Notes at the embeddedness of the minimal surface of Costa, Hoffman and Meeks

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Abstract

The existence of a complete, embedded minimal surface of genus one, with three ends and whose total Gaussian curvature satisfies equality in the estimate of Jorge and Meeks – which is therefore especially finite –, was a sensation in the middle eighties. From this moment on, the surface of Costa, Hoffman and Meeks has become famous all around the world, not only in the community of mathematicians. With this article, we want to fill a gap in the injectivity proof of Hoffman and Meeks, where there is a lack of a strict mathematical justification. Naturally, our paper is not intended to derogate their inimitably wonderful work. We exclusively argue topologically and do not use additional properties like differentiability or even holomorphy.

Keywords. Minimal surfaces, Costa, Hoffman, Meeks, local/global homeomorphisms, homotopy lifting.

MSC classification: 49Q05, 30F15, 51H05

IN MEMORY OF

MICHAEL GRÜTER

1 Introduction

As already stated, we suppose to have found a gap in the proof of HOFFMAN and MEEKS [3], where they showed that the surface, discovered three years before by COSTA is *embedded* into the three dimensional Euklidian space. The embeddedness of its *ends* has already been proved before in [2] with [4].
We are concered with proposition 3 in [3], the reader should acquaint himself with that article. We have a look at its proof on page 124. We cite the critical passage.

> '... We assert that $\pi \circ X : T_{\varepsilon,N} \to P$ is actually a submersion which is one-to-one on the boundary of $T_{\varepsilon,N}$. Since $T_{\varepsilon,N}$ is simply connected, this implies that $\pi \circ X$ is actually one-to-one on $T_{\varepsilon,N}$ ...'

The authors have concluded here from the injectivity of a map, that is just *locally injective* and *injective only at the boundary*, to the *total injectivity* on the whole closure of the domain, by virtue of the simple connectivity of the latter. We doubted this fact and we give now in section 3 a rigorous proof with the aid of actual elementary topological tools.
Of course, we contacted the authors, who were working in the United States during that time. They had been afflicted by doubts as well.
Maybe, there is another theorem that we and the original authors did not know. However: If so, we have got an alternative proof here in this article.

2 Basic facts from topology

In this section, we summarise some elementary facts from topology. The reader will find these theorems in every textbook about this subject. But it is sometimes a hard struggle in order to push forward into a non familiar area. The proofs that we give here are all basic and need not much previous knowledge.

The following theorem is clearly a highlight of topology, there are no further explanations necessary.

2.1 Theorem (Jordan curve theorem) *Every Jordan curve in the plane* $\mathbb{C}$, *i.e. the trace of a*

$$\text{continuous map } \gamma : [0,1] \to \mathbb{C}, \text{ s.t. } \gamma(0) = \gamma(1) \text{ and } \gamma|_{[0,1)} \text{ is injective}$$

separates $\mathbb{C}$ *into*

- *a bounded, simply connected domain* U_γ *(=: the interior of* γ*),*
- *an unbounded domain* U_γ^∞ *(=: the exterior of* γ*),*

and is the common boundary of both domains:

$$\mathbb{C} = U_\gamma \,\dot\cup\, \gamma\big([0,1]\big) \,\dot\cup\, U_\gamma^\infty, \qquad \partial U_\gamma = \gamma\big([0,1]\big) = \partial U_\gamma^\infty.$$

Proof: The reader finds one of the shortest proofs in Schmidt [6]. The simply connectedness of the interior is proved for example in Carathéodory [1], part II, chapter II 'jordan curves'. ■

2.2 Definition. Let E, B, Y be topological spaces and $p : E \to B$ a continuous map.

(i) A topological subspace $U \subseteq B$ is called *trivially covered by* p, iff there exists a discrete topological space F_U and a homeomorphism

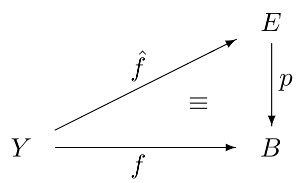
$$\Phi_U : p^{-1}(U) \to U \times F_U, \tag{1}$$

that is compatible with the projection $\pi_U : U \times F_U \to U$, i.e. with the property

$$p = \pi_U \circ \Phi_U \qquad \text{on} \quad p^{-1}(U). \tag{2}$$

(ii) A map $p : E \to B$ is called *covering*, iff for each $b \in B$ there is an open neighbourhood $U \subseteq B$ that is trivially covered by p.

