Hub Cover and Hub Center Problems

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Contents

Chapter 1

Introduction

1.1 Hub Location Problems

Hub location problems arise when it is desirable to transport commodities, that may be goods or passengers, from their origin to their destination. In general a direct transportation of these commodities can not be realized due to the fact that establishing such a network is extremely costly. As an alternative one uses a special logistic network with a so called hub-and-spoke structure. The hubs act as collection, consolidation, transfer and distribution points. The advantage of using the hubs is that by consolidating the flow, economies of scale can be achieved so that transferring flow between hubs is cheaper than the cost of moving flow to and from non-hub nodes. The spokes represent all non-hub nodes which are connected to one or more hubs, depending on whether we deal with single or multiple allocation.

It is usually assumed that the hubs are fully interconnected, therefore all commodities have to routed via at least one hub. Hence such a logistic network consists of two parts, namely the hub level and the spoke level. Figure 1.1 illustrates such a network whereas the squares are the hubs and the circles are the spokes.

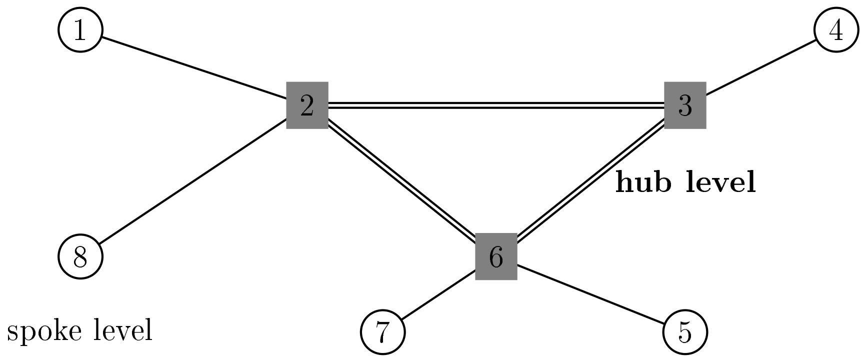


Figure 1.1: Example for a hub and spoke network.

In general in a hub location problem we deal with two different tasks. Firstly the *hub location*, where we have to choose certain nodes as hubs and secondly the *spoke allocation*, where we have to assign the spoke nodes to the hub nodes. For many hub location problems that are discussed in literature, both hub location and spoke allocation are known to be NP-hard.
Hub location problems have been studied quite thoroughly in the last decade, but only a few authors have considered hub covering problems. As for hub location problems, the task is to design a hub-and-spoke network. But a predefined delivery time limit β is given for transports between any origin destination pair (o-d pair).

1.2 Chapter Outline

The focus of this thesis is hub covering problems that are minimizing the number of hubs while making sure that all travel times between o-d pairs do not exceed a given time limit. The remainder of this thesis is partitioned into three parts.

Part I provides a general overview on basic definitions that are used throughout the thesis, presents the most common applications of hub location problems and model formulations for hub covering problems as well as the radius formulation for the p-hub center problem.
Chapter 2 introduces some basic mathematical notations and gives an overview of the different variants of the basic hub location problems that have been studied in the literature. Some typical application areas of hub location problems, which are air transportation, postal delivery services and telecommunications are introduced in **Chapter 3**. In **Chapter 4** the radius formulation for the uncapacitated single allocation p-hub center problem (USApHCP) and several formulations for hub covering problems are presented. Finally, some properties of the time limit β are described in detail in particular a lower bound for β, for which the hub covering problem is feasible, is presented.

The emphasis of **Part II** lies in the polyhedral description of the hub covering problem.
In **Chapter 5** we introduce some basic definitions of polyhedral theory and establish the dimension of the polyhedron. There, we also check all given constraints with regard to their facet-inducing property and derive two new classes of facet-defining inequalities. Then, we discuss the corresponding separation problems in **Chapter 6.**

Part III deals with the relation between hub covering and hub center problems and provides an algorithm for solving the USApHCP.
In **Chapter 7** we develop a new approach for solving the USApHCP based on a binary-search algorithm. Numerical results are presented and finally we summarizes the main results of this thesis in **Chapter 8.**

Appendix A illustrates how the new facet-defining inequalities are derived using the PORTA tool.

Part I

Basic Definitions, Model Formulations and Applications

Chapter 2

Basic Definitions

In this chapter we give some basic mathematical notations, which will be used for the remainder of this thesis. Furthermore some variants of hub location problem are presented.

2.1 Common Mathematical Notation

Let $G = (V, E)$ be a complete undirected graph with $|V| = n$, where the nodes represent the cities and the arcs transport links between these cities. For any o-d pair (i, j) the distances d_{ij} [1] are given. In general, the per unit cost c_{ij} is proportional to the distance. The distances d_{ij} are assumed to satisfy the following properties:

- $d_{ij} \geq 0 \; \forall \; i, j \in V$ (non-negativity)
- $d_{ij} = 0 \Leftrightarrow i = j$
- $d_{ij} = d_{ji} \; \forall \; i, j \in V$ (symmetry)
- $d_{ij} \leq d_{ik} + d_{kj} \; \forall \; i, k, j \in V$ (triangle inequality)

[1] d_{ij} not only represent the distance, but it can also be considered as the travel time.

Let $\alpha \in [0,1]$ be a discount factor on hub-to-hub links, that reflects consolidations of traffic. The more traffic is transported between hub nodes, the lower the per unit costs are, resulting from economies of scale. This comes from utilizing the capacities of the means of transportation more efficiently or using means of transportation with a higher capacity.
For instance, if d_{ij} represent travel times, α, if less than 1, might reflect time savings due to the usage of faster means of transportation. Hence an o-d path from i to j via two hubs k and m, causes cost $c_{ij} := d_{ik} + \alpha d_{km} + d_{jm}$. Of course i, k, m, j do not have to be different nodes.
Note that, due to the triangle inequality a connection of an o-d pair via more than two hubs is not taken into account.
The parameter β reflects a predetermined bound that imposes a deadline for travel time between any o-d pair. We refer to β as the *cover radius.*

Definition: Given a graph G with cost coefficients d_{ij}, β and α as defined above, the **hub covering problem** consists of choosing a minimum number of hubs within the node set V such that the travel time between any o-d pair is no more than the cover radius β.

Definition: Given a graph G with cost coefficients d_{ij} and α as defined above, the **discrete hub location problem** consists of choosing a set of hubs within the node set V and allocating all non-hub nodes to hub nodes such that a given function based on the arc costs d_{ij} is minimized.

Remark 2.1.1. It would be more precise to define a discount factor α for any pair of nodes as a decreasing function of the traffic shipped along link (k, m). But unfortunately it is difficult to handle the resulting objective function, so that most authors work with quantity-independent discount factor only. However, Wagner provided a model formulation for hub covering problems with quantity-dependent discount factors ([1]).

2.2 Variants of Hub Location Problems

There exist multiple variants of hub location problems that have been studied. We distinguish between different types of the objective function and constraints.

2.2.1 Objective

Definition: In the **hub center problem** the objective is to minimize the maximum transportation cost for all origin-destinations pairs.

Definition: In the **hub median problem** the objective is to minimize the overall transportation cost.

2.2.2 Constraints

Definition: In the **single allocation** hub location problem each spoke node is allocated to a single hub, i.e. each spoke node can send and receive commodities via only one single hub.

This kind of network profits from its simple structure, possible higher resource utilization and lower start-up costs. But for many applications such a network impose undesirable restrictions. As we will see in the next chapter, in case of air transportation, single allocation is not reasonable relating to transportation costs and convenience of the passengers. In order to avoid this, we introduce the multiple allocation hub problem.

Definition: In the **multiple allocation** hub location problem spoke nodes are allowed to be allocated to several hubs, i.e. each spoke node can send and receive commodities via more than one hub.

Definition: In the **capacitated** hub location problem there are capacity restriction on hub and/or spoke nodes and/or on the flows. Otherwise it is an **uncapacitated** hub location problem.

Furthermore, it can be distinguished, as in usual facility location, between p-hub location, where the number of hubs to be located is fixed to a number p (see e.g. [15]) and fixed charge hub location problems, where this number is unlimited, but a certain fixed cost has to be paid for establishing a hub node (see e.g. [2]).

Chapter 3

Applications

As a result of the Airline Deregulation Act in 1978 and the Bus Regulatory Reform Act in 1982, the geographic positioning of hubs is of crucial importance to the performance of the industry. Therefore hub location problems appear in a variety of applications including airlines, postal delivery networks and telecommunications. Here we provide an overview of each of these fields.

3.1 Air Transportation

As already stated hub location problems have their origin in the air passenger transportation. Due to the Airline Deregulation Act, the passenger airlines as well as freight airlines were permitted to choose the routes to serve and the fares to charge. As an effect economists and transportation experts carried out the advantage of a hub and spoke network.[1]

In such a network passengers from different origins are collected at a hub airport

[1] The hub and spoke networks for air passenger and air freight differ, since in the network for freight the overall performance and cost is paramount and in the passenger network we also have to consider the convenience of the passengers.

and then transported either to a second hub or directly to their destination. In order to keep the transportation costs as low as possible it is useful to allow multiple allocation, that is non-hub airports are connected with more than one hub airport, so that passengers with the same origin but different destinations could use different hub airports.

3.2 Postal Delivery

In a postal network hubs act not only as consolidation but as sorting center, that is, mail commodities of different postal districts are collected and sorted at hubs and distributed either to a further hub or to their final recipient, hence even mail commodities that are sent to its own district have to be shipped first to a hub and then on the same way back. In general each postal district is connected to its nearest hub, hence a capacitated or uncapacitated p-hub median problem with single allocation is generally the best model for this kind of application. Note that in case of express delivery the above model require more time to ship mails or parcels, therefore a model with multiple allocation performs better.

3.3 Telecommunication

In the area of telecommunication we have the widest range of applications for hub location problems, including telephone networks, computer communications, video conferences, etc. In a telecommunication network, hubs can be identified as electronic devices, concentrators, satellites and so on. In general the commodities are pieces of information, and the arcs represent different media-like telephone lines, fiber optic cables or the air.

Chapter 4

Model Formulations

In this chapter model formulations of hub center and hub covering problems are introduced. The most promising mathematical formulation for the USApHCP is given in Section 4.1.

In Section 4.2 we present different formulations for hub covering problems. There, we also investigate the given time limit β. To model the p-hub center as well as the hub covering problem we introduce a decision variable X_{ik} as follows:

$$X_{ik} = \begin{cases} 1 & \text{if } i \text{ is located to hub } k, \\ 0 & \text{otherwise.} \end{cases}$$

and

$$X_{kk} = \begin{cases} 1 & \text{if a hub is open at node } k, \\ 0 & \text{otherwise.} \end{cases}$$

Let r_k be a continuous variable representing the *radius* of hub k, that is the maximum distance between hub k and all spoke nodes.

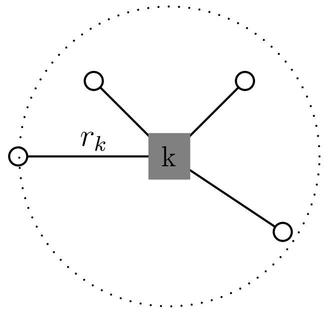


Figure 3.1: Example of a hub radius.

4.1 Hub Center Problems

In the hub center problem, the main objective is one of minimizing the maximum distance/cost between o-d pairs.

In 1994 Campbell [2] first presented a quadratic formulation for single allocation hub center problems and later a linearization of his previous formulation with the disadvantage of having a huge amount of additional variables. In 2000, Kara and Tansel [3] provided a linearization of Campbell's quadratic formulation, which clearly outperformed that of Campbell. However, the most promising formulation of the USApHCP was provided by Ernst, Hamacher, Jiang, Krishnamoorthy and Woeginger [4]. They developed a new formulation based on the *hub radius*, we refer to this formulation as USApHCP-r. Beside a branch-and-bound approach for exact solving, several heuristic methods have been considered by the authors, among those special heuristics for solving the allocational subproblem in the single allocation case ([4], [5]). This thesis will deal with the USApHCP-r and its optimal solution is used as the minimax delivery time needed for a given number of hubs .

USApHCP-r:

$$\min \quad z$$

$$\text{s.t.} \qquad z \geq r_k + r_m + \alpha \cdot d_{km} \qquad \forall k, m \in \{1, ..., n\} \tag{4.1}$$

$$r_k \geq d_{ik} X_{ik} \qquad \forall i, k \in \{1, ..., n\} \tag{4.2}$$

$$\sum_k X_{ik} = 1 \qquad \forall i \in \{1, ..., n\} \tag{4.3}$$

$$X_{ik} \leq X_{kk} \qquad \forall i, k \in \{1, ..., n\} \tag{4.4}$$

$$\sum_k X_{kk} = p \tag{4.5}$$

$$X_{ik} \in \{0, 1\} \qquad \forall i, k \in \{1, ..., n\} \tag{4.6}$$

$$r_k \geq 0 \qquad \forall k \in \{1, ..., n\} \tag{4.7}$$

Constraint (4.1) defines the objective function z as the maximum distance between any two hub nodes by using the respective hub radii. Constraint (4.2) sets the radius variable r_k to at (least) the maximum distance to k of a spoke node that is allocated to k. Constraint (4.3) ensures that every node is assigned to exactly one hub, (4.4) guarantees that node i can only be allocated to node k if k is a hub and (4.5) meets the condition of having exactly p hubs.

4.2 Hub Covering Problems

The hub covering problem (HCoP) has only be considered by a few authors. It has been introduced by Campbell [2]. He gave a quadratic formulation of the hub covering problem, which was later outperformed by Kara and Tansel's [3] linearization of the problem. While in the p-median and p-center hub problem

p is given a priori, p is now a variable and it is to be minimized under the restriction that the demand has to be met in a given time limit. The objective of minimizing the number of hubs while obeying a maximum time bound on travel time makes sense in view of the fact that establishing and operating hubs involve large amounts of set-up and operating costs.
For instance, consider the overnight package delivery where parcels have to be delivered within twelve hours. Another application is the transportation of perishable goods. The hub covering problem is also appropriate for air transportation passengers if we want to keep the dissatisfaction factor as low as possible.

