Semi-Simultaneous Flows and Binary Constrained (Integer) Linear Programs

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Abstract

Linear and integer programs are considered whose coefficient matrices can be partitioned into K consecutive ones matrices. Mimicking the special case of $K = 1$ which is well-known to be equivalent to a network flow problem we show that these programs can be transformed to a generalized network flow problem which we call *semi-simultaneous (se-sim) network flow problem.*

Feasibility conditions for se-sim flows are established and methods for finding initial feasible se-sim flows are derived. Optimal se-sim flows are characterized by a generalization of the negative cycle theorem for the minimum cost flow problem. The issue of improving a given flow is addressed both from a theoretical and practical point of view. The paper concludes with a summary and some suggestions for possible future work in this area.

Keywords: network flows, consecutive ones matrix, linear programming, integer programming

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1 Introduction

It is well known that efficient computer codes solving a *linear program (LP)*

$$\text{minimize } c^{\mathrm{T}}x \text{ subject to } x \in \{x \in \mathbb{R}^n : Ax = b, x \geq 0\} \tag{1}$$

defined by an integer matrix $A \in \mathbb{Z}^{m \times n}$ and two integer vectors $b \in \mathbb{Z}^m$ and $c \in \mathbb{Z}^n$ rely on the detection of hidden structures in the matrix A. If such a hidden structure is detected, combinatorial algorithms can be used to speed up any general purpose LP algorithm.

If A is, for instance, the (node-arc) incidence matrix of a digraph, then the resulting special instance of LP is a network flow problem, which can be solved with strongly polynomial algorithms (see, e.g., Ahuja et al. (1993), Bazaraa et al. (2005) or Hamacher and Klamroth (2006)). The validity of these algorithms also establishes the *integrality property*, i.e., the fact that such LPs always have an integer optimal solution. Hence, the resulting *integer program (IP)*

$$\text{minimize } c^{\mathrm{T}}x \text{ subject to } x \in \{x \in \mathbb{R}^n : Ax = b, x \geq 0\} \cap \mathbb{Z}^n \tag{2}$$

is also solvable in polynomial time.

If parts of A are incidence matrices, this information is used to improve the computation and updating of basis solutions, and, thus, to speed up LP codes. Obviously, if A is itself not an incidence matrix, the integrality property does in general no longer hold.

An extension of node-arc matrices - and thus an example of a hidden structure - is a coefficient matrix which is binary and has the *consecutive ones property (C1)*, i.e., in each row (or column) the ones occur in a single consecutive block. It is well-known (see, e.g., Nemhauser and Wolsey (1988) or Schrijver (2003)) that the corresponding linear program can be transformed into an equivalent network flow problem by m row subtractions. Thus, linear programs with C1 coefficient matrices are solvable in strongly polynomial time and have the integrality property.

In this paper, we consider the generalization of the preceding situation where the coefficient matrix A consists of $K > 1$ C1 matrices instead of just a single one. The resulting linear and integer programs are highly relevant, since they are of importance in various applications including cancer radiation planning (Baatar et al. (2005), Boland et al. (2004)) or the design of stops in public transportation (Schöbel et al. (2002), Poetranto et al. (2006)). To the best of our knowledge, this problem has not been considered in the literature. The closest paper we are aware of is Ruf and Schöbel (2004) where integer programs are considered with few blocks of ones per row.

In the next section, we introduce (integer) linear programs with a coefficient matrix A consisting of $K > 1$ C1 matrices (KC1, for short) and show - first via example - their relation to so-called *semi-simultaneous network flow* problems. We show that KC1, which has the integrality property for $K = 1$, loses this property already for $K = 2$. The relevance of KC1 for integer programs is emphasized by a result stating that every integer program with binary coefficient matrix is equivalent to KC1. In Section 3, we consider semi-simultaneous flows in more detail. We derive characterizations for feasibility, which are based on feasibility arguments for classical network problems, and give an analog to the well-known negative cycle result as optimality criterion. The resulting improvement procedure is investigated in the last subsection of Section 3 and used in algorithms proposed in Section 4. In the concluding section our results are summarized and several suggestions for further research related to simultaneous flow and graph theory problems are given.

2 Integer Linear Programs, Consecutive Ones Matrices, and Semi-Simultaneous Flows

In this section, we discuss the interrelation between a given integer linear program (IP) $\min\{c^{\mathrm{T}}x : Ax = b, x \geq 0 \text{ integer}\}$ with binary, $m \times n$ coefficient matrix A and consecutive ones matrices. Using this interrelation, we show that the dual of the linear programming relaxation of IP can be formulated as a semi-simultaneous network flow problem.

Definition 1. A binary matrix is a *consecutive ones (C1) matrix* if the ones occur consecutively in a single block in each row.

If we want to emphasize the fact that the ones are blocks in each row, we also call these matrices *row C1 matrices* and use the notion of *column C1 matrices*, if the ones of the matrix occur consecutively in a single block for each column. It is well-known (see, for instance, Nemhauser and Wolsey (1988) or Schrijver (2003)) that IPs which have column C1 matrices as coefficient matrices are equivalent to network flow problems and, thus, polynomially solvable. The purpose of our paper is to discuss the extension of this relation to general binary matrices.

Definition 2. Let $K \in \mathbb{N}$ be a given positive integer, let $A^k \in \mathbb{B}^{m \times n_k}$, $k = 1, \ldots, K$ be C1 matrices, let $b \in \mathbb{Z}^m_+$ be a nonnegative integer vector and let $c^k \in \mathbb{Z}^{n_k}$, $k = 1, \ldots, K$ be integer vectors. Then the *K consecutive ones integer program (KC1-IP)* is defined by

$$\begin{aligned} \text{minimize } & \sum_{k=1}^{K} {c^k}^{\mathrm{T}} x^k \\ \text{subject to } & \sum_{k=1}^{K} A^k x^k = b, \\ & x^k \geq 0 \text{ and integer for all } k = 1, \ldots, K. \end{aligned} \tag{3}$$

Proposition 1. *Any integer program with binary coefficient matrix and integer data is equivalent to KC1-IP for some $K \leq \lceil \frac{n}{2} \rceil$. The smallest K with this property can be found in polynomial time.*

Proof. Given KC1-IP, we use $A = (A^1 \ \ldots \ A^K) \in \mathbb{B}^{m \times n}$, $c = (c^1 \ \ldots \ c^K) \in \mathbb{Z}^n$ and $x = (x^1 \ \ldots \ x^K)$ with $n = \sum_{k=1}^{K} n_k$ to obtain the equivalent IP $\min\{c^{\mathrm{T}}x : Ax = b, x \geq 0 \text{ integer}\}$ with binary coefficient matrix A and integer data.

Conversely, given any IP with $A \in \mathbb{B}^{m \times n}$, we can always find some partition $A = (A^1 \ \ldots \ A^K)$ so that each A^k, $k = 1 \ldots, K$ is C1. Trivially, we can choose $K = n$ so that the matrix A is partitioned into its n column vectors, $A = (a^1 \ \ldots \ a^n)$, with $a^k \in \mathbb{B}^m$ being C1 for all $k = 1, \ldots, n$. In fact, K can be chosen so that $K \leq \lceil \frac{n}{2} \rceil$, as any matrix consisting of only two binary column vectors is necessarily C1 as well. To find the smallest K such that a binary IP is equivalent to KC1-IP, the polynomial decomposition algorithm of Baatar et al. (2005) can be applied. □

If $K = 1$, then the matrix A is itself C1 and hence totally unimodular. Therefore, we know that 1C1-IP is solvable by linear programming relaxation. The following example, however, shows that this property is already lost for $K = 2$. (Examples were independently found by Schöbel (2004) and Engau (2005))

Example 1. Choose $K = n_1 = n_2 = 2$, $m = 3$, $A^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{pmatrix}$, $A^2 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{pmatrix}$, $b = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, and $c^1 = c^2 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ with $x^{\mathrm{T}} = (x^1_1 \ x^1_2 \ x^2_1 \ x^2_2)$. Then, the 2C1-IP problem

$$\min\left\{(1 \ 3 \ 1 \ 3)x : \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, x \geq 0 \text{ integer}\right\} \tag{4}$$

has the optimal solution $x_1^1 = x_2^1 = x_1^2 = 0$, $x_2^2 = 1$ with objective 3. Its linear programming relaxation 2C1-LP, however, yields the improved, fractional solution $x_1^1 = x_2^1 = x_1^2 = 0.5$, $x_2^2 = 0$ with objective 2.5. Clearly, this also implies that the combined matrix $A = (A^1\ A^2)$ cannot be TU, which is verified easily.