(iii) Let $f : Y \to B$ be continuous. A *lifting of* f *in the covering* $p : E \to B$ is a continuous map $\hat{f} : Y \to E$ such that the diagram



commutes, i.e. with the property

$$p \circ \hat{f} = f.$$

2.3 Proposition (Uniqueness of liftings) *Let* E, B, Y *be topological spaces and* $\hat{f}_1$, $\hat{f}_2 : Y \to E$ *two liftings of* $f : Y \to B$ *in the covering* $p : E \to B$. *If* Y *is connected and if* $\hat{f}_1$ *and* $\hat{f}_2$ *are equal at at least one single point, there follows* $\hat{f}_1 = \hat{f}_2$.

Proof: We separate the whole space

$$Y = Y^* \,\dot{\cup}\, Y_*$$

into disjoint parts

$$Y^* := \{y \in Y : \ \hat{f}_1(y) = \hat{f}_2(y)\}, \qquad Y_* := \{y \in Y : \ \hat{f}_1(y) \neq \hat{f}_2(y)\}.$$

Let $y \in Y$ be arbitrary, U an open, trivially covered neighbourhood of $f(y)$ and

$$\Phi_U : p^{-1}(U) \to U \times F_U$$

a homeomorphism that is compatible with the projection π_U as in (1). Let π_{F_U} denote the projection of $U \times F_U$ on F_U. Then, there holds

$$\Phi_U(\hat{f}_i(\eta)) \overset{(2)}{=} \big((p \circ \hat{f}_i)(\eta), k_i\big) = \big(f(\eta), k_i\big) \qquad (i = 1, 2) \tag{3}$$

for all $\eta \in \hat{f}_i^{-1}\big(p^{-1}(U)\big)$, where

$$k_i := \pi_{F_U}\big(\Phi_U\big(\hat{f}_i(y)\big)\big).$$

Since the $U \times \{k_i\}$ are open with respect to the product topology and the maps Φ_U, $\hat{f}_1$, $\hat{f}_2$ are continuous,

$$D := \hat{f}_1^{-1}\left(\Phi_U^{-1}\big(U \times \{k_1\}\big)\right) \cap \hat{f}_2^{-1}\left(\Phi_U^{-1}\big(U \times \{k_2\}\big)\right)$$

is an open neighbourhood of y. We distinguish two cases.

(a) If $y \in Y^*$, there follows $k_1 = k_2$ and

$$\Phi_U\big(\hat{f}_1(\eta)\big) = \Phi_U\big(\hat{f}_2(\eta)\big)$$

for all $\eta \in D$ from (3), thus $\hat{f}_1(\eta) = \hat{f}_2(\eta)$, and consequently $D \subseteq Y^*$.

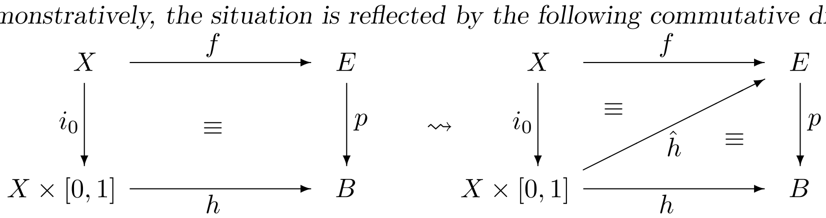
(b) If $y \in Y_*$, there follows $k_1 \neq k_2$ and, completey analogeously as above, $D \subseteq Y_*$.

