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Spherical Location Problems with Restricted Regions and Polygonal Barriers

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ABSTRACT

This thesis investigates the constrained form of the spherical Minimax location problem and the spherical Weber location problem. Specifically, we consider the problem of locating a new facility on the surface of the unit sphere in the presence of convex spherical polygonal restricted regions and forbidden regions such that the maximum weighted distance from the new facility on the surface of the unit sphere to m existing facilities is minimized and the sum of the weighted distance from the new facility on the surface of the unit sphere to m existing facilities is minimized . It is assumed that a forbidden region is an area on the surface of the unit sphere where travel and facility location are not permitted and that distance is measured using the great circle arc distance.

We represent a polynomial time algorithm for the spherical Minimax location problem for the special case where all the existing facilities are located on the surface of a hemisphere. Further, we have developed algorithms for spherical Weber location problem using barrier distance on a hemisphere as well as on the unit sphere.

Keywords: spherical location, spherical convex polygon, restricted and barrier regions, great circle arc, barrier distance

1. INTRODUCTION

1.1 Applications and Literature Survey

Facility location on the plane can be considered as to locate one or more new facilities among m given demand points (or existing facilities) on the plane. When we locate only one new facility, the optimality is achieved :

- 1. the sum of weighted distances from the new facility to the demand point is minimized,*
or
- 2. the maximum weighted distance from the new facility to the demand points is minimized or the minimum weighted distance from the new facility to the demand points is maximized.*

The weights usually represent the cost of delivery per unit distance, goods demanded, population, etc.. In each of these optimality approaches, the planar distances are used.

The first formulation is referred as "Classical weber problem [22]" or "median (minisum)" formulation of the problem and the second formulation is referred as "center (minimax/maximin" [22]) formulation of the problem.

These two formulations, of course, are still valid when all locations are on the surface of a sphere. For example, the problem of locating a store of emergency supplies for the relief of the consequence of natural or man-made disasters around the globe has the element of a minimax problem on the surface of the sphere.

When demand points are located on the plane, the maximin facility location problem is of little practical significance. That means, a facility can be located "at infinity" to maximize the minimum weighted distance. But, on a sphere, the maximum distance is one-half of the maximum circumference and, hence the problem is not trivial. Locating a facility as far as possible from a given set of missile bases can be given as an application. The objective of this problem would be the maximization of the time before the arrival of a missile.

However, all the demand points are spread all over the globe, planar distances are no longer suitable approximations in modelling. Therefore, many researchers consider spherical distances instead of planar distances to locate an appropriate location over the globe. Then the relevant location problem is as follows : Consider that all the demand points are located on the surface of a sphere with known associated weights. Then the problem to find an optimal location on the surface of the sphere is referred as "facility location on the sphere" or "**Spherical Location Problem(SphereLoc)**".

We consider single facility spherical location problems (**SphereLoc**) of the median and center type. I.e., we solve

$$\min_{X \in \mathcal{S}_0} f(X) := \sum_{i=1}^m w_i d(X, Ex_i) \quad \text{WeberSphereLoc}$$

and

$$\min_{X \in \mathcal{S}_0} h(X) := \max_{i=1}^m w_i d(X, Ex_i) \quad \text{CenterSphereLoc}$$

where Ex_1, Ex_2, \dots, Ex_m are given demand points (or existing locations) and X is the unknown location of a new facility. All locations lie on the unit sphere \mathcal{S}_0 and possible distance functions $d(X, Y)$ between points $X, Y \in \mathcal{S}_0$ are discussed in detail later on.

Applications of spherical location problems appear in military, civil, naval, commercial problems. These are becoming global in the sense that the distances involved are so large on the globe that planar distances are no longer suitable.

As an illustration of this spherical location problem, consider the following example: a product is to be distributed to 15 cities by air, as shown in Table 1.1, where each city is defined by its latitude and longitude. The weights are the functions of the overall demand. Our task is to find a best location for the factory in order to distribute the product with minimum cost.

Spherical location problems with the measuring distance on the surface of the sphere is the shortest length of arc (great circle distance) (see Definition 1.2.8), is more complex than its counterpart on the plane because its objective is not convex as the distance function is not convex on the surface of the sphere(see Theorem 1.2.1).

	City	Latitude	Longitude	Weights
1	London	51.5	0.4	0.12
2	Paris	48.9	2.3	0.07
3	Zurich	47.4	8.5	0.08
4	Rome	41.9	12.5	0.05
5	Copenhagen	55.7	12.6	0.08
6	Berlin	52.5	13.4	0.07
7	Stockholm	59.3	18.9	0.06
8	Athens	38.0	23.7	0.07
9	Ankara	39.9	32.8	0.05
10	Tel-Aviv	32.1	34.8	0.05
11	Moscow	55.7	37.7	0.05
12	Teheran	35.4	51.4	0.07
13	Bombay	18.9	72.8	0.03
14	Manila	14.6	121.0	0.05
15	Tokyo	35.6	139.7	0.10

Tab. 1.1: Latitudes, Longitudes and corresponding weights of 15 cities

In the literature various definitions of distances $d(X, Y)$ are used, which we will discuss in some details in sections 1.2 and 2.2. In the following brief summary of the literature we will use corresponding notations. Some of the results are further detailed in this thesis.

*Drezner and Wesolowsky [11], in 1978 considered the related problem where they used two ways of measuring distances (see section 2.2) and used iterative heuristic method for solving the **WeberSphereLoc** problem with shortest arc distance.*

A short overview on locating a facility on a sphere can be found in the text book of Robert F. Love, James G. Morris and G.O. Wesolowsky [22]. They consider the median problem where d is the shortest arc distance and present a Weiszfeld-like iterative procedure on the sphere.

A. A. Aly, D. C. Kay, D. W. Litwhiler [1], in 1978 worked on the spherical median problem with the shortest arc distance as distance measure. They found out that an optimal solution to this problem must lie within the spherical convex hull (Definition 1.2.14) of the demands points if the demand points are not located entirely on a great circle arc.

Zvi Drezner [12], in 1981 considered the case when all the demand points lie on a great circle arc and he proposed that the optimal solution occurs on a demand point in this situation (Theorem 2.1.4).

In 1979, Katz and Cooper [19] considered the problem, "Optimal Location on the Sphere ". They use three different metrics as distances between points on the surface of the sphere: (1) Euclidean ; (2) squared Euclidean distances; (3) geodesic or great circle distance.

Both, "Katz and Cooper [19]" and "Drezner and Wesolowsky [11]" propose Weiszfeld - like algorithms for finding an optimal facility location on a sphere. However, convergence has never been proposed.

In 1985, Zvi Drezner [14] proposed a convergent algorithm for the solution to the minisum location problem on the sphere with measuring distance on the surface of the sphere is the length of shortest arc. The proposed algorithm is presented in the section 2.3.

Drezner and Wesolowsky [13] dealt with minimax and maximin facility location problem on a sphere in 1983. First they propose an algorithm for finding a local minimax point using a non linear programming approach. Then they develop an algorithm to determine the global minimax points using the obtained local minimax points (see section 3.1).

In 1994, Xue [32] proposed a globally convergent algorithm to the minisum formulation of this problem with the shortest length of arc as the distance metric. In his paper, he proved the hull property of the problem, i.e., every global minimizer of the problem must lie within the spherical convex hull (Definition 1.2.14) of the existing facilities. Also, he presented optimality conditions for the spherical facility location problem in terms of the optimality conditions for the corresponding Euclidean facility location problem. Finally, a gradient

algorithm for solving the spherical facility location problem is presented and the global convergence of this algorithm was proved. He assumed that all of the existing facilities are included within a spherical circle (Definition 1.2.11) of radius $\pi/4$.

In 1994, Minnie H. Patel [24] dealt with the spherical minimax location problem and formulated the spherical location problem in the Cartesian coordinate system using Euclidean norm, instead of the spherical coordinate system using spherical arc distance measures. It is shown that minimizing the maximum of the spherical arc distances between the facility point and the demand points on the surface of the sphere is equivalent to minimizing the maximum of the corresponding Euclidean distances.

Pierre Hansen, B. Jaumard and S. Krau [18], in 1994, presented an exact and practically efficient algorithm for the WeberSphereLoc problem using a Branch-and-Bound approach. This is an extension of the continuous branch-and-bound algorithm for location of a facility in the plane, known as "Big Square Small Square (BSSS) [32]". Further, four ways to compute lower bounds are studied.

In 1996, A.K. Sakar, P.K. Chaudhuri [27] and in 1998, P.Das, N.R. Chakraborty, P.K. Chaudhuri [4] developed two algorithms for the equally-weighted CenterSphereLoc problem when all demand points lie on a hemisphere. Both yield an exact solution with the time complexity $O(n^2)$ in the worst case. The methods of these approaches are basically geometrical and do not require the use of the nonlinear programming techniques like most of the other papers. The difference between the two algorithms is that while the first algorithm in [27] heavily depends on properties of the spherical triangle (Definition 1.2.13), the second in [4] depends on the maximization of the Euclidean distance (for more details, see section 3.4).

P.Das, N.R. Chakraborti and P.K. Chaudhuri [5], in 1999 considered the CenterSphereLoc problem with respect to shortest arc distance. They assume that all the demand points are equally weighted and distributed over the sphere. The procedure they present is based on an enumeration techniques and determines global optimal solutions in a finite number of steps. This algorithm determines the exact solution of the global as well as the hemispher-

ical minimax location problem with the time complexity $O(n^3)$ (see section 3.2).

Kelly M. Betes [2], in 2001, analyze alternative solutions methodologies for the Weber (minisum) problem on the surface of the sphere.

Atsuo Suzuki [29] presents the results for (multi-) facility location problems on the sphere based on Voronoi diagrams. The problems which are discussed here are the p -median problem, the p -center problem and the competitive location problem. He assumes that all the demand points are spread continuously on the sphere.

Kokichi Sugihara [28] also uses on Voronoi diagrams as tools for space analysis. The concepts of the Voronoi diagram, various kinds of its generalizations and the methods for computing them are surveyed from a user point of view. Particular application of his studies on voronoi diagrams is to place them on a sphere, which will be useful for facility layout on the spherical surfaces.

Further, U.R. Dhar and J.R. Rao [7] in 1980 dealt with " A comparative study of three norms for facility location problem on spherical surfaces. "

In 1981, U.R. Dhar and J.R. Rao [8] studied "multi source location problem on a sphere" and in 1982, U.R. Dhar and J.R. Rao [9] considered the problem of locating more than one new facilities among existing facilities on surface of the sphere. The optimality of this problem is achieved when the sum of all weighted distances between new to new facilities and new to existing facilities is minimized with the measuring distance on the surface of a sphere is shortest length of arc. This problem is known as multi-facility spherical location problem.

Before, formulating of some solving methods for the spherical location problems, it is important to know whether or not all the demand points are on a hemisphere. In 1993, Mannie H. Patel, D.L. Nettles and S.J. Deutsch [23] represented a Linear-Programming-Based Method to determine this.

1.2 Review of Spherical Geometry

We assume that each point X which is considered in the following will lie on a unit sphere \mathcal{S}_0 and the point X is defined by its latitude ϕ and longitude θ and is denoted by $X = X(\phi, \theta)$ where $-\pi/2 \leq \phi \leq \pi/2$ and $0 \leq \theta \leq 2\pi$.

The Euclidean coordinates of point X are given by

$$\begin{aligned} x &= \cos \phi \cos \theta \\ y &= \cos \phi \sin \theta \quad \text{and} \\ z &= \sin \phi \end{aligned}$$

(see Figure 1.1) and it is denoted by $X = X(x, y, z)$.

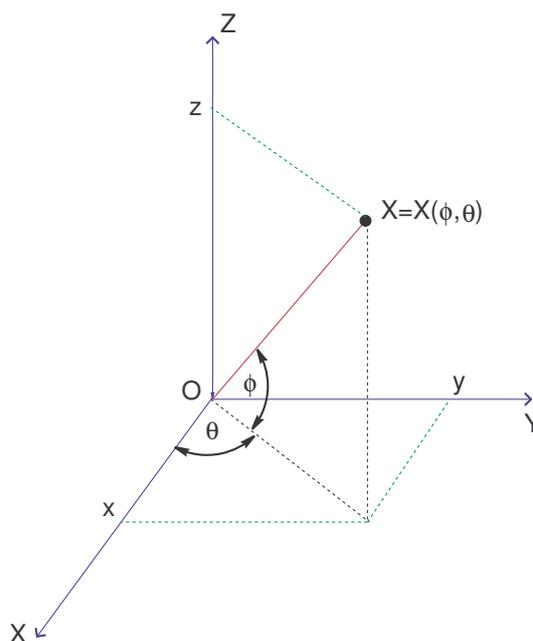


Fig. 1.1: Conversion of polar coordinates of a point $X = X(\phi, \theta)$ on the unit sphere to cartesian coordinates $X = X(x, y, z)$ where $-\pi/2 \leq \phi \leq \pi/2$ and $0 \leq \theta \leq 2\pi$

Definition 1.2.1: Latitude is a north-south measurement of position on the Earth. It is defined by the angle measured from horizontal plane located Earth's center that perpen-

dicular to the polar axis (see Definition 1.2.3). A circular line connecting all places of the same latitude is termed a **parallel** (see Figure 1.2).

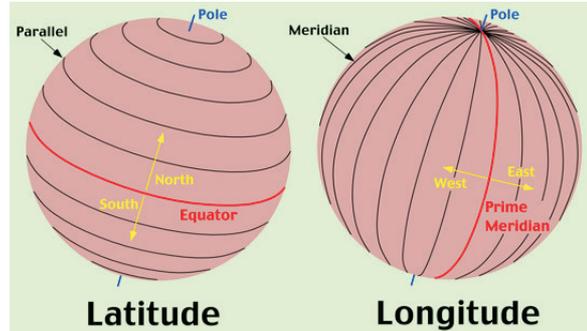


Fig. 1.2: Latitudes, longitude, meridian and prime meridian

Definition 1.2.2: **Longitude** is a west-east measurement of the Earth. It is defined by the angle measured from a vertical plane running through the polar axis and prime meridian (see definition 1.2.4). A circular line connecting all places of the same longitude is termed a **meridian** (see Figure 1.2).

Definition 1.2.3: **Polar axis** is a line drawn through the Earth around the planet rotates. The point at which the polar axis intercepts the Earth's surface in the Northern hemisphere is called the **North pole**. Likewise, the point at which the polar axis intercepts the Earth's surface in Southern hemisphere is called the **South pole** (see Figure 1.3).

Definition 1.2.4: The location from which meridians of longitude measured is called the **Prime meridian**. It has zero degrees of longitude. (see Figure 1.2)

Definition 1.2.5: Location on the Earth that has a latitude of 0 degrees is called the **Equator** (see Figure 1.3).

Definition 1.2.6: Every plane section of a sphere is a circle. The largest circle which can be drawn on the surface of a sphere is a circle whose plane passes through the center of the sphere. Such a circle is called a **great circle**. All other circles on the surface of the sphere are called **small circles** (see Figure 1.4).

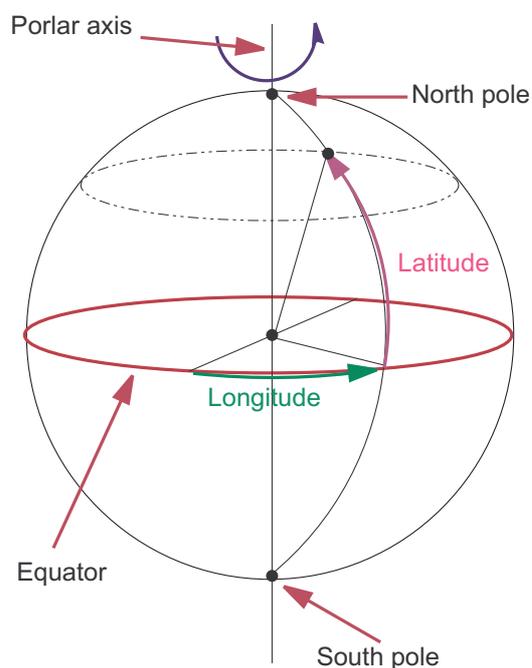


Fig. 1.3: Equator, polar axis, north pole and south pole

Definition 1.2.7: The **poles** of a great circle are the extremities of a diameter of the sphere that is perpendicular to the plane of the great circle. This diameter is also known as the **axis** of the great circle.

Note that the two poles for the a great circle are equidistant from its plane and the center of the sphere. The poles and axes of small circles are similarly defined. However, since the plane of a small circle does not contain the center of the sphere, its two poles are at a different distance from the plane of the small circle, one is nearer and the other is more distant. For convenience, refer to them as the nearer and distant poles of a small circle (see Figure 1.4).

Definition 1.2.8: The shortest distance between any two points on a sphere must be measured along the great circle passing through them and is the shorter of the two arcs between the points. This distance is known as the **great circle distance**, α or **shortest arc distance** (see Figure 1.5).

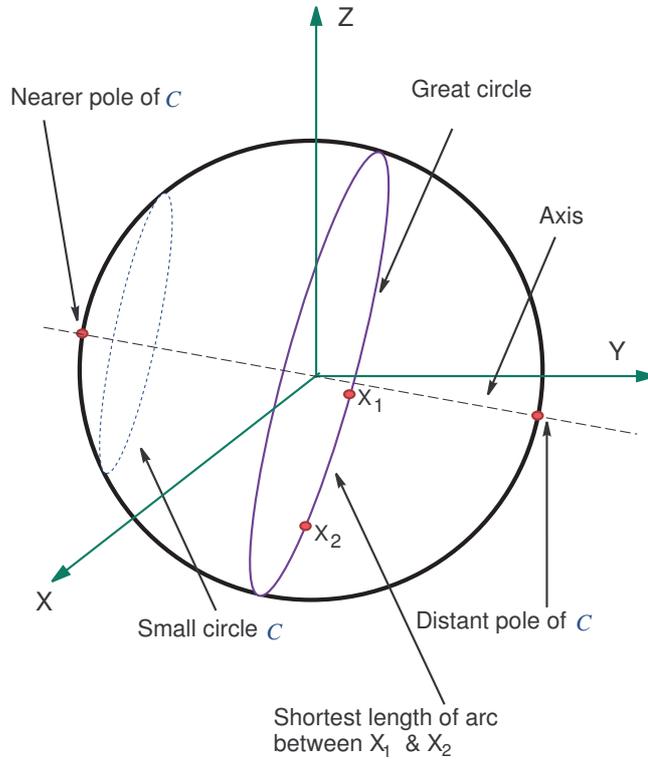


Fig. 1.4: Circles on a sphere

Note that arc length, $\alpha(X_1, X_2)$ (or $\text{arc}(X_1, X_2)$) between two points, X_1 and X_2 on the unit sphere is simply the angle (measured in radians) between the two rays emanating from the center of the sphere, one passing through X_1 and the other through X_2 .

The distances $d_1 : 4 \sin^2(\alpha/2)$ and $d_2 : \pi \sin^2(\alpha/2)$ may be used to approximate squared arc distance on a hemisphere and also rough approximation for arc distance (see [11]). The difference between d_1 and d_2 is only a multiplicative constant. In two figures (Figure 1.6, Figure 1.7), d_1 and d_2 are plotted against α (shortest length of arc). Note that when the distance between points is less than half the circumference of the sphere ($\alpha \leq \pi/2$), d_1 is a reasonably good approximation to the squared shortest arc distance. d_2 can be thought of as a rough approximation for α . Also, d_1 is exactly the squared Euclidean distance through the sphere.

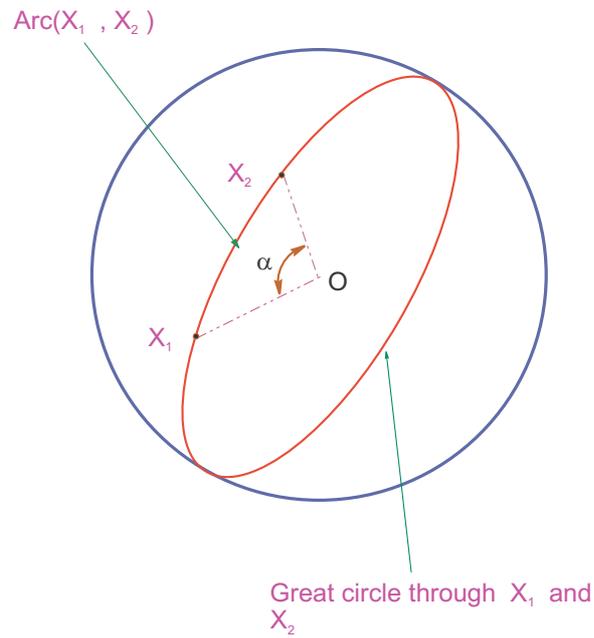


Fig. 1.5: Shortest length of arc between X_1 and X_2

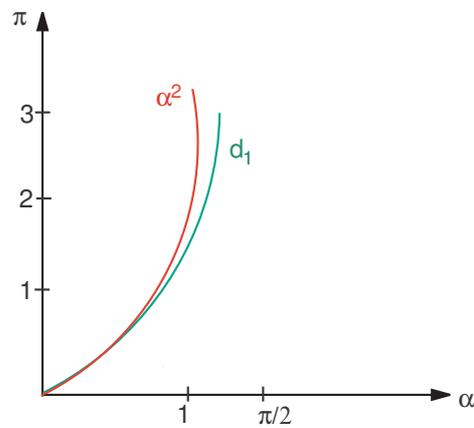


Fig. 1.6: The graph of d_1 Vs. α

Result 1.2.1: Given two points (See Figure 1.5) $X_1(\phi_1, \theta_1)$, $X_2(\phi_2, \theta_2)$ on \mathcal{S}_0 , the length of the shortest arc, $\alpha = arc(X_1, X_2)$ satisfies

$$\cos \alpha = \cos \phi_1 \cos \phi_2 \cos(\theta_1 - \theta_2) + \sin \phi_1 \sin \phi_2 \tag{1.1}$$

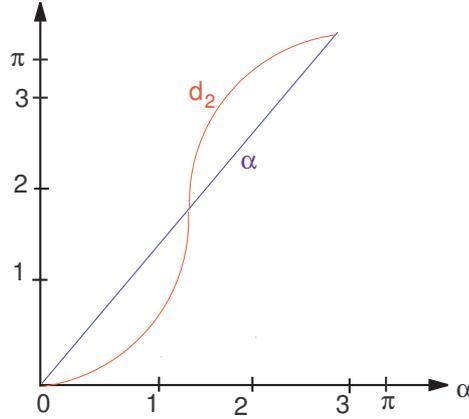


Fig. 1.7: The graph of d_2 Vs. α

Proof : Let $X_1(\phi_1, \theta_1)$ and $X_2(\phi_2, \theta_2)$ are two points on the surface of the sphere. Then, according to the cosines law for plane triangles, the Euclidean distance between X_1 and X_2 can be written as :

$$|X_1X_2|^2 = |OX_1|^2 + |OX_2|^2 - 2|OX_1||OX_2|\cos \alpha \quad (1.2)$$

where $|X_1X_2|^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2$. By applying the Euclidean coordinates of the points X_1 and X_2 to (1.2), with $|OX_1| = |OX_2| = 1$ (since X_1 and X_2 are on the surface of the unit sphere), we have the desired result. ■

Definition 1.2.9: The length of the great circle arc from any point on the circumference of a small circle to its nearer pole is called the **spherical radius** of the small circle.

Definition 1.2.10: The **antipode** of a given point is the point on the other side of the sphere on the line connecting the point with the center of the sphere. The antipode of $X(\phi, \theta)$ is $\bar{X}(-\phi, \theta \pm \pi)$.

Definition 1.2.11: A **spherical circle** $\mathcal{C}(X, \alpha)$ (see Figure 1.8) with a given **center** X and **radius** α is defined on a sphere by the locus of all points whose shortest arc distance from the center is equal to that radius. A circle divides the sphere into two parts; A point

is said to be within a circle only if the point and the center of the circle are included in the same part.

Definition 1.2.12: A **spherical disk** $\mathcal{D}(X, \alpha)$ (see Figure 1.8) is the set on the surface of the sphere which is formed by a spherical circle and its interior.

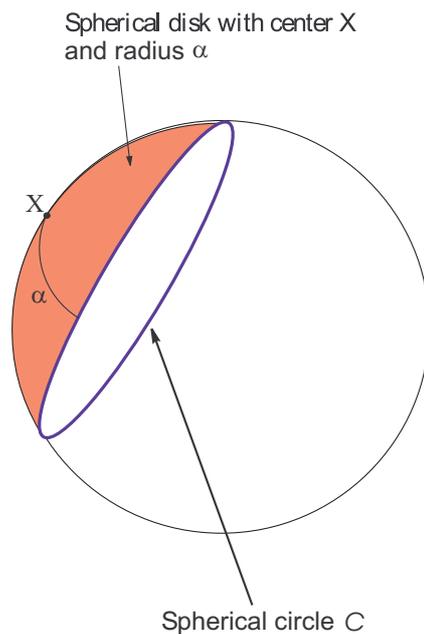


Fig. 1.8: Spherical circle and spherical disk

Definition 1.2.13: The surface area of a sphere that is bounded by arc segments of three great circles is called a **spherical triangle**(see Figure 1.9). A spherical triangle with two equal sides (or arc lengths) is called **isosceles** spherical triangle.

Result 1.2.2: (Median Formula) Consider the spherical triangle $X_1X_2X_3$. Let M be the mid point of the $arcX_1X_2$. Then the $arcX_3M$ satisfies the following formula :

$$\cos(arcX_3M) = \frac{\cos(\frac{arcX_1X_3+arcX_2X_3}{2}) \cos(\frac{arcX_1X_3-arcX_2X_3}{2})}{\cos(\frac{arcX_1X_2}{2})}. \quad (1.3)$$

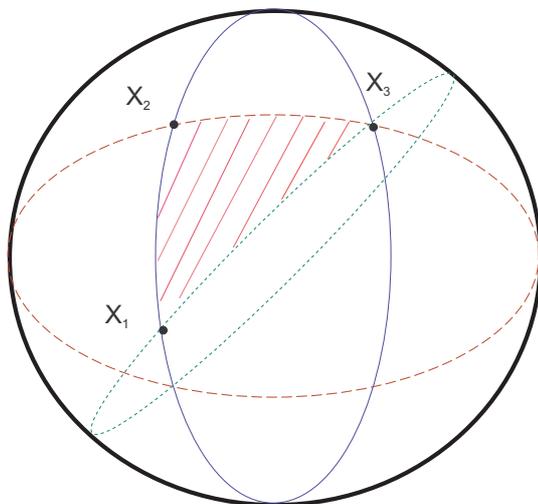


Fig. 1.9: Shaded area represents the spherical triangle with vertices X_1, X_2 and X_3

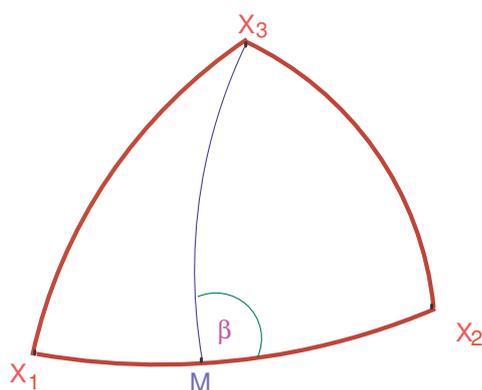


Fig. 1.10:

Proof Consider the Figure 1.10. Let β designate the angle X_3MX_2 . By cosine formula, we have

$$\cos \text{arc}X_2X_3 = \cos \text{arc}X_2M \cos \text{arc}X_3M + \sin \text{arc}X_2M \sin \text{arc}X_3M \cos \beta \quad (1.4)$$

$$\cos \text{arc}X_1X_3 = \cos \text{arc}X_1M \cos \text{arc}X_3M - \sin \text{arc}X_1M \sin \text{arc}X_3M \cos \beta \quad (1.5)$$

Multiply 1.4 by $\sin \text{arc}X_1M$, 1.5 by $\sin \text{arc}X_2M$, and add two. We get,

$$\begin{aligned} & \sin \operatorname{arc} X_1 M \cos \operatorname{arc} X_2 X_3 + \sin \operatorname{arc} X_2 M \cos \operatorname{arc} X_1 X_3 = \\ & \quad \sin(\operatorname{arc} X_1 M + \operatorname{arc} X_2 M) \cos \operatorname{arc} X_3 M \\ \Rightarrow \quad \cos \operatorname{arc} X_3 M &= \frac{\sin \operatorname{arc} X_2 M \cos \operatorname{arc} X_1 X_3 + \sin \operatorname{arc} X_1 M \cos \operatorname{arc} X_2 X_3}{\sin(\operatorname{arc} X_1 M + \operatorname{arc} X_2 M)} \end{aligned}$$

As $\operatorname{arc} X_1 M = \operatorname{arc} X_2 M = \frac{1}{2} \operatorname{arc} X_1 X_2$, we have

$$\begin{aligned} \cos \operatorname{arc} X_3 M &= \frac{\sin \frac{\operatorname{arc} X_1 X_2}{2} (\cos \operatorname{arc} X_1 X_3 + \cos \operatorname{arc} X_2 X_3)}{\operatorname{arc} X_1 X_2} \\ &= \frac{\cos(\frac{\operatorname{arc} X_1 X_3 + \operatorname{arc} X_2 X_3}{2}) \cos(\frac{\operatorname{arc} X_1 X_3 - \operatorname{arc} X_2 X_3}{2})}{\cos \frac{\operatorname{arc} X_1 X_2}{2}}. \end{aligned}$$

■

Definition 1.2.14: A **spherical convex set** is defined on the surface of a sphere as a set where for any two points of the set, the whole shortest arc connecting them is included in the set. The **spherical convex hull** of a set of points on the sphere is defined to be the smallest spherical convex set which contains the set of given points.

Definition 1.2.15: Let $\rho = \rho(X_1, X_2, \lambda)$ be a point on the shortest arc between X_1 and X_2 such that the distance between X_1 and ρ is $\lambda d(X_1, X_2)$ and between X_2 and ρ is $(1 - \lambda)d(X_1, X_2)$ for $\lambda \in [0, 1]$ where $d(X_1, X_2)$ is the shortest length of arc between X_1 and X_2 .

Definition 1.2.16: $f(X)$ is called a **spherical convex function** on a spherical convex set D of a sphere if for every $0 \leq \lambda \leq 1$ and $X_1, X_2 \in D$, we have

$$f(\rho(X_1, X_2, \lambda)) \leq (1 - \lambda)f(X_1) + \lambda f(X_2). \quad (1.6)$$

$f(X)$ is called a **strictly spherical convex function** if the inequality (1.6) is strict when $X_1 \neq X_2$ and $\lambda \in (0, 1)$.

Definition 1.2.17: A spherical location problem is in its **normal form** if it has only positive weights and there is no pair of demand points which are antipodes to each other.

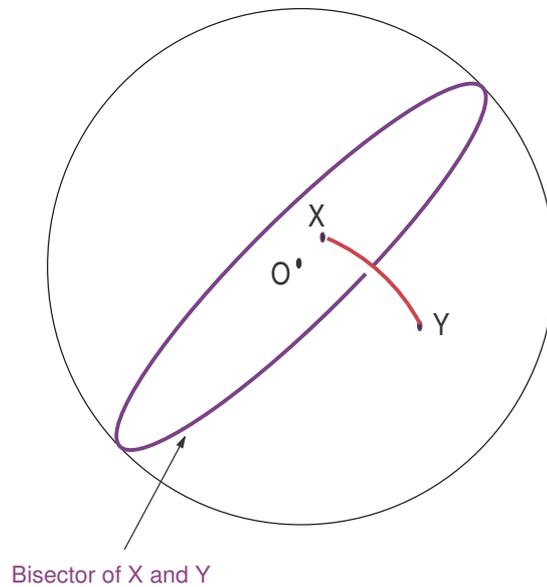


Fig. 1.11: The Bisector of X and Y

Definition 1.2.18: The **bisector** of spherical points X and Y defined with respect to the great circle distance is given by the great circle that perpendicularly passes through the mid-point of the great circular arc connecting X and Y ('perpendicularly' means that sufficiently small segments of the two great circles around the mid-point are orthogonal) (see Fig 1.11).

