An Improved ε-Constraint Method for Multiobjective Programming

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Abstract

In this paper we revisit one of the most important scalarization techniques used in multiobjective programming, the ε-constraint method. We summarize the method and present some criticism, namely the lack of results on properly efficient solutions and computational difficulties. We present two modifications that address this criticism. The improved ε-constraint method we propose combines both modifications.

Keywords: Multiobjective programming, scalarization, properly efficient solution, ε-constraint method.

1 Introduction

1.1 Basics

Multiobjective programming is a part of mathematical programming dealing with decision problems characterized by multiple and conflicting objective functions that are to be optimized over a feasible set of decisions. Such problems, referred to as multiobjective programs (MOPs), are commonly encountered in many areas of human activity including engineering, management, and others.

More precisely, let $\mathbb{R}^n$ and $\mathbb{R}^p$ be Euclidean vector spaces referred to as the decision space and the objective space. Let $X \subset \mathbb{R}^n$ be a non-emtpy and compact feasible set

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and let f be a vector-valued objective function $f : \mathbb{R}^n \to \mathbb{R}^p$ composed of p real-valued continuous objective functions, $f = (f_1, \ldots, f_p)$, where $f_k : \mathbb{R}^n \to \mathbb{R}$ for $k = 1, \ldots, p$. A multiobjective program (MOP) is given by

$$\begin{aligned} &\min\ (f_1(x), \ldots, f_p(x)) \\ &\qquad\qquad \text{subject to } x \ \in \ X. \end{aligned} \tag{MOP}$$

We usually assume that the set X is given implicitly in the form of constraints, i.e., $X := \{x \subseteq \mathbb{R}^n : g_j(x) \leq 0, j = 1, \ldots, l; h_j(x) = 0, j = 1, \ldots, m\}$. We define the set of all attainable points or objective vectors for all feasible solutions $x \in X$ in the objective space, $Y := f(X) \subset \mathbb{R}^p$.

The symbol “min” in the MOP is generally understood as finding optimal or preferred outcomes in Y and their pre-images in X, where the preference between the criterion vectors results from a binary relation defined on Y. The following notation is used to define binary relations on $\mathbb{R}^p$. For $y, y' \in \mathbb{R}^p$

- $y < y'$ denotes $y_k < y'_k$ for all $k = 1, \ldots, p$,
- $y \leqq y'$ denotes $y_k \leq y'_k$ for all $k = 1, \ldots, p$,
- and $y \leq y'$ denotes $y \leqq y'$ but $y \neq y'$.

The so-called Pareto concept of optimality is based on these binary relations and can be defined as in Definition 1. Let $\mathbb{R}^p_{\geqq} := \{y \in \mathbb{R}^p : y \geqq 0\}$.

Definition 1 *Consider the MOP. A point $x \in X$ is called*

1. *a weakly efficient solution if there is no $x' \in X$ such that $f(x') < f(x)$;*
2. *an efficient solution if there is no $x' \in X$ such that $f(x') \leq f(x)$;*
3. *a strictly efficient solution if there is no $x' \in X$, $x' \neq x$, such that $f(x') \leqq f(x)$.*

We denote the sets of weakly efficient solutions, efficient solutions, and strictly efficient solutions by X_{wE}, X_E, X_{sE}, respectively. We call their images weakly nondominated (weak Pareto) points and nondominated (Pareto) points, respectively. The latter are denoted by Y_{wN}, Y_N. Note that strictly efficient solutions correspond to unique efficient solutions, and therefore they do not have a counterpart in the objective space.

Additionally, Geoffrion (1968) (among many others) defines properly efficient solutions.

Definition 2 *A feasible solution $x \in X$ is said to be a properly efficient solution of the MOP if it is efficient and if there exists a scalar $M > 0$ such that for all $i, 1 \leq i \leq p$, and each $x' \in X$ satisfying $f_i(x') < f_i(x)$, there exists at least one $j, 1 \leq j \leq p$, such that $f_j(x') > f_j(x)$ and $(f_i(x) - f_i(x'))/(f_j(x') - f_j(x)) \leq M$.*

The set of all properly efficient solutions and properly nondominated outcomes (in the sense of Geoffrion) are denoted by X_{pE} and Y_{pN}.

The traditional approach to solving MOPs is by scalarization which involves formulating an MOP-related single objective program (SOP) by means of a real-valued scalarizing function typically being a function of the objective functions of the MOP, auxiliary scalar or vector variables, and/or scalar or vector parameters. Sometimes the feasible set of the MOP is additionally restricted by new constraint functions related to the objective functions of the MOP and/or the new variables introduced. For a survey on scalarizing (and non-scalarizing) techniques, the reader is referred to Ehrgott and Wiecek (2004).

In the following section we review a well-known scalarization technique, the so-called ε-constraint method, and list related theoretical results. This method has some practical and theoretical drawbacks which are discussed in Section 1.3. In the two subsequent sections we propose two modifications of the ε-constraint method as "measures of remedy" with respect to the main two drawbacks of the ε-constraint method. In Section 4 a combination of the modifications proposed in Section 2 and Section 3 is presented as the improved ε-constraint method. We present some conclusions in Section 5.

1.2 The ε-constraint Method

Chankong and Haimes (1983) propose the *ε-constraint method.* It is based on a scalarization where one of the objective functions is minimized while all the other objective functions are bounded from above by means of additional constraints.

$$\begin{aligned} \min f_k(x) & & \\ \text{subject to } f_i(x) &\leq \varepsilon_i \quad i \neq k \\ x &\in X, \end{aligned} \tag{$P_{\varepsilon_{-k}}$}$$

where $\varepsilon_{-k} = (\varepsilon_1, \ldots, \varepsilon_{k-1}, \varepsilon_{k+1}, \ldots, \varepsilon_p)^T \in \mathbb{R}^{p-1}$ and $k \in \{1, \ldots, p\}$. We shall denote the feasible set of the ε-constraint problem $P_{\varepsilon_{-k}}$ by

$$X_k^\varepsilon := \{x \in X : f_i(x) \leq \varepsilon_i, i \neq k\}.$$

Throughout this article, we assume that ε_{-k} is always chosen such that $P_{\varepsilon_{-k}}$ (or its modifactions) are feasible, i.e. $X_k^\varepsilon \neq \emptyset$.

Below we state the two main theorems about the ε-constraint method, see Chankong and Haimes (1983): Optimal solutions of $P_{\varepsilon_{-k}}$ are weakly-efficient and unique optimal solutions of $P_{\varepsilon_{-k}}$ are stictly efficient and efficient solutions of the MOP can be characterized using the ε-constraint method.