4.2.1 Quadratic Formulation

HCoP-Q:

$$min \quad \sum_k X_{kk}$$

$$\text{s.t.} \qquad \sum_k X_{ik} = 1 \qquad \forall i \in \{1,...,n\} \tag{4.8}$$

$$X_{ik} \leq X_{kk} \qquad \forall i,k \in \{1,...,n\} \tag{4.9}$$

$$X_{ik} \in \{0,1\} \qquad \forall i,k \in \{1,...,n\} \tag{4.10}$$

$$(d_{ik} + \alpha d_{km} + d_{jm}) X_{ik} X_{jm} \leq \beta \qquad \forall i,k,j,m \in \{1,...,n\} \tag{4.11}$$

Constraints (4.8) and (4.9) are standard, (4.11) make sure that the given time bound is met and (4.10) specifies the domain of the assignment variables X_{ik}. HCoP-Q is a binary program with n^2 binary variables and $n^4 + n^2 + n$ constraints. Due to constraint (4.11) this programm is nonlinear.
This formulation was proposed by Kara and Tansel [3]. They referred to an earlier

formulation by Campbell, who used instead of (4.11) the set covering constraints

$$\sum_{k,m} V_{ijkm} X_{ik} X_{jm} \geq 1 \qquad \forall i, j \in \{1...n\} \tag{4.12}$$

where

$$V_{ijkm} \begin{cases} 1 & d_{ik} + \alpha d_{km} + d_{jm} \leq \beta \ , \\ 0 & \text{otherwise.} \end{cases}$$

We refer to this as the *hub set covering problem* HSCoP.

4.2.2 Linearization

Campbell [2] gave the first linearization of HCoP-Q by replacing $X_{ik}X_{jm}$ by a four index variable Y_{ijkm}, but this approach highly increased the number of variables. Kara and Tansel [3] then present the following formulation, where they replaced the quadratic term by (4.13)

HCoP-Lin:

$$min \ \sum_{k} X_{kk}$$

s.t. $\qquad (4.8) - (4.10)$

$$(d_{ik} + \alpha d_{km}) X_{ik} + d_{mj} X_{jm} \leq \beta \qquad \forall i, k, j, m \in \{1, ..., n\} \tag{4.13}$$

Theorem 4.2.1. $[Kara\, and\, Tansel]$
Any feasible solution to HCoP-Q is a feasible solution to HCoP-Lin and vice versa.

Proof. Let $\hat{X}$ be a feasible solution to HCoP-Q. Since constraints (4.8) -(4.10) are common to both HCoP-Q and HCoP-Lin, it suffices to show that $\hat{X}$ is feasible to (4.13). Consider the constraint (4.13) associated with the quadruplet i, j, k, m. There are four cases to consider depending on the values of $\hat{X}_{ik}$ and $\hat{X}_{jm}$.

<u>case 1:</u> $\hat{X}_{ik} = 1,\ \hat{X}_{jm} = 1$

Then (4.11) and (4.13) give the same left hand side.

<u>case 2:</u> $\hat{X}_{ik} = 1,\ \hat{X}_{jm} = 0$

Then the left side of (4.13) is $d_{ik} + \alpha d_{km}$ and we must show that this is less than or equal to β. Either $\hat{X}_{mm} = 1$ or 0. If $\hat{X}_{mm} = 1$, constraint (4.11) for i, k, m, m yields $d_{ik} + \alpha d_{km} + d_{mm} \leq \beta$, which implies that (4.13) is satisfied.
If $\hat{X}_{mm} = 0$ then (4.8) implies that there exists an index $l \neq m$ such that $\hat{X}_{ml} = 1$. Constraint (4.11) for i, k, m, l yields $d_{ik} + \alpha d_{kl} + d_{lm} \leq \beta(*)$. Note that $0 \leq \alpha \leq 1$, and that $d_{ik} \leq d_{kl} + d_{lm}$, we have $d_{ik} + \alpha d_{kj} \leq d_{ik} + \alpha d_{kl} + d_{lm}$, which implies with $(*)$ that (4.13) is satisfied.

<u>case 3:</u> $\hat{X}_{ik} = 0,\ \hat{X}_{jm} = 1$

In this case the left hand side of (4.13) is d_{jm}. Since $\hat{X}_{ik} = 0$, constraint (4.8) implies that there exists an index $l \neq k$ such that $\hat{X}_{il} = 1$. Constraint (4.11) for i, j, l, m gives $\beta \geq d_{il} + \alpha d_{lm} + d_{jm}$ which implies that $d_{jm} \leq \beta$.

<u>case 4:</u> $\hat{X}_{ik} = 0,\ \hat{X}_{jm} = 0$

In this case both (4.11) and (4.13) give the same left hand side.

To prove the converse, observe that the left side of (4.11) is always less than or equal to the left side of (4.13), which implies that any feasible solution to HCoP-Lin is also feasible to HCoP-Q. □

Corollary 4.2.2. *An optimum solution to HCoP-Lin is also an optimum solution to HCoP-Q (and vice versa).*

Remark 4.2.3. HCoP-Lin uses the same number of variables, namely n^2, as HCoP-Q, hence there is no change in the dimension of the space. Moreover the feasible sets and the optimal sets are the same. Note that this is not the case for Campbell's HSCoP. Kara and Tansel give two new linearizations of HSCoP , which both require additional variables. Furthermore they stated that HCoP-Lin yields significantly faster solution times than all linearizations of HSCoP.

Recently Wagner [1] developed an improved model formulation for hub covering problems. The idea was to rule out some assignments to hubs in advance, that is this formulation requires not only less variables, but also less constraints. Wagner uses the notation of SAQI-W1 (single allocation with quantity independent discount factors), which we will adopted in the following.

If it is not possible to get to every j in time if i is assigned to k, then variable X_{ik} can be dropped. Therefore, we have to examine all "valid" assignments:

$$VA = \Big\{(i,k)|2d_{ik} \leq \beta \text{ and } d_{ik} + \alpha \underset{j}{max}(d_{kj}) \leq \beta \Big\}$$

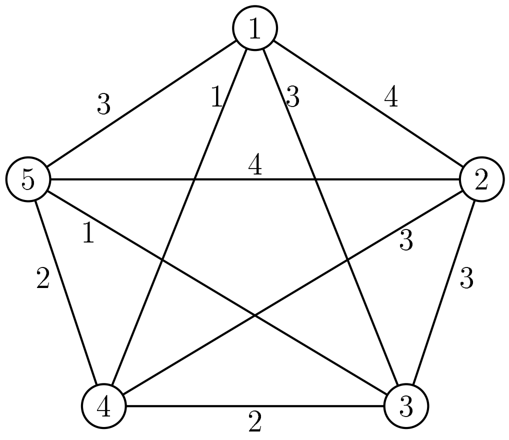
Let $|VA| := q \leq n^2$ (see Proposition 4.2.9).

For those i, j, k, m where $d_{ik} + \alpha d_{km} + d_{jm} \leq \beta$ constraints (4.13) are obsolete. But if this is not the case and $(i,k) \in VA$ as well as $(j,m) \in VA$ then the corresponding constraint can be tightened by prohibiting that X_{ik} and X_{jm} take a value of 1 concurrently, that is either $X_{ik} = 1$ or $X_{jm} = 1$, but not both at the same time. Therefore we define the set IA, which contains all pairs of incompatible assignments:

$$IA = \Big\{(i,k,j,m)|(i,k),(j,m) \in VA, i < j \text{ and } d_{ik} + \alpha d_{km} + d_{jm} > \beta\Big\}$$

Note that Wagner's definition of IA can be extended by $k \neq j$ and $i \neq m$. Otherwise we also obtain constraints of the form $X_{ik} + X_{kj} \leq 1$ and $X_{ik} + X_{ji} \leq 1$ that are useless anyway. Let $|IA| := t << n^4$.

Example 4.2.4. Consider the following network consisting of 5 cities and assume that we want to serve all of them within 6 hours. For the sake of simplicity we set α to 1.



The corresponding sets VA and IA are the following:

$VA = \{$ (1,1), (1,3), (1,4), (2,2), (2,3), (2,4), (3,3), (3,4), (3,5), (4,1), (4,3), (4,4), (4,5), (5,3), (5,4), (5,5) $\}$

$IA = \{$(1,3,2,4), (1,3,5,4), (2,3,4,1), (2,3,5,4)$\}$

Here we have reduced the number of variables from 25 of 16. Note that the smaller β the less variables we have. For instance we have delete variable X_{42}, since $d_{42} + 1 \cdot \underset{j}{max}(d_{2j}) \leq \beta$, that is $3 + 1 \cdot 4 \nleq 6$.

The number of constraints have been reduced from 655[1] to 25[2], which is an enormous improvement.

The above observations result in the following model:

[1] $n^4 + n^2 + n$

[2] $|IA| + |VA| + n$

SAQI-W1:

$$min \ \sum_k X_{kk}$$

$$\text{s.t.} \quad \sum_{k:(i,k)\in VA} X_{ik} = 1 \qquad \forall \quad i \in \{1, ..., n\} \tag{4.14}$$

$$X_{ik} \leq X_{kk} \qquad \forall \quad (i,k) \in VA \tag{4.15}$$

$$X_{ik} \in \{0,1\} \qquad \forall \quad (i,k) \in VA \tag{4.16}$$

$$X_{ik} + X_{jm} \leq 1 \qquad \forall \quad (i,k,j,m) \in IA \tag{4.17}$$

(4.14)-(4.16) is basically the same as (4.8)-(4.10). In (4.17) only incompatible assignments are forbidden. It is obvious that this formulation requires less variables and constraints.

Theorem 4.2.5. *Any feasible solution to HCoP-Lin is a feasible solution to SAQI-W1 and vice versa.*

Proof. Let $\hat{X}$ be a feasible solution to HCoP-Lin. Since $|VA| = q \leq n^2$ constraint (4.15) is satisfied for both HCoP-Lin and SAQI-W1, it suffices to show that $\hat{X}$ is feasible to (4.14) and (4.17).

First we need to show that $\hat{X}_{ik} = 0 \ \forall (i,k) \notin VA$. Assume that $\hat{X}_{ik} = 1 \ \forall (i,k) \notin VA$. Then $2d_{ik} > \beta$ or $d_{ik} + \alpha \max_j(d_{kj}) > \beta$, $(i,k) \notin VA$.

<u>case 1:</u> $2d_{ik} > \beta$, $(i,k) \notin VA$

Then

$$\begin{aligned}(d_{ik} + \alpha d_{kk})\hat{X}_{ik} + d_{ki}\hat{X}_{ik} &\overset{d_{kk}=0}{=} d_{ik} + d_{ki} \\ &\overset{symm.}{=} 2d_{ik} \\ &> \beta,\end{aligned}$$

which is a contradiction to the feasibility of $\hat{X}$.

<u>case 2:</u> $d_{ik} + \alpha \max_j(d_{kj}) > \beta$, $(i,k) \notin VA$

Let $d_{km} := \max_j(d_{kj})$. Then for i, k, m, m we have

$$\begin{aligned}(d_{ik} + \alpha d_{km})\hat{X}_{ik} + d_{mm}\hat{X}_{mm} &\overset{d_{kk}=0}{=} d_{ik} + \alpha d_{km} \\ &> \beta,\end{aligned}$$

which is again a contradiction to the feasibility of $\hat{X}$.

Thus, $\hat{X}_{ik} = 0 \ \forall (i,k) \notin VA$ and (4.14) is satisfied.

Next, we show that (4.17) is also satisfied. There are four cases to consider depending on the values of $\hat{X}_{ik}$ and $\hat{X}_{jm}$.

<u>case 1:</u> $\hat{X}_{ik} = 1$, $\hat{X}_{jm} = 1$

Then due to definition of IA $\hat{X}_{ik} + \hat{X}_{jm} \leq 1$ do not occur in SAQI-W1.

<u>case 2:</u> $\hat{X}_{ik} = 1$, $\hat{X}_{jm} = 0$

It is obvious that (4.17) is satisfied, since $1 \leq 1$.

<u>case 3:</u> $\hat{X}_{ik} = 0$, $\hat{X}_{jm} = 1$

Similar to case 2.

<u>case 4:</u> $\hat{X}_{ik} = 0$, $\hat{X}_{jm} = 0$

Constraint (4.17) holds, since $0 \leq 1$.

To prove the converse, observe that due to definition of IA (4.13) is always satisfied. Now let $\hat{X}$ be a solution to SAQI-W1. Since $\hat{X}_{ik} = 0 \ \forall (i,k) \notin VA$ we have

$$\sum_{k:(i,k)\notin VA} \hat{X}_{ik} = 0$$

$$\Rightarrow \sum_{k:(i,k)\in VA} \hat{X}_{ik} + \sum_{k:(i,k)\notin VA} \hat{X}_{ik} = 1$$

$$\Rightarrow \sum_{k=1}^{n} \hat{X}_{ik} = 1$$

which implies that any feasible solution to SAQI-W1 is also feasible for HCoP-Lin.

□

Corollary 4.2.6. *An optimum solution to SAQI-W1 is also an optimum solution to HCoP-Lin (and vice versa).*

Furthermore, this formulation can still be improved by aggregating some constraints of type (4.17).

Lemma 4.2.1. *Let* $M^{(i,k,j,\cdot)} := \{m : (i,k,j,m) \in IA\}$
and $M^{(i,\cdot,j,m)} := \{k : (i,k,j,m) \in IA\}$
Then in SAQI-W1 the following constraints

$$X_{ik} + X_{jm} \leq 1 \qquad \forall m \in M^{(i,k,j,\cdot)} \tag{4.18}$$

and

$$X_{ik} + X_{jm} \leq 1 \qquad \forall k \in M^{(i,\cdot,j,m)} \tag{4.19}$$

can be replaced by

$$X_{ik} + \sum_{m \in M^{(i,k,j,\cdot)}} X_{jm} \leq 1 \tag{4.20}$$

and

$$\sum_{k \in M^{(i,\cdot,j,m)}} X_{ik} + X_{jm} \leq 1 \quad \textit{respectively.} \tag{4.21}$$

Proof. We restrict ourself to prove that (4.18) can be replaced by (4.20). The other case can be proved analogously.

" $\Longrightarrow$ " Let $\hat{X}$ be a feasible solution to SAQI-W1.
We have to show that $\hat{X}_{ik} + \sum\limits_{m \in M^{(i,k,j,\cdot)}} \hat{X}_{jm} \leq 1$.
There are two cases to consider depending on the value of $\hat{X}_{ik}$.

<u>case 1:</u> $\hat{X}_{ik}$=1
In this case (4.18) implies that $\hat{X}_{jm} = 0 \; \forall m \in M^{(i,k,j,\cdot)}$. Thus $\sum\limits_{m \in M^{(i,k,j,\cdot)}} \hat{X}_{jm} = 0$, which implies that (4.20) is satisfied.

<u>case 2:</u> $\hat{X}_{ik}$=0
Then either $\hat{X}_{jm}$=1 or 0. The validity of (4.20) follows since $\sum\limits_{m \in M^{(i,k,j,\cdot)}} \hat{X}_{jm} \overset{(4.14)}{\leq} 1$.