Therefore, Y^* and Y_* are open. As Y is connected and there holds $Y^* \neq \emptyset$, there must hold $Y^* = Y$ and $Y_* = \emptyset$. Therfore, $\hat{f}_1 = \hat{f}_2$. ■

2.4 Proposition (Homotopy-lifting-property) *Let E, B, X be arbitrary topological spaces. Then every covering $p : E \to B$ has the* homotopy-lifting-property *for X. This means:*
If the maps $f : X \to E$, $h : X \times [0,1] \to B$ are continuous and $i_0 : X \to X \times [0,1]$, $x \mapsto (x, 0)$ is a map satisfying $h \circ i_0 = p \circ f$, then there exists exactly one continuous map $\hat{h} : X \times [0,1] \to E$ with the property

$$(i)\ p \circ \hat{h} = h, \qquad (ii)\ \hat{h} \circ i_0 = f.$$

Demonstratively, the situation is reflected by the following commutative diagram:



Especially, $\hat{h}$ is a lifting of the homotopy h.

Proof: We subdivide the proof into three parts.

FIRST PART (LOCAL VERSION): We show that for each $x \in X$ there exist an open neighbourhood V_x of x and a continuous map $\hat{h}^x : V_x \times [0,1] \to E$ with the property

$$(\mathrm{i}^x)\ p \circ \hat{h}^x = h \quad \text{auf } V_x \times [0,1], \qquad (\mathrm{ii}^x)\ \hat{h}^x \circ i_0 = f \quad \text{auf } V_x. \tag{4}$$

(a) Let $(x,t) \in X \times [0,1]$ be arbitrary, $U_{x,t} \subseteq B$ a trivially covered neighbourhood of $h(x,t) \in B$ and

$$\Phi_{x,t} : p^{-1}(U_{x,t}) \to U_{x,t} \times F_{x,t} \qquad (F_{x,t} \text{ discrete})$$

a homeomorphism compatible with the projection $\pi_{U_{x,t}}$ as in (1). Since h is continuous, we find an open neighbourhood $V_{x,t} \subseteq X$ of x and an interval $I_{x,t}$ that is open with respect to the unit interval $[0,1]$ such that

$$t \in I_{x,t} \qquad \text{and} \qquad h\big(V_{x,t} \times I_{x,t}\big) \subseteq U_{x,t}.$$

As $[0,1]$ is compact, there exist finitely many

$$t_1, \ldots, t_m \subseteq [0,1] \qquad \text{s.t.} \qquad [0,1] \subseteq I_{x,t_1} \cup \ldots \cup I_{x,t_m}.$$

Let $\varepsilon > 0$ be a Lebesgue number for this cover and $n \in \mathbb{N}$ such that $1/n < \varepsilon$. Then by definition of the Lebesgue number, for each $0 \le i \le n-1$, there exists a $j = j(i) \in \{1, \ldots, m\}$ such that

$$\begin{aligned} \big[i/n, (i+1)/n\big] &= \bar{B}_{1/n}\big((2i+1)/(2n)\big) \\ &\subseteq B_\varepsilon\big((2i+1)/(2n)\big) \\ &\subseteq I_{x,t_{j(i)}}. \end{aligned}$$

Now $V_x := V_{x,t_1} \cap \ldots \cap V_{x,t_m}$ is an open neighbourhood of x, that satisfies

$$h\Big(V_x \times \big[i/n, (i+1)/n\big]\Big) \subseteq h\left(V_{x,t_{j(i)}} \times I_{x,t_{j(i)}}\right) \subseteq U_{x,t_{j(i)}}. \tag{5}$$

(b) Let now $x \in X$ be fixed and V_x chosen as in (a). We define inductively $\hat{h}^x : V_x \times [0,1] \to E$ over

$$\big[0,0\big],\ \big[0,1/n\big],\ \big[0,2/n\big],\ \ldots,\ \big[0,(n-1)/n\big],\ \big[0,1\big].$$

Induction basis: Define $\hat{h}^x(\cdot,0) := f(\cdot)$ on V_x. Then here holds (4)(i^x) on $V_x \times [0,0]$ and (4)(ii^x).