We concentrate in the following on the linear, continuous problem KC1-LP and its dual KC1-LPD given by

$$\max \left\{ b^{\mathrm{T}}\pi : {A^k}^{\mathrm{T}}\pi \leq c^k \text{ for all } k = 1, \ldots, K, \pi \in \mathbb{R}^m \right\}. \tag{5}$$

Since the matrices A^k in (3) are row C1, the matrices ${A^k}^{\mathrm{T}}$ in (5) are column C1. Thus, after introducing nonnegative slack variables $\alpha^k \in \mathbb{R}_+^{n_k}$, we obtain the equality constraints

$$ {A^k}^{\mathrm{T}}\pi + I_{n_k}\alpha^k = c^k $$

which now can be transformed into K systems of flow conservation constraints in K underlying networks G^k, $k = 1, \ldots, K$. A solution of problem (5) thus corresponds to vectors $\pi \in \mathbb{R}^m$, $\alpha^1 \in \mathbb{R}_+^{n_1}, \ldots, \alpha^K \in \mathbb{R}_+^{n_K}$ so that for each $k = 1, \ldots, K$, the pair (α^k, π) establishes a feasible flow in network G^k with π maximizing the objective $b^{\mathrm{T}}\pi$. Since each such pair consists of an individual flow α^k and the common flow π that has to be chosen simultaneously for all K networks, we call the collection of all these flows a *semi-simultaneous network flow* and the associated problem the *semi-simultaneous network flow problem.*

Example 2. Consider the linear programming dual 2C1-LPD of Example 1 with $\pi^{\mathrm{T}} = (\pi_1\ \pi_2\ \pi_3)$, $\alpha^1 = (\alpha_1^1\ \alpha_2^1)$ and $\alpha^2 = (\alpha_1^2\ \alpha_2^2)$,

$$\max \left\{ \pi_1 + \pi_2 + \pi_3 : \left(\begin{smallmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \end{smallmatrix}\right)\pi + \alpha^1 = \left(\begin{smallmatrix} 1 \\ 3 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \end{smallmatrix}\right)\pi + \alpha^2 = \left(\begin{smallmatrix} 1 \\ 3 \end{smallmatrix}\right) \right\}. \tag{6}$$

To transform this problem into a semi-simultaneous network flow problem, first append an additional zero row to the constraints, and then subtract its preceding row from all but the first, yielding the two systems of flow conservation constraints

$$\left(\begin{smallmatrix} 0 & 1 & 1 \\ 1 & -1 & 0 \\ -1 & 0 & -1 \end{smallmatrix}\right)\pi + \left(\begin{smallmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{smallmatrix}\right)\alpha^1 = \left(\begin{smallmatrix} 1 \\ 2 \\ -3 \end{smallmatrix}\right) \text{ and } \left(\begin{smallmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & -1 \end{smallmatrix}\right)\pi + \left(\begin{smallmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{smallmatrix}\right)\alpha^2 = \left(\begin{smallmatrix} 1 \\ 2 \\ -3 \end{smallmatrix}\right). \tag{7}$$

Hence, 2C1-LPD is equivalent to finding a flow $(\alpha^1, \alpha^2, \pi)$ that maximizes $b^{\mathrm{T}}\pi$ subject to flow conservation at each node, flow capacities $\alpha^1, \alpha^2 \geq 0$ and identical partial flow π in both networks (see Figure 1). Notice that although the values of π_1, π_2 and π_3 are required to be identical in both parts of the semi-simultaneous network, the corresponding edges connect, in general, different nodes.

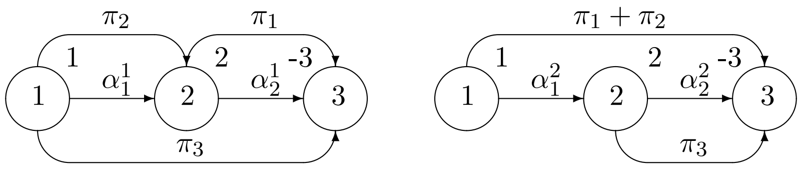

Figure 1: Semi-simultaneous flow corresponding to 2C1-LPD of Example 2 with common flow π and individual flows α^1 and α^2.

It is easy to see that this problem has the optimal fractional solution $\pi_1 = 1.5$, $\pi_2 = -0.5$ and $\pi_3 = 1.5$ with slack variables $\alpha_1^1 = 0$, $\alpha_2^1 = 1$, $\alpha_1^2 = 0$ and $\alpha_2^2 = 0.5$ and objective 2.5 - thus confirming the result of Example 1.

3 Semi-Simultaneous Network Flows

In this section, we formally define semi-simultaneous network flows as motivated in the previous section, derive feasibility and optimality conditions, and introduce improvement flows which will be useful in the algorithms of the subsequent section.

Definition 3. A *semi-simultaneous network (se-sim network)* is a collection $G = \left\{G^k = \left(V^k, D^k \cup E^k\right) : k = 1, \ldots, K\right\}$ of K individual networks G^k. The node sets are $V^k = \left\{v_1^k, v_2^k, \ldots, v_{n_k+1}^k\right\}$ and the arc sets are partitioned into the sets $D^k = \left\{e_1^k, e_2^k, \ldots, e_{n_k}^k\right\} = \left\{(v_1^k, v_2^k), (v_2^k, v_3^k), \ldots, (v_{n_k}^k, v_{n_k+1}^k)\right\}$ of *individual arcs* and $E^k = \{e_1, e_2, \ldots, e_m\}$ of *common arcs*.

In this paper, we only consider se-sim networks related to KC1-LP. Hence, in each individual network G^k the individual arcs $e^k \in D^k$ arise from the node-arc incidence matrix obtained from the slack identity matrix. They thus form the special structure of a Hamiltonian path

$$v_1^k \xrightarrow{e_1^k} v_2^k \xrightarrow{e_2^k} \cdots \xrightarrow{e_{n_k-1}^k} v_{n_k}^k \xrightarrow{e_{n_k}^k} v_{n_k+1}^k. \tag{8}$$

For any two nodes $v_i^k, v_j^k \in V^k$ with $i < j$, we define the node interval $\left[v_i^k, v_j^k\right] = \left\{v_i^k, v_{i+1}^k, \ldots, v_j^k\right\}$. For each individual network G^k, we let $g^k, h^k : D^k \cup E^k \to V^k$ be the tail and head functions, such that $g^k(e_i^k) = v_i^k$ and $h^k(e_i^k) = v_{i+1}^k$ for all individual arcs $e_i^k \in D^k$. For the common arcs $e \in E^k$, we denote $g^k(e_i) = g_i^k$ and $h^k(e_i) = h_i^k$ and then observe that by construction, g_i^k precedes h_i^k, or equivalently $g_i^k \neq h_i^k$ and $\left[g_i^k, h_i^k\right] \neq \emptyset$ in every network G^k. In particular, this implies that all networks G^k are free of loops, $\delta_+(v_i^k) \cap \delta_-(v_i^k) = \emptyset$ for all vertices $v_i^k \in V^k$, where $\delta_+(v_i^k) = \left(g^k\right)^{-1}\left(\{v_i^k\}\right) = \left\{e \in D^k \cup E^k : g^k(e) = v_i^k\right\}$ and $\delta_-(v_i^k) = \left(h^k\right)^{-1}\left(\{v_i^k\}\right) = \left\{e \in D^k \cup E^k : h^k(e) = v_i^k\right\}$ are the set of arcs leaving and arriving at node v_i^k, respectively.

Definition 4. Given a se-sim network $G = \left\{G^k : k = 1, \ldots, K\right\}$, a *semi-simultaneous flow (se-sim flow)* f in G is defined as a collection $f = \left\{f^k : k = 1, \ldots, K\right\}$ of K flows f^k for the individual networks G^k with identical flow value on all common arcs, i.e., $f^{k_1}(e_i) = f^{k_2}(e_i)$ for all $i = 1, \ldots, m$ and $k_1, k_2 = 1, \ldots, K$. For each individual network G^k, the flow f^k restricted to single and common arcs, $f^k|_{D^k}$ and $f^k|_{E^k}$, is called the *individual* and *common network flow* associated with f^k, respectively.

Given a flow f^k in some individual network G^k, we often use the notation $f^k(e_i) = \pi_i^k$ for all $i = 1, \ldots, m$ and $f^k(e_j^k) = \alpha_j^k$ for all $j \in 1, \ldots, n_k$, so that a collection $\left\{f^k = (\alpha^k, \pi^k) : k = 1, \ldots, K\right\}$ of K flows f^k for the individual networks G^k is semi-simultaneously feasible if and only if $\pi_i^{k_1} = \pi_i^{k_2}$ for all $i = 1, \ldots, m$ and $k_1, k_2 = 1, \ldots, K$. In this case, we also write $\pi^k = \pi$ and $f = (\alpha^1, \alpha^2, \ldots, \alpha^K, \pi)$, and in analogy to Section 2, we restrict the flow along individual arcs to be nonnegative, i.e., $\alpha^k \geq 0$, while the flow π on common arcs remains unrestricted.

The right-hand side vectors for the K systems of flow conservation constraints are defined by $c^k : V^k \to \mathbb{Z} : v_i^k \mapsto c^k(v_i^k) = c_i^k$ and satisfy $\sum_{i=1}^{n_k+1} c^k(v_i^k) = \sum_{i=1}^{n_k+1} c_i^k = 0$. Since we maximize we refer to the objective as *benefit*. Each individual arc $e_i^k \in D^k$ has a benefit $b(e_i^k) = b_i^k = 0$ and each common arc $e_i \in E^k$ a nonnegative benefit $b(e_i) = b_i \geq 0$. Then the *total benefit* of a se-sim network flow f is defined by $b(f) = \sum_{i=1}^{m} b(e_i) f(e_i) = \sum_{i=1}^{m} b_i \pi_i = b^{\mathrm{T}} \pi$.

Definition 5. Let $G = \{G^k : k = 1, \dots, K\}$ be a given se-sim network with benefits b and flow conservation right-hand-sides c^k for each individual network G^k. Then the *se-sim flow problem (SE-SIM-FLOP)* is defined as

$$\text{maximize } b(f) = \sum_{i=1}^{m} b(e_i) f(e_i) \tag{9}$$

$$\text{subject to}$$

$$\sum_{e \in \delta_+^k(v_i^k)} f^k(e) - \sum_{e \in \delta_-^k(v_i^k)} f^k(e) = c^k(v_i^k) \qquad i = 1, \dots, n_k + 1,\ k = 1, \dots, K, \tag{9a}$$

$$f^{k_1}(e_i) = f^{k_2}(e_i) \qquad i = 1, \dots, m, k_1,\ k_2 = 1, \dots, K, \tag{9b}$$

$$f^k(e_j^k) \geq 0 \qquad j = 1, \dots, n_k,\ k = 1, \dots, K. \tag{9c}$$

For fixed $k = 1, \dots, K$, the associated subproblem is called *individual network flow problem (IN-FLOP)*.

