TECHNISCHE UNIVERSITÄT
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FUNKTIONALANALYSIS
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Martin J. Fengler, Dominik Michel, Volker Michel

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Bericht 16 – Februar 2005

FACHBEREICH MATHEMATIK

Harmonic Spline–Wavelets on the 3–dimensional Ball and their Application to the Reconstruction of the Earth's Density Distribution from Gravitational Data at Arbitrarily Shaped Satellite Orbits

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Abstract

We introduce splines for the approximation of harmonic functions on a 3–dimensional ball. Those splines are combined with a multiresolution concept. More precisely, at each step of improving the approximation we add more data and, at the same time, reduce the hat–width of the used spline basis functions. Finally, a convergence theorem is proved. One possible application, that is discussed in detail, is the reconstruction of the Earth's density distribution from gravitational data obtained at a satellite orbit. This is an exponentially ill–posed problem where only the harmonic part of the density can be recovered since its orthogonal complement has the potential 0. Whereas classical approaches use a truncated singular value decomposition (TSVD) with the well–known disadvantages like the non–localizing character of the used spherical harmonics and the bandlimitedness of the solution, modern regularization techniques use wavelets allowing a localized reconstruction via convolutions with kernels that are only essentially large in the region of interest. The essential remaining drawback of a TSVD and the wavelet approaches is that the integrals (i.e. the inner product in case of a TSVD and the convolution in case of wavelets) are calculated on a spherical orbit, which is not given in reality. Thus, simplifying modelling assumptions, that certainly include a modelling error, have to be made. The splines introduced here have the important advantage, that the given data need not be located on a sphere but may be (almost) arbitrarily distributed in the outer space of the Earth. This includes, in particular, the possibility to mix data from different satellite missions (different orbits, different derivatives of the gravitational potential) in the calculation of the Earth's density distribution. Moreover, the approximating splines can be calculated at varying resolution scales, where the differences for increasing the resolution can be computed with the introduced spline–wavelet technique.

Key Words: inverse problem, regularization, multiresolution, spline–wavelets, Sobolev spaces, inner harmonics, spherical harmonics, ball, regular surface; CHAMP, GRACE, and GOCE satellite missions

AMS(2000) Classification: 41A15, 42C40, 45K05, 86A22.

*each: Geomathematics Group, Department of Mathematics, University of Kaiserslautern, P.O. Box 3049, D–67653 Kaiserslautern, Germany; Email: fengler/dmichel/michel@mathematik.uni-kl.de, Internet: http://www.mathematik.uni-kl.de/∼wwwgeo

1 Preliminaries and Introduction

As usual, the set of all positive integers is denoted by $\mathbb{N}$, where $\mathbb{N}_0$ represents the set of all non–negative integers. $\mathbb{R}$ stands for the set of all real numbers, such that $\mathbb{R}^n$ is the n–dimensional Euclidean vector space with the inner product $x \cdot y := \sum_{i=1}^{n} x_i y_i$ and the norm $|x| := \sqrt{x \cdot x}$ $(x = (x_1, ..., x_n)^{\mathrm{T}} \in \mathbb{R}^n)$.
The function spaces $\mathrm{L}^2(D)$ and $\mathrm{C}(D)$ represent the sets of all square–Lebesgue–integrable, respectively, continuous bounded functions from $D \subset \mathbb{R}^n$ into $\mathbb{R}$, where the elements of $\mathrm{L}^2(D)$ are, more precisely, equivalence classes of almost everywhere identical functions. $\mathrm{L}^2(D)$ equipped with the inner product

$$(F, G)_{\mathrm{L}^2(D)} := \int_D F(x)G(x)\,dx; \quad F, G \in \mathrm{L}^2(D);$$

is a Hilbert space (with norm $\|F\|_{\mathrm{L}^2(D)} := \sqrt{(F, F)_{\mathrm{L}^2(D)}}$, $F \in \mathrm{L}^2(D)$). Moreover, $\mathrm{C}(D)$ equipped with the norm

$$\|F\|_{\mathrm{C}(D)} := \sup_{x \in D} |F(x)|, \quad F \in \mathrm{C}(D),$$

is a Banach space and we have the relation

$$\|F\|_{\mathrm{L}^2(D)} \leq \sqrt{\lambda(D)}\, \|F\|_{\mathrm{C}(D)}, \quad F \in \mathrm{C}(D), \tag{1}$$

where $\lambda(D)$ is the n–dimensional Lebesgue measure of D, provided that $D \subset \mathbb{R}^n$ is Lebesgue measurable with finite measure.
Canonically, we denote by Δ the Laplace operator and by ∇ the gradient, such that $\nabla \otimes \nabla F$ stands for the Hessian of a twice continuously differentiable (scalar) function F.
Furthermore, a twice continuously differentiable function $F : D \to \mathbb{R}$ on an open domain $D \subset \mathbb{R}^3$ is called *harmonic* if it satisfies the Laplace equation

$$\Delta_x F(x) = \left(\left(\frac{\partial}{\partial x_1} \right)^2 + \left(\frac{\partial}{\partial x_2} \right)^2 + \left(\frac{\partial}{\partial x_3} \right)^2 \right) F(x) = 0, \quad x \in D.$$

If X and Y are Banach spaces then $\mathcal{L}(X, Y)$ denotes the space of all continuous, linear mappings from X into Y. A basic theorem of functional analysis says that a linear mapping $S : X \to Y$ between two Banach spaces $(X, \|.\|_X)$ and $(Y, \|.\|_Y)$ is continuous if and only if it is bounded, which means that the operator norm

$$\|S\|_{\mathcal{L}} := \sup_{x \in X,\, x \neq 0} \frac{\|Sx\|_Y}{\|x\|_X} = \sup_{x \in X, \|x\|_X = 1} \|Sx\|_Y = \sup_{x \in X, \|x\|_X \leq 1} \|Sx\|_Y$$

is finite.
The unit sphere in $\mathbb{R}^3$ is denoted by Ω. The restrictions of homogeneous harmonic polynomials in (x_1, x_2, x_3) of degree n to Ω are called *spherical harmonics* and are collected in the space $\mathrm{Harm}_n(\Omega)$. Correspondingly, we define the spaces

$$\mathrm{Harm}_{0..n}(\Omega) \;\; := \;\; \bigoplus_{i=0}^{n} \mathrm{Harm}_i(\Omega),$$

$$\mathrm{Harm}_{0..\infty}(\Omega) \quad := \quad \bigcup_{i=0}^{\infty} \mathrm{Harm}_{0..i}(\Omega).$$

It is well–known that $\dim(\mathrm{Harm}_n(\Omega)) = 2n+1$ for all $n \in \mathbb{N}_0$ and $\mathrm{Harm}_{0..\infty}(\Omega)$ is dense in $(\mathrm{L}^2(\Omega), \|.\|_{\mathrm{L}^2(\Omega)})$. Thus, we assume that a system $\{Y_{n,j}\}_{n\in\mathbb{N}_0;j=1,...,2n+1}$ is given such that $\{Y_{n,j}\}_{j=1,...,2n+1}$ is a complete $\mathrm{L}^2(\Omega)$–orthonormal system in $\mathrm{Harm}_n(\Omega)$ for all $n \in \mathbb{N}_0$. Since $\mathrm{Harm}_n(\Omega)$ is $\mathrm{L}^2(\Omega)$–orthogonal to $\mathrm{Harm}_m(\Omega)$ for $n \neq m$, such a system $\{Y_{n,j}\}_{n\in\mathbb{N}_0;j=1,...,2n+1}$ is automatically a complete orthonormal system of $\mathrm{L}^2(\Omega)$. Analogously, by defining $Y^B_{n,j}(x) := \frac{1}{\beta} Y_{n,j}\left(\frac{x}{|x|}\right)$ for some fixed $\beta > 0$, we obtain a complete orthonormal system $\{Y^B_{n,j}\}_{n\in\mathbb{N}_0;j=1,...,2n+1}$ of $\mathrm{L}^2(B)$, where B is the sphere with center 0 and radius β. Moreover, each system of that type ($Y^\Omega_{n,j} = Y_{n,j}$) satisfies the *addition theorem for spherical harmonics*

$$\sum_{j=1}^{2n+1} Y_{n,j}(\xi) Y_{n,j}(\eta) = \frac{2n+1}{4\pi} P_n(\xi \cdot \eta); \quad \xi, \eta \in \Omega;\ n \in \mathbb{N}_0; \tag{2}$$

where P_n is the *Legendre polynomial* of degree n, which is uniquely determined by the requirements

(i) P_n is a polynomial of degree n for all $n \in \mathbb{N}_0$.

(ii) $\int_{-1}^{1} P_n(t) P_m(t)\, dt = 0$ for all $n, m \in \mathbb{N}_0$ with $n \neq m$.

(iii) $P_n(1) = 1$ for all $n \in \mathbb{N}_0$.

For further details on spherical harmonics and on Legendre polynomials we refer to [10, 20].
In this paper we assume that the Earth has the shape of a ball with radius $\beta > 0$. The surface, i.e. the sphere with center 0 and radius β, here denoted by B, has an inner space B_{int} which is the open ball with radius β representing the Earth's interior. The outer space of B is denoted by $B_{\mathrm{ext}} := \mathbb{R}^3 \setminus \overline{B_{\mathrm{int}}}$. In contrast to the spherical situation, the harmonic functions on the ball B_{int} are not dense in $\mathrm{L}^2(B_{\mathrm{int}})$. More precisely, the space $\mathrm{Harm}(B_{\mathrm{int}})$ of all harmonic functions on B_{int} is a closed linear strict subspace of $\mathrm{L}^2(B_{\mathrm{int}})$. A complete orthonormal system in $\mathrm{Harm}(B_{\mathrm{int}})$ is given by the system of *inner harmonics*

$$H^B_{n,j}(x) := \sqrt{\frac{2n+3}{\beta^3}} \left(\frac{|x|}{\beta}\right)^n Y_{n,j}\left(\frac{x}{|x|}\right), \quad x \in B_{\mathrm{int}},$$

$n \in \mathbb{N}_0$, $j \in \{1, ..., 2n+1\}$ (see [16]).
The relation between the gravitational potential V of the Earth and the density distribution $\rho \in \mathrm{L}^2(B_{\mathrm{int}})$ is given by the Fredholm integral equation of first kind

$$(T\rho)(y) := \int_{B_{\mathrm{int}}} \frac{\rho(x)}{|x-y|}\, dx = V(y), \quad y \in B_{\mathrm{ext}}. \tag{3}$$