" $\Longleftarrow$ " Let $\hat{X}$ be a feasible solution to the reformulated SAQI-W1 with (4.20).
We have to show that $\hat{X}_{ik} + \hat{X}_{jm} \leq 1 \; \forall m \in M^{(i,k,j,\cdot)}$.

Since $\hat{X}_{ik} + \sum\limits_{m \in M^{(i,k,j,\cdot)}} \hat{X}_{jm} \leq 1$ it follows that

$$\hat{X}_{ik} + \hat{X}_{jm} \leq \hat{X}_{ik} + \sum_{m \in M^{(i,k,j,\cdot)}} \hat{X}_{jm} \leq 1 \quad \forall m \in M^{(i,k,j,\cdot)}.$$

$\square$

The constraint aggregation procedure is given in [1]. The procedure terminates if as many constraints (4.17) as possible are aggregated. In the following, the resulting model is referred to as SAQI-W2.

Example 4.2.7. Consider the following constraints of type (4.17):
$X_{12} + X_{34} \leq 1$
$X_{12} + X_{35} \leq 1$
$X_{12} + X_{36} \leq 1$

$X_{23} + X_{41} \leq 1$

$X_{25} + X_{41} \leq 1$

Applying the constraint aggregation yields

$X_{12} + X_{34} + X_{35} + X_{36} \leq 1$

$X_{23} + X_{25} + X_{41} \leq 1.$

For a given β the HCoP has not necessarily a solution. Therefore we have to find a lower bound for β for which a solution exists to the HCoP.

Theorem 4.2.8. *Let $G = (V, E)$ be a complete graph with $|V| = n \geq 2$ and the distances d_{ij} satisfy the triangle inequality.*

The HCoP has a solution if and only if $\beta \geq \alpha \underset{i,j}{max}\ (d_{ij})$.

Proof. "$\Longrightarrow$" Any feasible solution to HCoP satisfies

$\sum_{k:(i,k)\in VA} X_{ik} = 1\ \forall i$

$\Rightarrow \forall i \in \{1, ..., n\}\ \exists\ k\text{:}\ (i, k) \in VA$

$\Rightarrow \forall i \in \{1, ..., n\}\ \exists\ k\text{:}\ d_{ik} + \alpha \underset{j}{max}\ (d_{kj}) \leq \beta.$

Let $d_{rq} = \underset{i,j}{max}\ (d_{ij})$.
Then for r it exists a k such that $d_{rk} + \alpha \underset{j}{max}\ (d_{kj}) \leq \beta$.

Now we can conclude

$$\begin{array}{rcl}
\alpha \underset{i,j}{max}(d_{ij}) & = & \alpha\ d_{rq} \\
& \overset{\triangle-ineq.}{\leq} & \alpha\ (d_{rk} + d_{kq}) \\
& \leq & d_{rk} + \alpha\ d_{kq} \\
& \overset{d_{kq} \leq \underset{j}{max}(d_{kj})}{\leq} & d_{rk} + \alpha \underset{j}{max}\ (d_{kj}) \\
& \leq & \beta
\end{array}$$

"$\Longleftarrow$" Let $\beta \geq \alpha \max_{i,j} (d_{ij})$.

<u>claim:</u> $\forall k \in \{1, ..., n\}$: $(k,k) \in VA$

$2 \underbrace{d_{kk}}_{=0} \leq \beta$ ✓

It remains to show that

$\underbrace{d_{kk}}_{=0} + \alpha \max_{j} (d_{kj}) \leq \beta.$

$\forall k \in \{1, ..., n\}$: $\alpha \max_{j} (d_{kj}) \leq \alpha \max_{i,j} (d_{ij}) \overset{by\ assumption.}{\leq} \beta.$

Thus the HCoP has a feasible solution. □

Proposition 4.2.9. $|VA| = n^2 \iff \beta \geq 2 \max_{i,j} (d_{ij})$

Proof. "$\Longleftarrow$" Let $\beta \geq 2 \max_{i,j}(d_{ij})$.

$\forall i, k \in \{1, ..., n\}$:

$$\begin{aligned} d_{ik} + \alpha \max_{j} (d_{kj}) &\leq \max_{i,j} (d_{ij}) + \alpha \max_{i,j} (d_{ij}) \\ &\overset{\alpha \in [0,1]}{\leq} 2 \max_{i,j} (d_{ij}) \\ &\leq \beta \end{aligned}$$

$\Rightarrow VA = \{(i,k) : i \in \{1, ..., n\}, k \in \{1, ..., n\}\}$
$\Rightarrow |VA| = n^2$

"$\Longrightarrow$" Assume $\beta < 2 \max_{i,j} (d_{ij})$.

$\exists\ r, q$: $d_{rq} = \max_{i,j} (d_{ij})$

$\underset{2d_{rq} > \beta}{\overset{def.\ of\ VA}{\Rightarrow}} (r,q) \notin VA \Rightarrow |VA| < n^2$. Contradiction.

□

Part II

Polyhedral Analysis

Chapter 5

Facet Derivation

In this chapter we analyze the feasible integer polyhedron $\mathcal{P}_{SAQI-W2}$. First, we give some backround material about polyhedral theory in Section 5.1 that is needed throughout this chapter. Then, we determine the dimension of $\mathcal{P}_{SAQI-W2}$ in Section 5.2, which is an important result needed when deriving facets. Next, in Section 5.3, we present facets that are induced by the problem constraints and in Section 5.4 we present two new classes of facet-defining inequalities.

5.1 Basic Definitions of Polyhedral Theory

In this section we introduce some basic definitions and results of polyhedral theory. A thorough introduction to polyhedral theory can be found in [8].

Definition: A set $\mathcal{P} \subseteq \mathbb{R}^n$ is called a **polyhedron** if it is the intersection of a finite set of closed halfspaces $\mathcal{P} = \{x \in \mathbb{R}^n | Ax \leq b\}$. A bounded polyhedron is called a **polytope**. If $b = 0$ the polyhedron is a **cone**.

When dealing with polytopes the notion of affine independence is more suitable then linear independence because it is invariant under translations.

Definition: A set of points $\{x_1, ..., x_i\} \subseteq \mathbb{R}^n$ is **affinely independent** if the unique solution of $\sum_i \alpha_i \, x^i = 0$, $\sum_i \alpha_i = 0$ is $\alpha_i = 0 \; \forall i$.

Note that linear independence implies affine independence but the reverse is not necessarily true.

Definition: The **dimension** $dim \; \mathcal{P}$ of a polyhedron $\mathcal{P}$ is the maximum number of affinely independent points in $\mathcal{P}$ - 1.

Definition: Let A be a $m \times n$ matrix.
Then the **rank(**A**)** is the maximum number of linear independent rows or columns of A.

Now we relate the dimension of P to the rank of its equality matrix $(A^=, b^=)$.

Proposition 5.1.1. *Let* $\mathcal{P} = \{x \in \mathbb{R}^n | Ax = b\}$.
If $\mathcal{P} \neq \emptyset$[1] *then* $dim \; \mathcal{P} = n$ *- rank(*$A^=, b^=$*)*

Definition: An inequality $\pi x \leq \pi_0$ ($\pi \in \mathbb{R}^n$, $\pi_0 \in \mathbb{R}$) is a **valid inequality** for some set $X \subseteq \mathbb{R}^n$, if $\pi x \leq \pi_0$ is satisfied $\forall \; x \in X$.

Definition: Let $\mathcal{P} \subseteq \mathbb{R}^n$ be a polyhedron.

- $\mathcal{F}$ defines a **face** of $\mathcal{P}$ if $\mathcal{F} = \{x \in \mathcal{P} : \pi x = \pi_0\}$ for some valid inequality $\pi x \leq \pi_0$ of $\mathcal{P}$.

- $\mathcal{F}$ is a **proper face**, if $\mathcal{F}$ is a face and $\mathcal{F} \neq \emptyset$.

- $\mathcal{F}$ is a **facet**, if $\mathcal{F}$ is a face of $\mathcal{P}$ and dim $\mathcal{F} =$ dim $\mathcal{P} - 1$.

- If $\mathcal{F}$ is a face of $\mathcal{P}$ with $\mathcal{F} = \{x \in \mathcal{P} : \pi x = \pi_0\}$ then the valid inequality $\pi x \leq \pi_0$ **represents** the face .

[1]The result is still valid with the convention that if $\mathcal{P} = \emptyset$, then dim $\mathcal{P} = -1$

Remark 5.1.2. Each face is itself a polyhedron.

$$\text{Faces of dimension} \begin{cases} 0 & \text{are called } \textit{vertices} \text{ or } \textit{extreme points}, \\ 1 & \text{are called } \textit{edges}, \\ \text{d - 1} & \text{are called } \textit{facets}, \text{ with } d = dim\ \mathcal{P}. \end{cases}$$

Question: How can we show that $\pi x \leq \pi_0$ defines a facet of $\mathcal{P}$?

Approach 1: Find $dim\ \mathcal{P}$ affinely independent points satisfying $\pi x = \pi_0$.

The next approach is just an indirect but very useful way to verify affine independence of points.

Approach 2: Let $\mathcal{F} = \{P \in \mathcal{P} : \pi x = \pi_0\}$ and assume that all points in $\mathcal{F}$ also satisfy a further equation. We have to show that this equation is a linear combination of $\pi x = \pi_0$ and the equality set of $\mathcal{P}$. This is done by deriving dependencies between the coefficients by constructing points in $\mathcal{F}$ and inserting them into this further equation.

5.2 Dimension of the Polyhedron

Recall Wagners formulation of the hub covering problem given in [1]:

SAQI-W1

$$min \quad \sum_k X_{kk}$$

s.t.
$$\sum_{k:(i,k)\in VA} X_{ik} = 1 \qquad \forall i \in \{1, ..., n\} \tag{5.1}$$

$$X_{ik} \leq X_{kk} \qquad \forall (i,k) \in VA \tag{5.2}$$

$$X_{ik} \in \{0,1\} \qquad \forall (i,k) \in VA \tag{5.3}$$

$$X_{ik} + X_{jm} \leq 1 \qquad \forall (i,k,j,m) \in IA \tag{5.4}$$

Remember that by applying the aggregation procedure to IA we obtain SAQI-W2.

Definition:

- Let $\mathcal{X}_{SAQI-W2} := \{P = (X_{11}, .., X_{nn}) \in \mathbb{R}^q : \text{P feasible for } (SAQI-W2)\}$
  be the set of feasible solutions.

- Let $\mathcal{Z}_{SAQI-W2} := \{P \in \mathcal{X}_{SAQI-W2} : X_{ik} \in \{0,1\}, \forall (i,k) \in VA\}$
  be be the set of feasible integral points of $(SAQI-W2)$.

- Let $\mathcal{P}_{SAQI-W2} := conv(\mathcal{Z}_{SAQI-W2})$
  be the polyhedron defined by the integer solutions to $(SAQI-W2)$.

Theorem 5.2.1. *Let $\beta \geq \alpha \max_{i,j} (d_{ij})$ and $|VA| = q$.
Then dim $\mathcal{P}_{SAQI-W2} = q - n$.*

Proof. We have to show that there are $q - n + 1$ affinely independent points in $\mathcal{P}_{SAQI-W2}$.

Consider the following points:

(a) For $j \in \{1,..,n\}$ let $(j,l) \in VA, j \neq l$: $P = (X_{11}, ..., X_{nn}) \in \mathbb{R}^q$ with

$$X_{ik} = \begin{cases} 1 & i = s,\ \ k = s,\ \ s \in \{1,..,n\} \setminus \{j\}, \\ 1 & i = j,\ \ k = l, \\ 0 & \text{otherwise.} \end{cases}$$

Note that there exist $q - n$ elements $(j,l) \in VA$, $j \neq l$. Therefore we have $q - n$ points as described above.

(b) the point P with $X_{ss} = 1\ \forall s \in \{1, ..., n\}$ and all other values are 0.

These are $q - n + 1$ points in $\mathcal{P}_{SAQI-W2}$. They are affinely independent, since every vector occurring in (a) contains a '1' in a position where there have been only zeros in the other vectors. The point where all nodes are hubs is clearly independent of them. $\square$

Example 5.2.2. For $n = 4$ and $q = 12$ we obtain the following $q - n + 1$ affine independent points:

X_{11}	X_{12}	X_{13}	X_{22}	X_{23}	X_{24}	X_{31}	X_{32}	X_{33}	X_{42}	X_{43}	X_{44}
1	0	0	1	0	0	0	0	1	0	1	0
1	0	0	1	0	0	0	0	1	1	0	0
1	0	0	1	0	0	0	1	0	0	0	1
1	0	0	1	0	0	1	0	0	0	0	1
1	0	0	0	0	1	0	0	1	0	0	1
1	0	0	0	1	0	0	0	1	0	0	1
0	0	1	1	0	0	0	0	1	0	0	1
0	1	0	1	0	0	0	0	1	0	0	1
1	0	0	1	0	0	0	0	1	0	0	1

Corollary 5.2.3. *Let* $\beta \geq \alpha \max_{i,j}.\ (d_{ij})$
Then dim $\mathcal{P}_{HC-Lin} = n^2 - n$.

Proof. Follows since in Theorem 5.2.1 $q = n^2$ $\square$

Corollary 5.2.4. *If $|VA| = n^2$ then the dimension of SAQI-W2 and HC-Lin coincide.*

5.3 Elementary Facets

As we saw in the preceding section facets are proper faces of maximal dimension. Among all possible valid inequalities these inequalities which define facets are the strongest. Therefore the best we could possibly hope for is to know all valid inequalities that define facets, since then the feasible set of the LP-relaxation is the same as the convex hull of the integer feasible solutions. That is all extreme points are integer and thus an IP can be solved as a LP. Unfortunately determining all facets of a polyhedron is as hard as solving IP directly. However, even a few new added facets can improve the solution procedure.

In this section those inequalities given in the formulation of SAQI-W2 are investigated. We will examine the faces that are induced by constraints (5.1)-(5.4) and check which of them are facet defining.

Let $\mathcal{P} = \{x \in \mathbb{R}^n | Ax \leq b\}$ be some polyhedron and $S = \mathcal{P} \cap \mathbb{Z}^n$

Proposition 5.3.1. $[Nemhauser \;\; and \;\; Wolsey, [8]]$
If $\pi x \leq \pi_0$ is valid for S, it is also valid for $conv(S)$.

Theorem 5.3.2. $[Nemhauser \;\; and \;\; Wolsey, [8]]$
If $\pi x \leq \pi_0$ defines a face of dimension $k-1$ of $conv(S)$, there are k affinely independent points $x^1, ..., x^k \in S$ such that $\pi x^i = \pi_0$ for $i = 1, ..., k$.

