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Initial Temperature Reconstruction for a Nonlinear Heat Equation: Application to Radiative Heat Transfer.

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Abstract

Consider a cooling process described by a nonlinear heat equation. We are interested to recover the initial temperature from temperature measurements which are available on a part of the boundary for some time. Up to now even for the linear heat equation such problem has been usually studied as a nonlinear ill-posed operator equation, and regularization methods involving Fréchet derivatives have been applied. We propose a fast derivative-free iterative method. Numerical results are presented for the glass cooling process, where nonlinearity appears due to radiation.

1 Introduction

In many cooling processes (for example in glass production [15, 13], polymer processing [10, 9]) it is important to avoid big temperature differences inside the material. However, measurements of the temperature inside the material is usually extremely difficult and sometimes even impossible. On the other side, the boundary temperature measurements can be made much easier. Thus, we are faced with the problem of reconstructing the temperature inside the material from the temperature on its boundary.

We will consider a cooling process, which is modelled by the heat equation with a source term nonlinearly depended on the temperature. Since the temperature in the body is uniquely determined by the initial temperature, it is sufficient to reconstruct only the initial temperature.

Previously, reconstruction of the initial temperature from boundary measurements was considered only for the linear heat equation (see [10, 9, 6]) with source terms independent of the temperature. In [10, 9] the problem was formulated as a constrained minimization problem. Minimization was performed by the conjugate gradient method, resulting in the necessity to compute Fréchet derivatives. Extension of this idea to a nonlinear heat equation results in both theoretical and computational difficulties. In [6] the initial temperature was considered as the solution to some operator equation. Although the original equation was nonlinear, using a proper affine decomposition [4] it was reduced to a linear equation. We propose to use this reduction in constructing an iterative method for our problem. This method does not involve computation of derivatives. Each iteration consists of solving an ill-posed linear equation with a fixed operator, which has to be regularized. The proposed method does not depend on the concrete type of the nonlinear source term and can be easily applied for multidimensional problems.

The structure of the paper is as follows. In Section 2 we present the model for the cooling of a glass plate, which leads to a nonlinear heat equation in 1D due to heat radiation effects. We use this model as a prototype for a general nonlinear heat equation. We also formulate the nonlinear operator equation for the initial temperature. The proposed iterative method and regularization issues are presented in Section 3. Numerical results are presented in Section 4. Finally we make conclusions and give an outlook in Section 5.

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2 Problem Formulation

2.1 Mathematical model for the cooling of a glass plate

Let us first present the mathematical model for the cooling of a hot glass plate. Since the thickness of the plate is smaller then its height and width, it is reasonable to consider a spatial dependence of the plate temperature only on its thickness [16, 3, 8]. Thus, the temperature $T(z,t)$ is a function of the thickness $z \in [0,l]$ and time $t \in [0,t_f]$.

Note that for the processes involving high temperatures, heat transfer by radiation needs to be taken into account. The energy gain by radiation is characterized by the so-called intensity $I(\vec{r},t,\vec{s},\nu)$, which is, in general, a function of position $\vec{r}$, time t, direction $\vec{s}$ and frequency ν. In analogy we may assume that the intensity depends actually only on thickness z and direction coordinate $\mu \in [-1,1]$ along the thickness. The energy loss by radiation is characterized by Planck's function:

$$P(T,\nu) = \frac{2\,h_p\,\nu^3}{c^2}\,\left(e^{\frac{h_p\,\nu}{k_B\,T}} - 1\right)^{-1}$$

where h_p, k_B are Planck's and Boltzmann's constants, and c is the speed of light in the material.

In the heat equation the radiation effects are represented as an additional nonlinear source term:

$$c_m\,\rho_m\,\frac{\partial T}{\partial t} = k_h\,\frac{\partial^2 T}{\partial z^2} \;-\; 2\pi \int\limits_{[\nu_1,\infty)} \kappa(\nu) \left(\int\limits_{[-1,1]} \left(P(T,\nu) - I(z,t,\mu,\nu)\right)\,d\mu \right) d\nu. \tag{1}$$

where c_m, $\rho_m\,k_h$ are the specific heat, the density and the thermal conductivity of the material which are assumed to be constant. The function $\kappa(\nu)$ is the absorption coefficient defined in the so-called semitransparent frequency region $[\nu_1,\infty)$. Equation (1) is equipped with the following boundary and initial conditions:

$$\begin{array}{rcl} -k_h\,\frac{\partial T}{\partial z}(0,t) &=& \varepsilon\,\pi \int\limits_{[0,\nu_1]} (P(T_a,\nu) - P(T(0,t),\nu))d\nu \\ k_h\,\frac{\partial T}{\partial z}(l,t) &=& \varepsilon\,\pi \int\limits_{[0,\nu_1]} (P(T_a,\nu) - P(T(l,t),\nu))d\nu \\ T(z,0) &=& u(z) \end{array} \tag{2}$$

where ε is the hemispherical emissivity and T_a is the ambient temperature. In (2) we have neglected convective heat exchange along the boundary. This is justified for big temperatures, when radiative heat exchange is dominant.