Theorem 1 *For any $\varepsilon_{-k} \in \mathbb{R}^{p-1}$ the following statements hold.*

- *If $\hat{x} \in X$ is an optimal solution of $P_{\varepsilon_{-k}}$, then $\hat{x} \in X_{wE}$.*

- *If $\hat{x} \in X$ is a unique optimal solution of $P_{\varepsilon_{-k}}$, then $\hat{x} \in X_{sE}$.*

Theorem 2 *A solution $\hat{x} \in X$ is efficient if and only if it is an optimal solution of $P_{\varepsilon_{-k}}$ for every $k = 1, \ldots, p$, where $\varepsilon_i = f_i(\hat{x})$ for $i = 1, \ldots, p,\ i \neq k$.*

Thus, given a feasible solution of the MOP, checking this solution for efficiency can be done by checking for uniqueness of the solution in $P_{\varepsilon_{-k}}$ for some k or by solving p single-objective optimization problems $P_{\varepsilon_{-k}}, k = 1, \ldots, p$.

1.3 Criticism

The results stated in the previous section clearly show the main advantage of the ε-constraint problem: Every efficient solution can be found as optimal solution of some $P\varepsilon_{-k}$. This result is independent of the structure of X, i.e., it is also true for nonconvex and discrete optimization problems. This fact is very important, as it distinguishes the method from the weighted sum method of minimizing a convex combination of the objectives of the MOP: The latter is restricted to convex MOPs.

Multiobjective programming is a field of optimization driven by applications. In many real world decision making situations there is a need to consider multiple conflicting objectives. The bibliography by White (1990), and Chapters 20 – 23 in Figueira *et al.* (2004) provide ample evidence of this fact. Consequently, there is a need for efficient and effective techniques to solve multiobjective programmes in many areas. If a solution method is based on scalarization it is usually necessary to solve the scalarized problem repeatedly. It is therefore important to ask whether it can be solved with acceptable computational effort.

Ehrgott and Ryan (2002) report on a bicriteria set partitioning model in airline crew scheduling. They show that the problem $P_{\varepsilon_{-k}}$ cannot be solved within acceptable time, despite the use of sophisticated methods to solve single objective set partitioning problems. They mention that the addition of one ε-constraint (a knapsack type cutting plane) destroys the polyhedral structure of X and renders the efficient set partitioning techniques ineffective. Moreover, the results indicate that the bounds on objective values "misleads" the column generation subproblem into generating columns that appear beneficial, but result in a large number of iterations with little progress towards an optimal integer solution.

Another issue is related to properly efficient solutions. In practical situations, decision makers are usually interested in properly efficient solutions rather than just efficient ones. These solutions are characterized by finite trade-offs (or rates of substitution) between the objectives, i.e., an increase in one objective can be compensated by a finite decrease in another one. While all optimal solutions of a weighted sum problem with positive weights are always properly efficient (Geoffrion, 1968), the traditional ε-constraint method does not provide results on proper efficiency of optimal solutions. There are, however, results that relate proper efficiency to stability of P_{ε_k} and Karush-Kuhn-Tucker conditions for P_{ε_k}, see (Miettinen, 1999, pp 89-91) and references therein.

In Section 2 we address the second issue: By inclusion of slack variables that measure the gap between objective values $f_i(x)$ and bounds ε_i, $i \neq k$, we obtain statements on proper efficiency in a natural way. In Section 3 we present a modification that overcomes the computational difficulties associated with the ε-constraint method by relaxing the constraints $f_i(x) \leq \varepsilon_i$ and a penalty function type approach.

2 Modification for Proper Efficiency

A first modification of the ε-constraint method utilizes nonnegative slack variables $s_i, i \neq k$, which are added to the ε-constraints. The objective function equals the sum of the k^{th} objective function and the negative weighted sum of the slack variables. The scalarized problem $P^+_{\varepsilon_{-k}}$ can be formulated as

$$\begin{aligned} \min f_k(x) - \sum_{i \neq k} \lambda_i s_i & & & \\ \text{subject to } f_i(x) + s_i &\leq & \varepsilon_i & \quad i \neq k \\ s_i &\geq & 0 & \quad i \neq k \\ x &\in & X, & \end{aligned} \qquad \left(P^+_{\varepsilon_{-k}}\right)$$

where $\varepsilon_{-k} = (\varepsilon_1, \ldots, \varepsilon_{k-1}, \varepsilon_{k+1}, \ldots, \varepsilon_p)^T \in \mathbb{R}^{p-1}$ and $\lambda_i \geq 0, i \neq k$, are nonnegative weights.

As shown in the following, these slack variables provide information about proper efficiency of a solution. Thus, this first modification can be seen as a measure of remedy for one of the major drawbacks of the ε-constraint method. A first difference to the ε-constraint method is that the ε-constraints are always active at optimality.

Lemma 1 *Let $\lambda \geqq 0$. Then $P^+_{\varepsilon_{-k}}$ has an optimal solution $(\hat{x}, \hat{s})$ such that $f_i(\hat{x}) + \hat{s}_i = \varepsilon_i$ for all $i \neq k$. If $\lambda > 0$ then every optimal solution of $P^+_{\varepsilon_{-k}}$ satisfies $f_i(x) + \hat{s}_i = \varepsilon_i$ for all $i \neq k$.*

Proof:
Assume that $(\hat{x}, \hat{s})$ is an optimal solution of $P^+_{\varepsilon_{-k}}$, but there is some $j \in \{1, \ldots, p\} \setminus \{k\}$ such that $f_j(\hat{x}) + \hat{s}_j < \varepsilon_j$. Then for $\delta = \varepsilon_j - f_j(\hat{x}) > 0$ define

$$\tilde{s}_i := \begin{cases} \hat{s}_i & \text{for } i \in \{1, \ldots, p\} \setminus \{j, k\} \\ \hat{s}_i + \delta & \text{for } i = j. \end{cases}$$

$(\hat{x}, \tilde{s})$ is feasible for $P^+_{\varepsilon_{-k}}$. Since $\tilde{s}_j > \hat{s}_j$, we have

$$f_k(\hat{x}) - \sum_{i \neq k} \lambda_i \tilde{s}_i \leq f_k(\hat{x}) - \sum_{i \neq k} \lambda_i \hat{s}_i.$$

That means $(\hat{x}, \tilde{s})$ yields a better objective function value for $P^+_{\varepsilon_{-k}}$ than $(\hat{x}, \hat{s})$ (if $\lambda_j > 0$) or the same as $(\hat{x}, \hat{s})$ (if $\lambda_j = 0$). □

Before we start analyzing this modification theoretically, we want to gain some insight into the way this method works. Looking at the special case of $p = 2$ and exploiting Lemma 1 provides the tools for a visualization in the objective space. Consider an optimal solution $(\hat{x}, \hat{s}_1) \in \mathbb{R}^2 \times \mathbb{R}$ for a problem $P^+_{\varepsilon_{-2}}$.

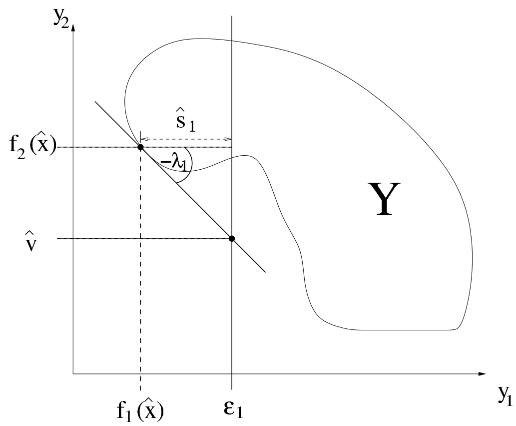

Figure 1: The modified ε-constraint method for a bicriteria example.

