TECHNISCHE UNIVERSITÄT
KAISERSLAUTERN
Fachbereich Mathematik

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INFLATION-LINKED PRODUCTS AND OPTIMAL INVESTMENT WITH MACRO DERIVATIVES

Vom Fachbereich Mathematik der Technischen Universität Kaiserslautern zur Verleihung des akademischen Grades Doktor der Naturwissenschaften (Doctor rerum naturalium, Dr. rer. nat.) genehmigte Dissertation

Datum der Disputation: 5. Dezember 2006

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TO MY DEAR MUM AND DAD

ABSTRACT

In this thesis diverse problems concerning inflation-linked products are dealt with. To start with, two models for inflation are presented, including a geometric Brownian motion for consumer price index itself and an extended Vasicek model for inflation rate. For both suggested models the pricing formulas of inflation-linked products are derived using the risk-neutral valuation techniques. As a result Black and Scholes type closed form solutions for a call option on inflation index for a Brownian motion model and inflation evolution for an extended Vasicek model as well as for an inflation-linked bond are calculated. These results have been already presented in Korn and Kruse (2004) [17]. In addition to these inflation-linked products, for the both inflation models the pricing formulas of a European put option on inflation, an inflation cap and floor, an inflation swap and an inflation swaption are derived.

Consequently, basing on the derived pricing formulas and assuming the geometric Brownian motion process for an inflation index, different continuous-time portfolio problems as well as hedging problems are studied using the martingale techniques as well as stochastic optimal control methods. These utility optimization problems are continuous-time portfolio problems in different financial market setups and in addition with a positive lower bound constraint on the final wealth of the investor. When one summarizes all the optimization problems studied in this work, one will have the complete picture of the inflation-linked market and both counterparts of market-participants, sellers as well as buyers of inflation-linked financial products. One of the interesting results worth mentioning here is naturally the fact that a regular risk-averse investor would like to sell and not buy inflation-linked products due to the high price of inflation-linked bonds for example and an underperformance of inflation-linked bonds compared to the conventional risk-free bonds. The relevance of this observation is proved by investigating a simple optimization problem for the extended Vasicek process, where as a result we still have an underperforming inflation-linked bond compared to the conventional bond.

This situation does not change, when one switches to an optimization of expected utility from the purchasing power, because in its nature it is only a change of measure, where we have a different deflator. The negativity of the optimal portfolio process for a normal investor is in itself an interesting aspect, but it does not affect the optimality of handling inflation-linked products compared to the situation not including these products into investment portfolio.

In the following, hedging problems are considered as a modeling of the other half of inflation market that is inflation-linked products buyers. Natural buyers of these inflation-linked products are obviously institutions that have payment obligations in the future that are inflation connected. That is why we consider problems of hedging inflation-indexed payment obligations with different financial assets. The role of inflation-linked products in the hedging portfolio is shown to be very important by analyzing two alternative optimal hedging strategies, where in the first one an investor is allowed to trade as inflation-linked bond and in the second one he is not allowed to include an inflation-linked bond into his hedging portfolio. Technically this is done by restricting our original financial market, which is made of a conventional bond, inflation index and a stock correlated with inflation index, to the one, where an inflation index is excluded.

As a whole, this thesis presents a wide view on inflation-linked products: inflation modeling, pricing aspects of inflation-linked products, various continuous-time portfolio problems with inflation-linked products as well as hedging of inflation-related payment obligations.

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ACKNOWLEDGMENTS

This Ph.D. thesis has greatly benefited from the comments and suggestions of several people. First of all, the author would like to thank his supervisor, Professor Dr. Ralf Korn, for his excellent assistance and guidance in the research process as well as in the preparation of this manuscript. The author also gratefully acknowledges Junior Professor Dr. Holger Kraft and Dr. Jürgen Amendinger for their valuable comments and suggestions. The author wishes to express sincere appreciation of the hospitality of the Graduate College "Mathematics and Practice" at the Department of Mathematics of the Technical University of Kaiserslautern, Germany. In addition, special thanks to Dr. Susanne Kruse whose professional knowledge and willingness to help was supportive during the process of making this scientific research.

The author also would like to thank Ph.D. Nikolai Botkin, Dr. Markus Junker and the group of financial mathematics at the Department of Mathematics of the Technical University of Kaiserslautern for their valuable input and intellectual support. Last but not least, I would like to show gratitude to dear Svitlana Stotska for her precious comments and excellent support in helping me to realize some important aspects in my research area as well as for her great encouragement during the whole Ph.D. research time. In addition, I want to thank my beloved parents for their exceptional support and excellent assistance during my whole life – you are just the best.

Kaiserslautern, 8 August 2006,

Taras Beletski

Chapter 1

INTRODUCTION

1 Nature of Inflation

Recent economic developments have shown that neither equity nor interest rates have been effective in protecting against purchasing power erosion. Due to the downturn, equity markets have no longer been able to provide an over-hedge to inflation, as has historically been the case, which showed that banks should be able to handle risk connected to cyclical fluctuations such as systematic credit risk or obsolescence due to inflation. Furthermore, the rate policy of the European Central Bank has not helped curb the very high inflation that many European countries have experienced. Consequently we have seen a decorrelation between inflation and other financial instruments, and a subsequent heightening in awareness by consumers, investors, and issuers that their inflation exposures need to be tackled with inflation protection, not with external proxy hedges. As a result, inflation has started trading as a separate asset class to interest rates and equities, and has experienced a sharp growth over the last years. A short discussion of the advantages of economic derivatives is given by Elhedery (2003) [10] where he underlines the importance of these financial products.