The bisector divides the sphere into two disjoint hemispheres.

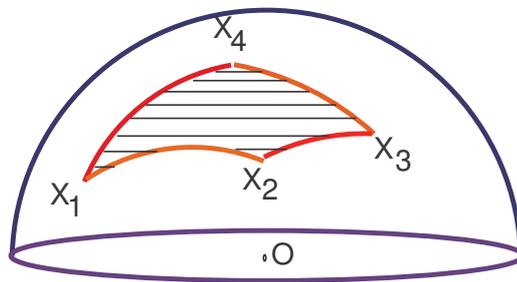


Fig. 1.12: Shaded area represents a spherical polygon on a hemisphere

Definition 1.2.19: A **spherical polygon** is a closed geometric figure on the surface of a sphere which is formed by the arcs of great circles. The spherical polygon is a generalization of the spherical triangle. A **spherical convex polygon** generated by points X_1, X_2, \dots, X_n is defined by the spherical polygon in which the lesser arc of a great circle passing through any two points in the spherical polygon is embedded in the spherical polygon. (see Figure 1.12)

The great circle arc segments of the spherical polygon are called the **edges** of the spherical polygon and a point at which two edges meet is called a **vertex** or **corner point** of the spherical polygon.

Definition 1.2.20: The **level set** and **level curves** of the objective function $h(X)$ in CenterSphereLoc with respect to the great circle arc distance, α is defined as follows :

$$\text{Level sets: } L_{\leq}(z) := \{X \in \mathcal{S}_0 : w_i \cdot \max_{i=1,2,\dots,m} \alpha(Ex_i, X) \leq z\}$$

$$\text{Level curves: } L_{=}(z) := \{X \in \mathcal{S}_0 : w_i \cdot \max_{i=1,2,\dots,m} \alpha(Ex_i, X) = z\}$$

The arc segments of the level set are called the **edges** of the level set. The end points of the edges are called the **vertices** or **corner points** of the level set.

Definition 1.2.21: Suppose f_k is an edge (or a facet) of $L_{\leq}(z)$ and $Ex_i \in \mathcal{E}_x$. The point P_{ik} is defined as the **projection point** of Ex_i on f_k if

$$(a) \ P_{ik} \in f_k$$

and

$$(b) \ \alpha(Ex_i, P_{ik}) = \min\{\alpha(Ex_i, X) : X \in f_k\}.$$

Result 1.2.3: (i) Since

$$\begin{aligned} L_{\leq}(z) &= \{X \in \mathcal{S}_0 : w_i \cdot \max_{i=1,2,\dots,m} \alpha(Ex_i, X) \leq z\} \\ &= \{X \in \mathcal{S}_0 : \alpha(Ex_i, X) \leq z/w_i \quad \forall i = 1, 2, \dots, m\} \\ &= \bigcap_{i=1,2,\dots,m} \{X \in \mathcal{S}_0 : \alpha(Ex_i, X) \leq z/w_i\}, \end{aligned}$$

we can write the level set as an intersections of m spherical disks $\mathcal{D}(Ex_i, z_i)$ centered at the existing facilities Ex_i , with spherical different radius $z_i = z/w_i; i = 1, 2, \dots, m$.

(ii) The level curve in this case is the boundary of intersections of the m spherical disks (that is the boundary of the level set).

(see Fig 1.13).

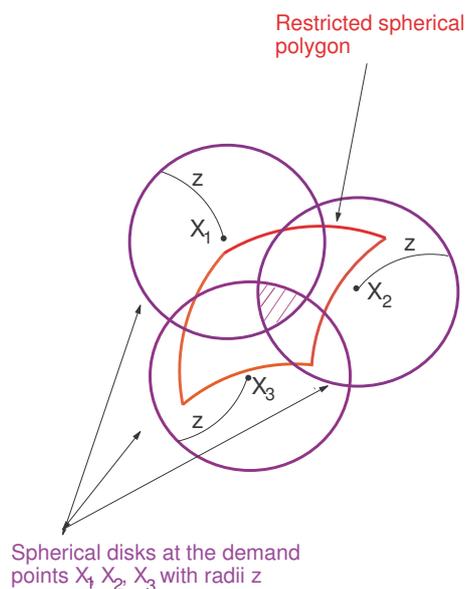


Fig. 1.13: Shaded area and the boundary of this region represents the level set and level curve respectively

Property 1.2.1: Some properties of spherical triangles [30]

- (a). The angles at the base of an isosceles spherical triangle (see definition 1.2.13) are equal.
- (b). If one angle of a spherical triangle is greater than another, the side opposite the greater angle is greater than the side opposite the lesser angle.
- (c). Any two sides of spherical triangle are together greater than the third side.

Theorem 1.2.1: [11] Points within a circle of radius less or equal to $\pi/4$ (spherical disk \mathcal{D}) on a unit sphere \mathcal{S}_0 , form a spherical convex set. The shortest arc distance from a given point X on \mathcal{S}_0 is a spherical convex function on a spherical disk of radius $\pi/2$ and center X . Every local minimizer of a spherical convex function on a spherical convex set is also a global minimizer.

Proof: *The convexity property of the spherical disk with radius less or equal $\pi/2$ is obvious. According to Figure 1.14, it is clear that the shortest arc between X_3 and X_4 is included within the spherical disk with spherical radius less than or equal $\pi/2$. This is true for any two points in this spherical disk.*

To prove the convexity of the shortest arc distance α from a given point X , we assume wlog that X is the north pole, i.e. $X = X(\pi/2, 0)$. Take any two points $X_1(\phi_1, \theta_1)$, $X_2(\phi_2, \theta_2)$ with $\phi_1, \phi_2 \geq 0$. Note that since α is continuous, it is enough to prove that :

$$\alpha[\rho(X_1, X_2, 0.5), X] \leq 1/2[\alpha(X_1, X) + \alpha(X_2, X)]$$

in order to prove convexity.

Then

$$\alpha(X_1, X) = \pi/2 - \phi_1$$

$$\alpha(X_2, X) = \pi/2 - \phi_2$$

$$\alpha[\rho(X_1, X_2, 0.5), X] = \pi/2 - \phi_0,$$

where ϕ_0 is the latitude of the center of the arc connecting X_1 and X_2 . By the median formula (1.3),

$$\sin \phi_0 = \sin\left(\frac{\phi_1 + \phi_2}{2}\right) \cos\left(\frac{\phi_1 - \phi_2}{2}\right) / \cos \frac{\alpha}{2}. \quad (1.7)$$

Using equation(1.1) :

$$\sin \phi_0 = \frac{\sin\left(\frac{\phi_1 + \phi_2}{2}\right)}{\left[1 - \frac{\sin^2[(\theta_1 - \theta_2)/2] \cos \phi_1 \cos \phi_2}{\cos^2[(\phi_1 - \phi_2)/2]}\right]^{1/2}}. \quad (1.8)$$

As, numerator of (1.8) is less than or equal 1 and $\phi_0 \leq \pi/2$,

$\phi_0 \geq \frac{\phi_1 + \phi_2}{2}$. Therefore

$$\pi/2 - \phi_0 \leq \frac{\pi/2 - \phi_1 + \pi/2 - \phi_2}{2}$$

and

$$\alpha[\rho(X_1, X_2, 0.5), X] \leq \frac{1}{2}[\alpha(X_1, X) + \alpha(X_2, X)].$$

Thus α is a convex function north of the equator.

Now we have to show that every local minimizer of a spherical convex set \mathcal{D} is also a global minimizer.

To prove this, suppose that X_1^* and X_2^* are different local minima. The arc connecting X_1^* and X_2^* is included in \mathcal{D} . We know that

$$f[\rho(X_1^*, X_2^*, \lambda)] \leq \lambda f(X_1^*) + (1 - \lambda)f(X_2^*), \forall \lambda \in (0, 1).$$

Now suppose that $f(X_1^*) < f(X_2^*)$. Then by replacing $f(X_1^*)$ with $f(X_2^*)$ in the above equation, we have

$$f[\rho(X_1^*, X_2^*, \lambda)] < \lambda f(X_2^*) + (1 - \lambda)f(X_2^*) = f(X_2^*)$$

for $\lambda > 0$ obviously close to 1.

This contradicts the statement that X_2^* is a local minimum. ■

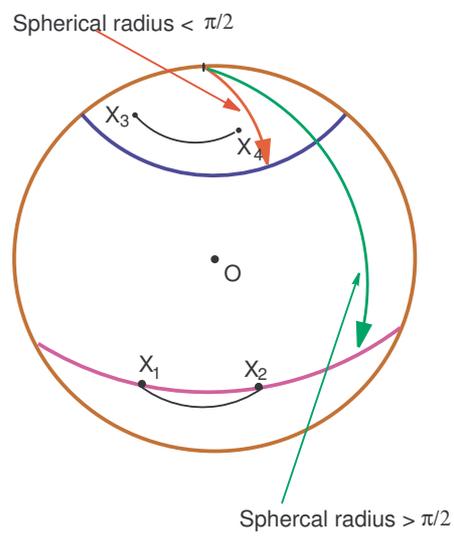


Fig. 1.14: Convexity of spherical disks

2. SPHERICAL WEBER PROBLEM

We assume that each model which is described in the following will deal with a unit sphere, \mathcal{S}_0 where the radius is equal to one. Every point X on the sphere is defined by its latitude ϕ and longitude θ and it is denoted by $X(\phi, \theta)$.

Consider m demand points (or existing facilities) $Ex_i, i = 1, 2, \dots, m$, on the surface of the sphere with associated weights w_i and some distance function $d(X, Y)$ measuring the distances between spherical points X and Y .

We consider **single facility spherical location problem (SphereLoc)** of the median type, i.e., we solve

$$\min_{X \in \mathcal{S}_0} f(X) := \sum_{i=1}^m w_i d(X, Ex_i) \quad \mathbf{WeberSphereLoc} \quad (2.1)$$

where X is the unknown location.

In the usual Weber problem, it is assumed that $w_i \geq 0$. In the **WeberSphereLoc**, we can omit this condition because this problem can be transformed into an equivalent "normal form" (see Definition 1.2.17) as follows :

A point with negative weight can be replaced by its antipode with weight $-w_i$ and from a pair of points which are antipodes to each other we can subtract the smaller weight, thus eliminating at least one of the points. This normal problem has the same minimal point as the original.

In this chapter, I would like to discuss the behavior of the objective function $f(X)$ of **WeberSphereLoc** problem and to represent different approaches to solve this problem.

2.1 Convexity of the Objective Function

We assume that the distance d of the objective function $f(X)$ is the shortest arc distance α . I.e.,

$$\min_{X \in S_0} f(X) := \sum_{i=1}^m w_i \text{arc}(X, Ex_i) = \sum_{i=1}^m w_i \alpha_i(X, Ex_i) \quad (2.2)$$

Theorem 2.1.1: [19] If all demand points of the normal form of a problem are included in a disk \mathcal{D} of radius $\pi/4$, then the objective function $f(X)$ is a spherical convex function on \mathcal{D} and attains its minimum in a unique point of \mathcal{D} .

Proof

$$\begin{aligned} \bullet \text{Demand points in } \mathcal{D} = \mathcal{D}(Y, \alpha) \text{ with } \alpha \leq \pi/4 &\Rightarrow \text{distance } \text{arc}(X_1, X_2) \leq \pi/2 \\ &\forall X_1, X_2 \in \mathcal{D} \\ &\stackrel{\text{Th.1,2.1}}{\Rightarrow} \text{arc}(X, Ex_i) \text{ convex } \forall X \in \mathcal{D} \\ &\Rightarrow w_i \cdot \text{arc}(X, Ex_i) \text{ convex} \\ &\forall X \in \mathcal{D} \\ &\Rightarrow f(X) \text{ convex } \forall X \in \mathcal{D} \end{aligned}$$

• \mathcal{D} convex, $f(X)$ convex $\stackrel{\text{Th.1,2.1}}{\Rightarrow} f(X)$ attains its minimum in a unique point of \mathcal{D} ■

Theorem 2.1.2: [8] The value $\pi/4$ in Theorem 2.1.1 is the maximum value of a radius that assures a unique minimum.

Proof We give an example of points in a disk of radius $\pi/4 + \epsilon$ (for every $\epsilon > 0$) containing two different local minima. The problem consists of three demand points with parameters ($\epsilon > 0$).

i	w_i	ϕ_i	θ_i
1	$1 + \epsilon^5$	$\pi/4$	ϵ
2	1	$\pi/4$	$-\epsilon$
3	ϵ^2	$\pi/4 - \epsilon$	π

By equation (2.13), for points 1 and 2 we get :

$$\frac{w_k}{[A_k^2 + B_k^2]^{1/2}} = 1 + (\sqrt{2} - 1)\epsilon^4 + O(\epsilon^5) \quad \text{for } k = 1, 2.$$

Which means that points $(\phi_1, \theta_1), (\phi_2, \theta_2)$ are local minima with different values of the objective function. This proves that $\pi/4$ is the largest possible radius that assures a unique minimum. ■

Theorem 2.1.3: [32] Suppose that all of the existing facilities are included within a spherical circle $\mathcal{C}(X_0, \pi/4)$ of center $X_0 = X_0(x_0, y_0, z_0)$ and radius $\pi/4$. Then every global minimizer of $f(X)$ must lie within the spherical convex hull of existing facilities.

Proof See Appendix.

Theorem 2.1.4: If all of the existing facilities lie on a great circle arc of length less than or equal to $\pi/2$, then one of the existing facilities is a global minimizer of the problem.

Proof. According to the Theorem 2.1.1, there is global minimizer on the spherical convex hull of the existing facilities. In this case, the spherical convex hull of the existing facilities is the great circle segment passing through all the existing facilities and having two of the existing facilities as ends. Straightening this great circle segment into a straight line segment, the spherical facility location problem is transformed into a equivalent one dimensional Euclidean facility location problem. Let the existing facilities be ordered (from one end to the other along the great circle segment) as $Ex_1', Ex_2', \dots, Ex_m'$. Find the integer t such that

$$\sum_{i=1}^{t-1} w_i < \frac{1}{2} \sum_{i=1}^m w_i \leq \sum_{i=1}^t w_i. \quad (2.3)$$

Then Ex_t is a global minimizer of the spherical facility location problem. ■

2.2 Weiszfeld Approach

The following approach duplicates the Weiszfeld procedure for planar, Euclidean location problems. It is due to Drezner and Wesolowsky [11] and can also be found in the textbook

Love and Morris [22].

Drezner and Wesolowsky considered the **WeberSphereLoc** problem where they used two ways of measuring distances. One is simply the shortest length of arc α . The other norm ($d_1 = 4 \sin^2(\alpha/2)$ and $d_2 = \pi \sin^2 \alpha/2$) may be used to approximate squared arc distance on a hemisphere and also rough approximation for arc distance (see the note under Definition 1.2.8). The difference between d_1 and d_2 is only a multiplicative constant. The optimum point using d_1 is always the same as the one using d_2 because for the purpose of optimizing location their problem, the constant is irrelevant.

Consider three distances α , d_1 and d_2 are as the measuring distances on the surface of the sphere. Then from equation (2.1), we can write the objective functions $F_\alpha[X(\phi, \theta)]$, $F_{d_1}[X(\phi, \theta)]$, $F_{d_2}[X(\phi, \theta)]$ with respect to the above distances as follows :

$$F_\alpha(X) = \sum_{i=1}^m w_i \alpha_i. \quad (2.4)$$

$$F_{d_1}(X) = 4 \sum_{i=1}^m w_i \sin^2(\alpha_i/2) \quad (2.5)$$

$$F_{d_2}(X) = \pi \sum_{i=1}^m w_i \sin^2(\alpha_i/2) \quad (2.6)$$

Let

$$F(X) = \sum_{i=1}^m w_i \sin^2(\alpha_i/2) \quad (2.7)$$

It is evident that the point that minimizes F is the same as that which minimizes F_{d_1} and F_{d_2} .

Property 2.2.1: The sum of the objective function evaluated at a point and at its antipode is a constant, and equal to $\pi \sum_{i=1}^m w_i$ in distances α and d_2 and, $4 \sum_{i=1}^m w_i$ in d_1 .

1. The shortest length of arc from the point X to the given demand point Ex_i is α_i

2. The shortest length of arc from the antipode \bar{X} of X to the demand point Ex_i is :

$$(\pi - \alpha_i)$$

3. Sum of the of the objective value at X and \bar{X} is

- in distance α : $\sum_{i=1}^m \alpha_i w_i + \sum_{i=1}^m (\pi - \alpha_i) w_i = \pi \sum_{i=1}^m w_i$
- in distance d_1 : $\sum_{i=1}^m 4 \sin^2(\alpha_i) w_i + \sum_{i=1}^m 4 \sin^2(\pi - \alpha_i) w_i = 4 \sum_{i=1}^m w_i$
- in distance d_2 : $\sum_{i=1}^m \pi \sin^2(\alpha_i) w_i + \sum_{i=1}^m \pi \sin^2(\pi - \alpha_i) w_i = \pi \sum_{i=1}^m w_i$

Property 2.2.2: A point is the minimum to a problem if and only if its antipode is the maximum.

Property 2.2.3: A point and its antipode with equal weights can be added to the problem without a change in the optimal location of the facility.

Let the point Ex_{m+1} with weight w_{m+1} . Now add the this point and its antipode with the same weight w_{m+1} to the set of demand points. Then the objective function is

$$\begin{aligned} f_{New}(X) &= \sum_{i=1}^m w_i \alpha_i + w_{m+1} \alpha_{m+1} + w_{m+1} (\pi - \alpha_{m+1}). \\ &= \sum_{i=1}^m w_i \alpha_i + \pi w_i. \end{aligned}$$

As πw_i is constant, the optimal location of the facility of $f_{New}(X)$ is the same the optimum of $f(X)$.

Property 2.2.4: A point with weight w_i can be replaced by its antipode with weight $-w_i$, without changing the optimal location of the facility.

By replacing the point Ex_j with weight w_j , we have the objective function

$$\begin{aligned} f^{New}(X) &= \sum_{i=1}^{j-1} w_i \alpha_i + (-w_j)(\pi - \alpha_j) + \sum_{i=j+1}^m w_i \alpha_i \\ &= \sum_{i=1}^m w_i \alpha_i - \pi w_j. \end{aligned}$$

As $-\pi w_j$ is constant, the optimal location of $f(X)$ will not change.

Computation of Stationary Points

Given two points $X = X(\phi, \theta)$ and $X_i = X_i(\phi_i, \theta_i)$, the shortest length of arc, $\alpha_i = \alpha_i(X, X_i)$ has the form (1.1)

$$\alpha_i = \arccos[\cos \phi \cos \phi_i \cos(\theta - \theta_i) + \sin \phi \sin \phi_i] \quad (2.8)$$

Now consider the solution of the extremal conditions for the objective functions $F(X)$ and $F_\alpha(X)$ using shortest length of arc α_i .

Then the partial derivatives of $F(X)$ are :

$$\frac{\partial F}{\partial \phi} = -\frac{1}{2} \sum_{i=1}^m w_i [-\sin \phi \cos \phi_i \cos(\theta - \theta_i) + \cos \phi \sin \phi_i] \quad (2.9)$$

$$\frac{\partial F}{\partial \theta} = \frac{1}{2} \cos \phi \sum_{i=1}^m w_i \cos \phi_i \sin(\theta - \theta_i). \quad (2.10)$$

Note that at the poles, $\cos \phi = 0$ and thus $\frac{\partial F}{\partial \theta} = 0$. This simply means that here a change in θ will not change the point. $\frac{\partial F}{\partial \phi} = \frac{\partial F}{\partial \theta} = 0$ yields an explicit solution and derived by:

$$\tan \theta = \frac{\sum_{i=1}^m w_i \cos \phi_i \sin \theta_i}{\sum_{i=1}^m w_i \cos \phi_i \cos \theta_i} \quad (2.11)$$

$$\frac{\tan \phi}{\sin \theta} = \frac{\sum_{i=1}^m w_i \sin \phi_i}{\sum_{i=1}^m w_i \cos \phi_i \sin \theta_i} \quad (2.12)$$

Equations (2.11) and (2.12) produce two solutions for θ and ϕ which are antipodes.

The following Theorem represents the conditions under which a demand point at $Ex_k(\phi_k, \theta_k)$ is a local optimum of $F_\alpha(X)$.

Theorem 2.2.1: [22] There is a local minimum at point Ex_k if and only if

$$w_k \geq (A_k^2 + B_k^2)^{1/2}, \quad (2.13)$$

where,

$$A_k = \sum_{i=1, i \neq k}^m \frac{w_i}{\sin \alpha_{ik}} [-\sin \phi_k \cos \phi_i \cos(\theta_k - \theta_i) + \cos \phi_k \sin \phi_i] \quad (2.14)$$

$$B_k = \sum_{i=1, i \neq k}^m \frac{w_i}{\sin \alpha_{ik}} \cos \phi_i \sin(\theta_i - \theta_k) \quad (2.15)$$

with

$$\alpha_{ik} = \arccos[\cos \phi_k \cos \phi_i \cos(\theta_k - \theta_i) + \sin \phi_k \sin \phi_i]$$

be the shortest arc distance between points Ex_i and Ex_k .

Proof Consider the objective function $F_\alpha(X) = \sum_{i=1}^m w_i \alpha_i$. It can be shown that for movement from point Ex_k :

$$\begin{aligned} dF_\alpha(X) &= w_k [(d\phi)^2 + \cos^2 \phi_k (d\theta)^2]^{1/2} \\ &- d\phi \sum_{i \neq k}^m w_i (-\sin \phi_k \cos \phi_i \cos(\theta_k - \theta_i) \\ &+ \cos \phi_k \sin \phi_i) / \sin \alpha_{ki} \\ &- d\theta \sum_{i \neq k}^m w_i (\cos \phi_k \cos \phi_i \sin(\theta_i - \theta_k)) / \sin \alpha_{ki}. \end{aligned}$$

For a local minimum, $dF_\alpha(X) > 0$, and hence, we must show

$$w_k ((d\phi)^2 + \cos^2 \phi_k (d\theta)^2)^{1/2} - A_k d\phi - B_k \cos \phi_k d\phi_k > 0.$$

Letting $L = d\theta \cos \phi_k / d\phi$, we have

$$|d\phi| w_k (1 + L)^{1/2} > d\phi (A_k + LB_k)$$

and so :

$$w_k > d\phi(A_k + LB_k)(1 + L^2)^{-1/2}/|d\phi|.$$

Note that $d\phi/|d\phi|$ is ± 1 . It can be shown that :

$$\begin{aligned} -(A_k^2 + B_k^2)^{1/2} &\leq (A_k + LB_k)/(1 + L^2)1/2 \\ &\leq (A_k^2 + B_k^2)^{1/2} \end{aligned}$$

and hence, the condition

$$w_k \geq (A_k^2 + B_k^2)^{1/2}$$

is necessary and sufficient for $dF_\alpha(X) > 0$ for every L . ■

We now consider the extremal conditions for the objective function F_α . The partial derivatives are:

$$\frac{\partial F_\alpha}{\partial \phi} = - \sum_{i=1}^m \frac{w_i}{\sin \alpha_i} [-\sin \phi \cos \phi_i \cos(\theta - \theta_i) + \cos \phi \sin \phi_i] \quad (2.16)$$

$$\frac{\partial F_\alpha}{\partial \theta} = \cos \phi \sum_{i=1}^m \frac{w_i}{\sin \alpha_i} \cos \phi_i \sin(\theta - \theta_i) \quad (2.17)$$

Further note that at the poles, $\cos \phi = 0$ and thus $\frac{\partial F_\alpha}{\partial \theta} = 0$. Solution of $\frac{\partial F_\alpha}{\partial \phi} = \frac{\partial F_\alpha}{\partial \theta} = 0$ yields :

$$\tan \theta = \frac{\sum_{i=1}^m \frac{w_i}{\sin \alpha_i} \cos \phi_i \sin \theta_i}{\sum_{i=1}^m \frac{w_i}{\sin \alpha_i} \cos \phi_i \cos \theta_i} \quad (2.18)$$

$$\frac{\tan \phi}{\sin \theta} = \frac{\sum_{i=1}^m \frac{w_i}{\sin \alpha_i} \sin \phi_i}{\sum_{i=1}^m \frac{w_i}{\sin \alpha_i} \cos \phi_i \sin \theta_i} \quad (2.19)$$

This is an implicit solution because ϕ and θ are used in the calculation of the α_i 's.

In equations (2.18) and (2.19), solutions are also come in pairs of antipodes. Once a solution is obtained, its antipode is also checked. Note also that (2.18) and (2.19) may give us saddle points as well as local minima or maxima.

Therefore, (2.18) and (2.19) can be used iteratively in a manner analogous to the Weiszfeld procedure, to find the solution if we are close enough to the local minimum or maximum.

The developed algorithm is as follows :

Algorithm 2.2.1: (Weiszfeld for WeberSphereLoc)

Step 1. Choose a starting point $\phi^{(0)}, \theta^{(0)}$. Set $k = 0$.

Step 2. Compute $\phi^{(k+1)}$ and $\theta^{(k+1)}$ by (2.18) and (2.19) using $\phi^{(k)}, \theta^{(k)}$ to calculate α_i .

Step 3. If $|\phi^{(k+1)} - \phi^{(k)}| + |\theta^{(k+1)} - \theta^{(k)}| > \epsilon$ go to step 2.

Step 4. $[\phi^{(k+1)}, \theta^{(k+1)}]$ and its antipode to get the minimal and maximal point.

Note that the sum of the objective function evaluated at a point and at its antipode is a constant and equal to $\pi \sum_{i=1}^m w_i$ in distance α (see Property 2.2.1). The solutions of the above algorithm come in pairs of antipodes and one of these points is a minimum point and according to the Property 2.2.3, the other point is the maximum for the problem.

That means, if $F_\alpha < \pi/2 \sum_{i=1}^m w_i$, then the point is a minimum.

Remark : There might be several local minima and we want to find the best of these. As a point and its antipode as starting points result same solution, we can choose starting points only in one hemisphere. Further, there are various strategies in choosing starting points: randomly, in a pattern, using the Norm d_1 or d_2 solution. In addition all demand points should be checked for local minima by equation (2.13).

2.3 Approximate Algorithm Using Candidate Lists [14]:

In this section we discuss an always convergent algorithm (Drezner [14]) for the **WeberSphereLoc** problem using a given candidate set of points on the surface of the sphere.

In this approach we assume that the **WeberSphereLoc** problem (2.1) is in the normal form (see Definition 1.2.17).

We construct here a lower bound on the optimal value of the objective function based on a given set of points on the surface of the sphere. Therefore consider a given set J of points on the surface of the sphere to construct a lower bound on the optimal value of the objective

function $f(X)$. Note that the set J is different from the set of demand points .

Let $f_j = f(X_j)$ and $d(X, X_j)$ be the distance between $X = (\phi, \theta)$ and $X_j = X_j(\phi_j, \theta_j)$ for $X_j \in J$, and let $w = \sum_{i=1}^m w_i$.

The triangle inequality implies for all $X \in \mathcal{S}_0$:

$$d(X, X_j) + d(X, Ex_i) \geq d(X_j, Ex_i).$$

Then we have

$$f_j - f(X) = \sum_{i=1}^m w_i [d(X_j, Ex_i) - d(X, Ex_i)] \leq \sum w_i d(X, X_j) = wd(X, X_j).$$

Thus, $f(X) \geq f_j - wd(X, X_j)$, and

$$f(X) \geq \max_{j \in J} \{f_j - wd(X, X_j)\}. \quad (2.20)$$

Let f^* be the optimal solution to problem (2.1). By equation (2.20): $f^* \geq f^0$ where :

$$f^0 = \min_{X \in \mathcal{S}_0} \{ \max_{j \in J} \{f_j - wd(X, X_j)\} \}. \quad (2.21)$$

Finding f^0 in (2.21) is the minimax single facility location problem which can be optimally solved [13]. Based on this observation, we can minimize $f(X)$ to within an ε of the optimal value of the objective function for any $\varepsilon > 0$.

The resulting algorithm can be represented as follows:

Algorithm 2.3.1: (Candidate list for **WeberSphereLoc**)

Step 1. Let J be any two point subset of the sphere and set $f^m := \min_{j \in J} \{f_j\}$.

Step 2. Compute f^0 by solving the minimization problem in (2.21). Add the solution point to J . Update f^m .

Step 3. If $f^0 < f^m - \varepsilon$, go to Step 2.

Step 4. stop with f^m as the approximate optimal solution.

Note that if the solution point in Step 2 is a point in J , then $f^0 = f^m$ and the algorithm terminates. But, when $f^0 < f^m - \varepsilon$, the solution to Problem (2.21) is at least ε/w from all points in J . Since there is a limit to the number of points, one can arrange on the sphere which are at least ε/w from each other. Then f^0 will get within an ε of f^m . Then for a give $\varepsilon > 0$, the algorithm must be finite.

2.4 Steepest Descent Algorithm for WeberSphereLoc [32]

Consider the WeberSphereLoc problem (2.1) in the following form :

$$f(X) = \sum_{i=1}^n w_i \arccos(Ex_i^T \cdot X). \quad (2.22)$$

where, the points $Ex_i, i = 1, 2, \dots, m$ and X are three dimensional points on the surface of the sphere and $\arccos(Ex_i^T X)$ is the shortest length of arc between Ex_i and X .

Note that the dot product $(Ex_i \cdot X)$ of Ex_i and X is equal to $\| Ex_i \| \| X \| \cos \alpha$, where $\| Ex_i \| = \| X \| = 1$ and $\alpha = \alpha(Ex_i, X)$ is the angle between two vectors Ex_i and X . So, we have $\alpha = \arccos(Ex_i \cdot X)$ or $\alpha = \arccos(Ex_i^T X)$.

This objective function (2.22) is defined only on the sphere \mathcal{S}_0 . We extend the domain of f to all $X \in \mathfrak{R}^3$ such that $X \neq 0$. For any nonzero $X \in \mathfrak{R}^3$, the function value at X is defined to be $f(\frac{X}{\|X\|})$. Then the objective function of **WeberSphereLoc** can be written as

$$F(X) = \sum_{i=1}^n w_i \cos^{-1}(X_i \frac{X}{\|X\|}) \quad (2.23)$$

and the only constraint is $\underline{X} \neq \underline{0}$.

In the following we will assume that all of the existing facilities of the WeberSphereLoc problem are include within a spherical disk of radius $\pi/4$. For convenience, we will assume that the center of this spherical disk of radius $\pi/4$ is $(0,0,1)$. Therefore, all the existing facilities are above the xy -plane. We proved that every global minimizer of this problem must lie within the spherical convex hull of the existing facilities (see Theorem 2.1.3).