Note that the feasible set of $P^+_{\varepsilon_{-2}}$ (in x variables) equals a subset of the feasible set of the BOP, namely X_2^{ε}. For a given feasible solution $(\bar{x}, \bar{s}_1)$, let

$$v = f_2(\bar{x}) - \lambda_1 \bar{s}_1$$

denote the objective function value associated with $(\bar{x}, \bar{s}_1)$. The level set of $v = f_2(x) - \lambda_1 s_1$ can be interpreted as a line in the s_1-f_2-space with slope $-\lambda_1$ passing through $(\bar{s}_1, f_2(\bar{x}))$.

Let $\hat{v} = f_2(\hat{x}) - \lambda_1 \hat{s}_1$ denote the optimal objective function value of $P^+_{\varepsilon_{-2}}$. By Lemma 1, the ε-constraints hold with equality at optimality, so $\hat{s}_1 = \varepsilon_1 - f_1(\hat{x})$.

If $\hat{s}_1 = 0$, we have $f_1(\hat{x}) = \varepsilon_1$ and $f_2(\hat{x}) = \hat{v}$.

Let us now assume that $\hat{s}_1 \neq 0$. Substituting $\hat{s}_1$ into the objective function of P_{ε_2} gives

$$-\lambda_1 = \frac{f_2(\hat{x}) - \hat{v}}{f_1(\hat{x}) - \varepsilon_1}.$$

The scalar weight λ_1 equals the negative slope of the line through $(f_1(\hat{x}), f_2(\hat{x}))$ and $(\varepsilon_1, \hat{v})$. For a visualization see Figure 1. The feasible set of the MOP is reduced by the additional ε-constraint forcing $f_1(x) \leq \varepsilon_1$. Then, a line with slope $-\lambda_1$ is parallely translated towards the origin until it supports the restricted Pareto set. The point of support is the nondominated point $f(\hat{x})$.

Depending on the choice of the weight vectors different results for $P^+_{\varepsilon_{-k}}$ can be derived.

Let us first consider weight vectors $\lambda \geqq 0$. We will show that we obtain weakly efficient solutions. We present an example which shows that optimal solutions of $P^+_{\varepsilon_{-k}}$ are not always efficient or in other words, a stronger result for $\lambda \geqq 0$ cannot be derived. Note that, of course, $P^+_{\varepsilon_{-k}}$ with $\lambda = 0$ corresponds to the original ε-constraint problem.

Proposition 1 *Let $(\hat{x}, \hat{s})$ be an optimal solution of $P^+_{\varepsilon_{-k}}$ with $\lambda \geqq 0$. Then $\hat{x}$ is a weakly efficient solution of the MOP.*

Proof:
Let $(\hat{x}, \hat{s})$ be an optimal solution of $P^+_{\varepsilon_{-k}}$ with $\lambda \geqq 0$. Assume that there is $x \in X$ such that $f(x) < f(\hat{x})$. Then

$$f_i(x) + \hat{s}_i < f_i(\hat{x}) + \hat{s}_i \leq \varepsilon_i \text{ for } i \neq k,$$

i.e., $(x, \hat{s})$ is feasible for $P^+_{\varepsilon_{-k}}$ and $f_k(x) < f_k(\hat{x})$. This implies

$$f_k(x) - \sum_{i \neq k} \lambda_i \hat{s}_i < f_k(\hat{x}) - \sum_{i \neq k} \lambda_i \hat{s}_i,$$

a contradiction to the optimality of $(\hat{x}, \hat{s})$ for $P^+_{\varepsilon_{-k}}$. $\square$

Optimal solutions of $P^+_{\varepsilon_{-k}}$ are indeed not necessarily efficient, even if $\lambda \neq 0$, as the next example shows.

Example 1 *Consider the following set of attainable objective vectors Y of a discrete MOP:*

$$Y = \{(16, 1, 2)^T, (1, 5, 4)^T, (1, 5, 5)^T, (17, 3, 1)^T\}.$$

Suppose $k = 1$, $\lambda = (\lambda_2, \lambda_3)^T = (1, 0)^T$ and $\varepsilon_{-1} = (10, 10)^T$. Note that $Y_{wN} = Y$. Let $z(y)$ denote the optimal objective function value of $P^+_{\varepsilon_{-k}}$ for $y \in Y$. Then

$$\begin{array}{rclcl} z\big((16,1,2)^T\big) &=& 16 - 9 &=& 7 \\ z\big((1,5,4)^T\big) &=& 1 - 5 &=& -4 \\ z\big((1,5,5)^T\big) &=& 1 - 5 &=& -4 \\ z\big((17,3,1)^T\big) &=& 17 - 7 &=& 10. \end{array}$$

So, both $(1, 5, 4)^T$ and $(1, 5, 5)^T$ yield optimal objective values of $P^+_{\varepsilon_{-1}}$. However, $(1, 5, 5)^T$ is only weakly nondominated.

Proposition 2 *Let $(\hat{x}, \hat{s})$ be an optimal solution of $P^+_{\varepsilon_{-k}}$ with $\lambda \geqq 0$. If $\hat{x}$ is unique then $\hat{x}$ is a strictly efficient solution of the MOP.*

Proof:
Assume that x is such that $f_k(x) \leq f_k(\hat{x}), k = 1, \ldots, p$. Then $(x, \hat{s})$ is also a feasible solution of $P^+_{\varepsilon_{-k}}$. Since the objective function value of $(x, \hat{s})$ is not worse than that of $(\hat{x}, \hat{s})$, uniqueness of $\hat{x}$ implies that $x = \hat{x}$. □

Propositions 1 and 2 and Example 1 show that the results of the original ε-constraint method cannot be strengthened by including slack variables s_i if $\lambda \geq 0$ is allowed: Optimal solutions of the ε-constraint problem are only weakly efficient in general. However, optimal solutions of the modified ε-constraint problem with positive weights are always efficient.

Theorem 3 *Let $(\hat{x}, \hat{s})$ be an optimal solution of $P^+_{\varepsilon_{-k}}$ with $\lambda > 0$. Then $\hat{x}$ is an efficient solution of the MOP.*

Proof:
Assume, that $(\hat{x}, \hat{s})$ is feasible for $P^+_{\varepsilon_{-k}}$, but there is $x \in X$ such that $f(x) \leq f(\hat{x})$ or, in other words, $f_i(x) \leq f_i(\hat{x}), i = 1, \ldots, p$, with strict inequality for at least one i. In the following, we distinguish two cases depending on where the strict inequality holds.

Case 1: $f_k(x) < f_k(\hat{x})$. Then $(x, \hat{s})$ is feasible for $P^+_{\varepsilon_{-k}}$. Furthermore,

$$f_k(x) - \sum_{i \neq k} \lambda_i \hat{s}_i < f_k(\hat{x}) - \sum_{i \neq k} \lambda_i \hat{s}_i$$

contradicts that $(\hat{x}, \hat{s})$ is optimal for $P^+_{\varepsilon_{-k}}$.

