces that are of particular significance in our approach. Let Σ be a regular surface. $\text{Pot}(\Sigma_{\text{int}})$ denotes the space of all functions $U \in \mathrm{C}^{(2)}(\Sigma_{\text{int}})$ satisfying Laplace's equation in Σ_{int}, while $\text{Pot}(\Sigma_{\text{ext}})$ denotes the space of all functions $U \in \mathrm{C}^{(2)}(\Sigma_{\text{ext}})$ satisfying Laplace's equation in Σ_{ext} and being regular at infinity (that is,

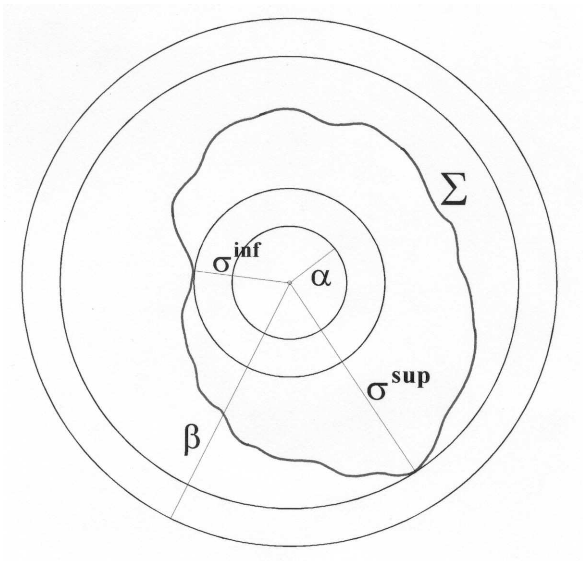


Figure 1: Regular surface (geometrical illustration)

$|U(x)| = O(|x|^{-1})$, $|(\nabla U)(x)| = O(|x|^{-2})$ for $|x| \to \infty$ uniformly with respect to all directions).

For $k = 0, 1, \ldots$ we denote by $\mathrm{Pot}^{(k)}(\overline{\Sigma_{\mathrm{ext}}})$ the space of all $U \in \mathrm{C}^{(k)}(\overline{\Sigma_{\mathrm{ext}}})$ such that $U|_{\Sigma_{\mathrm{ext}}}$ is of class $\mathrm{Pot}(\Sigma_{\mathrm{ext}})$.
In shorthand notation,

$$\begin{aligned} \mathrm{Pot}^{(k)}(\overline{\Sigma_{\mathrm{int}}}) &= \mathrm{Pot}(\Sigma_{\mathrm{int}}) \cap \mathrm{C}^{(k)}(\overline{\Sigma_{\mathrm{int}}}), \\ \mathrm{Pot}^{(k)}(\overline{\Sigma_{\mathrm{ext}}}) &= \mathrm{Pot}(\Sigma_{\mathrm{ext}}) \cap \mathrm{C}^{(k)}(\overline{\Sigma_{\mathrm{ext}}}). \end{aligned}$$

Let U be of class $\mathrm{Pot}^{(0)}(\overline{\Sigma_{\mathrm{ext}}})$. Then the maximum/minimum principle gives

$$\sup_{x \in \overline{\Sigma_{\mathrm{ext}}}} |U(x)| \leq \sup_{x \in \Sigma} |U(x)|.$$

In $\mathrm{C}^{(0)}(\Sigma)$ we have the inner product

$$(F, H)_{L^2(\Sigma)} = \int_\Sigma F(x) H(x) \, d\omega(x),$$

where $d\omega$ denotes the surface element. The inner product $(\cdot,\cdot)_{\mathrm{L}^2(\Sigma)}$ implies the norm $\|F\|_{\mathrm{L}^2(\Sigma)} = \left((F,F)_{\mathrm{L}^2(\Sigma)}\right)^{1/2}$. The space $(\mathrm{C}^{(0)}(\Sigma), (\cdot,\cdot)_{\mathrm{L}^2(\Sigma)})$ is a pre-Hilbert space. For every $F \in \mathrm{C}^{(0)}(\Sigma)$ we have the norm-estimate $\|F\|_{\mathrm{L}^2(\Sigma)} \leq \sqrt{\|\Sigma\|} \, \|F\|_{\mathrm{C}^{(0)}(\Sigma)}$, where $\|\Sigma\| = \int_\Sigma \, d\omega(x)$.

By $\mathrm{L}^2(\Sigma)$ we denote the *space of (Lebesgue) square-integrable functions* on the regular surface Σ. $\mathrm{L}^2(\Sigma)$ is a Hilbert space with respect to the inner product $(\cdot,\cdot)_{\mathrm{L}^2(\Sigma)}$ and a Banach space with respect to the norm $\|\cdot\|_{\mathrm{L}^2(\Sigma)}$. $\mathrm{L}^2(\Sigma)$ is the completion of $\mathrm{C}^{(0)}(\Sigma)$ with respect to the norm $\|\cdot\|_{\mathrm{L}^2(\Sigma)}$:

$$\overline{\mathrm{C}^{(0)}(\Sigma)}^{\|\cdot\|_{\mathrm{L}^2(\Sigma)}} = \mathrm{L}^2(\Sigma) \ .$$

3 Geodetic Background

The gravimetric determination of the geoid is a current research area in physical geodesy. It has become even more important, since the GPS techniques deliver accurate measurements with dense data coverage on continental areas. In particular, locally reflected approximation methods resulting in high wavelength geoidal reconstructions are of great significance.
For the convenience of the reader we recapitulate roughly the derivation of the linearized Molodensky problem.
The original problem of Molodensky can briefly be formulated as follows: Given, at all points of the geoid Σ^G, the gravity potential W and the gravity vector g to determine the surface Σ^G. It is clear by definition of the geoid, that W is constant on Σ^G, such that only a gauge value W_0 has to be given. Furthermore, we will not discuss in detail, how the gravity vector g is obtained on Σ^G from measurements on the real Earth's surface. For a detailed discussion of this downward continuation of g from the Earth's surface to the (apriori not known) geoid, the reader is referred to [16] or [17].
Our description of the linearized Molodensky problem essentially follows the course of [28].

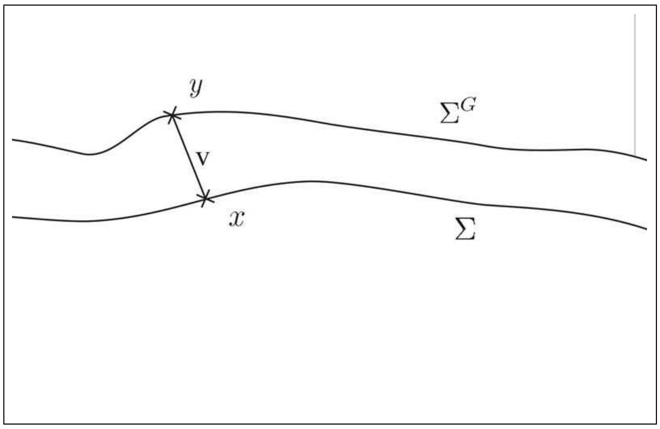


Figure 2: The telluroid Σ as an approximation of the geoid Σ^G.

The geoidal height determination is based on the fact that the actual geoid Σ^G is approximated by a regular surface Σ called the telluroid with known gravitational potential U in Σ_{ext}. For example, the telluroid can be chosen to be the EGM96-geoid. In this case, the normal potential is the EGM96 geopotential outside the EGM96-geoid. We assume that there exists a one-to-one correspondence between Σ^G and Σ (see Figure 2). W is the actual potential and U is an approximation of W called normal potential. We define $\gamma = \nabla U$ which is called the normal gravity and $g = \nabla W$ called the actual gravity which is given on Σ^G. Assume that, for given $x \in \Sigma$, the point $y \in \Sigma^G$ is the one associated to x by the one-to-one correspondence between Σ^G and Σ. The two points are connected by the vector $v = y - x$. An equivalent formulation of the classical Molodensky problem is now to determine the length of v, i.e., the distance of the geoid and the approximating telluroid along the one-to-one correspondence between Σ^G and Σ. To this end we introduce

$$\delta W = W|_{\Sigma^G} - U|_{\Sigma}$$
$$\delta g = g|_{\Sigma^G} - \gamma|_{\Sigma}$$

where δW is called the potential anomalie and δg is called gravity anomalie. Furthermore, we define the disturbing potential T by $T = W - U$, in Σ_{ext} , such that we have

$$\delta W = T|_{\Sigma^G} + U|_{\Sigma^G} - U|_{\Sigma} \tag{2}$$

$$\delta g = g|_{\Sigma^G} - \gamma|_{\Sigma} \, . \tag{3}$$

Next, we linearize these equations, i.e., we develop U and γ by Taylor expansion in terms of v and neglect terms of higher order in v which represents no substantial loss of accuracy if a sufficiently close telluroid is chosen. This procedure results in the approximations

$$\begin{aligned} U(y) &= U(x) + (\nabla U)(x) \cdot v, \quad y \in \Sigma^G,\, x \in \Sigma, \\ &= U(x) + \gamma(x) \cdot v \,, \end{aligned} \tag{4}$$

and

$$\gamma(y) = \gamma(x) + (\nabla\gamma)(x) \cdot v, \quad y \in \Sigma^G,\, x \in \Sigma \,, \tag{5}$$

where

$$\nabla\gamma = \left(\frac{\partial^2 U}{\partial x_i \partial x_j} \right)_{i,j=1,\dots,3} = M$$

Using this notation, the identity (5) results in

$$\gamma(y) = \gamma(x) + M(x)v, \quad y \in \Sigma^G,\, x \in \Sigma \, .$$

Finally, we get for the disturbing potential T by Taylor expansion in terms of v

$$T(y) = T(x) + (\nabla T)(x) \cdot v, \quad y \in \Sigma^G,\, x \in \Sigma \, . \tag{6}$$

Because T is already of first order in v, ∇T is of second order and the second term in (6) can be neglected such that we get

$$T(y) = T(x), \quad y \in \Sigma^G,\, x \in \Sigma \, .$$

Inserting these simplifications into (2) and (3) gives

$$\begin{aligned} \delta W &= T(x) + \gamma(x) \cdot v, \quad x \in \Sigma \\ \delta g &= g(y) - \gamma(y) + M(x)v, \quad y \in \Sigma^G,\, x \in \Sigma \, . \end{aligned}$$

Observing that

$$\begin{aligned} g(y) - \gamma(y) &= (\nabla W)(y) - (\nabla U)(y) \\ &= (\nabla T)(y), \quad y \in \Sigma^G, \\ &= (\nabla T)(x), \quad x \in \Sigma \end{aligned}$$

we finally arrive at

$$\delta W = T(x) + \gamma(x) \cdot v, \quad x \in \Sigma \tag{7}$$

$$\delta g = (\nabla T)(x) + M(x)v, \quad x \in \Sigma \, . \tag{8}$$

Equation (7) may be understood as a generalized Bruns formula. Actually it connects the disturbing potential T on the telluroid Σ with the geoid anomalies v, i.e., the anomalies between the geoid Σ^G and the telluroid Σ. If we assume that $M(x)$ is invertible for all $x \in \Sigma$, we get

$$v = M(x)^{-1}(\delta g - (\nabla T)(x)), \quad x \in \Sigma \, . \tag{9}$$

Inserting the identity (9) into equation (8) gives

$$T(x) - \gamma(x)(M(x))^{-1}(\nabla T)(x) = \delta W - \gamma(x) M(x)^{-1} \delta g \, . \tag{10}$$

This is the fundamental boundary condition of physical geodesy (see [28]).

Following [28] the vector $\gamma(x)(M(x))^{-1}$, for $x \in \Sigma$, can be shown in first order of v to be oriented in the direction of the exterior unit normal field ν on the telluroid Σ. More specifically,

$$\gamma(x)(M(x))^{-1} = -\frac{|x|}{2}\,\nu(x), \quad x \in \Sigma\,. \tag{11}$$

Inserting expression (11) into equation (10) therefore results in

$$\nu(x) \cdot (\nabla T)(x) + \frac{2}{|x|}\,T(x) = F(x), \quad x \in \Sigma, \tag{12}$$

where we have used the abbreviation $F(x) = \delta g + \frac{2}{|x|}\,\delta W$, for $x \in \Sigma$. The boundary condition (12) is rigorously equivalent to (10) transformed in an appropriate coordinate system. Summarizing our considerations we are led to the following boundary–value problem
Exterior Linearized Molodensky Problem (EMP)
Find $T \in \mathcal{C}^{(2)}(\Sigma_{ext}) \cap \mathcal{C}^{(1)}(\overline{\Sigma_{ext}})$ such that

$$\begin{aligned} \Delta T(x) &= 0 & x &\in \Sigma_{ext}, \\ \frac{\partial T}{\partial \nu}(x) + \lambda(x)T(x) &= F(x), & x &\in \Sigma, \\ T(x) &= O\left(\frac{1}{|x|}\right), & |x| &\to \infty\,. \end{aligned}$$

where λ, $F \in \mathcal{C}^{(0)}(\Sigma)$ are known functions on the regular surface Σ, i.e., $\lambda(x) = 2/\,|x|$, $x \in \Sigma$.
It should be noted that in the mathematical language the (linearized) Molodensky problem is a special Robin problem.