The inverse problem of determining ρ out of V is called gravimetry problem. It is well–known (see, for example, [11, 15, 16, 21]) that the $\mathrm{L}^2(B_{\mathrm{int}})$–orthogonal space of $\mathrm{Harm}(B_{\mathrm{int}})$,

$$\mathrm{Anharm}(B_{\mathrm{int}}) := \left\{ F \in \mathrm{L}^2(B_{\mathrm{int}}) \,\middle|\, (F, H)_{\mathrm{L}^2(B_{\mathrm{int}})} = 0 \text{ for all } H \in \mathrm{Harm}(B_{\mathrm{int}}) \right\},$$

of so–called *anharmonic functions* is the infinite–dimensional null space of the operator T. Thus, only the harmonic part of the Earth's density distributions can be recovered from gravitational data. For this reason, we restrict our attention in this paper to the treatment of the harmonic part of the solution of the integral equation (3). Concerning strategies for the treatment of the null space we refer to, for example, [1, 2, 3, 4, 11, 16, 18].
Let us have a closer look at Hadamard's criteria, according to which a problem is called *well–posed* if and only if a solution exists, is unique, and continuously depends on the given data. Otherwise it is called *ill–posed*. The inverse gravimetric problem given by (3) is ill–posed for the following reasons: Errors in measuring V can, for example, destroy the harmonicity of the function such that the problem becomes unsolvable. As we already mentioned above, the solution is not unique. Finally, the inverse operator of the restriction of T to $\mathrm{Harm}\,(B_{\mathrm{int}})$ (note that the restriction is necessary to gain invertibility) is not continuous. If V is sampled at a positive distance $\sigma - \beta > 0$ to the Earth, for example at a satellite orbit, then we have an exponentially ill–posed problem in the sense that the singular value $(T^\wedge(n))^{-1}$ of T^{-1}, given by $TH^B_{n,j} = T^\wedge(n)\frac{1}{\sigma}Y_{n,j}(\frac{\cdot}{|\cdot|})$ exponentially diverges to infinity as $n \to \infty$. This means that measuring errors, which become, in particular, for large degrees n relevant, are extremely strengthened in this inversion process.
The mentioned results of the discussion of Hadamard's criteria remain true if derivatives of V are given instead of V. For more details we refer to [18]. This exponential ill–posedness requires an adequate regularization procedure which is able to calculate a sequence of approximations to the harmonic density with the following requirements:

1. Every approximation must be computable even if the right hand side V is not in the image $\mathrm{im}\,(T)$ of the operator T.

2. Every approximation continuously depends on V.

3. The sequence of approximations converges in an appropriate topology (like $\mathrm{L}^2\,(B_{\mathrm{int}})$) to the exact harmonic solution of (3).

Some wavelet based regularization methods for this purpose have already been presented in [11, 15, 16, 17, 18, 19]. In this work we will, for the first time, introduce spline–wavelets as regularization method for the satellite–data based gravimetry problem motivated by an alternative spline–wavelet method for various problems in potential theory presented in [11]. Our new spline–wavelet approach has several advantages: First, it includes, in particular, the smoothing and best approximating properties of splines. Second, the satellite data can be arbitrarily distributed in the outer space of the Earth. This is an essential improvement, since the methods developed in [15, 16, 17, 18, 19] require data on a spherical domain and, though, the approach in [11] allows data given on a general regular surface, the theory assumes the knowledge of a complete orthonormal system of the L^2–space of that surface, which is usually not given for computational purposes.
In Section 2 we introduce the spline method based on inner harmonics on a ball in $\mathbb{R}^3$. They are contained in certain Sobolev spaces $\mathcal{H}$ which are given by a sequence $\{A_n\}_{n\in\mathbb{N}_0}$ that satisfies a summability condition. Furthermore, we discuss the special spline basis functions obtained for the gravimetry problem. Since the sequence $\{A_n\}_{n\in\mathbb{N}_0}$ allows us to control the had–width of

the basis functions we use this phenomenon to construct a multiscale spline concept in Section 3. There a family $\{\Phi_J^\wedge(n)\}_{n\in\mathbb{N}_0}$, $J \in \mathbb{N}_0$, of sequences generates a family of Sobolev spaces $\mathcal{H}_J$, $J \in \mathbb{N}_0$. Under some conditions on the values $\Phi_J^\wedge(n)$ these Sobolev spaces represent a multiresolution of the set of harmonic functions on the ball. The larger the scale J, the smaller is the hat–width of the spline basis functions. This is compensated by adding more and more data to the spline interpolation problem. Then we prove in Section 4 a convergence theorem for the limit $J \to \infty$. Finally, in Section 5 the results are summarized and an outlook is given.

2 Harmonic Splines on a Ball

In analogy to [7, 8, 9, 10], where the cases of a sphere and the outer space, respectively, are discussed, we introduce harmonic splines on B_{int}.

Definition 2.1 *Let $\{A_n\}_{n\in\mathbb{N}_0}$ be a real sequence. By $\mathcal{E} := \mathcal{E}(\{A_n\}; B_{\mathrm{int}})$ we denote the space of all $F \in \mathrm{Harm}\,(B_{\mathrm{int}})$ with*

$$\left(F, H_{n,j}^B\right)_{\mathrm{L}^2(B_{\mathrm{int}})} = 0 \textit{ for all } n \in \mathbb{N}_0 \textit{ with } A_n = 0$$

and

$$\sum_{\substack{n=0\\A_n\neq 0}}^{\infty} \sum_{j=1}^{2n+1} A_n^{-2} \left(F, H_{n,j}^B\right)^2_{\mathrm{L}^2(B_{\mathrm{int}})} < +\infty.$$

For $F, G \in \mathcal{E}$ we introduce the inner product

$$(F, G)_{\mathcal{H}(\{A_n\};B_{\mathrm{int}})} := \sum_{\substack{n=0\\A_n\neq 0}}^{\infty} \sum_{j=1}^{2n+1} A_n^{-2} \left(F, H_{n,j}^B\right)_{\mathrm{L}^2(B_{\mathrm{int}})} \left(G, H_{n,j}^B\right)_{\mathrm{L}^2(B_{\mathrm{int}})}. \tag{4}$$

The norm $\|.\|_{\mathcal{H}(\{A_n\};B_{\mathrm{int}})}$ is induced by $\|F\|_{\mathcal{H}(\{A_n\};B_{\mathrm{int}})} := \sqrt{(F, F)_{\mathcal{H}(\{A_n\};B_{\mathrm{int}})}}$.

Note that the Cauchy–Schwarz inequality shows that the inner product of (4) is always finite.

Definition 2.2 *The completion of $\mathcal{E}(\{A_n\}; B_{\mathrm{int}})$ with respect to $(.,.)_{\mathcal{H}(\{A_n\};B_{\mathrm{int}})}$ is denoted by the Sobolev space $\mathcal{H}(\{A_n\}; B_{\mathrm{int}})$. If no confusion is likely to arise, we will simply write $\mathcal{H}$ instead of $\mathcal{H}(\{A_n\}; B_{\mathrm{int}})$.*

Definition 2.3 *A real sequence $\{A_n\}_{n\in\mathbb{N}_0}$ is called summable if*

$$\sum_{n=0}^{\infty} A_n^2 \frac{(2n+3)(2n+1)}{4\pi\beta^3} < +\infty.$$

Assumption 2.4 *We always assume that the used sequences $\{A_n\}_{n\in\mathbb{N}_0}$ are summable.*

The summability of the sequence $\{A_n\}_{n\in\mathbb{N}_0}$ automatically guarantees that every element of the Hilbert space $\mathcal{H}(\{A_n\}; B_{\mathrm{int}})$ can be related to a continuous bounded function such that $\mathcal{H}(\{A_n\}; B_{\mathrm{int}}) \subset \mathrm{C}(B_{\mathrm{int}})$.

Lemma 2.5 (Sobolev Lemma) *Every $F \in \mathcal{H}(\{A_n\}; B_{\mathrm{int}})$ corresponds to a continuous function on B_{int}. Moreover, F is even harmonic on B_{int} and the Fourier series*

$$F = \sum_{n=0}^{\infty} \sum_{j=1}^{2n+1} \left(F, H_{n,j}^B\right)_{\mathrm{L}^2(B_{\mathrm{int}})} H_{n,j}^B$$

is uniformly convergent on B_{int}.

Proof: Application of the Cauchy–Schwarz inequality yields for $F \in \mathcal{H}(\{A_n\}; B_{\mathrm{int}})$ the estimate

$$\begin{aligned}
\left| \sum_{n=N}^{\infty} \sum_{j=1}^{2n+1} \left(F, H_{n,j}^B\right)_{\mathrm{L}^2(B_{\mathrm{int}})} H_{n,j}^B(x) \right| &= \left| \sum_{\substack{n=N \\ A_n \neq 0}}^{\infty} \sum_{j=1}^{2n+1} \left(F, H_{n,j}^B\right)_{\mathrm{L}^2(B_{\mathrm{int}})} A_n^{-1} A_n H_{n,j}^B(x) \right| \\
&\leq \left(\sum_{\substack{n=N \\ A_n \neq 0}}^{\infty} \sum_{j=1}^{2n+1} \left(F, H_{n,j}^B\right)^2_{\mathrm{L}^2(B_{\mathrm{int}})} A_n^{-2} \right)^{1/2} \left(\sum_{\substack{n=N \\ A_n \neq 0}}^{\infty} \sum_{j=1}^{2n+1} A_n^2 \, \frac{2n+3}{\beta^3} \left(\frac{|x|}{\beta}\right)^{2n} \left(Y_{n,j}\left(\frac{x}{|x|}\right)\right)^2 \right)^{1/2} \\
&\leq \|F\|_{\mathcal{H}(\{A_n\}; B_{\mathrm{int}})} \left(\sum_{\substack{n=N \\ A_n \neq 0}}^{\infty} A_n^2 \, \frac{(2n+3)(2n+1)}{4\pi\beta^3} \, P_n(1) \right)^{1/2} \underset{N\to\infty}{\longrightarrow} 0,
\end{aligned}$$

where the right hand side converges as $N \to \infty$ uniformly with respect to $x \in B_{\mathrm{int}}$ due to the summability condition. ■

Essential for the construction of the splines here is the existence of a reproducing kernel. This is also guaranteed by the summability of the sequence $\{A_n\}_{n \in \mathbb{N}_0}$. Note that a reproducing kernel is always unique if it exists.

Theorem 2.6 *$\mathcal{H}$ has a unique reproducing kernel $K_{\mathcal{H}} : B_{\mathrm{int}} \times B_{\mathrm{int}} \to \mathbb{R}$ satisfying*

(i) $K_{\mathcal{H}}(x, .), K_{\mathcal{H}}(., x) \in \mathcal{H}$ *for all* $x \in B_{\mathrm{int}}$.

(ii) $(K_{\mathcal{H}}(x, .), F)_{\mathcal{H}} = (K_{\mathcal{H}}(., x), F)_{\mathcal{H}} = F(x)$ *for all* $F \in \mathcal{H}$ *and all* $x \in B_{\mathrm{int}}$.