1.1 Definition of Inflation

The most common form of inflation discussed and referred to in the financial markets is the percentage increase/decrease of Consumer Price Indices (CPI) or Retail Price Indices (RPI). These indices are constructed from the price of a basket of goods and services deemed representative of the consumption patterns of households in a geographical area, and are published monthly by national statistics institutes. The Euroland Harmonized CPI, reflecting the rate of inflation in the Euro Area, is aggregated and published by the European Statistics Institute Eurostat.

Other variants of price indices focus on, or exclude, certain items, such as UK RPIX, which excludes mortgage interest payment, and Euroland CPI ex-tobacco, which excludes price changes (often due to taxes) on tobacco. Finally some indices concentrate on a category of the society rather than the whole community, as per the CPI for Urban Consumers in the US, and the CPI for employees and workers in Italy.

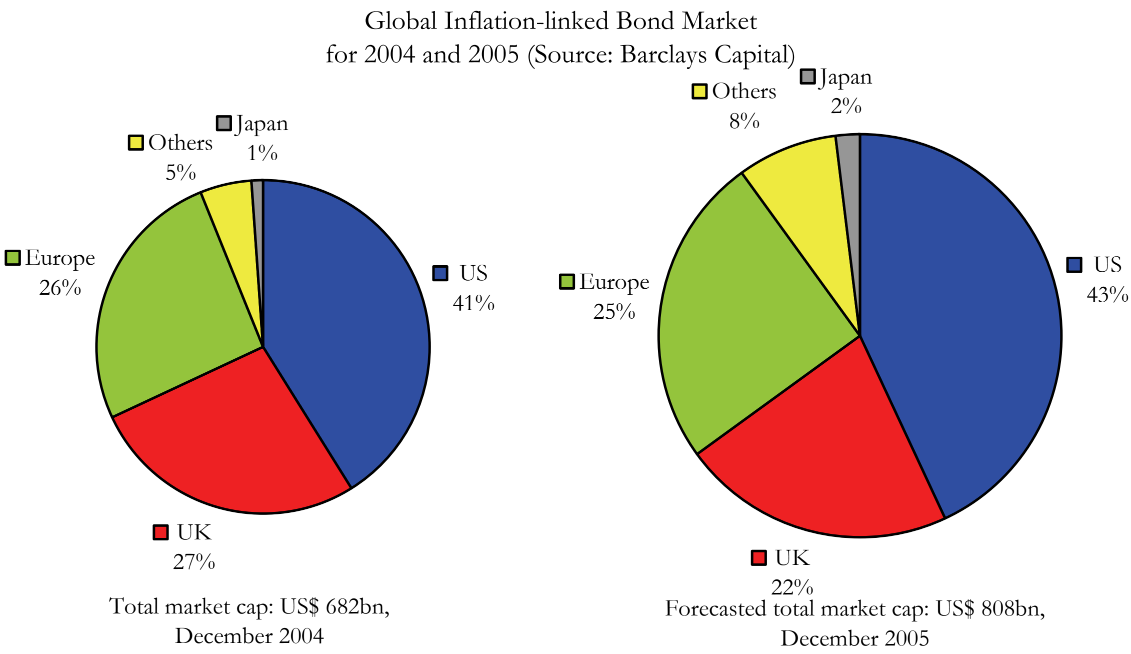

Figure 1.1 Global inflation-linked bond market value in 2004 and 2005, source Barclays Capital.

The major traded inflation indices are the Harmonized Euroland CPI ex-tobacco, UK RPI, French CPI ex-tobacco and US CPI for urban consumers, primarily as a result of the existence of sovereign bonds linked to these underlying indices.

1.2 Inflation-linked Government Bonds

Sovereign issuers were historically the first and main providers of inflation in the financial markets. Two main reasons led governments to issue inflation-linked bonds in several countries:

- To signal to the market the government's commitment to curb high inflation: although this reason is a rather historical one for European countries (mainly the UK in the eighties), it is still a major driver in countries experiencing high inflation, such as Mexico and South Africa.

- Asset liability management: government income is linked to inflation, either directly (e.g. through VAT), or through taxes linked to economic activity, as inflation is usually higher during an expansion phase of the economy (e.g. corporate tax).

The main sovereign issuers of inflation-linked bonds have been the US, UK, France, Sweden, Australia, Canada, New Zealand, South Africa, and recently Greece. The amount of inflation linkers outstanding exceeds US$ 279 billion for the US, the largest market so far (see Figure 1.1).

Most governments have been active in building a yield curve through issuing in different maturities. A broader maturity spectrum enables better duration management for buyers of inflation. At this point in time, a gap exists at the short-end of the Euroland inflation curve, as linkers issued by the French and Greek governments span the longer term.

Due to large size and issuance commitments, sovereign inflation bonds are being used as benchmarks for the computation of inflation expectations. They play an important role in providing the liquidity and transparency that the market needs to develop rapidly and attract new participants both on the supply side and on the investor side.