Remark. In the case of Σ to be a sphere the problem becomes the well-known Stokes problem (see [16] or [28]) and in the case of an ellipsoid it is called ellipsoidal Stokes problem (see [17] or [28]).

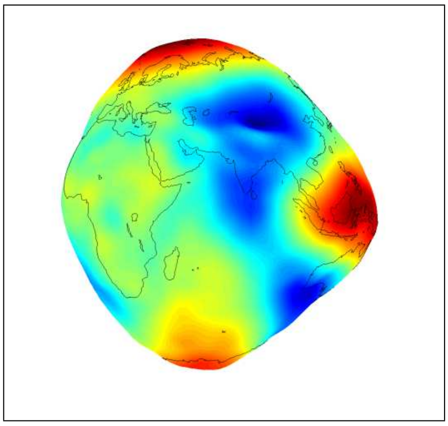


Figure 3: Low frequent approximation of the EGM96 geoid. The geoid undulations are inflated 10000 times. The linearized Molodensky problem can be solved on this telluroid as regular reference surface following the ideas of our approach. Due to the low discrepancy between the telluroid and the actual geoid, the errors caused by the linearizations in (4), (5) and (6) are small compared to the approach when the telluroid is taken to be a sphere or an ellipsoid.

4 Basic Concepts

Next we introduce some settings which are standard in potential theory (see, for example, [21], [25]) and which are needed to prove uniqueness and existence of the solution of the aforementioned exterior Molodensky problem (EMP).

4.1 Limit Formulae and Jump Relations

For $\tau \neq \sigma$ with $|\tau|, |\sigma|$ sufficiently small, the functions

$$(x,y) \mapsto \frac{1}{|x+\tau\nu(x)-(y+\sigma\nu(y))|}, \ (x,y) \in \Sigma \times \Sigma, \tag{13}$$

are continuous. Thus the *potential operators* $P(\tau,\sigma)$ defined by

$$P(\tau,\sigma)F(x) = \int_\Sigma F(y) \frac{1}{|x+\tau\nu(x)-(y+\sigma\nu(y))|} \ d\omega(y) \tag{14}$$

form mappings from $\mathrm{L}^2(\Sigma)$ into $\mathrm{C}^{(0)}(\Sigma)$ and are continuous with respect to $\|\cdot\|_{\mathrm{C}^{(0)}(\Sigma)}$. For all $\tau \neq \sigma$ the restrictions of $P(\tau,\sigma)$ on $\mathrm{C}^{(0)}(\Sigma)$ are bounded with respect to $\|\cdot\|_{\mathrm{L}^2(\Sigma)}$.
By formal operations we obtain for $F \in \mathrm{C}^{(0)}(\Sigma)$

$$P(\tau,0)F(x) = \int_\Sigma F(y) \frac{1}{|x+\tau\nu(x)-y|} \ d\omega(y) \tag{15}$$

($P(\tau,0)$*: operator of the single–layer potential on* Σ *for values on* $\Sigma(\tau)$),

$$\begin{aligned} P_{|2}(\tau,0)F(x) &= \frac{\partial}{\partial\sigma} P(\tau,\sigma)F(x)|_{\sigma=0} \qquad (16)\\ &= \int_\Sigma F(y) \left(\frac{\partial}{\partial\nu(y)} \frac{1}{|x+\tau\nu(x)-(y+\sigma\nu(y))|} \right)_{\sigma=0} d\omega(y) \\ &= \int_\Sigma F(y) \frac{\nu(y)\cdot(x+\tau\nu(x)-y)}{|x+\tau\nu(x)-y|^3} \ d\omega(y) \qquad (17) \end{aligned}$$

($P_{|2}(\tau,0)$*: operator of the double–layer potential on* Σ *for values on* $\Sigma(\tau)$).
The notation $P_{|i}$ indicates differentiation with respect to the i-th variable. Analogously we get

$$\begin{aligned} P_{|1}(\tau,0)F(x) &= \frac{\partial}{\partial\tau} P(\tau,\sigma)F(x)\big|_{\sigma=0}, \qquad (18)\\ &= -\int_\Sigma F(y) \frac{\nu(x)\cdot(x+\tau\nu(x)-y)}{|x+\tau\nu(x)-y|^3} \ d\omega(y) \qquad (19) \end{aligned}$$

and

$$P_{|2|1}(\tau,0)F(x) = \frac{\partial^2}{\partial\tau\partial\sigma} P(\tau,\sigma)F(x)\big|_{\sigma=0} \tag{20}$$

for the *operators of the normal derivatives.*
If $\tau=\sigma=0$, the kernels of the potentials have weak singularities. The integrals *formally* defined by

$$P(0,0)F(x) = \int_\Sigma F(y) \frac{1}{|x-y|} \ d\omega(y), \tag{21}$$

$$P_{|2}(0,0)F(x) = \int_\Sigma F(y) \frac{\partial}{\partial\nu(y)} \frac{1}{|x-y|} \ d\omega(y), \tag{22}$$

$$P_{|1}(0,0)F(x) = \frac{\partial}{\partial\nu(x)} \int_\Sigma F(y) \frac{1}{|x-y|}\, d\omega(y), \tag{23}$$

however, exist and define linear bounded operators in $\mathrm{L}^2(\Sigma)$. $P(0,0)$, $P_{|1}(0,0)$ and $P_{|2}(0,0)$ map $\mathrm{C}^{(0)}(\Sigma)$ into itself (see [21]). Furthermore, the operators are continuous (even compact) with respect to $\|\cdot\|_{\mathrm{C}^{(0)}(\Sigma)}$.
The operator $P(\tau,\sigma)^*$ satisfying

$$(F, P(\tau,\sigma)G)_{\mathrm{L}^2(\Sigma)} = (P(\tau,\sigma)^* F, G)_{\mathrm{L}^2(\Sigma)} \tag{24}$$

for all $F, G \in \mathrm{L}^2(\Sigma)$ is called the *adjoint operator* of $P(\tau,\sigma)$ with respect to $(\cdot,\cdot)_{\mathrm{L}^2(\Sigma)}$. According to Fubini's theorem it follows that

$$\begin{aligned} (F, P(\tau,\sigma)G)_{\mathrm{L}^2(\Sigma)} &= \int_\Sigma F(x) \left(\int_\Sigma \frac{G(y)}{|x+\tau\nu(x) - (y+\sigma\nu(y))|}\, d\omega(y) \right) d\omega(x) \\ &= \int_\Sigma G(y) \left(\int_\Sigma \frac{F(x)}{|x+\tau\nu(x) - (y+\sigma\nu(y))|}\, d\omega(x) \right) d\omega(y) \\ &= (P(\sigma,\tau)^* F, G)_{\mathrm{L}^2(\Sigma)} \ . \end{aligned} \tag{25}$$

By comparison we thus have

$$P(\tau,0)^* F(x) = P(\tau,\sigma)^* F(x)|_{\sigma=0} = \int_\Sigma F(y) \frac{1}{|y+\tau\nu(y) - x|}\, d\omega(y). \tag{26}$$

Analogously we can obtain expressions of $P_{|1}(\tau,0)^*$ and $P_{|2}(\tau,0)^*$.

The potential operators now enable us to give concise formulations of the classical *limit formulae* and *jump relations* in potential theory. Let I be the identity operator in $\mathrm{L}^2(\Sigma)$. Suppose that, for all sufficiently small values $\tau > 0, L_i^\pm(\tau), i = 1,2,3$, and $J_i(\tau), i = 1,...,6$, respectively, define the following operators:

$$\begin{aligned} \mathrm{L}_1^\pm(\tau) &= P(\pm\tau,0) - P(0,0), && (27)\\ \mathrm{L}_2^\pm(\tau) &= P_{|1}(\pm\tau,0) - P_{|1}(0,0) \pm 2\pi I, && (28)\\ \mathrm{L}_3^\pm(\tau) &= P_{|2}(\pm\tau,0) - P_{|2}(0,0) \mp 2\pi I, && (29) \end{aligned}$$

$$\begin{aligned} J_1(\tau) &= P(\tau,0) - P(-\tau,0), && (30)\\ J_2(\tau) &= P_{|1}(\tau,0) - P_{|1}(-\tau,0) + 4\pi I, && (31)\\ J_3(\tau) &= P_{|2}(\tau,0) - P_{|2}(-\tau,0) - 4\pi I, && (32)\\ J_4(\tau) &= P_{|2|1}(\tau,0) - P_{|2|1}(-\tau,0), && (33)\\ J_5(\tau) &= P_{|1}(\tau,0) + P_{|1}(-\tau,0) - 2P_{|1}(0,0), && (34)\\ J_6(\tau) &= P_{|2}(\tau,0) + P_{|2}(-\tau,0) - 2P_{|2}(0,0). && (35) \end{aligned}$$

Then, for $F \in \mathrm{C}^{(0)}(\Sigma)$, the main results of classical potential theory may be formulated by

$$\begin{aligned} \lim_{\substack{\tau\to 0\\ \tau>0}} \| L_i^\pm(\tau)F \|_{\mathrm{C}^{(0)}(\Sigma)} = 0, &\qquad \lim_{\substack{\tau\to 0\\ \tau>0}} \| J_i(\tau)F \|_{\mathrm{C}^{(0)}(\Sigma)} = 0, \\ \lim_{\substack{\tau\to 0\\ \tau>0}} \| L_i^\pm(\tau)^*F \|_{\mathrm{C}^{(0)}(\Sigma)} = 0, &\qquad \lim_{\substack{\tau\to 0\\ \tau>0}} \| J_i(\tau)^*F \|_{\mathrm{C}^{(0)}(\Sigma)} = 0. \end{aligned} \tag{36}$$

The relations (36) can be generalized to the Hilbert space $\mathrm{L}^2(\Sigma)$ (see [5], [22]):

THEOREM 4.1. *For all* $F \in \mathrm{L}^2(\Sigma)$

$$\begin{aligned} \lim_{\substack{\tau\to 0\\ \tau>0}} \| \mathrm{L}_i^\pm(\tau)F \|_{\mathrm{L}^2(\Sigma)} = 0, &\qquad \lim_{\substack{\tau\to 0\\ \tau>0}} \| J_i(\tau)F \|_{\mathrm{L}^2(\Sigma)} = 0, \\ \lim_{\substack{\tau\to 0\\ \tau>0}} \| \mathrm{L}_i^\pm(\tau)^*F \|_{\mathrm{L}^2(\Sigma)} = 0, &\qquad \lim_{\substack{\tau\to 0\\ \tau>0}} \| J_i(\tau)^*F \|_{\mathrm{L}^2(\Sigma)} = 0. \end{aligned} \tag{37}$$

A proof of Theorem 4.1 can be found in [13].

5 Existence and Uniqueness

Next, we discuss the well-posedness of the Molodensky boundary–value problem corresponding to a regular telluroidal surface Σ as introduced in Section 3. First, we will reformulate the problem in a short notation. Then we will show uniqueness and existence of the solution and at the end we will show the continuous dependency of the solution from the boundary data.

5.1 Formulation, Uniqueness and Existence

We begin with the reformulation of the boundary–value problem using the notation of Section 2.

