Functions preserving 2-series strict orders

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Abstract

In recent years a considerable attention was paid to an investigation of finite orders relative to different properties of their isotone functions [2,3]. Strict order relations are defined as strict asymmetric and transitive binary relations. Some algebraic properties of strict orders were already studied in [6]. For the class $\mathcal{K}$ of so-called 2-series strict orders we describe the partially ordered set $End\mathcal{K}$ of endomorphism monoids, ordered by inclusion. It is obtained that $End\mathcal{K}$ possesses a least element and in most cases defines a Boolean algebra. Moreover, every 2-series strict order is determined by its n-ary isotone functions for some natural number n.

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1 Definitions and notations

A `strict order relation` is a binary relation $\rho \subseteq A^2$, satisfying the following conditions:

1. (Strict) asymmetry: $(a,b) \in \rho \Longrightarrow (b,a) \notin \rho$
2. Transitivity: $(a,b),(b,c) \in \rho \Longrightarrow (a,c) \in \rho$

Instead of $(a,b) \in \rho$ it is often written $a <_\rho b$. If only one single relation ρ is in consideration, we denote $a < b$ instead of $a <_\rho b$. The n-ary order preserving or isotone functions are called `polymorphisms`. That is, for all $(a_1,b_1),\ldots,(a_n,b_n) \in \rho$ follows $(f(a_1,\ldots,a_n), f(b_1,\ldots,b_n)) \in \rho$. The set of all polymorphisms is designated by $Pol\rho$ and its subset of n-ary functions by $Pol^{(n)}\rho$, respectively. The monoid of unary polymorphisms is denoted by $End\rho$. A chain $\mathcal{C}$ is an order, in which any two elements are comparable, i.e. for distinct $a,b \in A$ it holds either $a <_\mathcal{C} b$ or $b <_\mathcal{C} a$.

Definition 1.1 *Let ρ be a strict order relation. $N^{\downarrow}_{\rho}(x)$ is defined as the supremum over all n, s.t. there is a path $x_1x_2\ldots x_{n-1}\,x$ in the Hasse diagram of ρ, ending at x. Dually, $N^{\uparrow}_{\rho}(x)$ is the supremum over all n, s.t. there is a path $x\,x_1x_2\ldots x_{n-1}$ starting at x. The cardinality of a maximum chain in ρ is designated by $c(\rho)$.*

Definition 1.2 *Let $\mathcal{K} = \boldsymbol{n{+}m}$ denote the class of all strict order relations $\rho \subseteq A^2$, consisting of a "scaffolding" ζ, composed by a n-element chain $\mathcal{C}_1 = (C_1;<)$ and a m-element chain $\mathcal{C}_2 = (C_2;<)$, s.t.*

$$A = C_1 \dot{\cup} C_2$$

and for every $\rho \in \boldsymbol{n{+}m}$ and $a \in A$ holds

$$N^{\downarrow}_{\rho}(a) = N^{\downarrow}_{\zeta}(a) \quad \textit{and}$$
$$N^{\uparrow}_{\rho}(a) = N^{\uparrow}_{\zeta}(a).$$

These relations are called `2-series strict order relations`.

Since the structure of the endomorphism monoids of the elements of $\boldsymbol{n{+}m}$ doesn't change, if one adds two elements 0 and 1 with $\forall x \in A : 0 \leq x \leq 1$, in the following example bounded strict orders are considered.

Example 1.3 *The class $\boldsymbol{2{+}2}$, where $C_1 = \{a_1, a_2\}$ and $C_2 = \{b_1, b_2\}$.*

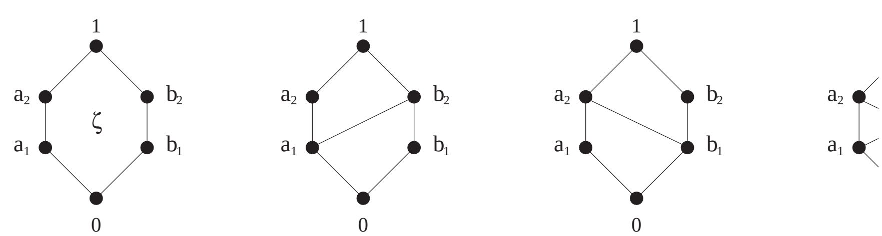
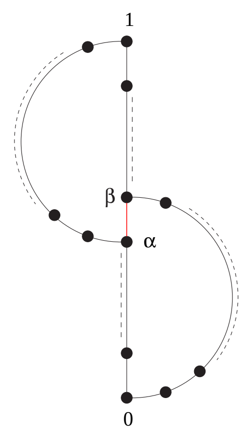


