Hub Cover and Hub Center Problems

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Abstract

Using covering problems (CoP) combined with binary search is a well-known and successful solution approach for solving continuous center problems. In this paper, we show that this is also true for center hub location problems in networks. We introduce and compare various formulations for hub covering problems (HCoP) and analyse the feasibility polyhedron of the most promising one. Computational results using benchmark instances are presented. These results show that the new solution approach performs better in most examples.

Key words: hub location, hub covering, integer programming, valid inequalities, facets

1 Introduction

Hub location problems arise when it is desirable to transport commodities (e.g., goods or passengers) between origin-destination (o-d) pairs. In general, a direct transportation of these commodities can not be realized due to the fact that establishing such a network is extremely costly. As an alternative, one uses a special logistic network with a so called hub-and-spoke structure where the hubs act as collection, consolidation, transfer and distribution points. The advantage of using hubs is that by consolidating the flow, economies of scale can be achieved so that transferring flow between hubs is cheaper than the cost of moving commodities directly between non-hub nodes (spokes). Spokes

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can be connected to one or more hubs, depending on whether we deal with single or multiple allocation. It is usually assumed that the hubs are fully interconnected, while spokes are not connected to each other. Therefore all commodities have to be routed via at least one hub. Hence such a logistic network consists of two parts, namely the hub level and the spoke level (see Figure 1).
In general, we deal in hub location problems with two different tasks. First, the *hub selection*, where we choose certain nodes as hubs, and secondly, the *spoke allocation*, where we assign the spoke nodes to the hub nodes. For many hub location problems which are discussed in the literature, both hub location and spoke allocation are known to be NP-hard.

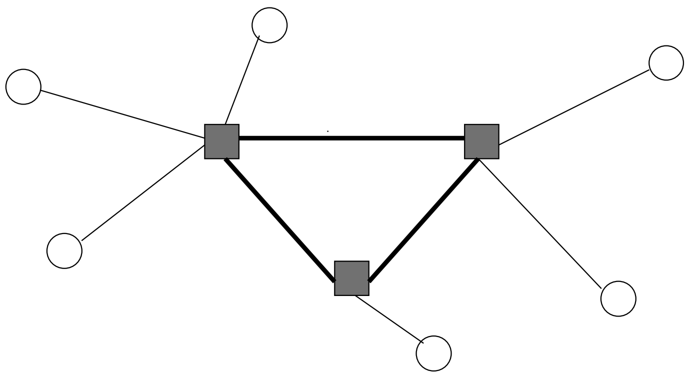

Fig. 1. Example of hub-and-spoke network (hubs are shown as squares, spokes as balls)

The hub location problem to be discussed in this paper is known as the *Uncapacitated Single Allocation p Hub Center Problem* (USApHCP). In this problem there are no capacity restrictions on the hubs or on the flow between arcs. We want to choose a fixed number p of the nodes to be hubs and allocate the spokes to exactly one of the chosen hubs in such a way that the maximum path between any o-d pair is minimized.

In 1994, Campbell [2] presented a quadratic formulation for USApHCP and a linearization of this formulation which has the drawback of many additional variables. In 2000, Kara and Tansel [10] provided a linearization of Campbell's quadratic formulation, which clearly outperformed that of Campbell. The most promising formulation of the USApHCP was provided in 2002 by Ernst et al. [4]. Their formulation is based on the concept of a *hub radius* and will be explained in more details in Section 2.
(Mixed) integer programming approaches based on these formulation solve the USApHCP exactly. Several heuristic methods were proposed in Ernst et al. ([4],[5]). The heuristics were analysed with respect to their worst case behaviour for the hub selection as well as for the allocational part of USApHCP.

In this paper we tackle USApHCP by iteratively solving *hub covering problems* (HCoP). In contrast to the p-center hub problem, where p is given a priori, p is in HCoP a variable. The objective is, in fact, to minimize p under the constraint that the demand has to be met within a given treshold path length β.
HCoP can, for instance, be used to model the overnight package delivery where parcels have to be at its destination within, say, twelve hours. Another application is the transportation of perishable goods. The hub covering problem is also an appropriate model for passenger air transportation with the goal to keep the dissatisfaction factor as low as possible.

The paper is organized as follows. In Section 2, we provide precise mathematical definitions of USApHCP and HCoP. Moreover, we review and improve several (mixed) integer programming formulations. Some polyhedral results related to the most promising formulation of HCoP are presented in Section 3. In Section 4, we propose the combined HCoP and binary search solution approach to USApHCP and discuss its computational performance. Section 5 concludes the paper with a short summary of our results and ideas for further research.

2 Model formulations

Let $G = (V, E)$ be a complete undirected graph in which each pair of nodes is connected by an arc $[i, j]$ with cost c_{ij}. We assume that the graph is symmetric, i.e., $c_{ij} = c_{ji}$, and satisfies the triangle inequality $c_{ij} \leq c_{ik} + c_{kj}$ for all $i, j, k = 1, \ldots, n$ (this can be done without loss of generality, since c_{ij} can be replaced by the shortest path distances between nodes i and j, otherwise).

