Isotone mappings of levelled strict orders

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Abstract

Strict order relations are defined as strict asymmetric and transitive binary relations. For classes of so-called levelled strict orders it is analyzed, under which conditions the endomorphism monoids of two relations coincide; in particular the case of direct sums of strict antichains is studied. Further, it is shown that these orders differ in their sets of binary order preserving functions.

1 Definitions and preliminaries

A `strict order relation` is a binary relation $\rho \subseteq A^2$, satisfying the following conditions:

1. (Strict) asymmetry: $(a,b) \in \rho \Longrightarrow (b,a) \notin \rho$
2. Transitivity: $(a,b),(b,c) \in \rho \Longrightarrow (a,c) \in \rho$

Instead of $(a,b) \in \rho$ it is often written $a <_\rho b$. If only one single relation ρ is in consideration, we denote $a < b$ instead of $a <_\rho b$. The n-ary order preserving or isotone functions are called `polymorphisms`. That is, for all $(a_1,b_1),\dots,(a_n,b_n) \in \rho$ follows $(f(a_1,\dots,a_n), f(b_1,\dots,b_n)) \in \rho$. The set of all polymorphisms is designated by $Pol\rho$ and its subset of n-ary functions by $Pol^{(n)}\rho$, respectively. The monoid of unary polymorphisms is denoted by $End\rho$. A chain $\mathcal{C}$ is an order, in which any two elements are comparable, i.e. for distinct $a,b \in A$ it holds either $a <_\mathcal{C} b$ or $b <_\mathcal{C} a$. A `maximum chain` in $(A;\rho)$ is a suborder $\mathcal{C}$ of $(A;\rho)$, s.t. for any other chain $\mathcal{C}'$ in $(A;\rho)$ holds $|\mathcal{C}'| \leq |\mathcal{C}|$.

Definition 1 *Let ρ be a strict order relation. $N_\rho^\downarrow(x)$ is defined as the supremum over all n, s.t. there is a path $x_1x_2\dots x_{n-1}\,x$ in the Hasse diagram of ρ, ending at x. Dually, $N_\rho^\uparrow(x)$ is the supremum over all n, s.t. there is a path $x\,x_1x_2\dots x_{n-1}$ starting at x. The cardinality of a maximum chain in ρ is designated by $c(\rho)$.*

Theorem 1 *Let $\rho \subseteq A^2$ be a strict order relation, $C = v_1 \ldots v_{c(\rho)}$ a maximum chain and $a \in A$ arbitrary. Then the following functions determine polymorphisms of ρ:*

$$\begin{aligned}\forall(x_1,\ldots,x_n):\ \theta_{\downarrow}[C,n](x_1,\ldots,x_n) &:= v_{min\{N_\rho^{\downarrow}(x_i)\,|\,1\le i\le n\}}.\\ \theta_{\uparrow}[C,n](x_1,\ldots,x_n) &:= v_{c(\rho)+1-min\{N_\rho^{\uparrow}(x_i)\,|\,1\le i\le n\}}.\end{aligned}$$

$$\theta[C,n,a](x_1,\ldots,x_n) := \begin{cases} v_{c(\rho)+1-min\{N_\rho^{\uparrow}(x_i)\,|\,1\le i\le n\}}, & \text{if } \forall i: a\le x_i,\\ v_{min\{N_\rho^{\downarrow}(x_i)\,|\,1\le i\le n\}} & \text{else.}\end{cases}$$

Proof: In the following let $(\alpha_i,\beta_i)\in\rho$, $i=1,\ldots,n$ and

$$\beta_k := arg\min\{N_\rho^{\downarrow}(\beta_i)\,|\,1\le i\le n\}$$

$$\begin{aligned}&\Longrightarrow\quad arg\min\{N_\rho^{\downarrow}(\alpha_i)\,|\,1\le i\le n\}\le\alpha_k<\beta_k\\ &\Longrightarrow\quad v_{\min\{N_\rho^{\downarrow}(\alpha_i)\,|\,1\le i\le n\}}<v_{\min\{N_\rho^{\downarrow}(\beta_i)\,|\,1\le i\le n\}}\\ &\Longrightarrow\quad \theta_{\downarrow}[C,n]\in Pol^{(n)}\rho.\end{aligned}$$

Dually, it is shown $\theta_{\uparrow}[C,n]\in Pol^{(n)}\rho$.