Figure 1: The class $\boldsymbol{2{+}2}$

Definition 1.4 *Let $\rho \in \boldsymbol{n{+}m}$. $N(\rho)$ denotes the set of all relations $\mu \in \boldsymbol{n{+}m}$ of the form $\mu = \rho \cup \{(\alpha_1, \beta_1), \ldots, (\alpha_r, \beta_r)\}, r \geq 0$ arbitrary, s.t. for the case $r \geq 1$ holds*

$$\begin{aligned} \forall\, 1 \leq i \leq r : N^{\uparrow}_{\rho}(\alpha_i) &> N^{\uparrow}_{\rho}(\beta_i) + 1 \ \textit{and} \\ \forall\, 1 \leq i \leq r : N^{\downarrow}_{\rho}(\beta_i) &> N^{\downarrow}_{\rho}(\alpha_i) + 1. \end{aligned}$$

It is not necessary to require that α_i and β_i are elements of different chains of the scaffolding. (Else it follows automatically $(\alpha_i, \beta_i) \in \rho$.)
Moreover, if there is no $(\alpha', \beta') \in \rho$ with $[\alpha' > \alpha$ and $\beta' \leq \beta]$ or $[\alpha' \geq \alpha$ and $\beta' < \beta]$, then the tuples (α, β) are called ***blind edges***:



Defining

$$B(\rho) \quad = \quad \{(a, b) \mid \text{Every unrefineable chain around } a \text{ and } b \text{ contains at least one blind edge}\},$$

$\tilde{\rho} := \rho \setminus B(\rho)$ arises from ρ by deletion of all blind edges in the Hasse diagram of ρ. In the following the chains of ζ are denoted by $C_1 = \{a_1, \ldots, a_n\}, a_1 < \ldots < a_n$ and $C_2 = \{b_1, \ldots, b_m\}, b_1 < \ldots < b_m$.

2 Endomorphism classes

We need the following

Lemma 2.1 *Let $\rho, \mu \in$ $\boldsymbol{n+m}$. Then it holds $End\rho = End\mu$ if and only if $\tilde{\rho} = \tilde{\mu}$.*

That is, the equivalence ε, defined by

$$[\rho]_\varepsilon = \{\mu \mid \mu \in N(\tilde{\rho})\},$$

divides the $\boldsymbol{n+m}$ -orders into their endomorphism classes.

Proof: Let $a \in C_1$ and $b \in C_2$. In the case $n \geq N_\rho^\downarrow(b)$ we define $a_b^\downarrow \in \mathcal{C}_1$ by the property $N_\rho^\downarrow(a_b^\downarrow) = N_\rho^\downarrow(b)$, in the case $n \geq N_\rho^\uparrow(b)$ the element $a_b^\uparrow \in \mathcal{C}_1$ is determined by the property $N_\rho^\uparrow(a_b^\uparrow) = N_\rho^\uparrow(b)$. Dually, the elements $b_a^\downarrow$ and $b_a^\uparrow$ are defined for the case $m \geq N_\rho^\downarrow(a)$ and $m \geq N_\rho^\uparrow(a)$, respectively.

"$\Longrightarrow$:" Let w.l.o.g. $(a, b) \in \rho \setminus \mu$. We may assume that (a, b) doesn't be a blind edge, that is, it holds $(a, b) \in \tilde{\rho}$. In the case $N_\rho^\downarrow(a) = N_\rho^\downarrow(b) - 1$ we define $f_{ab} \in End\rho \setminus End\mu$ by

$$f_{ab}(x) := \begin{cases} a_{N_\rho^\downarrow(x)-1}, & \text{if } x < b \\ x & \text{else.} \end{cases}$$

In the other case it holds $N_\rho^\uparrow(a) - 1 = N_\rho^\uparrow(b)$ and we define

$$f_{ab}(x) := \begin{cases} b_{m+2-N_\rho^\uparrow(x)}, & \text{if } x > a \\ x & \text{else.} \end{cases}$$

"$\Longleftarrow$:" Let in reversion be

$$a = \min_{x \in \mathcal{C}_1} \left\{ \exists y \in \mathcal{C}_2 \,\middle|\, (x, y) \in \rho \setminus \mu \right\}.$$

We have to show:

i) $f \in End\mu \Longrightarrow (f(a), f(b)) \in \rho$, and

ii) $g \in End\rho$ and $(\alpha, \beta) \in \mu \Longrightarrow (g(\alpha), g(\beta)) \neq (a, b)$.