The economy of scale is modeled by a discount factor $\alpha \in [0,1]$ on each of the hub-to-hub links. It can be interpreted as a speed-up or cost-decrease factor incurred by a higher usage of this link. Due to the triangle inequality any o-d path between nodes i and j which are allocated to hubs k and m, resepectively, has the length $d_{ij} = c_{ik} + \alpha c_{km} + c_{jm}$. Note that i, k, m, j do not have to be different nodes.

For given p with $1 \leq p \leq n$ two decisions have to be made

- the selection of a subset $\mathcal{H}$ of p nodes as hubs (*hub selection*) and
- the allocation of each *spoke* $i \in V \setminus \mathcal{H}$ to some hub $k \in \mathcal{H}$ (*allocation*)

In the *Uncapacitaced Single Allocation p Hub Center Problem (USApHCP)* both decisions have to be made while the *allocation problem* assumes that the set $\mathcal{H}$ of hubs has already been selected and only the allocation decision is required. The goal in both problems is to minimize the length d_{ij} of a longest path between nodes $i, j \in V$. Both were shown to be NP-hard in Ernst et al. [4].

2.1 Radius formulation of USApHCP

All formulations of USApHCP and the hub covering problem discussed in the next subsection use decision variables

$$x_{ik} = \begin{cases} 1 \text{ if } k \text{ is a hub} \\ 0 \text{ otherwise} \end{cases}$$

Note, that this definition implies that $x_{kk} = 1$ if and only if node i is a hub node. Ernst et al. [4] use in addition for each $k = 1, \ldots, n$ a continuous variable r_k representing the *radius* of hub k, i.e. the maximum cost between hub k and a spoke node allocated to k.

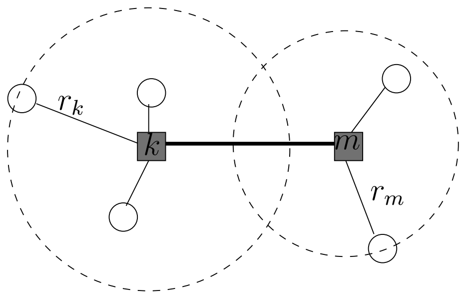

Fig. 2. Example of hubs and spokes with hub radii r_m and r_k and a path with length $r_k + r_m + \alpha\, c_{km}$.

The resulting model USApHCP is as follows.

$$\text{minimize} \quad z$$

subject to

$$z \geq r_k + r_m + \alpha\, c_{km} \quad \forall\, k, m \in \{1, \ldots, n\} \tag{1}$$

$$r_k \geq c_{ik} x_{ik} \quad \forall\, i, k \in \{1, \ldots, n\} \tag{2}$$

$$\sum_{k=1}^{n} x_{kk} = p \tag{3}$$

$$\sum_{k=1}^{n} x_{ik} = 1 \quad \forall\, i \in \{1, \ldots, n\} \tag{4}$$

$$x_{ik} \leq x_{kk} \quad \forall\, i, k \in \{1, \ldots, n\} \tag{5}$$

$$x_{ik} \in \{0,1\} \quad \forall\, i,k \in \{1,\ldots,n\} \tag{6}$$

$$r_k \geq 0 \qquad \forall k \in \{1,\ldots,n\} \tag{7}$$

Constraint (1) defines the objective function z as the maximum cost of a path between any two nodes by using the respective hub radii (see Figure 2). Constraint (5) and (2) make sure that node i can only be allocated to k, if k is a hub and if the cost c_{ik} between i and k is at most the radius r_k of k, respectively. With constraint (3) it is guaranteed that exactly p hubs are selected, while constraint (4) ensures that every node is assigned to exactly one hub.

2.2 Formulations of the Hub Covering Problem

In USApHCP, the number p of hubs is given and the length of a maximum path is to be minimized. In the *hub covering problem (HCoP)* the situation is reversed. A parameter β - the *cover radius* - which is an upper bound on the path length d_{ij} between any o-d pair is given. The goal is to find a hub set $\mathcal{H}$ with minimum cardinality $|\mathcal{H}|$ such that this bound can be obtained.

HCoP may not necessarily have a solution for given β. Proposition 1 characterizes values of β for which a solution exists to HCoP.