The intensity $I(z,t,\mu,\nu)$ satisfies the radiative transfer equation:

$$\left\{ \begin{array}{rcl} \mu\,\frac{\partial I}{\partial z} &=& \kappa(\nu)(P(T,\nu) - I) \\ I(0,t,\mu,\nu) &=& P(T_a,\nu), \text{ for } \mu > 0 \\ I(l,t,\mu,\nu) &=& P(T_a,\nu), \text{ for } \mu < 0 \end{array} \right. \tag{3}$$

2.2 Inverse problem for a general heat equation with a nonlinear source term

Let us write (1) and (2) in the following scaled and compact form:

$$\left\{ \begin{array}{rcl} \dfrac{\partial T}{\partial t} &=& \dfrac{\partial^2 T}{\partial z^2} \;-\; F(T) \\ \\ n(z_b)\,\frac{\partial T}{\partial z}(z_b,t) &=& B(T(z_b,t)), \text{ for } z_b \in \{0,l\} \\ T(z,0) &=& u(z) \end{array} \right. \tag{4}$$

where $n(z_b) = -1$ for $z_b = 0$, and $n(z_b) = 1$ for $z_b = l$. The well-posedness of (4) has been rigorously shown for some particular cases, see [2, 7] and the references therein. One possible numerical method for the solution of (4) can be found in [1].

If the initial temperature were known, we would be able to obtain the temperature inside the material for later time. But the measurements of the temperature inside a hot body are extremely difficult, and

thus the initial data $u(z)$ is unknown. Fortunately, we have boundary temperature measurements at hand. The boundary temperature is often known not on the whole boundary but only on some part of it. Since in 1D the boundary consists only of two points $\{0, l\}$, we assume that the temperature $D(t)$ is known at the point $z_b = 0$ in the time interval $[0, t_f]$. Let $R : \mathcal{L}^2(0, l) \to \mathcal{L}^2(0, t_f)$ denote a nonlinear operator which maps the initial temperature to the boundary temperature. Then, the initial temperature $u(z)$ satisfies the nonlinear operator equation

$$R\,u \;=\; D. \tag{5}$$

In case of source and flux terms in (4) not depending on the temperature, the operator equation for the initial temperature was studied in [6]. Since we will use this equation in the iterative method for the solution of (5), we state it for later reference. For this purpose examine the linear heat equation

$$\begin{cases} \dfrac{\partial T}{\partial t} &= \dfrac{\partial^2 T}{\partial z^2} - f(z,t) \\ n(z_b)\,\frac{\partial T}{\partial z}(z_b,t) &= b(z_b,t), \text{ for } z_b \in \{0,l\} \\ T(z,0) &= u^*(z) \end{cases}$$

Denote the corresponding boundary temperature at $z_b = 0$ as $d(t)$ for $t \in [0, t_f]$. Consider an operator $L : \mathcal{L}^2(0, l) \times \mathscr{F} \times \mathscr{B} \to \mathcal{L}^2(0, t_f)$ which maps functions $u^*(z) \in \mathcal{L}^2(0, l)$, $f(z, t) \in \mathscr{F}$, $b(z_b, t) \in \mathscr{B}$ to the function $d(t)$, where $\mathscr{F}$ and $\mathscr{B}$ are admissible function spaces for source and flux terms. So, the initial temperature $u^*(z)$ satisfies the operator equation

$$L(u^*, f, b) \;=\; d. \tag{6}$$

If it holds that $f \neq 0$ or $b \neq 0$, then this equation is nonlinear with respect to u^*. In the next section we show how one can transform it to a linear equation. This transformation will be used in the iterative method for the solution of (5).