Two cases are to be considered:

Case 1: $n \geq m$.

Ad i) Let $f \in End\mu$. Then the fact $\forall x : x \not<_\mu f(x)$ and $f(x) \not<_\mu x$ yields

$$(f(a), f(b)) \in \{(a, b), (a, a_b^\downarrow)\} \subseteq \rho.$$

Ad ii) Now let $g \in End\rho$ and $(\alpha, \beta) \in \mu$ with $(g(\alpha), g(\beta)) = (a, b)$. One observes

$$\begin{aligned} \alpha &\in \{a\} \,\dot\cup\, (\mathcal{C}_2 \cap [b_a^\uparrow, min\{b_a^\downarrow, b\}]) \text{ and} \\ \beta &\in \{b\} \,\dot\cup\, (\mathcal{C}_1 \cap [a_b^\downarrow, a_b^\uparrow]). \end{aligned}$$

Suppose that $\alpha \neq a$ and $\beta \neq b$ holds. Then it exists a non-blind edge $e = (a', b') \in \mathcal{C}_1 \times \mathcal{C}_2$ with $a \leq a'$ and $b' \leq b$ in ρ. It follows $e \in \mu$ and with $(a, b) \in \mu$ a contradiction.

Case 2: $n < m$. The proof is done as in the case $n > m$, considering the converse relations $\bar{\mu}$ and $\bar{\rho}$.

$\square$

Lemma 2.2 [4] *Let ρ be a 2-series strict order relation. Further, let μ be a strict order relation. Then it holds $End\rho = End\mu$ if and only if there is a permutation π, s.t. for their maximum chains $\mathcal{C}_\mu$ and $\mathcal{C}_\rho$ holds:*

$$\mathcal{C}_\mu = \{C_\pi \,|\, C \in \mathcal{C}_\rho\},$$

where C_π arises from C by $C_\pi := v_{\pi(1)} \dots v_{\pi(k)}$ with $C = v_1 \dots v_k$.

Now we are able to prove the next theorem, which gives answer to the question, under which conditions the inclusion of endomorphism monoids is fulfilled.

Theorem 2.3 *Let $\rho, \mu \in \boldsymbol{n{+}m}$ and $n \neq m$. Then it holds $End\rho \subseteq End\mu$ if and only if $\tilde{\rho} \subseteq \tilde{\mu}$.*

Proof:

"$\Longleftarrow$:" Because of lemma 2.1 it remains to study the case $\tilde{\rho} \subset \tilde{\mu}$. Let $(\alpha, \beta) \in \tilde{\mu}$. The function $f_{\alpha\beta}$ from the proof of the lemma fulfills $f_{\alpha\beta} \in End\mu \setminus End\rho$.
Now let $f \in End\rho$ and $(a, b) \in \mu \setminus \rho$ with $a \in \mathcal{C}_1$ and $b \in \mathcal{C}_2$. W.l.o.g. let hold the inequation $n = |\mathcal{C}_1| > |\mathcal{C}_2| = m$. (Else consider the relations $\bar{\mu}$ and $\bar{\rho}$ instead of ρ and μ.)

By lemma 2.1 it suffices to consider the case $(a, b) \in \tilde{\mu} \setminus \tilde{\rho}$. With lemma 2.2 and $\tilde{\rho} \subseteq \rho \cap \mu$ follows either $(f(a), f(b)) = (a, b)$ or $(f(a), f(b)) \in \mathcal{C}_2$ and hence $(f(a), f(b)) \in \mu$.

"$\Longrightarrow$:" It remains to analyze the case $End\rho \subset End\mu$. One observes that for $f \in End\mu \setminus End\rho$ and all pairs $(\xi, \varsigma) \in N(\mu) \times N(\rho)$ holds: $f \in End\xi \setminus End\varsigma$.
To get a contradiction we assume $\vartheta \in N(\rho) \setminus N(\mu)$. Then exists $(a, b) \in \vartheta$ with $\forall \xi \in N(\mu) : (a, b) \notin \xi$

$$\begin{aligned} &\Longrightarrow\ (a, b) \in \tilde{\rho} \setminus \tilde{\mu} \\ &\Longrightarrow\ \exists g \in End\rho \text{ and } (\alpha, \beta) \in \mu \text{ with } (g(\alpha), g(\beta)) = (a, b)^* \\ &\Longrightarrow\ g \notin End\mu. \end{aligned}$$

Thus it follows $g \in End\rho \setminus End\mu$ contradicting the assumption. $\square$

*E.g. the function f_{ab} possesses this property.