Note how each IN-FLOP can be characterized as a minimum cost flow problem by treating the negative benefit subject to minimization. Moreover, based on the chain structure of the underlying networks, observe that SE-SIM-FLOP can equivalently be written as

$$\text{maximize } b^{\mathrm{T}} \pi \tag{10}$$

$$\text{subject to}$$

$$\alpha_1^k + \sum_{\{i : g_i^k = v_1^k\}} \pi_i - \sum_{\{i : h_i^k = v_1^k\}} \pi_i = c_1^k, \qquad k = 1, \dots, K, \tag{10a}$$

$$\alpha_j^k + \sum_{\{i : g_i^k = v_j^k\}} \pi_i - \alpha_{j-1}^k - \sum_{\{i : h_i^k = v_j^k\}} \pi_i = c_j^k \qquad j = 2, \dots, n_k,\ k = 1, \dots, K, \tag{10b}$$

$$\sum_{\{i : g_i^k = v_{n_k+1}^k\}} \pi_i - \alpha_{n_k}^k - \sum_{\{i : h_i^k = v_{n_k+1}^k\}} \pi_i = c_{n_k+1}^k \qquad k = 1, \dots, K, \tag{10c}$$

$$\alpha_j^k \geq 0 \qquad j = 1, \dots n_k,\ k = 1, \dots, K. \tag{10d}$$

Proposition 2. *Any feasible se-sim network flow $f = \{f^k = (\alpha^k, \pi) : k = 1, \dots, K\}$ for SE-SIM-FLOP is uniquely determined by the common flow π.*

Proof. The proof follows for each individual network G^k by induction on the flow values α_i^k along the individual arcs $e_i^k = (v_i^k, v_{i+1}^k)$, $i = 1, \dots, n_k$. Let $k = 1, \dots, K$ be fixed and rewrite the flow conservation constraint (10a) at node v_1^k as

$$\alpha_1^k = c_1^k + \sum_{\{i : h_i^k = v_1^k\}} \pi_i - \sum_{\{i : g_i^k = v_1^k\}} \pi_i. \tag{11}$$

Hence, the flow α_1^k along individual arc e_1^k is uniquely determined by the flow π and c_1^k. Now assume that for any $j = 2, \dots, n_k$, the flow along individual arc e_{j-1}^k has been uniquely determined by the given common flow π. It then follows from the flow conservation constraint (10b) at node v_j^k that the individual flow α_j^k along individual arc e_j^k must satisfy

$$\alpha_j^k = c_j^k + \alpha_{j-1}^k + \sum_{\{i : h_i^k = v_j^k\}} \pi_i - \sum_{\{i : g_i^k = v_j^k\}} \pi_i, \tag{12}$$

so the flow α_j^k along individual arc e_j^k is also uniquely determined by the given common flow π. □

Note that we also could have argued that the individual flows α^k in networks G^k correspond to the slack variables in the LPR dual and thus are uniquely determined by the dual variable or common flow vector π.

3.1 Feasible se-sim flows

The goal of this section is to establish that the feasibility of IN-FLOP is necessary and sufficient for the feasibility of SE-SIM-FLOP. The first part of this statement is obvious.

Proposition 3. *Given a feasible se-sim flow $f = \left\{f^k : k = 1, \dots, K\right\}$ for SE-SIM-FLOP, then each of the flows f^k is feasible for its associated individual network flow problems IN-FLOP.*

To prove the converse we first show the following result.

Lemma 1. *Let SE-SIM-FLOP be given and let $k \in \{1, \dots, K\}$ be fixed. For the k^{th} IN-FLOP, let $f^k = (\alpha^k, \pi^k)$ be a feasible flow and let $\pi \le \pi^k$. Then the flow determined by π according to Proposition 2 is also feasible for IN-FLOP.*

Intuitively, this result follows from the special chain structure in the individual networks G^k: since the common flow along all common arcs is unrestricted, we can reduce the flow along some common arc e_i by the respective nonnegative flow difference $\pi^k - \pi$ while maintaining the flow balance constraints by increasing the flow along the individual arcs connecting its tail and head nodes g_i^k and h_i^k by the same amount. This idea is made more precise in the following proof.

Proof. If $\pi = \pi^k$, then nothing needs to be shown. Hence, assume that $\pi_i < \pi_i^k$ for some $i = 1, \dots, m$, and first assume that l is the only such index for which $\pi_l < \pi_l^k$, i.e., $\pi_i = \pi_i^k$ for all $i \ne l$. Define the new flow (β^k, ρ) by

$$\beta_j^k = \begin{cases} \alpha_j^k + \pi_l^k - \pi_l & \text{if } v_j^k \in [g_l^k, h_l^k), \\ \alpha_j^k & \text{if } v_j^k \notin [g_l^k, h_l^k), \end{cases} \tag{13}$$

and $\rho = \pi$. By feasibility of $f^k = (\alpha^k, \pi^k)$, equations (10b) and (10d) imply that

$$\alpha_j^k + \sum_{\{i: g_i^k = v_j^k\}} \pi_i^k - \alpha_{j-1}^k - \sum_{\{i: h_i^k = v_j^k\}} \pi_i^k = c_j^k \tag{14}$$

and $\alpha_j^k \ge 0$ for all $j = 1, \dots, n_k$. Since $\pi_l^k - \pi_l > 0$ by assumption, it then also follows that $\beta_j^k \ge 0$, and hence, all that remains to show feasibility of (β^k, ρ) is to verify the flow conservation constraints (10a-c). By setting $\beta_0^k = \beta_{n_k+1}^k = 0$ for $j = 1$ or $j = n_k + 1$, equations (10a+c) can be discussed as special cases of (10b),

$$\beta_j^k + \sum_{\{i: g_i^k = v_j^k\}} \rho_i - \beta_{j-1}^k - \sum_{\{i: h_i^k = v_j^k\}} \rho_i = c_j^k, \tag{15}$$

and now the proof distinguishes the following four cases.

1. Case $v_j^k \notin [g_l^k, h_l^k]$. Then $v_{j-1}^k \notin [g_l^k, h_l^k)$ and hence $\beta_j^k = \alpha_j^k$ and $\beta_{j-1}^k = \alpha_{j-1}^k$ by (13). Furthermore, $l \notin \{i : g_i^k = v_j^k\}$ and $l \notin \{i : h_i^k = v_j^k\}$ and thus $\rho_i = \pi_i = \pi_i^k$ for all $i \in \{i : g_i^k = v_j^k\} \cup \{i : h_i^k = v_j^k\}$. Therefore

$$\begin{aligned} & \beta_j^k + \sum_{\{i: g_i^k = v_j^k\}} \rho_i - \beta_{j-1}^k - \sum_{\{i: h_i^k = v_j^k\}} \rho_i \\ = \; & \alpha_j^k + \sum_{\{i: g_i^k = v_j^k\}} \pi_i^k - \alpha_{j-1}^k - \sum_{\{i: h_i^k = v_j^k\}} \pi_i^k = c_j^k. \end{aligned} \tag{15a}$$

2. Case $v_j^k = g_l^k$. Then $v_j^k \in [g_l^k, h_l^k)$ and $v_{j-1}^k \notin [g_l^k, h_l^k)$ and hence $\beta_j^k = \alpha_j^k + \pi_l^k - \pi_l$ and $\beta_{j-1}^k = \alpha_{j-1}^k$ by (13). Furthermore, $l \in \{i : g_i^k = v_j^k\}$ and $l \notin \{i : h_i^k = v_j^k\}$ and thus $\rho_i = \pi_i = \pi_i^k$ only for $i \in \{i \neq l : g_i^k = v_j^k\} \cup \{i : h_i^k = v_j^k\}$, while $\rho_l = \pi_l \neq \pi_l^k$. Therefore

$$\beta_j^k + \sum_{\{i:g_i^k=v_j^k\}} \rho_i - \beta_{j-1}^k - \sum_{\{i:h_i^k=v_j^k\}} \rho_i$$

$$= \left(\alpha_j^k + \pi_l^k - \pi_l\right) + \left(\sum_{\{i\neq l:g_i^k=v_j^k\}} \pi_i^k + \pi_l\right) - \alpha_{j-1}^k - \sum_{\{i:h_i^k=v_j^k\}} \pi_i^k \tag{15b}$$

$$= \alpha_j^k + \sum_{\{i:g_i^k=v_j^k\}} \pi_i^k - \alpha_{j-1}^k - \sum_{\{i:h_i^k=v_j^k\}} \pi_i^k = c_j^k. \tag{15c}$$