Theorem 5.3.2 states that for establishing the dimensionality of a face of $conv(S)$, it suffices to consider points of S, hence we can restrict the search of affinely independent points to integral points. Note that Theorem 5.3.2 can be adapted to the case that $\mathcal{P} = \mathcal{X}_{SAQI-W2}$ and $S = \mathcal{Z}_{SAQI-W2}$

Proposition 5.3.3. *For all $(i,k) \in VA$ and $i \neq k$ the following valid inequality is a facet of $\mathcal{P}_{SAQI-W2}$:*

$$X_{ik} \geq 0 \tag{5.5}$$

Proof. To show that (5.5) defines a facet of $\mathcal{P}_{SAQI-W2}$, we have to show that there exist $dim\ \mathcal{P}_{SAQI-W2} = q - n$ affinely independent points, that fulfill (5.5) with equality.

Consider the following points:

(a) For $j \in \{1,..,n\}$ let $(j,l) \in VA, j \neq l$ and $(j,l) \neq (i,k)$:
$P = (X_{11}, ..., X_{nn}) \in \mathbb{R}^q$ with

$$X_{ik} = \begin{cases} 1 & i = s,\ \ k = s,\ \ s \in \{1,..,n\} \setminus \{j\}, \\ 1 & i = j,\ \ k = l, \\ 0 & \text{otherwise.} \end{cases}$$

Remember that there exist $q - n$ elements $(j,l) \in VA,\ j \neq l$. Since $(j,l) \neq (i,k)$ we have $q - n - 1$ points as described above.

(b) the point P with $X_{ss} = 1\ \forall s \in \{1,...,n\}$ and all other values are 0.

These are $q - n - 1 + 1 = q - n$ points in $\mathcal{P}_{SAQI-W2}$ satisfying (5.5) with equality. They are affinely independent, since every vector occurring in (a) contains a '1' in a position where there have been only zeros in the other vectors. The point where all nodes are hubs is clearly independent of them. □

Proposition 5.3.4. *If* $|\{i : (k,i) \in VA,\ i \neq k\}| = 1$ *then the valid inequality*

$$X_{kk} \leq 1 \qquad \forall\,(k,k) \in VA \tag{5.6}$$

represents a facet of $\mathcal{P}_{SAQI-W2}$.

Proof. Similarly to the proof of Proposition 5.3.3 we show that there are *dim* $\mathcal{P}_{SAQI-W2} = q - n$ affinely independent points lying on the face $\mathcal{F} = \{P \in \mathcal{P}_{SAQI-W2} : X_{kk} = 1\}$ ($|\{i : (k,i) \in VA,\ i \neq k\}| = 1$).

Since $|\{i : (k,i) \in VA,\ i \neq k\}| = 1$, there exist $n - q - 1$ points with $X_{ss} = 1$ $\forall s \in \{1, ..., n\} \setminus \{j\}$ $j \neq k$ and $X_{jl} = 1$ for $(j,l) \in VA$ $j \neq l$.
Since $X_{kk} = 1$ for all points they lie in $\mathcal{F}$. In every of those vectors there is exactly one of the entries of the form X_{jl} ($(j,l) \in VA$ $j \neq k \neq l$) not equal to zero, so these $q-n-1$ points are affinely independent. Next we choose the point, where all nodes are hubs. Since all X_{jl} entries are zero in this vector, it is linearly independent of the above vectors. Thus, in total we have $q - n - 1 + 1 = q - n$ affinely independent points in $\mathcal{F}$. □

We notice that we use the same kind of points in the proof of Proposition 5.3.3 and 5.3.4. Having a closer look we can conclude that $X_{ki} \geq 0$ $(i \neq k)$ and $X_{kk} \leq 1$ (if $|\{i : (k,i) \in VA,\ i \neq k\}| = 1$) defines the same facet and hence exactly one of them is needed in a minimal description.

Corollary 5.3.5. *If* $|\{i : (k,i) \in VA,\ i \neq k\}| = 1$ *then (5.5) and (5.6) represent the same facet.*

Proof. Without loss of generality we prove this corollary for $(k,i) = (1,2)$.
Let $\mathcal{F}_{\mathcal{I}} = \{P \in \mathcal{P}_{SAQI-W2} : X_{12} = 0\}$ and $\mathcal{F}_{\mathcal{II}} = \{P \in \mathcal{P}_{SAQI-W2} : X_{11} = 1\}$.
We show $\mathcal{F}_{\mathcal{I}} = \mathcal{F}_{\mathcal{II}}$.

$\mathcal{F}_{\mathcal{I}} \subseteq \mathcal{F}_{\mathcal{II}}$: $X \in \mathcal{F}_{\mathcal{I}} \Rightarrow X_{12} = 0 \overset{|\{i:(k,i)\in VA,\ i\neq k\}|=1}{\Rightarrow} X_{11} = 1 \Rightarrow X \in \mathcal{F}_{\mathcal{II}}$

$\mathcal{F}_{\mathcal{II}} \subseteq \mathcal{F}_{\mathcal{I}}$: $X \in \mathcal{F}_{\mathcal{II}} \Rightarrow X_{11} = 1 \overset{\sum_{i:(i,k)\in VA} X_{ik}=1}{\Rightarrow} X_{12} = 0 \Rightarrow X \in \mathcal{F}_{\mathcal{I}}$ □

Proposition 5.3.6. *If* $|\{i : (k,i) \in VA,\ i \neq k\}| \geq 2$ *then (5.6) does not represent a facet of* $\mathcal{P}_{SAQI-W2}$.

Proof. All points lying on the face $X_{kk} = 1$ also satisfy $X_{ki} = 0\ \forall (k,i) \in VA$, $i \neq k$. □

For the inequalities considered so far, we did not need the set IA. Remember that IA contains all incompatible assignments (i,k,j,m) for which $d_{ik}+\alpha d_{km}+d_{jm} > \beta$ $((i,k),(j,m) \in VA)$. However, due to constraints (5.1) and (5.2) we are not allowed to assign node i to k if k is not a hub and to assign node i to two different hubs. In the remainder of this thesis we have to handle with both incompatible assignments, those induced by IA and those induced by (5.1) and (5.2). For convenience we define the following set:

$$\begin{aligned} IA^{(s,t)} := \{\ &(j,m) : (j,m,s,t) \in IA\ (j < s)\ [resp.\ (s,t,j,m) \in IA\ (j > s)]\} \cup \\ &\{(t,m) : (t,m) \in VA, m \neq t\} \cup \{(s,m) : (s,m) \in VA, m \neq s,t\ \} \\ &\cup \{(m,s) : (m,s) \in VA, m \neq s\ \}. \end{aligned}$$

Hence, $IA^{(s,t)}$ contains all elements of VA which take a value of zero whenever $X_{st} = 1$.

Example 5.3.7. Consider the following sets:

VA ={ (1,1), (1,3), (1,4), (2,2), (2,3), (2,4), (3,3), (3,4), (3,5), (4,1), (4,3), (4,4), (4,5), (5,3), (5,4), (5,5) }

IA = {(1,3,2,4), (1,3,5,4), (2,3,4,1), (2,3,5,4) }

Let $(s,t) = (1,3)$ then

$$IA^{(1,3)} = \{(2,4),(5,4)\} \cup \{(3,4),(3,5)\} \cup \{(1,4)\} \cup \{(4,1)\}$$

Proposition 5.3.8. *If* $\bigcap\limits_{\substack{i:(k,i)\in VA \\ i\neq k}} IA^{(k,i)} = \emptyset$ *then*

$$X_{kk} \geq 0 \qquad \forall\,(k,k) \in VA \tag{5.7}$$

represents a facet of $\mathcal{P}_{SAQI-W2}$.

Proof. We prove that (5.7) defines a facet (if $\bigcap\limits_{\substack{i:(k,i)\in VA \\ i\neq k}} IA^{(k,i)} = \emptyset$) using the indirect method. Let $\mathcal{F} = \{P \in \mathcal{P}_{SAQI-W1} : X_{kk} = 0\}$. We assume that there exists a further equation

$$\sum_{l} \sum_{s:(l,s)\in VA} \lambda_{ls}\, X_{ls} \;=\; \lambda_0 \tag{5.8}$$

that is satisfied by all points in $\mathcal{F}$. In order to show that (5.8) is a linear combination of $X_{kk} = 0$ and $\sum\limits_{k:(i,k)\in VA} X_{ik} = 1$ we need to show:

1. $\lambda_{ls} \stackrel{!}{=} \lambda_{ll} =: \lambda_l \;\forall l \neq k$
2. $\lambda_{ks} \stackrel{!}{=} \lambda_{kt} =: \lambda_k \;\forall s,t \neq k$

For step 1 consider the following two points in $\mathcal{F}$:

By assumption there exist at least one $i \neq k$ such that $(l,s) \notin IA^{(k,i)}$.

point 1: $X_{dd} = 1 \;\forall d \in \{1,...,n\} \setminus \{k,l\}$
$X_{ki} = 1 \;(k,i) \in VA$
$X_{ls} = 1$

point 2: $X_{dd} = 1 \;\forall d \in \{1,...,n\} \setminus \{k\}$
$X_{ki} = 1 \;(k,i) \in VA$
$X_{ll} = 1$

Inserting both points in (5.8) and compare the expressions we obtain $\lambda_{ls} = \lambda_{ll}$ $\forall l \neq k$.

Step 2 can be shown by using the points:

point 1: $X_{dd} = 1 \ \forall d \in \{1, ..., n\} \setminus \{k\}$
$X_{ks} = 1 \ (k,s) \in VA, \ s \neq k$

point 2: $X_{dd} = 1 \ \forall d \in \{1, ..., n\} \setminus \{k\}$
$X_{kt} = 1 \ (k,t) \in VA, \ t \neq k$

As above, inserting *point 1* and *2* yields $\lambda_{ks} = \lambda_{kt} \ \forall s, t \neq k$

It is easy to see that all constructed points lie in $\mathcal{F}$. Now, we can conclude

$$\begin{aligned} \lambda_0 &= \sum_l \sum_{s:(l,s) \in VA} \lambda_{ls} X_{ls} \\ &= \sum_{l \neq k} \sum_{s:(l,s) \in VA} \lambda_{ls} X_{ls} + \sum_{\substack{s:(k,s) \in VA \\ s \neq k}} \lambda_{ks} X_{ks} + \lambda_{kk} X_{kk} \\ \overset{1,2,X_{kk}=0}{\Longleftrightarrow} \ \lambda_0 - \sum_l \lambda_l &= 0. \end{aligned}$$

Thus (5.8) is, indeed, a linear combination of the given equations. □

Proposition 5.3.9. *The valid inequality*

$$X_{ik} \leq 1 \qquad \forall \, (i,k) \in VA \quad i \neq k \tag{5.9}$$

does not represent a facet of $\mathcal{P}_{SAQI-W2}$.

Proof. All points lying on the face $X_{ik} = 1$ also satisfy $X_{kk} = 1$ and $X_{im} = 0$ for $(i,m) \in VA$ and $m \neq k$. □

Proposition 5.3.10. *The valid inequality*

$$X_{ik} \leq X_{kk} \qquad \forall\,(i,k) \in VA \quad i \neq k \tag{5.10}$$

defines a facet of $\mathcal{P}_{SAQI-W2}$ *if*

$$IA^{(i,k)} \bigcap_{\substack{i:(k,i)\in VA \\ i\neq k}} IA^{(k,i)} = \emptyset.$$

Remark 5.3.11. Let $IA^{(i,k)} \bigcap_{i:(k,i)\in VA, i\neq k} IA^{(k,i)} = \{(r,q)\}$ and assume that $\mathcal{F} = \{P \in \mathcal{P}_{SAQI-W2} : X_{ik} = X_{kk}\}$ defines a facet. Then by Theorem 5.3.2 there exist $P_1, ..., P_{q-n}$ affinely independent points in $\mathcal{F} \cap \mathcal{Z}_{SAQI-W2}$. Now, assuming $X_{rq} = 0$ for all points in $\mathcal{F}$, we would obtain $dim\ \mathcal{F} < dim\ \mathcal{P}_{SAQI-W2} - 1$, which is a contradiction. Therefore it exists a point P_i with $X_{rq} = 1$. But then, since $IA^{(i,k)} \bigcap_{i:(k,i)\in VA, i\neq k} IA^{(k,i)} = \{(r,q)\}$ it follows $1 = lhs(5.10) = rhs(5.10) = 1$, which is a contradiction.

Inequality (5.10) has been shown to be a special case of a more general class of facet-defining inequalities presented in the next section. Hence we will not give the proof of Proposition 5.3.9 here, but in Theorem 5.4.2.

Proposition 5.3.12. *The valid inequality*

$$X_{ik} + X_{jm} \leq 1 \quad (i,k,j,m) \in IA \tag{5.11}$$

represent a facet of $\mathcal{P}_{SAQI-W2}$ *if*

$$IA^{(i,k)} \cap IA^{(j,m)} = \emptyset.$$

Proof. see Proposition 5.3.13 ∎

For the same reason as in Remark 5.3.11 it yields that whenever (k, j) or (j, i) $\in$ VA then obviously $X_{kj} = 0$ resp. $X_{ji} = 0$ for all points satisfying (5.11) with equality.

Lastly, we have to check the aggregation constraints to be facet-defining.

Proposition 5.3.13. *The valid inequalities*

$$X_{ik} + \sum_{m \in M^{(i,k,j,\cdot)}} X_{jm} \leq 1 \tag{5.12}$$

and

$$\sum_{k \in M^{(i,\cdot,j,m)}} X_{ik} + X_{jm} \leq 1 \tag{5.13}$$

represent facets of $\mathcal{P}_{SAQI-W2}$ *if*

$$IA^{(i,k)} \cap \Big(\bigcap_{m \in M^{(i,k,j,\cdot)}} IA^{(j,m)} \Big) = \emptyset$$

and

$$IA^{(j,m)} \cap \Big(\bigcap_{k \in M^{(i,\cdot,j,m)}} IA^{(i,k)} \Big) = \emptyset \quad \textit{respectively.}$$

We notice that (5.12) and (5.13) is either the same as (5.11) or dominates it.