With respect to the function $\theta[C,n,a]$, one has to consider two cases.
For this, the abbreviations $\tilde\alpha=(\alpha_1,\ldots,\alpha_n)$ and $\tilde\beta=(\beta_1,\ldots,\beta_n)$ are used.

<u>Case 1:</u> $\forall i:\alpha_i\ge a\Longrightarrow\forall i:\beta_i\ge a\Longrightarrow\Big(\theta[C,n,a](\tilde\alpha),\theta[C,n,a](\tilde\beta)\Big)\in\rho.$

<u>Case 2:</u> $\exists j:\alpha_j\not\ge a\Longrightarrow\theta[C,n,a](\tilde\alpha)=v_{\min\{N_\rho^{\downarrow}(\alpha_i)\,|\,1\le i\le n\}}.$

<u>Case 2.1:</u> $\exists r:\beta_r\not\ge a\Longrightarrow\Big(\theta[C,n,a](\tilde\alpha),\theta[C,n,a](\tilde\beta)\Big)\in\rho.$

<u>Case 2.2:</u> $\forall i:\beta_i\ge a\Longrightarrow\theta[C,n,a](\tilde\beta)=v_{c(\rho)+1-min\{N_\rho^{\uparrow}(\beta_i)\,|\,1\le i\le n\}}.$

If one defines $\beta_k := arg\min\{N_\rho^{\downarrow}(\beta_i)\,|\,1\le i\le n\}$ as it was done above, one gets

$$\begin{aligned}\theta[C,n,a](\tilde a)=v_{min\{N_\rho^{\downarrow}(\alpha_i)\,|\,1\le i\le n\}} &\le v_{N_\rho^{\downarrow}(\alpha_k)}\\ &< v_{c(\rho)+1-N_\rho^{\uparrow}(\beta_k)}=\theta[C,n,a](\tilde\beta).\end{aligned}$$

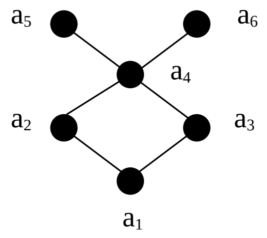
□

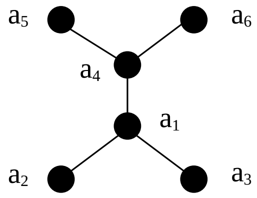
Definition 2 *Let ρ be a strict order relation and π a permutation over $\{1,\ldots,c(\rho)\}$. We define*

$$\rho_\pi := \Big\{(a,b)\,\Big|\,\pi(N_\rho^{\downarrow}(a))<\pi(N_\rho^{\downarrow}(b));(a,b)\in\rho \text{ or } (b,a)\in\rho\Big\}.$$

For example consider the following Hasse diagrams:

ρ





ρ_π

The relation ρ_π was got by permuting the first with the second level of ρ.
If π is of the form

$$\pi(N_\rho^\downarrow(a)) < \pi(N_\rho^\downarrow(b)) \Longleftrightarrow N_\rho^\downarrow(b) < N_\rho^\downarrow(a),$$

we get $\rho_\pi = \{(a,b) \mid (b,a) \in \rho\} =: \bar{\rho}$, the `converse relation` of ρ. Obviously, ρ_π i.g. doesn't determine a strict order, since the transitivity is vulnerated.
We define

$$\mathcal{C}_\rho := \{C \mid C \text{ is a maximum chain in } \rho\}.$$

Considering the above example yields $\mathcal{C}_\rho = \{C_1, C_2, C_3, C_4\}$, where

$$\begin{aligned} C_1 &= a_1a_2a_4a_5, \\ C_2 &= a_1a_2a_4a_6, \\ C_3 &= a_1a_3a_4a_5 \\ \text{and } C_4 &= a_1a_3a_4a_6. \end{aligned}$$