Exterior Molodensky Problem (EMP):
Given $F, \lambda \in \mathcal{C}^{(0)}(\Sigma)$, find $T \in \mathrm{Pot}^{(1)}(\overline{\Sigma_{\mathrm{ext}}})$ such that

$$\left(\frac{\partial T^+}{\partial \nu_\Sigma} + \lambda T^+\right)(x) = \lim_{\substack{\tau \to 0 \\ \tau > 0}} \Big(\nu(x) \cdot (\nabla T)(x + \tau\nu(x)) + \lambda(x) T(x + \tau\nu(x))\Big) = F(x), \quad x \in \Sigma.$$

We recall the role of layer potentials in the solution theory of the aforementioned boundary–value problem:

(**EMP**) The solution of the exterior Molodensky problem can be formulated in terms of a single layer potential

$$T(x) = \int_\Sigma \mu(y) \frac{1}{|x - y|} \, d\omega(y), \tag{38}$$

where $\mu \in \mathcal{C}^{(0)}(\Sigma)$ satisfies the integral equations

$$\frac{\partial T^+}{\partial \nu_\Sigma} + \lambda T^+ = (-2\pi I + P_{|1}(0,0) + \lambda P(0,0))\mu = F, \quad \text{on } \Sigma. \tag{39}$$

Since the operator $P_{|1}(0,0) + \lambda P(0,0) : \mathcal{C}^{(0)}(\Sigma) \to \mathcal{C}^{(0)}(\Sigma)$ is compact, the Fredholm alternative is applicable. Thus, (39) is uniquely solvable for all $F \in \mathcal{C}^{(0)}(\Sigma)$ which are orthogonal to the non-trivial solutions of the homogenous adjoint equation. We define

$$N(\Sigma, \lambda) = \{\tilde{\mu} \in \mathcal{C}^{(0)}(\Sigma) \,|\, (-2\pi I + P_{|2}(0,0) + \lambda P(0,0))\tilde{\mu} = 0\}.$$

By orthogonal decomposition we have

$$\mathcal{C}^{(0)}(\Sigma) = \overline{(N(\Sigma, \lambda))^\perp \oplus N(\Sigma, \lambda)}^{\|\cdot\|_{\mathcal{C}^{(0)}(\Sigma)}}$$

and obtain unique solvability of (39) for all $F \in (N(\Sigma, \lambda))^\perp$.

Uniqueness of the Molodensky problem has been extensively discussed in the geodetic literature (see [20], [28], [31], etc.). It has been shown for the case of the regular surface Σ to be a sphere Ω_R, $R > 0$, and for $\lambda = 2/R$, the space $N(\Omega_R, \lambda)$ is the linear span of spherical harmonics of degree 1 which is of dimension 3. It has also been shown that in the case of an arbitrary regular surface Σ the dimension of $N(\Sigma, \Lambda)$ is still 3 (see [28]).

5.2 Regularity Theorems

From the maximum/minimum principle of potential theory we already know that

$$\sup_{x \in \overline{\Sigma_{\mathrm{ext}}}} |T(x)| \le \sup_{x \in \Sigma} \left|T^+(x)\right| \tag{40}$$

holds for $T \in \mathrm{Pot}^{(0)}(\overline{\Sigma_{\mathrm{ext}}})$. Moreover, from the theory of integral equations it can be easily detected (see e.g. [27]) that there exists a constant C (dependent on Σ) such that for $T \in \mathrm{Pot}^{(1)}(\overline{\Sigma_{\mathrm{ext}}})$

$$\sup_{x \in \overline{\Sigma_{\mathrm{ext}}}} |T(x)| \leq \mathrm{C} \sup_{x \in \Sigma} \left| \frac{\partial T^+}{\partial \nu}(x) \right| \tag{41}$$

In what follows we want to verify analogous *regularity theorems* in the $\mathrm{L}^2(\Sigma)$-context for the Molodensky problem.

THEOREM 5.1. *Let T be of class $\mathrm{Pot}^{(1)}(\overline{\Sigma_{\mathrm{ext}}})$. Then, for every (sufficiently small) $\rho > 0$, there exists a constant $\mathrm{C}(= \mathrm{C}(k; K, \Sigma))$ such that*

$$\sup_{x \in K} \left| \left(\nabla^{(k)} T \right)(x) \right| \leq \mathrm{C} \left(\left\| \frac{\partial T^+}{\partial \nu} \right\|_{\mathrm{L}^2(\Sigma)} + \|\lambda\|_{\mathrm{L}^2(\Sigma)} \left\| T^+ \right\|_{\mathrm{L}^2(\Sigma)} \right) \tag{42}$$

for all $K \subset \Sigma_{\mathrm{ext}}$ with dist $(\overline{K}, \Sigma) \geq \rho > 0$ and for all $k \in \mathbb{N}_0$.

Proof. Recall that the exterior Molodensky problem (*EMP*) can be solved by (38), (39). The operator Q defined by

$$Q = -2\pi I + P_{|1}(0,0) + \lambda P(0,0)$$

and its adjoint operator Q^* with respect to $\|\cdot\|_{\mathrm{L}^2(\Sigma)}$ are bijective in the Banach space $(\mathrm{C}^{(0)}(\Sigma), \|\cdot\|_{\mathrm{C}^{(0)}(\Sigma)})$ (see e.g. [27]). By virtue of the open mapping theorem the operators Q and Q^{-1} are linear and bounded with respect to $\|\cdot\|_{\mathrm{C}^{(0)}(\Sigma)}$. Furthermore, $(Q^*)^{-1} = (Q^{-1})^*$. Therefore, by virtue of the technique due to P. Lax (1954) (cf. Theorem 4.1), Q and its inverse operator Q^{-1} are bounded with respect to $\|\cdot\|_{\mathrm{L}^2(\Sigma)}$.
Now, for all sufficiently small values $\rho > 0$ and all points $x \in K \subset \Sigma_{\mathrm{int}}$ with $\mathrm{dist}(\overline{K}, \Sigma) \geq \rho$, the Cauchy-Schwarz inequality gives

$$\begin{aligned} \left| \left(\nabla^{(k)} T \right)(x) \right| &= \left| \int_\Sigma \sigma(y) \left(\nabla_x^{(k)} \frac{1}{|x-y|} \right) \, d\omega(y) \right| \\ &\leq \left(\int_\Sigma \left| \nabla_x^{(k)} \frac{1}{|x-y|} \right|^2 \, d\omega(y) \right)^{\frac{1}{2}} \left(\int_\Sigma |\sigma(y)|^2 \, d\omega(y) \right)^{\frac{1}{2}}. \end{aligned} \tag{43}$$

This shows us that

$$\sup_{x \in K} \left| \left(\nabla^{(k)} T \right)(x) \right| \leq D \left(\int_\Sigma |\sigma(y)|^2 \, d\omega y) \right)^{\frac{1}{2}}, \tag{44}$$

where we have used the abbreviation

$$D = \sup_{x \in K} \left(\int_\Sigma \left| \nabla_x^{(k)} \frac{1}{|x-y|} \right|^2 \, d\omega(y) \right)^{\frac{1}{2}}. \tag{45}$$

However,

$$\sup_{x \in K} \left| \left(\nabla^{(k)} T \right)(x) \right| < D \left(\int_\Sigma \left| Q^{-1}(F)(y) \right|^2 \, d\omega(y) \right)^{\frac{1}{2}}. \tag{46}$$

Because of the boundedness of Q^{-1} with respect to $\|\cdot\|_{\mathrm{L}^2(\Sigma)}$ this tells us that

$$\sup_{x \in K} \left| \left(\nabla^{(k)} T \right)(x) \right| \leq \mathrm{C} \left(\int_\Sigma |F(y)|^2 \, d\omega(y) \right)^{\frac{1}{2}}, \tag{47}$$

with $\mathrm{C} = D \left\| Q^{-1} \right\|_{\mathrm{L}^2(\Sigma)}$. Hence, the statement (Theorem 5.1) is true. □

6 Multiscale Modelling in $(\mathrm{L}^2(\Sigma), \|\cdot\|_{\mathrm{L}^2(\Sigma)})$

Writing out the limit and jump relations (Theorem 4.1) we obtain the following corollary.

COROLLARY 6.1. *For* $F \in \mathrm{L}^2(\Sigma)$

$$\lim_{\substack{\tau\to 0\\ \tau>0}} \int_\Sigma \Phi_\tau^i(\cdot, y)\, F(y)\; d\omega(y) = A_i(F), \quad i = 1, \ldots, 7 \tag{48}$$

where we have set

$$A_i(F) = \begin{cases} F & , \quad i = 2,3,5,6 \\ 0 & , \quad i = 4,7 \\ \int_\Sigma \frac{1}{|\cdot - y|} F(y)\; d\omega(y) & , \quad i = 1 \\ \int_\Sigma \frac{\partial}{\partial\nu(\cdot)} \frac{1}{|\cdot - y|} F(y)\; d\omega(y) & , \quad i = 8 \\ \int_\Sigma \frac{\partial}{\partial\nu(y)} \frac{1}{|\cdot - y|} F(y)\; d\omega(y) & , \quad i = 9, \end{cases} \tag{49}$$

and

$$\begin{aligned}
\Phi^1_{\pm\tau}(x,y) &= \frac{1}{|x \pm \tau\nu(x) - y|}, \\
\Phi^2_{\pm\tau}(x,y) &= \frac{1}{2\pi}\left(\frac{(x \pm \tau\nu(x) - y)\cdot\nu(x)}{|x \pm \tau\nu(x) - y|^3} - \frac{(x-y)\cdot\nu(x)}{|x-y|^3}\right), \\
\Phi^3_{\pm\tau}(x,y) &= \frac{1}{2\pi}\left(\frac{(x \pm \tau\nu(x) - y)\cdot\nu(y)}{|x \pm \tau\nu(x) - y|^3} - \frac{(x-y)\cdot\nu(y)}{|x-y|^3}\right), \\
\Phi^4_{\tau}(x,y) &= \frac{1}{|x + \tau\nu(x) - y|} - \frac{1}{|x - \tau\nu(x) - y|} \\
\Phi^5_{\tau}(x,y) &= \frac{1}{4\pi}\left(\frac{(x + \tau\nu(x) - y)\cdot\nu(x)}{|x + \tau\nu(x) - y|^3} - \frac{(x - \tau\nu(x) - y)\cdot\nu(x)}{|x - \tau\nu(x) - y|^3}\right), \\
\Phi^6_{\tau}(x,y) &= \frac{1}{4\pi}\left(\frac{(x + \tau\nu(x) - y)\cdot\nu(y)}{|x + \tau\nu(x) - y|^3} - \frac{(x - \tau\nu(x) - y)\cdot\nu(y)}{|x - \tau\nu(x) - y|^3}\right), \\
\Phi^7_{\tau}(x,y) &= \frac{\nu(x)\cdot\nu(y)}{|x + \tau\nu(x) - y|} - \frac{\nu(x)\cdot\nu(y)}{|x - \tau\nu(x) - y|^3} \\
&\quad -3\frac{((x + \tau\nu(x) - y)\cdot\nu(y))((x + \tau\nu(x) - y)\cdot\nu(y))}{|x + \tau\nu(x) - y|^5} \\
&\quad +3\frac{((x - \tau\nu(x) - y)\cdot\nu(y))((x + \tau\nu(x) - y)\cdot\nu(y))}{|x - \tau\nu(x) - y|^5}, \\
\Phi^8_{\tau}(x,y) &= \frac{1}{2}\left(\frac{(x + \tau\nu(x) - y)\cdot\nu(x)}{|x + \tau\nu(x) - y|^3} + \frac{(x - \tau\nu(x) - y)\cdot\nu(x)}{|x - \tau\nu(x) - y|^3}\right), \\
\Phi^9_{\tau}(x,y) &= \frac{1}{2}\left(\frac{(x + \tau\nu(x) - y)\cdot\nu(y)}{|x + \tau\nu(x) - y|^3} + \frac{(x - \tau\nu(x) - y)\cdot\nu(y)}{|x - \tau\nu(x) - y|^3}\right),
\end{aligned}$$

$\tau > 0, (x,y) \in \Sigma \times \Sigma$.

6.1 Scaling and Wavelet Functions

For $\tau > 0$ and $i \in \{1, \ldots, 9\}$, the family $\{\Phi_\tau^i\}_{\tau>0}$ of kernels $\Phi_\tau^i : \Sigma \times \Sigma \to \mathbb{R}$ is called a Σ–*scaling function of type* i. Moreover, $\Phi_1^i : \Sigma \times \Sigma \to \mathbb{R}$ (i.e.: $\tau = 1$) is called the *mother kernel of the* Σ–*scaling function of type* i.

Correspondingly, for $\tau > 0$ and $i \in \{1, \ldots, 9\}$, the family $\{\Psi_\tau^i\}_{\tau>0}$ of kernels $\Psi_\tau^i : \Sigma \times \Sigma \to \mathbb{R}$ given by

$$\Psi_\tau^i(x, y) = -\alpha(\tau)^{-1} \frac{d}{d\tau} \Phi_\tau^i(x, y), \qquad x, y \in \Sigma, \tag{50}$$

is called a Σ–*wavelet function of type* i.
In the remainder of this paper we particularly choose $\alpha(\tau) = \tau^{-1}$ (of course, other weight functions than $\alpha(\tau) = \tau^{-1}$ can be chosen in (50)). Moreover, $\Psi_1^i : \Sigma \times \Sigma \to \mathbb{R}$ (i.e.: $\tau = 1$) is called the *mother kernel of the* Σ–*wavelet function of type* i.

The differential equation (50) is called the (scale continuous) Σ–*scaling equation of type* i.