Proof. We restrict ourself to prove that (5.12) represents a facet using the indirect method. The facet-defining property of (5.13) can be proven analogously. Now, let $\mathcal{F} = \{P \in \mathcal{P}_{SAQI-W2} : X_{ik} + \sum_{m \in M^{(i,k,j,\cdot)}} X_{jm} = 1\}$
and assume that all points in $\mathcal{F}$ satisfy an further equation

$$\sum_{l} \sum_{s:(l,s) \in VA} \lambda_{ls} \, X_{ls} \; = \; \lambda_0 \tag{5.14}$$

We show that (5.12) is linearly dependent on $X_{ik} + \sum_{m \in M^{(i,k,j,\cdot)}} X_{jm} = 1$ and $\sum_{k:(i,k) \in VA} X_{ik} = 1$, by showing that:

1. (a) $\lambda_{ls} \stackrel{!}{=} \lambda_{ll} =: \lambda_l \quad \forall (l,s) \in VA,\, l \neq i,j$

   (b) $\lambda_{is} \stackrel{!}{=} \lambda_{ii} =: \lambda_i \quad \forall (i,s) \in VA, s \neq k$

   (c) $\lambda_{js} \stackrel{!}{=} \lambda_{jj} =: \lambda_j \quad \forall (j,s) \in VA, s \notin M^{(i,k,j,\cdot)}$

2. $\lambda_{jm} - \lambda_j \stackrel{!}{=} \lambda_{ik} - \lambda_i =: \lambda \quad \forall m \in M^{(i,k,j,\cdot)}$

Now, we can conclude

$$\begin{aligned}
\lambda_0 &= \sum_l \sum_{s:(l,s) \in VA} \lambda_{ls} X_{ls} \\
&= \sum_{l \neq i,j} \sum_{s:(l,s) \in VA} \lambda_{ls} X_{ls} + \sum_{\substack{s:(i,s) \in VA \\ s \neq k}} \lambda_{is} X_{is} + \lambda_{ik} X_{ik} \\
&\quad + \sum_{s \notin M^{(i,k,j,\cdot)}} \lambda_{js} X_{js} + \sum_{m \in M^{(i,k,j,\cdot)}} \lambda_{jm} X_{jm} \\
\stackrel{1a-1c}{\iff} \lambda_0 - \sum_l \lambda_l &= (\lambda_{ik} - \lambda_i) X_{ik} + \sum_{m \in M^{(i,k,j,\cdot)}} (\lambda_{jm} - \lambda_j) X_{jm} \\
\stackrel{2}{\iff} \lambda_0 - \sum_l \lambda_l &= \lambda (X_{ik} + \sum_{m \in M^{(i,k,j,\cdot)}} X_{jm})
\end{aligned}$$

Next, we prove the required steps $1(a) - 2$ by constructing points in $\mathcal{F}$ and inserting them in (5.12).

claim 1(a) : $\quad \lambda_{ls} \stackrel{!}{=} \lambda_{ll} \; \forall (l,s) \in VA,\, l \neq i,j$

<u>case 1:</u> $\quad (l,s) \notin IA^{(i,k)}$

point 1: $X_{dd} = 1 \; \forall d \in \{1,...,n\} \setminus \{i,l\}$

$X_{ik} = 1$

$X_{ls} = 1$

point 2: $X_{dd} = 1\ \forall d \in \{1,...,n\} \setminus \{i\}$

$X_{ik} = 1$

case 2: $(l,s) \in IA^{(i,k)}$

By assumption there exists at least one (j,m) such that $(l,s) \notin IA^{(j,m)}, m \in M^{(i,k,j,\cdot)}$.

point 1: $X_{dd} = 1\ \forall d \in \{1,...,n\} \setminus \{j,l\}$

$X_{jm} = 1$ for $m \in M^{(i,k,j,\cdot)}$

$X_{ls} = 1$

point 2: $X_{dd} = 1\ \forall d \in \{1,...,n\} \setminus \{j\}$

$X_{jm} = 1$ for $m \in M^{(i,k,j,\cdot)}$

claim 1(b) : $\lambda_{is} \stackrel{!}{=} \lambda_{ii}\ \forall (i,s) \in VA,\ s \neq k$

Since $(i,s) \in IA^{(i,k)}$ there exists, by assumption, at least one (j,m) with $(i,s) \notin IA^{(j,m)}$, $m \in M^{(i,k,j,\cdot)}$.

point 1: $X_{dd} = 1\ \forall d \in \{1,...,n\} \setminus \{i,j\}$

$X_{is} = 1$

$X_{jm} = 1$ for $m \in M^{(i,k,j,\cdot)}$

point 2: $X_{dd} = 1\ \forall d \in \{1,...,n\} \setminus \{j\}$

$X_{jm} = 1$

claim 1(c) : $\lambda_{js} \stackrel{!}{=} \lambda_{jj}\ s \notin M^{(i,k,j,\cdot)}$

point 1: $X_{dd} = 1\ \forall d \in \{1,...,n\} \setminus \{i,j\}$

$X_{js} = 1$

$X_{ik} = 1$

point 2: $X_{dd} = 1\ \forall d \in \{1,...,n\} \setminus \{i\}$

$X_{ik} = 1$

Now, (5.12) can be written as

$$\lambda_0 - \sum_l \lambda_l = (\lambda_{ik} - \lambda_i)X_{ik} + \sum_{m \in AGG^{(i,k,j)}} (\lambda_{jm} - \lambda_j)X_{jm} \tag{5.15}$$

claim 2: $\lambda_{jm} - \lambda_j \stackrel{!}{=} \lambda_{ik} - \lambda_i \ \forall m \in M^{(i,k,j,\cdot)}$

point 1: $X_{dd} = 1 \ \forall d \in \{1, ..., n\} \setminus \{i\}$
$X_{ik} = 1$

point 2: $X_{dd} = 1 \ \forall d \in \{1, ..., n\} \setminus \{j\}$
$X_{jm} = 1 \ m \in M^{(i,k,j,\cdot)}$

By evaluating (5.15) at the above points we obtain $\lambda_{js} - \lambda_j = \lambda_{ik} - \lambda_i$, which concludes the proof.

It is easy to see that all constructed points lie in $\mathcal{F}$, since $1 = lhs(5.12) = rhs(5.12) = 1$. □

5.4 Non-elementary Facets

This section introduces two new classes of valid inequalities for $\mathcal{P}_{SAQI-W2}$ and sufficient conditions for these inequalities to be facet-inducing.

In order to deduce new valid inequalities for $\mathcal{P}_{SAQI-W2}$ the PORTA tool (Polyhedron Representation Transformation Algorithm [7]) has been used. A complete description how to find facets using PORTA is given in Appendix A.

By Proposition 5.3.10 we know that constraint '$X_{ik} \leq X_{kk}$' is only facet-defining if $IA^{(i,k)} \bigcap_{i:(k,i) \in VA, i \neq k} IA^{(k,i)} = \emptyset$. Next, we derive new facet-defining inequalities that generalize the above constraint.

Theorem 5.4.1. *Let* $(i,k) \in VA,\ k \neq i$ *and let* $|\{i : (k,i) \in VA,\ i \neq k\}| = 1$ *then*

$$X_{ik} + \sum_{\substack{m:(i,m)\in VA \\ m\neq i,k}} X_{im} \leq X_{kk} \tag{5.16}$$

is valid for $\mathcal{P}_{SAQI-W2}$ *and defines a facet if*

$$\bigcap_{\substack{m:(i,m)\in VA \\ m\neq i}} IA^{(i,m)} \cap IA^{(k,i)} = \emptyset$$

Proof. We only prove the validity of (5.16). The facet defining property will be shown for a more general case.

Assume $X_{kk} = 0$. Then, since for k only $(k,i) \in VA$, $X_{ki} = 1$ and thus $X_{is} = 0$ $\forall s : (i,s) \in VA$. Thus inequality (5.16) holds with equality.

Now, assume $X_{kk} = 1$. Then either $rhs(5.16) = 0$ (e.g. $X_{ii} = 1$) or $rhs(5.16) = 1$ (since $\sum_{k:(i,k)\in VA} X_{ik} = 1$). Thus inequality (5.16) holds. □

Note that the extended inequality (5.16) is either the same as the original inequality (5.10) (if $\sum_{\substack{m:(i,m)\in VA \\ m\neq k,i}} X_{im} = 0$) or dominates the original one.

Theorem 5.4.1 represents just a special case of the following theorem.

Theorem 5.4.2. (*Facet I*)
Let $(i,k) \in VA,\ k \neq i$ *and let*
$IA_j(\mathcal{F}) = \{m | (j,m) \in \bigcap_{\substack{r:(k,r)\in VA \\ r\neq k}} IA^{(k,r)} \cap IA^{(i,k)}\}$[2] $(j \in \{1,..,n\})$. *Then*

$$X_{ik} + \sum_{m\in IA_j(\mathcal{F})} X_{jm} \leq X_{kk} \tag{5.17}$$

[2]Note that (j,j) cannot lie in $\bigcap_{r:(k,r)\in VA} IA^{(k,r)} \cap IA^{(i,k)}$.

is valid for $\mathcal{P}_{SAQI-W2}$ and represents a facet of $\mathcal{P}_{SAQI-W2}$ if

$$\bigcap_{m \in IA_j(\mathcal{F})} IA^{(j,m)} \bigcap_{\substack{r:(k,r)\in VA \\ r\neq k}} IA^{(k,r)} \cap IA^{(i,k)} = \emptyset \tag{5.18}$$

Proof. First we prove the validity of (5.17).

Assume $X_{kk} = 0$. Then we have to allocate k to any s with $(k,s) \in VA$ which implies $X_{ik} = 0$. Since $m \in IA_j(\mathcal{F})$ we obtain that $\sum_{m \in IA_j(\mathcal{F})} X_{jm} = 0$. Thus inequality (5.17) holds with equality.

Now, assume $X_{kk} = 1$. Then either $rhs(5.17) = 0$ (e.g. $X_{ii} = 1$) or $rhs(5.17) = 1$, that is due to definition of $IA_j(\mathcal{F})$.

Now, we prove that (5.17) is facet-defining.

Let $\mathcal{F} = \{P \in \mathcal{P}_{SAQI-W2} : X_{ik} + \sum_{m \in IA_j(\mathcal{F})} X_{jm} = X_{kk}\}$ and assume that all points lying in $\mathcal{F}$ also satisfy

$$\sum_{l} \sum_{s:(l,s)\in VA} \lambda_{ls} X_{ls} \; = \; \lambda_0. \tag{5.19}$$

We have to show that (5.19) is linearly dependent on $\sum_{s:(l,s)\in VA} X_{ls}$ and $X_{ik} + \sum_{m \in IA_j(\mathcal{F})} X_{jm} - X_{kk} = 0$.

For this sake, we prove the following:

1. (*a*) $\lambda_{ls} \stackrel{!}{=} \lambda_{ll} =: \lambda_l \quad \forall (l,s) \in VA,\, l \neq i,k,j$

   (*b*) $\lambda_{is} \stackrel{!}{=} \lambda_{ii} =: \lambda_i \quad \forall (i,s) \in VA, s \neq k$

   (*c*) $\lambda_{ks} \stackrel{!}{=} \lambda_{kt} =: \lambda_k \quad \forall (k,s),(k,t) \in VA, s,t \neq k$

   (*d*) $\lambda_{js} \stackrel{!}{=} \lambda_{jj} =: \lambda_j \quad \forall s \notin IA_j(\mathcal{F})$

2. $\lambda_{jm} - \lambda_j \stackrel{!}{=} \lambda_{ik} - \lambda_i =: \lambda \quad \forall m \in IA_j(\mathcal{F})$

3. $\lambda \stackrel{!}{=} -(\lambda_{kk} - \lambda_k)$

With steps 1(a) - 3 being correct we can conclude:

$$\begin{aligned}
\lambda_0 &= \sum_{l} \sum_{s:(l,s)\in VA} \lambda_{ls} X_{ls} \\
&= \sum_{l\neq i,j,k} \sum_{s:(l,s)\in VA} \lambda_{ls} X_{ls} + \sum_{\substack{s:(i,s)\in VA \\ s\neq k}} \lambda_{is} X_{is} + \lambda_{ik} X_{ik} \\
&\quad + \sum_{s\notin IA_j(\mathcal{F})} \lambda_{js} X_{js} + \sum_{m\in IA_j(\mathcal{F})} \lambda_{jm} X_{jm} \\
&\quad + \sum_{\substack{s:(k,s)\in VA \\ s\neq k}} \lambda_{ks} X_{ks} + \lambda_{kk} X_{kk}
\end{aligned}$$

$$\stackrel{1a-1d}{\Longleftrightarrow} \lambda_0 - \sum_{l} \lambda_l = (\lambda_{ik} - \lambda_i) X_{ik} + \sum_{m\in IA_j(\mathcal{F})} (\lambda_{jm} - \lambda_j) X_{jm} + (\lambda_{kk} - \lambda_k) X_{kk}$$

$$\stackrel{3,4}{\Longleftrightarrow} \lambda_0 - \sum_{l} \lambda_l = \lambda (X_{ik} + \sum_{m\in IA_j(\mathcal{F})} X_{jm} - X_{kk})$$

which establishes that (5.19) is indeed a linear combination of the given equations.

Next, we prove the required steps $1(a) - 3$ by constructing points that lie in $\mathcal{F}$ and inserting them in (5.19).

claim 1(a) : $\quad \lambda_{ls} \stackrel{!}{=} \lambda_{ll} \qquad \forall (l,s) \in VA,\ l \neq i,k,j$

<u>case 1:</u> $\quad (l,s) \notin IA^{(i,k)}$

point 1: $X_{dd} = 1 \quad \forall d \in \{1,...,n\} \setminus \{i,l\}$

$X_{ik} = 1$

$X_{ls} = 1 \quad$ for $s : (l,s) \in VA$

point 2: $X_{ls} = 0$

$X_{ll} = 1$ and all other values as in *point 1*

These two points lie in $\mathcal{F}$, since for both points $rhs(5.17) = lhs(5.17) = 1$.