Induction step: Let $\hat{h}^x$ already be defined on $V_x \times [0, i/n]$, continuous and satisfying (4)(i^x) on $V_x \times [0, i/n]$. For

$$(\xi, t) \in V_x \times \big[i/n, (i+1)/n\big],$$

$h(\xi,\tau)$ is contained in the trivially covered neighbourhood $U_{x,t_{j(i)}}$ by virtue of (5). Then set

$$k := \pi_{F_{x,t(j)}}\big(\Phi_{x,t_{j(i)}}\big(\hat{h}^x(\xi, i/n)\big)\big) \in F_{x,t_{j(i)}},$$

and

$$\hat{h}^x(\xi,\tau) := \Phi^{-1}_{x,t_{j(i)}}\big(h(\xi,\tau) \times \{k\}\big).$$

Now $\hat{h}^x$ is continuously defined on $V_x \times [0,(i+1)/n]$ and satisfies (4)(i^x) on $V_x \times [0,(i+1)/n]$.

SECOND PART (EXISTENCE): Now we choose for each $x \in X$ a V_x and $\hat{h}^x$ as in the first part. For x_1, $x_2 \in X$ and arbitrary $\xi \in V_{x_1} \cap V_{x_2}$, there follows

$$(p \circ \hat{h}^{x_1})(\xi, \cdot) \overset{(4)(i)}{=} h(\xi, \cdot) \overset{(4)(i)}{=} (p \circ \hat{h}^{x_2})(\xi, \cdot),$$

and

$$\hat{h}^{x_1}(\xi, 0) \overset{(4)(ii)}{=} f(\xi) \overset{(4)(ii)}{=} \hat{h}^{x_2}(\xi, 0),$$

i.e. $\hat{h}^{x_i}(\xi, \cdot)$ are liftings of $h(\xi, \cdot)$, that coincide at one point, namely $\tau = 0$. Since $[0, 1]$ is connected, proposition 2.3 implies $\hat{h}^{x_1}(\xi, \cdot) = \hat{h}^{x_2}(\xi, \cdot)$, therefore $\hat{h}^{x_1} = \hat{h}^{x_2}$ on $(V_{x_1} \cap V_{x_2}) \times [0, 1]$, because ξ was arbitrarily chosen. Consequently, the map

$$h : X \times [0, 1] \to E, \ (\xi, t) \mapsto \hat{h}^x(\xi, t) \quad \text{for } \xi \in V_x$$

is well defined and has got all the desired properties.

THIRD PART (UNIQUENESS): Uniqueness follows from proposition 2.3. ■

The reader may find alternative proofs of the precedent two propositions in the book [7].

3 Two topological propositions

3.1 Proposition. *Let $f : X \to Y$ be a continuous, locally injective map of a compact metric space (X, d) into a topological space (Y, τ) and $A \subseteq X$ a closed set. If $f|_A$ is injective, there exists an open neighbourhood U with the property $A \subseteq U \subseteq X$, such that $f|_U$ is injective.*

Proof: We assume the converse. Then we consider

$$U_n := \bigcup_{x \in A} B^X_{1/n}(x) = \{x \in X : \ \mathrm{dist}(x, A) < 1/n\}$$

for $n \in \mathbb{N}$. These are X-open neighbourhoods of A, thus by assumption, there exist for each $n \in \mathbb{N}$ points

$$x^n_1, \, x^n_2 \in U_n \qquad \text{s.t.} \qquad x^n_1 \neq x^n_2, \qquad \text{but} \qquad f(x^n_1) = f(x^n_2).$$