3 A Derivative-free Iterative Method

To transform equation (6) to a linear one, we use the following affine decomposition:

$$L(u^*, f, b) = L(u^*, 0, 0) + L(0, f, b). \tag{7}$$

The operator $L(u^*, 0, 0)$ depends linearly on u^* and equation (6) becomes

$$L(u^*, 0, 0) = d - L(0, f, b). \tag{8}$$

This approach was first proposed in [4] and numerically applied in [6]. But as it is documented in [12], in the case when u^* satisfies inhomogeneous boundary condition b, the numerical solution of (8) yields approximations with wrong fluxes at the part of the boundary where there are no measurements. Thus, instead of (7) we propose another affine decomposition

$$L(u^*, f, b) = L(u^* - v(b), 0, 0) + L(v(b), f, b), \tag{9}$$

where $v(b)$ is some function satisfying

$$n(z_b)\frac{d}{dz}v(b)(z_b) = b(z_b, 0) \text{ for } z_b \in \{0, l\}$$

Using (9) we rewrite (6) as

$$L(u^* - v(b), 0, 0) = d - L(v(b), f, b).$$

Now consider the nonlinear heat equation (4). First, note that the functions $F, B, v(B)$ can be considered as functions of the initial temperature u. Indeed, let $S : \mathcal{L}^2(0, l) \to \mathcal{L}^2(0, t_f)$ denote a solution operator which maps the initial temperature to the solution of (4). Then we define the operators:

$$\begin{aligned} F(T(z,t))) = F(\,S(u(z))\,) &=: \mathbf{F}(u(z)), \\ B(T(z,t))) = B(\,S(u(z))\,) &=: \mathbf{B}(u(z)), \\ v(B) = v(B(\,S(u(z))\,)) &=: \mathbf{v}(u(z)). \end{aligned}$$

The operator equation (5) can be written using the operator L as follow

$$L(u, \mathbf{F}(u), \mathbf{B}(u)) \;=\; D.$$

Employing (9) this equation becomes

$$L(u - \mathbf{v}(u), 0, 0) = D - L(\mathbf{v}(u), \mathbf{F}(u), \mathbf{B}(u)).$$

With the notations $A\,w := L(w, 0, 0)$ and $G\,u := L(\mathbf{v}(u), \mathbf{F}(u), \mathbf{B}(u))$ this reads

$$A(u - \mathbf{v}(u)) = D - G\,u. \tag{10}$$

For the solution of (10) we propose the following iterative procedure:

1. Choose an initial guess u_0, e.g. $u_0(z) = D(0)$ for all $z \in [0, l]$.
2. For $k = 0, 1, 2...$
   - solve the linear operator equation
   $$A\,w_{k+1} = D - G\,u_k \tag{11}$$
   - add $\mathbf{v}(u_k)$ to w_{k+1}
   $$u_{k+1} = w_{k+1} + \mathbf{v}(u_k)$$

Remark. *Clearly, step 2 is repeated until some stopping criterion is satisfied.*

Note that the linear operator equation (11) is ill-posed [6]. Thus, it is extremely sensitive to the noise in the data D. Due to measurement, modeling and discretization errors, in practice we always have noisy data D_δ, such that $\|D - D_\delta\| \leq \delta$. Hence, special regularization methods need to be applied for the solution of (11) (for a general introduction to this subject see for example [5]). Moreover inverse problems arising from parabolic equations are exponentially or severely ill-posed [4]. Tikhonov regularization is appropriate for such problems [11]. Thus, in each iteration we actually solve

$$(\alpha I + A^*\,A)\,w_{k+1} = A^*(\,D_\delta - G\,u_k\,), \tag{12}$$

where α is a regularization parameter, I is the identity operator and A^* is the adjoint operator. To choose the regularization parameter α we use the so-called quasi-optimality criterion, first introduced in [14]. This criterion does not depend on the noise level δ and consists in the following steps:

1. Select a finite number of regularization parameters
   $$0 < \alpha_0 < \alpha_1 < \cdots < \alpha_m,$$
   which are part of a geometric sequence, i.e. $\alpha_i = \alpha_0\,q^i$ with $q > 1$.
2. For each α_i solve (12) and obtain solutions w_{k+1,α_i}.
3. Among $\{w_{k+1,\alpha_i}\}_{i=0}^m$ choose w_{k+1,α_j} such that
   $$\|w_{k+1,\alpha_j} - w_{k+1,\alpha_{j-1}}\| = \min\left\{\|w_{k+1,\alpha_i} - w_{k+1,\alpha_{i-1}}\|,\; i = 1, 2, \ldots, m\right\},$$
   where $\|\cdot\|$ denotes the norm in the space $\mathcal{L}^2(0, l)$.

The realization of the proposed method and numerical results are discussed in the next section.