DEFINITION 6.2. Let $\{\Phi_\tau^i\}_{\tau>0}$ be a Σ–scaling function of type i. Then the associated Σ–*wavelet transform of type* i is defined by $(WT)^{(i)} : \mathrm{L}^2(\Sigma) \to \mathrm{L}^2((0, \infty) \times \Sigma)$ with

$$(WT)^{(i)}(F)(\tau, x) = \int_\Sigma \Psi_\tau^i(x, y) F(y) \, d\omega(y) \ .$$

In accordance with our construction we have

$$\begin{aligned}
\Psi_\tau^1(x, y) &= \frac{\tau(x + \tau\nu(x) - y) \cdot \nu(x)}{|x + \tau\nu(x) - y|^3} \\
\Psi_\tau^2(x, y) &= \frac{-\tau}{2\pi} \left(\frac{1}{|x + \tau\nu(x) - y|^3} - 3\frac{((x + \tau\nu(x) - y) \cdot \nu(x))^2}{|x + \tau\nu(x) - y|^5} \right), \\
\Psi_\tau^3(x, y) &= \frac{\tau}{2\pi} \left(\frac{\nu(y) \cdot \nu(x)}{|x + \tau\nu(x) - y|^3} \right) \\
&\quad - \frac{3\tau}{2\pi} \left(\frac{((x + \tau\nu(x) - y) \cdot \nu(x))((x + \tau\nu(x) - y) \cdot \nu(y))}{|x + \tau\nu(x) - y|^5} \right), \\
\Psi_\tau^4(x, y) &= -\tau \left(\frac{(x + \tau\nu(x) - y) \cdot \nu(x)}{|x + \tau\nu(x) - y|^3} + \frac{(x - \tau\nu(x) - y) \cdot \nu(x)}{|x - \tau\nu(x) - y|^3} \right), \\
\Psi_\tau^5(x, y) &= \frac{-\tau}{4\pi} \left(\frac{1}{|x - \tau\nu(x) - y|^3} + \frac{1}{|x - \tau\nu(x) - y|^3} \right) \\
&\quad + \frac{3\tau}{4\pi} \left(\frac{((x + \tau\nu(x) - y) \cdot \nu(x))^2}{|x + \tau\nu(x) - y|^5} + \frac{((x - \tau\nu(x) - y) \cdot \nu(x))^2}{|x - \tau\nu(x) - y|^5} \right), \\
\Psi_\tau^6(x, y) &= \frac{-\tau}{4\pi} \left(\frac{\nu(x) \cdot \nu(y)}{|x + \tau\nu(x) - y|^3} + \frac{\nu(x) \cdot \nu(y)}{|x - \tau\nu(x) - y|^3} \right) \\
&\quad + \frac{3\tau}{4\pi} \left(\frac{((x + \tau\nu(x) - y) \cdot \nu(x))((x + \tau\nu(x) - y) \cdot \nu(y))}{|s + \tau\nu(x) - y|^5} \right. \\
&\quad + \left. \frac{((x - \tau\nu(x) - y) \cdot \nu(x))((x - \tau\nu(x) - y) \cdot \nu(y))}{|x - \tau\nu(x) - y|^5} \right),
\end{aligned}$$

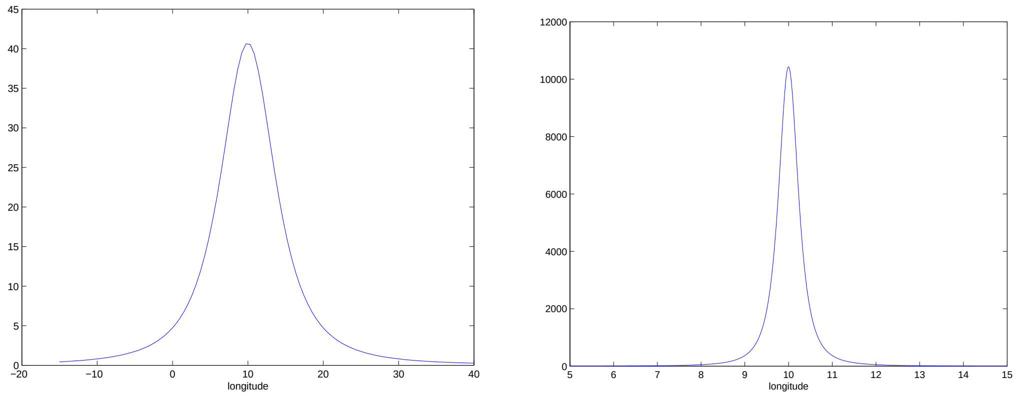
for $x, y \in \Sigma$. For simplicity, we omit the representations of $\Psi_\tau^7(x, y), \Psi_\tau^8(x, y)$ and $\Psi_\tau^9(x, y)$, $x, y \in \Sigma$, but the reader should note that they are available in an explicit representation.

6.2 Scale Continuous Reconstruction Formula

It is not difficult to see that the wavelets Ψ_τ^i, $i \in \{1, \ldots, 9\}$, behave like $O(\tau^{-1})$, hence, the convergence of the following integrals in the *reconstruction theorem* is guaranteed.

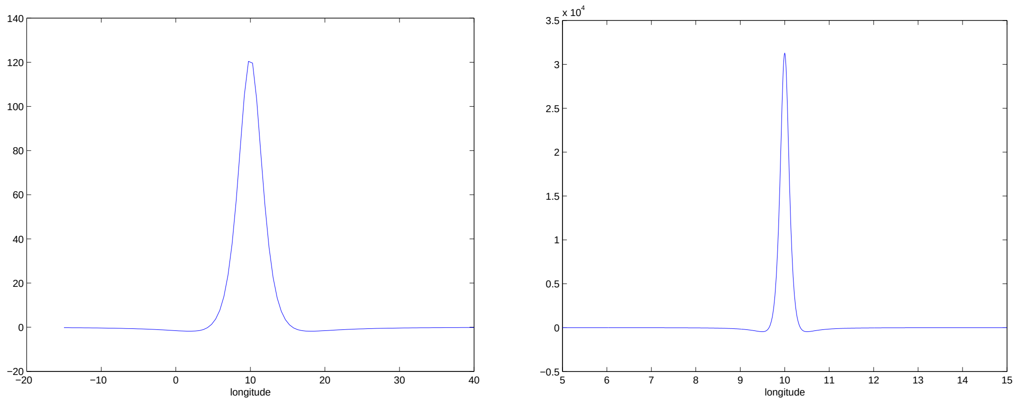
THEOREM 6.3. *Let* $\{\Phi_\tau^i\}_{\tau>0}$ *be a* Σ–*scaling function of type* i. *Suppose that* F *is of class* $\mathrm{L}^2(\Sigma)$. *Then the reconstruction formula*

$$\int_0^\infty (WT)^i(F)(\tau, \cdot) \frac{d\tau}{\tau} = A_i(F), \quad i = 1, \ldots, 7$$



(a) $\tau = 2^{-4}$ (b) $\tau = 2^{-8}$

Figure 4: Scaling–function Φ_τ^6 (sectional illustration) for two values of τ



(a) $\tau = 2^{-4}$ (b) $\tau = 2^{-8}$

Figure 5: Wavelet–function Ψ_τ^6 (sectional illustration) for two values of τ

holds in the sense of $\|\cdot\|_{\mathrm{L}^2(\Sigma)}$ *where* $A_i(F)$ *is defined in Corollary 6.1*

Proof. Let $R > 0$ be arbitrary. By observing Fubini's theorem and the identity

$$\Phi^i_R(x,y) = \int_R^\infty \Psi^i_\tau(x,y)\frac{d\tau}{\tau}, \qquad (x,y) \in \Sigma \times \Sigma,$$

we obtain

$$\begin{aligned}\int_R^\infty (WT)^i(F)(\tau,\cdot)\frac{d\tau}{\tau} &= \int_R^\infty \int_\Sigma \Psi^i_\tau(\cdot,y)F(y)\ d\omega(y)\frac{d\tau}{\tau} \qquad (51)\\ &= \int_\Sigma \int_R^\infty \Psi^i_\tau(\cdot,y)F(y)\ d\omega(y)\frac{d\tau}{\tau}\\ &= \int_\Sigma \Phi^i_R(\cdot,y)F(y)\ d\omega(y)\ .\end{aligned}$$

The limit $R \to 0$ in connection with Theorem 6.1 yields the desired result. □

Next, our interest is to reformulate the wavelet transform and the reconstruction theorem by use of dilated and shifted versions of the mother kernel. For that purpose we introduce the x–translation and the τ–dilation operator of a mother kernel as follows:

$$T_x : \Psi^i_1 \mapsto T_x\Psi^i_1 = \Psi^i_{1;x} = \Psi^i_1(x,\cdot), \qquad x \in \Sigma, \qquad (52)$$

$$D_\tau : \Psi^i_1 \to D_\tau\Psi^i_1 = \Psi^i_\tau, \quad \tau > 0\ . \qquad (53)$$

Consequently it follows that

$$T_x D_\tau \Psi^i_1 = T_x\Psi^i_\tau = \Psi^i_{\tau;x} = \Psi^i_\tau(x,\cdot), \qquad (54)$$

$i = 1, \ldots, 9$. In other words,

$$(WT)^i(F)(\tau;x) = \int_\Sigma \Psi_{\tau;x}(y)F(y)\ d\omega(y), \quad x \in \Sigma, \tau > 0\ . \qquad (55)$$

Moreover, we have the following limit results.

THEOREM 6.4. *For* $x \in \Sigma$ *and* $F \in \mathrm{L}^2(\Sigma)$

$$\lim_{\substack{R\to 0\\ R>0}} \int_\Sigma \Phi^i_{R;x}(y)F(y)\ d\omega(y) = A_i(F), \quad i = 1, \ldots, 7 \qquad (56)$$

Note that the properties of the Σ–wavelets of type i (analogously to variants of spherical wavelets developed in [9], [10]) do not presume the zero–mean property of Ψ^i_τ. The wavelets constructed in this way, therefore, do not satisfy a substantial condition of the Euclidean concept. However, it should be pointed out that a construction of wavelets possessing the zero-mean property (see [9]), is obvious and will not be discussed here.

6.3 Scale Discretized Reconstruction Formula

Until now we were concerned with a scale continuous approach to wavelets. In what follows, scale discrete Σ–scaling functions and wavelets of type i will be introduced. We start with the choice of a sequence which divides the continuous scale interval $(0,\infty)$ into discrete pieces. More explicitly, $(\tau_j)_{j\in\mathbb{Z}}$ denotes a sequence of real numbers satisfying

$$\lim_{j\to\infty} \tau_j = 0 \quad \text{and} \quad \lim_{j\to-\infty} \tau_j = \infty\ . \qquad (57)$$

For example, one may choose $\tau_j = 2^{-j}$, $j \in \mathbb{Z}$ (note that in this case, $2\tau_{j+1} = \tau_j$, $j \in \mathbb{Z}$).

Given a Σ–scaling function $\{\Phi^i_\tau\}_{\tau>0}$ of type i, then we clearly define the (scale) *discretized Σ–scaling function* of type i by $\{\Phi^i_{\tau_j}\}_{j\in\mathbb{Z}}$. In doing so, by Theorem 6.4, we immediately get the following result.

THEOREM 6.5. *For $F \in \mathrm{L}^2(\Sigma)$*

$$\lim_{j\to\infty} \int_\Sigma \Phi^i_{\tau_j}(\cdot, y)F(y)\ d\omega(y) = A_i(F), \quad i = 1,\ldots,7 \tag{58}$$

holds in the $\|cdot\|_{\mathrm{L}^2(\Sigma)}$–sense where $A_i(F)$ is defined in Corollary 6.1.

Our procedure canonically leads us to the following type of scale discretized wavelets.

DEFINITION 6.6. Let $\{\Phi_{\tau_j}\}_{j\in\mathbb{Z}}$ be a discretized Σ–scaling function of type i. Then the (scale) discretized Σ–*wavelet function of type* i is defined by

$$\Psi^i_{\tau_j}(\cdot,\cdot) = \int_{\tau_{j+1}}^{\tau_j} \Psi^i_\tau(\cdot,\cdot)\frac{d\tau}{\tau}, \quad j \in \mathbb{Z} \tag{59}$$

In connection with (50) it follows that

$$\Psi^i_{\tau_j}(\cdot,\cdot) = -\int_{\tau_{j+1}}^{\tau_j} \tau\frac{d}{d\tau}\Phi^i_\tau(\cdot,\cdot)\frac{d\tau}{\tau} = \Phi^i_{\tau_{j+1}}(\cdot,\cdot) - \Phi^i_{\tau_j}(\cdot,\cdot). \tag{60}$$

Formula (60) is called *(scale) discretized Σ–scaling equation of type i.*
Assume now that F is a function of class $\mathrm{L}^2(\Sigma)$. Observing the discretized Σ–scaling equation of type i we get for $J \in \mathbb{Z}$ and $N \in \mathbb{N}$

$$\int_\Sigma \Phi^i_{\tau_{J+N}}(\cdot,y)F(y)\ d\omega(y) = \int_\Sigma \Phi^i_{\tau_J}(\cdot,y)F(y)\ d\omega(y) + \sum_{j=J}^{J+N-1} \int_\Sigma \Psi^i_{\tau_j}(\cdot,y)F(y)\ d\omega(y)\ .$$

Therefore we are able to formulate the following corollary.