1.3 Other Inflation Sources

A number of other economic entities (in addition to the sovereigns) have also driven the market for inflation, by virtue of either being long or short the asset class, either directly or indirectly. The longs with direct exposure are those entities with revenues directly linked to inflation, such as state owned monopolies with tariffs linked to inflation, e.g. gas, water, electricity distribution, or rail infrastructure. Real-estate companies with rents linked to inflation fit in this category as well. The longs with an indirect exposure comprise sovereign issuers as discussed previously, but also regions, agencies or municipalities with the same revenue profile as governments. Finally, large retailers such as supermarkets have revenues linked to inflation.

The shorts with direct exposure include pension funds or insurance companies with liabilities indexed to inflation or wage indices, or companies and agencies with wage bills directly linked to consumer prices. The shorts with indirect exposure include retail aggregators, in particular where individual investors are sensitive to real money protection/purchasing power preservation.

The longs will be naturally driven to offset their exposure to inflation, for the purpose of removing the uncertainty associated with inflation revenues and replacing it with fixed revenues; they thus become the natural payers, or providers, of inflation. On the other hand, the shorts become the natural receivers of inflation.

1.4 Market of Inflation-linked Products

On the current market of inflation-linked products three major categories of inflation-linked products could be clearly distinguished: inflation-linked bonds, inflation swaps and inflation-structured products. Ben-Hamou (2003) [4].

1.4.1 *Inflation-linked Bonds*

Inflation-linked bonds enjoy great popularity as they offer natural protection against inflation accreting assets. This is a rapidly growing market with some government bond issue of more than 10 to 15 billion dollars. The most liquid markets are France with the OAT*i*, CADES*i* bonds and the OAT€*i*s (the Eurozone inflation-linked bonds), the US with the TIPS bonds and the UK with I/L Gilts. Various utilities and banks have also issued inflation linked corporate bonds. Because of the delay in publication of inflation reference numbers by the various statistics agencies, the inflation reference is lagged by some months (typically, 3 months for French OAT*i*s and OAT€*i*s, and 8 months for UK I/L Gilts).

Although liquidity and ability to asset swap inflation bonds have been growing, the inflation-linked bond market is still relatively small and suffers from its illiquidity. In reaction, governments have been increasing outstanding amount of their issue. Pension funds and insurers (especially in the UK) have swallowed some of the issues, narrowing considerably the market. In addition, risk appetite for asset swaps on inflation-linked bond is still low.

1.4.2 *Inflation Swaps*

The inflation market for these products is still not completely unified as it is still at the early development stage. Inflation swaps consist of an exchange of a payment stream growing at a fixed rate of interest, swap rate, for a payment stream developing at a floating rate, inflation rate. Inflation swap could have one of the following three forms: zero coupon inflation swap, revenue inflation swap and year-on-year inflation swap.

Inflation swaps are very attractive to client as they are highly customizable due to the existing ability to specify the appropriate inflation index, ability to match the risk profile in term of the structure, and great accounting benefit as this can easily qualify as a hedging trade. Zero coupon inflation swaps bear a substantial credit risk exposure compared to revenue inflation and year-on-year inflation swaps as it pays a single payment at maturity.

1.4.3 *Inflation-structured Products*

Inflation-structured products are tailor-made products addressing specific clients' preoccupations. For instance, common structures are:

- Standard vanilla like caps, floors and swaptions (swap options) on inflation.

- Event-trigger structures on inflation provide the for right to enter into an inflation-linked note depending on a third party asset like price of oil or a commodity index. Useful for utility companies and cheaper than standard structures.

- Yield enhancement or financing cost reducing inflation structure can offer substantially lower financing cost (on the liability side) or increase the yield (asset side) for investors that issue an inflation-linked note, selling the option to enter into a payer inflation swap. Useful for inflation-linked issuer or for inflation-linked investors.

- Callable inflation bonds offer the right to an investor to call back the inflation-linked issue at a set of early redemption dates to protect him against an inflation surge. As a matter of fact, this can just be seen as a Bermudan inflation swap.

- Spread option on inflation pays the spread between two inflation indexes, with some capped and or floor. Useful as a hedge against the inflation rates differential between different countries. Other slight variations of the inflation spread options are structures paying the maximum of two inflation rates.

- And many more exotics like quanto inflation structure, all the standards correlation hybrids products structured on inflation and other assets, like equity stock, foreign exchange, fixed income, commodity and credit indexes, and real capital guaranteed products, where the real notional is protected.

2 Overview of the Research Problem

The inflation market has experienced exponential growth over the last years, and the number of participants is growing by the day. Pension reforms in Europe should bring a new powerful impetus to this market. The challenge for the future is to increase liquidity and provide creative and flexible structured solutions that fit the specific needs of different participants; it is the diversity of the inflation market that will allow it to compete with, and complete, the very established interest rate market. Elhedery (2003) [10].