Furthermore, $K_{\mathcal{H}}$ is given by

$$\begin{aligned}
K_{\mathcal{H}}(x, y) &= \sum_{n=0}^{\infty} \sum_{j=1}^{2n+1} A_n^2 H_{n,j}^B(x) H_{n,j}^B(y) \\
&= \sum_{n=0}^{\infty} A_n^2 \, \frac{(2n+3)(2n+1)}{4\pi\beta^3} \, P_n\left(\frac{x}{|x|} \cdot \frac{y}{|y|}\right) \left(\frac{|x|\,|y|}{\beta^2}\right)^n.
\end{aligned}$$

The proof of this theorem is analogous to the proof in the spherical case (see, for example [10]).

Definition 2.7 *Let $\mathcal{F} := \{\mathcal{F}_1, ..., \mathcal{F}_N\} \subset \mathcal{L}(\mathcal{H}, \mathbb{R})$ be a linearly independent system of bounded linear functionals from $\mathcal{H}$ into $\mathbb{R}$. A function $S \in \mathcal{H}$ of the form*

$$S(x) = \sum_{k=1}^{N} a_k \mathcal{F}_k K_{\mathcal{H}}(.,x), \quad x \in B_{\mathrm{int}},$$

$a = (a_1, ..., a_N)^{\mathrm{T}} \in \mathbb{R}^N$, is called harmonic spline in $\mathcal{H}(\{A_n\}; B_{\mathrm{int}})$ relative to $\mathcal{F}$. Such splines are collected in the space $\mathrm{Spline}(\{A_n\}; \mathcal{F})$.

A harmonic spline interpolation problem can be formulated by a system of linearly independent functionals $\mathcal{F} = \{\mathcal{F}_1, ..., \mathcal{F}_N\} \subset \mathcal{L}(\mathcal{H}, \mathbb{R})$ and a vector $y = (y_1, ..., y_N)^{\mathrm{T}} \in \mathbb{R}^N$ as follows: Determine $S \in \mathrm{Spline}(\{A_n\}; \mathcal{F})$ such that

$$\mathcal{F}_i S = y_i \quad \text{for all } i = 1, ..., N$$

or, equivalently, determine $a \in \mathbb{R}^N$ such that

$$\sum_{j=1}^{N} a_j \mathcal{F}_i \mathcal{F}_j K_{\mathcal{H}}(.,.) = y_i \quad \text{for all } i = 1, ..., N.$$

This yields a linear equation system with the matrix

$$(\mathcal{F}_i \mathcal{F}_j K_{\mathcal{H}}(.,.))_{i,j=1,...,N} \tag{5}$$

which is positive definite according to the following considerations. First, we realize the validity of the following lemma as a consequence of a general theorem in [5], p. 318.

Lemma 2.8 *Let $\mathcal{F} \in \mathcal{L}(\mathcal{H}, \mathbb{R})$ be arbitrary. Then $y \mapsto \mathcal{F}_x K_{\mathcal{H}}(x,y)$ is in $\mathcal{H}$ and*

$$\mathcal{F}(F) = (F, \mathcal{F}_x K_{\mathcal{H}}(x,.))_{\mathcal{H}}$$

for all $F \in \mathcal{H}$.

Here, $\mathcal{F}_x K_{\mathcal{H}}(x,.)$ means that $\mathcal{F}$ is applied to the function $x \mapsto K_{\mathcal{H}}(x,y)$ where y is arbitrary but fixed. Finally, we get the following result.

Theorem 2.9 *Let $\mathcal{F} = \{\mathcal{F}_1, ..., \mathcal{F}_N\} \subset \mathcal{L}(\mathcal{H}, \mathbb{R})$ be a system of functionals. This system is linearly independent if and only if the matrix (5) is positive definite.*

Proof: Due to Lemma 2.8 we see that (5) is a Gram matrix since

$$(\mathcal{F}_i)_x (\mathcal{F}_j)_y K_{\mathcal{H}}(x,y) = ((\mathcal{F}_j)_y K_{\mathcal{H}}(.,y), (\mathcal{F}_i)_x K_{\mathcal{H}}(x,.))_{\mathcal{H}}.$$

By definition the linear independence of $\{(\mathcal{F}_i)_x K_{\mathcal{H}}(x,.)\}_{i=1,...,N}$ implies that

$$G(y) := \sum_{i=1}^{N} a_i (\mathcal{F}_i)_x K_{\mathcal{H}}(x,y) = 0 \quad \text{for all } y \in B_{\mathrm{int}} \Leftrightarrow a_i = 0 \quad \text{for all } i = 1, ..., N.$$

According to Lemma 2.8 this is equivalent to the statement that

$$(F,G)_{\mathcal{H}} = \sum_{i=1}^{N} a_i \mathcal{F}_i F = 0 \quad \text{for all } F \in \mathcal{H} \Leftrightarrow a_i = 0 \quad \text{for all } i = 1, ..., N.$$

This is true if and only if $\mathcal{F}$ is linearly independent. Since a Gram matrix is positive definite if and only if the corresponding system of vectors is linearly independent, the statement of the theorem is valid. ■

Consequently, the formulated spline interpolation problem is always uniquely solvable.

In case of the discussed gravimetry problem the following types of functionals are of particular interest.

$$\begin{aligned}
\mathcal{G}_k^{(0)} F &= \int_{B_{\mathrm{int}}} \frac{F(y)}{|y-x_k|}\,\mathrm{d}y, \quad x_k \in B_{\mathrm{ext}} \text{ fixed},\\
\mathcal{G}_k^{(1)} F &= -\left(\frac{x}{|x|} \cdot \nabla_x \int_{B_{\mathrm{int}}} \frac{F(y)}{|y-x|}\,\mathrm{d}y\right)\Bigg|_{x=x_k}, \quad x_k \in B_{\mathrm{ext}} \text{ fixed},\\
\mathcal{G}_k^{(2)} F &= \left(\frac{x}{|x|} \cdot \left(\left(\nabla_x \otimes \nabla_x \int_{B_{\mathrm{int}}} \frac{F(y)}{|y-x|}\,\mathrm{d}y\right) \frac{x}{|x|}\right)\right)\Bigg|_{x=x_k}, \quad x_k \in B_{\mathrm{ext}} \text{ fixed}. \qquad (6)
\end{aligned}$$

Due to, for example, [12] and [18] we have

$$\begin{aligned}
\mathcal{G}_k^{(0)} H_{n,j}^B &= \frac{4\pi}{2n+1}\sqrt{\frac{\beta^3}{2n+3}}\left(\frac{\beta}{|x_k|}\right)^n \frac{1}{|x_k|} Y_{n,j}\left(\frac{x_k}{|x_k|}\right),\\
\mathcal{G}_k^{(1)} H_{n,j}^B &= \frac{n+1}{|x_k|}\frac{4\pi}{2n+1}\sqrt{\frac{\beta^3}{2n+3}}\left(\frac{\beta}{|x_k|}\right)^n \frac{1}{|x_k|} Y_{n,j}\left(\frac{x_k}{|x_k|}\right),\\
\mathcal{G}_k^{(2)} H_{n,j}^B &= \frac{(n+1)(n+2)}{|x_k|^2}\frac{4\pi}{2n+1}\sqrt{\frac{\beta^3}{2n+3}}\left(\frac{\beta}{|x_k|}\right)^n \frac{1}{|x_k|} Y_{n,j}\left(\frac{x_k}{|x_k|}\right)
\end{aligned}$$

as well as the representation

$$\mathcal{G}_k^{(i)} F = \sum_{n=0}^{\infty}\sum_{j=1}^{2n+1} q_n^{(i)} \frac{4\pi}{2n+1}\sqrt{\frac{\beta^3}{2n+3}}\left(\frac{\beta}{|x_k|}\right)^n (F, H_{n,j}^B)_{\mathrm{L}^2(B_{\mathrm{int}})} \frac{1}{|x_k|} Y_{n,j}\left(\frac{x_k}{|x_k|}\right)$$

for all $F \in \mathrm{L}^2(B_{\mathrm{int}})$, where $q_n^{(i)}$ is polynomial in n with maximal degree 2 and $\{(F, H_{n,j}^B)_{\mathrm{L}^2(B_{\mathrm{int}})}\}_{n,j}$ is bounded due to the square–integrability of F. This series is obviously convergent since $|x_k| > \beta$ and $\|Y_{n,j}\|_{\mathrm{C}(\Omega)} \le \sqrt{(2n+1)(4\pi)^{-1}}$. The estimate

$$\left|\mathcal{G}_k^{(i)} F\right| \le \left(\sum_{\substack{n=0\\A_n\neq 0}}^{\infty}\sum_{j=1}^{2n+1}\left(q_n^{(i)} \frac{4\pi}{2n+1}\sqrt{\frac{\beta^3}{2n+3}}\left(\frac{\beta}{|x_k|}\right)^n \frac{1}{|x_k|} Y_{n,j}\left(\frac{x_k}{|x_k|}\right) A_n\right)^2\right)^{\frac{1}{2}}$$

$$\cdot \left(\sum_{\substack{n=0\\A_n\neq 0}}^{\infty} \sum_{j=1}^{2n+1} \left(F, H_{n,j}^B\right)^2_{\mathrm{L}^2(B_{\mathrm{int}})} A_n^{-2} \right)^{\frac{1}{2}}$$

$$= \left(\sum_{\substack{n=0\\A_n\neq 0}}^{\infty} \left(q_n^{(i)} \sqrt{\frac{\beta^3}{2n+3}} \left(\frac{\beta}{|x_k|}\right)^n \frac{1}{|x_k|}\right)^2 \frac{4\pi}{2n+1} A_n^2 \right)^{\frac{1}{2}} \|F\|_{\mathcal{H}},$$