Proposition 1 *Let $G = (V,E)$ be a complete graph with $|V| = n \geq 2$ and let the cost c_{ij} satisfy the triangle inequality. The HCoP has a solution if and only if $\beta \geq \alpha\ max_{i,j}\ (c_{ij})$.*

PROOF. If $\beta \geq \alpha\ max_{i,j}\ c_{ij}$, then $\mathcal{H} = V$ is a feasible solution with respect to β, since $d_{ij} = \alpha\ c_{ij} \leq \beta$.
If, conversely, $\beta < \alpha\ c_{ij}$, for some o-d pair (i,j), then

$$\begin{aligned} \beta &< \alpha\ c_{ij} \\ &\overset{\triangle-ineq.}{\leq} \alpha\ (c_{ik} + c_{km} + c_{mj}) \\ &\leq c_{ik} + \alpha\ c_{km} + c_{mj} \\ &= d_{ij}. \quad \square \end{aligned}$$

HCoP has only been considered by few authors. It was introduced by Campbell [2]. His quadratic formulation was later outperformed by Kara and Tansel's [9] linearization:

HCoP-KT:

$$\text{minimize} \quad \sum_{k=1}^{n} x_{kk}$$

subject to

$$(4) - (6)$$

$$(c_{ik} + \alpha\, c_{km})x_{ik} + c_{mj}x_{jm} \leq \beta \quad \forall\, i, k, j, m \in \{1, \ldots, n\} \tag{8}$$

Constraint (8) make sure that i and j can only be assigned to k and m, respectively, if the path cost between i and j does not exceed β.

In 2004, Wagner [13] developed an improved model formulation for the hub covering problem. His preprocessing rules out some hub allocations, such that his formulation requires not only less variables, but also less constraints.

The first preprocessing idea is to drop variable x_{ik}, if an allocation of i to k leads to a violation of constraint (8). Hence, only the following set VA of *valid allocations* needs to be considered.

$$VA = \Big\{(i,k) \mid \quad 2\, c_{ik} \leq \beta \text{ and } c_{ik} + \alpha\, max_j(c_{kj}) \leq \beta \;\Big\}$$

The second preprocessing idea is based on the observation that for i, j, k, m with $c_{ik} + \alpha c_{km} + c_{jm} \leq \beta$ constraint (8) is obsolete. Conversely, if this inequality does not hold, one can exclude the simultaneous allocation of i to k and of j to m. This can be achieved by prohibiting that x_{ik} and x_{jm} take a value of 1 concurrently, that is $x_{ik} = 1$ and $x_{jm} = 1$ must not hold at the same time. By the first preprocessing idea, one can obviously restrict this preprocessing to pairs $(i, k) \in VA$. Therefore Wagner defines the set IA, which contains all of these pairs of *incompatible assignments*:

$$IA = \Big\{(i,k,j,m) \mid \quad (i,k),(j,m) \in VA, i < j \text{ and } c_{ik} + \alpha\, c_{km} + c_{jm} > \beta\Big\}$$

Note that we assume in the definition of IA that $k \neq j$ and $i \neq m$. (Otherwise we also obtain useless constraints of the form $x_{ik} + x_{kj} \leq 1$ and $x_{ik} + x_{ji} \leq 1$.) In general, the cardinality $t := |IA| << n^4$.

The previous observations result in the following model:

HCoP-W1:

$$\text{minimize} \quad \sum_{k=1}^{n} x_{kk}$$

s.t.

$$\sum_{k:(i,k)\in VA} x_{ik} = 1 \quad \forall\, i \in \{1, \ldots, n\} \tag{9}$$

$$x_{ik} \leq x_{kk} \quad \forall\ (i,k) \in VA \tag{10}$$

$$x_{ik} \in \{0,1\} \quad \forall\ (i,k) \in VA \tag{11}$$

$$x_{ik} + x_{jm} \leq 1 \quad \forall\ (i,k,j,m) \in IA \tag{12}$$

The objective function minimizes the number of hubs. Constraints (9) - (11) are the same as (4) - (6). In (12) only incompatible assignments are forbidden.

This formulation can be further improved by aggregating some constraints of type (12). For instance, the constraints $x_{12} + x_{34} \leq 1$ and $x_{12} + x_{35} \leq 1$ can be replaced by $x_{12} + x_{34} + x_{35} \leq 1$ [2] without changing the solution space of the integer program. The constraint aggregation procedure develop by Wagner terminates if no constraint (12) can be further aggregated. In the following, we refer to the resulting model as HCoP-W2.

The last formulation (HCoP-r) for the HCoP which is based on the same radius concept introduced for USApHCP in Subsection 2.1 was proposed by Ernst et. al. [6] in 2005:

$$\text{minimize} \quad \sum_{k=1}^{n} x_{kk}$$

subject to

$$r_k + r_m + \alpha\, c_{km} \leq \beta \quad \forall\ k,m \in \{1,\ldots,n\} \tag{13}$$

$$r_k \geq c_{ik} x_{ik} \quad \forall\ i,k \in \{1,\ldots,n\} \tag{14}$$

$$\sum_{k=1}^{n} x_{ik} = 1 \quad \forall\ i \in \{1,\ldots,n\} \tag{15}$$

$$x_{ik} \leq x_{kk} \quad \forall\ i,k \in \{1,\ldots,n\} \tag{16}$$

$$x_{ik} \in \{0,1\} \quad \forall\ i,k \in \{1,\ldots,n\} \tag{17}$$

$$r_k \geq 0 \qquad \forall k \in \{1,\ldots,n\} \tag{18}$$

Constraint (13) plays the same role as constraint (8). Constraints (14) - (18) are the same as in the radius formulation of the USApHCP.