Case 2: $f_k(x) = f_k(\hat{x})$. Then $f_j(x) < f_j(\hat{x})$ for at least one $j \in \{1, \ldots, p\} \setminus \{k\}$. Thus for some $\delta_j > 0$ (x, s) with

$$s_i := \begin{cases} \hat{s}_i & \text{for } i \in \{1, \ldots, p\} \setminus \{j, k\} \\ \hat{s}_j + \delta_j & \text{for } i = j \end{cases}$$

is feasible for $P^+_{\varepsilon_{-k}}$. Furthermore, (x, s) yields a better objective function value than $(\hat{x}, \hat{s})$:

$$\begin{aligned} f_k(x) - \sum_{i \neq k} \lambda_i s_i &= f_k(x) - \sum_{i \neq j,k} \lambda_i \hat{s}_i - \lambda_j(\hat{s}_j + \delta_j) \\ &< f_k(x) - \sum_{i \neq k} \lambda_i \hat{s}_i \\ &= f_k(\hat{x}) - \sum_{i \neq k} \lambda_i \hat{s}_i. \end{aligned}$$

□

Below we state an easy to check sufficient condition for identifying properly efficient solutions among the solutions of $P^+_{\varepsilon_{-k}}$ with positive weights.

Theorem 4 *If $(\hat{x}, \hat{s})$ is an optimal solution of $P^+_{\varepsilon_{-k}}$ with $\lambda > 0$ and $\hat{s} > 0$ then $\hat{x}$ is a properly efficient solution of the MOP.*

Proof:
Let $(\hat{x}, \hat{s})$ be an optimal solution of $P^+_{\varepsilon_{-k}}$ and $\hat{s} > 0$.

From Theorem 3 we know that $\hat{x}$ is efficient. Since, at optimality, the ε-constraints hold with equality, we can rewrite the objective function as

$$f_k(\hat{x}) - \sum_{i \neq k} \lambda_i(\varepsilon_i - f_i(\hat{x})) = \sum_{i=1}^{p} \lambda_i f_i(\hat{x}) - \sum_{i \neq k} \lambda_i \varepsilon_i,$$

where we define $\lambda_k := 1$. Since the term $\sum_{i \neq k} \lambda_i \varepsilon_i$ is constant, we can interpret $\hat{x}$ as optimal solution of a weighted sum problem with weight coefficients strictly greater than zero, which has a restricted feasible set, namely the feasible set X_k^ε of $P^+_{\varepsilon_{-k}}$. By Geoffrion's theorem (Geoffrion, 1968) $\hat{x}$ is properly efficient for the MOP with feasible set X_k^ε.

Since $\hat{s} > 0$, properly efficient solution $\hat{x}$ of this restricted MOP is also a properly efficient solution of the MOP. □

So far results about optimal solutions of $P^+_{\varepsilon_{-k}}$ have been stated. We will now analyze how (weakly / properly) efficient solutions can be obtained by appropriate choices of parameters.

It follows directly from the ε-constraint method that weakly efficient solutions can be obtained with $\lambda = 0$.

Lemma 2 *Let $\hat{x}$ be efficient. Then there exist ε, $\hat{s}$ and $\lambda \geqq 0$ such that $(\hat{x}, \hat{s})$ is an optimal solution of $P^+_{\varepsilon_{-k}}$ for all $k \in \{1, \ldots, p\}$.*

This results follows immediately from Theorem 2 choosing $\varepsilon = f(\hat{x}), \hat{s} = 0$ and $\hat{\lambda} = 0$.

Next, we will show that any properly efficient solution can be obtained as an optimal solution of $P^+_{\varepsilon_{-k}}$ with positive weights.

Theorem 5 *Let $\hat{x}$ be properly efficient. Then, for every $k \in \{1, \ldots, p\}$ there are ε, $\hat{s}$, and $\lambda > 0$, such that $(\hat{x}, \hat{s})$ is an optimal solution of $P^+_{\varepsilon_{-k}}$.*

Proof:
Let $\hat{x} \in X_{pE}$ and $k \in \{1, \ldots, p\}$. Define $\varepsilon_i := f_i(\hat{x})$ for $i \neq k$. Thus, we can choose $\hat{s} = 0$.

Suppose there is no $x \in X$ and no $i \neq k$ such that $f_i(x) < f_i(\hat{x})$. Consider any feasible solution for $P^+_{\varepsilon_{-k}}$, i.e., any $x \in X_k^\varepsilon$. Then for all $i \neq k$ we have $f_i(x) = f_i(\hat{x})$ and, since $\hat{x} \in X_E$, $f_k(x) \geq f_k(\hat{x})$. Consequently, (x, s) does not yield a better objective function value for $P^+_{\varepsilon_{-k}}$ for any choice of $\lambda > 0$.

Let us therefore assume that there is $x \in X$ and $i \neq k$ such that $f_i(x) < f_i(\hat{x})$. Since $\hat{x} \in X_{pE}$ there exists $M > 0$ such that for all i and all $x \in X$ with $f_i(x) < f_i(\hat{x})$ there is at least one j with $f_j(x) > f_j(\hat{x})$ such that $(f_i(\hat{x}) - f_i(x))/(f_j(x) - f_j(\hat{x})) \leq M$. Moreover, if we only consider points x feasible for $P^+_{\varepsilon_{-k}}$, i.e., points $x \in X_k^\varepsilon$, in the definition of proper efficiency it follows that there is only one j, namely $j = k$, such that $f_j(x) > f_j(\hat{x})$ and $(f_i(\hat{x}) - f_i(x))/(f_j(x) - f_j(\hat{x})) \leq M$. Thus, since $x \in X_{pE}$, there is $M > 0$ such that for all i and all $x \in X_k^\varepsilon$ with $f_i(x) < f_i(\hat{x})$ it holds that $f_k(x) > f_k(\hat{x})$ and $(f_k(x) - f_k(\hat{x}))/(f_i(\hat{x}) - f_i(x)) \geq 1/M$.

Define weights $\lambda_i := 1/M(p-1)$. Note that $\lambda > 0$. We claim that $(\hat{x}, \hat{s})$ is optimal for $P^+_{\varepsilon_{-k}}$. Suppose there exists a feasible (x, s) with

$$f_k(x) - \sum_{i \neq k} \lambda_i s_i < f_k(\hat{x}) - \sum_{i \neq k} \lambda_i \hat{s}_i = f_k(\hat{x}). \tag{1}$$

Furthermore, we assume without loss of generality that (x, s) is optimal for $P^+_{\varepsilon_{-k}}$, i.e.,

$$f_i(x) + s_i = \varepsilon_i = f_i(\hat{x}) \quad i \neq k.$$

Substituting s_i, $i \neq k$, in (1) yields

$$f_k(x) - f_k(\hat{x}) < \sum_{i \neq k} \lambda_i (f_i(\hat{x}) - f_i(x)). \tag{2}$$

Inequality (2) is a contradiction as we will show in the following. By the choice of the weights and the consideration above, for any $x \in X_k^\varepsilon$ and any index i with $f_i(x) < f_i(\hat{x})$, it holds that

$$\lambda_i = \frac{1}{M(p-1)} \leq \frac{1}{p-1} \frac{f_k(x) - f_k(\hat{x})}{f_i(\hat{x}) - f_i(x)} \quad i \neq k.$$

Consequently, the inequality

$$\lambda_i (f_i(\hat{x}) - f_i(x)) \leq \frac{1}{p-1} (f_k(x) - f_k(\hat{x}))$$

is valid for any $x \in X_k^\varepsilon$ and any index $i \neq k$. Summing over all $i \neq k$ yields

$$\sum_{i \neq k} \lambda_i (f_i(\hat{x}) - f_i(x)) \leq f_k(x) - f_k(\hat{x}).$$

and the contradiction is obvious. □

The following example shows that $x \in X_E \setminus X_{pE}$ cannot necessarily be obtained with positive weights.