Next, we consider the optimality conditions for the spherical facility location problem in terms of the optimality conditions for the corresponding Euclidean facility location problem. To show this, let X be a point on the surface of the sphere which does not coincide with any of the existing facilities. Then $F(X)$ is differentiable at X , with gradient given by

$$\begin{aligned}\nabla F(X) &= \sum_{i=1}^m w_i \frac{-1}{\sqrt{1 - ((X/\|X\|)^T Ex_i)^2}} \frac{\|X\| Ex_i - X^T Ex_i (X/\|X\|)}{\|X\|^2} \\ &= \sum_{i=1}^m w_i \frac{X - \frac{Ex_i}{(X/\|X\|)^T Ex_i}}{\|X - \frac{Ex_i}{(X/\|X\|)^T Ex_i}\|}.\end{aligned}\quad (2.24)$$

If X coincides with one of existing facilities, Ex_t , then $F(X)$ is not differentiable at X . In this case, for any nonzero vector d , the directional derivative $F'(Ex_t; d)$ of $F(X)$ at point Ex_t in direction d is given by

$$F'(Ex_t; d) = d^T \sum_{i=1, i \neq t}^m w_i \frac{Ex_t - (Ex_i/Ex_t^T Ex_i)}{\|Ex_t - (Ex_i/Ex_t^T Ex_i)\|} + w_t \sqrt{\|d\|^2 - (Ex_t^T d)^2}.\quad (2.25)$$

Notice that all of the n points $\frac{Ex_i}{(X/\|X\|)^T Ex_i}, i = 1, 2, \dots, m$ lie on the plane which is tangent to the sphere, S at point X . For any given X on the surface of the sphere, define

$$Ex_i^X = \frac{Ex_i}{(X/\|X\|)^T Ex_i}, \quad i = 1, 2, \dots, m.\quad (2.26)$$

Now, we have a planner Euclidean facility location problem defined on the plane as follows :

$$\min \mathcal{F}_X(y) = \sum_{i=1}^m w_i \|y - Ex_i^X\|.\quad (2.27)$$

If X does not coincide with any of the Ex_i^X 's, then $\mathcal{F}_X(y)$ is differentiable at X with gradient given by

$$\nabla \mathcal{F}_X(X) = \sum w_i \frac{X - Ex_i^X}{\|X - Ex_i^X\|}.\quad (2.28)$$

If X coincides with Ex_t^X , the $\mathcal{F}_X(y)$ is not differentiable at X . In this case, for any nonzero vector d , the directional derivative $\mathcal{F}'_X(Ex_t; d)$ of $\mathcal{F}_X(y)$ at point Ex_t in direction d is given by

$$\mathcal{F}'_X(Ex_t; d) = d^T \sum_{i=1, i \neq t}^m w_i \frac{X - Ex_i^X}{\|X - Ex_i^X\|} + w_t \|d\|.\quad (2.29)$$

From the optimality conditions for the planner facility location problem [22], we have the optimality conditions for the planner location problem (2.27) as follows :

(i) An existing facility Ex_t is a global minimizer of (2.27) if and only if

$$\left\| \sum_{i=1, i \neq t}^m w_i \frac{Ex_t - \frac{Ex_i}{Ex_t^T Ex_i}}{\|Ex_t - (Ex_i/Ex_t^T Ex_i)\|} \right\| \leq w_t. \quad (2.30)$$

(ii) A smooth point X is a global minimizer of (2.27) if and only if

$$\sum_{i=1}^m w_i \frac{X - \frac{Ex_i}{(X/\|X\|)^T Ex_i}}{\|X - \frac{Ex_i}{(X/\|X\|)^T Ex_i}\|} = 0. \quad (2.31)$$

Note that the above optimality conditions are also the optimality conditions for the spherical facility location problem (2.22).

Theorem 2.4.1: An existing facility Ex_t is a global minimizer of (2.22) if and only if

$$\left\| \sum_{i=1, i \neq t}^m w_i \frac{Ex_t - \frac{Ex_i}{Ex_t^T Ex_i}}{\|Ex_t - (Ex_i/Ex_t^T Ex_i)\|} \right\| \leq w_t. \quad (2.32)$$

A smooth point X is a global minimizer of (2.22) if and only if

$$\sum_{i=1}^m w_i \frac{X - \frac{Ex_i}{(X/\|X\|)^T Ex_i}}{\|X - \frac{Ex_i}{(X/\|X\|)^T Ex_i}\|} = 0. \quad (2.33)$$

Proof Let us consider the non-smooth case first. Suppose that

$$\left\| \sum_{i=1, i \neq t}^m w_i \frac{Ex_t - \frac{Ex_i}{Ex_t^T Ex_i}}{\|Ex_t - (Ex_i/Ex_t^T Ex_i)\|} \right\| \geq w_t. \quad (2.34)$$

Let $d = -\sum_{i=1, i \neq t}^m w_i \frac{Ex_t - \frac{Ex_i}{Ex_t^T Ex_i}}{\|Ex_t - (Ex_i/Ex_t^T Ex_i)\|}$. Then $Ex_t^T d = 0$ because d is on the plane with Ex_t as its normal vector. Therefore, it follows from (2.25) that

$$F'(Ex_t; d) = d^T \sum_{i=1, i \neq t}^m w_i \frac{Ex_t - (Ex_i/Ex_t^T Ex_i)}{\|Ex_t - (Ex_i/Ex_t^T Ex_i)\|} + w_t \|d\| = \|d\| (w_t - \|d\|) < 0. \quad (2.35)$$

This means that d is a descent direction of $F(X)$ at point Ex_t . Therefore, Ex_t could not be a local minimizer. This proves that (2.32) is a necessary condition for Ex_t to be a minimizer of (2.22).

Now we have to prove that (2.32) is also a sufficient condition for the global optimality of Ex_t of the problem (2.22). Suppose that Ex_t is not a global minimizer of (2.22). Then there

exists a point Y within the spherical convex hull of the existing facilities such that $f(Y) < f(Ex_t)$. Since $f(X)$ is spherically convex, every point on the arc (Ex_t, Y) (except Ex_t) has a function value smaller than $f(Ex_t)$. Therefore, we may assume that $Ex_t^T Y \neq 0$ without loss of generality. Define $\bar{Y} = (Y/Ex_t^T Y)$. Then $F(\bar{Y}) = F(Y) < F(Ex_t)$. For any $\lambda \in (0, 1)$, define $\bar{\rho}(Ex_t, Y, \lambda) = \rho(Ex_t, Y, \lambda)/Ex_t^T \rho(Ex_t, Y, \lambda)$. Let $\beta = \arccos(Ex_t^T Y)$. Then for any $\lambda \in (0, 1)$, we have

$$\begin{aligned} F\left(\left(1 - \frac{\tan(\lambda\beta)}{\tan(\beta)}\right)Ex_t + \frac{\tan(\lambda\beta)}{\tan(\beta)}\bar{Y}\right) &= F(\bar{\rho}(Ex_t, Y, \lambda)) = f(\rho(Ex_t, Y, \lambda)) \quad (2.36) \\ &\leq (1 - \lambda)f(Ex_t) + \lambda f(Y) \\ &= F(Ex_t) + \lambda(F(\bar{Y}) - F(Ex_t)). \end{aligned}$$

This implies that $F'(Ex_t; d) \leq F(\bar{Y}) - F(Ex_t) < 0$, where $d = \bar{Y} - Ex_t$. Since $Ex_t^T d = 0$, it follows from (2.25) and (2.32) that $F'(Ex_t; d) \geq 0$. This contradiction proves that (2.32) is a sufficient condition for the optimality of Ex_t of (2.22).

Now consider the smooth case. It is clear that (2.33) is a necessary condition for X to be a minimizer of (2.22). Suppose that X is not a global minimizer of (2.22). Then there exists a point Y within the spherical convex hull of the existing facilities such that $f(Y) < f(X)$. As in the non-smooth case, we may assume that $X^T Y \neq 0$ without loss of generality. Define $\bar{Y} = Y/X^T Y$. Then $F(\bar{Y}) = F(Y) < F(X)$. For any $\lambda \in (0, 1)$, define $\bar{\rho}(X, Y, \lambda) = \rho(X, Y, \lambda)/X^T \rho(X, Y, \lambda)$. Let $\beta = \arccos(X^T Y)$. Then for any $\lambda \in (0, 1)$, we have

$$\begin{aligned} F\left(\left(1 - \frac{\tan(\lambda\beta)}{\tan(\beta)}\right)X + \frac{\tan(\lambda\beta)}{\tan(\beta)}\bar{Y}\right) &= F(\bar{\rho}(X, Y, \lambda)) = f(\rho(X, Y, \lambda)) \quad (2.37) \\ &\leq (1 - \lambda)f(X) + \lambda f(Y) \\ &= F(X) + \lambda(F(\bar{Y}) - F(X)). \end{aligned}$$

$\Rightarrow F'(X; d) \leq F(\bar{Y}) - F(X) < 0$, where $d = \bar{Y} - X$.

However, $F'(Ex_t; d)$ must be zero since $\nabla F(X) = 0$. This is a contradiction and proves the Theorem. ■

In the next step of this procedure, we will present an algorithm for solving the weber spherical facility location problem. The algorithm first checks if any of the existing facilities is a global minimizer of the problem. If this doesn't, the algorithm generates a sequence of

descent search directions and iteration points with decreasing function values.

The relevant algorithm is as follows:

Algorithm 2.4.1: Algorithm 3 (Descent algorithm for **WeberSphereLoc**)

Input: Existing facilities $Ex_i, i = 1, \dots, m$ contained in a spherical disk of radius $\alpha \leq \pi/4$.

Step 1. Find an existing facility Ex_t , such that $f(Ex_t) \leq f(Ex_i)$ for all $i = 1, 2, \dots, m$.

Check the optimality conditions for Ex_t . If Ex_t is an optimal solution, Stop.

Step 2. Let $d = -\sum_{i=1, i \neq t}^m w_i(Ex_t - Ex_i^{Ex_t})/\|Ex_t - Ex_i^{Ex_t}\|$

where, $Ex_i^{Ex_t} = \frac{Ex_i}{(Ex_t/\|Ex_t\|)^T Ex_i}$. Find a small step size $\beta > 0$ such that the point $Ex_t + \beta d$ lies in the convex hull of $Ex_i^{Ex_t}, i = 1, 2, \dots, m$, and that $X^1 = Ex_t + \alpha d/\|Ex_t + \alpha d\|$ has a function value less than $f(Ex_t)$. Let $k = 1$.

Step 3. Compute $Ex_i^{X^k}$ for $i = 1, 2, \dots, m$.

Compute $d^k = -\sum_{i=1}^m w_i(X^k - Ex_i^{X^k})/\|X^k - Ex_i^{X^k}\|$. If $d^k = 0$, Stop; Otherwise compute $\beta^k = \frac{1}{\sum_{i=1}^m w_i/\|X^k - Ex_i^{X^k}\|}$.

Step 4. Set $X^{k+1} = X^k + \beta^k d^k/\|X^k + \beta^k d^k\|$. If $f(X^{k+1}) \leq f(X^k) - 0.1\beta^k\|d^k\|^2$, then replace k with $k+1$ and goto Step 3; Otherwise replace β^k with $0.5\beta^k$ and goto Step 4.

Note that Step 1 and Step 2 are used to eliminate non smooth points from further consideration. Let Ex_t be an existing facility whose objective function value is minimum among all the existing facilities. If Ex_t satisfies the optimality condition (2.32), then it is also a global minimizer of the problem. If Ex_t does not satisfy the optimality condition (2.32), d computed in Step 2 is a descent direction of $f(X)$ at point Ex_t . Step 3 computes the search direction d^k , which is the negative of the gradient. If $d^k = 0$, then X^k satisfies the optimality condition (2.33), and therefore it is a global minimizer. If $d^k \neq 0$, then it is a descent direction and Step 4 finds a better location.

It is clear that the description of the algorithm that the whole iteration sequence $\{X^k\}$ lie in the spherical convex hull of the existing facilities.

In the next step, we will prove global convergence of the algorithm. In Lemma 2.4.1, we will prove that when the algorithm stops after a finite number of iterations, it stops at a global minimizer and if the algorithm does not stop after a finite number of iterations, then the **WeberSphereLoc** problem has a strictly spherical convex objective function and therefore has only one local minimizer (also global minimizer) which is inside of the spherical convex hull of the existing facilities.

In Lemma 2.4.2, we prove that every accumulation point of the infinite sequence generated by the algorithm is a global minimizer of the **WeberSphereLoc** problem.

Lemma 2.4.1: If Algorithm 3 stops at X^k after a finite number of iterations, then X^k is a global minimizer of the **WeberSphereLoc** problem. If the algorithm generates an infinite sequence $\{X^k\}$, then the objective function (2.22) is strictly spherical convex, and therefore, the problem has only one local minimizer (also a global minimizer) which is inside of the spherical convex hull of the existing facilities.

Proof If the algorithm stops in Step 1, then Ex_t must satisfy the optimality condition (2.32). Therefore, it is a non smooth global minimizer. If the algorithm stops in Step 3, then d^k must be zero. In this case, X^k satisfies the optimality condition (2.33). Therefore, it is a smooth global minimizer.

Now, we will consider the case that the algorithm generates an infinite sequence $\{X^k\}$. It follows from Theorem 2.1.4 that all of the existing facilities do not lie on a great circle segment. This implies that all the existing facilities lie within the spherical disk of radius less than $\pi/4$. It then follows that the objective function (2.22) is strictly spherical convex. Therefore, the **WeberSphereLoc** problem has only one local minimizer (also a global minimizer) which is inside of the spherical convex hull of the existing facilities. ■

Lemma 2.4.2: Let \bar{X} be an accumulation point of $\{X^k\}$, i.e., there is a subsequence $\{X_t^k\}$ which converges to \bar{X} . Then \bar{X} is a global minimizer of the **WeberSphereLoc** problem.

Proof Assume that \bar{X} is not a global minimizer. Let $\bar{d} = -\sum_{i=1}^m w_i(\bar{X} - Ex_i^{\bar{X}}) / \|\bar{X} - Ex_i^{\bar{X}}\|$. Since \bar{X} is not a global minimizer, $\bar{d} \neq 0$. Therefore, there exists a pos-

itive number $\bar{\beta} \leq \frac{1}{\sum_{i=1}^m w_i / \|\bar{X} - Ex_i^{\bar{X}}\|}$ such that for all $\beta \in (0, \bar{\beta}]$, we have

$$F(\bar{X} + \beta \bar{d}) \leq F(\bar{X}) - 0.2\beta \|\bar{d}\|^2 < F(\bar{X}) - 0.1\beta \|\bar{d}\|^2. \quad (2.38)$$

From the definition, we can easily prove that $\{d_t^k\}$ converges to \bar{d} and that $\{\sum_{i=1}^m \frac{w_i}{\|X^{k_t} - Ex_i^{X^{k_t}}\|}\}$ converges to $\{\sum_{i=1}^m \frac{w_i}{\|\bar{X} - Ex_i^{\bar{X}}\|}\}$. It is then follows from the continuity of $F(\cdot)$ at \bar{X} that there exists integers T and l such that for $t \geq T$, we have

$$\beta_t^k \triangleq \frac{1}{2^l} \frac{1}{\sum_{i=1}^m w_i / \|X^{k_t} - Ex_i^{X^{k_t}}\|} \in (0, \bar{\beta}], \quad (2.39)$$

and that

$$F(X^{k_t} + \gamma^{k_t} d^{k_t}) < F(\bar{X}) - 0.1\gamma^{k_t} \|d^{k_t}\|^2. \quad (2.40)$$

Therefore, for $t \geq T$, we have $\beta_t^k \geq \gamma^{k_t}$ and that

$$F(X^{k_t} + 1) \leq F(X^{k_t}) - 0.1\gamma^{k_t} \|d^{k_t}\|^2. \quad (2.41)$$

Since $\{f(X^k)\}$ is strictly decreasing and that $F(X)$ is continuous at \bar{X} , the sequence $\{f(X^k)\}$ converges to $f(\bar{X})$. Taking limit in (2.41) when t approaches ∞ , we get

$$F(\bar{X}) \leq F(\bar{X}) - 0.1 \frac{1}{2^l} \frac{1}{\sum_{i=1}^m w_i / \|\bar{X} - Ex_i^{\bar{X}}\|} \|\bar{d}\|^2 < F(\bar{X}). \quad (2.42)$$

This is a contradiction. ■

Theorem 2.4.2: Algorithm 2.4.1 either stops at a global minimizer after a finite number of iterations; or generates an infinite sequence $\{X^k\}$ which converges to a global minimizer of the **WeberSphereLoc** problem.

Combining the two lemmas 2.4.1 and 2.4.2, proves the Theorem. ■

2.5 Big Region-Small Region Algorithm [18]

In their paper, they discussed the unconstrained Weber problem and the constrained Weber problem on the sphere. The unconstrained Weber problem is simply the **WeberSphereLoc** problem which we are discussing in our article. In the constrained Weber problem

(WeberSphereLoc^{constraint}), the new facility X must belong to a given (not necessarily convex or connected) subset \mathcal{F} of the surface of the sphere \mathcal{S}_0 . This subset can usually be approximated with sufficient precision by a set of n spherical triangles $T_j : \mathcal{F} = \cup_{j=1}^n T_j$.

This constrained problem is the complement of the restricted spherical location problem because in the restricted problem, the new facility should not be positioned in a given set (not necessarily convex or connected) on \mathcal{S}_0 .

Now, we will discuss the algorithm for (WeberSphereLoc^{constraint}) problem. This algorithm is a generalization of the "Big Square - Small Square (BSSS)" algorithm [17] with new bounding rules. The BSSS algorithm proceeds by

- (i) partitioning the smallest square containing the set of possible locations (feasible set) into sub squares;
- (ii) computing a lower bound of the objective function for those sub squares that intersect the feasible set;
- (iii) deleting the sub squares for which the lower bound exceeds the value of the best existing solution; and
- (vi) iterating until the length of a side of a square is smaller than a given tolerance.

We refer the generalized algorithm for spherical Weber problem as Big Region - Small Region (BRSR) and this is based on branch - and- bound method in a continuous space. It proceeds as follows :

- (i) partitioning the surface of the sphere \mathcal{S}_0 into regions Q_i defined by two latitudes and two longitudes (we start with an initial partitioning of \mathcal{S}_0 into 8 equal regions);
- (ii) deleting those regions which do not intersect the feasible region \mathcal{F} ;
- (iii) computing lower bounds \underline{f}_i on f on the remaining regions Q_i and deleting those regions for which the lower bound is greater than or equal to the value f_{opt} of the best solution X_{opt} yet obtained;

- (iv) computing the value of a feasible point in each remaining region Q_i and updating f_{opt} and X_{opt} if a point with a smaller value than that of the incumbent is found;
- (v) choosing the remaining region Q_i with smallest lower bound \underline{f}_i and partitioning it into four new regions;
- (vi) iterating the tests on the new regions Q_i obtained until the relative error $\frac{f_{opt} - \underline{f}_i}{f_{opt}}$ is smaller than a given tolerance ϵ .

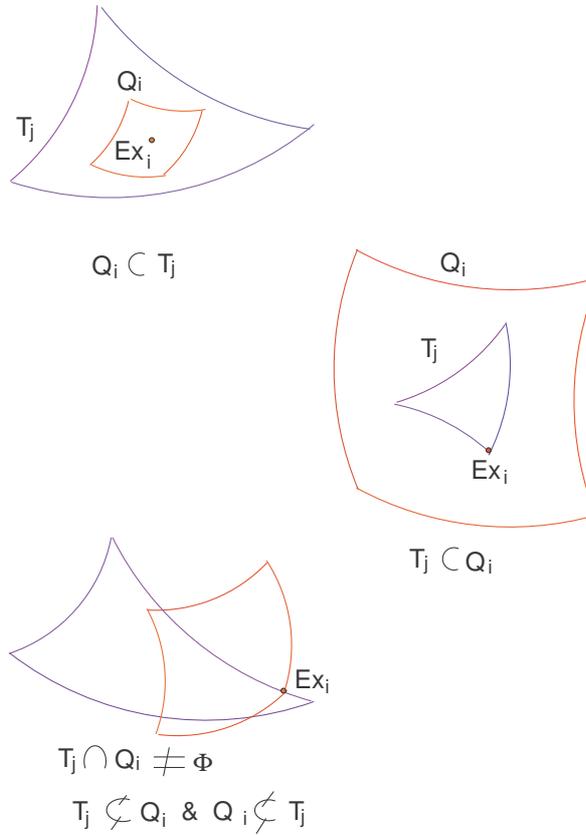


Fig. 2.1:

The detailed rules of (BRSR) are as follows:

a) Initialization $Q_1 \leftarrow S$;

$I \leftarrow \{1\}$; (I is the index set of unsolved subproblems)

$I_{new} \leftarrow \{1\}$; (I_{new} is the index set of subproblems for which a lower bound has not

been computed)

$X_{opt} \leftarrow$ randomly generated point in \mathcal{F}

(if one can be found, else $X_{opt} \leftarrow \infty$, i.e., a conventional value);

$f_{opt} \leftarrow f(X_{opt})$ if a point in \mathcal{F} has been found, else $f_{opt} \leftarrow \infty$;

b) Feasibility Test For all Q_i such that $i \in I_{new}$ do

compute $Q_i \cap \mathcal{F}$;

if $Q_i \cap \mathcal{F} = \emptyset$ delete i from I_{new} ;

EndFor;

c) Optimality Test) For all Q_i such that $i \in I_{new}$ do

compute a lower bound \underline{f}_i on $f(X)$ for $X \in Q_i$;

if $\underline{f}_i \geq f_{opt}$ delete i from I_{new} ;

EndFor;

(d) Improved Solution Test(see Figure 2.5)

For all Q_i such that $i \in I_{new}$ do

if, for some $j \in \{1, \dots, n\}$, $Q_i \subset T_j$,

compute the value $f(X_i)$ of the central point X_i of Q_i ;

if, for some $j \in \{1, \dots, n\}$, $T_j \subset Q_i$,

compute the value $f(X_i)$ of an arbitrary chosen extreme point X_i of T_j ;

else, for some $j \in \{1, \dots, n\}$ such that $T_j \cap Q_i \neq \emptyset$,

compute the value $f(X_i)$ of a point X_i on the boundaries of T_j and Q_i ; if $f(X_i) < f_{opt}$,

set $f_{opt} \leftarrow f(X_i)$ and $X_{opt} \leftarrow X_i$; EndFor;

add all indices $i \in I_{new}$ to I ;

(e) Branching and stopping conditions

If $I = \emptyset$, stop : The problem is infeasible; else select Q_i such that $\underline{f}_i = \min_{j \in I} \underline{f}_j$.

If $\frac{f_{opt} - \underline{f}_i}{\underline{f}_i} \leq \epsilon$: stop, an ϵ -optimal solution X_{opt} with value f_{opt} has been found, else partition Q_i into four new regions Q_{i_j} , $j = 1, 2, 3, 4$;

Remove i from I and set I_{new} equal to the set of indices of the new regions;

Return to b).

*The algorithmic scheme presented here can be simplified for the **WeberSphereLoc** problem as follows :*

(i) Step (b) is omitted (ii) Step (d) reduces to the first case.

It remains to specify how regions are partitioned. The easiest way to handle regions is to define them by a pair of latitudes and longitudes. Then four new regions of Q_i are obtained by taking as new boundaries the average of the two latitudes and the two longitudes.

3. SPHERICAL CENTER PROBLEM

As in the case of **WeberSphereLoc**, we assume that each model which is described in this chapter will deal with a unit sphere, \mathcal{S}_0 where the radius is equal to one. Every point X on the sphere is defined by its latitude ϕ and longitude θ and it is denoted by $X = X(\phi, \theta)$. Consider m demand points (or existing locations) $Ex_i, i = 1, 2, \dots, m$, on the surface of the sphere with associated weights w_i and some distance function $d(X, Y)$, which measures the distances between spherical points X and Y .

We consider single facility spherical location problem (**SphereLoc**) of the center type. I.e., we solve

$$\min_{X \in \mathcal{S}_0} h(X) := \max_{i=1}^m w_i d(X, Ex_i) \quad \text{CenterSphereLoc} \quad (3.1)$$

where X is the unknown location.

Unlike on the plane, the **CenterSphereLoc** problem (as well as **WeberSphereLoc**) has undesirable properties, such as non-convexity and non differentiability of the objective function at both the demand points and at the corresponding antipodal points, and restriction on the domain of the objective function. Analogous to the Theorem 1.2.1, if all the demand points are included within a spherical disk of radius $\pi/4$, then $h(X)$ is convex (and thus every local optimum is also global).

However, if it can be predetermined that all the demand points lie on a hemisphere, one can apply mathematical programming or geometrical solution methods for the minimax location problems in the Euclidean plane to solve the **CenterSphereLoc** problem.

In this chapter, we will discuss some solution approaches to solve the **CenterSphereLoc** problem on the unit sphere \mathcal{S}_0 as well as on a hemisphere.

3.1 An Iterative Procedure to find the Global Optimum for CenterSphereLoc [13]

Consider the **CenterSphereLoc** problem with great circle arc distance $\alpha_i = \text{arc}(Ex_i, X)$ (see (1.1)) between the demand point Ex_i and the new location X on the surface of the sphere. I.e., we want to minimize

$$h(X) = \max_{i=1}^m w_i \text{arc}(X, Ex_i) = \max_{i=1}^m w_i \alpha_i(X, Ex_i) \quad (3.2)$$

over all $X = X(\phi, \theta) \in \mathcal{S}_0$.

We can formulate the **spherical maximin** problem analogously.

The following Theorem shows that spherical maximin and minimax location problems are equivalent.

Theorem 3.1.1: [13]: Let the optimal solution to the spherical maximin problem be X^* . If a minimax problem is formed by replacing the demand points $Ex_i, i = 1, \dots, m$ with their antipodes $\bar{E}x_i, i = 1, \dots, m$ and by adding the constant $c = -\pi w_i$ to $w_i \bar{\alpha}_i$ then the optimal solution to this minimax problem is also X^* .

Proof: The distance $\bar{\alpha}_i$ between X and $\bar{E}x_i$ (the antipode of Ex_i) is $\pi - \alpha_i$ as any great circle containing Ex_i also contains $\bar{E}x_i$. Now consider the minimax problem :

$$\begin{aligned} \min_{X \in \mathcal{S}_0} \max_{i=1}^m w_i \bar{\alpha}_i + (\pi w_i) &= \min_{X \in \mathcal{S}_0} \max_{i=1}^m w_i (\pi - \alpha_i) - \pi w_i \\ &= \min_{X \in \mathcal{S}_0} \max_{i=1}^m (-) w_i \alpha_i \\ &= \min_{X \in \mathcal{S}_0} \{ (-) \min_{i=1}^m w_i \alpha_i \} \\ &= (-) \max_{X \in \mathcal{S}_0} \min_{i=1}^m w_i \alpha_i \end{aligned}$$

It follows the Theorem. ■

Theorem 3.1.2: [13]: Let X^{l-opt} be a local minimum of $h(X)$. Let \mathcal{E}'_x be the set of all i such that $h(X^{l-opt}) = w_i \alpha_i$. Then, if $\alpha_i < \pi/2$ for $i \in \mathcal{E}'_x$, then X^{l-opt} is the global minimum.

Proof

$\alpha_i < \pi/2 \quad \Rightarrow \quad$ all $Ex_i \in \mathcal{E}'_x$ in the hemisphere
centered at X^{l-opt}

$\xRightarrow{Th.1.2.1} \mathcal{E}'_x$ is a convex set

$\xRightarrow{Th.1.2.1} \alpha_i$ is convex on \mathcal{E}'_x

Then $h(X)$ is a convex function on \mathcal{E}'_x

$\xRightarrow{Th.1.2.1} X^{l-opt}$ is the global minimum. ■

Note that when $\alpha_i < \pi/2$, then all the demand points in \mathcal{E}'_x are in the hemisphere centered at X^{l-opt} . Further, the value of the objective function for the modified problem based on the demand points in \mathcal{E}'_x is only a lower bound for the value of the objective function for the problem based on all demand points.

Finding a local minimax point :

Here, we propose a method of finding a local minimum for $h(X)$ is a version of steepest descent for minimax problems. The proposed method is as follows :

Define

$$h_i(X) = w_i \alpha_i, \quad \text{for } i=1, \dots, m \tag{3.3}$$

Then we have

$$h(X) = \max_i \{h_i(X)\}$$

Also define

$$I_\epsilon(X) = \{i | h_i(X) \geq h(X) - \epsilon\}, \tag{3.4}$$

where ϵ is a small constant. Then construct the following quadratic programming problem in order to find a feasible vector $Y = (\phi_y, \theta_y)$ in the direction of the steepest descent of $h(X)$.

$$\begin{aligned} \text{minimize} \quad & u = \phi_y^2 + \theta_y^2, \\ \text{subject to} \quad & [\partial h_i / \partial \phi] \phi_y + [\partial h_i / \partial \theta] \theta_y \leq -1, \quad \text{for } i \in I_\epsilon(X). \end{aligned} \tag{3.5}$$

If $Y^* = (\phi_{y^*}, \theta_{y^*})$ is a feasible solution to the quadratic programming problem, it guarantees that Y^* and $\{Ex_i : i \in I_\epsilon(X)\}$ lie on a hemisphere.

Property 3.1.1: If there is no feasible solution (3.5) at X , then

$$h(X) - h(X^{l-opt}) \leq \epsilon,$$

where X^{l-opt} is a local minimax solution.

Property 3.1.2: If there is a feasible solution to (3.5) at X , then $Y^* = (\phi_{y^*}, \theta_{y^*})$, the optimal solution to (3.5), is a vector in the direction of the steepest descent of $h(X)$.

Therefore, if there is a feasible solution to (3.5), we can travel to a lower value of $h(X)$ along the great circle defined by Y^* .

Now, we have to find the global minimax point to problem. In the following we will explain the principles behind a procedure guaranteed to find the global minimax point :

Suppose that a local minimax point X^{l-opt} of $h(X)$ has been found. Let the intersection \mathcal{I} , of m spherical disks $\mathcal{D}_i, i = 1, 2, \dots, m$ with centers at points Ex_i and with radii $h(X^{l-opt})/w_i$. Note that \mathcal{I} may be disjoint. A better solution can be found in the set \mathcal{I} and if \mathcal{I} is formed by only of the points (and not arcs) the local minimax point is also the global one. Otherwise, if we obtain a starting point in an area of \mathcal{I} , the quadratic programming formulation (3.5) can be used to find better local minimax point. Note that this area is thereby "removed," If this process is repeated, the disk shrink, the finite number of areas in \mathcal{I} is reduced, and the global minimax point must eventually be found.

Now, we will propose an efficient method in order to obtain a starting point within the area of \mathcal{I} . Note that the area of \mathcal{I} must be bounded by arc segments cut from the circumferences of the spherical disks with centers Ex_i . Therefore, at least one such arc must be inside all other disks. Suppose that we start with the circle around any disk. Then we can check other circles to see if the first circle has an arc in its interior. If the intersection of such arc segments \mathcal{B} is not empty, then this intersection forms parts of a boundary of area of \mathcal{I} . Then, we can use the center of \mathcal{B} as the starting point for the quadratic programming improvement of the solution.

The algorithm for finding the optimal minimax solution is as follows:

Algorithm 3.1.1: (finding the global minimax point:)

Input: Set of existing facilities $\mathcal{E}_x = \{Ex_i; i = 1, 2, \dots, m : Ex_i \in \mathcal{S}_0\}$

Step 1 Choose a starting point.

Step 2 Use (3.5) to obtain a local minimax point X^{l-opt} .

Step 3 Using Theorem 3.1.2, check to see if X^{l-opt} is a global minimax point; if so terminate the procedure.

Step 4 Apply Algorithm 3.1.2 for the group \mathcal{E}'_x defined in Theorem 3.1.2. If \mathcal{I} has only points, terminate the procedure as X^{l-opt} is the global minimax point.

Step 5 Apply Algorithm 3.1.2 for the whole group of circles. If \mathcal{I} has only points now, X^{l-opt} is the global minimax point.

Step 6 Go to Step 2 with the starting point found by Algorithm 3.1.2 .

Let k be the number of spherical disks whose intersection we seek.

Algorithm 3.1.2: (finding an area of \mathcal{I}):

Step 1 set $i = 1, j = 2$.

Step 2 Define \mathcal{B} to be the entire circumference of circle i .

Step 3 If $i = j$, go to Step 7.

Step 4 Find that arc of circle i that is cut by disk j .

Step 5 Let \mathcal{B} be the intersection between the current \mathcal{B} and the arc formed in Step 4.

Step 6 If \mathcal{B} is empty and $i = k$, terminate the procedure: The intersection \mathcal{I} has no areas. If $I(arcs)$ is empty and $i < k$, set $i = i + 1$ and go to Step 2.

Step 7 If $j < k$ go to Step 3 with $j = j + 1$. If $j = k$ designate the center of any arc of \mathcal{B} as a new starting point in Algorithm 3.1.1 and terminate this procedure.

3.2 Enumeration Technique for Determining Global Optimum of CenterSphereLoc [5]

Here, we present an enumeration procedure of finding a minimax location of the **CenterSphereLoc** problem with the distant norm is the shortest arc distance on the surface of the sphere. This procedure determines global optimal solutions in a finite number of steps. In the following, we represent some notations and definitions which will be used in developing of the algorithm.