3 Polyhedral properties of the HCoP

In contrast to median and center hub problems, where some results on the polyhedral structure are known (see [1],[8]), no work has been done on the polyhedral analysis for the HCoP. We will use Wagner's second formulation,

[2] Note that (9) implies $x_{34} + x_{35} \leq 1$

HCoP-W2, as basis for our investigations. We first derive the dimension of the convex hull of all integer solutions of HCoP-W2. Then we examine the faces that are induced by constraints (9) - (12) and check which of them are facet defining. For this purpose we use well-known results from polyhedral theory and integer programming as can be found, for instance, in Nemhauser and Wolsey [12].

We use the following denotations:

- $\mathcal{X}_{\text{HCoP-W2}} := \{P = (x_{11}, .., x_{nn}) \in \mathbb{R}^q : \text{P feasible for HCoP-W2}\}$ is the set of feasible solutions of the LP relaxation of HCoP-W2.
- $\mathcal{Z}_{\text{HCoP-W2}} := \{P \in \mathcal{X}_{\text{HCoP-W2}} : x_{ik} \in \{0,1\}, \forall (i,k) \in VA\}$ is the set of feasible integral points of HCoP-W2.
- $\mathcal{P}_{\text{HCoP-W2}} := conv(\mathcal{Z}_{\text{HCoP-W2}})$ is the polyhedron obtained by the convex hull of $\mathcal{Z}_{\text{HCoP-W2}}$.

Since HCoP-W2 has $q := |VA|$ variables which have to satisfy the n linearly independent constraints (9), the dimension of $\mathcal{P}_{\text{HCoP-W2}}$ is at most $q - n$. The following dimension result shows that this is, indeed, the dimension of the polyhedron.

Theorem 2 *Let $\beta \geq \alpha \; max_{i,j} \; c_{ij}$ and let $q := |VA|$.*
Then $dim \; \mathcal{P}_{\text{HCoP-W2}} = q - n$.

PROOF. We have to show that there are $q-n+1$ affinely independent points in $\mathcal{P}_{\text{HCoP-W2}}$.
Consider the following points:
(a) For all $j \in \{1, .., n\}$ and for all $(j, l) \in VA$ with $j \neq l$ let $P = (x_{11}, ..., x_{nn}) \in \mathbb{R}^q$ be the point defined by

$$x_{ik} = \begin{cases} 1 \text{ if } i = k = s \quad \text{with } s \in \{1, .., n\} \setminus \{j\} \\ 1 \text{ if } i = j, \; k = l \\ 0 \text{ otherwise} \end{cases}$$

Since the pairs (j, j) are elements of VA, there exist $q - n$ elements $(j, l) \in VA$ with $j \neq l$. Therefore we have $q - n$ points of type (a). They are obviously affinely independent.
(b) The point P with $x_{ss} = 1 \; \forall s \in \{1, ..., n\}$ and all other components equal to 0. This point is clearly independent of the set of points of type (a).
Hence we have $q - n + 1$ affine independent point in $\mathcal{P}_{\text{HCoP-W2}}$. $\square$

Example 3 *For $n = 4$ and with VA as shown in the table ($|VA| = q = 12$), we obtain the following $q - n + 1$ affine independent points:*

x_{11}	x_{12}	x_{13}	x_{22}	x_{23}	x_{24}	x_{31}	x_{32}	x_{33}	x_{42}	x_{43}	x_{44}
1	0	0	1	0	0	0	0	1	0	1	0
1	0	0	1	0	0	0	0	1	1	0	0
1	0	0	1	0	0	0	1	0	0	0	1
1	0	0	1	0	0	1	0	0	0	0	1
1	0	0	0	0	1	0	0	1	0	0	1
1	0	0	0	1	0	0	0	1	0	0	1
0	0	1	1	0	0	0	0	1	0	0	1
0	1	0	1	0	0	0	0	1	0	0	1
1	0	0	1	0	0	0	0	1	0	0	1

Using the same idea as in the proof of Theorem 2 we get the next result.

Proposition 4 *For all* $(\hat{i}, \hat{k}) \in VA$ *and* $\hat{i} \neq \hat{k}$ *the valid inequality*

$$x_{\hat{i}\hat{k}} \geq 0 \tag{19}$$

represents a facet of $\mathcal{P}_{HCoP\text{-}W2}$.

PROOF. Use the same set of points as in the proof of Theorem 2, but delete the type (a) point with $(j, l) = (\hat{i}, \hat{k})$. Then all remaining $q - n - 1$ type (a) points satisfy $x_{\hat{i}\hat{k}} = 0$ and build together with the type (b) point $n - q$ affine independent points. $\square$

Proposition 5 *The valid inequality*

$$x_{kk} \leq 1 \quad \forall\, (k, k) \in VA \tag{20}$$

represents a facet of $\mathcal{P}_{HCoP\text{-}W2}$ *if and only if* $|\{i : (i, k) \in VA,\ i \neq k\}| = 1$.