Example 2 *Let* $p = 2$ *and* $X = \{x \in \mathbb{R}^2 : (x_1 - 1)^2 + (x_2 - 1)^2 \leq 1\}$ *with* $f(x) = x$. *Then* $(1,0)$ *and* $(0,1)$ *are efficient, but not properly efficient.*

The scalarization

$$\begin{aligned} \min x_2 - \lambda s & \\ \text{subject to } x_1 + s &\leq 1 \\ x &\in X \end{aligned}$$

is equivalent to (has the same optimal solution x *as)*

$$\min\{x_2 - \lambda(1 - x_1) : (x_1 - 1)^2 + (x_2 - 1)^2 = 1\}.$$

It is easy to see that for obtaining the point $(1,0)$*, the weight parameter* λ *has to equal zero.*

Note, however, that in order to obtain $(1,0)$*, we may also consider*

$$\begin{aligned} \min x_1 - \lambda s & \\ \textit{subject to } x_2 + s &\leq 0 \\ x &\in X. \end{aligned}$$

It is clear that $x_1 = 1, x_2 = 0, s = 0$ *is the unique solution of this problem for any* $\lambda \geq 0$.

Insertion of slack variables provides information enabling statements about proper efficiency. Nevertheless, numerical difficulties are not addressed in $P^+_{\varepsilon_{-k}}$: the structure-destroying character of the ε-constraints is not affected by the addition of slack variables. In the following section the ε-constraints are allowed to be violated thus resulting in numerical improvement.

3 Modification to Improve Computational Performance

As we have mentioned in Section 1, the problem $P_{\varepsilon_{-k}}$ may be extremely hard to solve in practice, in particular for discrete multiobjective problems. In order to address this problem, we "relax" the constraints $f_i(x) \leq \varepsilon_i$ by allowing them to be violated and penalizing any violation in the objective function. Thus, we consider the following problem

$$\begin{aligned} \min f_k(x) + \sum_{i \neq k} \mu_i s_i & \\ \text{subject to } f_i(x) - s_i &\leq \varepsilon_i \quad i \neq k \\ s_i &\geq 0 \quad i \neq k \\ x &\in X, \end{aligned} \qquad \left(P^-_{\varepsilon_{-k}}\right)$$

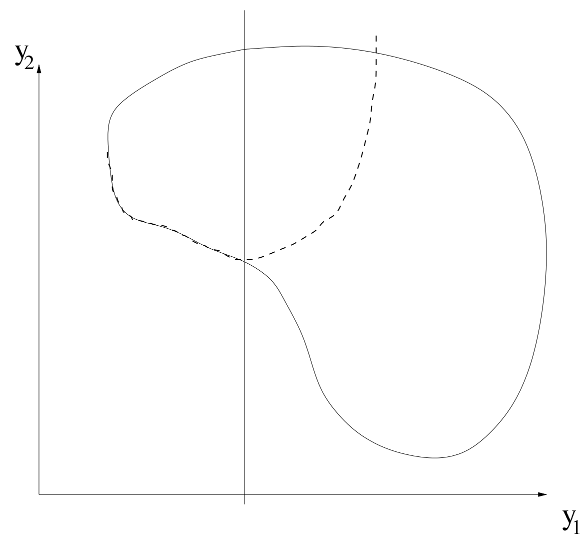

Figure 2: Feasible set and objective function of $P^{-}_{\varepsilon_{-k}}$.

where $\mu_i \geq 0, i \neq k$. The feasible set of $P^{-}_{\varepsilon_{-k}}$ in x variables is X, the feasible set of the MOP. Note that if $(\hat{x}, \hat{s})$ is an optimal solution, then we may w.l.o.g. assume $\hat{s}_i = \max\{0, \varepsilon_i - f_i(\hat{x})\}$.

In Figure 2 $P^{-}_{\varepsilon_{-2}}$ is illustrated for a bicriteria problem. The vertical line marks the value ε_1. The dotted line shows the objective function of $P^{-}_{\varepsilon_{-2}}$ as a function of component y_1 of nondominated points Y_N. The idea of the method is that, by penalizing violations of the constraint $f_1(x) \leq \varepsilon_1$, a minimum of the objective of $P^{-}_{\varepsilon_{-k}}$ is attained with the ε-constraint active when μ is chosen appropriately.

We obtain the following results:

Proposition 3 *Let $(\hat{x}, \hat{s})$ be an optimal solution of $P^{-}_{\varepsilon_{-k}}$ with $\mu \geqq 0$. Then $\hat{x}$ is a weakly efficient solution of the MOP.*

Proof:
Suppose $\hat{x}$ is not weakly efficient. Then there is some $x \in X$ such that $f_i(x) < f_i(\hat{x}), i = 1, \ldots, p$. Then $(x, \hat{s})$ is feasible for $P^{-}_{\varepsilon_{-k}}$ with an objective value that is smaller than that of $(\hat{x}, \hat{s})$. □

Under additional assumptions we get stronger results.

Proposition 4 *If $\hat{x}$ is unique in an optimal solution of $P^{-}_{\varepsilon_{-k}}$ with $\mu \geqq 0$, then $\hat{x}$ is a strictly efficient solution of the MOP.*

Proof:

Assume that x is such that $f_k(x) \leq f_k(\hat{x}), k = 1, \ldots, p$. Then $(x, \hat{s})$ is a feasible solution of $P^{-}_{\varepsilon_{-k}}$. Since the objective function value of $(x, \hat{s})$ is not worse than that of $(\hat{x}, \hat{s})$, uniqueness of $\hat{x}$ implies that $x = \hat{x}$. □

We remark that a result similar to Theorem 3 is not possible. Even if $\mu > 0$ an optimal solution of $P^{-}_{\varepsilon_{-k}}$ may be just weakly efficient, but not efficient.

Example 3 *Consider an MOP with* $X = \{(x_1, x_2) : x_1^2 + x_2^2 \leq 1\} + \mathbb{R}^2_{\geqq}$ *and* $f(x) = x$. *Let* $\varepsilon_1 > 1$. *Then* $(\hat{x}_1, \hat{x}_2, \hat{s}_1) = (\hat{x}_1, 0, 0)$ *is an optimal solution of* $P^{-}_{\varepsilon_{-2}}$ *for all* $1 \leq \hat{x}_1 \leq \varepsilon_1$. *If* $\hat{x}_1 > 1$ *this solution is weakly efficient, but not efficient. This result is independent of the choice of* μ.