$F \in \mathcal{H}$, in connection with the summability condition shows that each linear functional $\mathcal{G}_k^{(i)} : \mathcal{H} \to \mathbb{R}$ is continuous. Note that weaker conditions on $\{A_n\}$ than the summability condition such as $|A_n| \leq 1$ also suffice here to see the continuity of the functionals $\mathcal{G}_k^{(i)}$. This includes, in particular, the case $\mathcal{H}(\{1\}; B_{\mathrm{int}}) = \mathrm{Harm}\,(B_{\mathrm{int}})$. Hence, the matrix components have the form:

$$\mathcal{G}_i^{(l)}\mathcal{G}_k^{(m)} K_{\mathcal{H}}(.,.) = \sum_{n=0}^{\infty}\sum_{j=1}^{2n+1} A_n^2 \left(\mathcal{G}_i^{(l)} H_{n,j}^B\right)\left(\mathcal{G}_k^{(m)} H_{n,j}^B\right)$$

$$= \sum_{n=0}^{\infty} A_n^2 \frac{4\pi}{2n+1}\frac{\beta^3}{2n+3}\left(\frac{\beta^2}{|x_i|\,|x_k|}\right)^n P_n\left(\frac{x_i}{|x_i|}\cdot\frac{x_k}{|x_k|}\right)\cdot \begin{cases} \frac{1}{|x_i|\,|x_k|}; & l=0, m=0,\\ \frac{n+1}{|x_i|\,|x_k|^2}; & l=0, m=1,\\ \frac{n+1}{|x_i|^2|x_k|}; & l=1, m=0,\\ \frac{(n+1)^2}{|x_i|^2|x_k|^2}; & l=1, m=1,\\ \frac{(n+1)(n+2)}{|x_i|\,|x_k|^3}; & l=0, m=2,\\ \frac{(n+1)(n+2)}{|x_i|^3|x_k|}; & l=2, m=0,\\ \frac{(n+1)^2(n+2)}{|x_i|^2|x_k|^3}; & l=1, m=2,\\ \frac{(n+1)^2(n+2)}{|x_i|^3|x_k|^2}; & l=2, m=1,\\ \frac{(n+1)^2(n+2)^2}{|x_i|^3|x_k|^3}; & l=2, m=2, \end{cases}$$

$$= \sum_{n=0}^{\infty} A_n^2 \frac{4\pi}{2n+1}\frac{\beta^3}{2n+3}\left(\frac{\beta^2}{|x_i|\,|x_k|}\right)^n P_n\left(\frac{x_i}{|x_i|}\cdot\frac{x_k}{|x_k|}\right) \frac{(n+1)^{\delta_{l,1}+\delta_{m,1}}\left(n^2+3n+2\right)^{\delta_{l,2}+\delta_{m,2}}}{|x_i|^{1+l}|x_k|^{1+m}}.$$

In general, the name "spline" refers to the property of minimizing a certain non–smoothness measure among all interpolating functions. In the classical Euclidean case the natural cubic spline s minimizes the linearized deformation energy $\|s''\|_{\mathrm{L}^2}$. For the spherical spline approach (see, for instance, [7, 8, 9, 10]) and further spline concepts based on this spherical approach (such as the anharmonic splines in [16]) this optimization problem is transferred to the minimization of a different norm such as a Sobolev norm. Therefore, we obtain in our case the two typical minimum properties in analogy to the proofs in the references listed above.

Theorem 2.10 (1st Minimum Property) *Let $y \in \mathbb{R}^N$ be given and $\mathcal{F} = \{\mathcal{F}_1, ..., \mathcal{F}_N\} \subset \mathcal{L}(\mathcal{H}, \mathbb{R})$ be linearly independent. If S^* is the unique spline satisfying $\mathcal{F}_i S^* = y_i$ for all $i = 1, ..., N$, then S^* is the unique minimizer of*

$$\|S^*\|_{\mathcal{H}} = \min\left\{ \|F\|_{\mathcal{H}} \,|\, F \in \mathcal{H},\ \mathcal{F}_i F = y_i \,\forall\, i = 1, ..., N \right\}.$$

Theorem 2.11 (2nd Minimum Property, Best Approximation Property) *Let $F \in \mathcal{H}$ be given and $\mathcal{F} = \{\mathcal{F}_1, ..., \mathcal{F}_N\} \subset \mathcal{L}(\mathcal{H}, \mathbb{R})$ be linearly independent. If $S^* \in \mathrm{Spline}(\{A_n\}; \mathcal{F})$ is the unique spline satisfying $\mathcal{F}_i S^* = \mathcal{F}_i F$ for all $i = 1, ..., N$, then S^* is the unique minimizer of*

$$\|F - S^*\|_{\mathcal{H}} = \min \{ \|F - S\|_{\mathcal{H}} | \, S \in \mathrm{Spline}(\{A_n\}; \mathcal{F}) \} .$$

Thus, if F represents an unknown harmonic function on B_{int}, then the interpolating spline S^* represents the best possible approximation to F among all splines, measured with respect to the metric induced by the Sobolev norm $\| \cdot \|_{\mathcal{H}}$. Moreover, among all functions in $\mathcal{H}$ that fit to the known data y_i the spline S^* is the "smoothest" (in the $\|.\|_{\mathcal{H}}$–sense). Note that we use values $\mathcal{F}_i F$ related to the unknown function F to construct the approximation to F. For this purpose, we are in particular enabled to include various types of data into the calculations (e.g. 1st and 2nd radial derivative of the Earth's gravitational potential as derivable from recent satellite missions; spaceborne, airborne, and surface based data).

In analogy to [10] we are able to prove a Shannon Sampling Theorem.

Theorem 2.12 (Shannon Sampling Theorem) *Any spline function $S \in \mathrm{Spline}(\{A_n\}; \mathcal{F})$ is representable by its "samples" $\mathcal{F}_i S$ as*

$$S(x) = \sum_{k=1}^{N} (\mathcal{F}_k S) \, L_k(x), \quad x \in B_{\text{int}}, \tag{7}$$

where

$$L_k(x) = \sum_{j=1}^{N} a_j^{(k)} \mathcal{F}_j K_{\mathcal{H}}(x, \cdot), \quad x \in B_{\text{int}}, \tag{8}$$

with $a_j^{(k)}$ given as solutions of the linear equation systems

$$\sum_{j=1}^{N} a_j^{(k)} \mathcal{F}_i \mathcal{F}_j K_{\mathcal{H}}(\cdot, \cdot) = \delta_{i,k} \quad \forall\, i = 1, ..., N; \quad \forall\, k = 1, ..., N. \tag{9}$$

Proof: The set of equation systems in (9) guarantees that

$$\mathcal{F}_i L_k = \delta_{i,k},$$

such that

$$\mathcal{F}_i \left(\sum_{k=1}^{N} (\mathcal{F}_k S) \; L_k \right) = \sum_{k=1}^{N} \mathcal{F}_k S \, \mathcal{F}_i L_k = \mathcal{F}_i S$$

for all $i = 1, ..., N$. Thus, the uniqueness of the interpolating spline implies (7). ■

3 Spline–Wavelets

The idea of the new concept introduced here is to increase step by step the resolution of the approximating spline. We will then arrive at a multiresolution spline method. Typically, multiresolutions are generated by scaling functions, where wavelets provide band pass filters representing the step from one resolution to another. Here, we will use a weakened concept of a scaling function.
For alternative spline–wavelet approaches for different geomathematical problems we refer, for example, to [9] and [11].

Definition 3.1 *Let $\mathcal{F}$ represent a given, fixed (countable) system of linear and continuous functionals $\mathcal{F} = \{\mathcal{F}_1, \mathcal{F}_2, ...\} \subset \mathcal{L}(\mathcal{H}, \mathbb{R})$. Moreover, let $\mathcal{F}^{(N_J)} = \{\mathcal{F}_1, ..., \mathcal{F}_{N_J}\} \subset \mathcal{F} \subset \mathcal{L}(\mathcal{H}, \mathbb{R})$ be a linearly independent subsystem and $y^{(J)} \in \mathbb{R}^{N_J}$ be a given vector for every $J \in \mathbb{N}_0$ where $(N_J)_{J\in\mathbb{N}_0}$ is a monotonically increasing sequence of positive integers. Those notations are valid throughout this section.*

Definition 3.2 *If the family of sequences $\{\Phi_J^\wedge(n)\}_{n\in\mathbb{N}_0}$, $J \in \mathbb{N}_0$, satisfies the conditions*

(i) $0 \leq \Phi_J^\wedge(n) \leq \Phi_{J+1}^\wedge(n) \leq 1$ *for all* $n, J \in \mathbb{N}_0$,

(ii) $\{\Phi_J^\wedge(n)\}_{n\in\mathbb{N}_0}$ *is summable for all* $J \in \mathbb{N}_0$,

(iii) for every fixed $n \in \mathbb{N}_0$ *the sequence* $\{\Phi_J^\wedge(n)\}_{J\in\mathbb{N}_0}$ *is not identical to* 0, *i.e. there exists* j_n *such that* $\Phi_J^\wedge(n) > 0$ *for all* $J \geq j_n$,

then the elements of $\mathcal{V}_J := \mathrm{Spline}(\{\Phi_J^\wedge(n)\}; \mathcal{F}^{(N_J)})$ *are called spline–scaling functions.*

Example 3.3 *Various examples of such sequences are known from the spherical wavelet theory (cf. [10]). We distinguish bandlimited sequences such as the Shannon sequence*

$$\Phi_J^\wedge(n) = \begin{cases} 1, & 0 \leq n < 2^J, \\ 0, & n \geq 2^J \end{cases}$$

and the sequence generated by a cubic polynomial

$$\Phi_J^\wedge(n) = \begin{cases} \left(1 - 2^{-J}n\right)^2 \left(1 + 2^{1-J}n\right), & 0 \leq n < 2^J, \\ 0, & n \geq 2^J \end{cases}$$

from non–bandlimited sequences such as the Abel–Poisson sequence

$$\Phi_J^\wedge(n) = \exp\left(-R2^{-J}n\right), \quad R > 0,$$

the Gauß–Weierstraß sequence

$$\Phi_J^\wedge(n) = \exp\left(-R2^{-J}n\left(2^{-J}n + 1\right)\right), \quad R > 0,$$

and the Tykhonov–Philips sequence

$$\Phi_J^\wedge(n) = \frac{1}{1 + \gamma_{J,n}^2}$$

where in the last case $\gamma_{J,n}$ *must satisfy the requirements* *(i)* $\gamma_{J,n}^2 \geq \gamma_{J+1,n}^2$ *for all* $n, J \in \mathbb{N}_0$,

(ii) $\{\Phi_J^\wedge(n)\}_{n\in\mathbb{N}_0}$ *is summable for all fixed* $J \in \mathbb{N}_0$.