3. Case $v_j^k \in (g_l^k, h_l^k)$. Then $v_{j-1}^k \in [g_l^k, h_l^k)$ and hence $\beta_j^k = \alpha_j^k + \pi_l^k - \pi_l$ and $\beta_{j-1}^k = \alpha_{j-1}^k + \pi_l^k - \pi_l$ by (13). Furthermore, $l \notin \{i : g_i^k = v_j^k\}$ and $l \notin \{i : h_i^k = v_j^k\}$ and thus $\rho_i = \pi_i = \pi_i^k$ for all $i \in \{i : g_i^k = v_j^k\} \cup \{i : h_i^k = v_j^k\}$. Therefore

$$\beta_j^k + \sum_{\{i:g_i^k=v_j^k\}} \rho_i - \beta_{j-1}^k - \sum_{\{i:h_i^k=v_j^k\}} \rho_i$$

$$= \left(\alpha_j^k + \pi_l^k - \pi_l\right) + \sum_{\{i:g_i^k=v_j^k\}} \pi_i^k - \left(\alpha_{j-1}^k + \pi_l^k - \pi_l\right) - \sum_{\{i:h_i^k=v_j^k\}} \pi_i^k \tag{15d}$$

$$= \alpha_j^k + \sum_{\{i:g_i^k=v_j^k\}} \pi_i^k - \alpha_{j-1}^k - \sum_{\{i:h_i^k=v_j^k\}} \pi_i^k = c_j^k. \tag{15e}$$

4. Case $v_j^k = h_l^k$. Then $v_j^k \notin [g_l^k, h_l^k)$ and $v_{j-1}^k \in [g_l^k, h_l^k)$ and hence $\beta_j^k = \alpha_j^k$ and $\beta_{j-1}^k = \alpha_{j-1}^k + \pi_l^k - \pi_l$ by (13). Furthermore, $l \notin \{i : g_i^k = v_j^k\}$ and $l \in \{i : h_i^k = v_j^k\}$ and thus $\rho_i = \pi_i = \pi_i^k$ only for $i \in \{i : g_i^k = v_j^k\} \cup \{i \neq l : h_i^k = v_j^k\}$, while $\rho_l = \pi_l \neq \pi_l^k$. Therefore

$$\beta_j^k + \sum_{\{i:g_i^k=v_j^k\}} \rho_i - \beta_{j-1}^k - \sum_{\{i:h_i^k=v_j^k\}} \rho_i$$

$$= \alpha_j^k + \sum_{\{i:g_i^k=v_j^k\}} \pi_i^k - \left(\alpha_{j-1}^k + \pi_l^k - \pi_l\right) - \left(\sum_{\{i\neq l:h_i^k=v_j^k\}} \pi_i^k + \pi_l\right) \tag{15f}$$

$$= \alpha_j^k + \sum_{\{i:g_i^k=v_j^k\}} \pi_i^k - \alpha_{j-1}^k - \sum_{\{i:h_i^k=v_j^k\}} \pi_i^k = c_j^k. \tag{15g}$$

Note that the last equality in each of the four cases follows from (14) by feasibility of $f^k = (\alpha^k, \pi^k)$ for IN-FLOP.

Therefore, flow (β^k, ρ) satisfies the flow conservation constraints and is, thus, feasible for IN-FLOP. To show the lemma, we iteratively repeat this argument for all $i = 1, \dots, m$ with $\pi_i < \pi_i^k$. □

Using this result, we are now able to show that individual feasibility for all IN-FLOPs is sufficient for the feasibility of SE-SIM-FLOP.

Proposition 4. *Let SE-SIM-FLOP be given and assume that for each $k = 1, \dots, K$, there exists a feasible individual flow $f^k = \left(\alpha^k, \pi^k\right)$ for the associated IN-FLOP. Then there exists a feasible se-sim network flow for SE-SIM-FLOP.*

Proof. Let $\left\{f^k = \left(\alpha^k, \pi^k\right) : k = 1, \dots, K\right\}$ be a collection of feasible flows for the IN-FLOPs, and define a common flow $\pi = \pi^1 \wedge \pi^2 \wedge \ldots \wedge \pi^K$ as the componentwise minimum among all flows π^k, i.e., $\pi_i = \min\left\{\pi_i^k : k = 1, \dots, K\right\}$. Clearly $\pi \leq \pi^k$ for all $k = 1, \dots, K$, and hence, by Lemma 1, the common flow π determines feasible flows (β^k, π) in all individual networks G^k. Hence, the collection $f = \left\{\left(\beta^k, \pi\right) : k = 1, \dots, K\right\}$ of all such flows establishes a feasible se-sim network flow for SE-SIM-FLOP. □

By combining Propositions 3 and 4, we conclude that se-sim feasibility is equivalent to individual feasibility for each individual network.

Theorem 1. *SE-SIM-FLOP is feasible if and only if all IN-FLOPs are feasible.*

3.2 Optimal se-sim flows

Since each feasible se-sim flow defines a feasible flow for each IN-FLOP, we immediately obtain the following upper bound on the optimal objective value for SE-SIM-FLOP.

Proposition 5. *Let f be any feasible flow for SE-SIM-FLOP, and let $\left\{\hat{f}^k = (\hat{\alpha}^k, \hat{\pi}^k) : k = 1, \dots, K\right\}$ be a collection of optimal flows for the associated IN-FLOPs. Then $b(f) \leq \min\left\{b(\hat{f}^k) = b^{\mathrm{T}}\hat{\pi}^k : k = 1, \dots, K\right\}$.*

It also follows that, if one IN-FLOP is bounded, then SE-SIM-FLOP must be bounded. Furthermore, since SE-SIM-FLOP is a linear program, we can state the following sufficient condition for the existence of an optimal flow.

Proposition 6. *SE-SIM-FLOP has an optimal solution if it is feasible and if at least one associated IN-FLOP is bounded.*

The following example demonstrates, however, that the converse of Proposition 6 is, in general, not true.

Example 3. Consider the two networks in Figure 2 with zero right hand sides for the flow conservation constraints at all nodes and uniform benefits $b_1 = b_2 = b_3 = 1$ for all common arcs. Then every flow of the form $f^1 = (\alpha^1, \pi^1) = \left((0\ 0\ 0)^{\mathrm{T}}, (\rho\ \rho\ 0\ -\rho)^{\mathrm{T}}\right)$ with $\rho \in \mathbb{R}$ is feasible

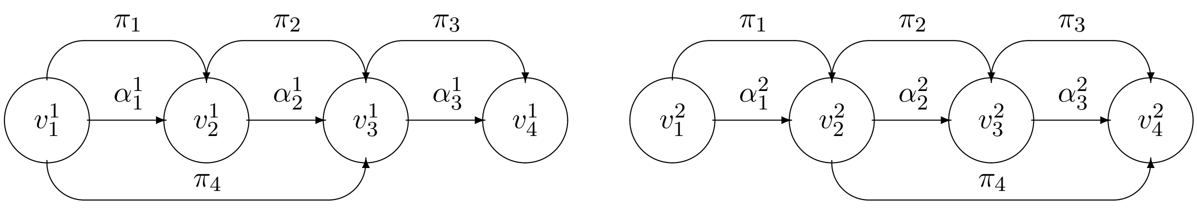

Figure 2: Illustration of Example 3 (semi-simultaneous boundedness)

for the first network and may achieve arbitrary benefit ρ. Similarly, for the second network, every flow $f^2 = (\alpha^2, \pi^2) = \left((0\ 0\ 0)^{\mathrm{T}}, (0\ \rho\ \rho\ -\rho)^{\mathrm{T}}\right)$ with $\rho \in \mathbb{R}$ is feasible with benefit ρ, so that both IN-FLOPs are unbounded. However, it is easily verified that the optimal se-sim flow is the zero flow with zero benefit.

In order for a given flow to be optimal, we can state the following sufficient condition.

Proposition 7. *Let SE-SIM-FLOP be given and let* $\left\{\hat{f}^k = \left(\hat{\alpha}^k, \hat{\pi}^k\right) : k = 1, \ldots, K\right\}$ *be a collection of optimal flows for the associated IN-FLOPs. If there exists an index* $j = 1, \ldots, K$ *such that* $\hat{\pi}^j \leq \hat{\pi}^k$ *for all* $k = 1, \ldots, K$*, then* $\pi = \hat{\pi}^j$ *determines an optimal se-sim network flow for SE-SIM-FLOP.*

Proof. Let $\hat{\pi}^j \leq \hat{\pi}^k$ for all $k = 1, \ldots, K$, or equivalently, $\hat{\pi}^1 \wedge \hat{\pi}^2 \wedge \ldots \wedge \hat{\pi}^K = \hat{\pi}^j$. Since all benefits are assumed to be nonnegative, it follows that $b^{\mathrm{T}}\hat{\pi}^j \leq b^{\mathrm{T}}\hat{\pi}^k$, and hence $b(f) \leq b^{\mathrm{T}}\hat{\pi}^j$ for all feasible se-sim flows f by Proposition 5. As was shown in the proof of Proposition 4, the common flow $\pi = \hat{\pi}^1 \wedge \hat{\pi}^2 \wedge \ldots \wedge \hat{\pi}^K = \hat{\pi}^j$ determines a feasible se-sim flow for SE-SIM-FLOP with maximal benefit $b^{\mathrm{T}}\pi = b^{\mathrm{T}}\hat{\pi}^j$, and hence the se-sim flow determined by $\pi = \hat{\pi}^j$ is optimal. □

Another sufficient optimality condition is based on se-sim residual flows.