The example shows that the problem here is the possible existence of weakly efficient solutions that satisfy the constraints $f_i(x) \leq \varepsilon_i, i \neq k$. If, however, all ε_i are chosen in such a way that no merely weakly efficient solution satisfies the ε-constraints, an optimal solution of $P^{-}_{\varepsilon_{-k}}$ with $\mu > 0$ will yield an efficient solution of the MOP.

Also, if $\mu > 0$, and in addition $P^{-}_{\varepsilon_{-k}}$ has an optimal solution $(\hat{x}, \hat{s})$ with $\hat{s} > 0$ we even obtain proper efficiency of $\hat{x}$.

Theorem 6 *If* $(\hat{x}, \hat{s})$ *is an optimal solution of* $P^{-}_{\varepsilon_{-k}}$ *with* $\mu > 0$ *and* $\hat{s} > 0$ *then* $\hat{x}$ *is a properly efficient solution of the MOP.*

Proof:

Observe that because $\mu > 0$ and $\hat{s} > 0$ we know that $\hat{s}_i = f_i(\hat{x}) - \varepsilon_i > 0, i \neq k$.

We first show that $\hat{x}$ is efficient. Assume there is $x \in X$ such that $f(x) \leq f(\hat{x})$. Then with s defined by $s_i := \max\{0, f_i(x) - \varepsilon_i\}$ we have that $s \leqq \hat{s}$. But from $f(x) \leq f(\hat{x})$ either $f_k(x) < f_k(\hat{x})$ or $s_i(x) < s_i(\hat{x})$ for some $i \neq k$, contradicting optimality of $(\hat{x}, \hat{s})$.

Furthermore, $\hat{s}_i = \varepsilon_i - f_i(\hat{x})$, which we can substitute in the objective function of $P^{-}_{\varepsilon_{-k}}$. This implies that $\hat{x}$ is an optimal solution of the problem

$$\begin{aligned} \min f_k(x) + \sum_{i \neq k} \mu_i(f_i(x) - \varepsilon_i) \\ \text{subject to } f_i(x) &\geq \varepsilon_i \quad i \neq k \\ x &\in X. \end{aligned}$$

Thus $\hat{x}$ is the optimal solution of a weighted sum problem with positive weights (and additional constraints). By Geoffrion's theorem (see Geoffrion (1968)) $\hat{x}$ is properly efficient for the MOP with added constraints $f_i(x) \geq \varepsilon_i, i \neq k$. However, since none of these constraints are active at the optimal solution $\hat{x}$, the condition of Definition 2 are also satisfied for the MOP, and $\hat{x}$ is a properly efficient solution of the MOP. □

We now turn to the problem of showing that (properly) efficient solutions of the MOP are optimal solutions of $P^{-}_{\varepsilon_{-k}}$ for appropriate choices of k, ε, and μ.

Lemma 3 *Let $\hat{x}$ be efficient. Then there exist $\varepsilon, \mu \geqq 0$ and $\hat{s}$ such that $(\hat{x}, \hat{s})$ is an optimal solution of $P^{-}_{\varepsilon_{-k}}$ for all $k \in \{1, \dots, p\}$.*

The lemma follows immediately from Theorem 2 by choosing $\varepsilon = f(\hat{x})$, $\hat{s} = 0$ and $\mu_i = \infty$ for all $i = 1, \dots, p$.

A more careful analysis shows that for properly efficient solutions, we can do without the infinite penalties.

Theorem 7 *Let $\hat{x}$ be a properly efficient solution of the MOP. Then, for every $k \in \{1, \dots, p\}$ there are $\varepsilon, \hat{s}, \mu^k$ with $\mu^k_i < \infty$ for all $i \neq k$ such that $(\hat{x}, \hat{s})$ is an optimal solution of $P^{-}_{\varepsilon_{-k}}$ for all $\mu \in \mathbb{R}^{p-1}$, $\mu \geqq \mu^k$.*

Proof:
We choose $\varepsilon_i := f_i(\hat{x}), i = 1, \dots, p$. Thus, we can choose $\hat{s} = 0$. Let $k \in \{1, \dots, p\}$. Because $\hat{x}$ is properly efficient there is $M > 0$ such that for all $x \in X$ with $f_k(x) < f_k(x^*)$ there is $i \neq k$ such that $f_i(x^*) < f_i(x)$ and $(f_k(x^*) - f_k(x))/(f_i(x) - f_i(x^*)) < M$.

We define μ^k by $\mu^k_i := \max(M, 0)$ for all $i \neq k$.

Let $x \in X$ and s be such that $s_i = \max\{0, f_i(x) - \varepsilon_i\} = \max\{0, f_i(x) - f_i(x^*)\} \quad i \neq k$, i.e., the smallest possible value it can take. We need to show that

$$f_k(x) + \sum_{i \neq k} \mu_i s_i \geq f_k(\hat{x}) + \sum_{i \neq k} \mu_i \hat{s}_i = f_k(\hat{x}). \tag{3}$$

First, we prove that we can assume $x \in X_E$ in (3). Otherwise there is $x' \in X_E$ with $f(x') \leq f(x)$ (note that due to our assumptions the MOP is externally stable (Sawaragi *et al.*, 1985) so that existence of x' is guaranteed) and s' with $s'_i = \max\{0, f_i(x') - \varepsilon_i\}$. Since $s' \leqq s$ we get that $f_k(x') + \sum_{i \neq k} \mu_i s'_i \leq f_k(x) + \sum_{i \neq k} \mu_i s_i$ for any $\mu \geqq 0$.

Now let $x \in X_E$. We consider the case $f_k(x) \geq f_k(\hat{x})$. Then

$$f_k(x) + \sum_{i \neq k} \mu_i s_i > f_k(\hat{x}) + 0 = f_k(\hat{x}) + \sum_{i \neq k} \mu^k_i \hat{s}_i$$

for any $\mu \geq 0$.

Now consider the case $f_k(x) < f_k(\hat{x})$ and let $I(x) := \{i \neq k : f_i(x) > f_i(\hat{x})\}$. As both $\hat{x}$ and x are efficient, $I(x) \neq \emptyset$. Furthermore, we can assume $s_i = 0$ for all $i \notin I(x), i \neq k$.

Let $i' \in I(x)$. Then

$$\begin{aligned}
f_k(x) + \sum_{i \neq k} \mu_i s_i &\geq f_k(x) + \sum_{i \neq k} \mu_i^k s_i \\
&\geq f_k(x) + \sum_{i \in I(x)} \frac{f_k(\hat{x}) - f_k(x)}{f_i(x) - f_i(\hat{x})} s_i \\
&\geq f_k(x) + \frac{f_k(\hat{x}) - f_k(x)}{f_{i'}(x) - f_{i'}(\hat{x})} s_{i'} \\
&= f_k(x) + \frac{f_k(\hat{x}) - f_k(x)}{f_{i'}(x) - f_{i'}(\hat{x})} (f_{i'}(x) - f_{i'}(\hat{x})) \\
&= f_k(\hat{x}) = f_k(\hat{x}) + \sum_{i \neq k} \mu_i \hat{s}_i.
\end{aligned}$$

This follows from $\mu_i \geq \mu_i^k$, the definition of μ_i^k, nonnegativity of all terms, $s_i = f_i(x) - f_i(\hat{x})$ for $i \in I(x)$ and $\hat{s} = 0$. □

We can also see, that for $x \in X_E \setminus X_{pE}$ finite values of μ are not sufficient.