Corollary 2.4 *The lattice of endomorphism monoids of the elements of* $\mathcal{K} = \boldsymbol{n{+}m}$, $m < n$, *is isomorphic to the power set lattice of a* $2(m-1)$*-element set; that is, from* $|M| = 2(m-1)$ *follows*

$$(\wp(M); \subseteq) \cong End\mathcal{K}.$$

Example 2.5 *In figure 2 the lattice of endomorphism classes of* $\boldsymbol{4{+}3}$*-orders is depicted. For each class* $End\rho$ *appears the representant* $\tilde{\rho} \in [\rho]_\varepsilon$ *of the corresponding order.*

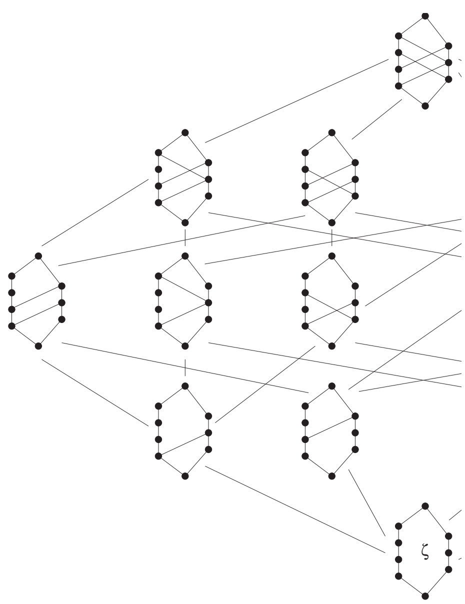


Figure 2: The class $\boldsymbol{4{+}3}$

Definition 2.6 *A relation* $\rho \subseteq A^2$ *is called* `rigid`*, if the identity mapping* $id(x) = x$ *is the only unary polymorphism, that is* $End\rho = \{id\}$*. Moreover, if for every natural number* n *the* n*-ary projections (or selector functions)* $e_i^n(x_1, \ldots, x_n), 1 \leq i \leq n$*, are the only* n*-ary polymorphisms, then* ρ *is called* `strongly rigid`*.*

In [9] it was shown by Z.Hedrlin et al. in 1965, that rigid relations exist on any set. In 1973, I.G.Rosenberg continued this work by presenting a strongly rigid binary relation on any $3 \leq n$-element set [8].

We transfer the concept of rigid relations to given classes $\mathcal{K}$ of relations.

Definition 2.7 *Let $\mathcal{K}$ be a class of relations. $\rho \in \mathcal{K}$ is called* `local rigid`*, if $End\rho \subseteq End\mu$ holds for all $\mu \in \mathcal{K}$. Moreover, if the inclusion $Pol\rho \subseteq Pol\mu$ holds for all $\mu \in \mathcal{K}$, then ρ is called* `local strongly rigid`*. If there exists $\rho \in \mathcal{K}$ with this property, we say "$\mathcal{K}$ admits local (strongly) rigid structures".*

Corollary 2.8 *Let $n \neq m$. Then $\mathcal{K} = \boldsymbol{n{+}m}$ admits local rigid structures.*

[***Proof:*** Obviously, every $\rho \in [\zeta]_\varepsilon$ is local rigid.]

Lemma 2.9 [4]*Let ρ and μ be strict order relations with $End\rho = End\mu$. Then it exists a permutation π, s.t. $\mathcal{C}_\mu = \{C_\pi \mid C \in \mathcal{C}_\rho\}$ holds.*

In the following the case $n = m$ is studied.