Theorem 2 *Let ρ and μ be strict order relations with $End\rho = End\mu$. Then it exists a permutation π, s.t. for their maximum chains $\mathcal{C}_\mu$ and $\mathcal{C}_\rho$ holds:*

$$\mathcal{C}_\mu = \{C_\pi \mid C \in \mathcal{C}_\rho\},$$

where C_π arises from C by $C_\pi := v_{\pi(1)} \ldots v_{\pi(k)}$ with $C = v_1 \ldots v_k$.

Proof: Let $C_1, C_2 \in \mathcal{C}_\rho$ and $l := c(\rho)$,

$$\begin{aligned} C_1 &= v_1 \ldots v_l \quad \text{and} \\ C_2 &= w_1 \ldots w_l. \end{aligned}$$

Then there are permutations π_1 and π_2 with $C_{1,\pi_1}, C_{2,\pi_2} \in \mathcal{C}_\mu$. We choose $f \in End\rho = End\mu$ with $Im(f) = C_1$. The function f is a retraction, since additionally $f \circ f = f$ is fulfilled. It follows $f(C_2) = C_1$ (maximum chains are mapped onto maximum chains) – e.g. consider the function $f(x) := \theta_\downarrow[C_1, 1](x)$ from theorem 1 – and it holds $\forall i : f(w_i) = v_i$. (Note that $f(w_1) < f(w_2) < \dots f(w_l)$.)

$$\begin{aligned} \Longrightarrow \quad & f(w_{\pi_2(i)}) = v_{\pi_2(i)} \\ \text{and} \quad & f(w_{\pi_2(i)}) = v_{\pi_1(i)}. \end{aligned}$$

Since $\forall i : v_{\pi_2(i)} = v_{\pi_1(i)}$, we get $\forall i : \pi_2(i) = \pi_1(i)$ and hence $\pi_2 = \pi_1$.

□

The conversion of theorem 2 is in general not true.

Definition 3 *A relation $\rho \subseteq A^2$ is called* `rigid`*, if the identity mapping $id(x) = x$ is the only unary polymorphism, that is $End\rho = \{id\}$. Moreover, if for every natural number n the n-ary projections (or selector functions) $e_i^n(x_1, \dots, x_n), 1 \leq i \leq n$, are the only n-ary polymorphisms, then ρ is called* `strongly rigid`*.*

In [5] it was shown by Z.Hedrlin et alt. in 1965, that rigid relations exist on any set. In 1973, I.G.Rosenberg continued this work by presenting a strongly rigid binary relation on any $3 \leq n$-element set [4].

Lemma 3 *Let ρ be a binary relation (not necessarily a strict order) and let $v, w \in A$ with the property that in the corresponding graph no arc belongs into v and no arc belongs out of w. Then for $f \in Pol^{(n)}\rho$ and $\{v, w\} \subseteq \{a_1, \dots, a_n\}$ the values $f(a_1, \dots, a_n)$ can be arbitrary chosen.*

[***Proof:*** Let $(a_1, b_1), \dots, (a_n, b_n) \in \rho$. None of the a_i's fulfills $a_i = w$ and none of the $b_i = v$.]
In the following we will call these elements maximum and minimum elements and their corresponding sets ρ_{min} or ρ_{max}, respectively.