Consider three points X_1, X_2 and X_3 on the surface of the sphere.

$X_1\hat{X}_2X_3 \equiv$ the spherical angle subtended from a point X_2 by the shorter arc, $\text{arc}(X_1X_3)$.

$\triangle X_1X_2X_3 \equiv$ the plane triangle with vertices at points X_1, X_2 and X_3 .

$\angle X_1, \angle X_2$ and $\angle X_3 \equiv$ angles of $\triangle X_1X_2X_3$.

The spherical angle $X_1\hat{X}_2X_3$ is measured as angle between two straight lines tangential at point X_2 to the two great circles, one passing through X_1 & X_2 and the other through X_2 & X_3 .

Definition 3.2.1: Given three distinct points, X_1, X_2 and X_3 on the surface of the sphere, $\mathcal{P}(X_1, X_2, X_3)$ denote the unique plane passing through the three points and bisecting the sphere(see Figure 3.1).

Definition 3.2.2: $\mathcal{C}(X_1, X_2, X_3)$ denotes the circle traced by the plane $\mathcal{P}(X_1, X_2, X_3)$ cutting through the sphere(see Figure 3.1).

Definition 3.2.3: Let X_1 and X_2 are not diametrically opposite. Denote the mid point of the (shorter) arc as the point P . Then, $\mathcal{C}(X_1, X_2)$ represents the small circle that goes through points X_1 and X_2 and has its nearer pole located at point P .

Definition 3.2.4: $\Gamma\mathcal{C}(X_1, X_2)$ and $\Gamma\mathcal{C}(X_1, X_2, X_3)$ denote the surface area of a sphere that contains the nearer pole and is bounded by $\mathcal{C}(X_1, X_2)$ and $\mathcal{C}(X_1, X_2, X_3)$, respectively.

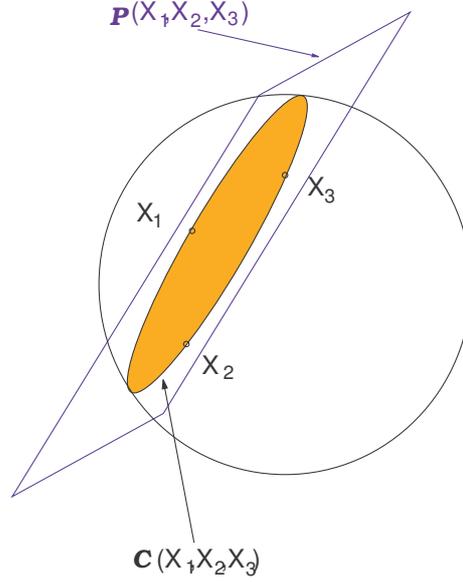


Fig. 3.1: Plane \mathcal{P} bisects the sphere

Definition 3.2.5: $RC(X_1, X_2)$ and $RC(X_1, X_2, X_3)$ denote the surface area of a sphere that contains the distant pole and is bounded by $\mathcal{C}(X_1, X_2)$ and $\mathcal{C}(X_1, X_2, X_3)$, respectively.

Next, We will represent some results regarding poles and small circles.

Lemma 3.2.1: Let P be the nearer pole of $\mathcal{C}(X_1, X_2, X_3)$, where $\triangle X_1 X_2 X_3$ is an acute triangle. Let $Q (\neq P)$ be any point on $\Gamma\mathcal{C}(X_1, X_2, X_3)$ and within the spherical triangle $X_1 X_2 X_3$. Then, the spherical radius of $\mathcal{C}(X_1, X_2, X_3)$ is greater than $\text{minimum}\{\text{arc}(QX_1), \text{arc}(QX_2), \text{arc}(QX_3)\}$.

Proof : See Appendix.

Lemma 3.2.2: Let P' be the distant pole of $\mathcal{C}(X_1, X_2, X_3)$ where $\triangle X_1 X_2 X_3$ is an acute triangle. Let Q_1 be a point on $RC(X_1, X_2, X_3)$ and $Q_1 \neq P'$. If Q_1 is sufficiently close to P' then $\text{maximum}\{\text{arc}(Q_1 X_1), \text{arc}(Q_1 X_2), \text{arc}(Q_1 X_3)\} > \text{arc}(X_1 P')$.

Proof : *See Appendix.*

Lemma 3.2.3: Let X_1, X_2 and X_3 be three different points on a unit sphere with $\angle X_3 > \pi/2$. Let P and P' be the nearer and distant poles of $\mathcal{C}(X_1, X_2, X_3)$ respectively. Then there exists a point Q , close to P' such that

$$\text{maximum}\{\text{arc}(X_1Q), \text{arc}(X_2Q), \text{arc}(X_3Q)\} < \text{arc}(X_1P') = \text{arc}(X_2P') = \text{arc}(X_3P').$$

Proof : *See Appendix.*

Corollary 3.2.1: $\mathcal{C}(X_1, X_2, X_3)$ may contain demand points other than X_1, X_2 , and X_3 . Assume that all other demand points lie in $RC(X_1, X_2, X_3) - \mathcal{C}(X_1, X_2, X_3)$. Then the distant pole of $\mathcal{C}(X_1, X_2, X_3)$ is not a solution of the spherical minimax problem if no triplet of demand points on $\mathcal{C}(X_1, X_2, X_3)$ forms an acute triangle.

Proof *In this case the demand points on $\mathcal{C}(X_1, X_2, X_3)$ lie on an arc of a semicircle. The result directly follows from Lemma 3.2.3.*

Theorem 3.2.1: (i) If $\triangle(X_1X_2X_3)$ is an acute and $\Gamma\mathcal{C}(X_1, X_2, X_3)$ may contain all demand points then the nearer pole of $\mathcal{C}(X_1, X_2, X_3)$ is the unique facility point.

(ii) If $\Gamma\mathcal{C}(X_1, X_2)$ contains all demand points, then the nearer pole of $\mathcal{C}(X_1, X_2)$ is the required facility point.

Proof *Let P be the nearer pole of $\mathcal{C}(X_1, X_2, X_3)$. As $\triangle(X_1X_2X_3)$ acute, we have*

$$\hat{X}_1 < \hat{X}_2 + \hat{X}_3, \hat{X}_2 < \hat{X}_1 + \hat{X}_3, \text{ and } \hat{X}_3 < \hat{X}_1 + \hat{X}_2.$$

Take any point X on $\Gamma\mathcal{C}(X_1, X_2, X_3)$. Join XP by the arc of the great circle. Since P is within the spherical triangle $X_1X_2X_3$, we have

$$X_1\hat{P}X_2 + X_2\hat{P}X_3 > \pi,$$

$$X_2\hat{P}X_3 + X_3\hat{P}X_1 > \pi,$$

and

$$X_3\hat{P}X_1 + X_1\hat{P}X_2 > \pi,$$

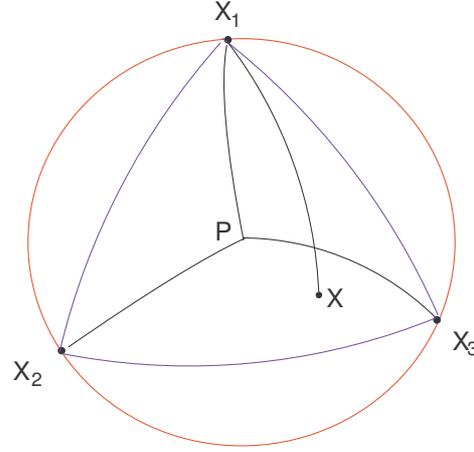


Fig. 3.2:

hence, we conclude that at least one of the spherical angles $X_1\hat{P}X$, $X_3\hat{P}X$ and $X_2\hat{P}X$ must be greater than $\pi/2$. In Figure (3.2), for example, $X_1\hat{P}X > \pi/2$, and consequently from the spherical triangle X_1PX , $\text{arc}(X_1X) > \text{arc}(X_1P)$.

Similarly,

$X_3\hat{P}X > \pi/2 \Rightarrow \text{arc}(X_3X) > \text{arc}(X_3P)$ and $X_2\hat{P}X > \pi/2 \Rightarrow \text{arc}(X_2X) > \text{arc}(X_2P)$.

This implies that P is the unique facility point.

Consider the spherical circle $\mathcal{C}(X_1, X_2)$ with nearer pole S .

From the spherical triangle PX_1X_2 , we have by the Property 1.2.1 (c),

$$\begin{aligned} \text{arc}(X_1P) + \text{arc}(X_2P) &> \text{arc}(X_1X_2) = \text{arc}(X_1S) + \text{arc}(SX_2) \\ \Rightarrow 2 \cdot \text{arc}(X_1P) &> 2 \cdot \text{arc}(X_1S) \\ \Rightarrow \text{arc}(X_1P) &> \text{arc}(X_1S). \end{aligned}$$

This implies that there exists a small circle $\mathcal{C}(X_1, X_2)$ of a smaller spherical radius than $\text{arc}(X_1P)$ such that all the demand points are contained on $\Gamma\mathcal{C}(X_1, X_2)$. That is, the nearer pole S of $\mathcal{C}(X_1, X_2)$ is the required facility point.

Corollary 3.2.2: $\mathcal{C}(X_1, X_2, X_3)$ may contain demand points other than X_1, X_2 , and X_3 .

Assume that all other demand points lie in $\Gamma\mathcal{C}(X_1, X_2, X_3) - \mathcal{C}(X_1, X_2, X_3)$. If all triplets of demand points on $\mathcal{C}(X_1, X_2, X_3)$ form obtuse triangle, then the nearer pole of $\mathcal{C}(X_1, X_2, X_3)$ is not the required facility point.

Proof : *The result follows from Theorem 3.2.1.*

Theorem 3.2.2: If there exists a triplet (X_1, X_2, X_3) of demand points such that

- (i) $\triangle(X_1X_2X_3)$ is acute,
- (ii) The center of the sphere and all demand points lie on the same side of $\mathcal{P}(X_1, X_2, X_3)$,
and
- (iii) (X_1, X_2, X_3) generates the plane closest to the center of the sphere, then the distant pole of $\mathcal{C}(X_1, X_2, X_3)$ is the required facility point.

Proof : *From Lemma 3.2.3, we know that the triplet of points forming an obtuse triangle cannot yield an optimal solution. Further, Lemma 3.2.2 represents that the distant pole of the small circle defined by a triplet satisfying (i) and (ii) is a local minimum and (iii) implies the optimality.*

Theorem 3.2.3: If $RC(X_1, X_2) - \mathcal{C}(X_1, X_2)$ contains all demand points other than X_1, X_2 , then the distant pole of $\mathcal{C}(X_1, X_2)$ cannot be a minimax location.

Proof : *See Appendix.*

Corollary 3.2.3: Assume $\mathcal{C}(X_1, X_2)$ contains a demand point(s) other than P_1 and P_2 . Let all the demand points lie on $RC(X_1, X_2) - \mathcal{C}(X_1, X_2)$. If not triplet of demand points on $\mathcal{C}(X_1, X_2)$ forms an acute triangle, then the distant pole of $\mathcal{C}(X_1, X_2)$ cannot be a facility point.

Proof : *The results follows from Theorem 3.2.3.*

Lemma 3.2.2 shows that if all the demand points lie in $RC(X_1, X_2, X_3)$, every point in a small neighborhood of distant pole, P' has an objective function value that is greater than

the one at P' . Thus P' is locally optimal. In the case that all the demand points lie in a hemisphere, Theorem 3.2.1 discuss the solution to the required problem when all demand points lie in a hemisphere and Theorem 3.2.2 characterizes a solution when all the demand points are distributed all over the sphere.

Next, will present the developed algorithm for solving the spherical minimax location problem.

In the following algorithm, we consider that $\mathcal{E}_x = \{Ex_i : i = 1, \dots, m\}$ denote the set of demand points and $Ex_k, Ex_l,$ and Ex_m be three distinct element of \mathcal{E}_x .

Then define the following :

$l(Ex_k, Ex_l, Ex_m)$: the Euclidean distance from the center of the sphere to the center of the circle $\mathcal{C}(Ex_k, Ex_l, Ex_m)$.

$$u(Ex_k, Ex_l, Ex_m) = \begin{cases} 0 & \text{if } \triangle Ex_k Ex_l Ex_m \text{ is obtuse;} \\ 1 & \text{otherwise.} \end{cases}$$

$$v(Ex_k, Ex_l, Ex_m) = \begin{cases} 0 & \text{if points lie on both sides of } \mathcal{P}(Ex_k, Ex_l, Ex_m) ; \\ 1 & \text{otherwise.} \end{cases}$$

This algorithm bellow examines all possible pairs of demand points to find minimax locations. To prevent a pair of demand points, (Ex_i, Ex_j) being examined twice, the following rules are imposed to update the indices of the pair to be examined next.

Rule 1. If $j < m$, then set $i = i$ and $j = j + 1$

Rule 2. If $j = m$ and $i < m - 1$, then set $i = i + 1$ and $j = j + 1$.

Together with the above definitions and two rules, the algorithm can be presented as follows :

Algorithm 5 (An algorithm for CenterSphereLoc problem)

Input The set $\mathcal{E}_x = \{Ex_i : i = 1, \dots, m\}$ of demand points on the unit sphere.

Initialization. Set $i = 1, j = 2, Opt^* = \emptyset, k = 1, l = 2, l_{best} = 1$. Go to step 1.

Step 1. If $\Gamma\mathcal{C}(Ex_k, Ex_l)$ contains every other demand points, stop and nearer pole of $\mathcal{C}(Ex_k, Ex_l)$ is the minimax location. Otherwise, go to Step 2.

Step 2. If $i = (m - 1)$, stop and every point in Opt^* is a minimax location. Otherwise, go to Step 3.

Step 3. Let Ex_p and Ex_q be two demand points other than Ex_k and Ex_l such that $\mathcal{P}(Ex_k, Ex_l, Ex_p)$ and $\mathcal{P}(Ex_k, Ex_l, Ex_q)$ yield the minimum and the maximum, respectively, inclination with the plane $\Gamma(Ex_k, Ex_l)$.

If $u(Ex_k, Ex_l, Ex_r) = 1$ and all the demand points lie on $\Gamma\mathcal{C}(Ex_k, Ex_l, Ex_r)$ for $r = p$ or $r = q$, then stop and the nearer pole of $\mathcal{C}(Ex_k, Ex_l, Ex_r)$ is the minimax location. Otherwise, go to Step 4.

Step 4. For $r = p$ and $r = q$, do one of the following:

If $u(Ex_k, Ex_l, Ex_r) = 1, v(Ex_k, Ex_l, Ex_r) = 1$ and $l(Ex_k, Ex_l, Ex_r) = l_{best}$, then add the distant pole of $\mathcal{C}(Ex_k, Ex_l, Ex_r)$ to Opt^* .

If $u(Ex_k, Ex_l, Ex_r) = 1, v(Ex_k, Ex_l, Ex_r) = 1$ and $l(Ex_k, Ex_l, Ex_r) < l_{best}$, then set $l_{best} = l(Ex_k, Ex_l, Ex_r)$ and replace Opt^* with a set that contains only the distant pole of $\mathcal{C}(Ex_k, Ex_l, Ex_r)$.

Update i and j according to the two rules and set $Ex_k = Ex_i$ and $Ex_l = Ex_j$. Go to Step 1.

If the algorithm stops in Step 1, the Theorem 3.2.1 guarantees that the nearer pole of $\mathcal{C}(Ex_k, Ex_l)$ is the optimum location. The set Opt is formed by the distance poles of $\mathcal{C}(Ex_k, Ex_l, Ex_r)$. If the algorithm terminates in Step 2, Theorem 3.2.3 justifies that the points Ex_k, Ex_l , and $Ex_r; r = p, q$ on $\mathcal{C}(Ex_k, Ex_l, Ex_r)$ forms an acute triangle and this justifies the optimality of every point in Opt . Consider the Step 3. The plane $\mathcal{P}(Ex_k, Ex_l)$ divides the sphere into two disjoint surfaces. If $Ex_m \in \Gamma\mathcal{C}(Ex_k, Ex_l) - \mathcal{C}(Ex_k, Ex_l)$, then $Ex_k \hat{E}x_m Ex_l$ is obtuse and then the poles of $\mathcal{C}(Ex_k, Ex_l, Ex_m)$ can not be a optimal location. When $Ex_m \in \mathcal{RC}(Ex_k, Ex_l) - \mathcal{C}(Ex_k, Ex_l)$ then $Ex_k \hat{E}x_m Ex_l$ is acute. If, in addition, every demand points lie on $\Gamma\mathcal{C}(Ex_k, Ex_l, Ex_m)$, then by Theorem 3.2.1, the nearer pole of $\mathcal{C}(Ex_k, Ex_l, Ex_m)$ is the optimal location. Otherwise, Step 4 examines the

possibility to have a distance pole of $\mathcal{C}(Ex_k, Ex_l, Ex_m)$ as a optimal location.

Further, in Step 3, we are looking for a third demand point Ex_m on $\mathcal{C}(Ex_k, Ex_l)$ in such a way that all the other demand points lie on one side of the plane $\mathcal{P}(Ex_k, Ex_l, Ex_m)$. There are no more than $(m - 2)$ planes that pass through demand points Ex_k, Ex_l , and another point Ex_m in $RC(Ex_k, Ex_l)$. Among these planes, at most two planes can have all the demand points other than Ex_k, Ex_l , and Ex_m all on one side. These two planes are the ones that yield the minimum and maximum inclinations with the plane $\mathcal{C}(Ex_k, Ex_l)$.

3.3 Algorithm Based on Factored Secant Update Technique [24]

In this paper, the author discussed **CenterSphereLoc** problem in the cartesian coordinate system using the Euclidean norm. He justified that minimizing the maximum of the shortest arc distances between the facility and the demand points on the unit sphere is equivalent to minimizing the maximum of the corresponding Euclidean distances. Using the Karush-Kuhn-Tucker (KKT) necessary optimality conditions, he obtained a set of non-linear equations which can be solved by a method of factored secant update technique (see [6]). He made attention for the following special cases :

1. All the demand points are on a hemisphere

and

2. One or more point-antipodal point(s) are included in the set of demand points.

3.3.1 The Behavior of the Euclidean Distances in Spherical Location Problems

Here, we will show that minimizing the maximum of the shortest arc distance between the facility to be located and the demand points is equivalent to minimizing the maximum of the corresponding Euclidean distances.

From (1.1), we have

$$\alpha = \arccos\{\cos \phi_1 \cos \phi_2 \cos(\theta_1 - \theta_2) + \sin \phi_1 \sin \phi_2\} \quad (3.6)$$

be the shortest arc distance between two points $X_1 = X_1(\phi_1, \theta_1)$ and $X_2 = X_2(\phi_2, \theta_2)$ on \mathcal{S}_0 . Let d be the corresponding Euclidean distance between these two points. Since α is also the angle between the two lines drawn from the center of the sphere to two points X_1 and X_2 , (see Figure 3.3)

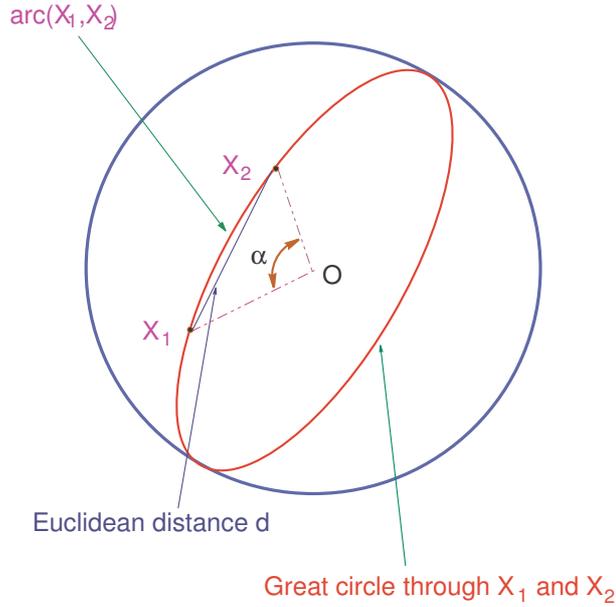


Fig. 3.3:

$$\begin{aligned}
 d^2 &= \|OX_1\|^2 + \|OX_2\|^2 - 2 \times \|OX_1\| \times \|OX_2\| \cos \alpha \\
 d^2 &= \|1\|^2 + \|1\|^2 - 2 \times \|1\| \times \|1\| \cos \alpha \\
 d^2 &= 2 - 2 \cos \alpha \\
 \alpha &= \arccos\left(1 - \frac{d^2}{2}\right), 0 \leq d \leq 2
 \end{aligned} \tag{3.7}$$

The equation 3.7 shows that there is a one-to-one correspondence between α and d . In addition, α is an increasing function of d . This means that finding the minimax point using the great circle distance, α is equivalent to finding minimax point using the Euclidean distance, d .

3.3.2 Formulation of the Problem with Euclidean Distance

A mathematical formulation of the **CenterSphereLoc** problem with Euclidean distance is as follows :

$$\min \quad H \quad (3.8)$$

$$\text{subject to} \quad (x_i - x_0)^2 + (y_i - y_0)^2 + (z_i - z_0)^2 \leq H \quad \text{for } i = 1, \dots, m \quad (3.9)$$

$$x_0^2 + y_0^2 + z_0^2 = 1 \quad (3.10)$$

where

m is the number of existing facilities ,

(x_i, y_i, z_i) are the cartesian coordinates of the existing facilities Ex_i ,

(x_0, y_0, z_0) are the coordinates of a point X_0 on \mathcal{S}_0 ,

H is the variable that measures the maximum of the squares of the Euclidean distances from X_0 to the existing facility Ex_i .

Now we consider the corresponding KKT necessary optimality conditions for the the min-*imax* problem (3.8) - (3.10).

$$\sum_{i=1}^m \lambda_i = -1 \quad (3.11)$$

$$\left(\mu + \sum_{i=1}^m \lambda_i\right)x_0 = \sum_{i=1}^m \lambda_i x_i \quad (3.12)$$

$$\left(\mu + \sum_{i=1}^m \lambda_i\right)y_0 = \sum_{i=1}^m \lambda_i y_i \quad (3.13)$$

$$\left(\mu + \sum_{i=1}^m \lambda_i\right)z_0 = \sum_{i=1}^m \lambda_i z_i \quad (3.14)$$

$$(3.15)$$

$$\lambda_i s_i = 0 \quad \text{for } i = 1, \dots, n \quad (3.16)$$

$$(x_i - x_0)^2 + (y_i - y_0)^2 + (z_i - z_0)^2 - F + s_i^2 = 0 \quad \text{for } i = 1, \dots, n \quad (3.17)$$

$$x_0^2 + y_0^2 + z_0^2 - 1 = 0 \quad (3.18)$$

$$\lambda_i + p_i^2 = 0 \quad \text{for } i = 1, \dots, n \quad (3.19)$$

where

λ_i is the Lagrange multiplier corresponding to the constraint set (3.9) ,

μ is the Lagrange multiplier corresponding to constraint (3.10),

s_i are the slack variables of inequality (3.9),

p_i are the slack variables of the non positivity conditions on λ_i .

The set of equations (3.11)- (3.19) are the set of nonlinear equations which can be solved by using the method of factored secant update with a finite difference approximation to the Jacobian.

3.3.3 Some Examples for Solving CenterSphereLoc

In order to apply the theory which we discussed here, we consider three examples :

1. when the demand points are on a hemisphere and at least one point-antipodal point pair is included in the set of demand points,
2. when the demand points are on a hemisphere and no point - antipodal point pair is included in the set of demand points,
3. when the demand points are not on a hemisphere.

Example 1

We consider 17 points all located in the Northern Hemisphere. Each point's latitude, longitude, and the corresponding Cartesian coordinates are included in Table 3.1. The last two points form a point - antipodal point pair on the equator.

For this example, a minimax point can be obtained quickly as follows (see [23]):

- Select a demand point $E x_i = (x_i, y_i, z_i)$ whose antipode $\bar{E} x_i = (-x_i, -y_i, -z_i)$ is also included in the set \mathcal{E}_x of demand points.

	City	Latitude	Longitude	x	y	z
1	London	51.5 N	0.4 E	0.6025	0.0043	0.7826
2	Paris	48.9 N	2.3 E	0.6568	0.0264	0.7536
3	Zurich	47.4 N	8.5 E	0.6694	0.1000	0.7361
4	Rome	41.9 N	12.5 E	0.7267	0.1611	0.6678
5	Copenhagen	55.7 N	12.6 E	0.5500	0.1229	0.8261
6	Berlin	52.5 N	13.4 E	0.5922	0.1411	0.7934
7	Stockholm	59.3 N	18.9 E	0.4830	0.1654	0.8600
8	Athens	38.0 N	23.7 E	0.7216	0.3167	0.6157
9	Ankara	39.9 N	32.8 E	0.6449	0.4156	0.6415
10	Tel-Aviv	32.1 N	34.8 E	0.6956	0.4835	0.5314
11	Moscow	55.7 N	37.7 E	0.4459	0.3446	0.8261
12	Teheran	35.4 N	51.4 E	0.5085	0.6370	0.5793
13	Bombay	18.9 N	72.8 E	0.2798	0.9038	0.3239
14	Manila	14.6 N	121.0 E	-0.4984	0.8295	0.2521
15	Tokyo	35.6 N	139.7 E	-0.6201	0.5260	0.5820
16	Point 16	0.0	30.0 E	0.8660	0.5000	0.0000
17	Point 17	0.0	150.0 W	-0.8660	-0.5000	-0.0000

Tab. 3.1: Latitudes, Longitudes, and corresponding Cartesian coordinates of 17 points. Points 16 and 17 form a point-antipodal point pair on the Equator

- Consider a plane passing through the points $(0,0,0)$, Ex_i , and $\bar{E}x_i$ such that the remaining points Ex_j , with $j \neq i$ lie on one side of the plane.
- If such a plane exists, then all the points including the point-antipodal pair are on a hemisphere.
- To check whether such a plane exists, we can solve the following linear programming problem with dummy objective g :

$$\begin{aligned}
\max \quad & g & (3.20) \\
\text{s.t.} \quad & ax_i + by_i + cz_i = 0 \\
& ax_j + by_j + cz_j \leq 0 \quad \text{for all } (x_j, y_j, z_j) \neq \pm(x_i, y_i, z_i) \\
& g \leq 1 \\
& a, b, c \quad \text{are unrestricted in sign.}
\end{aligned}$$

- Consider a solution (a, b, c) obtained by solving the linear programming formulation (3.20)
- This vector is normal to the plane $ax + by + cz = 0$ that divides the unit sphere into hemispheres such that the demand points lie on a hemisphere.
- This vector is directed towards the hemisphere that does not contain any of the demand points.
- Then the minimax point is given by

$$x_0 = -\frac{a}{\sqrt{a^2 + b^2 + c^2}}; y_0 = -\frac{b}{\sqrt{a^2 + b^2 + c^2}}; z_0 = -\frac{c}{\sqrt{a^2 + b^2 + c^2}} \quad (3.21)$$

- This minimax point is simply the center of the spherical disk with radius $\pi/2$ (the hemisphere which bears all the demand points).

As the linear programming formulation (3.20) has multiple optimal solutions whenever the great circle that divides the hemispheres contains only the point-antipodal point pair, this minimax point may not be unique. In this case the minimax location problem will have multiple solutions with the same maximum spherical distance $\pi/2$ from the minimax point to the demand points.

Using the method mention above, example 1 gives the minimax point $(-0.463, 0.803, 0.376)$.

The same problem is also solved using the KKT conditions (3.11) - (3.19) iteratively. It gives the minimax point $(-0.356, 0.616, 0.703)$.

These two solutions confirms that multiple solutions are possible for this problem.

Example 2

In this example, we consider the first 15 points of Table 3.1 all located in the Northern Hemisphere. This problem is solved by using KKT conditions (3.11) - (3.19) and it gives a unique globally optimal solution whenever the demand points lie on a hemisphere.

Example 3

	City/Point	Latitude ϕ	Longitude θ	x	y	z
1	Point 1	56.2 N	23.4 E	0.5105	0.2209	0.8310
2	Point 2	25.0 N	9.1 W	0.8949	-0.1433	0.4226
3	Point 3	7.0 S	43.2 E	0.7235	0.6794	-0.1219
4	Point 4	12.8 N	45.0 W	0.6895	-0.6895	0.2215
5	Point 5	0.0	100.5 E	-0.1822	0.9832	0.0000
6	Point 6	27.0 N	84.5 W	0.0854	-0.8869	0.4540
7	Point 7	9.5 S	110.3 W	0.3422	-0.9250	-0.1650
8	Point 8	32.5 S	87.0 E	0.0411	0.8422	-0.5373
9	Point 9	30.0 S	60.0 W	0.4330	-0.7500	-0.5000
10	Point 10	60.0 N	60.0 W	0.2500	-0.4330	0.8660
11	Point 11	45.0 N	75.0 E	0.1830	0.6830	0.7071
12	Point 12	85.0 N	0.0	0.0872	0.0000	0.9962
13	Point 13	15.0 S	130.0 W	-0.6209	-0.7399	-0.2588
14	Point 14	60.0 N	115.0 E	-0.2113	0.4532	0.8660

Tab. 3.2: Latitudes, Longitudes, and corresponding Cartesian coordinates of 14 points spread over the entire globe.

In this example, we consider the situation when all the demand points are not on a hemisphere. The table 3.2 represents 14 points with each point's latitudes, longitudes, and the corresponding Cartesian coordinates .

It gives different locally optimal solutions in each different starting values. The minimum value obtained among all of these locally optimal solutions may be a globally optimal solution. Next, we add the fifteenth in Table 3.2. This point is the antipode of Point 3 in this table. The KKT conditions (3.11) - (3.19) is solved with a same starting vales and it gives the same optimal solution as that obtained for the 14 points in Table 3.2. This confirms that adding an antipode of one of the demand points may not always alter a locally optimal solution.

Note: For the first example, the KKT necessary optimality conditions (3.11)-(3.19) need not be solved. The vector (a, b, c) that is normal to the plane $ax + by + cz = 0$ passing through the center $(0, 0, 0)$ of the sphere and dividing the plane into hemispheres such that the demand points are on a hemisphere, is directed towards the hemisphere that does not contain any of the points. The optimal location of **CenterSphereLoc** is then the normalized vector $(-a, -b, -c)$. For the other two examples, KKT conditions (3.11)-(3.19) need to be solved. There are $(3n + 5)$ equations involved in these (3.11)-(3.19). Thus the number of equations increases by 3 whenever a new demand point is added. Also it should be mentioned that the resulting KKT system of equations, (3.11)-(3.19), is very nonlinear.

3.4 Geometrical Approaches for CenterSphereLoc Problem on a Hemisphere [27],[4]

Consider the **CenterSphereLoc** problem on a hemisphere with equal weights. In both methods, we use the shortest arc distance as the measure of distance on the hemisphere. The solution method which is described in [27] depends heavily on properties of the spherical triangles. The second approach which is described in [4] is based on the properties of a plane triangle. During the development of these algorithms, we use the notations which are described in the section 3.2.

In the first step, we describe the Sakar - Chaudhuri[27] algorithm is as follows:

Let $\mathcal{E}_x = \{Ex_i, i \in I \text{ and } I = \{1, 2, \dots, m\}\}$ be the set of demands points on the surface of a hemisphere. The basic idea of this approach is to cover \mathcal{E}_x by a portion of a sphere bounded by a small circle. The next step consists of reducing the radius of this circle so that

demand points continue to remain within the portion of the sphere bounded by this circle. The algorithm is designed in such a manner that each iteration at least one demand point could be eliminated and no future iteration would need any information about this point.

Algorithm 3.4.1: (algorithm based on the properties of the spherical triangles)

Input $\mathcal{E}_x = \{Ex_i : i \in I = \{1, \dots, m\}\}$ be the set of existing facilities in a hemisphere S_0^H .