Theorem 2.10 *Let $\mathcal{K} = \boldsymbol{n{+}n}$. We define the disjoint partition $\mathcal{K} = \mathcal{K}_s \dot\cup \mathcal{K}_u$ by*

$$\begin{aligned} \mathcal{K}_s &= \{\rho \in \mathcal{K} \mid (a_i, b_j) \in \rho \Longleftrightarrow (a_j, b_i) \in \rho\} \text{ and} \\ \mathcal{K}_u &= \{\rho \in \mathcal{K} \mid \exists i, j : (a_i, b_j) \in \rho, (b_i, a_j) \notin \rho\}. \end{aligned}$$

Further, let $\rho, \mu \in \mathcal{K}$. Then the following conditions are equivalent:

i) *$End\rho \subseteq End\mu$*

ii) *$\tilde\rho \subseteq \tilde\mu$, and if $\exists r : (a_r, b_{r+1}), (b_r, a_{r+1}) \in \tilde\rho$, then the following implications hold (conditions of symmetry):*

$$\begin{aligned} \textbf{\textit{a)}}\ \ & \nexists i, j \geq r \quad \text{with } (a_i, b_j) \in \rho,\ (b_i, a_j) \notin \rho \\ & \qquad\qquad \text{or } (a_i, b_j) \notin \rho,\ (b_i, a_j) \in \rho \\ \Longrightarrow & \nexists i, j \geq r \quad \text{with } (a_i, b_j) \in \mu,\ (b_i, a_j) \notin \mu \\ & \qquad\qquad \text{or } (a_i, b_j) \notin \mu,\ (b_i, a_j) \in \mu. \end{aligned}$$

$$\begin{aligned} \textbf{\textit{b)}}\ \ & \nexists i, j \leq r \quad \text{with } (a_i, b_j) \in \rho,\ (b_i, a_j) \notin \rho \\ & \qquad\qquad \text{or } (a_i, b_j) \notin \rho,\ (b_i, a_j) \in \rho \\ \Longrightarrow & \nexists i, j \leq r \quad \text{with } (a_i, b_j) \in \mu,\ (b_i, a_j) \notin \mu \\ & \qquad\qquad \text{or } (a_i, b_j) \notin \mu,\ (b_i, a_j) \in \mu. \end{aligned}$$

Proof:

i) $\Longrightarrow$ ii): Let $(a_i, b_{i+1}) \in \tilde{\rho}$ be arbitrary chosen. The mapping $f_{a_i,b_{i+1}}$ is an endomorphism of ρ and with $(a_i, a_{i+1}) \in \mu$ follows

$$(a_i, b_{i+1}) = (f(a_i), f(a_{i+1})) \in \tilde{\mu}$$

and hence $\tilde{\rho} \subseteq \tilde{\mu}$. If there is no r with $(a_r, b_{r+1}), (b_r, a_{r+1}) \in \rho$, it remains nothing to show. Else exist $r_1 < \ldots < r_k, k \geq 1$, with this property. W.l.o.g. let $r_i + 1 < r_{i+1}$ for all $i \in \{1, \ldots, k\}$. We define

$$\begin{aligned} \rho_0 &:= \{(a,b) \mid (a,b), (b, b_{r_1+1}) \in \rho\} \text{ and} \\ \rho_j &:= \{(a,b) \mid (a,b), (a_{r_j}, a), (b, b_{r_{j+1}+1}) \in \rho\} \quad \text{for } 1 \leq j \leq k. \end{aligned}$$

Analogously $\mu_0, \ldots, \mu_k$ are defined.

Now we check the conditions of symmetry. For this, assume for some $i \in \{1, \ldots, n\}$:

$$\begin{aligned} \nexists r_i \leq s, t \leq r_{i+1} \quad \text{with } & (a_s, b_t) \in \rho_i,\ (b_s, a_t) \notin \rho_i \\ \text{or } & (a_s, b_t) \notin \rho_i,\ (b_s, a_t) \in \rho_i. \end{aligned}$$

Suppose that $\exists r_i \leq s, t \leq r_{i+1}$ with $(a_s, b_t) \in \mu_i$, $(b_s, a_t) \notin \mu_i$
or $(a_s, b_t) \notin \mu_i$, $(b_s, a_t) \in \mu_i$.

Then we are able to define a mapping $\tilde{f} \in End\rho_i \setminus End\mu_i$ by

$$\begin{aligned} \tilde{f}(a_l) &= b_l \text{ and} \\ \tilde{f}(b_l) &= a_l \end{aligned}$$

for $r_i \leq l \leq r_{i+1}$, which can be extended to an endomorphism $f \in End\rho \setminus End\mu$ as follows:

$$f(x) := \begin{cases} \tilde{f}(x), & x \in \{a_{r_i}, \ldots, a_{r_{i+1}}\} \cup \{b_{r_i}, \ldots, b_{r_{i+1}}\}, \\ x & \text{else.} \end{cases}$$

This yields a contradiction to the condition $End\rho \subseteq End\mu$.

ii) $\Longrightarrow$ i): We differ between two cases **A)** $\mu \in \mathcal{K}_s$ and **B)** $\mu \in \mathcal{K}_u$.