By choosing an appropriate subsequence – by virtue of the compactness of X – and relabeling if necessary, we may assume

$$x^n_1 \overset{n\to\infty}{\longrightarrow} x_1, \qquad x^n_2 \overset{n\to\infty}{\longrightarrow} x_2 \tag{6}$$

with certain elements x_1, $x_2 \in X$. Because of

$$\mathrm{dist}(x^n_j, A) \overset{n\to\infty}{\longrightarrow} 0$$

there must hold x_1, $x_2 \in A$, since A is closed. Due to the continuity of f, we conclude

$$f(x_1) \overset{\infty\leftarrow n}{\longleftarrow} f(x^n_1) = f(x^n_2) \overset{n\to\infty}{\longrightarrow} f(x_2),$$

thus $x := x_1 = x_2$, since the restriction $f|_A$ is injective according to our premise. Due to the local injectivity of f, we find an X-open neighbourhood V of x, on which f is injective. But if we choose $n_0 \in \mathbb{N}$ large enough, we may achieve that $x^{n_0}_1$, $x^{n_0}_2 \in V$ because of the convergence (6). And, by virtue of

$$x^{n_0}_1 \neq x^{n_0}_2, \qquad f(x^{n_0}_1) = f(x^{n_0}_2),$$

we have a contradiction to the injectivity of $f|_V$. ■

3.2 Proposition. *Let $\Omega \subseteq \mathbb{C}$ be open, G a bounded simply connected domain such that $\bar{G} \subseteq \Omega$ and*

$$f : \Omega \to \mathbb{C}$$

a local homeomorphism that is injective on the boundary ∂G. Then there holds:

(i) f is injective on whole $\bar{G}$.

(ii) There exists an open neighbourhood $G \subseteq \tilde{\Omega} \subseteq \Omega$ such that

$$f|_{\tilde{\Omega}} : \tilde{\Omega} \to f(\tilde{\Omega})$$

is a global homeomorphism.

Proof: ∂G is the trace of a continuous simply connected curve. As $f|_{\partial G}$ is injective,

$$f(\partial G) \subseteq \mathbb{C} \qquad \text{is a Jordan curve}$$

as well. This separates the plane $\mathbb{C}$ into an interior U and an exterior U^∞, by virtue of the Jordan curve theorem 2.1, and is common boundary of both domains

$$\mathbb{C} = U \mathbin{\dot\cup} f(\partial G) \mathbin{\dot\cup} U^\infty, \qquad \partial U = \partial U^\infty = f(\partial G). \tag{7}$$

PART A. The image $f(G)$ is

- open (since local homeomorphisms are open maps),
- bounded (as a subset of the compactum $f(\bar{G})$),
- connected (as continuous image of a connected set).

We show, that $f(G)$ is even *simply* connected by verifying the equalities

$$f(\partial G) = \partial f(G), \qquad U = f(G). \tag{8}$$

Herein $f(G)^c := \mathbb{C} \setminus f(G)$ denotes the complement of $f(G)$ in the complex plane $\mathbb{C}$. We proceed in five steps.

(1) There holds

$$\partial f(G) \subseteq f(\partial G). \tag{9}$$

For that purpose, let $w \in \partial f(G)$. Thus, there exits a sequence $(z_n) \subseteq G$ such that $f(z_n) \xrightarrow{n} w$. As G is relatively compact, there exists $z \in \bar{G} = G \mathbin{\dot\cup} \partial G$ such that $z_n \xrightarrow{n} z$. By virtue of the continuity of f, we conclude $f(z_n) \xrightarrow{n} f(z)$, especially $w = f(z)$. We assume $z \in G$. Then there exists an open neighbourhood $V \subseteq G$ of z such that $f|_V$ is injective. Then $f(V)$ is an open neighbourhood of $f(z)$. But then $w \in f(V) \subseteq f(G)$, in contradiction to $w \in \overline{f(G)} \setminus f(G)$, where we note that $f(G)$ is open. Thus $z \in \partial G$, and therefore $w = f(z) \in f(\partial G)$.