Proposition 8. *Let SE-SIM-FLOP be given and let* f *and* $\bar{f}$ *be two given feasible se-sim network flows with* $b(f) < b(\bar{f})$*. Then the difference flow* $\tilde{f} = \bar{f} - f$ *defines a se-sim flow with positive residual benefit* $b(\tilde{f}) = b(\bar{f}) - b(f) > 0$ *in the semi-simultaneous residual network* $G_f = \left\{G_f^k : k = 1, \ldots, K\right\}$*, where* G_f^k *denotes the residual network of network* G^k *induced by the flow* f^k*.*

Proof. Let $f = \left\{f^k = (\alpha^k, \pi) : k = 1, \ldots, K\right\}$ and $\bar{f} = \left\{\bar{f}^k = (\bar{\alpha}^k, \bar{\pi}) : k = 1, \ldots, K\right\}$ be the given flows and let $\tilde{f} = \left\{\tilde{f}^k = \left(\tilde{\alpha}^k, \tilde{\pi}\right) : k = 1, \ldots, K\right\}$ with $\tilde{f}^k = \bar{f}^k - f^k = (\bar{\alpha}^k - \alpha^k, \bar{\pi} - \pi)$. By the assumptions of the proposition, it immediately follows that $b(\tilde{f}) = b(\bar{f} - f) = b(\bar{f}) - b(f) > 0$. In order to verify that $\tilde{f}$ is a feasible flow in each individual residual network G_f^k it remains to show that $\tilde{f}$ satisfies the flow conservation as given in (10a-c) with vanishing right-hand side. Then, by setting $\alpha_0^k = \alpha_{n_k+1}^k = 0$ for the special cases $j = 1$ and $j = n_k + 1$ in (10a+c), it is sufficient to consider (10b), so

$$
\begin{aligned}
& \tilde{\alpha}_j^k + \sum_{\{i: g_i^k = v_j^k\}} \tilde{\pi}_i - \tilde{\alpha}_{j-1}^k - \sum_{\{i: h_i^k = v_j^k\}} \tilde{\pi}_i && (16) \\
= \; & \bar{\alpha}_j^k - \alpha_j^k + \sum_{\{i: g_i^k = v_j^k\}} (\bar{\pi}_i - \pi_i) - (\bar{\alpha}_{j-1}^k - \alpha_{j-1}^k) - \sum_{\{i: h_i^k = v_j^k\}} (\bar{\pi}_i - \pi_i) && (16a) \\
= \; & \bar{\alpha}_j^k + \sum_{\{i: g_i^k = v_j^k\}} \bar{\pi}_i - \bar{\alpha}_{j-1}^k - \sum_{\{i: h_i^k = v_j^k\}} \bar{\pi}_i - \left(\alpha_j^k + \sum_{\{i: g_i^k = v_j^k\}} \pi_i - \alpha_{j-1}^k - \sum_{\{i: h_i^k = v_j^k\}} \pi_i\right) && (16b) \\
= \; & c_j^k - c_j^k = 0, && (16c)
\end{aligned}
$$

where the last step follows from the fact that both f and $\bar{f}$ are feasible for SE-SIM-FLOP. □

Definition 6. Let SE-SIM-FLOP be given and let f be a feasible se-sim flow. The se-sim flow $\tilde{f}$ is called an *improvement (flow)* for f if $\tilde{f}$ is feasible in the se-sim residual network G_f with positive residual benefit $b(\tilde{f}) > 0$.

The next result shows that improvement flows take over the role of negative cycles in the theory of classical network flow theory.

Theorem 2. *Let SE-SIM-FLOP be given and $\hat{f}$ be a feasible se-sim flow. Then $\hat{f}$ is optimal if and only if there does not exist an improvement.*

Proof. If $\hat{f}$ is optimal, then $b(\hat{f}) \geq b(f)$ for all other flows f and thus, by definition and Proposition 8, there cannot exist an improvement.

Conversely, for any other se-sim flow f Proposition 8 implies that the flow $\tilde{f} = \hat{f} - f$ is a feasible se-sim flow in the residual network $G_{\hat{f}}$ which - by the assumption of the theorem - has a residual benefit $b(\tilde{f}) = b(\hat{f}) - b(f) = b(\hat{f} - f) \geq 0$. Hence, the se-sim flow $\hat{f}$ is optimal. □

3.3 Improvement flows

Since the benefit of a se-sim flow is only depending on its flow along common arcs, we are interested to know by how much this flow can be further increased.

Definition 7. Let SE-SIM-FLOP be given and let $f = \left\{f^k = (\alpha^k, \pi) : k = 1, \ldots, K\right\}$ be a given feasible se-sim flow. Then the *arc potential* of common arc e_i with respect to f is defined by

$$\phi_i = \min\left\{\alpha_j^k : v_j^k \in [g_i^k, h_i^k), j = 1, \ldots, n_k, k = 1, \ldots, K\right\}. \tag{17}$$

By (10d) we know that $\phi_i \geq 0$ for all common arcs e_i. If $\phi_i > 0$ for some arc e_i, then this means that in each individual network G^k, all individual arcs connecting that common arc's tail g_i^k and head h_i^k have positive flow value. Then, provided e_i has a positive flow benefit $b_i > 0$, we can improve the overall flow value by increasing the flow along arc e_i while correspondingly reducing the flows along its associated individual arcs.

Proposition 9. *Let SE-SIM-FLOP be given and let $f = \left\{f^k = (\alpha^k, \pi) : k = 1, \ldots, K\right\}$ be a feasible se-sim flow. If there exists a common arc e_l with associated nonzero benefit $b_l > 0$, and positive arc potential $\phi_l > 0$, then there exists an improvement flow for f.*

Proof. Given that the arc potential associated with common arc e_l is positive,

$$\phi_l = \min\left\{\alpha_j^k : v_j^k \in [g_l^k, h_l^k), j = 1, \ldots, n_k k = 1, \ldots, K\right\} > 0, \tag{18}$$

let $\tilde{\pi}_i = \phi_l$ for $i = l$ and zero otherwise, so in particular $\tilde{\pi} \geq 0$ and $b^{\mathrm{T}}\tilde{\pi} = b_l\tilde{\pi}_l = b_l\phi_l > 0$. Now let $\tilde{f}$ be the se-sim flow in the residual network G_f determined by $\tilde{\pi}$. To verify that $\tilde{f}$ is an improvement flow for f, we need to show that the flow $\hat{f} = \left\{\hat{f}^k = (\hat{\alpha}^k, \hat{\pi}) : k = 1, \ldots, K\right\}$ defined by $\hat{f} = f + \tilde{f}$ is semi-simultaneously feasible. Flow conservation can easily be derived from Proposition 8. Since $\tilde{f}$ is a residual se-sim flow in G_f, we can, similar to the proof of Lemma 1, verify that

$$\hat{\alpha}_j^k = \begin{cases} \alpha_j^k - \phi_l & \text{if } v_j^k \in [g_l^k, h_l^k), \\ \alpha_j^k & \text{if } v_j^k \notin [g_l^k, h_l^k), \end{cases} \text{ and } \hat{\pi}_i = \begin{cases} \pi_i + \phi_l & \text{if } i = l, \\ \pi_i & \text{if } i \neq l. \end{cases} \tag{19}$$

Hence, it only remains to show that the flow capacities $\hat{\alpha}_j^k \geq 0$ are satisfied. First note that $\hat{\alpha}_j^k = \alpha_j^k \geq 0$ for all $v_j^k \notin [g_l^k, h_l^k)$, since f is feasible. Further, if $v_j^k \in [g_l^k, h_l^k)$, then α_j^k is considered in the computation of the minimum in (18), i.e., $\hat{\alpha}_j^k = \alpha_j^k - \phi_l \geq 0$. □

Although easily detectable, a positive arc potential is sufficient, but not necessary for further improvement.

Example 4. Consider the individual network shown in Figure 3. The initial feasible flow $f = (\alpha, \pi) = \left((1\ 0\ 1)^{\mathrm{T}}, (0\ 0)^{\mathrm{T}}\right)$ is given on the left and has an associated benefit of $b(f) = b^{\mathrm{T}}\pi = 1\pi_1 + 2\pi_2 = 0$. Note that both arc potentials are zero, $\phi_1 = \min\{\alpha_1, \alpha_2\} = \min\{1, 0\} = 0$ and $\phi_2 = \min\{\alpha_2, \alpha_3\} = \min\{0, 1\} = 0$, and hence, there does not exist a direct improvement for any of the common arcs. However, observe that the feasible flow indicated in the right, $\hat{f} = (\hat{\alpha}, \hat{\pi}) = \left((2\ 0\ 0)^{\mathrm{T}}, (-1\ 1)^{\mathrm{T}}\right)$, has flow benefit of $b(\hat{f}) = b^{\mathrm{T}}\hat{\pi} = 1\hat{\pi}_1 + 2\hat{\pi}_2 = -1 + 2 = 1$ and can be found by the improvement flow $\tilde{f} = (\tilde{\alpha}, \tilde{\pi}) = \left((1\ 0\ -1)^{\mathrm{T}}, (-1\ 1)^{\mathrm{T}}\right)$.

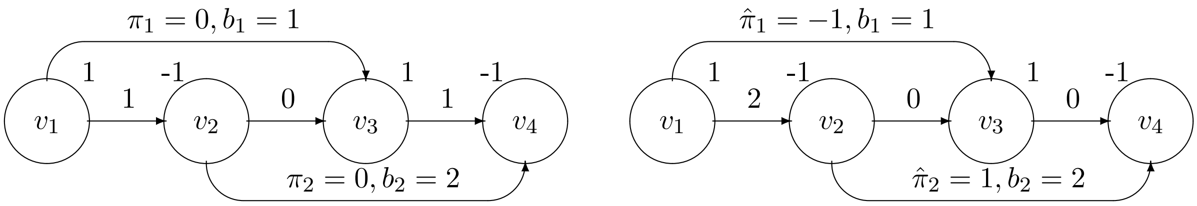

Figure 3: Illustration of Example 4 (flow improvement with zero arc potentials)

If all arc potentials are zero, then increasing the flow along some common arc is only possible while simultaneously decreasing the flow along others.