<u>case 2:</u> $(l, s) \in IA^{(i,k)}$ and $(l, s) \notin IA^{(k,r)}$, $r \neq k$

point 1: $X_{dd} = 1 \quad \forall d \in \{1, ..., n\} \setminus \{k, l\}$

$X_{kr} = 1$ for some $(k, r) \in VA$

$X_{ls} = 1$

point 2: $X_{ls} = 0$

$X_{ll} = 1$ and all other values as in *point 1*

<u>case 3:</u> $(l, s) \in IA^{(i,k)} \bigcap\limits_{\substack{r:(k,r) \in VA \\ r \neq k}} IA^{(k,r)}$

By assumption, there exists at least one (j, m) such that $(l, s) \notin IA^{(j,m)}$, $m \in IA_j(\mathcal{F})$

point 1: $X_{dd} = 1 \quad \forall d \in \{1, ..., n\} \setminus \{j, l\}$

$X_{jm} = 1$ $m \in IA_j(\mathcal{F})$

$X_{ls} = 1$

point 2: $X_{ls} = 0$

$X_{ll} = 1$ and all other values as in *point 1*

claim 1(b) : $\lambda_{is} \stackrel{!}{=} \lambda_{ii} \qquad \forall (i, s) \in VA, s \neq k$

<u>case 1:</u> $(i, s) \notin IA^{(k,r)}$, $r \neq k$

point 1: $X_{dd} = 1 \quad \forall d \in \{1, ..., n\} \setminus \{i, k\}$

$X_{kr} = 1$ $(k, r) \in VA$

$X_{is} = 1$

point 2: $X_{is} = 0$

$X_{ii} = 1$ and all other values as in *point 1*

Both points satisfy $rhs(5.17) = lhs(5.17) = 0$ and hence lie in $\mathcal{F}$.

case 2: $\quad (i,s) \in \bigcap_{\substack{r:(k,r)\in VA \\ r\neq k}} IA^{(k,r)}$ and $(i,s) \notin IA^{(j,m)}$, $m \in IA_j(\mathcal{F})$

point 1: $X_{dd} = 1 \quad \forall d \in \{1,...,n\} \setminus \{i,j\}$

$X_{jm} = 1 \; m \in IA_j(\mathcal{F})$

$X_{is} = 1$

point 2: $X_{is} = 0$

$X_{ii} = 1$ and all other values as in *point 1*

claim 1(c) : $\quad \lambda_{ks} \overset{!}{=} \lambda_{kt} \qquad \forall (k,s) \in VA,\; s,t \neq k$

point 1: $X_{dd} = 1 \quad \forall d \in \{1,...,n\} \setminus \{k\}$

$X_{ks} = 1 \; (k,s) \in VA$

point 2: $X_{dd} = 1 \quad \forall d \in \{1,...,n\} \setminus \{i\}$

$X_{kt} = 1 \; (k,t) \in VA$

It is obvious that both points satisfying (5.17) with equality.

claim 1(d) : $\quad \lambda_{js} \overset{!}{=} \lambda_{jj} \qquad \forall s \notin IA_j(\mathcal{F})$

If $(j,s) \notin IA^{(i,k)}$ and $s \notin IA_j(\mathcal{F})$ consider the following points in $\mathcal{F}$:

point 1: $X_{dd} = 1 \quad \forall d \in \{1,...,n\} \setminus \{i,j\}$

$X_{js} = 1 \quad s : (j,s) \in VA, s \neq j, s \notin IA_j(\mathcal{F})$

$X_{ik} = 1$

point 2: $X_{jj} = 1$

$X_{js} = 0$ and all other elements as in *point 1*

otherwise

$\forall (j,s) \in IA^{(i,k)}\ \exists r \neq k : (j,s) \notin IA^{(k,r)}$. Consider the points:

point 1: $X_{dd} = 1 \quad \forall d \in \{1,...,n\} \setminus \{k,j\}$

$X_{js} = 1 \qquad (j,s) \in VA,\ s \neq j,\ s \notin IA_j(\mathcal{F})$

$X_{kr} = 1$

point 2: $X_{js} = 0$

$X_{jj} = 1$ and all other elements as in *point 1*

With steps $1(a) - 3$ (5.19) can be reformulate to

$$\lambda_0 - \sum_l \lambda_l = (\lambda_{ik} - \lambda_i) X_{ik} + \sum_{m \in IA_j(\mathcal{F})} (\lambda_{jm} - \lambda_j) X_{jm} + (\lambda_{kk} - \lambda_k) X_{kk} \quad (5.20)$$

claim 2: $\lambda_{jm} - \lambda_j \overset{!}{=} \lambda_{ik} - \lambda_i\ \forall m \in IA_j(\mathcal{F})$

point 1: $X_{dd} = 1 \quad \forall d \in \{1,...,n\} \setminus \{j\}$

$X_{jm} = 1 \quad m \in IA_j(\mathcal{F})$

point 2: $X_{ik} = 1$

$X_{jm} = 0$ and all other elements as in *point 1*

point 1 and $2 \in \mathcal{F}$, since $rhs(5.17) = lhs(5.17) = 1$

By evaluating (5.19) at these two points, we obtain $\lambda_{jm} - \lambda_j = \lambda_{ik} - \lambda_i\ \forall m \in IA_j(\mathcal{F})$.

claim 3: $\lambda \overset{!}{=} -(\lambda_{kk} - \lambda_k)$

With step 2 being correct (5.20) reduces to

$$\lambda_0 - \sum_l \lambda_l = \lambda X_{ik} + \lambda \sum_{m \in IA_j(\mathcal{F})} X_{jm} - (\lambda_{kk} - \lambda_k) X_{kk} \tag{5.21}$$

Now, consider

$$\begin{aligned} \textit{point 1: } X_{dd} &= 1 \quad \forall d \neq k \\ X_{kl} &= 1 \ (k,l) \in VA,\ l \neq k \end{aligned}$$

$(rhs(5.17) = lhs(5.17) = 0)$

$$\begin{aligned} \textit{point 2: } X_{dd} &= 1 \quad \forall d \neq i \\ X_{ik} &= 1 \end{aligned}$$

$(rhs(5.17) = lhs(5.17) = 1)$

Plugging in these points in (5.21) yields

$$\textit{point 1:} \quad \lambda_0 - \sum_l \lambda_l = 0$$

$$\textit{point 2:} \quad \lambda_0 - \sum_l \lambda_l = \lambda + (\lambda_{kk} - \lambda_k)$$

Thus $\lambda = -(\lambda_{kk} - \lambda_k)$ which concludes the proof.

□

Remark 5.4.3. Actually inequality (5.17) can be further generalized by replacing $\sum_{m \in IA_j(\mathcal{F})} X_{jm}$ by sums that contain all variables which take a value of 1 concurrently, and lie in $\bigcap_{r:(k,r) \in VA} IA^{(k,r)} \cap IA^{(i,k)}$. It is obvious, that the determination of all those mentioned variables is too time consuming. Therefore, we consider only one sum with fixed j.

Theorem 5.4.4. (*Facet II*)
Let $(i,b) \in VA$ $(i \neq b)$, $(j,m) \in IA^{(i,b)}$ $(j \neq i)$, $A_i := \{a : (i,a) \in VA, a \neq b\}$ *and*
$\mathcal{F}^r_{A_i} := \{a \in A_i\colon (r,a) \in IA^{(j,m)} \cap IA^{(i,b)}\}$ $(r \neq i)$.
Then

$$\sum_{a \in A_i} X_{ia} \geq X_{jm} + \sum_{a \in \mathcal{F}^r_{A_i}} X_{ra} \tag{5.22}$$

is valid for $\mathcal{P}_{SAQI-W2}$ *and represents a facet of* $\mathcal{P}_{SAQI-W2}$ *if*

$$IA^{(j,m)} \cap IA^{(i,b)} \bigcap_{a \in \mathcal{F}^r_{A_i}} IA^{(r,a)} = \emptyset \tag{5.23}$$

Proof. As in the previous proof we first prove the validity of (5.22).

Assume $\sum\limits_{a \in A_i} X_{ia} = 0$. Then, of course $X_{ib} = 1$. Since $(j,m) \in IA^{(i,b)}$ and $a \in \mathcal{F}^r_{A_i}$ we have that $rhs(5.22) = 0$. Thus (5.22) holds with equality.

Now, assume $\sum\limits_{a \in A_i} X_{ia} = 1$. Due to $\sum\limits_{s:(r,s) \in VA} X_{rs} = 1$ and $a \in \mathcal{F}^r_{A_i}$ we obtain that either $rhs(5.22) = 0$ or 1.

Next, we prove that (5.22) is a facet of $\mathcal{P}_{SAQI-W2}$.

Let $\mathcal{F} = \{P \in \mathcal{P}_{SAQI-W2} : \sum\limits_{a \in A_i} X_{ia} = X_{jm} + \sum\limits_{a \in \mathcal{F}^r_{A_i}} X_{ra}\}$ and assume that

$$\sum_{l} \sum_{s:(l,s) \in VA} \lambda_{ls} \, X_{ls} \; = \; \lambda_0 \tag{5.24}$$

holds for all points in $\mathcal{F}$.

In order to show that (5.24) is a linear combination of the given equations we prove the following steps:

1. (a) $\lambda_{ls} \stackrel{!}{=} \lambda_{ll} =: \lambda_l \quad \forall (l,s) \in VA,\ l \neq i,j,r$

   (b) $\lambda_{js} \stackrel{!}{=} \lambda_{jj} =: \lambda_j \quad \forall (j,s) \in VA, s \neq m$

   (c) $\lambda_{rs} \stackrel{!}{=} \lambda_{rr} =: \lambda_r \quad \forall s : (r,s) \in VA, s \notin \mathcal{F}^r_{A_i}$

   (d) $\lambda_{ib} =: \lambda_i$

2. $\lambda_{jm} - \lambda_j \stackrel{!}{=} \lambda_{ra} - \lambda_r =: \lambda \ \ \forall a \in \mathcal{F}^r_{A_i}$

3. $\lambda \stackrel{!}{=} -(\lambda_{ia} - \lambda_i) \ \ \forall a \in A_i$

With $1(a) - 3$ being correct we can reformulate (5.24):

$$
\begin{aligned}
\lambda_0 &= \sum_{l=1}^{n} \sum_{s:(l,s)\in VA} \lambda_{ls} X_{ls} \\
&= \sum_{l\neq i,j,r} \sum_{s:(l,s)\in VA} \lambda_{ls} X_{ls} + \sum_{\substack{s:(j,s)\in VA\\ s\neq m}} \lambda_{js} X_{js} + \lambda_{jm} X_{jm} \\
&\quad + \sum_{a\in\mathcal{F}^r_{A_i}} \lambda_{ra} X_{ra} + \sum_{\substack{s\notin\mathcal{F}^r_{A_i}\\ (r,s)\in VA}} \lambda_{rs} X_{rs} \\
&\quad + \sum_{a\in A_i} \lambda_{ia} X_{ia} + \lambda_{ib} X_{ib} \\
\stackrel{1a-1d}{\Longleftrightarrow} \lambda_0 - \sum_l \lambda_l &= (\lambda_{jm} - \lambda_j) X_{jm} + \sum_{a\in\mathcal{F}^r_{A_i}} (\lambda_{ra} - \lambda_r) X_{ra} + \sum_{a\in A_i} (\lambda_{ia} - \lambda_i) X_{ia} \\
\stackrel{2,3}{\Longleftrightarrow} \lambda_0 - \sum_l \lambda_l &= \lambda (X_{jm} + \sum_{a\in\mathcal{F}^r_{A_i}} X_{ra} - \sum_{a\in A_i} X_{ia}).
\end{aligned}
$$

Thus, equation (5.24) is indeed a linear combination of the given equations.

Next, we give the proofs for $1(a) - 3$.

claim 1(a) : $\quad \lambda_{ls} \stackrel{!}{=} \lambda_{ll} \quad \forall (l,s) \in VA,\ l \neq i,j,r$

<u>case 1:</u> $\quad (l,s) \notin IA^{(i,b)}$

point 1: $X_{dd} = 1 \quad \forall d \in \{1, .., n\} \setminus \{i, l\}$
$X_{ib} = 1$
$X_{ls} = 1$

point 2: $X_{ls} = 0$
$X_{ll} = 1$ and all other values as in *point 1*

These two points lie in $\mathcal{F}$, since for both points $rhs(5.22) = lhs(5.22) = 0$.

<u>case 2:</u> $(l, s) \in IA^{(i,b)}$ and $(l, s) \notin IA^{(j,m)}$

point 1: $X_{dd} = 1 \quad \forall d \in \{1, .., n\} \setminus \{j, l\}$
$X_{jm} = 1$
$X_{ls} = 1$

point 2: $X_{ls} = 0$
$X_{ll} = 1$ and all other values as in *point 1*

point 1 and 2 satisfy $rhs(5.22) = lhs(5.22) = 1$

<u>case 3:</u> $(l, s) \in IA^{(i,b)} \cap IA^{(j,m)}$

Hence, by assumption there exists $a \in \mathcal{F}^r_{A_i} : (l, s) \notin IA^{(r,a)}$

point 1: $X_{dd} = 1 \quad \forall d \in \{1, .., n\} \setminus \{l, r\}$
$X_{ra} = 1$
$X_{ls} = 1$

point 2: $X_{ls} = 0$
$X_{ll} = 1$ and all other values as in *point 1*

Both points lie in $\mathcal{F}$, since $rhs(5.22) = lhs(5.22) = 1$

claim 1(b) : $\lambda_{js} \overset{!}{=} \lambda_{jj} \quad \forall (j, s) \in VA,\ s \neq m$

<u>case 1:</u> $(j, s) \notin IA^{(i,b)}$

point 1: $X_{dd} = 1 \quad \forall d \in \{1, .., n\} \setminus \{i, j\}$

$X_{ib} = 1$

$X_{js} = 1$

point 2: $X_{js} = 0$

$X_{jj} = 1$ and all other values as in *point 2*

These two points lie in $\mathcal{F}$, since for both points $rhs(5.22) = lhs(5.22) = 0$.

<u>case 2:</u> $(j, s) \in IA^{(i,b)}$ and $(l, s) \notin IA^{(r,a)}$, $a \in \mathcal{F}^r_{A_i}$

point 1: $X_{dd} = 1 \quad \forall d \in \{1, .., n\} \setminus \{j, r\}$

$X_{ra} = 1$

$X_{js} = 1$

point 2: $X_{js} = 0$

$X_{jj} = 1$ and all other values as in *point 1*

point 1 and *2* satisfy $rhs(5.22) = lhs(5.22) = 1$

claim 1(c) : $\lambda_{rs} \overset{!}{=} \lambda_{rr} \quad \forall (r, s) \in VA,\ s \notin \mathcal{F}^r_{A_i}$

<u>case 1:</u> $(r, s) \notin IA^{(i,b)}$

point 1: $X_{dd} = 1 \quad \forall d \in \{1, .., n\} \setminus \{i, r\}$

$X_{ib} = 1$

$X_{rs} = 1$

point 2: $X_{rs} = 0$

$X_{rr} = 1$ and all other values as in *point 1*

These two points lie in $\mathcal{F}$, since for both points $rhs(5.22) = lhs(5.22) = 0$.