(2) There holds

$$U^\infty \cap f(G) = \emptyset. \tag{10}$$

Otherwise, there exists a $z \in G$ such that $f(z) \in U^\infty$. As U^∞ is unbounded, there exists a continuous function $\gamma : [0,1) \to U^\infty$ such that $\gamma(0) = f(z)$ and $\lim_1 \gamma = \infty$. Then we have

$$t^* := \sup \big\{ t \in [0,1) : \ \gamma(t) \in f(G) \big\} < 1,$$

since $f(G)$ is bounded. Let $(t_n) \subseteq [0, t^*)$ such that $t_n \xrightarrow{n} t^*$ and $z_n \in G$ such that $f(z_n) = \gamma(t_n)$. Since G is relatively compact, we may assume that the (z_n) converge in $\bar{G}$. Because of the continuity of f there holds $\gamma(t_n) \xrightarrow{n} \gamma(t^*)$. Consequently, there holds

i. on the one hand $\gamma(t^*) \in \overline{f(G)}$,

ii. on the other hand $\gamma\big((t^*,1)\big) \subseteq \mathbb{C}\setminus f(G)$, thus $\gamma(t^*) \in \overline{\mathbb{C}\setminus f(G)}$.

We have derived the contradiction

$$\gamma(t^*) \in \overline{f(G)} \cap \overline{\mathbb{C}\setminus f(G)} \cap U^\infty = \partial f(G) \cap U^\infty \overset{(9)}{\subseteq} f(\partial G) \cap U^\infty \overset{(7)}{=} \emptyset.$$

(3) There holds

$$f(G) \cap U \neq \emptyset, \tag{11}$$

because (7) and (10) imply $f(G) \subseteq f(\partial G) \,\dot\cup\, U$. But $f(G) \subseteq f(\partial G)$ is impossible, since $f(G)$ is open.

(4) There holds

$$f(\partial G) \subseteq \partial f(G). \tag{12}$$

Otherwiese, there exists $w \in f(\partial G)$ such that $w \notin \partial f(G)$. Then there exists a ball $B = B_\varepsilon(w)$ such that either $B \subseteq f(G)$ or $B \subseteq f(G)^c$.

i. $B \subseteq f(G)$ implies the contradiction $\emptyset \neq B \cap U^\infty \subseteq f(G) \cap U^\infty = \emptyset$, since B is a neighbourhood of $w \in f(\partial G) = \partial U^\infty$.

ii. In the case $B \subseteq f(G)^c$, we connect a point of $U \cap f(G)$ – which exists by (11)! – by a continuous curve $\gamma : [0,1] \to U$ to w. Through a completely analogeous supremum argument as in part A(2), we find a point in $\partial f(G) \cap U$. And we have derived the contradiction

$$\emptyset \neq \partial f(G) \cap U \overset{(9)}{\subseteq} f(\partial G) \cap U \overset{(7)}{=} \emptyset.$$

(5) From the equality $\partial f(G) = f(\partial G)$, there now follows easily $U = f(G)$ in (8).

So, both set equalities in (8) are proved.

Part B. We show that

$$F := f\big|_G : G \to f(G) \tag{13}$$

is a covering. To this end, let $u \in f(G)$ be arbitrarily given. Then $F^{-1}(\{u\})$ is finite, since otherwise the elements would have a cluster point in the the compact set $\bar G$, which would contradict the local injectivity of f. Let

$$F^{-1}\big(\{u\}\big) = \big\{z_1, \ldots, z_n\big\}$$

with pairwise distinct z_j. Choose $\delta > 0$ such that

- all $B_\delta(z_j) \subseteq G$,
- all $f\big|_{B_\delta(z_j)}$ are local homeomorphisms,
- the $B_\delta(z_j)$ are pairwise disjoint.

Then let $\varepsilon > 0$ be such that $B_\varepsilon(u) \subseteq U$ and

$$F^{-1}\big(B_\varepsilon(u)\big) \subseteq B_\delta(z_1) \,\dot\cup\, \ldots \,\dot\cup\, B_\delta(z_n). \tag{14}$$