Initialization. Choose any point X on the surface of the hemisphere which contains all the demand points. Let Ex_k be the farthest demand point from X . Denote this point by A .

$$I \leftarrow I - \{k\}.$$

Let A_i be a point on the great circle $arc(AX)$ such that :

$arc(AA_i) = arc(A_iEx_i)$, where $i \in I$. Denote the point A_i for which XA_i is minimum by Y and corresponding index by k . Let this demand point Ex_k be denoted by B . $I \leftarrow I - \{k\}$.

Step 1. If all the demand points lie on $\Gamma C(A, B)$, then the nearer pole P of $\mathcal{C}(A, B)$ is the required facility point. Stop.

Else $X \leftarrow Y$, and go to step 2.

Step 2. Let D be the mid point of the $arc(AB)$. Find a point A_i on the great circle $arc(XD)$ such that $arc(A_iA) = arc(A_iEx_i)$, $i \in I$. Denote the point A_i for which the $arc(XA_i)$ is minimum by Y and the corresponding index by k . Let the demand point Ex_k be denoted by C .

If $\hat{A} < \hat{B} + \hat{C}$, $\hat{B} < \hat{A} + \hat{C}$ and $\hat{C} < \hat{A} + \hat{B}$, then the nearer pole P of $\mathcal{C}(A, B, C)$ is the required facility point. Stop.

Else go to Step 3.

Step 3. If $Ex_k \hat{E}x_i Ex_j > Ex_i \hat{E}x_j Ex_k + Ex_j \hat{E}x_k Ex_i$, where $Ex_i, Ex_j, Ex_k \in \{A, B, C\}$ and i, j, k are all different, then Ex_i is excluded from all future iterations. Denote the points Ex_j and Ex_k by A and B respectively. $I \leftarrow I - \{i\}$ and repeat Step 1.

If the Algorithm 3.4.1 stops in Step 1, and Step 2 then Theorem 3.2.1 guaranties that the optimality of the nearer poles of $\mathcal{C}(A, B)$ and $\mathcal{C}(A, B, C)$ respectively.

In order to explain the next algorithm[4] which is based on the properties of planner triangle, first we will consider the following lemma.

Lemma 3.4.1: Let X_1 and X_2 are any two points on the surface of the sphere that do not contain the ends of a diameter of the sphere. Let X_3 be an any point on the surface of the sphere such that $X_3 \notin \Gamma\mathcal{C}(X_1, X_2)$. Then $\angle X_1X_3X_2$ is acute.

Proof: Construct the sphere, S' with $\mathcal{C}(X_1, X_2)$ as a great circle. Clearly all the points of $\Gamma\mathcal{C}(X_1, X_2) - \mathcal{C}(X_1, X_2)$ lie within S' and all the points of $S_0 - \Gamma\mathcal{C}(A, B)$ lie outside S' . Now X_3 is a point which is lie outside of S' . It is obvious that $\angle X_1X_3X_2$ is an acute angle.

Corollary 3.4.1: Consider any three points X_1, X_2 and X_3 on the surface of the sphere such that $\triangle X_1X_2X_3$ is an acute triangle and $\mathcal{C}(X_1, X_2, X_3)$ is a small circle. Let O' be the center of $\mathcal{C}(X_1, X_2, X_3)$. Further assume that Y be a point on the surface of the sphere with $Y \notin \Gamma\mathcal{C}(X_1, X_2, X_3)$. Then $O'Y > O'X_1 = O'X_2 = O'X_3$.

Proof: Let S'' be the sphere of which $\mathcal{C}(X_1, X_2, X_3)$ is a great circle. Since Y is outside of S'' , the proof is immediately follows from the Lemma 3.4.1.

Then we represent the Das - Chakraborti [4] algorithm as follows:

Algorithm 3.4.2: (Algorithm based on the properties of the planner triangles)

Input $\mathcal{E}_x = \{Ex_i : i \in I = \{1, \dots, m\}\}$ be the set of existing facilities contained in a spherical disk of radius $\alpha \leq \pi/4$.

Initialization. Take any two demand points Ex_i and Ex_j . Go to Step 1.

Step 1. If all demands points lie on $\Gamma\mathcal{C}(Ex_i, Ex_j)$, then nearer pole of $\mathcal{C}(Ex_i, Ex_j)$ is the required facility point. Stop.

Else choose a demand point, say Ex_k , not in $\Gamma\mathcal{C}(Ex_i, Ex_j)$ such that $\angle Ex_iEx_kEx_j$

is minimum. Goto Step 2.

Step 2. If all the demand points lie on $\Gamma\mathcal{C}(Ex_i, Ex_j, Ex_k)$ and $\triangle Ex_i Ex_k Ex_j$ is an acute triangle then stop. The nearer pole of $\mathcal{C}(Ex_i, Ex_k, Ex_j)$ is the required facility point. Else goto Step 3.

Step 3. If $\triangle Ex_i Ex_k Ex_j$ is not an acute triangle then call the extremities of the largest side of the triangle by Ex_i and Ex_j . Return to Step 1.

Else find a demand point, Ex_l , in $S_0 - \Gamma\mathcal{C}(Ex_i, Ex_j, Ex_k)$ such that the distance of Ex_l from the center of the $\mathcal{C}(Ex_i, Ex_j, Ex_k)$ is maximum. Go to Step 4.

Step 4. Find the maximum distance of Ex_l from Ex_i, Ex_j, Ex_k . Denote the point having a maximum distance from Ex_l by Ex_i and rename the other two points by Ex_j and Ex_k . Denote the minimum $\{\angle Ex_i Ex_j Ex_l, \angle Ex_i Ex_k Ex_l\}$ by $\angle Ex_i Ex_j Ex_l$.

If $\angle Ex_i Ex_j Ex_l$ is greater than or equal to right angle,

then $Ex_j \leftarrow Ex_l$ and repeat Step 1.

Else $Ex_k \leftarrow Ex_l$ and return to Step 2.

The optimality conditions in Step 1 and Step 2 in this algorithm are directly follows from the Theorem 3.2.1.

*It is clear that the optimal solution of the **CenterSphereLoc** problem on hemispherical surface is the nearer pole of $\mathcal{C}(Ex_i, Ex_j)$ or $\mathcal{C}(Ex_i, Ex_j, Ex_k)$ whenever all the demand points lie on the $\Gamma\mathcal{C}(Ex_i, Ex_j)$ or $\Gamma\mathcal{C}(Ex_i, Ex_j, Ex_k)$. This simply says that a hemispherical minimax location reduces to finding a small circle of maximum radius on the surface of the sphere which contains either two demand points at the end of a diameter or the three demands points forming an acute triangle such that all the demand points lie on one side of the plane of the small circle and the center of the sphere on the other side.*

4. RESTRICTED SPHERICAL CENTER LOCATION PROBLEM

Given set a $\mathcal{E}_x = \{Ex_i; i = 1, 2, \dots, m\}$ of m demand points on the surface of a sphere with associated weights $w_i; i = 1, 2, \dots, m$, our goal is to find a location for a new facility in order to minimize the maximum weighted distance to the demand points with respect to a given distance of measure.

That is, we are looking for a point X^* on the surface of the sphere in which

$$\min_{X \in \mathcal{S}_0} \max_{i=1,2,\dots,m} w_i d(X, Ex_i) \quad \text{CenterSphereLoc} \quad (4.1)$$

is attained. Here $d(X, Y)$ is the distance between two points X and Y on the surface of the sphere and \mathcal{S}_0 denote the surface area of the sphere.

In practical situations, X^* will not be a feasible location. That means, there will be some regions in which the placement of a new facility is forbidden, but transportation is still possible. These regions often referred to as forbidden (or restricted) regions. These can be used to model, for example, state parks, lakes or other protected areas, or regions where the geographic characteristics are not allowed to construct the desired new facility. Therefore, finding an optimal solution(s) X^R of **CenterSphereLoc** problem can be considered as a "restricted facility location problem" on the spherical surface. This problem is known as "**Restricted Spherical Center (or minimax) (RestrictedCenterSphereLoc) problem**" (see Figure 4.1).

We assume here that , some spherical polygon (Definition 1.2.19) \mathcal{R} is given such that the new facility location X is not allowed to be contained in the interior, $int(\mathcal{R})$ of \mathcal{R} .

i.e., we want to solve

$$\min_{X \in \mathcal{F}} \max_{i=1,2,\dots,m} w_i d(X, Ex_i) \quad \text{RestrictedCenterSphereLoc} \quad (4.2)$$

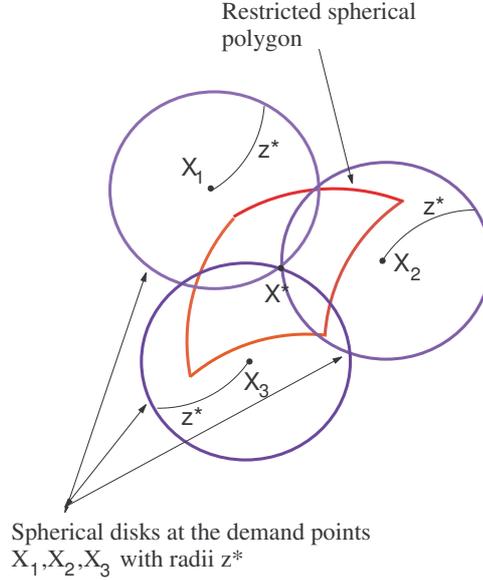


Fig. 4.1: Optimal location X^* is located in a restricted region \mathcal{R}

where $\mathcal{F} := \mathcal{S}_0 \setminus \text{int}(\mathcal{R})$.

In the following section, we restrict our problem to the special case where all the demand points lie on the surface of a hemisphere. Unrestricted version of this problem can be solved using some known methods (see [4], [26], [27]).

4.1 Basic Results for Hemispherical CenterSphereLoc Problem using Level Sets and Level Curves

Now, consider the hemispherical location problem with the shortest length of arc (great circle distance)(see Definition 1.2.8) as the distance of measure d and a convex spherical polygon (see Definition 1.2.19) as a restricted polygon, \mathcal{R} . Further we assume that $w_i = 1; \quad \forall i = 1, \dots, m$.

i.e., we want to solve

$$\min_{X \in \mathcal{F}} h(X) = \max_{i=1,2,\dots,m} \alpha(X, Ex_i) \quad (4.3)$$

where $\mathcal{F} := \mathcal{S}_0^H \setminus \text{int}(\mathcal{R})$.

Here $\alpha(X, Y)$ is the great circle arc distance between two points X and Y on the surface of the sphere and \mathcal{S}_0^H denote the surface area of a hemisphere.

Let X^* be the unique optimal solution of **CenterSphereLoc** problem and $X^{\mathcal{R}}$ be any optimal location of the **RestrictedCenterSphereLoc** problem. Further, let z^* and $z^{\mathcal{R}}$ represent the corresponding optimal objective values, respectively.

That is,

$$z^* = \max_{i=1,2,\dots,m} \alpha(X^*, Ex_i)$$

and

$$z^{\mathcal{R}} = \max_{i=1,2,\dots,m} \alpha(X^{\mathcal{R}}, Ex_i) \quad \text{etc.}$$

If $X^* \in \mathcal{S}_0^H \setminus \text{int}(\mathcal{R})$, then $X^{\mathcal{R}} = X^*$ and the restricted problem is trivially solved. Therefore, we assume that $X^* \in \text{int}(\mathcal{R})$.

If $X^* \in \text{int}(\mathcal{R})$, the following Theorem shows that $X^{\mathcal{R}}$ should be lie on the boundary, $\partial\mathcal{R}$ of the restricted polygon, \mathcal{R} .

Theorem 4.1.1: If the set of optimal locations of **CenterSphereLoc**, $\text{opt}^* \subseteq \text{int}(\mathcal{R})$ then the set of optimal locations, $\text{opt}^*(\mathcal{R})$ of the hemispherical **RestrictedCenterSphereLoc** problem is a subset of the boundary of \mathcal{R} (i.e., $\text{opt}^*(\mathcal{R}) \subseteq \partial\mathcal{R}$)

Proof Let $X^* \in \text{opt}^*$ and $X \notin \mathcal{R}$. Now we have to show $X \notin \text{opt}^*(\mathcal{R})$.

Since $\text{opt}^* \subseteq \text{int}(\mathcal{R})$ and $X \notin \mathcal{R}$, we know $h(X^*) < h(X)$.

Choose any δ such that $X^B = \delta X^* + (1 - \delta)X \in \partial\mathcal{R}$.

Since $h(X)$ is a convex function on the surface of the hemisphere, we have

$$h(X^B) = h(\delta X^* + (1 - \delta)X) \leq \delta h(X^*) + (1 - \delta)h(X).$$

$$\Rightarrow h(X^B) < \delta h(X) + (1 - \delta)h(X) = h(X).$$

I.e., there exists a point X^B on the $\partial\mathcal{R}$ which is better than X . This means $X \notin \text{opt}^*(\mathcal{R})$. ■

In the following, we will show that how can the optimal solutions $X^{\mathcal{R}}$ be characterized using level curves and level sets (see Definition 1.2.20).

Lemma 4.1.1: $z^{\mathcal{R}}$ is the optimal value for the restricted hemispherical center location problem if and only if $z^{\mathcal{R}} = \min\{z \in \mathbb{R} : L_{=}(z) \setminus \text{int}(\mathcal{R}) \neq \emptyset\}$.

Proof " \implies " *Let $z^{\mathcal{R}}$ be optimal. Then there exists $X^{\mathcal{R}}$ with $h(X^{\mathcal{R}}) = z^{\mathcal{R}}$ and $X^{\mathcal{R}} \notin \text{int}(\mathcal{R})$.*

$\implies X^{\mathcal{R}} \in L_{=}(z^{\mathcal{R}}) \setminus \text{int}(\mathcal{R}) \neq \emptyset$.

Assume $\exists \tilde{z} < z^{\mathcal{R}}$ s.t. $L_{=}(\tilde{z}) \setminus \text{int}(\mathcal{R}) \neq \emptyset$.

Then choose $\tilde{X} \in L_{=}(\tilde{z}) \setminus \text{int}(\mathcal{R})$ feasible and $h(\tilde{X}) < h(X^{\mathcal{R}}) = z^{\mathcal{R}}$. This is a contradiction for the optimality of $X^{\mathcal{R}}$. This implies

$z^{\mathcal{R}} = \min\{z \in \mathbb{R} : L_{=}(z) \setminus \text{int}(\mathcal{R}) \neq \emptyset\}$.

" \longleftarrow " *Let $z^{\mathcal{R}} = \min\{z \in \mathbb{R} : L_{=}(z) \setminus \text{int}(\mathcal{R}) \neq \emptyset\}$.*

Take $X \in L_{=}(z^{\mathcal{R}}) \setminus \text{int}(\mathcal{R})$ with $h(X) = z^{\mathcal{R}}$.

We have to show X is optimal :

Suppose X is not optimal. I.e., $\exists \tilde{X}$ s.t. $\tilde{z} = h(\tilde{X}) < h(X) = z^{\mathcal{R}}$ and $\tilde{X} \notin \text{int}(\mathcal{R})$.

$\implies L_{=}(\tilde{z}) \setminus \text{int}(\mathcal{R}) \neq \emptyset$. *This is a contradiction since $\tilde{z} < z^{\mathcal{R}}$.*

$\implies X$ is optimal. ■

Therefore, if $X^ \in \text{int}(\mathcal{R})$, we need to increase z^* until the boundary of the level set touches the boundary of the restricted region. The following Theorem presents the conditions which needs to be considered when we increase the value of z^* .*

Theorem 4.1.2: $z^{\mathcal{R}}$ is the optimal objective value of the restricted hemispherical center location problem if and only if

(1) $L_{\leq}(z^{\mathcal{R}}) \subseteq \mathcal{R}$ and

(2) $L_{=}(z^{\mathcal{R}}) \cap \partial\mathcal{R} \neq \emptyset$

Proof " \implies " : *Let $z^{\mathcal{R}}$ is optimal. Take $X \in \text{opt}^*(\mathcal{R})$ with $h(X) = z^{\mathcal{R}}$; i.e., $X \in L_{=}(z^{\mathcal{R}})$.*

Theorem 4.1.1 $\implies X \in \partial\mathcal{R}$.

Then we have $\partial\mathcal{R} \cap L_{=}(z^{\mathcal{R}}) \neq \emptyset$.

$$\begin{aligned}
 \text{Lemma 4.1.1} \quad &\Rightarrow L_=(z) \setminus \text{int}(\mathcal{R}) = \emptyset \quad \forall z < z^{\mathcal{R}}. \\
 &\Rightarrow L_=(z^{\mathcal{R}}) \setminus \mathcal{R} = \emptyset \\
 &\Rightarrow L_=(z^{\mathcal{R}}) \subseteq \mathcal{R} \\
 &\Rightarrow \cup_{z \leq z^{\mathcal{R}}} L_=(z) \subseteq \mathcal{R} \\
 &\Rightarrow L_{\leq}(z^{\mathcal{R}}) \subseteq \mathcal{R}.
 \end{aligned}$$

" \Leftarrow " : Let $L_=(z^{\mathcal{R}}) \cap \partial\mathcal{R} \neq \emptyset$ and $L_{\leq}(z^{\mathcal{R}}) \subseteq \mathcal{R}$.

$\Rightarrow L_=(z^{\mathcal{R}}) \setminus \text{int}(\mathcal{R}) \neq \emptyset$ but

$L_=(z) \setminus \text{int}(\mathcal{R}) = \emptyset \quad \forall z < z^{\mathcal{R}}$.

Lemma 4.1.1 $\Rightarrow z^{\mathcal{R}}$ is optimal. ■

Note that the optimal value z^* of the unrestricted problem is the smallest value z with

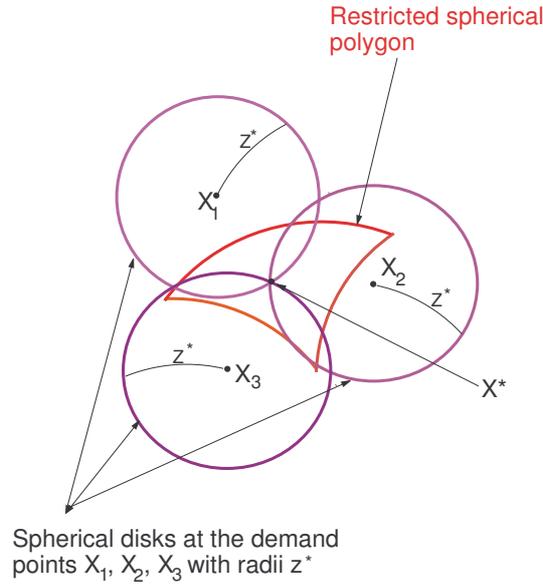


Fig. 4.2: X^* Represents the unique optimal location

$L_{\leq}(z) \neq \emptyset$. In this case $L_{\leq}(z^*) = \{X^*\}$ (see Figure 4.2). For $z > z^*$, $L_{\leq}(z)$ is an area in the hemisphere which is bounded by great circle arc segments (see Figure 4.3).

If $X^* \in \text{int}(\mathcal{R})$ is not feasible for the hemispherical **CenterSphereLoc** problem we need

to increase z^* until conditions (1) and (2) of Theorem 4.1.2 are satisfied. Since $L_{\leq}(z^{\mathcal{R}})$

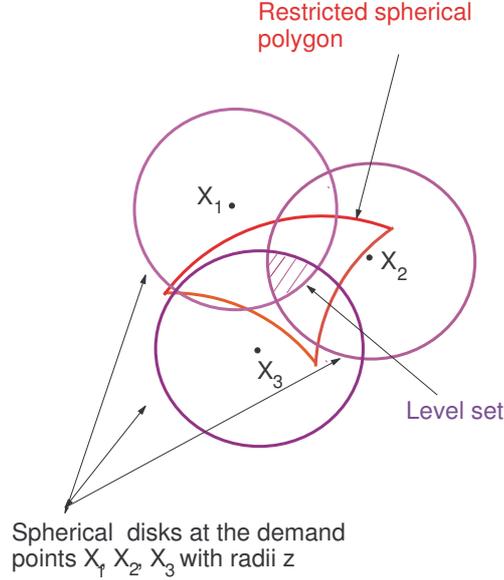


Fig. 4.3: $L_{\leq}(z)$ is the shaded area.

can be expressed as intersections of the spherical disks $\mathcal{D}(Ex_i, z)$ centered at the existing facilities Ex_i with radii z (see Result 1.2.3), the level set touches the boundary of the restricted region \mathcal{R} in two different ways as shown in Figures 4.4 and 4.5. Now, therefore we can identify the optimal solutions for the hemispherical **RestrictedCenterSphereLoc** problem as follows:

Suppose that the restricted set \mathcal{R} is a convex spherical polygon with facets f_1, f_2, \dots, f_k .

Theorem 4.1.3: If $X^* \in \text{int}(\mathcal{R})$, then there exists an optimal solution $X^{\mathcal{R}}$ to hemispherical **RestrictedCenterSphereLoc** problem with objective value

$z^{\mathcal{R}} = \max_{i=1,2,\dots,m} \alpha(Ex_i, X^{\mathcal{R}})$ and $z^{\mathcal{R}} > z^*$, which satisfies :

(a) $X^{\mathcal{R}} \in \partial\mathcal{R} \cap \text{Bisector}(Ex_i, Ex_j), i, j \in \{1, 2, \dots, m\}$. (see Figure 4.5.),

or

(b) $X^{\mathcal{R}}$ is a projection point of Ex_i on $f_k, k = 1, 2, \dots, K; i = 1, 2, \dots, m$. (see Figure 4.4).

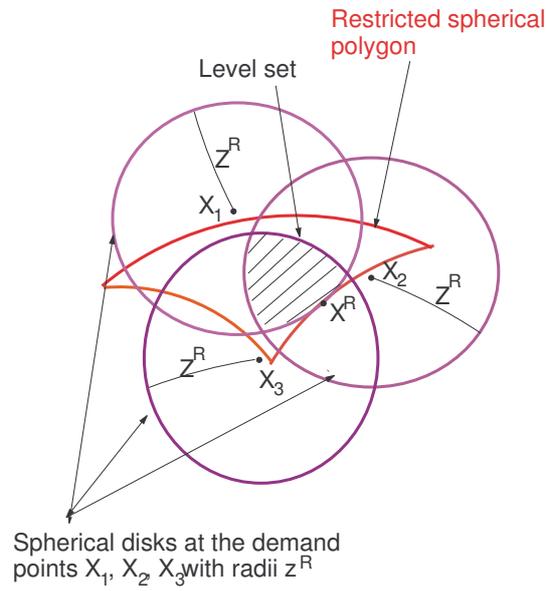


Fig. 4.4: An edge of the level set $L_{\leq}(z^R)$ touches a facet of \mathcal{R}

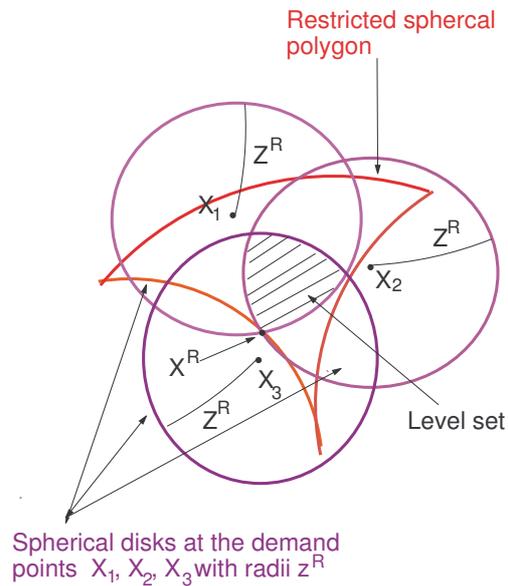


Fig. 4.5: A corner point of the level set $L_{\leq}(z^R)$ touches an edge of \mathcal{R}

Proof Theorem 4.1.2 implies that $L_{\leq}(z^{\mathcal{R}}) \subseteq \mathcal{R}$ and $L_{=}(z^{\mathcal{R}}) \cap \partial\mathcal{R} \neq \emptyset$. Since \mathcal{R} is convex spherical polygon, the intersection of spherical circles (i.e., level curve) touches \mathcal{R} from inside either at a corner point of $L_{\leq}(z^{\mathcal{R}})$ or an edge of \mathcal{R} is tangent to $L_{\leq}(z^{\mathcal{R}})$.

Case(a) A corner point (see Figure 4.5):

X_{ij} is a corner point of $L_{\leq}(z^{\mathcal{R}})$ if and only if there exists Ex_i, Ex_j such that $X_{ij} \in \mathcal{C}(Ex_i, z^{\mathcal{R}}) \cap \mathcal{C}(Ex_j, z^{\mathcal{R}})$. Hence $\alpha(Ex_i, X_{ij}) = \alpha(Ex_j, X_{ij})$ and $X_{ij} \in \text{Bisector}(Ex_i, Ex_j)$.

Case(b) An edge f_k of \mathcal{R} is tangent to $L_{\leq}(z^{\mathcal{R}})$ (see Figure 4.4):

X_{ik} is tangency point iff f_k touches one of the spherical circles; i.e., there exists $i \in \{1, 2, \dots, m\}$ such that f_k is tangent to $\mathcal{C}(Ex_i, z^{\mathcal{R}})$. i.e., X_{ik} is a projection point from Ex_i onto f_k . ■

4.2 Polynomial Algorithm for Hemispherical RestrictedCenterSphereLoc Problem

Theorem 4.1.3 characterizes the candidates for being optimal locations of the restricted problem.

Algorithm 4.2.1:

Input: $\{Ex_i : i = 1, 2, \dots, m\}$, the set of existing facilities.

\mathcal{R} : Convex spherical polygon with facets f_1, f_2, \dots, f_K .

Output: $Opt^{\mathcal{R}}$: set of all optimal locations.

$z^{\mathcal{R}}$: optimal objective value.

Step 1. Solve the unrestricted **CenterSphereLoc** problem to get the optimal location X^* with objective value z^* .

Step 2 If $X^* \notin \text{int}(\mathcal{R})$ then **output** $\text{Opt}^{\mathcal{R}} = \{X^*\}$
 Else : goto Step 3.

Step 3. Calculate

$$A = \{(P_{ik}, z) : P_{ik} \text{ is a projection point from } Ex_i \text{ onto } f_k, i = 1, 2, \dots, m; k = 1, 2, \dots, K, z = \alpha(P_{ik}, Ex_i)\}.$$

$$B = \{(P_{ij}, z) : P_{ij} \text{ is intersection point of bisector } (Ex_i, Ex_j) \text{ with } \partial\mathcal{R}, i, j = 1, 2, \dots, m, \ \& \ z = \alpha(P_{ij}, Ex_i)\}.$$

Step 4. For all $(P_{ij}, z) \in A \cup B$ with $z > z^*$, test:

if $L_{\leq}(z) \subseteq \mathcal{R}$ and $L_{=}(z) \cap \partial\mathcal{R} \neq \emptyset$. If this is the case

Output : $\text{Opt}^{\mathcal{R}} = L_{=}(z) \cap \partial\mathcal{R}, z^{\mathcal{R}} = z$.

In Case (a) of the Theorem 4.1.3, if the number of intersection points of the Bisector(Ex_i, Ex_j) with $\partial\mathcal{R}$ is two or less, they are included in the candidate list. As there are $m(m-1)/2$ bisectors of existing facilities, we will have maximum $m(m-1)$ intersection points in this case. There are $m \times K$ projection points of the existing facilities $Ex_i; i = 1, 2, \dots, m$ to the K facets of \mathcal{R} in Case (b) of Theorem 4.1.3.

The complexity of the Algorithm 4.2.1 is dominated by Step 1 and Step 4. The complexity of Step 4 is $O(m^3) + O(m^2K)$. If we solve the unrestricted hemispherical center location problem with the polynomial time algorithm, Algorithm 3.4.2, we get overall complexity of $O(m^3) + O(m^2K)$.

4.2.1 Computation of a Projection Point P_{ik} from Ex_i onto f_k

Suppose $X_k(1)$ and $X_k(2)$ be the two end points of edge f_k of the restricted polygon \mathcal{R} .

- *Let $X_k(1)$ and $X_k(2)$ be two unit vectors pointing from the center of the sphere towards points $X_k(1)$ and $X_k(2)$.*

- Take cross-product of $X_k(1)$ and $X_k(2)$ and normalize the result to get a vector G :

$$G = (X_k(1) \times X_k(2)) / |X_k(1) \times X_k(2)|.$$
- G is normal to the plane of the great circle joining $X_k(1)$ and $X_k(2)$.
- Now take the cross-product of G with Ex_i , the unit vector corresponding to point Ex_i :

$$F = G \times Ex_i$$
- F is perpendicular to Ex_i , so the great circle it defines passes through Ex_i . It is also perpendicular to G , so the great circle it defines is perpendicular to the great circle defines by G .
- Now take the cross-product of F and G and normalize the result to get a vector:

$$N = \frac{F \times G}{\|F \times G\|}.$$
- one of $\pm N$ is the projection point P_{ik} of the point Ex_i to f_k .
- $\pm N$ are antipodal points.

4.2.2 Computation of Intersection Points I_{ij} of Perpendicular Bisector M_{ij} of Ex_i and Ex_j with $\partial\mathcal{R}$

As the restricted region \mathcal{R} is formed by intersecting great circles, an edge of \mathcal{R} is a great circle segment. Also note that the bisector of Ex_i and Ex_j is a great circle. Therefore, we have to look the intersection of two great circles in order to get intersection points of bisectors with $\partial\mathcal{R}$.

Procedure of finding intersection points of two great circles

1. Let M be the mid point of the great circle $\text{arc}(Ex_i, Ex_j)$.
2. Take cross-product A , of Ex_i and Ex_j . This vector is normal to the plane of great circle passing through Ex_i and Ex_j .
3. Take cross-product B , of A with M . This vector is normal to the plane of great circle passing through A and M .
4. Let $X_k(m)$, $m = 1, 2$ be unit vectors pointing from the center of the sphere towards the end points $X_k(m)$, $m = 1, 2$ of the edge f_k of \mathcal{R} .
5. Now take cross-product C , of $X_k(1)$ and $X_k(2)$.
6. $\frac{\pm B \times C}{\|B \times C\|}$ are the intersection points of M_{ij} and f_k .

Note : The candidate intersections are antipodal points.

4.3 Hemispherical CenterSphereLoc Problem with Weights $w_i(> 0) \neq 1$

In this case, the level sets of the objective function can be defined as follows :

$$\begin{aligned}
 L \leq (z) &= \{X \in \mathcal{S}_0 : \max_{i=1,2,\dots,m} w_i \alpha(Ex_i, X) \leq z\} \\
 &= \{X \in \mathcal{S}_0 : \alpha(Ex_i, X) \leq z/w_i \quad \forall i = 1, 2, \dots, m\} \\
 &= \bigcap_{i=1,2,\dots,m} \{X \in \mathcal{S}_0 : \alpha(X, Ex_i) \leq z/w_i\}.
 \end{aligned}$$

That is, the level set can be obtained by intersecting all the spherical disks $\mathcal{D}(Ex_i, \frac{z}{w_i})$ with centers Ex_i and radii $\frac{z}{w_i}$, $i = 1, 2, \dots, m$. It is clear that spherical disks $\mathcal{D}((Ex_i, \frac{z}{w_i}))$ have different sizes.

As in the case of weights $w_i = 1$, the set of possible locations for the hemispherical **RestrictedCenterSphereLoc** problem with weights $w_i > 0 (\neq 1)$, consists of all projection points of existing facilities to the facets of \mathcal{R} and of all corner points of $L_{\leq}(z^R)$ (see Theorem 4.1.2, Figures 4.4 and 4.5), even if $\mathcal{D}((Ex_i, \frac{z}{w_i}))$ have different sizes.

Therefore, we have to check:

(i) all projection points X_{iq} from existing facility Ex_i to any facet f_q ,

and

(ii) all points X which satisfy $w_i\alpha(Ex_i, X) = w_j\alpha(Ex_j, X)$ for any pair of existing facilities Ex_i and Ex_j .

That means, to get corner points we have to calculate intersection points of $\{X \in S : w_i\alpha(Ex_i, X) = w_j\alpha(Ex_j, X)\}$ with ∂R for all $i < j; i, j \in \{1, 2, \dots, m\}$.