Theorem 4 *There are rigid, but no strongly rigid strict order relations.*

Proof: The rigid strict orders are exactly the chains. Every strict order, in particular a chain $\mathcal{C}$, possesses a minimum and a maximum element. Using the foregoing lemma, for $n \geq 2$ there is a n-ary polymorphism $f \in Pol^{(n)}\mathcal{C}$ being no projection. □

We conclude this section with an easy observation: *Let ρ be a strict order relation and $f \in End\rho$. Then $\forall a \in A : a \not<_\rho f(a)$ and $a \not>_\rho f(a)$.*

2 Levelled strict orders

Definition 4 *A strict order relation ρ is called* `levelled strict order` *relation, if for all $a \in A$ holds:*

$$N_\rho^\downarrow(a) + N_\rho^\uparrow(a) = c(\rho) + 1.$$

That is, every element is lying in a maximum chain of ρ.

For these relations the conversion of theorem 2 is also valid.

Theorem 5 *Let ρ be a levelled strict order relation. Further, let μ be a strict order relation. Then it holds $End\rho = End\mu$ if and only if there is a permutation π, s.t. $\mathcal{C}_\mu = \{C_\pi \mid C \in \mathcal{C}_\rho\}$.*

[***Proof:*** "$\Longleftarrow$": Every $f \in Pol\rho$ is level-preserving, that is

$$\forall x : N_\rho^\downarrow(f(x, \ldots, x)) = N_\rho^\downarrow(x).$$

Particularly for $f \in End\rho$ we obtain

$$\forall x : N_\rho^\downarrow(f(x)) = N_\rho^\downarrow(x).$$

"$\Longrightarrow$": Follows directly from theorem 2.]

Definition 5 *Let $A = A_1 \cup A_2$ a disjoint partition and $\rho_i \subseteq A_i^2, i \in \{1, 2\}$, strict order relations. The* `direct sum` *$P_1 \oplus P_2 = (A_1 \cup A_2; \rho)$ of strict orders $P_1 = (A_1; \rho_1)$ and $P_2 = (A_2; \rho_2)$ is defined as follows:*

$$(a, b) \in \rho :\Longleftrightarrow (a, b) \in \rho_1 \cup \rho_2 \cup (A_1 \times A_2).$$

An (strict) `antichain` *is an (strict) order, s.t. any two elements are incomparable.*

In the following corollary the set of all permutations over a n-element set is designated by S_n.

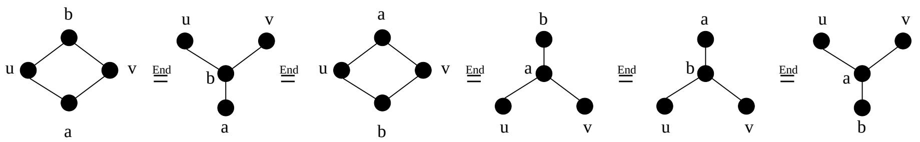
Corollary 6 *Let* $A = \cup_{i=1}^{n} A_i$ *and* $\mathcal{A}_i = (A_i; <)$, $i \in \{1, \ldots, n\}$, *strict antichains. We define*

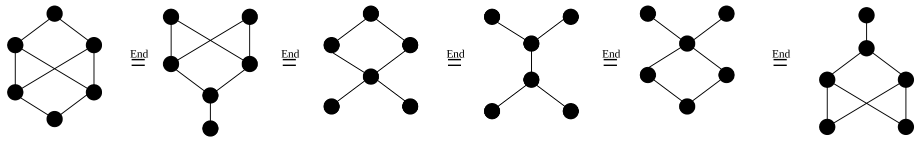
$$\mathcal{P}[A_1, \ldots, A_n] := \left\{ P \;\middle|\; P = \bigoplus_{i=1}^{n} \mathcal{A}_{\pi(i)}, \pi \in S_n \right\}.$$

Let $P_1 = (A; \rho_1)$ *and* $P_2 = (A; \rho_2)$ *be direct sums of antichains. Then it holds* $End P_1 = End P_2$ *iff* $P_1, P_2 \in \mathcal{P}[A_1, \ldots, A_n]$ *for an appropriate partition* $[A_1, \ldots, A_n]$ *of* A.