Example 4 *Let $p = 2$ and $X = \{x \in \mathbb{R}^2 : (x_1 - 1)^2 + (x_2 - 1)^2 \leq 1\}$ with $f(x) = x$. Then $(1, 0)$ and $(0, 1)$ are efficient, but not properly efficient. The scalarization*

$$\begin{aligned}
&\min x_2 + \mu s \\
\text{subject to } x_1 - s &\leq 0 \\
x &\in X
\end{aligned}$$

is equivalent to (has the same optimal solution x as)

$$\min\{x_2 + \mu x_1 : (x_1 - 1)^2 + (x_2 - 1)^2 = 1\}.$$

It is easy to see that the unique optimal solution is given by $x_1 = 1 - \sqrt{1 - \frac{1}{\mu+1}}$ and it is necessary that $\mu \to \infty$ to get $x_1 \to 0$.

Note, however, that in order to obtain $(0, 1)$, we can also consider

$$\begin{aligned}
&\min x_1 + \mu s \\
\textit{subject to } x_2 - s &\leq 1 \\
x &\in X.
\end{aligned}$$

It is clear that $x_1 = 0, x_2 = 1, s = 0$ is an optimal solution of this problem for any $0 \leq \mu \leq \infty$.

Since constraints on objective values are relaxed in a manner similar to penalty function methods in nonlinear programming, the computational effort to solve $P^-_{\varepsilon_{-k}}$ is less than that to solve $P_{\varepsilon_{-k}}$. Ehrgott and Ryan (2002) report on the success of this appproach in their application when solving large bicriteria set partitioning problems arising in airline crew scheduling.

4 The Improved ε-constraint Method

Let us now consider a combination of the two modifications from Section 2 and 3, that is,

$$\begin{aligned} \min f_k(x) - \sum_{i \neq k} \lambda_i s_i^+ + \sum_{i \neq k} \mu_i s_i^- & & \\ \text{subject to } f_i(x) + s_i^+ - s_i^- &\leq \varepsilon_i \quad i \neq k \\ x &\in X \\ s_i^+, s_i^- &\geq 0 \quad i \neq k, \end{aligned} \qquad \left(P_{\varepsilon_{-k}}^{comb}\right)$$

where $\lambda_i, \mu_i \geq 0, i \neq k$.

Given a feasible solution $(x, s^+, s^-) \in \mathbb{R}^p \times \mathbb{R}^{p-1} \times \mathbb{R}^{p-1}$ of $P_{\varepsilon_{-k}}^{comb}$, the slack and surplus variables s_i^+ and s_i^-, $i \neq k$, might be changed simultaneously by an amount $\alpha_i \in \mathbb{R}$,

$$s_i^+ := s_i^+ + \alpha_i \quad i \neq k \tag{4}$$

$$s_i^- := s_i^- + \alpha_i \quad i \neq k \tag{5}$$

without affecting the feasibility of the i^{th} ε-constraint, $i \neq k$. We want to investigate how large α_i can be chosen without affecting the overall feasibility of the solution. For sake of simplicity, we restrict our consideration to one $i \in \{1, \ldots, p\}$.

When changing the slack and surplus variables as in (4) and (5), the objective function value is changed by $(\mu_i - \lambda_i)\alpha_i$. If $\mu_i - \lambda_i < 0$, α_i can be chosen arbitrarily large without affecting the nonnegativity restriction on s_i^+ or s_i^-, respectively. As a consequence, the objective function value decreases arbitrarily and therefore $P_{\varepsilon_{-k}}^{comb}$ is unbounded.

Let us therefore consider the case when $\mu_i - \lambda_i \geq 0$. Then, α_i can be chosen to be

$$\alpha_i := -\min\{s_i^+, s_i^-\}$$

to improve the objective function value without violating the nonnegativity constraints. In this case, s_i^+ or s_i^- equals zero after the update.

If $\mu_i - \lambda_i = 0$, changing s_i^+ or s_i^- does not affect the objective function value.

Thus, we get the following lemma.

Lemma 4 *1. If there is an $i \neq k$ such that $\mu_i - \lambda_i < 0$, $P_{\varepsilon_{-k}}^{comb}$ is unbounded.*

2. If $\mu - \lambda \geq 0$, then there is always an optimal solution of $P_{\varepsilon_{-k}}^{comb}$ such that $s_i^+ s_i^- = 0$, $i \neq k$, or in other words, there is a partition $I \dot{\cup} \bar{I}$ of $\{1, \ldots, p\} \setminus \{k\}$ such that $s_i^+ = 0$ for all $i \in I$ and $s_i^- = 0$ for all $i \in \bar{I}$.

Note that since $s_i^+ = s_i^- = 0$ if $f_i(x) = \varepsilon_i$ this partition is not necessarily unique. We shall from now on assume that $\mu - \lambda \geq 0$.

Analogously to Sections 2 and 3, we obtain the following results most of which follow directly from results of the preceeding sections.

Proposition 5 *Let $(\lambda, \mu) \geqq 0$. If $(\hat{x}, \hat{s}^+, \hat{s}^-)$ is an optimal solution of $P^{comb}_{\varepsilon_{-k}}$ then $\hat{x}$ is a weakly efficient solution of the MOP.*

The result follows from Section 2, Proposition 1 or Section 3, Proposition 3, respectively.

With the additional assumption of uniqueness of the solution, $\hat{x}$ is even strictly efficient.

Proposition 6 *Let $(\lambda, \mu) \geqq 0$. Let $(\hat{x}, \hat{s}^+, \hat{s}^-)$ be an optimal solution of $P^{comb}_{\varepsilon_{-k}}$. If $\hat{x}$ is unique then $\hat{x}$ is a strictly efficient solution of the MOP.*

In Section 2 positive weight vectors yield efficient solutions. Despite the lack of an analogon in Section 3, a similar result can again be obtained for $P^{comb}_{\varepsilon_{-k}}$.

Theorem 8 *Let $\lambda > 0$ and $\mu > 0$. Let $(\hat{x}, \hat{s}^+, \hat{s}^-)$ be an optimal solution of $P^{comb}_{\varepsilon_{-k}}$. Then $\hat{x}$ is an efficient solution of the MOP.*

Proof:
Suppose $\hat{x} \notin X_E$. Then there is $x \in X$ with $f(x) \leq f(\hat{x})$ and at least one index $j \in \{1, \ldots, p\}$ such that $f_j(x) < f_j(\hat{x})$.