Then we can apply the above algorithm by changing the set B in step 3 as follows :

$$\begin{aligned}
 B' &= \{(X, z) : X \text{ is a intersection point of the set} \\
 &\quad \{X \in S : w_i\alpha(X, Ex_i) = w_j\alpha(Ex_j, X)\} \quad \text{with} \\
 &\quad \partial R, i, j \in \{1, 2, \dots, m\}; z = w_i\alpha(X, Ex_i) = w_j\alpha(X, Ex_j)\}.
 \end{aligned}$$

5. SPHERICAL LOCATION PROBLEMS WITH POLYGONAL BARRIERS

In development of spherical location models we deal with a geometric representation of the problem, and the geographical reality has to be incorporated into this representation. In almost every real-life situation we have to deal with restrictions and constraints of various types. As restricted or forbidden regions (see Chapter 4) in the context of spherical location models, there are many areas in which the placement of a new facility and transportation are completely forbidden or even impossible. These regions (or areas) often referred to as barrier regions. To give some examples of possible barrier regions, consider military areas, mountain ranges and lakes on the globe.

Consider a finite set of convex, closed and piecewise disjoint barrier regions $\{B_1, \dots, B_N\}$ on the surface of the sphere. We consider the union of these barrier regions by $\mathcal{B} := \bigcup_{i=1}^N B_i$ and the finite set of extreme points and facets of \mathcal{B} by $Ext(\mathcal{B})$ and $Facet(\mathcal{B})$, respectively. The interior of these barrier regions is forbidden for the placement of a new facility, and additionally, travelling through $int(\mathcal{B})$ is prohibited. Thus the feasible region \mathcal{F} on the spherical surface for new locations and for travelling is given by

$$\mathcal{F} = \mathcal{S}_0 \setminus int(\mathcal{B}).$$

Further, we assume that the measure of distance on the surface of the sphere \mathcal{S}_0 is length of shortest arc (or great circle distance), $\alpha = \alpha(X, Y)$ for all $X, Y \in \mathcal{S}_0$.

Definition 5.0.1: Given two points $X, Y \in \mathcal{F}$ the barrier distance $\alpha_{\mathcal{B}}(X, Y)$ with respect to α is the length of a shortest path between X and Y not intersecting the interior of a barrier region.

A permitted X - Y path with length $\alpha_{\mathcal{B}}(X, Y)$ will be called a α -shortest permitted path. Further, we call two points X and Y in \mathcal{F} α -visible if they satisfy $\alpha_{\mathcal{B}}(X, Y) = \alpha(X, Y)$, i.e.,

the distance between X and Y is not lengthened by the barrier regions.

Given set a $\mathcal{E}_x = \{Ex_i; i = 1, 2, \dots, m\}$ of m demand points on the surface of a sphere with associated weights (or demands) $w_i > 0; i = 1, 2, \dots, m$, spherical center location (**CenterSphereLoc**) problem and spherical Weber location (**WeberSphereLoc**) problem with polygonal barriers can be formulated respectively with this barrier distance $\alpha_{\mathcal{B}}(X, Y)$, as

minimize

$$\begin{aligned} h_{\mathcal{B}}(X) &= \max_{i=1,2,\dots,m} w_i \alpha_{\mathcal{B}}(X, Ex_i) && \text{BarrierCenterSphereLoc} && (5.1) \\ \text{s.t.} & X \in \mathcal{F} \end{aligned}$$

and

minimize

$$\begin{aligned} f_{\mathcal{B}}(X) &= \sum_{i=1,2,\dots,m} w_i \alpha_{\mathcal{B}}(X, Ex_i) && \text{BarrierWeberSphereLoc} && (5.2) \\ \text{s.t.} & X \in \mathcal{F}. \end{aligned}$$

Note that the shortest arc distance, α is not convex. Further, the barrier distance $\alpha_{\mathcal{B}}(X, Y)$ is in general not-convex and therefore $f_{\mathcal{B}}$ and $h_{\mathcal{B}}$ are also not convex functions.

5.1 Shortest Paths in the Presence of Barrier Regions

Definition 5.1.1: The set of points $Y \in \mathcal{F}$ that are not α -visible from a point $X \in \mathcal{F}$ is called the **shadow of \mathbf{X}** with respect to α , i.e.,

$$\text{shadow}_{\alpha}(X) := \{Y \in \mathcal{F} : \alpha_{\mathcal{B}}(X, Y) > \alpha(X, Y)\}.$$

(See Figure 5.1).

The following results shows that for any two points $X, Y \in \mathcal{F}, X \neq Y$ there always exists a α -shorted permitted path connecting X and Y that is a piecewise shortest arc path with

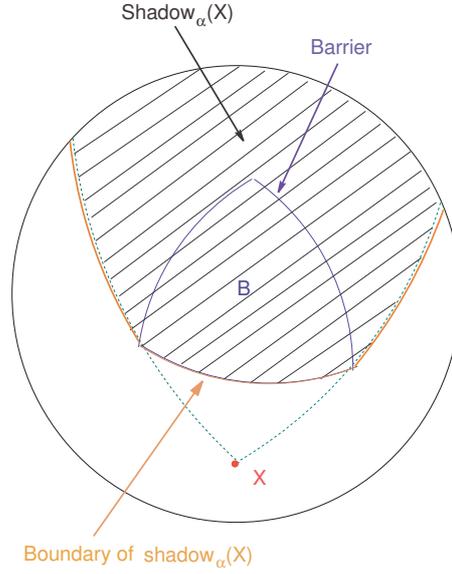


Fig. 5.1: Shaded area represents the shadow of X

breaking points only in extreme points of a barrier region.

Lemma 5.1.1: Let $\alpha = \alpha(X, Y)$ be the shortest arc distance between X and Y , where $X, Y \in \mathcal{F}$. Then there exists a α -shortest permitted path SP connecting X and Y with the following property.

Property 5.1.1: : SP is a piecewise shortest arc length path with breaking points only on extreme points of barrier regions.

Proof Let $X, Y \in \mathcal{F}$ and let SP be α - shortest permitted path connecting X and Y in \mathcal{F} that satisfies Property 5.1.1. Then consider two consecutive arc segments $\text{arc}(X_j, X_{j+1})$ and $\text{arc}(X_{j+1}, X_{j+2})$ on SP. Without loss of generality they may be assumed not to be curvilinear as otherwise X_{j+1} would be irrelevant and could be deleted.

Let X' and X'' denote points on arcs $\text{arc}(X_j, X_{j+1})$ and $\text{arc}(X_{j+1}, X_{j+2})$ respectively at an arbitrary small distance $\varepsilon > 0$ from X_{j+1} .

The path composed of arcs $\text{arc}(X_j, X')$, $\text{arc}(X', X'')$, $\text{arc}(X'', X_{j+2})$ is strictly shorter than the path composed of $\text{arc}(X_j, X_{j+1})$ and $\text{arc}(X_{j+1}, X_{j+2})$ due to a property that any two

sides of a spherical triangle are together greater than the third side.

As the latter path is feasible by the hypothesis, the former one can only be non-feasible for all positive ε if X_{j+1} is a vertex of a barrier region with a arc segment crossing $\text{arc}(X', X'')$ (see Figure 5.2). ■

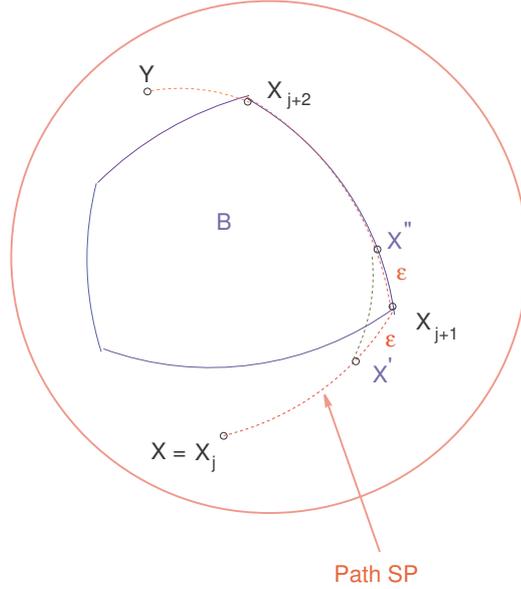


Fig. 5.2: Shortest path SP for proof of Lemma 5.1.1

Therefore, using Property 5.1.1 in Lemma 5.1.1, the barrier distance $\alpha_B(X, Y); X, Y \in \mathcal{F}$ can be calculated with respect to a so-called intermediate point $I_{X,Y}$, i.e., a breaking point on a α -shortest permitted path so that $I_{X,Y}$ is α -visible from Y . Note also that if X and Y are α -visible then the intermediate point $I_{X,Y}$ equals X .

Corollary 5.1.1: Let $\alpha = \alpha(X, Y)$ be the shortest arc distance between $X, Y \in \mathcal{F}$. Furthermore, let SP be a α -shortest permitted $X - Y$ path with Property 5.1.1 and the point $I_{X,Y} \neq Y$ be a breaking point on SP that is α -visible from Y . Then

$$\alpha_B(X, Y) = \alpha_B(X, I_{X,Y}) + \alpha(I_{X,Y}, Y). \quad (5.3)$$

Note that the intermediate points $I_{X,Y}$ are not necessarily unique.

Definition 5.1.2: The boundary of $shadow_\alpha(X)$,

$$\begin{aligned} \partial(shadow_\alpha(X)) &:= \{Y \in \mathcal{F} : \mathcal{D}(Y, \varepsilon) \cap shadow_\alpha(Y) \neq \emptyset \\ &\text{and } \mathcal{D}(Y, \varepsilon) \not\subseteq shadow_\alpha(Y) \quad \forall \varepsilon > 0\}, \end{aligned} \quad (5.4)$$

where $\mathcal{D}(Y, \varepsilon)$ is a spherical disk with center Y and radius $\varepsilon > 0$.

Note that the $\partial(shadow_\alpha(X))$ is a connected set of shortest length of arcs on the surface of the sphere.

Obviously, those parts of $\partial(shadow_\alpha(X))$ that are of the boundary of a barrier region are also shortest length of arcs on the spherical surface. For all other parts of $\partial(shadow_\alpha(X))$, consider a point Y on $\partial(shadow_\alpha(X))$ and let $I_{X,Y}$ be an intermediate point on a α -visible shortest permitted $X - Y$ path with Property 5.1.1. Note that in this case Y is α -visible from X . If all the points Z on the line segment starting at $I_{X,Y}$ passing through Y and ending as soon as it intersects the interior of a barrier region are α -visible from X .

5.2 Reducing the Non-convex BarrierSphereLoc Problem to a Set of Sub problems

Here, we consider a partitioning of the feasible region \mathcal{F} into finite set of subregions using the grid \mathcal{G}_α on the surface of the sphere.

The grid \mathcal{G}_α is defined by the boundaries of the shadows of all existing facilities $Ex_i, i = 1, 2, \dots, m$ and of all extreme points $Ext(\mathcal{B})$ of the barrier region \mathcal{B} , plus all the facets $Facet(\mathcal{B})$ of the barrier regions, i.e.,

$$\mathcal{G}_\alpha := \left\{ \bigcup_{X \in \mathcal{E}_x \cup Ext(\mathcal{B})} \partial(shadow_\alpha(X)) \right\} \cup Facet(\mathcal{B}) \quad (5.5)$$

Since the barriers are convex polygons and also the boundary of $shadow_\alpha(X)$ is set of arc segments for all $X \in \mathcal{F}$, the grid \mathcal{G}_α consists of a finite set of shortest length of arc segments in \mathcal{F} .

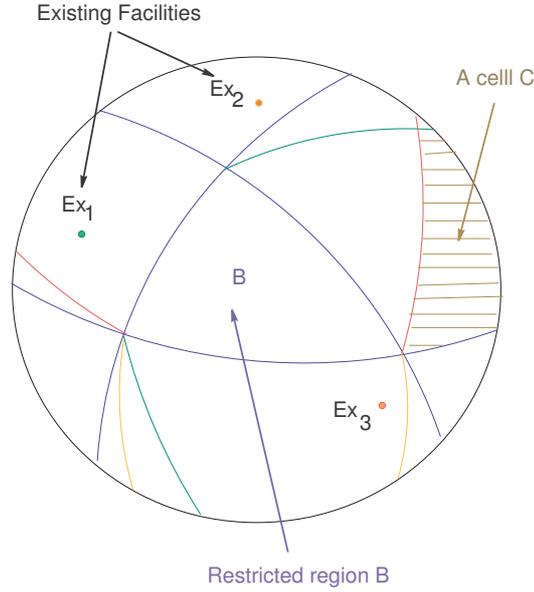


Fig. 5.3: The grids on the surface of the hemisphere

Definition 5.2.1: A cell of grid \mathcal{G}_α is a smallest set (not necessarily convex or closed) polygon not intersected by an arc segment in \mathcal{G}_α (see Figure 5.3).

We denote the set of cells of \mathcal{G}_α as $\mathcal{C}(\mathcal{G}_\alpha)$.

To see how the barrier distance defines from an existing facility to a point X in a cell C , we consider a following example with three existing facilities and one barrier region with four extreme points $t_j; j = 1, \dots, 4$ (see Figure 5.4). Then the barrier distance from X to Ex_2 , $\alpha_B(X, Ex_2)$ can be calculated as

$$\alpha_B(X, Ex_2) = \alpha_B(Ex_2, I_2) + \alpha(I_2 + X) \quad \forall X \in C$$

where $\alpha_B(Ex_2, I_2) = \alpha(Ex_2, t_1) + \alpha(t_1, I_2)$ and $I_2 = I_{Ex_2, X} = t_2$.

Therefore, we can generally consider a cell $C \in \mathcal{C}(\mathcal{G}_\alpha)$ and let $X \in C$. So if we let $I_i := I_{Ex_i, X}, i = 1, 2, \dots, m$ is an intermediate point on a α -shorted permitted $X - Ex_i$ -path with Property 5.1.1 that is α -visible from X , then the barrier distance between X and the existing facility $Ex_i, i = 1, 2, \dots, m$ can be written as

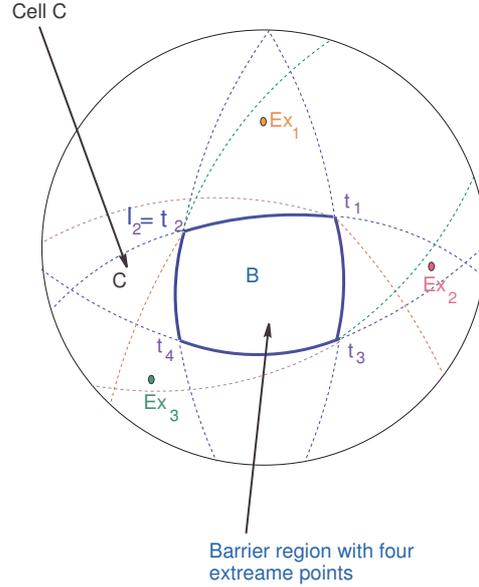


Fig. 5.4: The grids for the example problem with one barrier region and three existing facilities

$$\alpha_{\mathcal{B}}(X, Ex_i) = \alpha(X, I_i) + \alpha_{\mathcal{B}}(I_i, Ex_i) \quad \forall X \in \mathcal{C} \quad (5.6)$$

A visibility graph (as proposed in Butt and Cavalier [3]) can be used to determine distances between the facilities and all those points that are candidates of intermediate points on a α -shorted permitted path between an existing facility and a point $X \in \mathcal{F}$. Let the node set of this visibility graph G is $V(G) := \mathcal{E}_x \cup \text{Ext}(\mathcal{B})$ and arc set of G is $E(G)$, where $E(G)$ consists of all the arcs that connect two nodes v_i, v_j in $V(G)$ if the corresponding nodes on the surface of the sphere (hemisphere) are α -visible and have the distance $\alpha(v_i, v_j)$. In figure 5.5, an example is given for the case that single barrier region presents in the location problem

Then the barrier distance $\alpha_{\mathcal{B}}(Ex_i, X)$ between an existing facility $Ex_i \in \mathcal{E}_x$ and a point $X \in \mathcal{F}$ can be now calculated as

$$\alpha_{\mathcal{B}}(Ex_i, X) = \alpha_G(Ex_i, I_{Ex_i, X}) + \alpha(I_{Ex_i, X}, X), \quad (5.7)$$

where $\alpha_G(Ex_i, I_{Ex_i, X})$ denotes the length of a shortest path between Ex_i and the interme-

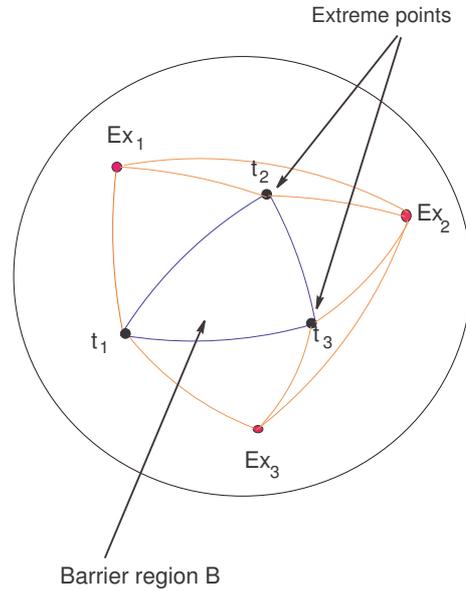


Fig. 5.5: The visibility graph for an example problem where a single barrier region is presents on the surface of the hemisphere

diate point $I_{Ex_i, X}$ in the visibility graph G .

Thus for any $X \in C$, barrier distance $h_X(X)$ from X to all the existing facilities can be calculated using (5.6).

Hence, we could find the optimal facility locations for (5.1) and (5.2) within C by solving subproblems which are defined on C .

In the rest of this section, we will focus only on the **BarrierWeberSphereLoc** problem. All the arguments which are made on this problem, are analogously true for the **Barrier-CenterSphereLoc** problem.

Now consider the **BarrierWeberSphereLoc** problem. For any $X \in C$ sum of the weighted distances, $f_X(X)$ from X to all the existing facilities can be calculated using the barrier distance, (5.6) as follows:

$$\begin{aligned}
 & \text{minimize} \\
 (\mathbf{SP}^1) \quad f_X(X) &= \sum_{i=1}^m w_i \{ \alpha_{\mathcal{B}}(Ex_i, I_i) + \alpha(I_i, X) \} & (5.8) \\
 & \text{s.t.} \quad X \in C. & (5.9)
 \end{aligned}$$

Because, $\alpha_{\mathcal{B}}(I_i, Ex_i)$ is a constant we can reformulate the objective function (5.9) as

$$\begin{aligned}
 \min f_X(X) = \{ \tilde{f}_X(X) &= \sum_{i=1}^m w_i \alpha(I_i, X) \} + \text{const} & (5.10) \\
 & \text{s.t.} \quad X \in C.
 \end{aligned}$$

where

$$\text{const} = \sum_{i=1}^m w_i \alpha_{\mathcal{B}}(Ex_i, I_i) \tag{5.11}$$

$$\tag{5.12}$$

and we can solve \mathbf{SP}^1 , by equivalently solving

$$\begin{aligned}
 & \text{minimize} \\
 (\mathbf{SP}^2) \quad \tilde{f}_X(X) &= \sum_{i=1}^m w_i \alpha(X, I_i) \\
 & \text{s.t.} \quad X \in C & (5.13)
 \end{aligned}$$

Further, if we relax the constraint of \mathbf{SP}^2 , then we have the following unconstraint problem :

$$\begin{aligned}
 & \text{minimize} \\
 \mathbf{SP}^3 \quad \tilde{f}_X(X) &= \sum_{i=1}^m w_i \alpha(X, I_i) & (5.14)
 \end{aligned}$$

$$\tag{5.15}$$

Note that \mathbf{SP}^3 is simply a WeberSphereLoc problem with existing facilities $I_i; i = 1, 2, \dots, m$.

Corollary 5.2.1: Let $C \in \mathcal{C}(\mathcal{G}_\alpha)$ be a cell and let $X \in C$ a feasible solution for the **BarrierWeberSphereLoc** problem. Then

$$f_{\mathcal{B}}(X) = \sum_{i=1}^m w_i \alpha_{\mathcal{B}}(X, Ex_i) = f_X(X), \quad (5.16)$$

$$\begin{aligned} \text{where } f_X(Y) &:= \sum_{i=1}^m w_i \{\alpha(Y, I_1) + c_1, \dots, \alpha(Y, I_m) + c_m\}, \\ &= f(\alpha(Y, I_1) + c_1, \dots, \alpha(Y, I_m) + c_m) \quad Y \in \mathcal{S}_0 \end{aligned} \quad (5.17)$$

$$\text{and } c_i := \alpha_{\mathcal{B}}(I_i, Ex_i), \quad i = 1, \dots, m. \quad (5.18)$$

and where $I_i := I_{Ex_i, X} \neq X; i = 1, 2, \dots, m$ is an intermediate point on a α -shortest permitted $X - Ex_i$ -path with *Property 5.1.1* that is α -visible from X .

According to the *Corollary 5.2.1*, the **BarrierWeberSphereLoc** problem can be reduced to a finite set of corresponding unconstrained (or **WeberSphereLoc**) problems with the shortest arc distance as the measure of distance.

5.3 BarrierWeberSphereLoc Problem on the Surface of a Hemisphere

As a result that the **WeberSphereLoc** problem on the surface of a hemisphere is a convex problem, the function, $f_X(Y)$ which is defined in *Corollary 5.2.1*, is also convex on the surface of the hemisphere since it can be interpreted as the composition of the convex nondecreasing function f and the convex functions $\alpha(Y, I_i) + c_i; i = 1, 2, \dots, m$, where c_i is a constant not depending on choice of Y .

Lemma 5.3.1: Let $C \in \mathcal{C}(\mathcal{G}_\alpha)$ be a cell and let $X \in C$. Then

$$F_X(Y) \geq F_Y(Y) \quad \forall Y \in C, \quad (5.19)$$

where F_X and F_Y are defined according to (5.17) and (5.18) and the intermediate points $I_m, m \in \mathcal{M}$ are chosen such that they are α -visible from X and Y respectively.

Proof Let $F_X(Y) = f(\alpha(Y, I_1) + c_1, \dots, \alpha(Y, I_m) + c_m)$, where $c_i = \alpha_{\mathcal{B}}(I_i, Ex_i)$ and the intermediate points $I_i = I_{Ex_i, X}$ are chosen such that they are α -visible from all points

in $C, i = 1, 2, \dots, m$. Due to the spherical triangle inequality, $\alpha(Y, I_i) + c_i = \alpha_{\mathcal{B}}(Y, I_i) + \alpha_{\mathcal{B}}(I_i, Ex_i) \geq \alpha_{\mathcal{B}}(Y, Ex_i)$ holds for all $i = 1, 2, \dots, m$ and $Y \in C$. Then

$$\begin{aligned} F_X(Y) &= f(\alpha(Y, I_1) + c_1, \dots, \alpha(Y, I_m) + c_m) \\ &\geq f(\alpha_{\mathcal{B}}(Y, Ex_1), \dots, \alpha_{\mathcal{B}}(Y, Ex_m)) \\ &= F_Y(Y). \end{aligned}$$

■

Theorem 5.3.1: Let $C \in \mathcal{C}(\mathcal{G}_\alpha)$ be a cell and let $X_{\mathcal{B}}^* \in C$ be an optimal solution of the **BarrierWeberSphereLoc** problem. Then $X_{\mathcal{B}}^*$ is an optimal solution to the corresponding convex problem

$$\begin{aligned} \min \quad & F_{X_{\mathcal{B}}^*}(Y) \\ \text{s.t.} \quad & Y \in C, \end{aligned} \tag{5.20}$$

where $F_{X_{\mathcal{B}}^*}(Y)$ is defined according to (5.17) and (5.18) and the intermediate points $I_i, i = 1, 2, \dots, m$ are chosen such that they are α -visible from $X_{\mathcal{B}}^*$.

Proof Let $X_{\mathcal{B}}^* \in C$, $F_{X_{\mathcal{B}}^*}(Y)$ be defined according to (5.17) and (5.18), and let $I_i, i = 1, 2, \dots, m$ be the corresponding intermediate points on α -shortest permitted $Ex_i - X_{\mathcal{B}}^*$ paths, satisfying the the property Property 5.1.1, that are α -visible from all points in C . Lemma 5.19 implies that

$$F_{X_{\mathcal{B}}^*}(Y) \geq F_Y(Y) = f_{\mathcal{B}}(Y) \tag{5.21}$$

holds for all $Y \in C$. Using Corollary 5.2.1 and the assumption that $X_{\mathcal{B}}^*$ is an optimal solution of **BarrierWeberSphereLoc** problem, we obtain

$$F_{X_{\mathcal{B}}^*}(Y) \geq f_{\mathcal{B}}(Y) \geq f_{\mathcal{B}}(X_{\mathcal{B}}^*) = F_{X_{\mathcal{B}}^*}(X_{\mathcal{B}}^*) \quad \forall Y \in C.$$

■

Theorem 5.3.1 implies that **BarrierWeberSphereLoc** problem on a hemisphere can be reduced to a finite set of convex subproblems within each cell in $\mathcal{C}(\mathcal{G}_\alpha)$ even though the

original objective function $f_{\mathcal{B}}(X)$ is in general non-convex within the cells.

If an optimal solution $X_{\mathcal{B}}^*$ of **BarrierWeberSphereLoc** problem is located in the interior of a cell, the following result proves that this solution can be found by solving a finite set of convex subproblems with the objective function $F_X(Y)$ defined according to (5.17) and (5.18).

Theorem 5.3.2: Let $C \in \mathcal{C}(\mathcal{G}_\alpha)$ be a cell and let $X_{\mathcal{B}}^* \in \text{int}(C)$ an optimal solution of **BarrierWeberSphereLoc** problem with barrier distance $\alpha_{\mathcal{B}}$. Then $X_{\mathcal{B}}^*$ is an optimal solution to the corresponding convex problem

$$\begin{aligned} \min \quad & F_{X_{\mathcal{B}}^*}(Y) \\ \text{s.t} \quad & Y \in \mathcal{S}_0^H \end{aligned} \tag{5.22}$$

where $F_{X_{\mathcal{B}}^*}(Y)$ is defined according to (5.17) and (5.18) and the intermediate points $I_i, i = 1, 2, \dots, m$ are chosen such that they are α -visible from $X_{\mathcal{B}}^*$.

Proof Let $X_{\mathcal{B}}^* \in \text{int}(C)$. Since $X_{\mathcal{B}}^* \in C$, Theorem 5.3.1 implies that $X_{\mathcal{B}}^*$ minimizes $F_{X_{\mathcal{B}}^*}$ in the cell C . Using the fact that $F_{X_{\mathcal{B}}^*}(Y)$ is convex function of Y on a hemisphere and that $X_{\mathcal{B}}^* \in \text{int}(C)$, we can conclude that $X_{\mathcal{B}}^*$ minimize the $F_{X_{\mathcal{B}}^*}(Y)$ on a hemisphere. ■

Theorem 5.3.3: Let C be a cell in $\mathcal{C}(\mathcal{G}_\alpha)$ and $X_{\mathcal{B}}^*$ be a global optimal solution to the convex problem

$$\begin{aligned} \min \quad & F_X(Y) \\ \text{s.t} \quad & Y \in \mathcal{S}_0^H \end{aligned} \tag{5.23}$$

where $F_X(Y)$ is defined according to (5.17) and (5.18) and the intermediate points $I_i, i = 1, 2, \dots, m$ are chosen such that they are α -visible from any $X \in C$. If $X_{\mathcal{B}}^* \in \text{int}(C)$, then $X_{\mathcal{B}}^*$ is at least a local optimal solution to the **BarrierWeberSphereLoc** problem on a hemisphere.

Proof First, given that $X_{\mathcal{B}}^* \in \text{int}(C)$, it is clear that $F_X(X_{\mathcal{B}}^*)$ is a lower bound to the optimal objective value of (5.20); that is $F_X(X_{\mathcal{B}}^*) \leq F_X(Y)$ for each $Y \in C$.

That is $X_{\mathcal{B}}^*$ is the global optimal solution of the convex subproblem which is defined on C . Therefore, there exists an ϵ -neighborhood of $X_{\mathcal{B}}^*$, $N_\epsilon(X_{\mathcal{B}}^*) \subset \text{int}(C)$, such that $F_X(X_{\mathcal{B}}^*) \leq F_X(Y)$ for each $Y \in N_\epsilon(X_{\mathcal{B}}^*)$.

But since

$$N_\epsilon(X_{\mathcal{B}}^*) \subset \text{int}(C) \subset \mathcal{G}_\alpha,$$

it follows that $f_{\mathcal{B}}(X_{\mathcal{B}}^*) = F_X(X_{\mathcal{B}}^*) \leq F_X(Y) = f_{\mathcal{B}}(Y)$ for each

$$Y \in N_\epsilon(X_{\mathcal{B}}^*) = \mathcal{G}_\alpha \cup N_\epsilon(X_{\mathcal{B}}^*) \quad (5.24)$$

This complete the proof, since (5.24) defines a local optimal solution of **BarrierWebwer-SphereLoc**. ■

5.3.1 Iterative Spherical Convex Hull

According to Theorems 5.3.1 and 5.3.2, it is clear that there are some relationship between **SphereLoc** problems and **BarrierSphereLoc** problems on a hemisphere. Therefore, some of the general properties of **SphereLoc** can be transferred to the **BarrierSphereLoc** problems. As an example, the optimal locations of **WeberSphereLoc** and **CenterSphereLoc** problems on a hemisphere lie within the spherical convex hull of the existing facilities (see 2.1.3). An analogous property can be proven for the **BarrierSphereLoc** problems by defining an iterative spherical convex hull I^{convex} of the existing facilities and the barrier regions.

Definition 5.3.1: Let \mathcal{B} be the union of a finite set of closed convex and pairwise disjoint spherical polygons on a hemisphere. **Iterative convex hull** I^{convex} is defined as the smallest spherical convex hull in the surface of the hemisphere such that

$$\{E x_i; i = 1, 2, \dots, m\} \subset I^{\text{convex}} \quad \text{and} \quad \partial I^{\text{convex}} \cap \text{int}(\mathcal{B}) = \emptyset.$$

(see Figure 5.6).

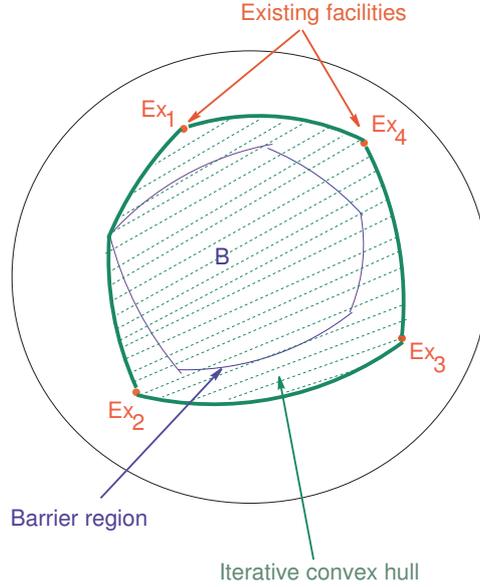


Fig. 5.6: The shaded area represents the iterative spherical convex hull

Theorem 5.3.4: Let $X_{\mathcal{B}}^* \notin \mathcal{G}_\alpha$ be an optimal solution for the **BarrierWeberSphereLoc** problem on a hemisphere .

If for all corresponding **WeberSphereLoc** subproblems with objective function F_X as defined in (5.17) and (5.18), the set of optimal solutions is contained in the spherical convex hull of the existing facilities, then

$$X_{\mathcal{B}}^* \in (I^{\text{convex}} \cap \mathcal{F}).$$

Proof Let $X_{\mathcal{B}}^*$ be an optimal solution of **BarrierWebwerSphereLoc** such that $X_{\mathcal{B}}^* \in \text{int}(C)$ for some cell $C \in \mathcal{C}(\mathcal{G}_\alpha)$.