Theorem 3. *Let SE-SIM-FLOP be given and let f be any feasible but not optimal se-sim flow. Then, for any $\rho > 0$, there exists an improvement flow $\tilde{f} = \left\{\tilde{f}^k = (\tilde{\alpha}^k, \tilde{\pi}) : k = 1, \ldots, K\right\}$ with $\tilde{\pi}_i \geq -\rho$ for all $i = 1, \ldots, m$.*

Proof. Let $f = \left\{f^k = (\alpha^k, \pi) : k = 1, \ldots, K\right\}$ be the given flow and let $\hat{f} = \left\{\hat{f}^k = (\hat{\alpha}^k, \hat{\pi}) : k = 1, \ldots, K\right\}$ be any other flow with $b(f) < b(\hat{f})$. The difference flow $\bar{f} = \hat{f} - f$ by $\bar{f} = \left\{\bar{f}^k = (\bar{\alpha}^k, \bar{\pi}) : k = 1, \ldots, K\right\} = \left\{(\hat{\alpha}^k - \alpha^k, \hat{\pi} - \pi) : k = 1, \ldots, K\right\}$, is by Proposition 8 a se-sim flow in the se-sim residual network G_f which improves f, since $b(\bar{f}) = b(\hat{f} - f) = b(\hat{f}) - b(f) > 0$.

If $\bar{\pi}_i \geq 0$ for all $i = 1, \ldots, m$, the theorem holds with $\tilde{f} = \bar{f}$ and $\tilde{\pi} = \bar{\pi}$. Otherwise, there exists $\bar{\pi}_i < 0$ and $p = \max\{-\bar{\pi}_i : \bar{\pi}_i < 0, i = 1, \ldots, m\} > 0$. We define the se-sim residual flow $\tilde{f}$ by scaling $\bar{f}$ with the scalar $\frac{\rho}{p}$, i.e., $\tilde{f} = \left(\frac{\rho}{p}\right)\bar{f}$ with $b(\tilde{f}) = b\left(\frac{\rho}{p}\bar{f}\right) = \left(\frac{\rho}{p}\right) b(\bar{f}) > 0$ and

$$\tilde{\pi}_i^k = \left(\frac{\rho}{p}\right)\bar{\pi}_i^k \begin{cases} \geq 0 & \text{if } \bar{\pi}_i \geq 0, \\ \geq -\rho & \text{if } \bar{\pi}_i < 0. \end{cases} \tag{20}$$

Hence, $\tilde{\pi}_i \geq -\rho$ for all $i = 1, \ldots, m$. □

Theorem 3 suggests the following heuristic for detection of improvements: first, we reduce the flow along all common arcs by some amount ρ and send this flow along the associated individual arcs instead. Although this temporarily decreases the objective value, it also results in positive arc potentials for all common arcs of at least ρ. Hence, based on some selection strategy, we then may iteratively select common arcs for improvement according to Proposition 9. With each such improvement, we also reduce the flow along the associated individual arcs and therefore, after updating the arc potentials with each improvement, will eventually reduce all of them back to zero. This process can be iterated and, as based merely on the repeated computation and selection of arc potentials, accomplished very efficiently. The remaining question, however, is which selection strategy to choose to maximize the overall benefit.

4 Algorithms and Examples

We start with a generic procedure for the se-sim network flow problem which can be interpreted as generalization of cycle canceling algorithm of classical network flow theory.

Procedure 1. Solving SE-SIM-FLOP:

1. Given SE-SIM-FLOP, find a feasible se-sim flow f.
   IF no feasible flow exists, STOP – the problem is infeasible.

2. Given feasible se-sim flow f, find an improvement $\tilde{f}$.
   IF no further improvement exists, STOP – f is optimal.

3. Update $f := f + \tilde{f}$ and repeat step 2.

The validity of Procedure 1 follows from the results of the previous sections. Here we discuss some implementation details of the first two steps and give several examples.

4.1 Checking feasibility of SE-SIM-FLOP

Theorem 1 together with Proposition 3 and 4 suggests Procedure 2 for finding an initial feasible se-sim flow.

Procedure 2. Finding a feasible se-sim flow:

1.1. Given SE-SIM-FLOP, find a feasible flow f^k for each individual network G^k.
IF one of the individual problems is infeasible, STOP - SE-SIM-FLOP is infeasible.

1.2. Combine the flows f^k, $k \in [1, K]$ into a se-sim network flow.

A naive way to implement this generic approach is to apply standard max flow algorithms to each of the individual networks $G_k, k = 1, \ldots, n_k$. By taking advantage of the special network structure of SE-SIM-FLOP we can, however, solve the individual feasibility problem with a worst case complexity of $O(mn_k)$ using Algorithm 1.

Algorithm 1 Finding a feasible flow $f^k = \{\alpha^k, \pi^k\}$ in an individual network

```
α_1^k = c_1^k
for j = 2 to n_k do
    α_j^k = α_{j-1}^k + c_j^k
end for
for i = 1 to m do
    π_i^k = min { α_j^k : v_j^k ∈ [g_i^k, h_i^k) }
    for all j ∈ { j : v_j^k ∈ [g_i^k, h_i^k) } do
        α_j^k = α_j^k − π_i^k
    end for
end for
```

Proposition 10. *Algorithm 1 is correct. In particular, if at termination*

1. $\alpha_j^k \geq 0$ *for all* $j = 1, \ldots, n_k$*, then the flow* $f^k = (\alpha^k, \pi^k)$ *is individually feasible;*

2. $\alpha_j^k < 0$ *for some* $j = 1, \ldots, n_k$*, then the individual problem is infeasible.*

The complexity of the algorithm can be bounded by $O(mn_k)$.

Proof. To show correctness, first observe that during the first **for** loop, the flow α^k is iteratively assigned so that the flow conservation constraints (10a-c) are satisfied at all nodes $v_j^k, j = 1, \ldots, n_k$, using flow along individual arcs only. During the second **for** loop, the flow π^k on common arc is assigned to guarantee $\alpha^k \geq 0$. If there remains some $\alpha_j^k < 0$, then $\alpha_j^k = \sum_{i=1}^{j} c_i^k < 0$ by definition of α_j^k in the first **for** loop and, together with $\sum_{i=1}^{n_k} c_i = 0$, it follows that $\sum_{i=j+1}^{n_k} c_i^k > 0$. Moreover, from the second **for** loop, we see that $j \notin [g_i^k, h_i^k)$ for all $i = 1, \ldots, m$, and thus there does not exist a common arc along which the overflow into nodes $\left\{v_{j+1}^k, v_{j+1}^k, \ldots, v_{n_k}^k\right\}$ flow could be sent back to $\left\{v_1^k, \ldots, v_j^k\right\}$. Therefore, if at termination $\alpha_j^k < 0$ for some $j = 1, \ldots, n_k$, the problem is infeasible.

The claimed complexity bound follows from the fact that after the first **for** loop of order $O(n_k)$, the algorithm performs m outer loops. In each of these loops a minimum among at most n_k values needs to be found and at most n_k values need to be updated. Thus, the overall complexity is $O(n_k) + O(m)\left(O(n_k) + O(n_k)\right) = O(mn_k)$. □

Example 5. Consider the se-sim network given in Figure 4a, formed by $K = 2$ individual networks with $n_1 = 4$ and $n_2 = 5$ individual arcs and $m = 3$ common arcs. After applying Algorithm 1 through Figures 4b-d, we obtain the two individually feasible flows shown in Figure 4e.

If a collection $f = \left\{f^k = (\alpha^k, \pi^k) : k = 1, \ldots, K)\right\}$ of K feasible individual flows has been found by Algorithm 1, Proposition 4 suggests Algorithm 2 in order to combine these flows to a feasible se-sim network flow.

Algorithm 2 Combining individual to a se-sim flow

```
for i = 1 to m do
  π = min {π_i^k : k ∈ [1, K]}
  for k = 1 to K do
    for all j ∈ {j : v_j^k ∈ [g_i^k, h_i^k)} do
      α_j^k = α_j^k + π_i^k − π_i
    end for
    π_i^k = π
  end for
end for
```

Proposition 11. *Algorithm 2 is correct and its complexity can be bounded by $O(mn)$.*

Proof. The correctness is established in the proof of Proposition 4. The complexity bound follows from the observation that the algorithm performs m outer loops. In each of these loops a minimum needs to be found among K values and at most $\sum_{k=1}^{K} n_k = n$ values α_j^k and K values π_i^k need to be updated. Using the fact that $K \leq n$ and combining then yields a total complexity of $O(m)\left(K + O(n) + K\right) = O(mn)$. □

Example 6. Starting from Figure 4e in Example 5 with $f^1 = (\alpha^1, \pi^1) = \left((2\ 0\ 0\ 1)^{\mathrm{T}}, (-1\ 3\ 0)^{\mathrm{T}}\right)$ and $f^2 = (\alpha^2, \pi^2) = \left((0\ 0\ 2\ 4\ 0)^{\mathrm{T}}, (2\ -1\ -1)^{\mathrm{T}}\right)$, Algorithm 2 determines the common flow $\pi = \pi^1 \wedge \pi^2 = (-1\ -1\ -1)^{\mathrm{T}}$, illustrated in Figure 5.

Combining Propositions 10 and 11 yields the overall complexity for finding a feasible flow for SE-SIM-FLOP.