COROLLARY 6.7. *Let $\{\Phi^i_{\tau_j}\}_{j\in\mathbb{Z}}$ be a (scale) discretized Σ–scaling function of type i. Then the multiscale representation of a function $F \in \mathrm{L}^2(\Sigma)$*

$$\sum_{j=-\infty}^{+\infty} \int_\Sigma \Psi^i_{\tau_j}(\cdot,y)F(y)\ d\omega(y) = A_i(F), \quad i = 1,\ldots,7 \tag{61}$$

holds in the $\|\cdot\|_{\mathrm{L}^2(\Sigma)}$–sense where $A_i(F)$ is defined in Corollary 6.1.

Corollary 6.7 admits the following reformulation.

COROLLARY 6.8. *Under the assumption of Corollary 6.7*

$$P^i_{\tau_J}(F) + \sum_{j=J}^{+\infty} \int_\Sigma \Psi^i_{\tau_j}(\cdot,y)F(y)\ d\omega(y) = A_i(F), \quad i = 1,\ldots,7 \tag{62}$$

for every $J \in \mathbb{Z}$ (in the sense of the $\|\cdot\|_{\mathrm{L}^2(\Sigma)}$–norm), where $P^i_{\tau_J}(F)$ is given by

$$P^i_{\tau_J}(F) = \int_\Sigma \Phi^i_{\tau_J}(\cdot,y)F(y)\ d\omega(y)\ .$$

Continuous Formulation		**Discrete Formulation**
	Scaling Functions	
$\Phi^i_\tau(x,y)$	$\Rightarrow$	$\Phi^i_J(x,y) = \Phi^i_{\tau_J}(x,y)$
	Wavelets	
$\Psi^i_\tau(x,y) = -\tau \frac{d}{d\tau}\Phi^i_\tau(x,y)$	$\Rightarrow$	$\Psi^i_j(x,y) = \Phi^i_{j+1}(x,y) - \Phi^i_j(x,y)$
	Convergence Relations	
$F = \lim\limits_{\tau\to 0}(F * \Phi^i_\tau)$	$\Rightarrow$	$F = \lim\limits_{J\to\infty}(F * \Phi^i_J)$
$F = \int\limits_0^\infty (F * \Psi^i_\tau)\frac{1}{\tau}\,d\tau$	$\Rightarrow$	$F = \sum\limits_{j=-\infty}^{\infty}(F * \Psi^i_j)$

Table 1: Comparison of continuous and discrete formulation

The scale discretized Σ–wavelets allow the following formulation

$$T_x D_{\tau_j} \Psi^i_1 = T_x \Psi^i_{\tau_j} = \Psi^i_{\tau_j;x} = \Psi^i_{\tau_j}(x,\cdot) \tag{63}$$

for $i = 1,\ldots,9$ and $x \in \Sigma$.

The *(scale) discretized Σ–wavelet transform of type i* is defined by

$$(WT)^i : \mathrm{L}^2(\Sigma) \mapsto \left\{ H : \mathbb{Z}\times\Sigma \to \mathbb{R} \Big| \sum_{j=-\infty}^{\infty} \int_\Sigma (H(j;y))^2 \; d\omega(y) < \infty \right\}$$

with

$$(WT)^i(F)(\tau_j;x) = \int_\Sigma \Psi^i_{\tau_j;x}(y)F(y)\; d\omega(y) \; .$$

THEOREM 6.9. *Let $\{\Phi^i_{\tau_j}\}_{j\in\mathbb{Z}}$ be a (scale) discretized Σ–scaling function of type i. Then, for all $F \in \mathrm{L}^2(\Sigma)$, the reconstruction formula*

$$\sum_{j=-\infty}^{+\infty} (WT)^i(F)(\tau_j;\cdot) = A_i(F), \quad i = 1,\ldots,7 \tag{64}$$

holds in $\|\cdot\|_{\mathrm{L}^2(\Sigma)}$–sense.

Comparing this result with the continuous analogue Theorem 6.3 we notice that the subdivision of the continuous scale interval $(0,\infty)$ into discrete pieces means substitution of the integral over τ by an associated discrete sum. A comparison of the discrete formulation and the continuous formulation can be found Table 1.

6.4 Scale and Detail Spaces

As in the spherical theory of wavelets (see [7], [8]), the operators $R^i_{\tau_j}$, $P^i_{\tau_j}$ defined by

$$R^i_{\tau_j}(F) = \int_\Sigma \Psi^i_{\tau_j}(\cdot,y)F(y)\; d\omega(y), \quad F \in \mathrm{L}^2(\Sigma), \tag{65}$$

$$P^i_{\tau_j}(F) = \int_\Sigma \Phi^i_{\tau_j}(\cdot,y)F(y)\; d\omega(y), \quad F \in \mathrm{L}^2(\Sigma) \tag{66}$$

may be understood as band pass and low pass filter, respectively. The *scale spaces* $\mathcal{V}^i_{\tau_j}$ and the *detail spaces* $W^i_{\tau_j}$ *of type* i are defined by

$$\mathcal{V}^i_{\tau_j} = P^i_{\tau_j}\left(\mathrm{L}^2(\Sigma)\right) = \left\{P^i_{\tau_j}(F)\middle|F \in \mathrm{L}^2(\Sigma)\right\}, \tag{67}$$

$$\mathcal{W}^i_{\tau_j} = R^i_{\tau_j}\left(\mathrm{L}^2(\Sigma)\right) = \left\{R^i_{\tau_j}(F)\middle|F \in \mathrm{L}^2(\Sigma)\right\}, \tag{68}$$

respectively. From the identity

$$\int_\Sigma \Phi^i_{\tau_{J+1}}(\cdot, y)F(y)\ d\omega(y) = \int_\Sigma \Phi^i_{\tau_J}(\cdot, y)F(y)\ d\omega(y) + \int_\Sigma \Psi^i_{\tau_J}(\cdot, y)F(y)\ d\omega(y) \tag{69}$$

i.e.

$$P^i_{\tau_{J+1}}(F) = P^i_{\tau_J}(F) + R^i_{\tau_J}(F) \tag{70}$$

for all $J \in \mathbb{Z}$ it easily follows that

$$\mathcal{V}^i_{\tau_{j+1}} = \mathcal{V}^i_{\tau_j} + \mathcal{W}^i_{\tau_j}\ . \tag{71}$$

However, it should be remarked that the sum (71) generally is neither direct nor orthogonal. The equation (71) may be interpreted in the following way: The set $\mathcal{V}^i_{\tau_j}$ contains a $P^i_{\tau_j}-$filtered version of a function belonging to the class $\mathrm{L}^2(\Sigma)$. The lower the scale, the stronger the intensity of filtering. By adding '$R^i_{\tau_j}-$details' contained in the space $\mathcal{W}^i_{\tau_j}$ the space $\mathcal{V}^i_{\tau_{j+1}}$ is created, which consists of a filtered versions at resolution $j+1$. Obviously, for $i = 2, 3, 5, 6$,

$$\overline{\bigcup_{j=-\infty}^{\infty} \mathcal{V}^i_{\tau_j}}^{\|\cdot\|_{\mathrm{L}^2(\Sigma)}} = \mathrm{L}^2(\Sigma)\ .$$

Our purpose is to establish a multiresolution analysis for the $\Sigma-$wavelet function.

Definition 6.10. A family of subspaces $\{V^i_\tau(\Sigma)\}_{\tau\in(0,\infty)} \subset \mathrm{L}^2(\Sigma)$, $i \in \{1, \dots, 9\}$, is called a *multiresolution* analysis if it satisfies the following properties:

(i) $\{0\} \subset V^i_\tau(\Sigma) \subset V^i_{\tau'}(\Sigma) \subset \mathrm{L}^2(\Sigma)$ for $0 \le \tau' \le \tau \le \infty$,

(ii) $\{\lim_{\tau\to\infty}\left(\int_\Sigma \Phi^i_\tau(., y)F(y)\ \ d\omega(y)\right)|F \in \mathrm{L}^2(\Sigma)\} = \{0\}$,

(iii) $\overline{\{F \in \mathrm{L}^2(\Sigma)|F \in V^i_\tau(\Sigma) \text{ for some } \tau \in (0,\infty)\}}^{\|\cdot\|_{\mathrm{L}^2(\Sigma)}} = \mathrm{L}^2(\Sigma)$.

The following lemma summarizes results which were listed in the previous section.

Lemma 6.11. *For the scale spaces* V^i_τ, $i = 5, 6$, *of the* $\Sigma-$*scaling function of type* 5 *and* 6 *defined in* (67), *respectively, the following statements are true:*

(i) $V^i_\tau \subset \mathrm{L}^2(\Sigma)$ *for all* $\tau \in (,\infty)$,

(ii) $\{\lim_{\tau\to\infty}\left(\int_\Sigma \Phi^i_\tau(., y)F(y)\ \ d\omega(y)\right)|F \in \mathrm{L}^2(\Sigma)\} = \{0\}$,

(iii) V^i_τ *is a linear subspace of* $\mathrm{L}^2(\Sigma)$,

(iv) $\overline{\{F \in \mathrm{L}^2(\Sigma)|F \in V^i_\tau(\Sigma) \textit{ for some } \tau \in (0,\infty)\}}^{\|\cdot\|_{\mathrm{L}^2(\Sigma)}} = \mathrm{L}^2(\Sigma)$.

Proof. Statement (i) is clear by the definition of the scale spaces. Moreover, statement (ii) follows from the fact that the $\Sigma-$scaling functions of type 5 and 6 tend to 0 for $\tau \to \infty$. Finally, property (iii) is a result of the linearity of the integral, while (iv) has been shown in the last section. □

It should be noted that for the case of Σ to be a sphere property (i) of Definition 6.10 has been shown in [13] by transcribing the scaling functions and wavelets generated by layer potentials into a nomenclature of the spherical wavelet theory. Spherical wavelets generated by layer potentials of this type will be discussed in a future master thesis in the context of solving the Stokes problem of physical geodesy.

6.5 A Tree Algorithm

Next we deal with some aspects of scientific computing (for a similar approach in spherical theory see [12]). We are interested in a *pyramid scheme* for the (approximate) recursive computation of the integrals $P^i_{\tau_j}(F)$ and $R^i_{\tau_j}(F)$ starting from an initial approximation of a given function $F \in \mathrm{L}^2(\Sigma)$. The tree algorithm (pyramid scheme) is based on the existence of a 'reproducing kernel function' on the regular surface Σ.
A pyramid scheme is a tree algorithm with the following ingredients. Starting from a sufficiently large $J \in \mathbb{N}$ such that for all $x \in \Sigma$

$$P^i_{\tau_J}(F) \simeq \sum_{k=1}^{N_J} a_k^{N_j} \Phi^i_{\tau_J}(x, y_k^{N_J}) \simeq A_i(F), \quad i = 1, \ldots, 7 \tag{72}$$

we want to calculate coefficients

$$a^{N_j} \in \mathbb{R}^{N_j}, a^{N_j} = \left(a_1^{N_j}, \ldots, a_{N_j}^{N_j}\right)^T, \quad j = J_0, \ldots, J,$$

such that the following statements hold true:

1. The vectors a^{N_j}, $j = J_0, \ldots, J-1$, are obtainable by recursion starting from the vector a^{N_J}.
2. For $j = J_0, \ldots, J$

$$P^i_{\tau_j}(F)(x) = \int_\Sigma \Phi^i_{\tau_j}(x,y) F(y)\ d\omega(y) \simeq \sum_{k=1}^{N_j} a_k^{N_j} \Phi^i_{\tau_j}\left(x, y_k^{N_j}\right) \ .$$

For $j = J_0 + 1, \ldots, J$

$$R^i_{\tau_{j-1}}(F)(x) = \int_\Sigma \Psi^i_{\tau_{j-1}}(x,y) F(y)\ d\omega(y) \simeq \sum_{k=1}^{N_{j-1}} a_k^{N_{j-1}} \Psi^i_{\tau_{j-1}}\left(x, y_k^{N_{j-1}}\right) \ .$$

(the symbol '$\simeq$' always means that the error is assumed to be negligible).