Ad A) Let $f \in End\rho$ and $(\alpha, \beta) \in \mu$. We may assume $(\alpha, \beta) = (a_i, b_j) \in \mathcal{C}_1 \times \mathcal{C}_2$. By lemma 2.9 we get

$$f(\alpha) \in \{a_i, b_i\} \text{ and } f(\beta) \in \{a_j, b_j\}.$$

With
$$\{(a_i,a_j),(b_i,b_j),(a_i,b_j),(b_i,a_j)\} \subseteq \mu$$
follows $(f(\alpha), f(\beta)) \in \mu$ and within $f \in End\mu$ as claimed.

Ad B) Obviously, it does also hold $\rho \in \mathcal{K}_u$.

Case B.1) $\nexists r : (a_r, b_{r+1}), (b_r, a_{r+1}) \in \rho$.
Let be $(a_i, b_j) \in \mu$ blind edges, $(b_i, a_j) \notin \mu$ and $f \in End\rho$. Suppose that $(f(a_i), f(b_j)) = (b_i, a_j)$. Then $f(\mathcal{C}_1) = \mathcal{C}_2$ and $f(\mathcal{C}_2) = \mathcal{C}_1$ is fulfilled, contradicting $\rho \in \mathcal{K}_u$, and one obtains $f \in End\mu$.

Case B.2) $\exists r : (a_r, b_{r+1}), (b_r, a_{r+1}) \in \rho$.
We consider the relations $\rho_0, \ldots, \rho_k$ and $\mu_0, \ldots, \mu_k$, which were defined in the other direction of the proof. By the use of the conditions of symmetry *ii.a)* and *ii.b)* the pairs (μ_i, ρ_i) in the case $\mu_i \in \mathcal{K}_u^i$ can be treated as the pair (μ, ρ) in B.1) or in the case $\mu_i \in \mathcal{K}_s^i$ as the pair (μ, ρ) in A). □

Corollary 2.11 *The class* $\boldsymbol{n+n}$ *admits local rigid structures.*

[***Beweis:*** ρ is local rigid if and only if $\rho \in \mathcal{K}_u$ and $\tilde{\rho} = \zeta$.]

By now, we only considered endomorphisms. To study n-ary polymorphisms, we need a generalization of the concept of blind edges.

3 Polymorphisms

Definition 3.1 *Let* $\rho \in \boldsymbol{n+m}$. *The tuple* (α, β) *is called* ***k-blind edge****, if* α *and* β *are elements of different chains of the scaffolding and additionally holds:*

$$\begin{array}{rrcl} i) & N_\rho^\uparrow(\alpha) & > & N_\rho^\uparrow(\beta) + k \\ & \text{and } N_\rho^\downarrow(\beta) & > & N_\rho^\downarrow(\alpha) + k, \end{array}$$

$$\begin{array}{rrrl} ii) & \nexists(\alpha', \beta') \in \rho & \text{with} & \alpha' \geq \alpha, \beta' < \beta \\ & & \text{or} & \alpha' > \alpha, \beta' \geq \beta. \end{array}$$

Lemma 3.2 *Let $\mathcal{K} = \boldsymbol{n{+}m}$. Then the following mapping $\theta : A^2 \longrightarrow A$ defines a binary polymorphism of ζ.*

$$\theta(x_1, x_2) := \begin{cases} a_2, & \text{if } (x_1, x_2) = (a_1, b_{m-1}) \\ a_3, & \text{if } (x_1, x_2) = (a_i, b_m) \text{ for an } i \geq 2 \\ 0, & \text{if } \exists j : x_j = 0 \\ 1, & \text{if } \exists j : x_j = 1 \\ x_2 & \text{else.} \end{cases}$$

Proof: Let $(\alpha_i, \beta_i) \in \zeta$ for $i = 1, 2$. W.l.o.g. let $\alpha_1 \neq 0 \neq \alpha_2$ and $\beta_1 \neq 1 \neq \beta_2$. (Else with $f(\tilde{\alpha}) = 0$ or $f(\tilde{\beta}) = 1$ it follows directly $(f(\tilde{\alpha}), f(\tilde{\beta})) \in \zeta$.)