Theorem 4. *The feasibility problem of SE-SIM-FLOP can be solved in $O(mn)$ using Algorithms 1 and 2.*

4.2 Generalization of cycle canceling algorithm

Theorem 2 suggests to successively determine improvement flows, similar to successive cycle-canceling algorithms for solving minimum cost flow problems. The following example shows that the detection of improvements may be easy if the situation of Proposition 9 applies.

Example 7. Starting from Figure 5 in Example 6 with feasible se-sim flow $f = (\alpha^1, \alpha^2, \pi) = \left((2\ 4\ 1\ 2)^{\mathrm{T}}, (3\ 0\ 2\ 4\ 0)^{\mathrm{T}}, (-1\ \ -1\ \ -1)^{\mathrm{T}}\right)$, the three arc potentials are $\phi_1 = \min\left\{\alpha^1_1, \alpha^1_2, \alpha^1_3, \alpha^2_1\right\} = 1$, $\phi_2 = \min\left\{\alpha^1_2, \alpha^2_2, \alpha^2_3, \alpha^2_4\right\} = 0$ and $\phi_3 = \min\left\{\alpha^1_3, \alpha^1_4, \alpha^2_3, \alpha^2_4, \alpha^2_5\right\} = 0$. Hence, since we obtain a positive arc potential for common arc e_1, the initial flow f can be increased along this arc by $\phi_1 = 1$. Note that the new flow in Figure 6 improves the previous one in Figure 5 if $b_1 > 0$.

The next example illustrates that in contrast to the case of non-simultaneous network flows, finding improvements for SE-SIM-FLOP may require the detection of multiple cycles in each individual residual network.

Example 8. The two individual networks shown in Figure 7a are almost identical except that common arcs e^2 and e^4 are swapped. In particular, both cases require that one unit of flow has to be sent from node v_2 to node v_3 and from v_5 to v_6.

In the initial se-sim flow shown in Figure 7a, this flow is carried only along individual arcs with a benefit of zero. It can be improved in two ways. In the first network, the flow from v^1_2 to v^1_3 can also be sent along common arc e_1 back to node v^1_1 and then along common arc e_2 to node v^1_3. Similarly, the flow from v^1_5 to v^1_6 can also be sent along common arc e_3 back to node v^1_4 and then along common arc e_4 to node v^1_6, both yielding a benefit of one. Alternatively, the two positive benefit cycles $v^1_3 \xrightarrow{e^1_2} v^1_2 \xrightarrow{e_1} v^1_1 \xrightarrow{e_2} v^1_3$ and $v^1_6 \xrightarrow{e^1_5} v^1_5 \xrightarrow{e_3} v^1_4 \xrightarrow{e_4} v^1_6$ can be detected in the residual of the first network, both yielding a benefit of one. Analogously, in the second network, the flow from v^2_2 to v^2_3 can also be sent along common arc e_1 back to node v^2_1 and then along common arc e_4 to node v^2_3, and similarly, the flow from v^2_5 to v^2_6 can also be sent along common arc e_3 back to node v^2_4 and then along common arc e_1 to node v^2_6, again yielding an improved benefit of one, respectively. Correspondingly, the two associated cycles in the residual network are given by $v^2_3 \xrightarrow{e^2_2} v^2_2 \xrightarrow{e_1} v^2_1 \xrightarrow{e_4} v^2_3$ and $v^2_6 \xrightarrow{e^2_5} v^2_5 \xrightarrow{e_3} v^2_4 \xrightarrow{e_2} v^2_6$. Note that each of these cycles involves different common arcs, and hence that there does not exist a se-sim improvement using only one of the two cycles in each individual network. However, by simultaneous combination of all above cycles, a se-sim improvement can be found and improves the flow benefit by two. The resulting flow is shown in Figure 7b.

Due to the remaining difficulty of finding multiple cycles that combine to a feasible se-sim improvement, Theorem 3 motivates the following heuristic for finding a se-sim improvement.

Procedure 3. Finding a se-sim improvement:

3.1. Reduce the current flow f on all common arcs by some given amount $\rho > 0$.

3.2. Compute the arc potentials ϕ_i and choose any common arc e_i with $\phi_i > 0$.
IF all $\phi_i = 0$, compute the benefit $b(\bar{f})$ of the new flow $\bar{f}$ and GOTO 3.4.

3.3. Increase the flow along arc e_i by its arc potential ϕ_i and REPEAT 3.2.

3.4. IF $b(\bar{f}) > b(f)$, update the best flow by setting $f = \bar{f}$ and REPEAT 3.1.
ELSE STOP – the algorithm terminates with flow f as best solution found.

Algorithms 3, 4 and 5 are derived directly from equations (13) in Lemma 1 and (17) and (19) in Proposition 9.

Algorithm 3 Flow reduction along common arcs

> **for** $i = 1$ **to** m **do**
>   **for** $k = 1$ **to** K **do**
>   **for all** $j \in \left\{ j : v_j^k \in \left[g_i^k, h_i^k\right) \right\}$ **do**
> $\alpha_j^k = \alpha_j^k + \rho$
>   **end for**
>   **end for**
> $\pi_i = \pi_i - \rho$
> **end for**

Algorithm 4 Computation of arc potentials ϕ_i

> **for** $i = 1$ **to** m **do**
>   **for** $k = 1$ **to** K **do**
> $\phi_i^k = \min \left\{ \alpha_j^k : v_j^k \in \left[g_i^k, h_i^k\right) \right\}$
>   **end for**
> $\phi_i = \min \left\{ \phi_i^k : k \in [1, K] \right\}$
> **end for**

Algorithm 5 Improvement of common arc e_i by arc potential ϕ_i

> **for** $k = 1$ **to** K **do**
>   **for all** $j \in \left\{ j : v_j^k \in \left[g_i^k, h_i^k\right) \right\}$ **do**
> $\alpha_j^k = \alpha_j^k - \phi_i$
>   **end for**
> **end for**
> $\pi_i = \pi_i + \phi_i$

Example 9. Again consider the individual network in Figure 3 from Example 4. After an initial flow reduction of $\rho = 1$ on all common arcs, the new flow is given in the network on the left in Figure 8. In particular, note that the reduction of 1 unit increases the flow along the middle individual arc by 2. Also observe that the new benefit has decreased from 0 to -3. The new arc potentials can be computed as $\phi_1 = \min\{\alpha_1, \alpha_2\} = \min\{2, 2\} = 2$ and $\phi_2 = \min\{\alpha_2, \alpha_3\} = \min\{2, 2\} = 2$, and hence allow for subsequent improvement along either of the two common arcs. In particular, by choosing the second common arc, we obtain the same optimal flow as before in Example 4. Clearly, all arc potentials again are reduced to zero.

As pointed out earlier, the success of Procedure 3 depends significantly on which common arcs e_i are chosen for improvement. Two alternative Greedy strategies are proposed in Procedure 4.

Procedure 4. Greedy strategies for finding a se-sim improvement:

3.2.1 Among all common arcs e_i which have positive arc potential $\phi_i > 0$, choose the arc with maximum benefit value b_i,

$$b_i \geq b_j \text{ for all } j = 1, \ldots, m.$$

3.2.1′ Among all common arcs e_i which have positive arc potential $\phi_i > 0$, choose the arc with maximum improvement benefit,

$$b_i \cdot \phi_i \geq b_j \cdot \phi_j \text{ for all } j = 1, \ldots, m.$$

It is not difficult, however, to construct examples for which both Greedy strategies are non-optimal. Nevertheless, based on an implementation of the above algorithms and a comparison of the results obtained with the optimal LP solutions on several randomly generated test problems, we found that on most small instances ($K \leq 3$, $m \leq 10$, $n_k \leq 20$) we were still able to find the optimal solution using the procedure with Greedy selection strategies. The difference between our non-exact solutions and the optimal solutions increased significantly with increased problem size.

Therefore, the question if it is possible to define a simple optimal selection strategy still remains open, although its affirmative answer is rather unlikely.

5 Conclusion

We considered linear programs whose coefficient matrices consist of K matrices with the row consecutive ones (C1) property. While the special case of $K = 1$ has the integrality property this is no longer true for $K \geq 2$. On the other hand, the transformation to a network flow problem which is well-known for $K = 1$ has an analog for general K by introducing semi-simultaneous (se-sim) flows. For the latter, feasibility can be tested in polynomial time by a reduction to a sequence of feasibility tests for classical network flows. As an analogue to the negative cycle theorem an optimality criterion based on the notion of improvement semi-simultaneous flows is proved. We finally showed how these results translate into algorithms for solving semi-simultaneous flows defined by linear programs with consecutive ones coefficient matrices.

Various additional research questions are motivated by the results of this paper.

Instead of starting with coefficient matrices which have the row C1 property, one may want to consider binary matrices that are column consecutive ones, or - equivalently - by partitioning a binary matrix $A \in \mathbb{B}^{m \times n}$ into matrices that are column consecutive one, $A = (A^1 \ \ldots \ A^L)$ with $A^l \in \mathbb{B}^{m_l \times n}$, $l = 1, \ldots, L$ and $\sum_{l=1}^{L} m_l = m$. The resulting LP is equivalent to

$$\begin{aligned} \text{minimize } & \sum_{l=1}^{L} c^{\mathrm{T}} x \\ \text{subject to } & A^l x = b^l \text{ for all } l = 1, \ldots, L, \\ & x \in \mathbb{R}^n_+. \end{aligned} \tag{21}$$

and the equality constraints $A^l x = b^l$ can directly be transformed into L systems of flow conservation constraints in L underlying networks G^l, $l = 1, \ldots, L$. A solution of problem (21) then corresponds to a vector $x \in \mathbb{Z}^n_+$ so that for each $l = 1, \ldots, L$, the vector x establishes a feasible flow in each of the networks G^l while minimizing the objective $c^{\mathrm{T}} x$. In contrast to the semi-simultaneous flows considered in this paper, in this approach there are no individual flows. We can therefore call x a *simultaneous flow*.