In the scheme we base the numerical integration on certain approximate formulae associated to known weights $w_k^{N_j} \in \mathbb{R}$ and prescribed knots $y_k^{N_j} \in \Sigma$, $j = J_0, \ldots, J$. This may be established, for example, by transforming the integrals over the regular surface Σ to integrals over the (unit) sphere Ω in case an explicit transformation $\Theta : \Omega \to \Sigma$ is given (see [6], [26]). Note that j denotes the scale of the discretized scaling function, N_j is the number of integration points to the accompanying scale j, and k denotes the index of the integration knot within the integration formulae under consideration, i.e.:

$$P^i_{\tau_j}(F)(x) \simeq \sum_{k=1}^{N_j} w_k^{N_j} F\left(y_k^{N_j}\right) \Phi^i_{\tau_j}\left(x, y_k^{N_j}\right), \tag{73}$$

$j = J_0, \dots J$,

$$R^i_{\tau_{j-1}}(F)(x) \simeq \sum_{k=1}^{N_{j-1}} w_k^{N_{j-1}} F\left(y_k^{N_{j-1}}\right) \Psi^i_{\tau_{j-1}}\left(x, y_k^{N_{j-1}}\right), \tag{74}$$

$j = J_0 + 1, \dots, J$.

The pyramid scheme – as every recursive implementation – is divided into two parts, the initial step and the recursion step, here called the pyramid step.

Initial Step. For a suitable large integer J, $P^i_{\tau_J}(x)$ is sufficiently close to the right hand side of (72) for all $x \in \Sigma$. Thus we simply get by (73)

$$a_k^{N_J} = w_k^{N_J} F(y_k^{N_J}), \quad k = 1, \dots, N_J \ . \tag{75}$$

Pyramid Step. The essential idea for the development of our recursive scheme is the existence of a (symmetric) kernel function $\Xi^i_j : \Sigma \times \Sigma \to \mathbb{R}$ such that

$$\Phi^i_{\tau_j}(x,y) \simeq \int_\Sigma \Phi^i_{\tau_j}(z,x) \Xi^i_j(y,z) \ d\omega(z) \tag{76}$$

and

$$\Xi^i_j(x,y) \simeq \int_\Sigma \Xi^i_j(z,x) \Xi^i_{j+1}(y,z) \ d\omega(z) \tag{77}$$

for $j = J_0, \dots, J$.

Since our scaling functions are non–bandlimited, the scale spaces $V^i_{\tau_j}$ are infinite–dimensional. This leads us to choose the functions Ξ_j, for example, to be equal to

$$\Xi^i_j = \Phi^l_{\tau_{J+L}}, \quad j = J_0, \dots, J; \ l \in \{2,3,5,6\} \ .$$

for suitable $L \in \mathbb{N}_0$. By virtue of the approximate integration rules on the sphere we thus get

$$\begin{aligned} \int_\Sigma \Phi^i_{\tau_j}(\cdot, y) F(y) \ d\omega(y) &\simeq \int_\Sigma \Xi^i_j(y,z) \int_\Sigma \Phi^i_{\tau_j}(\cdot, z) F(y) \ d\omega(z) \ d\omega(y) \\ &\simeq \int_\Sigma \Phi^i_{\tau_j}(\cdot, z) \int_\Sigma \Xi^i_j(y,z) F(y) \ d\omega(y) \ d\omega(z) \\ &\simeq \sum_{k=1}^{N_j} a_k^{N_j} \Phi^i_{\tau_j}(\cdot, y_k^{N_j}) \end{aligned} \tag{78}$$

for $j = J_0, \dots, J-1$, where we have set

$$a_k^{N_j} = w_k^{N_j} \int_\Sigma \Xi^i_j\left(y_k^{N_j}, y\right) F(y) \ d\omega(y) \tag{79}$$

for $j = J_0, \dots, J-1$ and $k = 1, \dots, N_j$. Hence, in connection with (78), we find

$$\begin{aligned} a_k^{N_j} &= w_k^{N_j} \int_\Sigma \Xi^i_j\left(y_k^{N_j}, y\right) F(y) \ d\omega(y) \\ &\simeq w_k^{N_j} \int_\Sigma \int_\Sigma \Xi^i_{j+1}(z,y) \Xi^i_j(y_k^{N_j}, z) \ d\omega(z) F(y) \ d\omega(y) \\ &\simeq w_k^{N_j} \sum_{l=1}^{N_{j+1}} w_l^{N_{j+1}} \Xi^i_j\left(y_k^{N_j}, y_l^{N_{j+1}}\right) \int_\Sigma \Xi^i_{j+1}\left(y_l^{N_{j+1}}, y\right) F(y) \ d\omega(y) \\ &= w_k^{N_j} \sum_{l=1}^{N_{j+1}} w_l^{N_{j+1}} \Xi_j\left(y_k^{N_j}, y_l^{N_{j+1}}\right) a_l^{N_{j+1}} \ . \end{aligned} \tag{80}$$

for $j = J-1, \ldots, J_0$ and $k = 1, \ldots, N_j$.

We see that the coefficients $a_k^{N_{J-1}}$ can be calculated recursively from $a_l^{N_J}$ for the initial level J, $a_k^{N_{J-2}}$ can be deduced from $a_l^{N_{J-1}}$, etc. Finally, we get as a reconstruction scheme

$$P^i_{\tau_j}(F) \simeq \sum_{k=1}^{N_j} a_k^{N_j} \Phi^i_{\tau_j}\left(\cdot, y_k^{N_j}\right), \quad j = J_0, \ldots, J, \tag{81}$$

$$R^i_{\tau_{j-1}}(F) \simeq \sum_{k=1}^{N_{j-1}} a_k^{N_{j-1}} \Psi^i_{\tau_{j-1}}\left(\cdot, y_k^{N_{j-1}}\right), \quad j = J_0+1, \ldots, J\ . \tag{82}$$

Note that the coefficients a^{N_J} in the initial step do not depend on the choice of $\Xi^i_J = \Phi^l_{\tau_{J+L}}$. Furthermore, the functions Ξ^i_j, $j = J_0, \ldots, J_{-1}$, can be chosen independently of the scaling function $\{\Phi^i_{\tau_j}\}_{j\in\mathbb{Z}}$ used in (81) and (82).

Table 2: Pyramid Scheme (Tree Algorithm)

Initial step: For J sufficiently large

$$a_k^{N_J} = w_k^{N_J}\, F(y_k^{N_J}), \quad k = 1, \ldots, N_J$$

Pyramid step: For $j = J_0, \ldots, J-1$ and $k = 1, \ldots, N_j$

$$a_k^{N_j} = w_k^{N_j} \sum_{l=1}^{N_{j+1}} \Xi^i_j\left(y_k^{N_j}, y_l^{N_{j+1}}\right) a_l^{N_{j+1}}$$

In conclusion, the above considerations lead us to the following decomposition and reconstruction scheme:

$$\begin{array}{ccccccccccccc}
F & \rightarrow & a^{N_J} & \rightarrow & a^{N_{J-1}} & \rightarrow & \ldots & \rightarrow & a^{N_{J_0+1}} & \rightarrow & a^{N_{J_0}} & \rightarrow & P^i_{\tau_{J_0}}(F) \\
 & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \\
 & & R^i_{\tau_J}(F) & & R^i_{\tau_{J-1}}(F) & & & & R^i_{\tau_{J_0+1}}(F) & & R^i_{\tau_{J_0}}(F) & &
\end{array}$$

(decomposition scheme)

$$\begin{array}{ccccccccccccc}
a^{N_{J_0}} & & & & a^{N_{J_0+1}} & & & & a^{N_{J-1}} & & & & \\
\downarrow & & & & \downarrow & & & & \downarrow & & & & \\
R^i_{\tau_{J_0}}(F) & & & & R^i_{\tau_{J_0+1}}(F) & & & & R^i_{\tau_{J-1}}(F) & & & & \\
 & \searrow & & & & \searrow & & & & \searrow & & & \\
P^i_{\tau_{J_0}}(F) & \rightarrow & + & \rightarrow & P^i_{\tau_{J_0+1}} & \rightarrow & + & \rightarrow & P^i_{\tau_{J-1}}(F) & \rightarrow & + & \rightarrow & P_{\tau_J}(F)
\end{array}$$

(reconstruction scheme).

The numerical effort of a pyramid step can drastically be reduced by use of a panel–clustering method (e.g. fast multipole procedures as developed by [18]). In doing so, the evaluations take advantage of the localizing structure of the kernels Ξ^i_j. Roughly spoken, the kernel is split into a near field and a far field component. The far field component is approximated by a certain expression obtaining the 'low frequency contributions'. For the points close to the evaluation position the evaluation uses the exact near field of the kernel. For the remaining points, the approximate far field contributions are put together.

The numerical results (based on the types $i = 5, 6$) presented in the Diploma thesis [26] illustrate the efficiency and economy in applications of our wavelet method for different types of regular surfaces (e.g. sphere, ellipsoid, Cassini's surfaces, etc). Figure 7 demonstrates the functionality of the multiscale analysis. The mechanism is as follows: To a scale–reconstruction at scale $J = 5$ the detail–structure at scale $J = 5$ has to be added to get the scale–reconstruction at scale $J = 6$. This can be done globally as shown in Figure 7 or locally as shown in Figure 8 without getting any oscillations because of the space–localizing properties of the scaling functions.
Finally three important applications of wavelet decomposition and reconstruction should be mentioned:

(i) The 'zoom–in' property allows a local high–scale reconstruction of fine structure based on global data. For the evaluation of a functional value under consideration, only wavelet coefficients close to the point have to be taken into account. This aspect of functional evaluation enables us to derive local features within a global model. This is demonstrated in Figure 6 by a reconstruction of the EGM96–geopotential model [24] on the reference ellipsoid from discrete data in local areas (for example, South–America).

(ii) Because of the space localizing property of scaling functions and wavelets it is possible to perform a local multiresolution analysis. This is done in Figure 8 using the EGM96–geopotential model over Italy on the ellipsoidal telluroid.

(iii) In Figure 9 the detection of a high frequency phenomena is demonstrated. We added within the EGM96–model a mass point lying $63km$ under the (ellipsoidal) Earth's surface and at 80^o West and 30^o South to the EGM96 disturbing potential model. 50% of the total energy of this mass point are located in a spherical cap with a diameter of 8^o. It is well known that phenomena with such short wavelengths cannot be detected with the spherical or ellipsoidal harmonic techniques known in the literature. A maximum spherical harmonic degree of $N = 300$ would be necessary to resolve such small scale phenomena.

In conclusion, as mentioned in our introduction, three essential features are incorporated in this way of thinking about wavelets generated by layer potentials, namely the basis property, the zoom–in ability, and fast computation. In particular, these facts justify the characterization of our wavelets as 'building blocks' that enable fast decorrelation of data given on a regular surface.