PROOF. $|\{i : (i, k) \in VA,\ i \neq k\}| = 1$ implies the equivalency of $x_{kk} = 1$ and $x_{ik} = 0$. Hence, (19) and (20) represent the same facet. If, on the other hand, $|\{i : (i, k) \in VA,\ i \neq k\}| \geq 2$, all points lying on the face $x_{kk} = 1$ also satisfy $x_{ik} = 0 \quad \forall (i, k) \in VA,\ i \neq k$. $\square$

Proposition 6 *The valid inequality*

$$x_{ik} \leq 1, \quad \forall\, (i, k) \in VA\,, i \neq k \tag{21}$$

does not represent a facet of $\mathcal{P}_{HCoP\text{-}W2}$.

PROOF. All points lying on the face $x_{ik} = 1$ also satisfy $x_{kk} = 1$ and $x_{im} = 0$ for $(i, m) \in VA$ and $m \neq k$. $\square$

For the next group of inequalities we consider the set IA of incompatible allocations, i.e., $IA = \Big\{(i, k, j, m) \mid (i, k), (j, m) \in VA, i < j$ and $c_{ik} + \alpha\, c_{km} + c_{jm} > \beta\Big\}$. Other incompatibility constraints are (9) and (10) which exclude allocation to non-hubs and multiple allocation, respectively. In the remainder of this section we have to handle both types of incompatibility. We therefore consider the set $IA^{(s,t)}$ of all elements of VA which attain a value of zero whenever $x_{st} = 1$:

$IA^{(s,t)} := \{\ (j, m)\ :\ (j, m, s, t) \in IA$ for $j < s$ or $(s, t, j, m) \in IA$ for $j > s\} \cup \{(t, m) : (t, m) \in VA, m \neq t\} \cup \{(s, m) : (s, m) \in VA, m \neq s, t\ \} \cup \{(m, s) : (m, s) \in VA, m \neq s\ \}$.

Example 7 *Consider the following sets:*

VA =\{ *(1,1), (1,3), (1,4), (2,2), (2,3), (2,4), (3,3), (3,4), (3,5), (4,1), (4,3), (4,4), (4,5), (5,3), (5,4), (5,5)* \}

IA = \{*(1,3,2,4), (1,3,5,4), (2,3,4,1), (2,3,5,4)* \}

For $(s, t) = (1, 3)$ *we get*

$IA^{(1,3)} = \{(2, 4), (5, 4)\} \cup \{(3, 4), (3, 5)\} \cup \{(1, 4)\} \cup \{(4, 1)\}$

The sets $IA^{(s,t)}$ can be used to characterize another class of facets.

Proposition 8 *Given* $k \in \{1, \dots, n\}$ *such that* $\bigcap_{\substack{i:(k,i)\in VA \\ i \neq k}} IA^{(k,i)} = \emptyset$. *Then*

$$x_{kk} \geq 0 \tag{22}$$

represents a facet of $\mathcal{P}_{HCoP\text{-}W2}$.

PROOF. We use the indirect method. Let $\mathcal{F} = \{P \in \mathcal{P}_{\text{HCoP-W2}} : x_{kk} = 0\}$ and assume that there exists an equation

$$\sum_{l} \sum_{s:(l,s)\in VA} \lambda_{ls}\, x_{ls} = \lambda_0 \tag{23}$$

which is satisfied by all points in $\mathcal{F}$. We show that (23) is a linear combination of $x_{kk} = 0$ and $\sum_{k:(i,k)\in VA} x_{ik} = 1$ by establishing

1. $\lambda_{ls} = \lambda_{ll} =: \lambda_l \quad \forall l \neq k$
2. $\lambda_{ks} = \lambda_{kt} =: \lambda_k \quad \forall s, t \neq k$

For step 1 consider the following two points in $\mathcal{F}$:

By assumption there exist at least one $i \neq k$ such that $(l, s) \notin IA^{(k,i)}$.

$$\begin{aligned} \textit{point 1:}\ & x_{dd} = 1 \quad \forall\, d \in \{1, ..., n\} \setminus \{k, l\} \\ & x_{ki} = 1 \quad (k, i) \in VA \\ & x_{ls} = 1 \end{aligned}$$

$$\begin{aligned} \textit{point 2:}\ & x_{dd} = 1 \quad \forall\, d \in \{1, ..., n\} \setminus \{k\} \\ & x_{ki} = 1\ (k, i) \in VA \\ & x_{ll} = 1 \end{aligned}$$

Inserting both points in (23) and comparing the expressions we obtain $\lambda_{ls} = \lambda_{ll}$ $\forall l \neq k$.