Note that in case of the Abel–Poisson reproducing kernel

$$\begin{aligned} K_{\mathcal{H}(\{\Phi_J^\wedge(n)\};B_{\mathrm{int}})}(x,y) &= \sum_{n=0}^{\infty} \exp\left(-R2^{-J}n\right)^2 \frac{(2n+3)(2n+1)}{4\pi\beta^3} P_n\left(\frac{x}{|x|}\cdot\frac{y}{|y|}\right)\left(\frac{|x|\,|y|}{\beta^2}\right)^n \\ &= \frac{1}{4\pi\beta^3}\sum_{n=0}^{\infty}(2n+3)(2n+1)\left(\exp\left(-R2^{1-J}\right)\frac{|x|\,|y|}{\beta^2}\right)^n P_n\left(\frac{x}{|x|}\cdot\frac{y}{|y|}\right) \end{aligned}$$

we can derive a closed representation. From [10] we know that

$$\sum_{n=0}^{\infty}(2n+1)h^n P_n(t) = \frac{1-h^2}{(1+h^2-2ht)^{3/2}}$$

for all $t \in [-1,1]$ *and all* $h \in [0,1[$. *By differentiating with respect to* h *and observing the uniform convergence of the summandwise derived series with respect to* $h \in [0,h_0]$, $0 < h_0 < 1$, *we obtain*

$$\sum_{n=1}^{\infty}(2n+1)nh^{n-1}P_n(t) = \frac{-5h+h^3+h^2t+3t}{(1+h^2-2ht)^{5/2}}$$

such that

$$\sum_{n=0}^{\infty}(2n+3)(2n+1)h^n P_n(t) = \frac{3-10h^2+8h^3t-h^4}{(1+h^2-2ht)^{5/2}}$$

for all $t \in [-1,1]$ *and all* $h \in [0,1[$. *Consequently, we get*

$$K_{\mathcal{H}(\{\exp(-R2^{-J}n)\};B_{\mathrm{int}})}(x,y) = \frac{1}{4\pi\beta^3}\frac{3-10\tilde{h}^2+8\tilde{h}^3t-\tilde{h}^4}{(1+\tilde{h}^2-2\tilde{h}t)^{5/2}},\ \tilde{h} = \exp\left(-R2^{1-J}\right)\frac{|x|\,|y|}{\beta^2},\ t = \frac{x}{|x|}\cdot\frac{y}{|y|};$$

$x, y \in B_{\mathrm{int}}$.
Another special case is given by

$$\Phi_J^\wedge(n) = \begin{cases} \frac{1}{\sqrt{n(2n+3)(2n+1)}}\exp\left(-R2^{-J}n\right), & n \geq 1, \\ \frac{1}{\sqrt{3}}, & n = 0, \end{cases}$$

which will be called here the modified Abel–Poisson sequence, since the corresponding reproducing kernel

$$K_{\mathcal{H}(\{\Phi_J^\wedge(n)\};B_{\mathrm{int}})}(x,y) = \frac{1}{4\pi\beta^3} + \frac{1}{4\pi\beta^3}\sum_{n=1}^{\infty}\frac{1}{n}\exp\left(-R2^{1-J}n\right)\left(\frac{|x|\,|y|}{\beta^2}\right)^n P_n\left(\frac{x}{|x|}\cdot\frac{y}{|y|}\right)$$

may, according to [13], be written in the closed representation

$$\begin{aligned} K_{\mathcal{H}(\{\Phi_J^\wedge(n)\};B_{\mathrm{int}})}(x,y) = & \frac{1}{4\pi\beta^3}\left(1-\log\left(\frac{1}{2}\left(1-\frac{|x|\,|y|}{\beta^2}\left(\frac{x}{|x|}\cdot\frac{y}{|y|}\right)\exp\left(-R2^{1-J}\right)\right.\right.\right. \\ & \left.\left.\left.+\sqrt{1-2\,\frac{|x|\,|y|}{\beta^2}\left(\frac{x}{|x|}\cdot\frac{y}{|y|}\right)\exp\left(-R2^{1-J}\right)+\frac{|x|^2|y|^2}{\beta^4}\exp\left(-R2^{2-J}\right)}\right)\right)\right) \\ = & \frac{1}{4\pi\beta^3}\left(1-\log\left(\frac{1}{2}\left(1-\frac{x\cdot y}{\beta^2}\exp\left(-R2^{1-J}\right)\right.\right.\right. \\ & \left.\left.\left.+\sqrt{1-2\frac{x\cdot y}{\beta^2}\exp\left(-R2^{1-J}\right)+\frac{|x|^2|y|^2}{\beta^4}\exp\left(-R2^{2-J}\right)}\right)\right)\right). \end{aligned}$$

Note that it might be reasonable (but not necessary) to require that $\lim_{J\to\infty}\Phi_J^\wedge(n)=1$ for all $n\in\mathbb{N}_0$. In this case the modified Abel–Poisson kernel is not applicable but the Abel–Poisson kernel itself is. To the knowledge of the authors no non–bandlimited kernel for the gravimetry problem has been known before.

The Sobolev spaces defined by the symbol $\{\Phi_J^\wedge(n)\}$ represent a multiresolution analysis as the following theorem shows.

Theorem 3.4 *Let $\{\Phi_J^\wedge(n)\}_{n\in\mathbb{N}_0}$, $J\in\mathbb{N}_0$, satisfy the conditions of Definition 3.2. Then the Sobolev spaces $\mathcal{H}_J := \mathcal{H}(\{\Phi_J^\wedge(n)\};B_{\mathrm{int}})$ satisfy the properties*

(i) $\mathcal{H}_J\subset\mathcal{H}_{J+1}\subset\mathcal{H}(\{\varphi(n)\};B_{\mathrm{int}})$ for all $J\in\mathbb{N}_0$,

(ii) $\overline{\bigcup_{J\in\mathbb{N}_0}\mathcal{H}_J}^{\|\cdot\|_{\mathcal{H}(\{\varphi(n)\};B_{\mathrm{int}})}} = \mathcal{H}(\{\varphi(n)\};B_{\mathrm{int}})$,

where $\varphi(n) := \lim_{J\to\infty}\Phi_J^\wedge(n)$ for every $n\in\mathbb{N}_0$.

Proof: We first mention that for fixed $n\in\mathbb{N}_0$ the sequence $\{\Phi_J^\wedge(n)\}_{J\in\mathbb{N}_0}$ is monotonically increasing and bounded and, therefore, convergent. Moreover, the third requirement for a scaling function implies in combination with the monotonicity that this limit $\varphi(n)$ is greater than 0.
Now let $F\in\mathcal{H}_J$ be an arbitrary element of the Sobolev space at scale J. Hence, the condition

$$\sum_{\substack{n=0\\ \Phi_J^\wedge(n)\neq 0}}^{\infty}\sum_{j=1}^{2n+1}\left(\Phi_J^\wedge(n)\right)^{-2}\left(F,H_{n,j}^B\right)^2_{\mathrm{L}^2(B_{\mathrm{int}})}<+\infty$$

must be satisfied. Due to the monotonicity of the symbol we conclude that

$$\sum_{\substack{n=0\\ \Phi_{J+1}^\wedge(n)\neq 0}}^{\infty}\sum_{j=1}^{2n+1}\left(\Phi_{J+1}^\wedge(n)\right)^{-2}\left(F,H_{n,j}^B\right)^2_{\mathrm{L}^2(B_{\mathrm{int}})}\leq\sum_{\substack{n=0\\ \Phi_J^\wedge(n)\neq 0}}^{\infty}\sum_{j=1}^{2n+1}\left(\Phi_J^\wedge(n)\right)^{-2}\left(F,H_{n,j}^B\right)^2_{\mathrm{L}^2(B_{\mathrm{int}})}<+\infty$$

such that $F \in \mathcal{H}_{J+1}$. Note that $(F, H^B_{n,j})_{\mathrm{L}^2(B_{\mathrm{int}})} = 0$ for all $n \in \{m \in \mathbb{N}_0 \,|\, \Phi^\wedge_J(m) = 0\} \supset \{m \in \mathbb{N}_0 \,|\, \Phi^\wedge_{J+1}(m) = 0\}$. In analogy, we obtain that $F \in \mathcal{H}(\{\varphi(n)\}; B_{\mathrm{int}})$ since $\varphi(n) \geq \Phi^\wedge_J(n)$. Finally, let $F \in \mathcal{H}(\{\varphi(n)\}; B_{\mathrm{int}})$ be arbitrary. Hence, F satisfies

$$\sum_{n=0}^{\infty} \sum_{j=1}^{2n+1} \varphi(n)^{-2} \left(F, H^B_{n,j}\right)^2_{\mathrm{L}^2(B_{\mathrm{int}})} < +\infty. \tag{10}$$

After defining the sequence $\{G_J\}_{J \in \mathbb{N}_0}$ of functions by

$$G_J := \sum_{n=0}^{\infty} \sum_{j=1}^{2n+1} \frac{\Phi^\wedge_J(n)}{\varphi(n)} \left(F, H^B_{n,j}\right)_{\mathrm{L}^2(B_{\mathrm{int}})} H^B_{n,j}$$

we observe that $G_J \in \mathcal{H}_J$ for all $J \in \mathbb{N}_0$ since

$$\sum_{\substack{n=0 \\ \Phi^\wedge_J(n) \neq 0}}^{\infty} \sum_{j=1}^{2n+1} \Phi^\wedge_J(n)^{-2} \left(G_J, H^B_{n,j}\right)^2_{\mathrm{L}^2(B_{\mathrm{int}})} \leq \sum_{n=0}^{\infty} \sum_{j=1}^{2n+1} \varphi(n)^{-2} \left(F, H^B_{n,j}\right)^2_{\mathrm{L}^2(B_{\mathrm{int}})} < +\infty.$$

Moreover, we observe that the series of the Parseval identity

$$\|F - G_J\|^2_{\mathcal{H}(\{\varphi(n)\}; B_{\mathrm{int}})} = \sum_{n=0}^{\infty} \sum_{j=1}^{2n+1} (\varphi(n))^{-2} \left(1 - \frac{\Phi^\wedge_J(n)}{\varphi(n)}\right)^2 \left(F, H^B_{n,j}\right)^2_{\mathrm{L}^2(B_{\mathrm{int}})}$$

converges uniformly with respect to $J \in \mathbb{N}_0$ due to (10) and the fact that $0 \leq \frac{\Phi^\wedge_J(n)}{\varphi(n)} \leq 1$. Consequently, we obtain

$$\lim_{J \to \infty} \|F - G_J\|^2_{\mathcal{H}(\{\varphi(n)\}; B_{\mathrm{int}})} = \sum_{n=0}^{\infty} \sum_{j=1}^{2n+1} (\varphi(n))^{-2} \lim_{J \to \infty} \left(1 - \frac{\Phi^\wedge_J(n)}{\varphi(n)}\right)^2 \left(F, H^B_{n,j}\right)^2_{\mathrm{L}^2(B_{\mathrm{int}})} = 0.$$