Another interesting topic is to generalize the concept of se-sim flows to more general se-sim networks. In our paper, the se-sim networks were defined by the underlying KC1 linear program. This implies by Equation 8 that the individual arcs in each of the individual networks have a Hamiltonian path structure. If we allow an arbitrary set of individual arcs, an interesting question is to characterize structures which allow a duplication of the feasibility, optimality and improvement results of Section 3.

Even more general, the topic of this paper motivates research on *simultaneous graph theory* problems defined on a collection $G = \{G^k = (V^k, D^k \cup E^k) : k = 1, \ldots, K\}$ of directed or undirected individual graphs G^k. Feasibility and optimality characterizations of (semi-)simultaneous shortest path, spanning tree, matching, etc. are interesting in their own right, but may also be of practical interest in the same way as the (semi-)simultaneous network flow theory developed in this paper.

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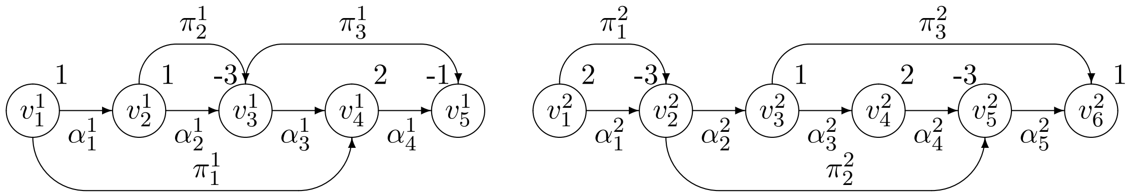

Figure 4a: Illustration of Example 5 (initial networks without flows)

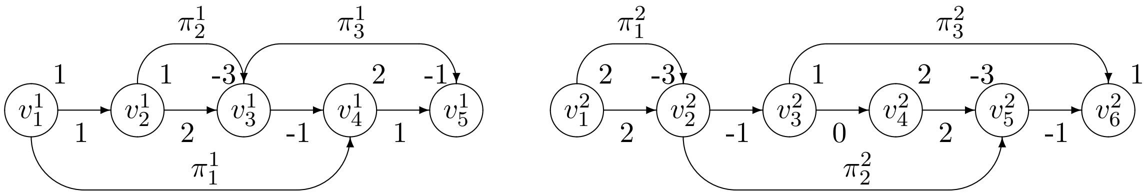

Figure 4b: Illustration of Example 5 (assignment of individual arc flows α^k)

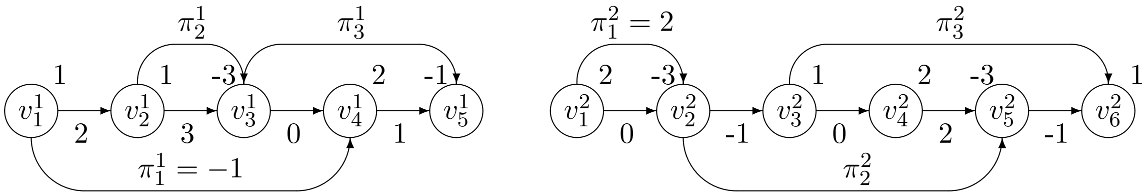

Figure 4c: Illustration of Example 5 (assignment of common arc flow π_1^k)

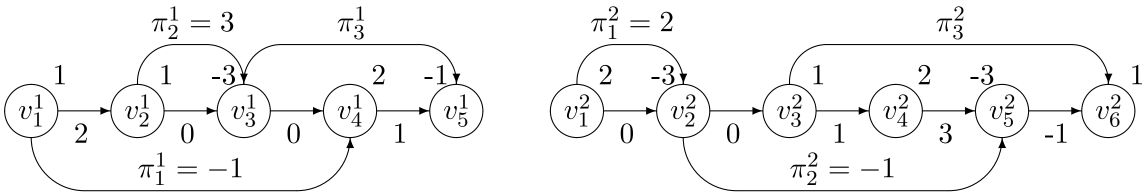

Figure 4d: Illustration of Example 5 (assignment of common arc flow π_2^k)

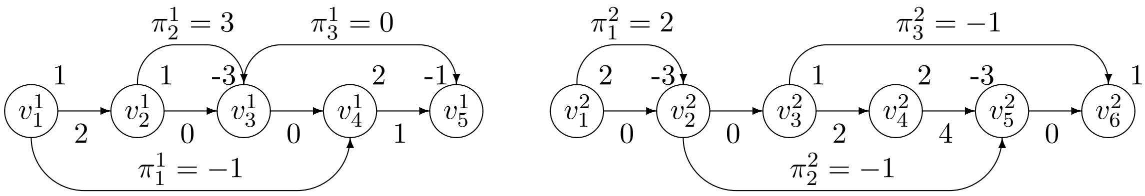

Figure 4e: Illustration of Example 5 (assignment of common arc flow π_3^k)

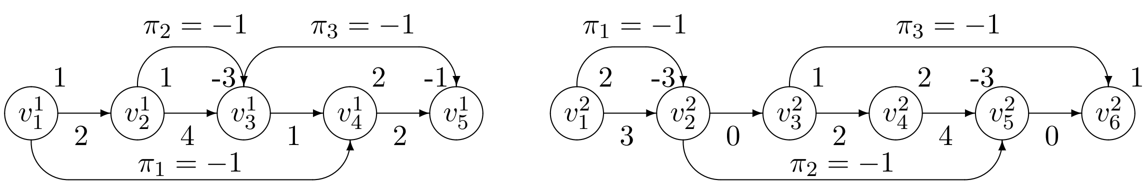

Figure 5: Illustration of Example 6 (se-sim network flow)

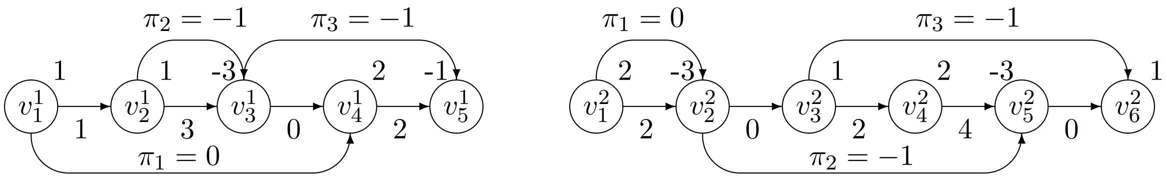

Figure 6: Illustration of Example 7 (improvement along e_1 by $\phi_1 = 1$)

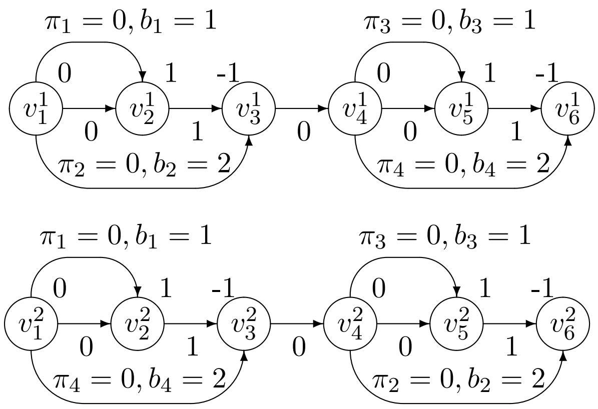

Figure 7a: Illustration of Example 8 (initial se-sim flow)

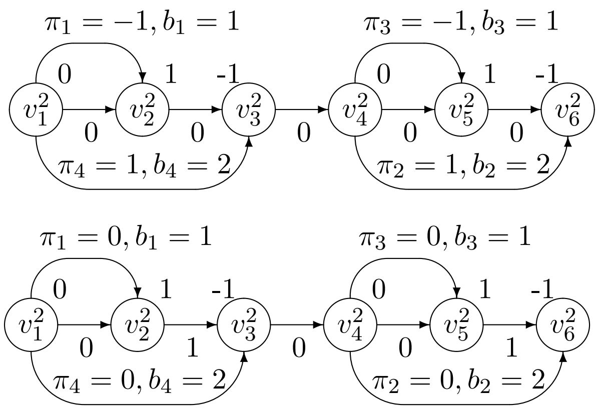

Figure 7b: Illustration of Example 8 (improvement along multiple cycles)

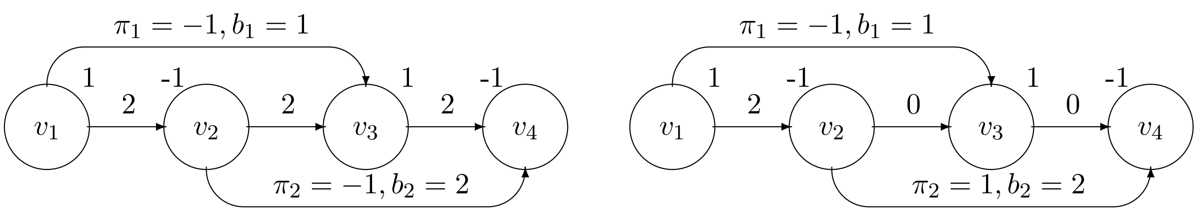

Figure 8: Illustration of Example 9 (heuristic approach for finding improvements)