Such an ε exists, since otherwise, we could find for each sequence $(\varepsilon_k)_{k\in\mathbb{N}}$ in $(0,\infty)$, that converges to zero, a sequence

$$(\xi_k)_{k\in\mathbb{N}} \subseteq \bigcap_{j=1}^{n} \big(G \setminus B_\delta(z_j)\big)$$

such that $F(\xi_k) \in B_{\varepsilon_k}(u)$. We may assume – maybe by choosing an appropriate subsequence – that $\xi := \lim_k \xi_k \in \bar{G}$ exists, since $\bar{G}$ is compact. There follows

$$f(\xi) = \lim_k f(\xi_k) = \lim_k F(\xi_k) = u.$$

Since $\xi \in \partial G$ is impossible due to (8), there follows $\xi \in F^{-1}(\{u\})$, which cannot be true, as the limit must satisfy $|\xi - z_j| \geq \delta$.
Now, $B_\varepsilon(u)$ is a trivially covered neighbourhood of u, since it is easy to see with the aid of (14) that the map

$$F^{-1}\big(B_\varepsilon(u)\big) \to B_\varepsilon(u) \times \big\{1, \ldots, n\big\}, \quad z \mapsto \big(f(z), j\big) \quad \text{if } z \in B_\delta(z_j)$$

is a homeomorphism compatible with the projection

$$B_\varepsilon(u) \times \big\{1, \ldots, n\big\} \ni (\zeta, j) \mapsto \zeta \in B_\varepsilon(u).$$

Part C. Now we show the injectivity of F from (13). If this would not be the case, there exist

$$z_0 \neq z_1 \in G \qquad \text{s.t.} \qquad w := F(z_0) = F(z_1).$$

Since G is pathwise connected, we may connect z_0 to z_1 by a continuous curve

$$\gamma : [0,1] \to G, \qquad \gamma(0) = z_0, \quad \gamma(1) = z_1.$$

Then, $F \circ \gamma$ is a closed curve in $f(G)$ with the property

$$(F \circ \gamma)(0) = w = (F \circ \gamma)(1).$$

As U is simply connected due to part A, there exists a homotopy $h : [0,1] \times [0,1] \to f(G)$ with the property

$$h(t,0) = (F \circ \gamma)(t), \qquad h(t,1) = w, \qquad h(0,s) = w = h(1,s). \tag{15}$$

Since F is a covering by virtue of part B, proposition 2.4 gives us a lifting $\hat{h} : [0,1] \times [0,1] \to G$ with

$$\text{(i) } F\big(\hat{h}(t,s)\big) = h(t,s), \qquad \text{(ii) } \hat{h}(t,0) = \gamma(t). \tag{16}$$

1. On the one hand, this implies

   $$F\big(\hat{h}(0,s)\big) \overset{(16)}{=} h(0,s) \overset{(15)}{=} w \overset{(15)}{=} h(1,s) \overset{(16)}{=} F\big(\hat{h}(1,s)\big).$$

   As F is injective on a neighbourhood of z_0 resp. z_1, there follows

   $$\hat{h}(0,s) = z_0, \qquad \hat{h}(1,s) = z_1.$$

2. On the other hand, we have

   $$F\big(\hat{h}(t,1)\big) \overset{(16)}{=} h(t,1) \overset{(15)}{=} w.$$

   Therefore, $\hat{h}(\cdot,1)$ is a path from z_0 to z_1 with $F\big(\hat{h}(\cdot,1)\big) \equiv w$.

But this contradicts the local injectivity of F. Therefore, F has to be totally injective.

Part D. We prove the assertions (i) and (ii) of the proposition.

(i) The injectivity of $f|_{\bar{G}}$ follows directly from the injectivity of the functions $f|_{\partial G}$, which is granted by premise, and $F = f|_G$, see part C, together with (8) from part A.

(ii) Let $h := \operatorname{dist}(\bar{G}, \partial\Omega)$. Then,

$$\Omega' := \{z \in \mathbb{C} : \operatorname{dist}(z, \bar{G}) < h\} \qquad \text{is open with} \qquad \Omega' \subset\subset \Omega.$$

The assertion now follows easily with help of proposition 3.1.

The proof is now finished. ■

With the aid of proposition 3.2, the cited conclusion in section 1 is in fact possible, no doubt. The reader should verify this.

3.3 Theorem. *The surface of Costa [2], Hoffman and Meeks [3] is – in fact – embedded into the three dimensional Euklidian space.*

3.4 Exercise. Prove proposition 3.2 for higher dimensions.

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