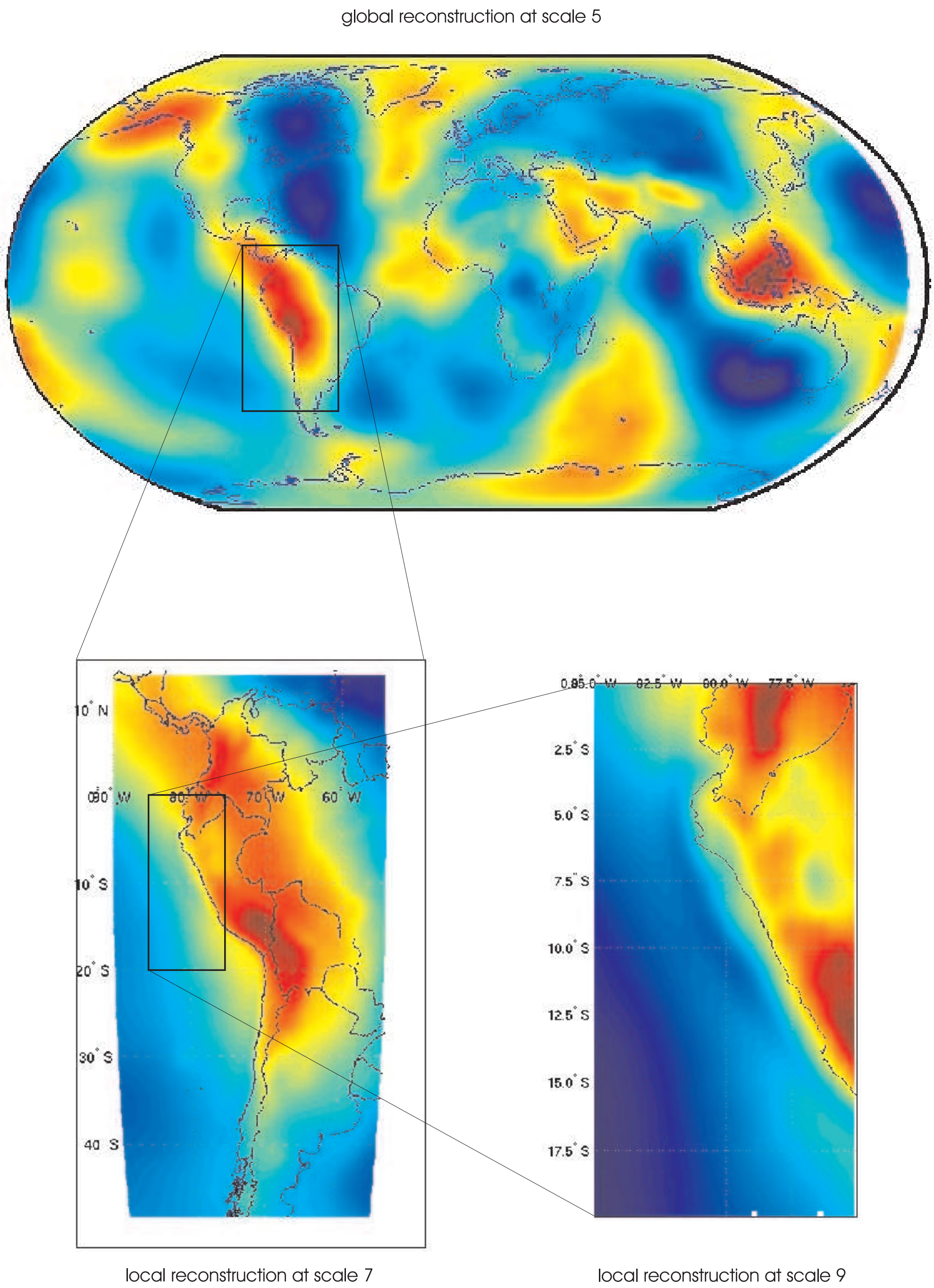
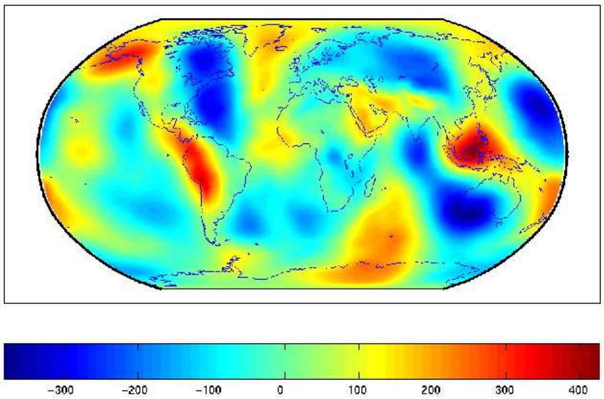
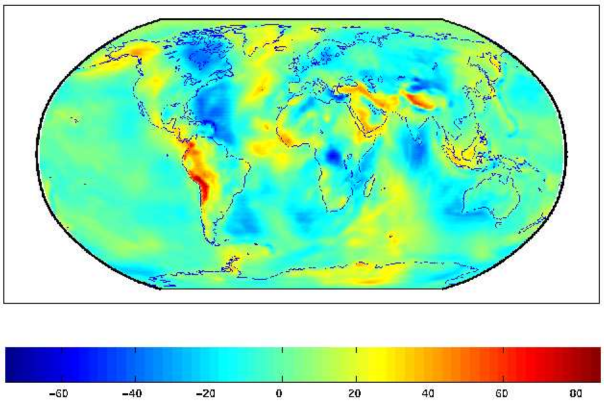


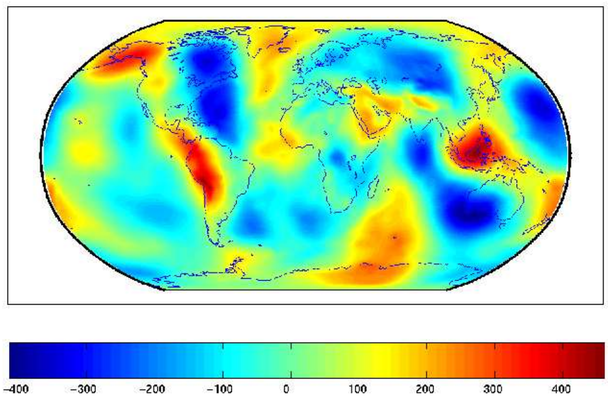
Figure 6: Illustration of the zoom–in property. In order to reconstruct a function on a local area, only data in a certain neighborhood of this area are used. Since global high–scale reconstruction of fine structure is very time–consuming, only the area of interest is reconstructed which can be done with a considerably fewer effort.



(a) global scale–reconstruction at scale $J = 5$

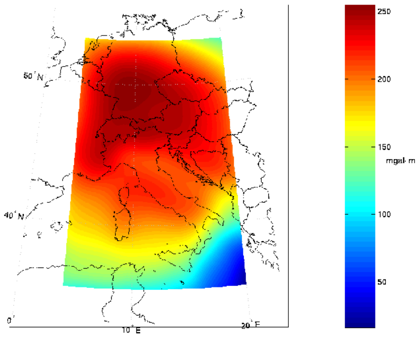


(b) global detail–reconstruction at scale $J = 5$

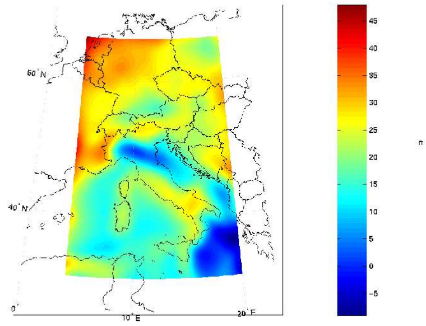


(c) global scale–reconstruction at scale $J = 6$

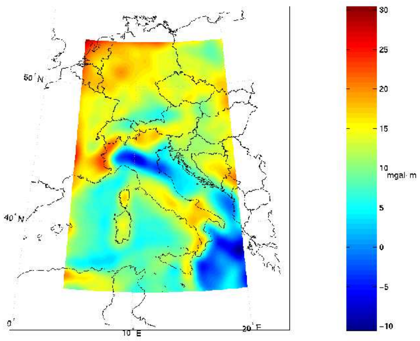
Figure 7: Scale–reconstruction at scale $J = 6$ (c) consists of detail–reconstruction at scale $J = 5$ (b) added to scale–reconstruction at scale $J = 5$ (a).



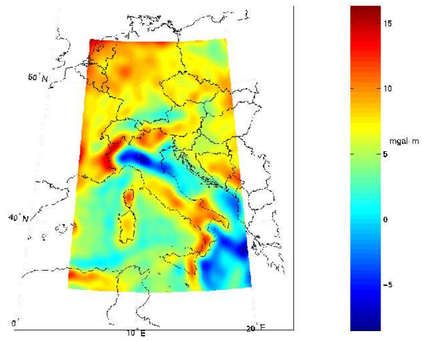
(a) local scale reconstruction with $\Sigma-$scaling function $\Phi^6_{\tau_j}$ at scale $j = 6$



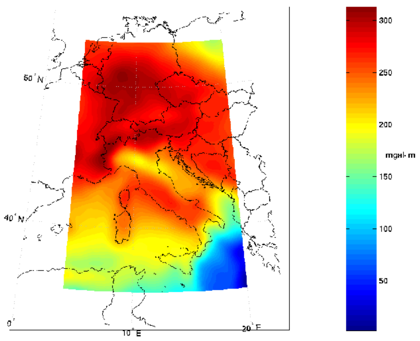
(b) local wavelet reconstruction with $\Sigma-$wavelet $\Psi^6_{\tau_j}$ at scale $j = 6$



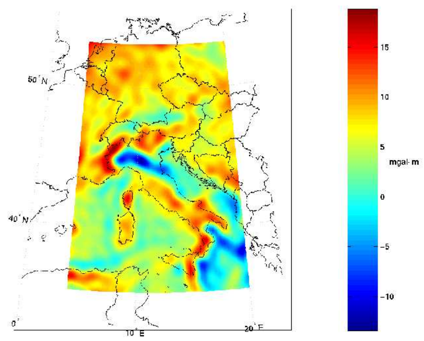
(c) local wavelet reconstruction with $\Sigma-$wavelet $\Psi^6_{\tau_j}$ at scale $j = 7$



(d) local wavelet reconstruction with $\Sigma-$wavelet $\Psi^6_{\tau_j}$ at scale $j = 8$

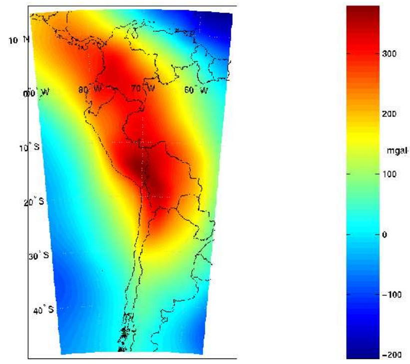


(e) local scale reconstruction with $\Sigma-$scaling function $\Phi^6_{\tau_j}$ at scale $j = 9$

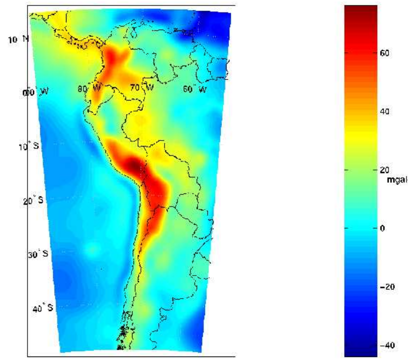


(f) error of the local scale reconstruction at scale $j = 9$ to the EGM96 model

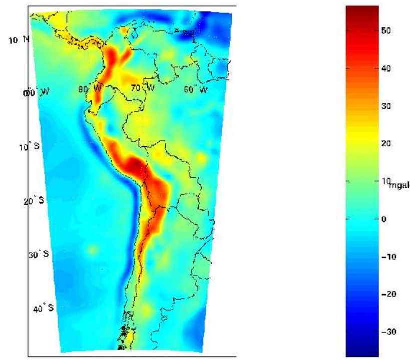
Figure 8: Local multiresolution analysis with $\Sigma-$scaling functions and wavelets of the EGM96–geopotential model over Italy.



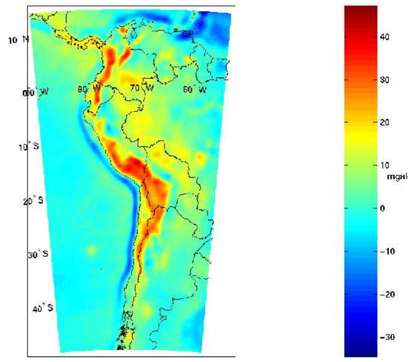
(a) local reconstruction with Σ−scaling function $\Phi^6_{\tau_j}$ at scale $j = 5$



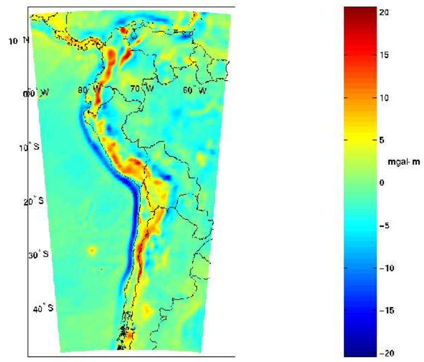
(b) local reconstruction with Σ−wavelet function $\Psi^6_{\tau_j}$ at scale $j = 5$



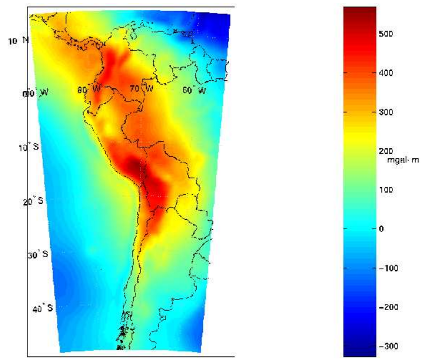
(c) local reconstruction with Σ−wavelet function $\Psi^6_{\tau_j}$ at scale $j = 6$



(d) local reconstruction with Σ−wavelet function $\Psi^6_{\tau_j}$ at scale $j = 7$



(e) local reconstruction with Σ−scaling function $\Psi^6_{\tau_j}$ at scale $j = 8$



(f) local reconstruction with Σ−scaling function $\Phi^6_{\tau_j}$ at scale $j = 9$

Figure 9: Detection of high frequency perturbation within a local area of the EGM96–geopotential model. The buried mass point at 80^o West, 30^o South is clearly detected, especially in the wavelet reconstruction at scale 8.