Step 2 can be shown by using the points:

$$\begin{aligned} \textit{point 1:}\ & x_{dd} = 1 \quad \forall\, d \in \{1, ..., n\} \setminus \{k\} \\ & x_{ks} = 1 \quad (k, s) \in VA,\ s \neq k \end{aligned}$$

$$\begin{aligned} \textit{point 2:}\ & x_{dd} = 1 \quad \forall\, d \in \{1, ..., n\} \setminus \{k\} \\ & x_{kt} = 1 \quad (k, t) \in VA,\ t \neq k \end{aligned}$$

As above, inserting points 1 and 2 yields $\lambda_{ks} = \lambda_{kt}$ $\forall s, t \neq k$

It is easy to see that all constructed points lie in $\mathcal{F}$. Now, we can conclude

$$\begin{aligned} \lambda_0 &= \sum_l \sum_{s:(l,s)\in VA} \lambda_{ls} x_{ls} \\ &= \sum_{l \neq k} \sum_{s:(l,s)\in VA} \lambda_{ls} x_{ls} + \sum_{\substack{s:(k,s)\in VA \\ s \neq k}} \lambda_{ks} x_{ks} + \lambda_{kk} x_{kk} \end{aligned}$$

i.e. $\lambda_0 - \sum_l \lambda_l = 0$. Thus (23) is, indeed, a linear combination of the given equations. $\square$

Note that in Proposition 8 $\bigcap_{i:(k,i)\in VA, i\neq k} IA^{(k,i)} = \emptyset$ is sufficient for $x_{kk} = 0$ to be facet defining. Assume that $\bigcap_{i:(k,i)\in VA, i\neq k} IA^{(k,i)} = \{(r, q)\}$ then all points lying on the face (22) also satisfy $x_{rq} = 0$. This argumentation can be adapted to prove the remaining propositions. (The reader is refered to the online version of Meyer's diploma thesis [11] for more details.)

Proposition 9 *The valid inequality*

$$x_{ik} \leq x_{kk} \quad \forall\, (i,k) \in VA,\; i \neq k \tag{24}$$

defines a facet of $\mathcal{P}_{HCoP\text{-}W2}$ *if*

$$IA^{(i,k)} \cap \Big(\bigcap_{\substack{i:(k,i)\in VA \\ i\neq k}} IA^{(k,i)} \Big) = \emptyset.$$

Proposition 10 *The valid inequality*

$$x_{ik} + x_{jm} \leq 1 \quad (i,k,j,m) \in IA \tag{25}$$

represent a facet of $\mathcal{P}_{HCoP\text{-}W2}$ *if*

$$IA^{(i,k)} \cap IA^{(j,m)} = \emptyset.$$

The last result establishes criteria for the aggregation constraints to be facet-defining.

Proposition 11 *Let* $M^{(i,k,j,\cdot)} := \{m : (i,k,j,m) \in IA\}$
and let $M^{(i,\cdot,j,m)} := \{k : (i,k,j,m) \in IA\}$*. The valid inequalities*

$$x_{ik} + \sum_{m \in M^{(i,k,j,\cdot)}} x_{jm} \leq 1 \tag{26}$$

and

$$\sum_{k \in M^{(i,\cdot,j,m)}} x_{ik} + x_{jm} \leq 1 \tag{27}$$

represent facets of $\mathcal{P}_{HCoP\text{-}W2}$ *if*

$$IA^{(i,k)} \cap \Big(\bigcap_{m \in M^{(i,k,j,\cdot)}} IA^{(j,m)} \Big) = \emptyset$$

and

$$IA^{(j,m)} \cap \Big(\bigcap_{k \in M^{(i,\cdot,j,m)}} IA^{(i,k)} \Big) = \emptyset, \quad \textit{respectively.}$$

Notice that (26) and (27) are either the same as (25) or dominate it. This confirms that HCoP-W2 is, in general, a much stronger formulation than HCoP-W1, except when there are no constraints to be aggregated. We can, therefore, expect that HCoP-W2 outperforms HCoP-W1.

4 A new algorithm for solving USApHCP and its numerical performance

4.1 Binary search algorithm

The new algorithm, called BS(HCoP), for solving USApHCP consists in a combination of binary search and the iterative solution of hub center problems. Let $c^*(\beta)$ be the optimal solution of HCoP with respect to some given bound β. Note that $c^*(\beta)$ is nonincreasing in β, i.e. $\beta' \geq \beta$ implies $c^*(\beta') \leq c^*(\beta)$. BS(HCoP) performs binary search on β in which the HCoP is solved until the minimax cost is reached for given number of hubs p.

By Proposition 1, the binary search can be started with $L_{start} := \alpha\ max_{i,j}\ c_{ij}$ as lower bound (possibly requiring that all nodes are hubs, i.e., $p = n$). As upper bound $U_{start} := 2\ max_{i,j}\ c_{ij}$ can be used, since $\mathcal{H} = \{k\}$ will yield a hub center objective which is at most that large.