Case 1: $j = k$. Then $(\hat{x}, \hat{s}^+, \hat{s}^-)$ is feasible for $P^{comb}_{\varepsilon_{-k}}$ and yields a better objective function value which contradicts the optimality of $\hat{x}$.

Case 2: $j \neq k$. Several subcases can occur:

Subcase 2.1: $f_j(x) < f_j(\hat{x}) \leq \varepsilon_j$

Then the contradiction follows from Section 2, Theorem 3.

Subcase 2.2: $\varepsilon_j \leq f_j(x) < f_j(\hat{x})$

Note that $\hat{s}_j^- > 0$. There is $\delta > 0$ such that $f_j(x) + \delta = f_j(\hat{x})$. It follows that $f_j(x) - (\hat{s}_j^- - \delta) = f_j(\hat{x}) - \hat{s}_j^- \leq \varepsilon_j$. Since $\delta = f_j(\hat{x}) - f_j(x) \leq f_j(\hat{x}) - \varepsilon_j = \hat{s}_j^-$ we can define

$$s_i^- := \begin{cases} \hat{s}_i^- - \delta & \text{for } i = j, \\ \hat{s}_i^- & \text{for } i \neq j \end{cases}$$

and $(x, \hat{s}^+, s^-)$ is feasible for $P^{comb}_{\varepsilon_{-k}}$ and yields a better objective function value than $(\hat{x}, \hat{s}^+, \hat{s}^-)$.

Subcase 2.3: $f_j(x) < \varepsilon_j < f_j(\hat{x})$

Since $\hat{x}$ is optimal, we assume without loss of generality that $\hat{s}_j^+ = 0$ and $\hat{s}_j^- > 0$. Define

$$s_i^+ := \begin{cases} \varepsilon_i - f_i(x) & \text{for } i = j, \\ \hat{s}_i^+ & \text{for } i \neq j \end{cases}$$

and

$$s_i^- := \begin{cases} 0 & \text{for } i = j, \\ \hat{s}_i^- & \text{for } i \neq j \end{cases}$$

This definition of (x, s^+, s^-) yields a better objective function value which contradicts the optimality of $(\hat{x}, \hat{s}^+, \hat{s}^-)$. □

The proof shows that in fact not necessarily all λ_j and μ_j have to be positive. $\hat{x}$ will also be efficient if $\lambda_j > 0$ for all j such that $f_j(\hat{x}) \leq \varepsilon_j$ and $\mu_j > 0$ otherwise.

Theorem 9 *Let $\lambda > 0$ and $\mu > 0$. If $(\hat{x}, \hat{s}^+, \hat{s}^-)$ is an optimal solution of $P_{\varepsilon-k}^{comb}$ and if there is a partition $I \cup \bar{I}$ of $\{1, \ldots, p\} \setminus \{k\}$ such that $\hat{s}_i^+ > 0$ for $i \in I$ and $\hat{s}_i^- > 0$ for $i \in \bar{I}$, then $\hat{x}$ is a properly efficient solution of the MOP.*

Proof:
That $\hat{x}$ is efficient follows from Theorem 8. Because $(\lambda, \mu) > 0$ we know that

$$\hat{s}_i^+ = \begin{cases} \varepsilon_i - f_i(\hat{x}) & i \in I \\ 0 & i \in \bar{I} \end{cases}$$

$$\hat{s}_i^- = \begin{cases} f_i(\hat{x}) - \varepsilon_i & i \in \bar{I} \\ 0 & i \in I \end{cases}$$

Therefore the objective value of $P_{\varepsilon-k}^{comb}$ is

$$f_k(\hat{x}) + \sum_{i \in I} \lambda_i (f_i(\hat{x}) - \varepsilon_i) + \sum_{i \in \bar{I}} \mu_i (f_i(\hat{x}) - \varepsilon_i)$$

and $\hat{x}$ is in fact an optimal solution of the weighted sum problem

$$\begin{aligned} \min f_k(x) + \sum_{i \in I} \lambda_i f_i(x) + \sum_{i \in \bar{I}} \mu_i f_i(x) & & & \\ \text{subject to } f_i(x) & \leq \varepsilon_i & i \in I & \quad (6) \\ f_i(x) & \geq \varepsilon_i & i \in \bar{I} & \quad (7) \\ x & \in X. & & \end{aligned}$$

By Geoffrion's theorem we have that $\hat{x}$ is a properly efficient solution of the MOP $\min_{x \in X}$ $(f_1(x), \ldots, f_p(x))$ with additional constraints (6) and (7). Since none of these constraints are active at the optimal solution $\hat{x}$, it is also properly Pareto optimal for the original MOP. □

Lemma 5 *Let $\hat{x}$ be an efficient solution of the MOP. Then there exist $\lambda \geqq 0$, $\mu \geqq 0$, ε and $\hat{s}^+$, $\hat{s}^-$ such that $(\hat{x}, \hat{s}^+, \hat{s}^-)$ is an optimal solution of $P_{\varepsilon-k}^{comb}$ for all $k = \{1, \ldots, p\}$.*

The result follows immediately from the analogous results of Sections 2 and 3.

Theorem 10 *Let $\hat{x}$ be properly efficient for the MOP. Then for every $k \in \{1, \ldots, p\}$ there exist $\lambda > 0$, $\mu > 0$ with $\mu_i < \infty$ for all i, ε and $\hat{s}^+$, $\hat{s}^-$ such that $(\hat{x}, \hat{s}^+, \hat{s}^-)$ is an optimal solution of $P^{comb}_{\varepsilon_{-k}}$.*

The theorem follows immediately from Section 2, Theorem 5 and Section 3, Theorem 7.

5 Conclusion

In this paper we have proposed modifications of the well-known ε-constraint scalarization technique for multiobjective programming. With these modifications we are able to prove results on proper efficiency of optimal solutions. Additionally, the new formulation can have advantages in computational performance, in particular for multiobjective integer programming problems.

In the following table we summarize the results obtained for the improved ε-constraint method with scalarization $P^{comb}_{\varepsilon_k}$.

Paramters and conditions	Implication for $\hat{x}$	Result
$(\lambda, \mu) \geqq 0$	$\hat{x} \in X_{wE}$	Proposition 5
$(\lambda, \mu) \geqq 0$ and $\hat{x}$ unique	$\hat{x} \in X_{sE}$	Proposition 6
$(\lambda, \mu) > 0$	$\hat{x} \in X_E$	Theorem 8
$(\lambda, \mu) > 0$ and $\hat{s}^+ + \hat{s}^- > 0$	$\hat{x} \in X_{pE}$	Theorem 9
Type of solution	**Optimality of for $P^{comb}_{\varepsilon_k}$**	**Result**
$\hat{x} \in X_E$	$\exists$ $(\varepsilon, \hat{s}^+, \hat{s}^-)$ so that $(\hat{x}, \hat{s}^+, \hat{s}^-)$ optimal $\forall k$	Lemma 5
$\hat{x} \in X_{pE}$	$\forall k$ $\exists$ $(\varepsilon, \hat{s}^+, \hat{s}^-)$ so that $(\hat{x}, \hat{s}^+, \hat{s}^-)$ optimal	Theorem 10

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