2.1 Background

Inflation has been extensively studied in economics, especially in the field of actuarial sciences and of course in macro economics. Pioneering work in macro economics on inflation can be traced back to Fisher (1930) [11], "The Theory of Interest", where Fisher forms his famous hypothesis that the nominal interest rates should vary closely with the movements in expected inflation. This hypothesis connects two distinct parts of economy – inflation rates express changes in the supply-demand conditions on the commodity spot market while nominal interest rates reflect differences in supply-demand conditions on the money market. Several empirical studies support Fisher's hypothesis by either analyzing market data on inflation-linked bonds in the UK, the US or Canada or using available survey data (Ang and Bekaert (2003) [2] and Nielsen (2003) [19]).

In their empirical study Ang and Bekaert (2003) [2] investigate the connection between nominal and real interest rates and inflation. Changes in nominal interest rates must be due to either movements in real interest rates, expected inflation, or the inflation risk premium. They develop a term structure model with regime switches, time-varying prices of risk and inflation to identify these components of the nominal yield curve. They find that the unconditional real rate curve is fairly flat at 1.44%, but slightly humped. In one regime, the real term structure is steeply downward sloping. Real rates (nominal rates) are pro-cyclical (counter-cyclical) and inflation is negatively correlated with real rates. An inflation risk premium that increases with the horizon fully accounts for the generally upward sloping nominal term structure. Ang and Bekaert (2003) [2] find that expected inflation drives about 80% of the variation of nominal yields at both short and long maturities, but during normal times, all of the variation of nominal term spreads is due to expected inflation and inflation risk. Due to the used data on nominal yields and inflation they conclude that expected inflation mainly drives the variation of nominal yields.

Instead of using market information on inflation-linked bonds Nielsen (2003) [19] extracts inflation expectations from the Consumer Survey, which is conducted by the European Commission for the European Union. The suitability of different probability distributions is analyzed to allow for different skewness and kurtosis deviating from the normal distribution, but empirical analysis suggests that no improvement in forecasting is gained by choosing a distribution different from the normal one.

Empirical research does not support the hypothesis that rates of inflation are constant over time or that there exists a long-run mean towards which current rates will regress over time, which does not rule out the possibility of long-run reversion to a time-dependent mean. The variability of yearly measured inflation rates varied widely over the last century, there was some evidence that high variability can be associated with periods of high inflation. From an economic point of view analyses of interest rate volatility can also be used to gauge inflation rate variability, since the level of interest rates provides an indicator of inflation expectations. Beside Fisher's hypothesis there exists a variety of different models describing the relation between nominal interest, real interest and inflation such as the Taylor rule and the forward rate rule.

Gerlach and Schnabel (1999) [12] examine the validity of the so-called Taylor rule that nominal interest rates are closely related to average output gaps (which is the difference between the economy's actual output and the level of production it could achieve with existing labor, capital and technology without putting sustained upward pressure on inflation) and inflation rates. They come to the conclusion that if the European Central Bank were to conduct monetary policy using the Taylor rule during the period from 1990 to 1998, it would in fact not deviate much from past (weighted) interest rate setting behavior in the countries forming the European Monetary Union area.

Forward interest rates have become popular indicators of inflation expectations. The usefulness of this indicator depends on the relative volatility and the correlation of inflation expectations and expected real interest rates. In his paper Söderlind (1995) [22] introduces the forward rate rule saying that there exists a linear relation between the expected inflation and the forward rates under the assumption of constant real interest rates. He studies US and UK data, using a range of different tools and data sets, and detects that the forward rate rule performs reasonably well, in spite of significant movements in the expected real interest rate. Söderlind (1995) [22] suggests that the reason for such a good performance is that the "noise" that movements in the expected real interest rate add to the inflation expectations and in this way is balanced by a tendency for expected real interest rates and inflation expectations to move in opposite directions.

In their paper Söderlind and Svensson (1997) [23] selectively survey recent methods to extract information about market expectations from asset prices for monetary policy purposes. Traditionally, interest rates and forward exchange rates have been used to extract expected means of future interest rates, exchange rates and inflation. These methods have been refined to rely on implied forward interest rates, so as to extract expected future time-paths. More recently, methods have been designed to extract not only the means, but also the whole (risk neutral) probability distribution from a set of option prices.

In their paper Atta-Mensah and Yuan (1998) [3] look at the forward rate rule introduced in Söderlind (1995) [22] that a linear relation between the expected inflation and the forward rates gives an optimal estimate for the expected inflation. This relation between forward rates and inflation values is used to extract expected future inflation from Canadian forward rates assuming a joint normal distribution between inflation expectations and expected real interest rates and fixed risk premiums, which prove to be small. The authors find that Canadian real interest rates are quite volatile, especially at the shorter end. Using daily observed forward rates they show how to construct a term structure of expected inflation. In addition they give an event study that following Central Bank's announcements expectations in future inflation immediately adapt.

In the field of actuarial sciences Wilkie (1986) [24] introduced a stochastic asset model, which models the random behavior of various economic series (including price and wage inflation, short and long interest rates, share yields and dividends, and exchange rates) over time. The motivation for developing the Wilkie model was the fact that under the alternative random walk model, inflation and interest rates have no "normal" level and can therefore reach and remain at very high (e.g. 30%) or very low levels (e.g. 0.5%) for long periods. History shows this to be unrealistic: instead both interest rates and inflation exhibit strong autoregressive behavior, under which "normal" levels do exist (for example 4% for price inflation and 7% for long term interest rates) around which actual levels fluctuate, with departures having a tendency to be corrected. The Wilkie model incorporates such autoregressive or mean-reverting behavior within its projections of economic series, and therefore recognizes that if inflation suddenly becomes "high" (e.g. at 30%), then it is more likely to come down from that level in future than to rise. In contrast, under the random walk level, inflation would be assumed to be as likely to rise as to fall from a high level, even from an extremely high level (e.g. 300%). Mathematically the Wilkie model is based on ARIMA models. Because of the unsatisfactory behavior of economic series (in particular of interest rates and price inflation over anything other than very short time periods) under the random walk model, the Wilkie model (and variants based on it) has become the most widely used stochastic asset model for actuarial use.