Algorithm BS(HCoP))
Input: Complete graph $G = (V, E)$ with $|V| = n$, costs c_{ij}, discount factor α, number p of required hubs, and stopping criterion ε.
Output: Hub set $\mathcal{H}$ with $|\mathcal{H}| = p$ and allocation of spokes to hubs such that the length of the maximal path in the hub and spoke network is minimized.
Steps:
(1) Let $z := max_{i,j}\ c_{ij}$, $U := 2z$, $L := \alpha\ z$, $\beta := (L + U)/2$.
(2) Solve HCoP with respect to β to obtain $c^*(\beta)$.
(3) If $c^*(\beta) \leq p$, set $U := \beta$, else, set $L := \beta$.
(4) If $U - L < \varepsilon$, STOP. Else, set $\beta := (L + U)/2$ and go to (2).

Obviously, smaller tolerances ε lead to better solution qualities. For ε which are small enough the set of hubs will no longer change in the binary search, such that optimal solutions for USApHCP are obtained. The complexity of BS(HCoP) depends on the problem HCoP, which is known to be NP-hard. Clearly, the algorithm requires $\lceil log_2(\frac{U_{start} - L_{start}}{\varepsilon})\rceil$ iterations to return a solution for USApHCP.

Note that the optimal objective values of two USApHCPs with respect to different values p_1 and p_2 may coincide even if $p_1 > p_2$. Since Algorithm BS(HCoP) minimizes in each iteration for any given β the number of hubs, the USApHCP with respect to p_1 will always output p_2 as number of hubs. This is highly desirable, since establishing hubs is usually related to some fixed cost. If the range of p is needed, for which an objective value of USApHCP computed in Algorithm BS(HCoP) is optimal, this can be computed by some obvious, minor modifications of the algorithm.

4.2 Computational Results

In the following, we present the results of our computational test, where we compare the performances of Algorithm BS(HCoP) with the radius formulation of USApHCP. All numerical tests were carried out on a dual Xeon machine with 3.2 GHz and 3 GB RAM. The presolving algorithms (e.g. determining VA, IA) were coded in C++ and compiled with g++ (GCC) 3.3.3. We used the built-in branch and bound routine of Ilog Cplex 9.0 to solve the integer programs.

The programs were tested with the AP (Australian Post) and CAB (Civil Aeronautics Board) data sets, which are considered to be benchmarks by most researchers in the hub location area. For the CAB data we created different instances by choosing subsets with $n \in \{15, 20, 25\}$, $p \in \{2, 3, 4\}$ and $\alpha \in \{0.25, 0.5, 0.75\}$. For the AP data we created instances with $n \in \{30, 40, 50\}$, $p \in \{2, 3, 4, 5\}$ and $\alpha = 0.75$.

We tested two versions of BS(HCoP) - BS(HCoP-W2) and BS(HCoP-r), in which the HCoP-W2 and HCoP-r formulation is used to solve the hub covering problem, respectively.

Table 1 shows representative results for BS(HCoP-W2), BS(HCoP-r) and the solution of USApHCP by a mixed integer program for small, medium and large CAB problems. A tolerance value of $\varepsilon = 10^{-3}$ was used, which was shown to be small enough to warrant optimality of the BS(HCoP-W2) and BS(HCoP-r) solution in all instances.
Column 1 of the table specifies the test problem $n.p.\alpha$ of CAB, where n is the number of nodes in the graph, p is the required number of hubs, and α is the discount factor. In column 2 the optimal objective value is given. The remaining three columns present the computation times (in seconds) needed to solve the problem to optimality when using BS(HCoP-W2), BS(HCoP-r) and (the mixed integer program for) USApHCP, respectively. For BS(HCoP-W2), t_{cplex} is the Cplex computation time whereas t_{all} also includes the time needed for the preprocessing, i.e. to determine VA and IA. Table 2 shows results for large AP problems with a tolerance of 10^{-4}. It is organized as Table 1 except that the discount factor α is always 0.75 and thus deleted in column 1.

The tests indicate that BS(HCoP-W2) outperforms BS(HCoP-r) in 35 out of 37 test instances. In most of the 35 instances, BS(HCoP-W2) is about 4 times (t_{cplex}) resp. 3 times (t_{all}) faster than BS(HCoP-r). Hence, we restrict ourselves in the following to compare the performance of USApHCP only with BS(HCoP-W2).

Both, the new approach and the radius formulation of the hub center problem find optimal solutions very quickly. However, for almost all instances, BS(HCoP-W2) needs less CPU time than USApHCP. Although in some cases of $\alpha = 0.5$ BS(HCoP-W2) needs more computation time than the radius formulation. On average, the CPU time (t_{cplex}) used by USApHCP is more than twice the CPU time of BS(HCoP-W2).

The results from Table 2 indicate that the binary search algorithm performs extremely well for problems with 30 and 40 nodes. Comparing problems with 50 nodes, we see that BS(HCoP-W2) needs less CPU time for AP50.4 and AP50.5. However, the USApHCP performs better for AP50.2 and AP50.3. The extreme long CPU time for the 50.3 AP example are at this time not fully understood.