<u>case 2:</u> $(r, s) \in IA^{(i,b)}$ and $(r, s) \notin IA^{(j,m)}$

point 1: $X_{dd} = 1 \quad \forall d \in \{1,..,n\} \setminus \{j,r\}$

$X_{jm} = 1$

$X_{rs} = 1$

point 2: $X_{rs} = 0$

$X_{rr} = 1$ and all other values as in *point 1*

point 1 and *2* satisfy $rhs(5.22) = lhs(5.22) = 1$

Inserting the above constructed points, step by step, in (5.24) we obtain the required dependencies and (5.24) reduces to

$$\lambda_0 - \sum_l \lambda_l = (\lambda_{jm} - \lambda_j)X_{jm} + \sum_{a \in \mathcal{F}^r_{A_i}} (\lambda_{ra} - \lambda_r)X_{ra} + \sum_{a \in A_i} (\lambda_{ia} - \lambda_i)X_{ia} \tag{5.25}$$

claim 2: $\lambda_{jm} - \lambda_j \overset{!}{=} \lambda_{ra} - \lambda_r \quad \forall a \in \mathcal{F}^r_{A_i}$

Consider the following points in $\mathcal{F}$:

point 1: $X_{dd} = 1 \quad \forall d \in \{1,..,n\} \setminus \{j\}$

$X_{jm} = 1$

point 2: $X_{dd} = 1 \quad \forall d \in \{1,..,n\} \setminus \{r\}$

$X_{ra} = 1$ (r,s) with $a \in \mathcal{F}^r_{A_i}$

Evaluating these two points at (5.25) yields $\lambda_{jm} - \lambda_j = \lambda_{ra} - \lambda_r$ and we can reformulate (5.25) to

$$\lambda_0 - \sum_l \lambda_l = \lambda(X_{jm} + \sum_{a \in \mathcal{F}^r_{A_i}} X_{ra}) + \sum_{a \in A_i} (\lambda_{ia} - \lambda_i)X_{ia} \tag{5.26}$$

claim 3: $\lambda \overset{!}{=} -(\lambda_{ia} - \lambda_i) \; \forall a \in A_i$

case 1: $(i,a) \notin IA^{(j,m)}$, $a \in A_i$

point 1: $X_{dd} = 1 \quad \forall d \in \{1, .., n\} \setminus \{i\}$

$X_{ib} = 1$

point 2: $X_{dd} = 1 \quad \forall d \in \{1, .., n\} \setminus \{i, j\}$

$X_{jm} = 1$

$X_{ia} = 1$

<u>case 2:</u> $(i, a) \in IA^{(j,m)}$, $a \in A_i$

By assumption there exists a $p \in \mathcal{F}^r_{A_i} : (i, a) \notin IA^{(r,p)}$

point 1: same as in case 1 of claim 3.

point 2: $X_{dd} = 1 \quad \forall d \in \{1, .., n\} \setminus \{i, r\}$

$X_{rp} = 1$

$X_{ia} = 1$

Inserting these points in (5.26):

$$\textit{point 1:} \quad \lambda_0 - \sum_l \lambda_l = 0$$

$$\textit{point 2:} \quad \lambda_0 - \sum_l \lambda_l = \lambda + (\lambda_{ia} - \lambda_i)$$

Thus $\lambda = -(\lambda_{ia} - \lambda_i)$, which completes the proof. □

Example 5.4.5. Again we consider the hub covering instance of Example 4.2.4. Recall the sets VA and IA:

VA ={ (1,1), (1,3), (1,4), (2,2), (2,3), (2,4), (3,3), (3,4), (3,5), (4,1), (4,3), (4,4), (4,5), (5,3), (5,4), (5,5) }

IA = {(1,3,2,4), (1,3,5,4), (2,3,4,1), (2,3,5,4)}

Let $(i, b) = (4, 1)$ and

$IA^{(4,1)}$= {(1,3), (1,4), (2,3), (2,4), (3,4), (4,3), (4,5), (5,4) }

Facet Derivation

$A_4 := \{3,4,5\}$

and choose $(1,3) \in IA^{(4,1)}$, hence

$IA^{(1,3)}$= { (1,4), (2,4), (5,4), (3,4), (3,5), (4,1) }

Let w.l.o.g $r = 2$

$\mathcal{F}^2_{A_4} := \{ \ 4 \ \}$.

Hence, we obtain that

$$X_{43} + X_{44} + X_{45} \geq X_{13} + X_{24} \tag{5.27}$$

is valid for $\mathcal{P}_{SAQI-W2}$.

If

$$IA^{(1,3)} \cap IA^{(4,1)} \bigcap_{a \in \mathcal{F}^2_{A_4}} IA^{(2,a)} = \emptyset$$

then (5.27) defines a facet.

Therefore we need to determine the required sets:

$IA^{(1,3)}$= { (1,4), (2,4), (5,4), (3,4), (3,5), (4,1) }

$IA^{(4,1)}$= {(1,3), (1,4), (2,3), (4,3), (4,5) }

$IA^{(2,4)}$ = { (1,3), (2,3), (4,3), (4,5) }

It is easy to see that the intersection of these sets is indeed the empty set. Therefore (5.27) represents a facet of $\mathcal{P}_{SAQI-W2}$.

Chapter 6

Separation

As shortly stated in Chapter 1, branch-and-bound and branch-and-cut strategies are the most powerful approaches for solving hub location problems. The newly derived facet defining inequalities are added to the problem, and a branch-and-cut algorithm is used. However, just adding these new inequalities to the original formulation of SAQI-W2 is highly inefficient. The only ones needed are those that cut off a current non-integral solution. In order to identify violated inequalities, or *cutting planes*, we have to solve the corresponding *separation problem:*

Given a point $\hat{X}^*$ with $0 \leq \hat{X}^*_{ij} \leq 1$ and a family of valid inequalities for SAQI-W2. Find out if $\hat{X}^*$ satisfies all those inequalities, and, if not, find some inequalities that are violated by $\hat{X}^*$.

Enclosing cutting planes can benefit a branch-and-bound algorithm by more quickly exposing integer solutions and by causing nodes to be fathomed through tighter LP relaxations. However, the use of each cutting plane incurs computational costs in time and memory. These costs result not only from the generation of the cut and the need to perform simplex pivots with more rows but also from the possibility that the cutting plane may negatively affect the sparsity of the matrix. Hence, it is not obvious how useful a particular cutting plane or set of cutting planes will be.

6.1 Separating Facet I

Let $\hat{X}^*$ denote a fractional solution to $\mathcal{X}_{SAQI-W2}$. To find a violated inequality (5.17), we have to determine, for a given (i,k), the set $IA_j(\mathcal{F})$ $(j \in \{1,..,n\})$, check whether (5.18) and

$$X^*_{ik} + \sum_{m \in IA_j(\mathcal{F})} X^*_{jm} > X^*_{kk} \tag{6.1}$$

is satisfied. If (5.18) and (6.1) hold, we include the corresponding inequality (5.17) in the constraint set of SAQI-W2 and solve the new LP relaxation. Otherwise we branch.

Since there exist $q-n$ variables X_{ik} $(i \neq k)$ and $j \in \{1,..,n\}$ we have at most $(q-n)n$ inequalities $X^*_{ik} + \sum_{m \in IA_j(\mathcal{F})} X^*_{jm} \leq X^*_{kk}$ representing the class of *Facet I*. Hence a simple enumeration procedure will solve the corresponding separation problem efficiently.

Next, we give an example of how such an inequality (5.17) can be violated if X^* is non-integral.

Example 6.1.1. Consider a hub covering instance with $n = 6$ and assume that w.l.o.g $(i,k) = (2,3) \in VA$, $IA_5(\mathcal{F}) = \{1{,}6\}$ and (4.18) holds. Then, a point $\hat{X}^*$ with $X^*_{23} = X^*_{51} = 0$, $X^*_{56} = 1$ and $X^*_{33} = 0.5$ satisfies $X^*_{23} \leq X^*_{33}$, but $X^*_{23} + X^*_{51} + X^*_{56} = 1 > 0.5 = X^*_{33}$.

6.2 Separating Facet II

Recall the inequalities representing $Facet\ II$:

$$\sum_{a \in A_i} X_{ia} \geq X_{jm} + \sum_{a \in \mathcal{F}^r_{A_i}} X_{ra} \tag{6.2}$$

With $(i,b) \in VA$ $(i \neq b)$, $(j,m) \in IA^{(i,b)}$ $(j \neq i)$, $A_i := \{a : (i,a) \in VA, a \neq b\}$ and $\mathcal{F}^r_{A_i} := \left\{a \in A_i\colon\ (r,a) \in IA^{(j,m)} \cap IA^{(i,b)}\right\}$ $(r \neq i)$.
Remember that the above inequality represents a facet if

$$IA^{(j,m)} \cap IA^{(i,b)} \bigcap_{a \in \mathcal{F}^r_{A_i}} IA^{(r,a)} = \emptyset. \tag{6.3}$$

As for $Facet\ I$, we consider the separation for $Facet\ II$ for each $(i,b) \in VA$ separately. For each $(i,b) \in VA$ $(b \neq i)$ we have at most $q-n-1$ $(j,m) \in IA^{(i,b)}$ and $r \in \{1,..,n\} \setminus \{i\}$. Hence, there are at most $(q-n)(q-n-1)(n-1) = \mathcal{O}(n^5)$ inequalities (5.29) representing the class of $Facet\ II$. Thus, similar to the separation of $Facet\ I$, we use an enumeration procedure to detect inequalities (5.29) that are violated by some $\hat{X}^*$.

Part III

Relation between Hub Covering and Hub Center Problems

Chapter 7

A new approach solving the USApHCP

In this chapter, we fist recall some results of Chapter 4. By using these results we provide an algorithm for solving the USApHCP. Implementation details and computational experiments are presented, which show that the new solution approach performs better than the USApHCP.

7.1 Algorithm

Recall the two main results of Section 4.2.

1. the HCoP has a solution if and only if $\beta \geq \alpha \max_{i,j} (d_{ij})$.

2. $|VA| = n^2 \Longleftrightarrow \beta \geq 2 \max_{i,j} (d_{ij})$

Using these results we are able to construct an algorithm for solving the USApHCP.

Hub covering problems have an inverse relationship to hub center problems, hence hub covering problems may prove important in solving hub center problems, since many algorithms for solving p-center problems require solving a sequence of set covering problems.

Our main objective of this thesis is to solve the USApHCP by using the HCoP. The idea is to use a very common optimization technique called **binary search** in which the HCoP is solved until the minimax distance is reached for a given number of hubs p. The HCoP is defined as a subroutine *cover* in which the optimal objective function, that is the number of hubs, is returned.

BS(HCoP) - Algorithm
Input: $\lvert V\rvert = n$, $\lvert H\rvert = p$, distancematrix d_{ij}, α, ε
Output: β
Step 1: $z := \max_{i,j} d_{ij}$
$\beta := 2z$
$s := \alpha z$
$a := (s + \beta)/2$
Step 2: if $\beta - s < \varepsilon$ then STOP
Step 3: if $cover(a) \leq p$ then
$\beta := a$
else
$s := a$
Step 4: $a := (s + \beta)/2$ go to *Step 2*

A small deviation of at most ε of β is allowed, since the β values are real numbers and so it is difficult to compare these solutions exactly. It is obvious that the smaller the tolerance the better the solution quality. This algorithm requires

$\lceil log_2(\frac{z}{\varepsilon})\rceil$ iterations to return the solution. Since SAQI-W2 is the best known formulation so far, we use SAQI-W2 in the above algorithm and denote the corresponding version of BS(HCoP) with BS(SAQI-W2).

7.2 Implementation Details

The preprocessing (e.g. determining VA, IA) and BS(HCoP) are implemented in the Mosel programming language of Xpress MP. The following implementation details are given in the Mosel syntax.

First recall the definition of VA and IA:

$$VA = \left\{(i,k)|2d_{ik} \leq \beta \text{ and } d_{ik} + \alpha \underset{j}{max}(d_{kj}) \leq \beta \right\}$$

$$IA = \left\{(i,k,j,m)|(i,k),(j,m) \in VA, i < j \text{ and } d_{ik} + \alpha d_{km} + d_{jm} > \beta\right\}$$

In VA, the term $\alpha \underset{j}{max}(d_{kj})$ can be treated as a constant, hence it only has to be determined once for all k. For this purpose let

```
alphamaxd(k):= alpha * max(j in RN)d(k,j).
```

Since all constraints of SAQI-W2 are dependent on VA, we define

```
va(i,k):= 2 * d(i,k) <= beta and d(i,k) + alphamax(k) <= beta
```

as boolean.

The set of incompatible assignments can then be easily determined by "conditional forall"-loops which results in much less indices to enumerate:

```
RN = 1..n

forall(i in RN,k in RN | va(i,k) ,j in RN, m in RN |
                         va(j,m) and i<j and
                         k<>j and i<>m and
                         d(i,k) + alpha * d(k,m) + d(m,j) > beta)

          X(i,k) +  X(j,m) <= 1
```

The implementation of the aggregation procedure follows in essence the description in [1].

The algorithm BS(HCoP) is implemented as described in the preceding section. There, it has been stated that the HCoP is defined as a subroutine called *cover*. Note that decision variables and constraints defined in a subroutine are not accessible from the model after the end of the subroutine, but their definition is not removed from the optimization problem. Therefore we have to reset the constraints created by *cover* before leaving the subroutine, otherwise the problem would keep growing.

In Section 8.2 we describe a method for updating VA and IA in every iteration of BS(SAQI-W2)

7.3 Computational Results

In the following, we present the computational behavior of BS(SAQI-W2) and compare the performance with this of the USApHCP.

Unfortunately, it turns out that the cuts derived in Section 6 do not improve the CPU time. One reason for this is that the new inequalities have been separated through enumeration, which is quite time consuming. Hence, instead of a branch-and-cut algorithm, we use the built-in branch-and-bound routine of Xpress-IVE to solve the integer programs on an AMD Athlon PC with 1.67 GHz and 512 MB RAM.

We test our programs with the AP and CAB data sets, which are considered to be a benchmark by most researchers in the hub location area. For the CAB data we create different instances by choosing different subsets with $n \in \{15, 20, 25\}$, $p \in \{2, 3, 4\}$ and $\alpha \in \{0.25, 0.5, 0.75\}$. For the AP data we create instances with $n \in \{30, 40, 50\}$, $p \in \{2, 3, 4, 5\}$ and $\alpha = 0.75$. Note that it is possible that for (n, p_1) and (n, p_2) with $p_1 > p_2$ the optimal objective value of the USApHCP coincide. Since we minimize the number of hubs in the HCoP we cannot obtain two different number of hubs for the same β (see example below). However this effect only occur for larger p. But in practical applications the number of hubs should not be to large, since then the benefits of a hub-and-spoke network would be reduced or diminished. Future research should include a more thorough examination of this effect.