Brown and Schäfer (1994) [7] estimate real term structures from cross-sections of British government index-linked bond prices. The Cox, Ingersoll, and Ross (1985) [8] model is then fitted to the same data; the model closely approximates the shapes of the directly-estimated term structures. In contrast to similar studies of the nominal term structure, the long-term zero-coupon yield is quite stable, as the CIR model predicts, and in common with previous studies, the level of implied short rate volatility corresponds well with time series estimates. The other parameters, however, are often highly correlated and intertemporal parameter stability is rejected.

Woodward (1990) [25] estimates real interest rates and inflation expectations from market-observable data without assuming anything about the form of agents' expectation-formation mechanism, but considering the different taxation of conventional and indexed securities in the UK and using comparison by duration instead of maturity. He also analyzes the term structure of real interest rates and inflation expectations for a variety of maturities and how this structure shifts over time. Woodward (1990) [25] argues that in order to exploit the data on inflation-linked bonds fully, more observations need to accumulate. He concludes from the examined sample that the real interest rates can not be assumed constant and suggests that the inflation risk premium is relatively small.

2.2 Research Objectives

The central topic of this thesis is the concept of inflation-linked products. Due to the fact that these financial derivatives are newly introduced and form the new asset class, there is hardly any theoretical construction for modeling these products mathematically. Therefore, the main goal of this thesis is to approach the task of creating such a framework and solving the usual setup of problems for inflation-linked products that are already solved in classic theory of financial mathematic for other asset classes, interest rates and equity.

To start with, the appropriate stochastic model for inflation process is to be constructed, since inflation is the main driving component in the inflation-linked products. The examination of stochastic models is restricted to a geometric Brownian motion model for consumer price index (CPI) and extended Vasicek model for inflation rate. Basing on the introduced stochastic models the pricing formula for inflation-linked bonds is to be introduced with a help of a European call option on inflation. These results are in line with Korn and Kruse (2004) [17]. In addition, a European put option on inflation, an inflation cap and floor, an inflation swap and an inflation swaption are priced with a closed form solutions except for an inflation swaption under the extended Vasicek process for inflation rate.

Having the price processes for inflation derivatives, the question of optimal portfolio process arises. For that reason, the subject of optimal investment polices in different investment opportunities is analyzed under the geometric Brownian motion model except for the second problem. In the first case it is assumed that an investor has access to the market where an inflation-linked bond (risky asset) and a regular bond (risk-free asset) are traded. The second problem is analogical to the first one only here we assume an extended Vasicek process for inflation rate. In the third case the financial market is extend by introducing a risky stock as an investment opportunity for the investor; the negative correlation between an inflation-linked bond and a stock is assumed. Finally, in the fourth case the financial market is assumed to be the same as in the first case, an inflation-linked bond and a regular bond, but in the optimization problem a positive lower bound constraint on final wealth is introduced.

The analogical analysis is conducted for the continuous-time problem of optimization of the expected utility from the purchasing power of an investor under the geometric Brownian motion model. In addition to the continuous-time portfolio problems in this thesis the hedging of inflation-related future payment obligations is studied. Also for the hedging problems we assume the geometric Brownian motion model for inflation index. Furthermore, the question of including inflation-linked products into investment portfolio as well as hedging portfolio is addressed.

2.3 Research Problem

Even though inflation is of great interest for insurance economics and macro economics, there are not so many suitable models for financial pricing problems around. So far there is a range of publications, which focus on modeling the inflation process by statistical time series models such as ARIMA. In addition, inflation can be modeled by equilibrium models in quite different frameworks, such as for example discussed in Broll, Schweimayer and Welzel (2003) [6]. They use the industrial organization approach to the microeconomics of banking, which is based on a game theoretic mathematical model, augmented by uncertainty and risk aversion to examine credit derivatives and macro derivatives as instruments to hedge credit risk for a large commercial bank. In a partial-analytic framework Broll, Schweimayer and Welzel (2003) [6] distinguish between the probability of default and the loss given default, model different forms of derivatives, and derive hedge rules and strong and weak separation properties between deposit and loan decisions on the one hand and hedging decisions on the other. They also suggest how bank-specific macro derivatives could be designed from common macro indices which serve as underlyings of recently introduced financial products.

A couple of publications of Schweimayer *et al.* (2000) [20] and (2003) [21] give a basic idea on how economic derivatives could be priced using ARMA-GARCH models and how these can be used for risk management. Schweimayer and Wagatha (2000) [20] discuss the possible use of economic derivatives in the risk management in terms of an interconnectional banking framework. They are giving criteria how to choose the best underlying macro economic index depending on your portfolio structure and make advances on a possible product styling, for example in the case of a floor on unemployment and on a real estate index as well as a swap with two floating legs. In his Ph.D. thesis Schweimayer (2003) [21] mainly focuses on economic derivatives and their particular use in hedging macro economic risks connected to financial markets. He explains the design and choice of the derivatives and underlyings and their possible pricing by using an ARMA-GARCH model to cover the corresponding underlyings such as for example an inflation index.