This implies the second property of the multiresolution analysis. ■

This result means that we can obtain a sequence of approximating splines $S_J \in \mathcal{V}_J$ where each spline S_J is the smoothest function of $\mathcal{H}_J$ that satisfies the given equations $\mathcal{F}_n S_J = y_n^{(J)}$; $n = 1, ..., N_J$. Since the kernel and, therefore, usually also the basis functions become more and more localizing as the symbol $\Phi^\wedge_J(n)$ increases with respect to J, the resolution is expected to increase and, thus, more data have be taken into account ($N_J \leq N_{J+1}$). This is illustrated by Figure 1. There we plot in the left column the spline basis functions (see (6) for the definition of the functionals $\mathcal{G}_k^{(m)}$)

$$\begin{aligned}
\mathcal{G}_k^{(m)} K_{\mathcal{H}_J}(.,x) &= \sum_{n=0}^{\infty} \sum_{j=1}^{2n+1} \left(\Phi^\wedge_J(n)\right)^2 \left(\mathcal{G}_k^{(m)} H^B_{n,j}\right) H^B_{n,j}(x) \\
&= \sum_{n=0}^{\infty} \left(\Phi^\wedge_J(n)\right)^2 \left(\frac{|x|}{|x_k|}\right)^n \frac{1}{|x_k|^{1+m}} P_n\left(\frac{x_k}{|x_k|} \cdot \frac{x}{|x|}\right) (n+1)^{\delta_{m,1}} \left(n^2 + 3n + 2\right)^{\delta_{m,2}}
\end{aligned}$$

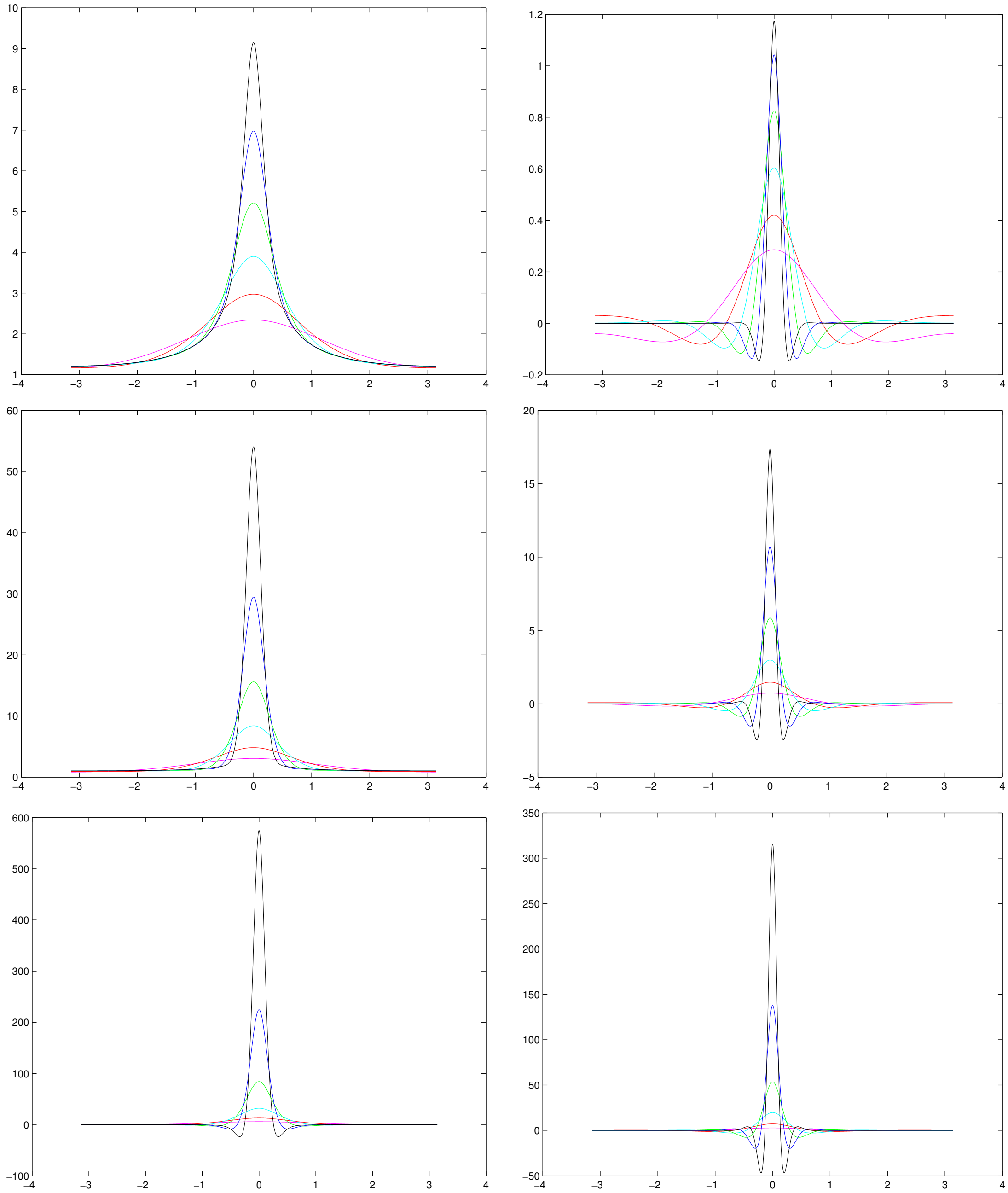

Figure 1: Plot of the spline basis for different scales (magenta: $J = 2$, red: $J = 4$, cyan: $J = 6$, green: $J = 8$, blue: $J = 10$, and black: $J = 12$) in the left column and the difference of spline bases of consecutive scales in the right column, the calculations have been performed for the 0th, 1st, and 2nd derivative (see 1st, 2nd, and 3rd row, respectively); see text for details.

for the different orders $m \in \{0, 1, 2\}$. Note that m refers to the order of the radial derivative of the gravitational potential given at the location x_k, which is here assumed to be 400 km above the Earth's surface (with radius 6371 km). The kernel is plotted for $x \in B$ in dependence of $t \in [-\pi, \pi]$ with $\frac{x_k}{|x_k|} \cdot \frac{x}{|x|} = \cos t$. In the right column we plot the difference kernels $\mathcal{G}_k^{(m)}(K_{\mathcal{H}_{J+1}}(.,x) - K_{\mathcal{H}_J}(.,x))$ which can be interpreted in an abstract sense as a spline–wavelet basis. In each case we used the Tykhonov–Philips sequence with $\gamma_{J,n}^2 = 2^{-J} n^3$ for $\Phi_J^\wedge(n)$. The series of the kernels were truncated at degree 1000 and were calculated via the Clenshaw algorithm (see [6] for details on the algorithm).
Note that the additional requirement $\lim_{J\to\infty} \Phi_J^\wedge(n) = 1$ mentioned above implies that the "limit space" $\mathcal{H}(\{\varphi(n)\}; B_{\text{int}})$ is $\text{Harm}(B_{\text{int}})$, i.e. the set of all harmonic functions on B_{int}. Be aware of the fact that $\{\varphi(n)\}_{n\in\mathbb{N}_0}$ need not be summable.

The approximating harmonic spline at scale $J \in \mathbb{N}_0$ may now be calculated by solving the system of linear equations given by

$$\sum_{k=1}^{N_J} a_k^{(J)} \mathcal{F}_l \mathcal{F}_k K_{\mathcal{H}_J}(.,.) = y_l^{(J)}; \quad l = 1, ..., N_J.$$

The corresponding spline is then given by

$$S_J(x) = \sum_{k=1}^{N_J} a_k^{(J)} \mathcal{F}_k K_{\mathcal{H}_J}(.,x).$$

The differences $S_{J+1} - S_J$ can be interpreted as spline–wavelets.
Note that the described method represents a regularization of the ill–posed problem.

Theorem 3.5 *Let $S_J \in \mathcal{V}_J$ be the unique spline satisfying the conditions $\mathcal{F}_l S_J = y_l^{(J)}$; $l = 1, ..., N_J$; then the spline continuously depends on the given data.*

Proof: This result is an immediate consequence of the linearity of the relation and the finite dimension of the involved spaces. ■

4 A Convergence Theorem

By adding more and more data and decreasing the hat–width of the basis functions at the same time we obtain a sequence of approximating splines. We will prove in this section that this sequence converges to the underlying unknown function. Here we will use the notations introduced in Section 3. Before we can prove the main result we have to verify some lemmata.

Lemma 4.1 *The dual spaces $\mathcal{H}_J^* := \mathcal{L}(\mathcal{H}_J, \mathbb{R})$, $J \in \mathbb{N}_0$, satisfy*

$$\mathcal{H}_\infty^* \subset \mathcal{H}_{J+1}^* \subset \mathcal{H}_J^* \subset \mathcal{H}_0^*$$

for all $J \in \mathbb{N}_0$, where $\mathcal{H}_\infty := \mathcal{H}(\{\varphi(n)\}; B_{\text{int}})$.

Proof: A functional $\mathcal{T} \in \mathcal{H}^*_{J+1}$ is a linear mapping $\mathcal{T} : \mathcal{H}_{J+1} \to \mathbb{R}$ which is bounded, i.e.

$$\|\mathcal{T}\|_{\mathcal{H}^*_{J+1}} = \sup_{\substack{\|F\|_{\mathcal{H}_{J+1}} \leq 1 \\ F \in \mathcal{H}_{J+1}}} |\mathcal{T}F| < +\infty.$$

Obviously, its restriction $\mathcal{T}\,|_{\mathcal{H}_J} : \mathcal{H}_J \to \mathbb{R}$ is also linear and satisfies

$$\|\mathcal{T}\|_{\mathcal{H}^*_J} = \sup_{\substack{\|F\|_{\mathcal{H}_J} \leq 1 \\ F \in \mathcal{H}_J}} |\mathcal{T}F| \leq \sup_{\substack{\|F\|_{\mathcal{H}_{J+1}} \leq 1 \\ F \in \mathcal{H}_J}} |\mathcal{T}F| \leq \sup_{\substack{\|F\|_{\mathcal{H}_{J+1}} \leq 1 \\ F \in \mathcal{H}_{J+1}}} |\mathcal{T}F| = \|\mathcal{T}\|_{\mathcal{H}^*_{J+1}} < +\infty,$$

since $\|F\|_{\mathcal{H}_{J+1}} \leq \|F\|_{\mathcal{H}_J}$ for all $F \in \mathcal{H}_J$ and $\mathcal{H}_J \subset \mathcal{H}_{J+1} (\subset \mathcal{H}_\infty)$, such that the unit ball in $\mathcal{H}_J$ is a subset of the unit ball in $\mathcal{H}_{J+1}$. ■

Lemma 4.2 *For all $F \in \mathcal{H}_{\tilde{J}}$, $\tilde{J} \in \mathbb{N}_0$, we have*

$$\lim_{\substack{J \to \infty \\ J \geq \tilde{J}}} \|F\|_{\mathcal{H}_J} = \|F\|_{\mathcal{H}_\infty}.$$