4 Numerical Results

For the numerical solution of (12) we use a Galerkin discretization. It requires a collection of finite dimensional subspaces $\{X_N\}$ such that $\dim X_N = N$ and $\overline{\bigcup_{N=1}^{\infty} X_N} = \mathcal{L}^2(0,l)$. Let $\{\varphi_i^N\}_{i=1}^N$ be a basis of X_N. Then for a fixed discretization parameter N the approximate solution $w_{k+1}^N = \sum_{j=1}^N c_j\,\varphi_j^N$ of (12) can be found by solving the linear system

$$\alpha \sum_{j=1}^N c_j\,(\varphi_j,\varphi_i) + \sum_{j=1}^N c_j\,(A\varphi_j, A\varphi_i) = (D_\delta - G\,u_k, A\varphi_i), \quad i = 1,...,N. \tag{13}$$

We choose the subspace X_N as the set of piecewise linear splines with N equidistant knots on $[0,l]$.

For a given initial temperature $\hat{u}(z)$ we generated the data $D(t)$ by the numerical solution of the coupled system (1)-(3). Thus, we have the values of $D(t)$ at a finite number of points $\{t_s\} \subset [0,t_f]$. The noise is simulated as in [9, 10], i.e.

$$D_\delta(t_s) = D(t_s) + \delta_r\,\xi_s,$$

where $\{\xi_s\}$ are independent random variables with uniform distribution over $[-1,1]$.

We present numerical results, which show the typical behavior of the method. The chosen initial temperature $\hat{u}$ has a symmetric profile with values between 800 K and 1200 K.

Let $\{u_k^N\}$ be the sequence of approximations for a fixed discretization N. The errors $\|\hat{u} - u_k^N\|$ with respect to the iteration number k for $(\delta_r = 0, N = 41)$, $(\delta_r = 2, N = 41)$, and $(\delta_r = 10, N = 11)$ are plotted in Figure 1. One observes fast convergence of approximating sequence $\{u_k^N\}$. Note that the smaller the noise level, the better is final approximation.

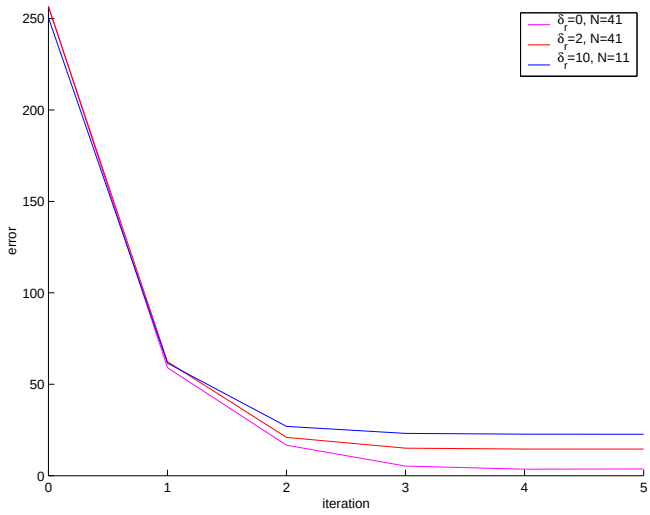


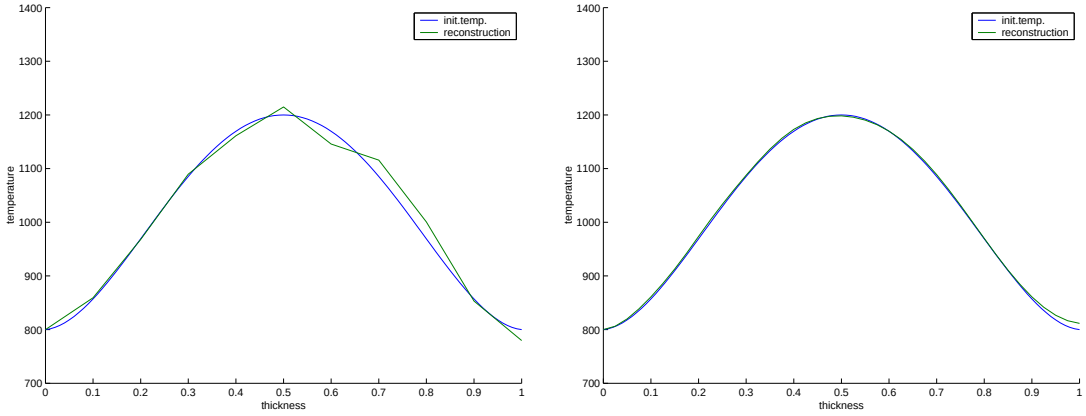
Figure 1: Behavior of the error $\|\hat{u} - u_k^N\|$ for different discretization and noise levels.