The most suitable model for financial modeling purposes seems to be the one proposed in Jarrow and Yildirim (2003) [13] where they set up a three-factor Heath-Jarrow-Morton (HJM) model to price US Treasury Inflation-Protected Securities (TIPS) and a European call option on the related consumer price index, the CPI-U, in an arbitrage-free, complete market model. In their three-factor HJM model both the nominal and real interest rates follow a Vasicek model and the inflation index is given by a geometric Brownian motion. First, using the market prices of TIPS and ordinary US Treasury securities, both the nominal and real zero-coupon bond price curves are obtained using standard coupon-bond price stripping procedures. Next, a three-factor arbitrage-free term structure model is fit to the time series evolutions of the CPI-U and the nominal and real zero-coupon bond price curves. Then, using these estimated term structure parameters, the validity of the HJM model for pricing TIPS is confirmed via its hedging performance. Furthermore, the usefulness of the pricing model is illustrated by valuing call options on the inflation index.

2.4 Key Results

As a whole, this thesis presents a wide view on inflation-linked products: inflation modeling, pricing aspects of inflation-linked products, various continuous-time portfolio problems with inflation-linked products as well as hedging of inflation-related payment obligations. For both suggested models for inflation the pricing formulas of inflation-linked products are derived using the risk-neutral valuation techniques. As a result Black and Scholes type closed form solutions for a call option on inflation index for a Brownian motion model and inflation evolution for an extended Vasicek model as well as for an inflation-linked bond are calculated. In addition to these inflation-linked products, for the both inflation models the pricing formulas of a European put option on inflation, an inflation cap and floor, an inflation swap and an inflation swaption are derived.

Consequently, basing on the derived pricing formulas and assuming the geometric Brownian motion process for an inflation index, different continuous-time portfolio problems as well as hedging problems are studied using the martingale techniques as well as stochastic optimal control methods. When one summarizes all the optimization problems studied in this work, one will have the complete picture of the inflation-linked market and both counterparts of market-participants, sellers as well as buyers of inflation-linked financial products. One of the interesting results worth mentioning here is naturally the fact that a regular risk-averse investor would like to sell and not buy inflation-linked products due to the high price of inflation-linked bonds for example and an underperformance of inflation-linked bonds compared to the conventional risk-free bonds. The relevance of this observation is proved by investigating a simple optimization problem for the extended Vasicek process, where as a result we still have an underperforming inflation-linked bond compared to the conventional bond.

This situation does not change, when one switches to an optimization of expected utility from the purchasing power, because in its nature it is only a change of measure, where we have a different deflator. The negativity of the optimal portfolio process for a normal investor is in itself an interesting aspect, but it does not affect the optimality of handling inflation-linked products compared to the situation not including these products into investment portfolio.

In the following, we consider problems of hedging future inflation-indexed payment obligations with different financial assets and calculate the minimal quadratic hedging errors in various market setups. The role of inflation-linked products in the hedging portfolio is shown to be very important by analyzing two alternative optimal hedging strategies, where in the first one an investor is allowed to trade as inflation-linked bond and in the second one he is not allowed to include an inflation-linked bond into his hedging portfolio.

2.5 Structure of the Thesis

This thesis is organized as follows. The next chapter reviews the mathematical preliminaries on the theoretical background on inflation process. The third chapter contains preliminary studies including models for inflation index and pricing results for inflation-linked products under assumption of the presented inflation models. The fourth chapter presents the main studies and results in the following way. Section 6 deals with continuous-time portfolio problems for investor's wealth process. In section 7 the analogical continuous-time portfolio problems are studied but for the purchasing power of an investor. Hedging problems are analyzed in section 8. Finally, section 9 concludes.

Chapter 2

MATHEMATICAL PRELIMINARIES

3 Basics on Inflation and Inflation-linked Bonds

We first start by defining some basics on inflation, inflation rate, as well as by giving its mathematical definition. Also the concept of inflation-linked bonds is formalized. In addition, a short overview on how the inflation index is calculated in practice is given.

3.1 Inflation Rate

Inflation is defined as an index measuring the economic evolution of prices. The inflation rate is calculated as a relative change of consumer price index (CPI) or retail price index (RPI). Therefore, the simple inflation rate $i_S(.)$ for the time interval $[t_0, t]$ is calculated as

$$i_S(t_0, t) = \frac{I(t) - I(t_0)}{I(t_0)}, \tag{3.1.1}$$

where $I(t)$ is consumer price index at time t. On can easily see that by defining simple inflation rate in the way of (3.1.1), we have the following relation for $t_0 \leq s_1 \leq \ldots \leq s_n \leq t$

$$\frac{I(t)}{I(t_0)} = 1 + i_S(t_0, t) = (1 + i_S(t_0, s_1)) \cdot \ldots \cdot (1 + i_S(s_n, t)). \tag{3.1.2}$$