In the case $(\alpha_1, \alpha_2) = (a_1, b_{m-1})$ one obtains $\beta_1 \in \mathcal{C}_1$ and $\beta_2 \in \{b_m, a_n\}$.

i) $(\beta_1, \beta_2) = (a_j, b_m)$ for some $j \geq 2$. Then $(f(\tilde{\alpha}), f(\tilde{\beta})) = (a_2, a_3) \in \zeta$.

ii) $(\beta_1, \beta_2) = (a_j, a_n)$ for an $j \geq 2$. Then $(f(\tilde{\alpha}), f(\tilde{\beta})) = (a_2, a_n) \in \zeta$.

In the case $(\alpha_1, \alpha_2) = (a_j, b_m)$ for some $j \geq 2$ follows $\beta_2 = 1$ and hence

$$(f(\tilde{\alpha}), f(\tilde{\beta})) = (a_j, 1) \in \zeta.$$

In all other cases it is obtained $(f(\tilde{\alpha}), f(\tilde{\beta})) = (\alpha_2, \beta_2) \in \zeta$.

$\square$

Theorem 3.3 *Let $n \neq m$. The class $\boldsymbol{n{+}m}$ doesn't admit local rigid structures.*

[***Proof:*** It suffices to consider ρ with $\tilde{\rho} = \zeta$, since the class of ζ contains the only rigid relations. We define $\rho \in \boldsymbol{n{+}m}$ by

$$\rho := \zeta \cup \{(a_1, b_{m-1}), (a_1, b_m)\}.$$

Then the mapping θ from lemma 3.2 is a binary polymorphism $\theta \in Bin\zeta \setminus Bin\rho$, since

$$\begin{array}{rcl} (a_1, b_{m-1}), (b_{m-1}, b_m) & \in & \rho, \text{ but} \\ (\theta(a_1, b_{m-1}), \theta(b_{m-1}, b_m)) & = & (a_2, b_m) \notin \rho.] \end{array}$$

That the above theorem is also valid for the case $n = m$, follows immediately from

Theorem 3.4 *For every 2-series strict order relation ρ there is a natural number m, s.t. ρ is uniquely determined by its m-ary polymorphisms $Pol^{(m)}\rho$.*

Proof: Let $\rho, \mu \in \boldsymbol{n{+}m}$ and $(a_r, b_s) \in \rho \setminus \mu$ be a k-blind edge for some $k \geq 1$. We define the mapping $\eta_{a_r b_s} : A^{s-1} \longrightarrow A$ by

$$\eta_{a_r b_s}(x_1, \ldots, x_{s-1}) := \begin{cases} a_r, & \text{if } (x_1, \ldots, x_{s-1}) = (b_1, \ldots, b_{s-1}) \\ 0, & \text{if } \exists i : x_i = 0 \\ x_{s-1} & \text{else.} \end{cases}$$

Then $\eta_{a_r b_s}$ is a polymorphism of ρ:
Let $(\alpha_i, \beta_i) \in \rho$, $i = 1, \ldots, s-1$. We have to consider three cases:

i) $(\alpha_1, \ldots, \alpha_{s-1}) = (b_1, \ldots, b_{s-1})$. It follows

$$\begin{aligned} f(\alpha_1, \ldots, \alpha_{s-1}) &= a_r, \\ \eta_{a_r b_s}(\beta_1, \ldots, \beta_{s-1}) &= \beta_{s-1} \end{aligned}$$

with $b_{s-1} < \beta_{s-1}$. It holds **1)** $\beta_{s-1} \in \mathcal{C}_2$, i.e. $\beta_{s-1} \geq b_s$ or **2)** it exists an element $a \in \mathcal{C}_1$ with $(\beta_{s-1}, a) \in \rho$ and hence $(a_r, a) \in \rho$.
In both cases 1) and 2) follows $(\eta_{a_r b_s}(\tilde{\alpha}), \eta_{a_r b_s}(\tilde{\beta})) = (a_r, \beta_{s-1}) \in \rho$.