Suppose that $X_{\mathcal{B}}^* \notin I^{\text{convex}}$. Wlog, we assume that there exists no barrier in $\mathcal{S}_0^H \setminus I^{\text{convex}}$, since this assumption does not increase the objective value of any point $X \in (I^{\text{convex}} \cap \mathcal{F})$.

Theorem 5.3.2 $\implies X_{\mathcal{B}}^*$ is an optimal solution of problem (5.22) with respect to some intermediate points $I_i \in \{Ex_i; i = 1, 2, \dots, m\} \cup \text{Ext}(\mathcal{B})$ for $i = 1, 2, \dots, m$. This problem is an **WeberSphereLoc** problem with the objective function F_X and thus $X_{\mathcal{B}}^* \in \text{conv}\{I_i : i = 1, 2, \dots, m\} \cap \mathcal{F}$. Since I^{convex} is the spherical convex hull of all existing facilities and all

barrier regions, we can conclude that

$$\text{conv}\{I_i : i = 1, 2, \dots, m\} \cap \mathcal{F} \subseteq \text{conv}(\{Ex_i; i = 1, 2, \dots, m\} \cup \text{Ext}(\mathcal{B})) \cap \mathcal{F} \subseteq I^{\text{convex}} \cap \mathcal{F}.$$

■

5.3.2 Line Search Procedure on a Hemispherical Surface

Suppose $X = (x_1, y_1, z_1)$ and $Y = (x_2, y_2, z_2)$ are two points (= position vectors) on the unit sphere \mathcal{S}_0 . To find the great circle that passes through X and Y , let

$$\begin{aligned} W &= Y - \text{Proj}_X(Y) = Y - \frac{X \cdot Y}{X \cdot X} X \\ &= Y - (X \cdot Y)X \quad \text{since } X \cdot X = 1^2. \end{aligned}$$

The vector W is perpendicular to X , but its length may not be one.

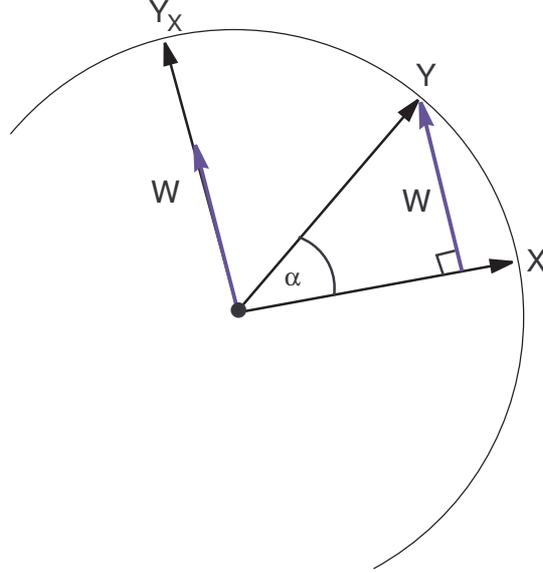


Fig. 5.7: Great circle that passes through X and Y

Thus, we re-scale to obtain a vector Y_X of the form

$$Y_X = \frac{1}{\|W\|} W \quad (5.25)$$

If we now define the curve

$$X(t) = \cos(t)X + \sin(t)Y_X \quad (5.26)$$

then $X''(t) = -X(t)$, which implies that the acceleration of $X(t)$ is normal to the sphere. Moreover, because X and Y_X are orthogonal, we have

$$\begin{aligned} X(t) \cdot X(t) &= \cos^2(t)X \cdot X + 2\sin(t)\cos(t)X \cdot Y_X + \sin^2(t)Y_X \cdot Y_X \\ &= 1^2 \cos^2(t) + 0 + 1^2 \sin^2(t) \\ &= 1^2 \end{aligned}$$

Thus $\|X(t)\| = 1$ for all t , which implies that $X(t)$ is on the unit sphere. As a result, $X(t) = \cos(t)X + \sin(t)Y_X$ is the great circle that passes through both X and Y .

Indeed, if we let

$$\alpha = \alpha(X, Y) = \arccos(X \cdot Y) \quad (5.27)$$

then it can be shown that $X(0) = X$ and $X(\alpha) = Y$.

That means, given two points X and Y on the surface of the unit sphere, any point $X(t)$ on the great circle arc, $\text{arc}(X, Y)$, has the following parametric form :

$$X(t) = \cos(t)X + \sin(t)Y_X = (x_t, y_t, z_t) \quad (5.28)$$

where $t \in [0, \alpha]$.

Suppose that g is a convex function on the surface of a hemisphere \mathcal{S}_0 . As an example, g may be *WeberSphereLoc* or *CtenterSphereLoc* problems on \mathcal{S}_0 . Now, our goal is to minimize g on grids \mathcal{G}_α .

That is, we want to minimize

$$\begin{aligned} &g(X(t)) \\ \text{s.t.} & X(t) \in \text{arc}(X, Y) \end{aligned} \quad (5.29)$$

where $X, Y \in \mathcal{S}_0$, $t \in [0, \alpha]$, with $\alpha(X, Y) = \arccos(X \cdot Y)$ and $X(t)$ is defined as (5.28).

Line search procedure on the great circle arc

Consider two points X and Y on the surface of the unit sphere. Let $X(t)$ be any point on the great circle arc, $\text{arc}(X, Y)$. Then $X(t)$ has the form (5.28) where $t \in [0, \alpha]$ and α is defined as (5.27).

Now consider the line search procedure to minimize $g(X(t))$ subject to $0 \leq t \leq \alpha$. As we don't know the exact solution of the minimum of g over $[0, \alpha]$ on the great circle arc, $\text{arc}(X, Y)$, the interval $[0, \alpha]$ is called interval of uncertainty.

During the search procedure if we can exclude points of this interval that do not contain the minimum, then the interval on uncertainty is reduced.

The following Theorem shows that if the function $g(X(t))$ is spherical convex then the interval of uncertainty can be reduced by evaluating g at two points within the interval.

Theorem 5.3.5: Let $g(X(t))$ be convex over the $\text{arc}(X, Y)$ with the interval of uncertainty $[0, \alpha]$. Let $\lambda, \mu \in [0, \alpha]$ such that $\lambda < \mu$. If $g(X(\lambda)) > g(X(\mu))$, then $g(X(z)) \geq g(X(\mu))$ for all $z \in [0, \lambda)$. If $g(X(\lambda)) \leq g(X(\mu))$, then $g(X(z)) \geq g(X(\lambda))$ for all $z \in (\mu, \alpha]$.

Proof Suppose that $g(X(\lambda)) > g(X(\mu))$ and let $z \in [0, \lambda)$.

By contradiction, suppose that $g(X(z)) < g(X(\mu))$. Since λ can be written as a convex combination of z and μ , and by the convexity of g , we have

$$\begin{aligned} g(X(\lambda)) = g(\beta X(z) + (1 - \beta)X(\mu)) &\leq \beta g(X(z)) + (1 - \beta)g(X(\mu)) \\ &< \beta g(X(\mu)) + (1 - \beta)g(X(\mu)) = g(X(\mu)) \end{aligned}$$

contradicting $g(X(\lambda)) > g(X(\mu))$. Hence, $g(X(z)) \geq g(X(\mu))$. The second part of the theorem can be proved similarly.

Remark From the Theorem 5.3.5, if $g(X(\lambda)) > g(X(\mu))$, then the new interval of uncertainty is $[\lambda, \alpha]$ under the convexity of g . On the other hand, if $g(X(\lambda)) \leq g(X(\mu))$, the

interval of uncertainty is $[0, \mu]$ (see figure 5.8).

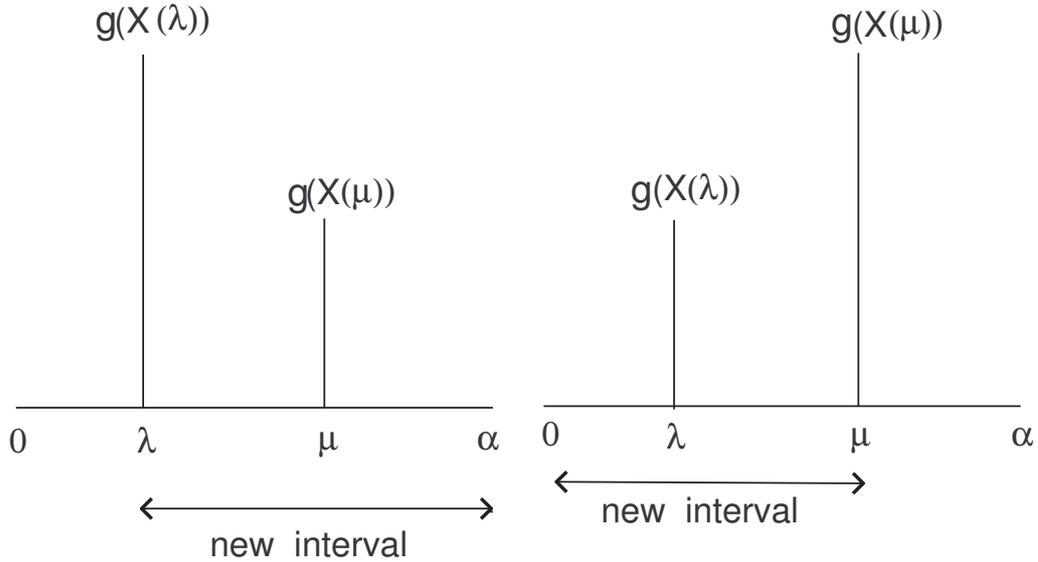


Fig. 5.8: Reducing the interval of uncertainty

The Fibonacci search

Suppose $g(X(t))$ is convex on the great circle arc $\text{arc}(X, Y)$ over a bounded interval $[0, \alpha]$. This procedure makes two functional evaluations at the first iteration and then only one evaluation at each of the subsequent iterations. During this procedure, the interval of uncertainty varies from one iteration to another.

Consider the Fibonacci sequence $\{F_\nu\}$ defined as follows:

$$\begin{aligned} F_\nu &= F_\nu + F_{\nu-1}, \nu = 1, 2, \dots \\ F_0 &= F_1 = 1 \end{aligned} \tag{5.30}$$

At each iteration k , suppose that the interval of uncertainty is $[a_k, b_k]$. Consider the two points λ_k , and μ_k given below, where n is the number of functional evaluations planned.

$$\lambda_k = a_k + \frac{F_{n-k-1}}{F_{n-k+1}}(b_k - a_k), k = 1, 2, \dots, n-1 \quad (5.31)$$

$$\mu_k = a_k + \frac{F_{n-k}}{F_{n-k+1}}(b_k - a_k), k = 1, 2, \dots, n-1 \quad (5.32)$$

By Theorem 5.3.5, the new interval of uncertainty is given by $[\lambda_k, b_k]$ if $g(X(\lambda_k)) > g(X(\mu_k))$ and is given by $[a_k, \mu_k]$ if $g(X(\lambda_k)) \leq g(X(\mu_k))$.

Case 1 : If $g(X(\lambda_k)) > g(X(\mu_k))$

From (5.31) and letting $\nu = n - k$ in (5.30), we get

$$\begin{aligned} b_{k+1} - a_{k+1} &= b_k - \lambda_k \\ &= b_k - a_k - \frac{F_{n-k-1}}{F_{n-k+1}}(b_k - a_k) \\ &= b_k - a_k - \left(1 - \frac{F_{n-k}}{F_{n-k-1}}\right)(b_k - a_k) \\ &= \frac{F_{n-k}}{F_{n-k+1}}(b_k - a_k) \end{aligned} \quad (5.33)$$

Case 2 : If $g(X(\lambda_k)) \leq g(X(\mu_k))$

$$\begin{aligned} b_{k+1} - a_{k+1} &= \mu_k - a_k \\ &= a_k + \frac{F_{n-k}}{F_{n-k+1}}(b_k - a_k) - a_k \\ &= \frac{F_{n-k}}{F_{n-k+1}}(b_k - a_k) \end{aligned} \quad (5.34)$$

Thus in both case, the interval of uncertainty is reduced by the factor $\frac{F_{n-k}}{F_{n-k+1}}$.

Now consider iteration $k + 1$.

Suppose $g(X(\lambda_k)) > g(X(\mu_k))$. Then by Theorem 5.3.5, $a_{k+1} = \lambda_k$, and $b_{k+1} = b_k$.

By replacing k with $k + 1$ in (5.31), we get

$$\begin{aligned}
\lambda_{k+1} &= a_{k+1} + b_{k+1} - a_{k+1}) \frac{F_{n-k-2}}{F_{n-k}} \\
&= \lambda_k + \frac{F_{n-k-2}}{F_{n-k}} (b_k - \lambda_k)
\end{aligned} \tag{5.35}$$

Substituting for λ_k from (5.31), we have

$$\begin{aligned}
\lambda_{k+1} &= a_k + \frac{F_{n-k-1}}{F_{n-k+1}} (b_k - a_k) + \frac{F_{n-k-2}}{F_{n-k}} (b_k - a_k - \frac{F_{n-k-1}}{F_{n-k+1}} (b_k - a_k)) \\
&= a_k + \frac{F_{n-k-1}}{F_{n-k+1}} (b_k - a_k) + \frac{F_{n-k-2}}{F_{n-k}} (1 - \frac{F_{n-k-1}}{F_{n-k+1}}) (b_k - a_k)
\end{aligned} \tag{5.36}$$

Letting $\nu = n - k$ in (5.30), we have $1 - \frac{F_{n-k-1}}{F_{n-k+1}} = \frac{F_{n-k}}{F_{n-k+1}}$.

Then from (5.36), we have

$$\lambda_{k+1} = a_k + \frac{(F_{n-k-1} + F_{n-k-2})}{F_{n-k+1}} (b_k - a_k).$$

Now letting $\nu = n - k - 1$ in (5.30), we have $F_{n-k} = F_{n-k-1} + F_{n-k-2}$. Then from the above equation we have

$$\lambda_{k+1} = a_k + \frac{F_{n-k}}{F_{n-k+1}} (b_k - a_k) = \mu_k$$

Similarly, if $g(X(\lambda_k)) \leq g(X(\mu_k))$, we can show that $\mu_{k+1} = \lambda_k$.

Thus at iteration $k + 1$, either $\lambda_{k+1} = \mu_k$ or $\mu_{k+1} = \lambda_k$. Thus in either case only one observation is needed at iteration $k + 1$.

To summarize, at the first iteration two observations are made and at each subsequent iteration only one observation is necessary.

Thus, at the end of iteration $n - 2$, we have to complete $n - 1$ functional evaluations. Further, for $k = n - 1$, it follows from (5.31) and (5.32), that $\lambda_{n-1} = \mu_{n-1} = \frac{1}{2}(a_{n-1} + b_{n-1})$.

Since either $\lambda_{n-1} = \mu_{n-2}$, or $\mu_{n-1} = \lambda_{n-2}$, theoretically no new observations are to be made at this stage. However, in order to further reduce the interval of uncertainty, the last observation is placed slightly to the right or the left of the midpoint $\lambda_{n-1} = \mu_{n-1}$, so that $\frac{1}{2}(b_{n-1} - a_{n-1})$ is the length of the final interval of uncertainty $[a_n, b_n]$.

The Fibonacci method requires that the total number of observations n chosen beforehand. This is because of the placement of the observations is given by (5.31) and (5.32) and, hence is dependent on n . From (5.33) and (5.34), the length of the interval of uncertainty is reduced at iteration k by the factor $\frac{F_{n-k}}{F_{n-k+1}}$. Hence, at the end of $n-1$ iteration, where n total observations have been made, the length of the interval of uncertainty is reduced from $b_1 - a_1$ to $b_n - a_n = (b_1 - a_1)/F_n$. Therefore n must be chosen such that $(b_1 - a_1)/F_n$ reflects the accuracy required.

Algorithm for the Fibonacci search method

The following is a summary of the Fibonacci search method for minimizing spherical convex function on a great circle arc segment over the interval $[0, \alpha]$.

Algorithm 5.3.1:

Input: X, Y : two points on the surface of the hemisphere with the length of the $\text{arc}(X, Y) = \alpha$ (see (5.27)).

Output: X^* : optimal location.

Z^* : optimal objective value.

Step 0: $l > 0$: allowable final length of uncertainty

$\epsilon > 0$: distinguishability constant

$[a_1, b_1]$: initial interval of uncertainty

n : number of observations to be taken such that $F_n > (b_1 - a_1)/l$

Step 1: Let $\lambda_1 = a_1 + (F_{n-2}/F_n)(b_1 - a_1)$ and

$\mu_1 = a_1 + (F_{n-1}/F_n)(b_1 - a_1)$.

Evaluate $g(X(\lambda_1))$ and $g(X(\mu_1))$, let $k = 1$, and goto Step 2.

Step 2: If $g(X(\lambda_k)) > g(X(\mu_k))$, goto Step 3, and if $g(X(\lambda_k)) \leq g(X(\mu_k))$, goto Step 4.

Step 3: Let $a_{k+1} = \lambda_k$ and $b_{k+1} = b_k$. Furthermore, let $\lambda_{k+1} = \mu_k$, and let $\mu_{k+1} = a_{k+1} + (\frac{F_{n-k-1}}{F_{n-k}})(b_{k+1} - a_{k+1})$. If $k = n - 2$, goto Step 6; Otherwise, evaluate $g(X(\mu_{k+1}))$ and goto Step 5.

Step 4: Let $a_{k+1} = a_k$ and $b_{k+1} = \mu_k$. Furthermore, let $\mu_{k+1} = \lambda_k$, and let $\lambda_{k+1} = a_{k+1} + \frac{F_{n-k-2}}{F_{n-k}}(b_{k+1} - a_{k+1})$. If $k = n - 2$, goto Step 6; Otherwise evaluate $g(X(\mu_{k+1}))$ and goto Step 5.

Step 5: Replace k by $k + 1$ and goto Step 2.

Step 6: Let $\lambda_n = \lambda_{n-1}$, and $\mu_n = \mu_{n-1} + \epsilon$. If $g(X(\lambda_n)) > g(X(\mu_n))$, let $a_n = \lambda_n$ and $b_n = b_{n-1}$. Otherwise, if $g(X(\lambda_n)) \leq g(X(\mu_n))$, let $a_n = a_{n-1}$ and $b_n = \lambda_n$. Stop; the optimal solution X^* lies in the interval $[a_n, b_n]$ with $X^* \in \text{arc}(X(a_n), X(b_n))$.

5.4 Algorithm for BarrierWeberSphereLoc Problem on a Hemisphere

According to the result of Theorem 5.3.2, the **BarrierWeberSphereLoc** problem can be reduced to a set of convex **WeberSphereLoc** subproblems. In this situation, two different cases may occur. An optimal solution $X_{\mathcal{B}}^*$ of **BarrierWeberSphereLoc** may be located

(a). on the grid \mathcal{G}_α ,

or

(b). in the interior of a cell $C \in \mathcal{C}(\mathcal{G}_\alpha)$.

Therefore, a two step algorithm can be suggested to solve the **BarrierWeberSphereLoc** as follows. In the first step, a line search procedure on great circle arcs (see Section 5.3.2) can be applied on each arc segment of the grid \mathcal{G}_α . In the second step, a local minimum can be found (see Theorem 5.3.3) in the interior of a cell in $\mathcal{F} \setminus \mathcal{G}_\alpha$ by solving convex subproblems (5.22) for all feasible reformulations $f_{\mathcal{B}}(Y) = F_X(Y)$ of the objective function. For each solution Y^* , $f_{\mathcal{B}}(Y^*) = F_X(Y^*)$ has to be verified to test the feasibility of Y^* .

Algorithm 5.4.1:

Input: $\mathcal{E}_x = \{Ex_i : i = 1, 2, \dots, m\}$, the set of existing facilities.

\mathcal{B} : Convex spherical polygon with sets of extreme points $Ext(\mathcal{B})$ and facets $Facet(\mathcal{B})$.

Output: $Opt_{\mathcal{B}}^*$: set of all optimal locations.

$Z_{\mathcal{B}}^*$: optimal objective value.

Step 1: Construct the grid \mathcal{G}_{α} .

Step 2: Find the minimum of the problem (5.2) on grid \mathcal{G}_{α} .

Step 3: For all feasible reformulations of the objective function, i.e., for all feasible assignments of intermediate points to the existing facilities,

(a) Find an optimal solution Y^* of the corresponding unrestricted problem

$$\min F_X(Y), \quad Y \in \mathcal{S}_0^H.$$

(b) If $f_{\mathcal{B}}(Y^*) = F_X(Y^*)$, the solution Y^* is a candidate for an optimal solution.

Step 4: Determine the set of global minima from the candidate set found in Steps 2 and 3.

The time complexity of Steps 1 and Step 2 of Algorithm 5.4.1 depends on the size of the grids \mathcal{G}_{α} and thus on the number of existing facilities and the number of extreme points of the barrier regions. Therefore, the number of intersection points in \mathcal{G}_{α} is bounded by $O((|\mathcal{E}_x| + |Ext(\mathcal{B})|)^2 \cdot |Ext(\mathcal{B})|^2)$, and the number of cells in \mathcal{G}_{α} is bounded by $O((|\mathcal{E}_x| + |Ext(\mathcal{B})|)^2 \cdot |Ext(\mathcal{B})|^2)$.

The overall time complexity of Algorithm 5.4.1 is in general dominated by Step 3. If no additional information is available to reduce the number of possible assignments of existing facilities to intermediate points, the number of subproblems is exponential in the number of existing facilities and in the number of extreme points of the barrier regions. Thus, the Algorithm 5.4.1 is computationally expensive when no additional information is available on the structure of the problem and hence a heuristic strategy can alternatively be applied. Instead of evaluating all the theoretically possible assignments of existing facilities to intermediate points, a sample set S of points can be constructed in $I^{convex} \cap \mathcal{F}$. For an example

this sample set S can be constructed by choosing the grid points of an equidistant grid in I^{convex} or by choosing specific points on the visibility grid \mathcal{G}_α . All the points in this sample set can be used as starting points to determine F_X for the unconstrained location problem (5.22). As in Algorithm 5.4.1, the corresponding optimal solution Y^* of F_X can be put in the candidate set if Y^* is feasible, i.e., if $f_{\mathcal{B}}(Y^*) = F_X(Y^*)$.

Algorithm 5.4.2:

Input: $\mathcal{E}_x = \{Ex_i : i = 1, 2, \dots, m\}$, the set of existing facilities.

w_i : Associated weights.

\mathcal{B} : Convex spherical polyhedron with sets of extreme point $Ext(\mathcal{B})$ and facets $Facet(\mathcal{B})$.

Output: $Opt_{\mathcal{B}}^*$: set of all optimal locations.

$Z_{\mathcal{B}}^*$: optimal objective value.

Step 1: Construct the grid \mathcal{G}_α .

Step 2: Find the minimum of the problem (5.2) on grid \mathcal{G}_α .

Step 3: Define a sample set S of grid points in I^{convex} .

Step 4: For each grid point $X \in S$

(a) Find an optimal solution X^* of the corresponding unrestricted problem

$$\min f_X(Y), \quad Y \in \mathcal{S}_0^H.$$

(b) If $f_{\mathcal{B}}(X^*) = f_X(X^*)$, the solution X^* is a candidate for an optimal solution.

Step 5: Determine the set of global minima from the candidate set found in Steps 2 and 4.

5.5 BarrierWeberSphereLoc Problem on the Surface of the Unit Sphere

According to the Corollary 5.2.1 in Section 5.2, **BarrierWeberSphereLoc** problem can be reduced to a finite set of corresponding unconstrained (or **WeberSphereLoc**) problems with the shortest arc distance α as the measure of distance.

As the objective function $f_{\mathcal{B}}(X)$ of the **WeberSphereLoc** problem is in general non-convex within the cells, the resulting corresponding subproblems are also in general non-convex. Therefore, the difficulty of the problem is not reduced as in the case where the existing facilities lie on a hemisphere.

Theorem 5.5.1: Let $C \in \mathcal{C}(\mathcal{G}_\alpha)$ be a cell and let $X \in C$. Let $X_{\mathcal{B}}^*$ represents the global optimal solution to the non convex problem

$$\begin{aligned} & \text{minimize} \\ & F_X(Y) \\ & \text{s.t. } Y \in \mathcal{S}_0 \end{aligned} \tag{5.37}$$

where $F_X(Y)$ is defined according to (5.17) and (5.18) and the intermediate points $I_i, i = 1, 2, \dots, m$ are chosen such that they are α -visible from X .

Then $F_X(X_{\mathcal{B}}^*)$ is a lower bound to the optimal objective value of

$$\begin{aligned} & \text{minimize} \\ & F_X(Y) \\ & \text{s.t. } Y \in C. \end{aligned} \tag{5.38}$$

That is

$$F_X(X_{\mathcal{B}}^*) \leq F_X(Y) \quad \forall Y \in C.$$

Further, if $X_{\mathcal{B}}^* \in C$, or equivalently, if $X_{\mathcal{B}}^*$ is a feasible solution to the problem (5.38), then $X_{\mathcal{B}}^*$ is the best optimal solution (5.38). ■

Theorem 5.5.2: Let $X_{\mathcal{B}}^*$ represent the global solution to the problem (5.37). If $X_{\mathcal{B}}^* \in \text{int}(C)$, then $X_{\mathcal{B}}^*$ is at least a local optimal solution to **BarrierWeberSphereLoc** problem

Proof First, given that $X_{\mathcal{B}}^* \in \text{int}(C)$, we know from Theorem 5.5.1 that

$$F_X(X_{\mathcal{B}}^*) \leq F_X(Y) \quad \text{for each } Y \in C.$$

That is, $X_{\mathcal{B}}^*$ is the global optimal solution to (5.38). Therefore, there exists an ε -neighborhood of $X_{\mathcal{B}}^*$, $N_\varepsilon(X_{\mathcal{B}}^*) \subset \text{int}(C)$, such that

$$f_X(X_{\mathcal{B}}^*) \leq f_X(Y) \quad \text{for each } Y \in N_\varepsilon(X_{\mathcal{B}}^*).$$

But since

$$N_\varepsilon(X_{\mathcal{B}}^*) \subset \text{int}(C) \subset \mathcal{G}_\alpha,$$

it follows that $F_X(X_{\mathcal{B}}^*) \leq F_X(Y)$ for each

$$Y \in N_\varepsilon(X_{\mathcal{B}}^*) = \mathcal{G}_\alpha \cap N_\varepsilon(X_{\mathcal{B}}^*) \quad (5.39)$$

This complete the proof since (5.39) defines a local optimal solution to **BarrierWeberSphereLoc** problem. ■

Heuristic Algorithm for BarrierWeberSphereLoc problem

From the visibility graph $G(V, E)$ (see section 5.2) on S_0 , we can easily define the shortest path from each existing facility location $Ex_i; i = 1, \dots, m$ to X in a cell, C . From these paths, we can then determine the visible nodes I_i in the shortest-permitted $Ex_i - X$ -path for $i = 1, \dots, m$. Now suppose that minimizing (5.37) (i.e., solving **WeberSphereLoc** problem with existing facilities $I_i, i = 1, \dots, m$ and weights, $w_i, i = 1, \dots, m$), results in the optimal location $X_{\mathcal{B}}^*$. From Theorem 5.5.1, we know that if $X_{\mathcal{B}}^* \in C$, then $X_{\mathcal{B}}^*$ is a global facility location in C . And from Theorem 5.5.2, if $X_{\mathcal{B}}^* \in \text{int}(C)$, then $X_{\mathcal{B}}^*$ must also be at least a local optimal solution to the **BarrierWeberSphereLoc** problem.

We can verify that $X_{\mathcal{B}}^* \in \text{int}(C)$ by showing that the distance functions, or equivalently the visible nodes, associated with X and $X_{\mathcal{B}}^*$ are not only identical, but unique. If the distance functions are not unique (i.e., there are at least two paths to $X_{\mathcal{B}}^*$ from some existing facility location Ex_i , such that the lengths of the paths are equivalent), then $X_{\mathcal{B}}^*$ is on the boundary of C (∂C). If $X_{\mathcal{B}}^* \in \partial C$, then an ε -neighborhood may also contain points which are elements of adjacent regions. Therefore, in this case, to be assured of a local optimal

solution to the **BarrierWeberSphereLoc**, we must also verify that $X_{\mathcal{B}}^*$ is a local optimal solution in each adjacent region for which $X_{\mathcal{B}}^* \in C$. Based on this, we propose the following heuristic algorithm for the barrier weber problem on the spherical surface.

Algorithm 5.5.1:

Input: $\mathcal{E}_x = \{Ex_i : i = 1, 2, \dots, m\}$, the set of existing facilities.

w_i : Associated weights

\mathcal{B} : Convex spherical polyhedron with sets of extreme point $Ext(\mathcal{B})$ and facets $Facet(\mathcal{B})$.

Output: $Opt_{\mathcal{B}}^*$: set of all optimal locations.

$Z_{\mathcal{B}}^*$: optimal objective value.

Step 1: Construct the grid \mathcal{G}_α . Choose a cell C and initial point $X^0 \in C$

Step 2: Find the minimum $X_{\mathcal{B}}^*$ of the problem (5.37).

Step 3: If :

- (a) $X_{\mathcal{B}}^* \notin C$, then choose an another cell. Go to Step2.
- (b) $X_{\mathcal{B}}^* \in \partial C$, then for each adjacent region for which $X_{\mathcal{B}}^* \in C$, reapply the Algorithm 5.5.1 using $X_{\mathcal{B}}^*$ as the initial point.
- (c) $X_{\mathcal{B}}^* \in int(C)$, then STOP: $X_{\mathcal{B}}^*$ is a local optimal facility location to the **BarrierWeberSphereLoc**.

6. NUMERICAL RESULTS

We develop the code in Visual C++ 6.0 for the Algorithm 4.2.1 which is presented in Section 4.2. The code is implemented on a computer AMD Athlon(tm)XP 1500+ at 1.34 GHZ.

First, consider the following example with fifteen existing facilities (cities) and four extreme pointed spherical polygon as the existing restricted region in the Northern hemisphere. Tables 6.1 and 6.2 below list the latitude and longitude as well as the corresponding Cartesian coordinates of these fifteen cities and of the extreme points of the restricted spherical polygons respectively.

The algorithm generates the optimal location for **RestrictedCenterSphereLoc** problem in the Northern hemisphere with the cartesian coordinates $(0.6019, -0.5504, 0.5784)$ and with the corresponding latitude and longitude $(35.34N, 42.43W)$. The corresponding optimal objective value is 0.9064. The intersection point of the spherical bisector of the 8th and the 12th existing facilities with facet generated by the 1st and the 2nd extreme points of the given spherical polygon is the required facility point. The CPU time of the algorithm for this example is 5.0 seconds.

Note that the unrestricted **CenterSphereLoc** problem is solved by applying the polynomial time algorithm, Algorithm 3.4.2.

Consider now 10 sets containing 10, 20, 30, 40, 50, 60, 70, 80, 90 and 100 demand points distributed randomly over the Northern hemisphere and 3 sets containing 3, 4, and 5 extreme pointed spherical polygons for each data set such that the optimal location for the **CenterSphereLoc** problem in the hemisphere is contained within the spherical polygons. Each of the above sets is randomly generated ten times. Table 6.3 shows the average computation time (in seconds) of the Algorithm 4.2.1. Figure 6.1 shows distribution of

	Latitude, ϕ	Longitude, θ	x	y	z
1	51.5N	0.4E	0.6225	0.0043	0.7826
2	48.9N	2.3E	0.6568	0.0264	0.7536
3	47.5N	8.5E	0.6694	0.1000	0.7361
4	41.9N	12.5E	0.7267	0.1611	0.6678
5	55.7N	12.6E	0.5500	0.1229	0.8261
6	52.5N	13.4E	0.5922	0.1411	0.7934
7	59.3N	18.9E	0.4830	0.1654	0.8600
8	38.0N	23.7E	0.7216	0.3167	0.6157
9	39.9N	32.8E	0.6449	0.4156	0.6415
10	32.1N	34.8E	0.6956	0.4835	0.5314
11	55.7N	37.7E	0.4459	0.3446	0.8261
12	35.4N	51.4E	0.5058	0.6370	0.5793
13	18.9N	72.8E	0.2798	0.9038	0.3239
14	14.6N	121.0E	-0.4984	0.8295	0.2521
15	35.6N	139.7E	-0.6201	0.5260	0.5820

Tab. 6.1: Latitudes, Longitudes and corresponding Cartesian coordinates of 15 cities

	Latitude, ϕ	Longitude, θ	x	y	z
1	41.96N	46.73W	0.5096	-0.5414	0.6686
2	28.47N	84.80E	0.0796	0.8754	0.4767
3	35.54N	104.33W	-0.7883	-0.2014	0.5813
4	18.72N	26.62W	0.8466	-0.4243	0.3209

Tab. 6.2: Latitudes, Longitudes and corresponding Cartesian coordinates of the extreme points of the restricted spherical polygon

the CPU time of the algorithm according to the increasing number of demand points and the shapes of the restricted regions.