Noticing the analogously between inflation rate and interest rate theory, continuously compounded inflation rate $i_C(t_0, t)$ on the time interval $[t_0, t]$ can be defined as a solution to the following equation

$$\frac{I(t)}{I(t_0)} = e^{i_C(t_0,t)(t-t_0)}, \tag{3.1.3}$$

which actually gives us

$$i_C(t_0, t) = \frac{\ln(I(t)) - \ln(I(t_0))}{(t - t_0)}. \tag{3.1.4}$$

Therefore, we define instantaneous inflation rate $i(t)$ at time t similar to the way they define instantaneous short rate in the interest rate theory, i.e.

$$i(t) = \lim_{s\to t} i_C(t,s) = \lim_{s\to t} \frac{\ln(I(s)) - \ln(I(t))}{(s-t)} = \frac{d\ln(I(t))}{dt}, \tag{3.1.5}$$

consequently we have the inflation evolution of the form

$$\frac{I(t)}{I(t_0)} = \exp\left(\int_{t_0}^{t} i(s)ds\right). \tag{3.1.6}$$

In the forthcoming chapters we will model the instantaneous inflation rate (3.1.5), and therefore we will use a shortened notation to the term "instantaneous inflation rate", where we drop the word "instantaneous".

3.2 Fisher's Equation

The most known result from microeconomics in the area of inflation is Irvin Fisher's "Theory of Interest" (1930) [11]. A well-known Fisher's equation (3.2.1) illustrates the effect of the monetary policy on the level of nominal interest rates; it states that the difference between the nominal and real interest rates equals to the expected inflation rate

$$\tilde{r}_N(t) = \tilde{r}_R(t) + E[i_S(0,t)], \tag{3.2.1}$$

where $\tilde{r}_N(t)$ is the nominal effective interest rate for the bond maturing at time t, $E[i_S(0,t)]$ is the expected (simple) inflation rate for the time horizon t and $\tilde{r}_R(t)$ is the real effective interest rate for the bond with maturity t, which corresponds to the growth of real purchasing power in the case of investment with the nominal effective interest rate $\tilde{r}_N(t)$.

To get Fisher's equation (3.2.1) formally one writes the nominal effective interest rate $\tilde{r}_N(t)$ as a combination of the real effective interest rate $\tilde{r}_R(t)$ and the expected (simple) inflation $E[i_S(0,t)]$ in the following way

$$1 + \tilde{r}_N(t) = (1 + \tilde{r}_R(t))(1 + E[i_S(0,t)]) = 1 + \tilde{r}_R(t) + E[i_S(0,t)] + \tilde{r}_R(t)E[i_S(0,t)], \tag{3.2.2}$$

assumes the term $\tilde{r}_R(t)E[i_S(0,t)]$ to be relatively small and neglects it.

If one assumes the real interest rates to be constant, according to Fisher's equation (3.2.1) the nominal interest rates follow the movements of expected inflation rate, which is supported by the empirical studies of i.a. Ang and Bekaert (2003) [2] and Nielsen (2003) [19]. This means that in the mean the market participants make reasonably consistent predictions about the forthcoming inflation rate in a sense that in average the ex post real interest rate could be obtained with the help of the ex ante nominal interest rate.

3.3 Inflation-linked Bond

The structure of introduced inflation-linked bonds varies a little depending on the issuer. In this thesis studied inflation-linked bonds have the most typical structure, i.e. the same structure as the ones issued by Agency France Trésor, OAT€*i* (for detailed information see Agency France Trésor (2002) [1]).

Inflation-linked bonds are bonds with a fixed real interest rate. Their principal is both guaranteed at par and protected from inflation. This is due to the indexation of these bonds on a daily inflation reference. For instance, OAT€*i*s are indexed on the harmonized consumer price index for the Eurozone, excluding tobacco, which is published by Eurostat monthly. The annual coupon, called a "real coupon", is likewise protected against inflation, since it is calculated as a fixed percentage of the index-linked principal.

In other words, inflation-linked bonds pay a regular fixed coupon paid on a notional that accretes at the inflation rate. The final notional is also accreting at the inflation rate, although it is additionally floored at par. To summarize, payments made by bond issuer are the following ones.

1. Inflation-protected principal (guaranteed redemption at par) is linked to the consumer price index, providing the holder with protection from inflation. This indexation can be calculated every day according to a daily price index reference, but is only paid out by the issuer at the moment of the bond's redemption, on the basis of the following formula

$$F \max\left[\frac{I(T)}{I(t_0)}, 1\right], \tag{3.3.1}$$

   where F is the face value, $I(T)$ is the price index at the maturity date $T > 0$ and $I(t_0)$ is the level of the fixed price index at the reference time $t_0 \leq 0$, referred as the "base index". Usually the reference date is set to the bond's creation date, i.e. $t_0 = 0$. From the formula above one can clearly see that in the case of deflation throughout the life of the bond, the redemption amount equals the face value.