Note that for a strict order relation ρ the condition "to be a direct sum of antichains" is the same as "$x < y \iff N_\rho^\downarrow(x) < N_\rho^\downarrow(y)$". Hence, corollary 6 can also be formulated by the use of the concept of ρ_π (see definition 2). Direct sums of at most two-element disjoint antichains are known as towers.

Example *The following figures show endomorphism classes of a four-element and a six-element strict tower, respectively. In the second example the diagrams are depicted up to isomorphism and the markings of their elements are omitted.*





To refine the endomorphism classes of levelled strict orders, we extend our investigations to n-ary polymorphisms and obtain

Theorem 7 *Let* ρ *be a levelled strict order relation and* μ *be a strict order relation with* $Pol^{(n)}\rho = Pol^{(n)}\mu$ *for a natural number* $n \geq 2$. *Then* $\mathcal{C}_\mu = \mathcal{C}_\rho$ *or* $\mathcal{C}_\mu = \mathcal{C}_{\bar{\rho}}$.

Proof: Let $Pol^{(n)}\rho = Pol^{(n)}\mu$ for a natural number $n \geq 2$ and

$$N^{\downarrow}_{\rho}(\alpha) = N^{\downarrow}_{\rho}(\beta) \Longleftrightarrow N^{\downarrow}_{\mu}(\alpha) = N^{\downarrow}_{\mu}(\beta). \qquad (*)$$

It follows that α and β remain in the same levels. By lemma 3 we may assume that the maximum and minimum elements of ρ and μ coincide. Now let w.l.o.g.

$$\rho_{min} = \{x \mid N^{\downarrow}_{\rho}(x) = 1\} = \{x \mid N^{\downarrow}_{\mu}(x) = 1\} = \mu_{min}.$$

(Else the converse relation $\bar{\mu}$ is considered instead of μ.)
To get a contradiction, we assume that there exist a, b with the property

$$N^{\downarrow}_{\rho}(a) < N^{\downarrow}_{\rho}(b) \text{ and } N^{\downarrow}_{\mu}(a) > N^{\downarrow}_{\mu}(b).$$

We choose a polymorphism $f \in Pol^{(n)}\rho$ satisfying the condition

$$\forall(x_1, \ldots, x_n) : N^{\downarrow}_{\rho}(f(x_1, \ldots, x_n)) = \max_i\{N^{\downarrow}_{\rho}(x_i)\}.$$

[Such functions exist, e.g. the function $\theta_{\uparrow}[C, n]$ from theorem 1, corresponding to a maximum chain $C = v_1 \ldots v_{c(\rho)}$. It is easy to verify that $\theta_{\uparrow}[C, n] \in Pol^{(n)}\rho$ holds.]
Let $o \in \rho_{min} = \mu_{min}$ and a', b' be chosen with the property $(o, b'), (b', a') \in \mu$, where $N^{\downarrow}_{\rho}(a') = N^{\downarrow}_{\rho}(a)$ and $N^{\downarrow}_{\rho}(b') = N^{\downarrow}_{\rho}(b)$. It follows

$$(f(o, b', \ldots, b'), f(b', a', \ldots, a')) \in \{x \mid N^{\downarrow}_{\rho}(x) = b\}^2$$

and by $(*)$ also $f \notin Pol^{(n)}\mu$, contradicting the assumption.

□

The concluding corollary is an immediate consequence of theorem 7.

Corollary 8 *Let $P_1 = (A; \rho_1)$ be a direct sum of strict antichains and $P_2 = (A; \rho_2)$ be a strict order. Then $Pol^{(2)}\rho_1 = Pol^{(2)}\rho_2$ iff $\rho_1 = \rho_2$ or $\rho_1 = \bar{\rho}_2$.*

That is, direct sums of strict antichains are – up to dual isomorphism – uniquely determined by their binary isotone functions.

References

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