Example 7.3.1. Consider an instance with (n, p) = (30,4) and (30,5). Both have an optimal objective value of 48.5632. Solving (30,5) with BS(SAQI-W2) will give us 4 hubs because the formulation of SAQI-W2 tries to minimize the number of hubs. For instances, we encounter the same problem with (40,5) and (40,10) as well as for (50,5) and (50,10).

Table 1 shows results for the USApHCP and BS(SAQI-W2) for medium and large CAB problems with a tolerance ε of 10^{-3}, which has shown to be small enough that both the USApHCP and BS(SAQI-W2) achieve the same objective value. The columns can be interpreted as follows. Column 1 and 2 specify the number of nodes n and the discount factor α. In column 3 and 4 the number of hubs p and the optimal objective value is given. The last two columns present the computation times (in seconds) needed for the solution process to solve the problem to optimality when using the USApHCP and BS(SAQI-W2). The computation is terminated whenever the CPU time is more than 4000 seconds. This is indicated by a "-" entry for the computation time.

Table 2 shows results for large AP problems with a tolerance of 10^{-4}. It is organized as Table 1 except the column for the discount factor α.

From table 1 it can be seen that the USApHCP performs better for small and medium sized problems and a small number of hubs p. As p gets larger, the new approach seems to provide faster CPU times. For instance, if $n \geq 20$ and $p \geq 3$ BS(SAQI-W2) needs less CPU time than the USApHCP. Table 2 illustrates that for $n \geq 30$ even for $p \geq 2$ BS(SAQI-W2) seems to be more efficient compared with the formulation of Ernst *et al.*[4].

We also compare the computational results for BS(SAQI-W2) with the *Incremental Heuristic* for solving the USApHCP given in [5]. Although the new approach needs more CPU time than the heuristic, the solution quality of BS(SAQI-W2) is much better even if $\varepsilon = 1$

Note that the hub set obtained by BS(SAQI-W2) contains sometimes $p+1$ elements instead of p. This effect occurs when β is exactly the boundary for which the hub covering problem gives either p or $p+1$ hubs. Then to get the correct hub set, just solve SAQI-W2 once with $\lceil \beta \rceil$. Since SAQI-W2 can be solved very quickly this has no effect on the efficiency compared to USApHCP. For instance, consider an AP problem with $n = 40$, $p = 4$ and $\alpha = 0.75$. BS(SAQI-W2) requires 50.191 sec. to solve this problem with an optimal objective value of 52.2653, whereas USApHCP cannot be solved to optimality within 6000 sec. However the hub set contains 5 elements instead of 4. Solving SAQI-W2 with $\lceil \beta \rceil$=52.266 requires 0.187 sec. to obtain the optimal solution of 4 hubs. Thus even an additional solving of SAQI-W2 do not reduce the efficiency of BS(SAQI-W2).

n	α	p	obj.	hub set	USApHCP	BS(SAQI-W2)
15	0.75	2	2343.4	{12,13}	1.625	5.501
15	0.5	2	2160.75	{5, 13}	1.593	2.903
15	0.25	2	2059.04	{5, 8}	3.031	3.471
15	0.75	3	2086.13	{3,12,13}	4.812	1.562
15	0.5	3	1760.15	{3, 12, 13}	4.015	5.218
15	0.25	3	1760.15	{5,12,13}	4.812	4.983
15	0.75	4	1979.01	{3, 11, 12, 14}	3.735	1.565
15	0.5	4	1530.41	{2, 11, 12, 14}	12.078	1.643
15	0.25	4	1361.42	{2, 11, 12, 14}	9.39	2.904
20	0.75	2	2444.89	{13, 19}	6.437	10.81
20	0.5	2	2224.11	{5, 19}	9.797	18.56
20	0.25	2	1933.42	{5, 19}	5.937	6.89
20	0.75	3	2187.63	{13,17,19}	32.672	7.594
20	0.5	3	1871.24	{2, 13, 19}	26.437	25.749
20	0.25	3	1635.37	{2, 13, 19}	19.015	4.752
20	0.75	4	2086.13	{12, 13, 17, 19}	29.875	4.53
20	0.5	4	1650.81	{11, 14, 17, 19}	37.625	9.203
20	0.25	4	1361.42	{2, 11, 14, 19}	67.453	6.483
25	0.75	2	2675.88	{7, 9}	33.703	10.658
25	0.5	2	2480.64	{8, 21}	27.844	97.406
25	0.25	2	2194.52	{21, 22}	15.688	19.749
25	0.75	3	2500.24	{6, 8, 16}	100.765	13.393
25	0.5	3	2218.32	{12, 21, 23}	154.063	53.094
25	0.25	3	2001.65	{5, 19, 23}	125.219	30.157
25	0.75	4	2372.12	{19, 21, 22, 23}	251.95	14.185
25	0.5	4	2045.65	{12, 13, 20, 23}	327.203	48.077
25	0.25	4	1703.61	{9, 16, 19, 23}	469.047	20.296

Table 1: Numerical results of BS(SAQI-W2) for CAB problems

n	p	obj.	hub set	USApHCP	BS(SAQI-W2)
30	2	55.8204	$\{6, 8\}$	38.657	16.624
30	3	49.3919	$\{6, 9, 25\}$	229.235	4.001
30	4	48.5632	$\{6, 9, 14, 25\}$	362.797	3.794
40	2	61.6825	$\{8, 10\}$	710.218	277.72
40	3	58.1298	$\{1, 8, 11\}$	-	118.156
40	4	52.2653	$\{1, 8, 12, 13\}$	-	50.191
40	5	49.7412	$\{1, 8, 12, 19, 33\}$	-	65.123
50	2	65.5234	$\{10, 13\}$	-	938.514
50	3	60.1321	$\{3, 9, 23, 24\}$	-	1079.17
50	4	52.9058	$\{1, 10, 15, 42\}$	-	151.407
50	5	50.7079	$\{1, 10, 15, 41, 42\}$	-	134.033

Table 2: Numerical results of BS(SAQI-W2) for large AP problems

Chapter 8

Conclusions and Outlook

This chapter summarizes the results and contributions of this thesis and presents possible future research topics.

8.1 Summary of the Main Results

In this thesis we discussed a formulation for the hub covering problem provided by Wagner [1]. In Chapter 4 we investigated the time limit β. In particular, we showed that the hub covering problem is feasible if and only if $\beta \geq \alpha \max_{i,j} (d_{ij})$. In Chapter 5 we derived the dimension of $\mathcal{P}_{SAQI-W2}$, checked the given problem constraints with regard to their facet-inducing property and derived two new facet-defining inequalities. We presented a new approach, based on the binary search algorithm, to solve the USApHCP in Chapter 7. This algorithm turned out to be efficient for $n \geq 30$ (or $n < 30$ and $p \geq 3$). Unfortunately, if the optimal objective value of the USApHCP for some different number of hubs p coincide, the hub set obtained by BS(HCoP) always yield the hubs for the smallest p. Nevertheless this effect only occurs for increasing number of hubs, which does not happen much in practice because, as already stated, a large number of hubs reduces the efficiency of a hub-and-spoke network.

8.2 Further Research

8.2.1 Hub Covering Problems

Updating VA and IA

As already stated in Section 7.1 the idea of BS(SAQI-W2) is to solve SAQI-W2 as long as the minimax distance for a given number of hubs p is reached. Though BS(SAQI-W2) performs well for large-scale instances, it is worth concentrating on the improvement of the algorithm. This should include an efficient computation of VA and IA. Instead of determining a new VA and IA in every iteration for a changed β, the two sets could just be updated. A promising method might be to sort the distances in increasing order in $\mathcal{O}(n^2 logn)$ time. Denote the point where for the first time $d_{ik} > \beta$ with w $(\leq n^2)$. The determination of VA has then a complexity of $\mathcal{O}(w)$ instead of $\mathcal{O}(n^2)$ since the condition $d_{ik} + alphamax(k) \leq \beta$ has to be checked only for the first w entries of the ordered array. Note that the sorting algorithm must be applied only once. Moreover, the implementation of the sorting algorithm assumes that the array to be sorted is one dimensional, hence the distancematrix d_{ij} has to be transformed into an one-dimensional array. This is done by the following assignment:

$$d_{(i-1)*n+j} := d_{i,j}.$$

If the current β is greater than the last β, we can adopt the last VA and add the new (i, k)'s. The same procedure can be applied to IA. However our implementation of this method did not produce the expected improvement since the update is only possible if all constraints are saved after the end of the subroutine, and as already stated in Section 7.2, to prevent the increase of the problem, we reset the constraints before leaving the subroutine.

Based on our implementation, no clear statement can be made whether this suggested method improves BS(SAQI-W2).

Separation

Concerning the separation problem of the class of $FacetII$ that have been derived in Section 5.4, an enumeration procedure was suggested. However, with at most $\mathcal{O}(n^5)$ inequalities that represent the class of $FacetII$ this procedure is extremely inefficient. Hence, there is need to develop a better separation algorithm.

Formulation

As we were completing this thesis, Ernst et.al [16] proposed a new formulation for the single allocation hub covering problem based on the radius formulation of the USApHCP :

USAHCoP-r:

$$min \quad \sum_k X_{kk}$$

$$\text{s.t.} \quad \sum_k X_{ik} = 1 \qquad \forall \quad i \in \{1, ..., n\} \tag{8.1}$$

$$X_{ik} \leq X_{kk} \qquad \forall \quad i, k \in \{1, ..., n\} \tag{8.2}$$

$$d_{ik} X_{ik} \leq r_k \qquad \forall \quad i, k \in \{1, ..., n\} \tag{8.3}$$

$$r_k + r_m + \alpha d_{km} \leq \beta \qquad \forall \quad k, m \in \{1, ..., n\} \tag{8.4}$$

$$X_{ik} \in \{0, 1\} \qquad \forall \quad i, k \in \{1, ..., n\} \tag{8.5}$$

$$r_k \geq 0 \qquad \forall \quad k \in \{1, ..., n\} \tag{8.6}$$

First investigations have shown that including this formulation in the binary-search algorithm only works efficiently if the preprocessing of Xpress-IVE is switched off. But then BS(USAHCoP-r) performs worse than BS(SAQI-W2). Hence, it could be of interest if Wagner's preprocessing can be adapted to USAHCoP-r.

8.2.2 Hub Location Problems

As mentioned at the beginning of this thesis, the task of hub location problems is to design an optimal hub-and-spoke network on a complete graph $G = (V, E)$ with a discrete node set V. The nodes correspond to spoke and potential hub locations. Hence, in a discrete (or local) hub location problem the location of hubs is only allowed in the nodes. Future research could deal with a so called *absolute case*, where we allow the location of hubs not only in the nodes but on edges (see Figure 6.1).

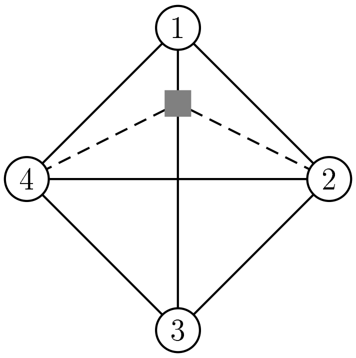


Figure 6.1: Network where a hub is located on an edge.

The distance between the spoke-to-hub connection (dashed lines in Figure 6.1) is now defined as the shortest path to the hub via the given arcs of the network. It would be of interest to know whether the optimal solution of the absolute case coincide with the one of the discrete hub location problem.

Appendix A

PORTA

This appendix illustrates how we have deduced the new facet classes in Section 5.4.

Porta is a collection of routines for analyzing polytopes and polyhedra. The polyhedra are either given as the convex hull of a set of points plus (possibly) the convex cone of a set of vectors, specified in a '.poi'-file, or as a system of linear equations and inequalities, specified in a '.ieq'-file.

For our purpose we need the following routines:

`fctp`: checks whether a set of inequalities given in a '.ieq'-file is facet inducing for a polyhedron given by a '.poi'-file. If this is not the case, points and rays which are not valid are output into a file. Points and rays that satisfy the inequality with equality are also determined and output into a file; for these points and rays the dimension is computed.

`traf`: carries out the transformation between the two polyhedron representations, the direction is determined by the input filename suffix'.poi' or '.ieq'.

`vint`: enumerates all integral points within given bounds that are valid for a linear system of inequalities and equations. The lower and upper bounds for each component must be specified in the input file by the keywords LOWER_BOUNDS and UPPER_BOUNDS.

More information about PORTA and a complete list of available routines is given in [7].

Before writing the equations and inequalities in an '.ieq'-file, we must transform doubly indexed variables into singly indexed ones, since only variables named $x\{i\}$ $i \in \{1,..,q = |VA|\}$ are allowed. Next, we illustrate the format of the '.ieq'-files needed to represent a polyhedron and an inequality that is checked for facet-inducing property by using example 3.2.4.

`example.ieq`

```
DIM = 16

VALID
0 1 0 0 1 0 1 0 0 0 1 0 0 1 0 0

LOWER_BOUNDS
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0

UPPER_BOUNDS
1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1

INEQUALITIES_SECTION
(1) x1 +x2 +x3 = 1
(2) x4 +x5 +x6 = 1
(3) x7 +x8 +x9 = 1
(4) x10 +x11 +x12 +x13 = 1
(5) x14 +x15 +x16 = 1
(6) x2 + x6<= 1
(7) x2 + x15<= 1
(8) x5 + x10<= 1
(9) x5 + x15<=1
(10) x2 - x7<= 0
    ⋮
(20) x15 - x12<= 0
(21) x1>= 0
(22) x1 <= 1
    ⋮
(51) x16>= 0
(52) x16 <= 1
END
```

```
facet.ieq

DIM = 16

VALID
0 1 0 0 1 0 1 0 0 0 1 0 0 1 0 0

INEQUALITIES_SECTION
x2 - x7<= 0
END
```

After creating example.ieq we use the function `vint` [1] to determine all integral points that are valid for the given polyhedron. This is done by the following program call in the command window: `vint example.ieq`. The output is written into a file named example.poi. With `traf example.poi`, we obtain a file named example.poi.ieq that contains all facet-defining inequalities for the polyhedron and give, together with equations, a minimal linear description of the polyhedron. Thus, facet-defining inequalities can be deduced by calculating those '.poi.ieq'-files for different small hub covering problems and identifying patterns among the results.

Checking whether $X_{13} \leq X_{33}$ (x2 - x7 $\leq$ 0) is facet-defining for our problem instance can be accomplished by `fctp facet.ieq example.poi`.

[1] The use of this function is important since we deal with binary variables and solely interested in points that only contain variables with values either 1 or 0.

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