The discretization parameter N has a significant influence on the reconstruction. In fact, the discretization brings additional noise. First, instead of the operator equation (12) we solve the discrete system (13). Secondly, the functions $A\varphi_i^N$ and the scalar products in (13) can be computed only approximately. Thus, the discretization and the noise need to be balanced which is illustrated by the numerical examples in Figure 2. If no random noise (i.e. $\delta_r = 0$) is added, then increasing N will give better results. In the presence of the noise the numerical results suggest, that there is a strong nonmonotonic dependence of the reconstruction accuracy on the discretization. Thus, criteria are needed for choosing the discretization adaptively depending on the data noise.

5 Conclusions

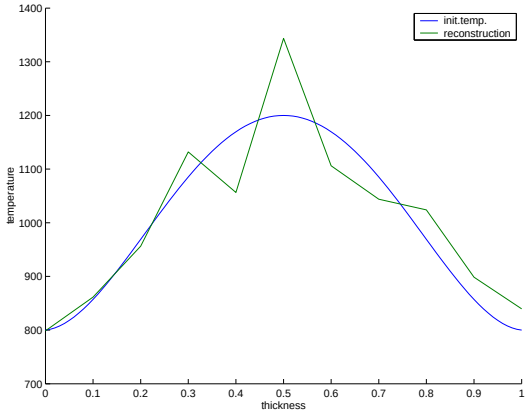
We presented a fast method of the initial temperature reconstruction for nonlinear heat equations with source terms. Its numerical performance was checked in 1D for the nonlinear model of a glass plate cooling. Further research will concentrate on:

- a rigorous analysis of the convergence of the proposed method;
- the development of criteria for the choice of discretization parameters;
- the implementation of the proposed method for 3D cooling processes.

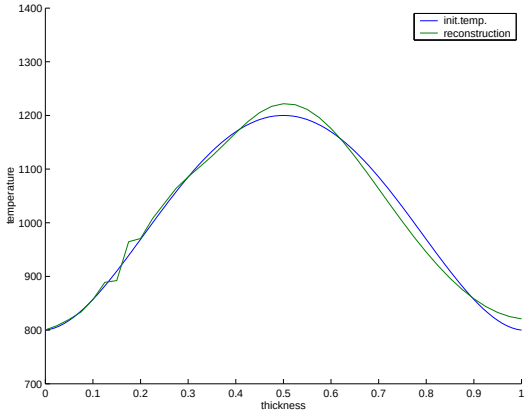


$\delta_r = 0,\ N = 11$

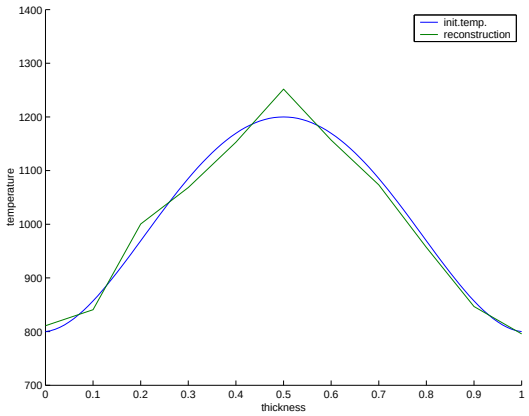
$\delta_r = 0,\ N = 41$



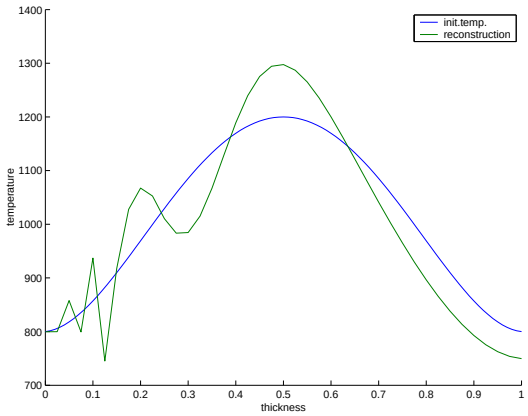
$\delta_r = 2,\ N = 11$



$\delta_r = 2,\ N = 41$



$\delta_r = 10,\ N = 11$



$\delta_r = 10,\ N = 41$

Figure 2: Examples of the reconstructions for different random noise and different discretizations.

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