Proof: Obviously, the series (note that $(F, H^B_{n,j})_{\mathrm{L}^2(B_{\mathrm{int}})} = 0$ if $\Phi^\wedge_{\tilde{J}}(n) = 0$)

$$\begin{aligned}
\|F\|^2_{\mathcal{H}_J} &= \sum_{n=0}^{\infty} \sum_{j=1}^{2n+1} p_J(n) \left(F, H^B_{n,j}\right)^2_{\mathrm{L}^2(B_{\mathrm{int}})} \\
&\leq \sum_{\substack{n=0 \\ \Phi^\wedge_{\tilde{J}}(n) \neq 0}}^{\infty} \sum_{j=1}^{2n+1} p_{\tilde{J}}(n) \left(F, H^B_{n,j}\right)^2_{\mathrm{L}^2(B_{\mathrm{int}})} \\
&= \sum_{n=0}^{\infty} \sum_{j=1}^{2n+1} p_{\tilde{J}}(n) \left(F, H^B_{n,j}\right)^2_{\mathrm{L}^2(B_{\mathrm{int}})} < +\infty,
\end{aligned}$$

$J \geq \tilde{J}$, where we use the abbreviation

$$p_J(n) = \begin{cases} 0, & \text{if } \Phi^\wedge_J(n) = 0 \\ \left(\Phi^\wedge_J(n)\right)^{-2}, & \text{if } \Phi^\wedge_J(n) \neq 0 \end{cases},$$

is uniformly convergent with respect to $J \geq \tilde{J}$. Hence,

$$\begin{aligned}
\lim_{\substack{J \to \infty \\ J \geq \tilde{J}}} \|F\|^2_{\mathcal{H}_J} &= \sum_{n=0}^{\infty} \sum_{j=1}^{2n+1} \left(\lim_{J \to \infty} p_J(n)\right) \left(F, H^B_{n,j}\right)^2_{\mathrm{L}^2(B_{\mathrm{int}})} \\
&= \sum_{n=0}^{\infty} \sum_{j=1}^{2n+1} (\varphi(n))^{-2} \left(F, H^B_{n,j}\right)^2_{\mathrm{L}^2(B_{\mathrm{int}})} \\
&= \|F\|^2_{\mathcal{H}_\infty}.
\end{aligned}$$

■

Lemma 4.3 *Every $\mathcal{H}_J^*$, $J \in \mathbb{N}$, is dense in $\mathcal{H}_0^*$. Moreover, $\mathcal{H}_\infty^*$ is also dense in $\mathcal{H}_0^*$.*

Proof: Let $\mathcal{T} \in \mathcal{H}_0^*$ be given. Due to the Riesz representation theorem there exists a function $F_\mathcal{T} \in \mathcal{H}_0 \subset \mathcal{H}_\infty$ such that $\mathcal{T}G = (F_\mathcal{T}, G)_{\mathcal{H}_0}$ for all $G \in \mathcal{H}_0$. Now let

$$\tilde{F}_\mathcal{T}^{(N)} := \sum_{\substack{n=0\\ \Phi_0^\wedge(n)\neq 0}}^{N} \sum_{j=1}^{2n+1} \left(\frac{\varphi(n)}{\Phi_0^\wedge(n)}\right)^2 \left(F_\mathcal{T}, H_{n,j}^B\right)_{\mathrm{L}^2(B_{\mathrm{int}})} H_{n,j}^B \in \mathcal{H}_0$$

be given for all $N \in \mathbb{N}$. Then we obtain

$$\begin{aligned}
\left|\left(\tilde{F}_\mathcal{T}^{(N)}, G\right)_{\mathcal{H}_\infty}\right| &= \left|\sum_{\substack{n=0\\ \Phi_0^\wedge(n)\neq 0}}^{N} \sum_{j=1}^{2n+1} \frac{1}{(\Phi_0^\wedge(n))^2} \left(F_\mathcal{T}, H_{n,j}^B\right)_{\mathrm{L}^2(B_{\mathrm{int}})} \left(G, H_{n,j}^B\right)_{\mathrm{L}^2(B_{\mathrm{int}})}\right| \\
&\leq \left(\sum_{\substack{n=0\\ \Phi_0^\wedge(n)\neq 0}}^{N} \sum_{j=1}^{2n+1} \frac{(\varphi(n))^2}{(\Phi_0^\wedge(n))^4} \left(F_\mathcal{T}, H_{n,j}^B\right)^2_{\mathrm{L}^2(B_{\mathrm{int}})}\right)^{\frac{1}{2}} \left(\sum_{n=0}^{\infty} \sum_{j=1}^{2n+1} \frac{1}{(\varphi(n))^2} \left(G, H_{n,j}^B\right)^2_{\mathrm{L}^2(B_{\mathrm{int}})}\right)^{\frac{1}{2}}
\end{aligned}$$

for all $G \in \mathcal{H}_\infty$. Hence, the operators

$$\begin{aligned}
\mathcal{T}^{(N)} : \mathcal{H}_\infty &\to \mathbb{R} \\
G &\mapsto \left(\tilde{F}_\mathcal{T}^{(N)}, G\right)_{\mathcal{H}_\infty},
\end{aligned}$$

$N \in \mathbb{N}$, are linear and bounded and satisfy

$$\begin{aligned}
\left\|\mathcal{T} - \mathcal{T}^{(N)}\right\|_{\mathcal{H}_0^*} &= \sup_{\substack{G\in\mathcal{H}_0\\ G\neq 0}} \frac{\left|\left(\mathcal{T} - \mathcal{T}^{(N)}\right) G\right|}{\|G\|_{\mathcal{H}_0}} \\
&= \sup_{\substack{G\in\mathcal{H}_0\\ G\neq 0}} \left| \sum_{\substack{n=0\\ \Phi_0^\wedge(n)\neq 0}}^{\infty} \sum_{j=1}^{2n+1} \left(\Phi_0^\wedge(n)\right)^{-2} \left(F_\mathcal{T}, H_{n,j}^B\right)_{\mathrm{L}^2(B_{\mathrm{int}})} \left(G, H_{n,j}^B\right)_{\mathrm{L}^2(B_{\mathrm{int}})} \right. \\
&\qquad \left. - \sum_{\substack{n=0\\ \Phi_0^\wedge(n)\neq 0}}^{N} \sum_{j=1}^{2n+1} \left(\Phi_0^\wedge(n)\right)^{-2} \left(F_\mathcal{T}, H_{n,j}^B\right)_{\mathrm{L}^2(B_{\mathrm{int}})} \left(G, H_{n,j}^B\right)_{\mathrm{L}^2(B_{\mathrm{int}})} \right| \frac{1}{\|G\|_{\mathcal{H}_0}} \\
&\leq \sup_{\substack{G\in\mathcal{H}_0\\ G\neq 0}} \left(\sum_{\substack{n=N+1\\ \Phi_0^\wedge(n)\neq 0}}^{\infty} \sum_{j=1}^{2n+1} \left(\Phi_0^\wedge(n)\right)^{-2} \left(F_\mathcal{T}, H_{n,j}^B\right)^2_{\mathrm{L}^2(B_{\mathrm{int}})} \right)^{\frac{1}{2}} \\
&\qquad \cdot \left(\sum_{\substack{n=N+1\\ \Phi_0^\wedge(n)\neq 0}}^{\infty} \sum_{j=1}^{2n+1} \left(\Phi_0^\wedge(n)\right)^{-2} \left(G, H_{n,j}^B\right)^2_{\mathrm{L}^2(B_{\mathrm{int}})} \right)^{\frac{1}{2}} \frac{1}{\|G\|_{\mathcal{H}_0}}
\end{aligned}$$

$$\leq \left(\sum_{\substack{n=N+1 \\ \Phi_0^\wedge(n)\neq 0}}^{\infty} \sum_{j=1}^{2n+1} \left(\Phi_0^\wedge(n)\right)^{-2} \left(F_{\mathcal{T}}, H^B_{n,j}\right)^2_{\mathrm{L}^2(B_{\mathrm{int}})} \right)^{\frac{1}{2}} .$$

Consequently, $\lim_{N\to\infty} \|\mathcal{T} - \mathcal{T}^{(N)}\|_{\mathcal{H}_0^*} = 0$ where $\mathcal{T}^{(N)} \in \mathcal{H}_\infty^*$ for every $N \in \mathbb{N}$. ■

Now we can prove a convergence theorem for the multiresolution spline method.

Theorem 4.4 (Convergence Theorem) *Let $F \in \bigcup_{J\in\mathbb{N}_0} \mathcal{H}_J$ be a given function, $\mathcal{F} = \{\mathcal{F}_i\}_{i\in\mathbb{N}}$ be a linearly independent system of linear and continuous functionals in $\mathcal{H}_\infty^*$ such that* $\mathrm{span}\{\mathcal{F}_i\}_{i\in\mathbb{N}}$ *is dense in $\mathcal{H}_\infty^*$. Let there exist subsystems $\mathcal{F}^{(N_J)} := \{\mathcal{F}_1, ..., \mathcal{F}_{N_J}\} \subset \mathcal{F}$, $J \in \mathbb{N}_0$, with $N_J \leq N_{J+1}$ for all $J \in \mathbb{N}_0$ and $\lim_{J\to\infty} N_J = \infty$. Moreover, let the sequence of spline–scaling functions $(S_J)_{J\in\mathbb{N}_0}$ be given by*

$$\begin{aligned} S_J &\in \mathcal{V}_J = \mathrm{Spline}\left(\{\Phi_J^\wedge(n)\}; \mathcal{F}^{(N_J)}\right) \\ \mathcal{F}_i S_J &= \mathcal{F}_i F \text{ for all } i = 1, ..., N_J. \end{aligned} \tag{11}$$

Then

$$\lim_{J\to\infty} \|S_J - F\|_{\mathcal{H}_\infty} = 0.$$

Proof: 1) Let $\mathcal{T} \in \mathcal{H}_0^*$ and $\varepsilon > 0$ be arbitrary. Due to Lemma 4.3 there exists $\tilde{\mathcal{T}} \in \mathcal{H}_\infty^*$ with $\|\mathcal{T} - \tilde{\mathcal{T}}\|_{\mathcal{H}_0^*} < \frac{\varepsilon}{2}$. For $\tilde{\mathcal{T}}$ there exists a linear combination $\sum_{i=1}^{\tilde{N}} c_i \mathcal{F}_i$ with

$$\left\| \tilde{\mathcal{T}} - \sum_{i=1}^{\tilde{N}} c_i \mathcal{F}_i \right\|_{\mathcal{H}_\infty^*} < \frac{\varepsilon}{2}.$$

Thus,

$$\begin{aligned} \left\| \mathcal{T} - \sum_{i=1}^{\tilde{N}} c_i \mathcal{F}_i \right\|_{\mathcal{H}_0^*} &\leq \left\| \mathcal{T} - \tilde{\mathcal{T}} \right\|_{\mathcal{H}_0^*} + \left\| \tilde{\mathcal{T}} - \sum_{i=1}^{\tilde{N}} c_i \mathcal{F}_i \right\|_{\mathcal{H}_0^*} \\ &\leq \left\| \mathcal{T} - \tilde{\mathcal{T}} \right\|_{\mathcal{H}_0^*} + \left\| \tilde{\mathcal{T}} - \sum_{i=1}^{\tilde{N}} c_i \mathcal{F}_i \right\|_{\mathcal{H}_\infty^*} \\ &< \varepsilon. \end{aligned}$$

Consequently, $\mathrm{span}\{\mathcal{F}_i\}_{i\in\mathbb{N}}$ is also dense in $\mathcal{H}_0^*$.