7 Multiscale Modelling of Boundary Value Problems

(EMP) For given $F \in \mathrm{C}^{(0)}(\Sigma)$, the solution $T \in \mathrm{Pot}^{(1)}(\overline{\Sigma_{\mathrm{ext}}})$ with $\frac{\partial T^+}{\partial \nu_\Sigma} = F$ of the exterior Molodensky problem (EMP) can be written as layer potential (38), where the single layer $\mu \in \mathcal{C}^{(0)}(\Sigma)$ satisfies the integral equation

$$-2\pi\mu(x) + \int_\Sigma \left(\Phi^8_{\tau_L}(x,y) + \lambda(x)\Phi^1_{\tau_L}(x,y)\right)\mu(y)\ d\omega(y) \simeq F(x) \tag{83}$$

for all $x \in \Sigma$ ($L \in \mathbb{N}$ sufficiently large). An approximation of scale J

$$\hat{P}^i_{\tau_J}(Q)(x) = \sum_{l=1}^{N_J} \hat{a}^{N_J}_l \Phi^i_{\tau_J}\left(x, \hat{y}^{N_J}_l\right), \quad x \in \Sigma \tag{84}$$

(with $i \in \{2,3,5,6\}$, $\hat{a}^{N_J}_l \in \mathbb{R}$, $\hat{y}^{N_J}_l \in \Sigma$, $l = 1,\dots,N_J$ and $J, N_J \in \mathbb{N}$ sufficiently large) is deducible from (83) by solving a system of linear equations obtained by an appropriate approximation method such as collocation, Galerkin procedure, least squares approximation, etc .

For solving the linear systems fast multipole methods (FMM) are applicable (see e.g. [18]). The aforementioned observations concerning the (exterior) Molodensky problem lead us to tree algorithms with the following ingredients:

Starting from $\hat{a}^{N_J} \in \mathbb{R}^{N_J}, \hat{a}^{N_J} = \left(\hat{a}^{N_J}_1, \dots, \hat{a}^{N_J}_{N_J}\right)^T$, the coefficients

$$\hat{a}^{N_j} \in \mathbb{R}^{N_j}, \hat{a}^{N_j} = \left(\hat{a}^{N_j}_1, \dots, \hat{a}^{N_j}_{N_j}\right)^T, j = J_0, \dots, J-1, \tag{85}$$

are determined such that the following rules hold true:

1. The vectors $\hat{a}^{N_j}, j = J_0, \dots, J-1$ are given by recursion (see Section 6.5)

$$\hat{a}^{N_j}_k = w^{N_j}_k \sum_{l=1}^{N_{j+1}} \Xi^i_j\left(\hat{y}^{N_j}_k, y^{N_{j+1}}_l\right) \hat{a}^{N_{j+1}}_l. \tag{86}$$

2. For $j = J_0, \dots J$

$$\hat{P}^i_{\tau_j}(Q)(x) \simeq \sum_{k=1}^{N_j} \hat{a}^{N_j}_k \Phi^i_{\tau_j}\left(x, \hat{y}^{N_j}_k\right), \quad x \in \Sigma\ . \tag{87}$$

   For $j = J_0 + 1, \dots J$

$$\hat{R}^i_{\tau_{j-1}}(Q)(x) \simeq \sum_{k=1}^{N_j} \hat{a}^{N_j}_k \Psi^i_{\tau_{j-1}}\left(x, \hat{y}^{N_j}_k\right), \quad x \in \Sigma \tag{88}$$

   where

$$\hat{R}^i_{\tau_{j-1}}(Q)(x) = \hat{P}^i_{\tau_j}(Q)(x) - \hat{P}^i_{\tau_{j-1}}(Q)(x)\ . \tag{89}$$

THEOREM 7.1. *Let Σ be a regular surface such that* (1) *holds true. For given $F \in \mathcal{C}^{(0)}(\Sigma)$, let U be the potential of class $\mathrm{Pot}^{(1)}(\overline{\Sigma_{\mathrm{ext}}})$ with $\frac{\partial T^+}{\partial \nu_\Sigma} + \lambda T^+ = F$. The function $F_J \in \mathcal{C}^{(0)}(\Sigma)$*

given by

$$\begin{aligned} F_J(x) &= \sum_{l=1}^{N_{J_0}} \hat{a}_l^{N_{J_0}} \left(-2\pi \Phi^i_{\tau_{J_0}} \left(x, \hat{y}_l^{N_{J_0}} \right) + \int_\Sigma \Phi^8_{\tau_L}(x,y) \Phi^i_{\tau_{J_0}} \left(y, \hat{y}_l^{N_{J_0}} \right) d\omega(y) \right. \\ & \qquad \left. \lambda(x) \int_\Sigma \Phi^1_{\tau_L}(x,y) \Phi^i_{\tau_{J_0}} \left(y, \hat{y}_l^{N_{J_0}} \right) d\omega(y) \right) \\ &+ \sum_{j=J_0}^{J-1} \sum_{l=1}^{N_{j+1}} \hat{a}_l^{N_{j+1}} \left(-2\pi \Psi^i_{\tau_{j+1}} \left(x, \hat{y}_l^{N_{j+1}} \right) + \int_\Sigma \Phi^8_{\tau_L}(x,y) \Psi^i_{\tau_{j+1}} \left(y, \hat{y}_l^{N_{j+1}} \right) d\omega(y) \right. \\ & \qquad \left. \lambda(x) \int_\Sigma \Phi^1_{\tau_L}(x,y) \Psi^i_{\tau_{j+1}} \left(y, \hat{y}_l^{N_{j+1}} \right) d\omega(y) \right) \end{aligned} \tag{90}$$

$x \in \Sigma$, *represents a* $J-$*scale approximation of* $F \in \mathcal{C}^{(0)}(\Sigma)$ *in the* $\|\cdot\|_{\mathrm{L}^2(\Sigma)}-$*sense, while* $U_J \in \mathrm{Pot}^{(0)}(\overline{\Sigma_{\mathrm{ext}}})$ *given by*

$$\begin{aligned} U_J(x) &= \sum_{l=1}^{N_{J_0}} \hat{a}_l^{N_{J_0}} \int_\Sigma \Phi^i_{\tau_{J_0}} \left(y, \hat{y}_l^{N_{J_0}} \right) \frac{1}{|x-y|} d\omega(y) \\ &+ \sum_{j=J_0}^{J-1} \sum_{l=1}^{N_{j+1}} \hat{a}_l^{N_{j+1}} \int_\Sigma \Psi^i_{\tau_j} \left(y, \hat{y}_l^{N_{j+1}} \right) \frac{1}{|x-y|} d\omega(y) \end{aligned}$$

represents a $J-$*scale approximation of* U *in the* $\|\cdot\|_{\mathcal{C}^{(0)}(\overline{K})}-$*sense for every* $K \subset \Sigma_{\mathrm{ext}}$ *with* $\mathrm{dist}(\overline{K}, \Sigma) > 0$. *Furthermore*

$$\sup_{x \in \overline{K}} \left| \nabla^{(k)} U(x) - \nabla^{(k)} U_J(x) \right| \leq C \, \|F - F_J\|_{\mathrm{L}^2(\Sigma)} \tag{91}$$

for all $k \in \mathbb{N}_0$.

In other words, the tree algorithm developed above uses an approximation method by solving a linear system for the initial step and integration rules with known weights and knots for the subsequent pyramid steps.

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References

[1] D. Colton, R. Kress, *Integral Equation Methods in Scattering Theory* (John Wiley & Sons, New York 1983).

[2] W. Dahmen, Wavelet and Multiscale Methods for Operator Equations, Acta Numerica, Cambridge University Press, (1997) 55-228.

[3] P.J. Davis, *Interpolation and Approximation* (Blaisdell Publishing Company, 1963).

[4] R. Klees, M. van Gelderen, C. Lage, C. Schwab, Fast Numerical Solution of the Linearized Molodensky Problem, Research Report No. 99-16, Seminar f"ur Angewandte Mathematik, ETH Z"urich, 1999.

[5] W. Freeden, On the Approximation of External Gravitational Potential With Closed Systems of (Trial) Functions, Bull. Géod. **54,** (1980) 1-20.

[6] W. Freeden, Über die Gausssche Methode zur angenährten Berechnung von Integralen, Math. Meth. in the Appl. Sci. **2,** (1980) 397-409.

[7] W. Freeden, M. Schreiner, R. Franke, A Survey on Spherical Spline Approximation, Surv. Math. **7,** (1996) 29-85.

[8] W. Freeden, U. Windheuser, Spherical Wavelet Transform and Its Discretization, Adv. Comput. Math. **5,** (1996) 51-94.

[9] W. Freeden, T. Gervens, M. Schreiner, *Constructive Approximation on the Sphere (With Applications to Geomathematics)* (Oxford Science Publications, Clarendon 1998).

[10] W. Freeden, *Multiscale Modelling of Spaceborn Geodata*, (B.G. Teubner, Stuttgart, Leipzig 1999).

[11] W. Freeden, K. Hesse, Spline Modelling of Geostrophic Flow: Theoretical and Algorithmic Aspects, Bericht der Arbeitsgruppe Technomathematik Nr 250, TU Kaiserslautern, 2002.

[12] W. Freeden, T. Maier, On Multiscale Denoising of Spherical Functions: Basic Theory and Numerical Aspects. Electronic Transactions on Numerical Analysis (ETNA), **14,** (2002) 40-62.

[13] W. Freeden, C. Mayer, Wavelets Generated by Layer Potentials. Applied Computational Harmonic Analysis (ACHA), **14,** (2003), 195-237.

[14] W. Freeden, V. Michel, Multiscale Potential Theory (With Applications to Geoscience), Birkhäuser, Boston, 2004.

[15] M. Gutting, Multiscale Gravitational Field Modeling from Oblique Derivatives, Diploma Thesis, Department of Mathematics, Geomathematics Group, University of Kaiserslautern, 2002.

[16] W. A. Heiskanen, H. Moritz, *Physical Geodesy*, (W. H. Freeman and Company 1967).

[17] Z. Martinec, *Boundary-value Problems for Gravimetric Determination of a Precise Geoid*, (Springer, Berlin, Heidelberg, New York 1999).

[18] O. Glockner, *On Numerical Aspects of Gravitational Field Modeling from SST and SGG by Harmonic Splines and Wavelets (With Application to CHAMP Data)*, Ph.D-Thesis, Department of Mathematics, Geomathematics Group, Shaker, Aachen, 2001.

[19] K. Hesse, *Domain Decomposition Methods in Multiscale Geopotential Determination from SST and SGG*, Ph.D-Thesis, Department of Mathematics, Geomathematics Group, Shaker, Aachen, 2003.

[20] L. Hörmander, *The Boundary Problems of Physical Geodesy*, Arch. Rat. Mech. Anal., **62**, 1-52, 1976.

[21] O.D. Kellogg, *Foundations of Potential Theory* (Frederick Ungar Publishing Company, 1929).

[22] H. Kersten, Grenz- und Sprungrelationen für Potentiale mit Quadratsummierbarer Flächenbelegung, Resultate der Mathematik **3,** (1980) 17-24.

[23] P.D. Lax, Symmetrizable Linear Transformations, Comm. Pure Appl. Math. **7,** (1954) 633-647.

[24] F.G. Lemoine, S.C. Kenyon, J.K. Factor, R.G. Trimmer, N.K. Pavlis, D.S. Chinn, C.M. Cox, S.M. Klosko, S.B. Luthcke, M.H. Torrence, Y.M. Wang, R.G. Williamson, E.C. Pavlis, R.H. Rapp, T.R. Olson, *The Development of the Doint NASA GSFC and NIMA Geopotential Model EGM96*, NASA/TP-1998-206861, (1998).

[25] E. Martensen, *Potentialtheorie* (B.G. Teubner, Stuttgart, 1968).

[26] C. Mayer, *Potential Operators, Jump Relations and Wavelets* (Diploma Thesis, Department of Mathematics, Geomathematics Group, University of Kaiserslautern, 2001).

[27] S.G. Michlin, *Lehrgang der Mathematischen Physik, 2nd edition* (Akademie Verlag, Berlin 1975).

[28] H. Moritz, *Recent Developments in the Geodetic Boundary Value Problem*, Report of the Department of Geodetic Science No. 266, Ohio State University, Columbus, Ohio, 1977.

[29] C. Müller, *Spherical Harmonics, Lecture Notes in Mathematics, 17,* (Springer, Berlin 1966).

[30] C. Müller, *Foundations of the Mathematical Theory of Electromagnetic Waves* (Springer, Berlin 1969).

[31] J. Otero, *A Uniqueness Theorem for a Robin Boundary Value Problem in Physical Geodesy*, Quart. of Appl. Math., **56**2, 245-257, 1998.

[32] S. Ritter, The Nullfield Method for the Ellipsoidal Stokes Problem, J. of Geodesy **72,** (1998) 101-106.

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