No. of Demand points	Runtime(in seconds) with the polygon having		
	3 extreme points	4 extreme points	5 extreme points
10	0.03	0.03	0.04
20	0.07	0.35	1.04
30	1.69	2.96	4.26
40	4.27	6.41	8.48
50	7.70	10.84	13.94
60	11.81	16.21	20.85
70	16.57	22.31	28.28
80	21.67	29.66	37.67
90	28.06	38.09	47.92
100	34.87	46.87	61.01

Tab. 6.3: Average CPU time (in seconds) for 10 different set of demand points with 3 different shapes of restricted spherical polygons

Further, some test runs for samples of 200, 300,400, 500 and 1000 of demand points with same shape of of restricted regions were tested and the computational time of these samples are included in the Table 6.4. Visual Version C++ 6.0 is used on the same computer for computation.

Now, we represent some results for the **BarrierWeberSphereLoc** problem using the developed algorithms, Algorithm 5.4.1 and Algorithm 5.4.2. Consider again the 15 existing cities given in Table 6.1 and a single barrier region with 4 extreme points which is given in Table 6.2 in the northern hemisphere. We developed the C++ codes for the Algorithm 5.4.1 and (0.5662,0.6490,0.9088) was resulted as the optimal location for the hemispherical Weber location problem with the optimal objective value 2.9542. The computational time in this example is 56.36 seconds.

In this solution approach, as we are considering all possible feasible assignments of existing facilities to intermediate points, this is computationally expensive. Therefore, Instead of evaluating all the theoretically possible assignments of existing facilities to intermediate

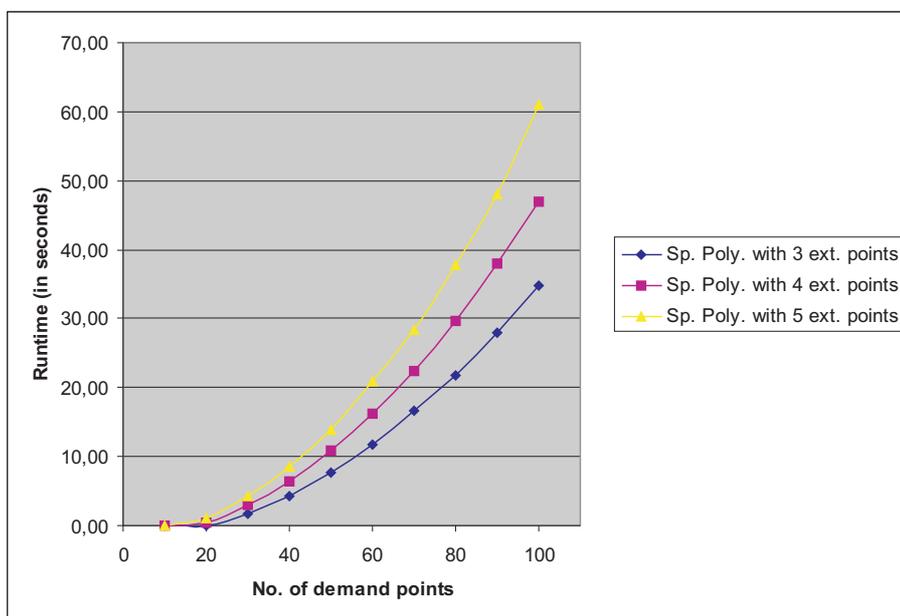


Fig. 6.1: Distribution of CPU time of the Algorithm 4.2.1

No. of Demand points	Runtime(in seconds) with the polygon having		
	3 extreme points	4 extreme points	5 extreme points
200	145.129	193.919	242.589
300	327.110	429.598	537.623
400	579.453	772.291	952.529
500	914.265	956.796	1509.06
1000	1670.27	1967.98	2296.730

Tab. 6.4: CPU time (in seconds) for large sets of demand points with 3 different shapes of restricted spherical polygons. * : these samples were not tested.

points, a suitable sample set S of points can be specified in $I^{convex} \cap \mathcal{F}$ to apply the developed Algorithm 5.4.2.

Now we consider the same hemispherical **BarrierWeberSphereLoc** problem with 15 existing cities and the single barrier for applying Algorithm 5.4.2 on the selected sample set S . Consider all the spherical triangles which are generated by the existing facilities and the extreme points of the barrier in which the extreme points of each spherical triangle that are α -visible from each other. Then a sample set S for this problem can be formed by randomly generated points from these spherical triangles. The Algorithm 5.4.2 generated the same location (0.5662, 0.6490, 0.9088) as the new facility for the **BarrierWeberSphereLoc** problem on the Northern hemisphere with same objective value in 14.8 seconds.

To see the distribution of run time of the Algorithm 5.4.2, 10 randomly generated set of demand points on the Northern hemisphere with 5, 10, 15, 20, 25, 30, 35, 40, 45 and 50 points with a spherical triangle as the polygonal barrier. The Algorithm was tested 5 times on each sample set and the resulted run time of the Algorithm in each case is given by the following Table, 6.5.

	Number of demand points	Run time (in seconds)
1	5	0.502
2	10	372.813
3	15	782.671
4	20	1247.643
5	25	1941.756
6	30	2875.903
7	35	4143.572
8	40	6241.743
9	45	8732.904
10	50	11995.761

Tab. 6.5: Average CPU time (in seconds) of the Algorithm 5.4.2

Figure 6.2 shows the distribution of run time of the Algorithm 5.4.2 in increasing number of demand points.

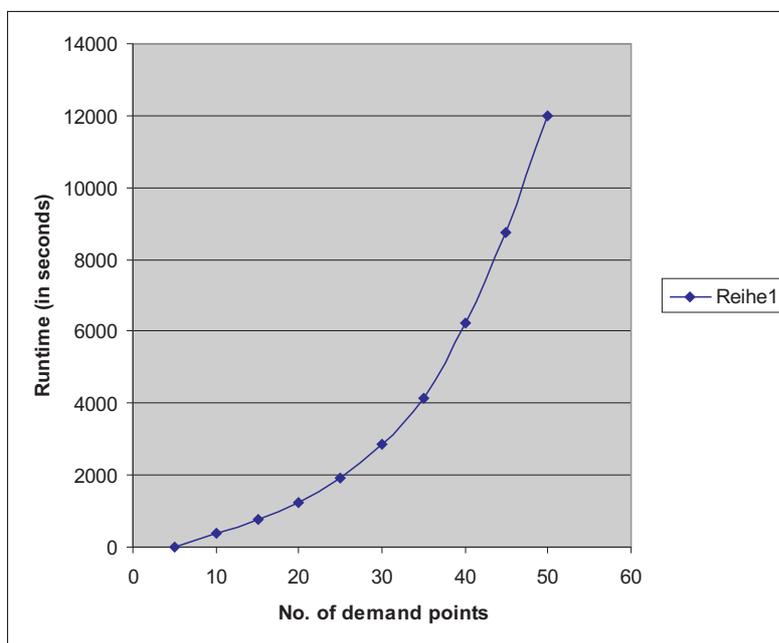


Fig. 6.2: Distribution of CPU time of the Algorithm 5.5.1

Further, some test runs on the sample sets of 100, 200 and 500 demand points were tested with Algorithm 5.5.1, and 38465.091, 107231.742 and 344362.056 seconds respectively were resulted as the CPU time.

Now, we will present some computational result for the **WeberSphereLoc** problem using the developed Algorithm 5.5.1 in section 5.5. The test sample sets with 10,20,30,40, 50, 60, 70, 80, 90 and 100 demand points on the sphere were generated randomly. A spherical triangle is exposed in to the sets of demand points in each case as the polygonal barrier. The Algorithm is tested 5 times on each case and the resulted run time is shown in the following Table 6.6. This algorithm was also tested for large samples of 200, 300, 400, 500 and 1000 demand points. The required run time for these samples are shown in the Table 6.7

Figure 6.3 shows distribution of the CPU time of the algorithm according to the increasing number of demand points.

	Number of demand points	Run time (in seconds)
1	10	13.299
2	20	20.671
3	30	28.281
4	40	36.874
5	50	47.939
6	60	62.017
7	70	79.437
8	80	103.771
9	90	137.003
10	100	184.423

Tab. 6.6: Average CPU time (in seconds) of the Algorithm 5.5.1

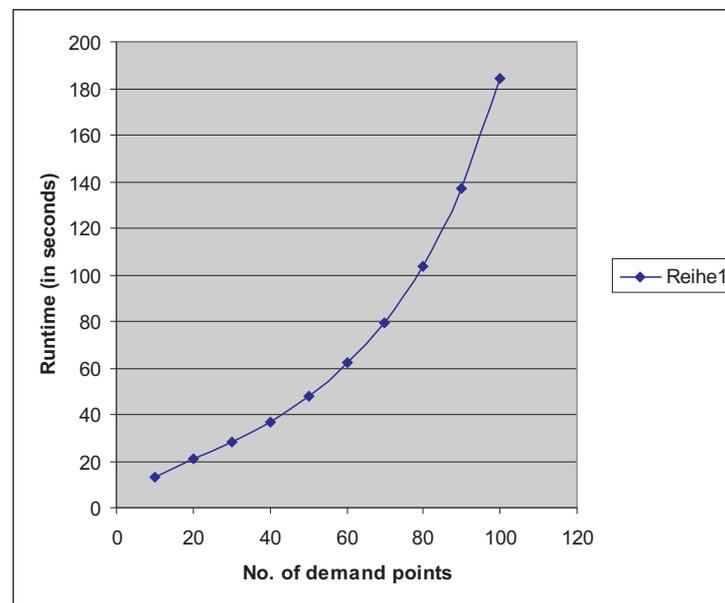


Fig. 6.3: Distribution of CPU time of the Algorithm 5.5.1

	Number of demand points	Run time (in seconds)
1	200	237.589
2	300	541.423
3	400	988.472
4	500	1668.106
5	1000	3496.450

Tab. 6.7: CPU time (in seconds) of the Algorithm 5.5.1

7. CONCLUSIONS AND FUTURE RESEARCH

*In this thesis, different solution approaches for the spherical center location (**CenterSphereLoc**) problem and the spherical weber location (**WeberSphereLoc**) problem have been investigated and unified presentation has been provided. Furthermore, as a new idea, the concepts "Restricted and Barrier regions" have been exposed in to spherical location problems and some solution strategies of these restricted and barrier spherical location (**RestrictedSphereLoc** and **BarrierSphereLoc** respectively) problems have been presented.*

*Basically, throughout this work the great "circle arc distance" (shortest arc length) on the surface of the unit sphere has been used in developing the mathematical models for the **RestrictedSphereLoc** and **BarrierSphereLoc** problems.*

*As a result that the great circle arc distance is non convex function on the surface of the sphere, some restrictions have to be made on the feasible region in order to discuss the solution criteria for the restricted center sphere location (**RestrictedCenterSphereLoc** problem. Therefore, In Chapter 4, some basic results for the hemispherical **RestrictedCenterSphereLoc** problem have been developed using the concept, "level sets" and "level curves" and a resulted "polynomial time " algorithm has been developed. In this algorithm, all the demand weights, $w_i, i = 1, 2, \dots, m$ have been assigned to be equal to one. When the demand weights are $w_i > 0$ but $w_i \neq 1$, a solution approach has been discuss in section 4.3. In this situation the weighted bisectors on the surface of the unit sphere have to be used to obtain the optimal location for the new facility.*

*A solution strategy for the **BarrierSphereLoc** problems has been presented using partitioning the feasible region into some subsets with the help of visibility phenomena in Chapter 5. Here, the concept "barrier distance" has been used in developing algorithms*

in both "spherical" and "hemispherical" cases. Also, "visibility graph" and a "line search procedure" on the spherical surface have been discussed in this Chapter.

In Chapter 6, some numerical results for the developed algorithms for both **(Restricted-CenterSphereLoc** and **BarrierSphereLoc** problems have been included. According to these results, the algorithm which has been developed for the hemispherical **Restricted-CenterSphereLoc** problem gives the solution for the new location in polynomial time.

One can observe that the running time of the Algorithms 5.4.1, 5.4.2 and 5.5.1 is highly dependent on the number of existing facilities and on the no of extreme points of the barrier regions.

The possible future research work is to propose different distance norms on the surface of the sphere to have better solution with better CPU time. Further, if we have different distance norm, like l_∞ distance norm in Euclidean space, one can partition the surface area of the sphere into two regions and then can apply both arc distance and the newly defined distance in each region to have another algorithmic approach for the **BarrierSphereLoc** problem.

Further, in radiation therapy, when the target volume has been irradiated in three dimensional way, the problem is to find better radiation therapy planing can be considered as a restricted or barrier spherical location problem on the spherical surface. Finally, finding weighted bisectors on the surface of the unit is also still a open problem for the future work.

8. APPENDIX

Proof for Theorem 2.1.3 *For the convenience, we will assume that the center of the spherical circle is $(0,0,1)$. Therefore, all of the existing facilities are above the XY -plane. Let $X^* = X^*(x^*, y^*, z^*)$ be a global minimizer of $f(X)$.*

Claim 1: $z^* \geq 0$.

Suppose $z^ < 0$ and $X' = X'(x^*, y^*, -z^*)$*

All the existing facilities are above XY - plane $\Rightarrow \text{arc}(X', Ex_i) < \text{arc}(X^, Ex_i) \quad \forall i = 1, 2, \dots, m$.*

$\Rightarrow f(X') < f(X^)$.*

This contradicts the global optimality assumption of X^ .*

Now, assume that X^ is on or above the XY -plane.*

If X^ is in the spherical convex hull of the existing facilities, we are done. Therefore, suppose that X^* is not in the spherical convex hull of the existing facilities.*

Claim 2: *There must be at least one existing facility, say Ex_j such that $\text{arc}(X^*, Ex_j) \leq \pi/4$. This is true because otherwise $f((0,0,1)) < f(X^*)$ and this contradicts the global optimality assumption of X^* .*

Let P be the orthogonal projection of X^ onto the convex cone generated by the existing facilities.*

$\stackrel{\text{Claim 1}}{\Rightarrow} P \neq O = O(0,0,0)$.

We will show that $f(P) < F(X^)$:*

Let \mathcal{P} be the plane passing through P and orthogonal to line segment OP . Let Ex'_i be the intersection of ray OEx_i with \mathcal{P} for all $i = 1, 2, \dots, m$.

We want to show that for any i ,

$$\angle X^* O E x'_i > \angle P O E x'_i. \quad (8.1)$$

Ex_i is above the XY -plane $\Rightarrow Ex'_i$ must also be above the XY -plane.

X^ is on or above the XY -plane $\Rightarrow P$ must also be on or above the XY -plane.*

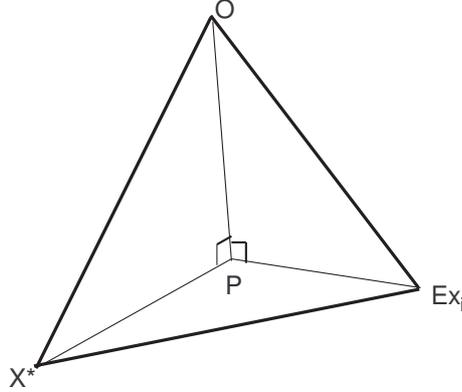


Fig. 8.1: The projection P of X^* has a better function value than X^*

Therefore, $\angle X^*OEx'_i, \angle POEx'_i \in [0, \pi)$. Therefore, we only need to prove

$$\cos(\angle POEx'_i) > \cos(\angle X^*OEx'_i). \quad (8.2)$$

Since P is the projection of X^* onto a convex cone and that Ex'_i is a point in that cone, we know that from convex analysis, $\angle X^*PEx'_i \geq 90^\circ$. Therefore, $(X^*Ex'_i)^2 \geq (X^*P)^2 + (Ex'_iP)^2$.

Since (Figure 8.1) $\cos(\angle POEx'_i) = OP/OEx'_i$, $\cos(\angle X^*OEx'_i) = ((OX^*)^2 + (OEx'_i)^2 - (X^*Ex'_i)^2)/(2 \cdot OX^*)$, we only need to prove

$$2 \cdot OX^* \cdot OP \geq (OX^*)^2 + (OEx'_i)^2 - (X^*Ex'_i)^2. \quad (8.3)$$

Again from Figure 8.1 , we have $(OX^*)^2 = (OP)^2 + (X^*P)^2$ and $(OEx'_i)^2 = (OP)^2 + (Ex'_iP)^2$. Therefore, inequality (8.3) is true and the Theorem is proved. ■

Proof for Theorem 2.2.1 Consider the objective function $f(X) = \sum_{i=1}^n w_i \alpha_i$. It can be shown that for movement from point X_k :

$$\begin{aligned}
df(X) &= w_k[(d\phi)^2 + \cos^2 \phi_k (d\theta)^2]^{1/2} \\
&- d\phi \sum_{i \neq k} w_i (-\sin \phi_k \cos \phi_i \cos(\theta_k - \theta_i) \\
&+ \cos \phi_k \sin \phi) / \sin \alpha_{ki} \\
&- d\theta \sum_{i \neq k} w_i (\cos \phi_k \cos \phi_i \sin(\theta_i - \theta_k)) / \sin \alpha_{ki}.
\end{aligned}$$

For a local minimum, $df(p) > 0$, and hence, we must show

$$w_k((d\phi)^2 + \cos^2 \phi_k (d\theta)^2)^{1/2} - A_k d\phi - B_k \cos \phi_k d\phi_k > 0.$$

Letting $L = d\theta \cos \phi_k / d\phi$, we have

$$|d\phi| w_k (1 + L)^{1/2} > d\phi (A_k + LB_k)$$

and so :

$$w_k > d\phi (A_k + LB_k) (1 + L^2)^{-1/2} / |d\phi|.$$

Note that $d\phi / |d\phi|$ is ± 1 . It can be shown that :

$$\begin{aligned}
-(A_k^2 + B_k^2)^{1/2} &\leq (A_k + LB_k) / (1 + L^2)^{1/2} \\
&\leq (A_k^2 + B_k^2)^{1/2}
\end{aligned}$$

and hence, the condition

$$w_k \geq (A_k^2 + B_k^2)^{1/2}$$

is necessary and sufficient for $df(p) > 0$ for every L .

■

Proof for Lemma 3.2.1 Consider the Figure 8.2. O' denote the center of the circle $\mathcal{C}(X_1, X_2, X_3)$. Then \bar{X}_1, \bar{X}_2 and \bar{X}_3 are the points on the circumference of the circle that are diametrically opposite of X_1, X_2 and X_3 respectively. Since $\triangle X_1 X_2 X_3$ is acute, points X_2 and X_3 cannot lie on the same side of the line joining X_1 and \bar{X}_1 . The same is true for points X_3 and X_1 and the line joining X_2 and \bar{X}_2 , and points X_1 and X_2 and the line joining X_3 and \bar{X}_3 . Let X be any point of the circumference of $\mathcal{C}(X_1, X_2, X_3)$, then obviously

$$\text{minimum}\{\angle X_1 O' X, \angle X_2 O' X, \angle X_3 O' X\} < \pi/2.$$

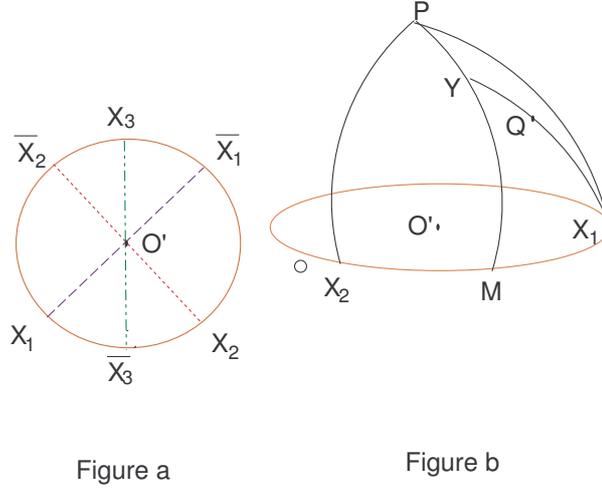


Figure a

Figure b

Fig. 8.2:

Extend the arc from P passing through Q to meet the circumference of $\mathcal{C}(X_1, X_2, X_3)$ at point X . Without loss of generality, assume that

$$\angle X_1 O' X < \pi/2, \text{ i.e., } \widehat{Q P X_1} < \pi/2 \text{ and } \angle X_1 O' X \leq \angle X_2 O' X.$$

If Q lies on $\text{arc}(P X_1)$, then the proof is complete. When Q does not lie on $\text{arc}(P X_1)$, let M be the midpoint of the shorter arc segment between points X_1 and X_2 on the circumference of $\mathcal{C}(X_1, X_2, X_3)$ (see Figure 8.2 b). Construct two great circle arcs, one joining points P and M and the other joining points X_1 and Q . Extend $\text{arc}(X_1 Q)$ to meet $\text{arc}(P M)$ at point Y .

By construction $\text{arc}(X_2 M)$ and $\text{arc}(X_1 M)$ are the same. Since P is the nearer pole of $\mathcal{C}(X_1, X_2, X_3)$, $\text{arc}(X_1 P)$ and $\text{arc}(X_2 P)$ are also the same. Thus, spherical triangles $X_1 M P$ and $X_2 M P$ are congruent and $\widehat{X_1 M P} = \pi/2$.

Then from Article 42 in [30],

$$\cos(\text{arc}(P X_1)) = \cos(\text{arc}(P M)) \cos(\text{arc}(X_1 M)) \quad (8.4)$$

and

$$\cos(\text{arc}(Y X_1)) = \cos(\text{arc}(Y M)) \cos(\text{arc}(X_1 M)). \quad (8.5)$$

Now, using the result $\text{arc}(PM) > \text{arc}(TM)$, and (8) and (8),

$$\text{arc}(PX_1) > \text{arc}(YX_1) \geq \text{arc}(QX_1).$$

Since $\text{arc}(PX_1)$ is the spherical radius of $\mathcal{C}(X_1, X_2, X_3)$, we receive

$$\text{arc}(XX_1) > \text{minimum}\{\text{arc}(QX_1), \text{arc}(QX_2), \text{arc}(QX_3)\}.$$

■

Proof for Lemma 3.2.2 Let P be the nearer pole of $\mathcal{C}(X_1, X_2, X_3)$. Let Q be the diametrically opposite point to Q_1 . Obviously, Q is on $\Gamma\mathcal{C}(X_1, X_2, X_3)$ and $P \neq Q$. Since Q_1 is sufficiently close to P' and P is in the spherical triangle $X_1X_2X_3$, Q must be in the spherical triangle as well. Assume that

$$\text{arc}(QX_1) = \text{minimum}\{\text{arc}(QX_1), \text{arc}(QX_2), \text{arc}(QX_3)\}.$$

From Lemma 3.2.1, it we have $\text{arc}(QX_1) < \text{arc}(PX_1)$. Construct two great circle arcs, one joining X_1 to P' and the other joining X_1 to Q_1 . Since P and Q are diametrically opposite of P' and Q_1 respectively, we have

$$\text{arc}(X_1P) + \text{arc}(X_1P') = \pi = \text{arc}(X_1Q) + \text{arc}(X_1Q_1).$$

Now $\text{arc}(QP_1) < \text{arc}(PX_1) \implies \text{arc}(X_1P') < \text{arc}(X_1Q_1)$.

$$\implies \text{arc}(X_1P') < \text{maximum}\{\text{arc}(X_1Q_1), \text{arc}(X_2Q_1), \text{arc}(X_3Q_1)\}.$$

■

Proof for Lemma 3.2.3

Refer figure 8.3. P and P' are nearer and distant poles of the small circle $\mathcal{C}(X_1, X_2, X_3)$. M denotes the mid point of $\text{arc}(X_1X_2)$. Take a point Q , in an arbitrary small neighborhood of P' on the great circle $\text{arc}(PMP')$. Construct $\text{arc}(QX_3)$, $\text{arc}(QX_2)$, $\text{arc}(X_3P')$, and $\text{arc}(X_2P')$. Now draw the great circle $\text{arc}(PX_1P')$. Since P' is the distant pole of small circle $\mathcal{C}(X_1, X_2, X_3)$, we have

$$\text{arc}(X_1P') = \text{arc}(X_2P') = \text{arc}(X_3P') \quad (8.6)$$

Hence spherical triangles X_1MP' and X_2MP' are congruent and

$$X_2\hat{M}P' = X_1\hat{M}P' = \pi/2. \quad (8.7)$$

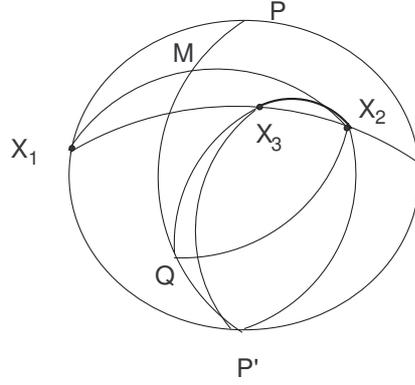


Fig. 8.3:

$\angle X_3 > \pi/2$ implies that X_3 lies on the (shorter) arc, $\text{arc}(X_1X_2)$ of small circle $\mathcal{C}(X_1, X_2, X_3)$. Without loss of generality, assume that X_2 and X_3 lie on the same hemisphere defined by the great circle passing through P, M , and P' . In the spherical triangle X_2X_3P' , $\text{arc}(X_2P') = \text{arc}(X_3P')$. Therefore, Property 1.2.1(a) of spherical triangles (see section 1.2) \Rightarrow

$$X_3\hat{X}_2P' = X_2\hat{X}_3P' \quad (8.8)$$

Consider the property that two great circles intersect at points which are diametrically opposite and the assumptions that Q, X_3, X_2 , and P' lie on the same hemisphere. Then

$$X_3\hat{X}_2P' = X_3\hat{X}_2Q + Q\hat{X}_2P' \Rightarrow X_3\hat{X}_2P' > X_3\hat{X}_2Q \quad \text{and} \quad (8.9)$$

$$X_2\hat{X}_3Q = X_2\hat{X}_3P' + Q\hat{X}_3P' \Rightarrow X_2\hat{X}_3Q > X_2\hat{X}_3P' \quad (8.10)$$

From Property 1.2.1(b), and results (8.8) through (8.8) \implies

$$\text{arc}(X_2Q) > \text{arc}(X_3Q) \quad (8.11)$$

Using (8.7), from spherical triangles X_2MQ and X_2MP' we have

$$\cos(\text{arc}(X_2Q)) = \cos(\text{arc}(X_2M)) \cos(\text{arc}(MQ)) \quad \text{and} \quad (8.12)$$

$$\cos(\text{arc}(X_2P')) = \cos(\text{arc}(X_2M)) \cos(\text{arc}(MP')) \quad (8.13)$$

Since $\text{arc}(MQ) < \text{arc}(MP')$, (8.12) and (8.13) \Rightarrow

$$\text{arc}(X_2Q) < \text{arc}(X_2P') \quad (8.14)$$

Since spherical triangles X_1MQ and X_2MQ are congruent, we have

$$\text{arc}(X_1Q) = \text{arc}(X_2Q) \quad (8.15)$$

Combining (8.6), (8.11), (8.14) and (8.15), we have

$$\text{maximum}\{\text{arc}(X_1Q), \text{arc}(X_2Q), \text{arc}(X_3Q)\} < \text{arc}(X_1P') = \text{arc}(X_2P') = \text{arc}(X_3P').$$

■

Proof for Theorem 3.2.3 P and P' in Figure 8.4 represent the nearer and distant pole of $\mathcal{C}(X_1, X_2)$. Consider the great circle $PX_1P'X_2$. Now construct the great circle arc joining P and P' through the mid point, M , of the smaller great circle arc, $\text{arc}(X_1X_2)$. For any demand point $Ex_i (\neq X_1 \text{ or } X_2)$,

$$\text{arc}(Ex_iP') < \text{arc}(X_2P') = \text{arc}(X_1P').$$

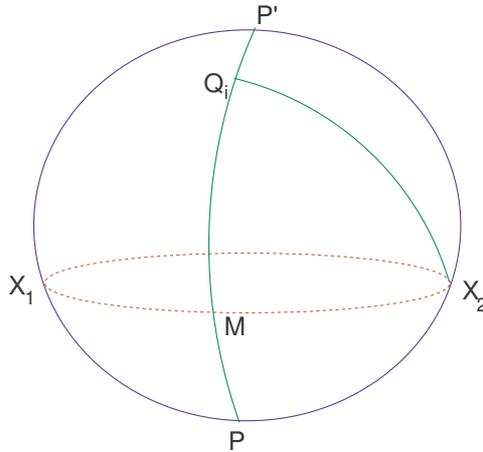


Fig. 8.4:

In particular, there exists a sufficiently small $\epsilon > 0$ such that

$$\text{arc}(Ex_iP') < \text{arc}(X_2P') - 2\epsilon. \quad (8.16)$$

Let Q_i be a point on the arc PMP' that is sufficiently near P' so that

$$\text{arc}(Ex_iQ_i) < \text{arc}(Ex_iP') + \epsilon. \quad (8.17)$$

From (8.16) and (8.17),

$$\text{arc}(Ex_iQ_i) < \text{arc}(X_2P') - \epsilon. \quad (8.18)$$

Since triangles X_1PQ_i and X_2PQ_i are congruent,

$$\text{arc}(X_1Q_i) = \text{arc}(X_2Q_i). \quad (8.19)$$

In the spherical triangle $X_2P'Q_i$, $X_2\hat{P}'Q_i = \pi/2$. The cosine rule gives

$$\cos(\text{arc}(X_2Q_i)) = \cos(\text{arc}(X_2P')) \cos(\text{arc}(Q_iP')).$$

Note that $\text{arc}(X_2P') > \pi/2$ as all the demand points are not on a hemisphere. Together this fact and the assumption that Q_i lies on the arc PMP' and is in the ϵ -neighborhood of P' , we have

$$\cos(\text{arc}(X_2Q_i)) > \cos(\text{arc}(X_2P')).$$

Hence,

$$\text{arc}(X_2Q_i) < \text{arc}(X_2P'). \quad (8.20)$$

Since, $\lim_{Q_i \rightarrow P'} \text{arc}(X_2Q_i) = \text{arc}(X_2P')$, there exists a small neighborhood around P' such that if Q_i is in this neighborhood, then

$$\text{arc}(X_2Q_i) > \text{arc}(X_2P') - \epsilon. \quad (8.21)$$

Therefore, it follows from (8.18), (8.20) and (8.21) that

$$\text{arc}(Ex_iQ_i) < \text{arc}(X_2Q_i) < \text{arc}(X_2P'). \quad (8.22)$$

The results (8.17) through (8.21) are not only true for Q_i but also for any Q on the arc $P'Q_i$, i.e.,

$$\text{arc}(Ex_iQ) < \text{arc}(X_2Q) < \text{arc}(X_2P'). \quad (8.23)$$

Therefore, corresponding to each demand point Ex_i , there exists a point Q_i on the arc $P'M$ such that an inequality of the type (8.22) holds. let

$$\text{arc}(P'Q) = \text{minimum}\{\text{arc}(Ex_iQ_i) : Ex_i (\neq X_1 \text{ or } X_2) \text{ is any demand point}\}.$$

However, (8.23) implies that the distances from Q to each demand point are shorter than the distance from P' to X_2 . Thus, P' cannot be a minimax location. ■

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