2) We prove the property for $F \in \mathcal{H}_{\tilde{J}}$ for an arbitrary but fixed scale $\tilde{J} \in \mathbb{N}_0$, see [9], pp. 128 for a related spherical case without multiresolution.

2a) We prove the weak convergence of S_J to F with respect to $(.,.)_{\mathcal{H}_\infty}$. Let $\mathcal{T} \in \mathcal{H}_\infty^*$ be arbitrary. We have to show that $\mathcal{T} S_J \to \mathcal{T} F$ for $J \to \infty$. Let $\varepsilon > 0$ be given. Since $\mathrm{span}\{\mathcal{F}_i\}_{i\in\mathbb{N}}$ is dense in $\mathcal{H}_\infty^*$, there exists a finite linear combination $\tilde{\mathcal{T}} = \sum_{i=1}^N b_i \mathcal{F}_i \in \mathcal{H}_\infty^*$ ($\subset \mathcal{H}_J^*$ for all $J \in \mathbb{N}_0$) such that $\|\mathcal{T} - \tilde{\mathcal{T}}\|_{\mathcal{H}_\infty^*} \leq \varepsilon$. Now let J_0 be sufficiently large such that $N_{J_0} \geq N$ and $J_0 \geq \tilde{J}$. Then we have for all $J \geq J_0$ due to (11)

$$\tilde{\mathcal{T}} S_J = \sum_{i=1}^N b_i \mathcal{F}_i S_J = \sum_{i=1}^N b_i \mathcal{F}_i F = \tilde{\mathcal{T}} F.$$

Hence, Lemma 2.8 and the Cauchy–Schwarz inequality imply that

$$\begin{aligned} |\mathcal{T} S_J - \mathcal{T} F| &= \left| \mathcal{T} S_J - \tilde{\mathcal{T}} S_J - \left(\mathcal{T} F - \tilde{\mathcal{T}} F \right) \right| \\ &= \left| \left(S_J, \left(\mathcal{T} - \tilde{\mathcal{T}} \right)_x K_{\mathcal{H}_J}(x,.) \right)_{\mathcal{H}_J} - \left(F, \left(\mathcal{T} - \tilde{\mathcal{T}} \right)_x K_{\mathcal{H}_J}(x,.) \right)_{\mathcal{H}_J} \right| \\ &\leq \left(\|S_J\|_{\mathcal{H}_J} + \|F\|_{\mathcal{H}_J} \right) \left\| \left(\mathcal{T} - \tilde{\mathcal{T}} \right)_x K_{\mathcal{H}_J}(x,.) \right\|_{\mathcal{H}_J} \end{aligned}$$

for all $J \geq J_0$. Note that

$$\begin{aligned} \left\| \left(\mathcal{T} - \tilde{\mathcal{T}} \right)_x K_{\mathcal{H}_J}(x,.) \right\|_{\mathcal{H}_J}^2 &= \left(\left(\mathcal{T} - \tilde{\mathcal{T}} \right)_x K_{\mathcal{H}_J}(x,.), \left(\mathcal{T} - \tilde{\mathcal{T}} \right)_x K_{\mathcal{H}_J}(x,.) \right)_{\mathcal{H}_J} \\ &= \left(\mathcal{T} - \tilde{\mathcal{T}} \right)_y \left(\mathcal{T} - \tilde{\mathcal{T}} \right)_x K_{\mathcal{H}_J}(x,y) \\ &\leq \left\| \mathcal{T} - \tilde{\mathcal{T}} \right\|_{\mathcal{H}_J^*} \left\| \left(\mathcal{T} - \tilde{\mathcal{T}} \right)_x K_{\mathcal{H}_J}(x,.) \right\|_{\mathcal{H}_J} \end{aligned}$$

such that

$$\left\| \left(\mathcal{T} - \tilde{\mathcal{T}} \right)_x K_{\mathcal{H}_J}(x,.) \right\|_{\mathcal{H}_J} \leq \left\| \mathcal{T} - \tilde{\mathcal{T}} \right\|_{\mathcal{H}_J^*} \leq \left\| \mathcal{T} - \tilde{\mathcal{T}} \right\|_{\mathcal{H}_\infty^*}.$$

We conclude from the 1st minimum property that for all $J \geq J_0$

$$|\mathcal{T} S_J - \mathcal{T} F| \leq 2\|F\|_{\mathcal{H}_J} \left\| \mathcal{T} - \tilde{\mathcal{T}} \right\|_{\mathcal{H}_\infty^*} \leq 2\|F\|_{\mathcal{H}_{\tilde{J}}}\, \varepsilon.$$

Hence, $\lim_{J\to\infty} \mathcal{T} S_J = \mathcal{T} F$ for all $\mathcal{T} \in \mathcal{H}_\infty^*$.

2b) We prove that $\lim_{J\to\infty} \|S_J\|_{\mathcal{H}_\infty} = \|F\|_{\mathcal{H}_\infty}$. Without loss of generality we may assume that $F \neq 0$ since otherwise $\|S_J\|_{\mathcal{H}_J} \leq \|F\|_{\mathcal{H}_J} = 0$ implies that $S_J = 0$ for all J ($\geq \tilde{J}$). Without loss of generality we may further assume that a given value $\varepsilon > 0$ is sufficiently small such that $0 < \varepsilon < \|F\|_{\mathcal{H}_{\tilde{J}}}$. According to a well–known fact from functional analysis (see, for example, [22], p. 91) we have

$$\|F\|_{\mathcal{H}_\infty} = \sup_{\substack{\mathcal{T} \in \operatorname{span}\{\mathcal{F}_i\}_{i\in\mathbb{N}} \\ \|\mathcal{T}\|_{\mathcal{H}_\infty^*} \leq 1}} |\mathcal{T} F|.$$

We, therefore, find $\tilde{\mathcal{T}} = \sum_{i=1}^N d_i \mathcal{F}_i \in \mathcal{H}_\infty^*$ ($\subset \mathcal{H}_J^*$ for all $J \in \mathbb{N}_0$) with $\|\tilde{\mathcal{T}}\|_{\mathcal{H}_\infty^*} \leq 1$ and $\|F\|_{\mathcal{H}_\infty} \leq |\tilde{\mathcal{T}} F| + \varepsilon$. Moreover, due to Lemma 4.2 there exists $J_0 \geq \tilde{J}$ such that for all $J \geq J_0$ the inequality $0 \leq \|F\|_{\mathcal{H}_J} - \|F\|_{\mathcal{H}_\infty} \leq \varepsilon$ holds. Thus, if we choose J_1 such that $J_1 \geq J_0$ and $N_{J_1} \geq N$ we obtain for all $J \geq J_1$ the result

$$\begin{aligned} \|F\|_{\mathcal{H}_\infty} &\leq \left| \tilde{\mathcal{T}} F \right| + \varepsilon = \left| \tilde{\mathcal{T}} S_J \right| + \varepsilon \leq \left\| \tilde{\mathcal{T}} \right\|_{\mathcal{H}_\infty^*} \|S_J\|_{\mathcal{H}_\infty} + \varepsilon \\ &\leq \|S_J\|_{\mathcal{H}_\infty} + \varepsilon \leq \|S_J\|_{\mathcal{H}_J} + \varepsilon \leq \|F\|_{\mathcal{H}_J} + \varepsilon \leq \|F\|_{\mathcal{H}_\infty} + 2\varepsilon \end{aligned}$$

such that

$$\lim_{J\to\infty} \|S_J\|_{\mathcal{H}_\infty} = \|F\|_{\mathcal{H}_\infty}.$$

2c) Combining 2a) and 2b) we see that

$$\lim_{J\to\infty} \|S_J - F\|_{\mathcal{H}_\infty} = 0$$

for all $F \in \bigcup_{J\in\mathbb{N}_0} \mathcal{H}_J$. ■

Note that it is enough to find a sufficiently large $\tilde{J}$ such that $F \in \mathcal{H}_{\tilde{J}}$, where $\bigcup_{J\in\mathbb{N}_0} \mathcal{H}_J$ is dense in $\mathcal{H}_\infty$. Thus, we have in practice a large enough set of approximatable functions.

5 Conclusions

A multiscale spline interpolation method for harmonic functions on the 3–dimensional ball was introduced. We discussed, in particular, its applicability to an exponentially ill–posed problem in geophysics. With this tool we are enabled to determine approximations to the harmonic mass density distribution of the Earth out of gravitational information given outside the Earth.
Essential advantages of this approach are the following features: The method represents a regularization, i.e. every spline continuously depends on the given pointwise information. Moreover, due to the development of a spline–wavelet method we obtain a multiresolution analysis of Sobolev spaces. Each spline is the smoothest among all Sobolev space elements that produce the same gravitational data. Furthermore, the method can be better adapted to the data situation than present wavelet methods, since the data need not be located on a sphere (as approximation to the satellite orbit) any more and different types of data, in particular, different derivatives, can be combined. In addition, the resolution of the obtained solution can locally be varied by increasing the data points in the corresponding area. The larger the scale of the Sobolev space, the smaller is the hat–width of the basis function. Thus, the scale can be chosen in accordance to the density of the data grid. Finally, a new convergence theorem was proved.
In future research numerical aspects should be discussed in detail. Moreover, the method should also be extendable to vectorial and tensorial data since for various cases singular value decompositions of the involved operators are known (see, for example, [18, 19]). Moreover, it should be noted that the method basically only requires a complete orthonormal system in a Hilbert space and, therefore, appears to be applicable to various other problems.

6 Acknowledgments

The authors gratefully acknowledge the support by the Forschungsschwerpunkt Mathematik und Praxis and the DFG–Graduiertenkolleg Mathematik und Praxis at the University of Kaiserslautern. Moreover, the members of the Geomathematics Group at the Department of Mathematics of the University of Kaiserslautern deserve special thanks for valuable discussions.

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TECHNISCHE UNIVERSITÄT KAISERSLAUTERN

Informationen:

Prof. Dr. W. Freeden
Prof. Dr. E. Schock
Fachbereich Mathematik
Technische Universität Kaiserslautern
Postfach 3049
D-67653 Kaiserslautern
E-Mail: freeden@mathematik.uni-kl.de
schock@mathematik.uni-kl.de