ii) $(\beta_1, \ldots, \beta_{s-1}) = (b_1, \ldots, b_{s-1})$. It follows $\alpha_1 = 0$ and hence

$$(\eta_{a_r b_s}(\tilde{\alpha}), \eta_{a_r b_s}(\tilde{\beta})) = (0, a_r) \in \rho.$$

iii) $\tilde{\alpha} \neq (b_1, \ldots, b_{s-1}) \neq \tilde{\beta}$. It follows

$$(\eta_{a_r b_s}(\tilde{\alpha}), \eta_{a_r b_s}(\tilde{\beta})) = (\alpha_{s-1}, \beta_{s-1}) \in \rho.$$

On the other side one obtains $\eta_{a_r b_s} \notin Pol^{(s-1)}\mu$, being a consequence of

$$\begin{aligned} (b_1, b_2), \ldots, (b_{s-1}, b_s) &\in \mu \text{ and} \\ (\eta_{a_r b_s}(b_1, \ldots, b_{s-1}), \eta_{a_r b_s}(b_2, \ldots, b_s)) = (a_r, b_s) &\notin \mu. \end{aligned}$$

□

Corollary 3.5 *The class $\boldsymbol{n{+}n}$ doesn't admit local strongly rigid structures.*

[***Proof:*** We have to consider the set

$$\mathcal{S} := \left\{\rho \,\middle|\, \rho \in \mathcal{K}_u \text{ and } \tilde{\rho} = \zeta\right\}$$

of all local rigid relations. Since $\zeta \notin \mathcal{S}$, two different relations $\rho_1, \rho_2 \in \mathcal{S}$ differ in at least two blind edges $(a_{r_1}, b_{s_1}) \in \rho_1 \setminus \rho_2$ and $(a_{r_2}, b_{s_2}) \in \rho_2 \setminus \rho_1$. (W.l.o.g. the tuples were assumed to be elements of $\mathcal{C}_1 \times \mathcal{C}_2$.) For the mappings

$$\eta_{a_{r_i} b_{s_i}} : A^{s_i - 1} \longrightarrow A,\ i \in \{1, 2\}$$

from theorem 3.4 holds:

$$\begin{aligned} & \eta_{a_{r_1} b_{s_1}} \in Pol^{(s_1-1)}\rho_1 \setminus Pol^{(s_1-1)}\rho_2 \\ \text{and} \quad & \eta_{a_{r_2} b_{s_2}} \in Pol^{(s_2-1)}\rho_2 \setminus Pol^{(s_2-1)}\rho_1 .] \end{aligned}$$

It is possible to generalize the concept of 2-series strict orders in a natural way to k-series strict orders of the form $\mathcal{K} = \sum_{i=1}^{k} \boldsymbol{n_i}$ (see [5]). Then $End\mathcal{K}$ in general does not be isomorphic to a BOOLEan algebra and $\mathcal{K}$ admits no local rigid structures and hence no local strongly rigid structures, too. If a BOOLEan algebra $\mathcal{B}$ appears as a direkt product of an even number of two-element BOOLEan algebras, then it is possible to find a class $\mathcal{K}$ of k-series strict orders with $End\mathcal{K} \cong \mathcal{B}$.

References

[1] A.Fearnley, "A strongly rigid binary relation", *Acta Sci. Math.* **61**, *No.*1-4 (1995), pp. 35-41.

[2] Z.Fueredi, I.G.Rosenberg, "Orders admitting an isotone majority operation", *Mult.-Valued Log.* **3**, *No.*1 (1998), pp. 39-53.

[3] B.Larose, L.Zadori, "Algebraic properties and dismantlability of finite posets", *Discrete Math.* **163**, *No.*1-3 (1997), pp. 89-99.

[4] R.Lenz, "Isotone mappings of levelled strict orders", *Preprint Nr. 331, University of Kaiserslautern,* 2002.

[5] R.Lenz, "Algebraisch abgeschlossene Klassen graphentreuer Funktionen", *PhD-thesis, University of Kaiserslautern,* 2002.

[6] O.Lueders, D.Schweigert, “Strictly order primal algebras”, *Acta Math. Univ. Comen., New Ser.* **63**, *No.*2 (1994), pp. 275-284.

[7] A.Pultr, V.Trnkova, “Combinatorial algebraic and topological representations of groups, semigroups and categories”, *North-Holland Math. Libr.* 1980.

[8] I.G.Rosenberg, “Strongly rigid relations”, *Rocky Mountain J. Math.* **3** (1973), pp. 631-639.

[9] P.Vopěnka, A.Pultr, Z.Hedrlin, “A rigid relation exists on any set”, *Comment. Math. Univ. Carolinae* **6** (1965), pp. 149-155.

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