We also compared the computational results for BS(HCoP-W2) with the *Incremental Heuristic* for solving the USApHCP given in Ernst et al. [5]. Although the new approach needs more CPU time than the heuristic, the solution quality of BS(HCoP-W2) is much better, even if $\varepsilon = 1$.

5 Conclusion and Outlook

In this paper, a binary search algorithm BS(HCoP) to solve the USApHCP was proposed which is based on the inverse relationship between p-hub-center and hub-covering problems. For the HCoP we analyzed the feasibility polyhedron and identified several classes of facet defining valid inequalities. A computational study using the CAB and AP benchmark data sets tested the computational performance of BS(HCoP). The two most efficient hub covering formulations and the radius formulation of USApHCP were compared with each other. All algorithms found the optimal solutions. With respect to the computing time, our tests demonstrated that BS(HCoP) performs better in most examples.

Our current work involves further improvements of BS(HCoP-W2) by iteratively updating the sets VA and IA instead of determining them in each iteration from scratch. Of course, any formulation of the HCoP that outperforms that of Wagner leads to a better performance of BS(HCoP). Current research, therefore, focusses on the identification of additional facets of the integer polyhedron and on resulting improved branch-and-cut techniqes.

Prob	Obj	BS(HCoP-W2) t_{cplex}/t_{all}	BS(HCoP-r)	USApHCP
10.2.75	1759.13	0.04 / 0.1	0.1	0.07
10.2.5	1728.49	0.06 / 0.13	0.13	0.17
10.2.25	1476.07	0.06 / 0.12	0.14	0.36
10.3.75	1538.46	0.05 / 0.1	0.19	0.05
10.3.5	1286.03	0.05 / 0.1	0.15	0.1
10.3.25	1119.53	0.05 / 0.09	0.09	0.25
10.4.75	1377.38	0.04 / 0.09	0.08	0.08
10.4.5	1047.62	0.05 / 0.09	0.1	0.09
10.4.25	858.216	0.04 / 0.1	0.12	0.11
15.2.75	2343.4	0.08 / 0.16	0.23	0.26
15.2.5	2160.75	0.23 / 0.28	0.9	0.27
15.2.25	2059.04	0.1 / 0.19	0.84	0.62
15.3.75	2086.13	0.07 / 0.12	0.21	0.27
15.3.5	1760.15	0.09 / 0.16	0.49	0.27
15.3.25	1760.15	0.08 / 0.16	1.02	0.47
15.4.75	1979.01	0.06 / 0.11	0.14	0.17
15.4.5	1530.41	0.07 / 0.12	0.22	0.28
15.4.25	1361.42	0.2 / 0.26	0.29	0.25
20.2.75	2444.89	0.32 / 0.46	1.5	0.72
20.2.5	2224.11	0.82 / 0.97	3.04	1.5
20.2.25	1933.42	0.5 / 0.7	1.44	1.41
20.3.75	2187.63	0.33 / 0.41	2.15	1
20.3.5	1871.24	1.91 / 2.01	3.41	1.16
20.3.25	1635.37	0.23 / 0.3	0.84	1.53
20.4.75	2086.13	0.1 / 0.21	0.37	0.86
20.4.5	1650.81	0.73 / 0.82	2.61	0.82
20.4.25	1361.42	1.43 / 1.55	2.31	1.18
25.2.75	2675.88	0.94 / 1.14	2.35	2.32
25.2.5	2480.64	14.47 / 14.74	24.15	2.9
25.2.25	2194.52	1.69 / 1.92	6.4	3.05
25.3.75	2500.24	0.82 / 0.67	10.6	2.21
25.3.5	2218.32	15.64 / 15.85	11	3.2
25.3.25	2001.65	14.16 / 14.34	11.92	4.24
25.4.75	2372.12	1.02 / 1.21	14.63	3.21
25.4.5	2045.65	3.27 / 3.48	14.93	3.22
25.4.25	1703.61	2 / 2.1	7.41	3.47

Table 1: Numerical results for CAB problems

Prob	Obj	BS(HCoP-W2) t_{cplex} / t_{all}	BS(HCoP-r)	USApHCP
30.2	55.8204	0.53 / 0.89	0.57	4.81
30.3	49.3919	0.24 / 0.51	0.5	4.8
30.4	48.5632	0.66 / 0.93	0.56	4.98
40.2	61.6825	9.68 / 10.97	16.8	13.66
40.3	58.1928	5.49 / 6.46	18.54	23.53
40.4	52.2653	2.92 / 3.49	7.25	13.63
40.5	49.7412	4.28 / 4.79	15.35	14.61
50.2	65.5234	239.02 / 242	118	39.46
50.3	60.1321	290.27 / 292.62	622.05	116.55
50.4	52.9058	7.96 / 9.32	24.74	46.27
50.5	50.7079	6.18 / 7.37	13.87	35.89

Table 2: Numerical results for AP problems

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