2. Inflation-linked coupon payments are fixed rates of the index-linked principal. Inflation-linked bonds pay out an annual coupon calculated as a fixed percentage of the index-linked principal. This fixed percentage, also called the "real coupon", is determined at the time of issue and remains fixed for the rest of the life of the bond. The coupon that the holder receives depends, thus, on the real coupon, as well as on the ratio between the daily inflation reference on the date of coupon payment and the daily inflation reference on the bond's initial reference date, according to the following formula

$$\sum_{i=1}^{n} c_i F \frac{I(t_i)}{I(t_0)} = \sum_{i=1}^{n} C_i \frac{I(t_i)}{I(t_0)}, \tag{3.3.2}$$

where c_i is a real coupon, C_i coupon payment before adjustment to the inflation and $0 < t_1 \leq \ldots \leq t_n = T$ for $i = 1, \ldots, n$.

Consequently, the payments that the holder of inflation-linked bond receives during its life are

$$\sum_{i=1}^{n} C_i \frac{I(t_i)}{I(t_0)} + F \max\left[\frac{I(T)}{I(t_0)}, 1\right]. \tag{3.3.3}$$

In this thesis we will most use an inflation-linked bond with the following structure for the illustration purposes

$$\begin{aligned} T &= 20, \qquad F = 100, \\ C_i &= 2, \qquad t_i = 1, 2, \ldots, 20, \\ I(t_0) &= 100. \end{aligned}$$

3.4 Calculating the Daily Inflation Reference

Due to the fact that statistics offices publish an official consumer price index with some delay, the index used for inflation-linked products is actually lagged by some months. In the case of French OAT€is the lag is three months, and the daily inflation reference I on the day d of the month m is calculated by linear interpolation of the two harmonized monthly price indices $HICP$ (excluding tobacco) published by Eurostat, according to the following formula

$$I_{m,d} = HICP_{m-3} + \frac{d-1}{M}\left(HICP_{m-2} - HICP_{m-3}\right), \tag{3.4.1}$$

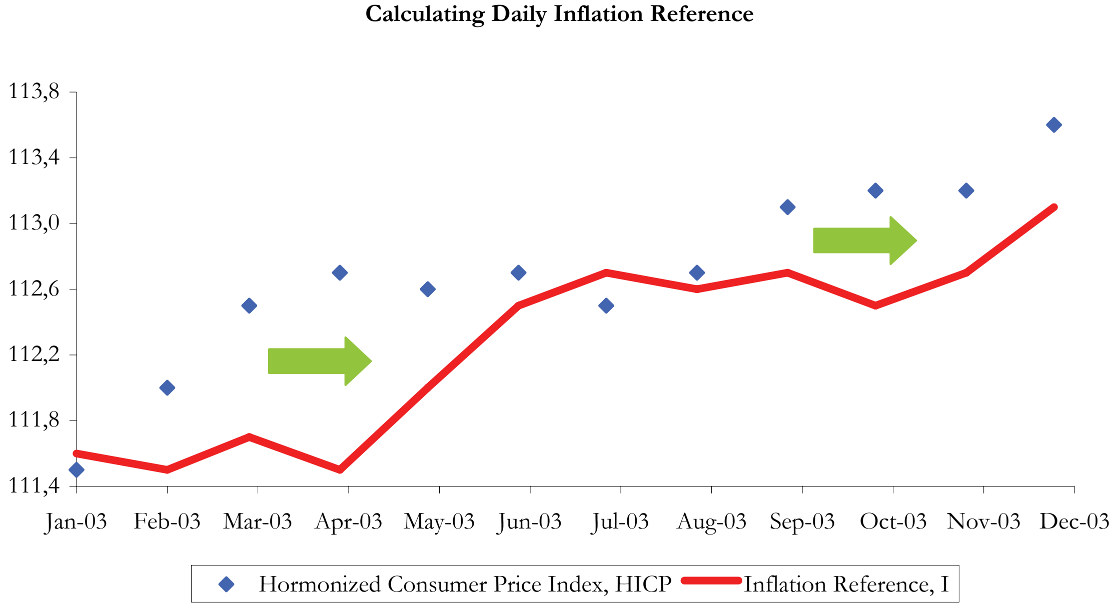

Figure 3.1 Calculating the daily inflation reference from the HICP (excluding tobacco) consumer price index.

where M is the number of days in the month m. This way of calculating the daily inflation reference is justified by the fact that the $HICP$ of the month $m-2$ is always published before the end of the month $m-1$. It is therefore available for calculating the reference index I by the first day of the month m. It is particularly worth noting that the daily inflation reference for the first day of the month m is actually the $HICP$ of the month $m-3$. That is why this index is said to have a three-month lag. The method of computing the daily inflation reference according to the formula (3.4.1) is visualized in Figure 3.1.

Chapter 3

PRELIMINARY STUDIES

4 Inflation Models

In general, there are two types of approaches for modeling the inflation: macro-economic based models and option pricing based models. In this thesis the latter ones will be applied: geometric Brownian motion for consumer price index and extended Vasicek model for inflation rate.

Macro-economic based models describe and quantify the impact of economic variables like the level of nominal interest rates, the output gap and the unemployment to describe the fundamentals that may affect the inflation level. These models are extremely useful at forecasting the inflation level and are widely used by central banks. The typical example of a macro-economic based model is a stochastic model based on the Taylor rules.

Option pricing based models ignore any fundamental impact but rather take the dynamic of the inflation for granted and aim at providing option prices based on the assumed dynamics. These models are widely used by banks to price complex derivatives and to offer hedging solutions for them. For instance, proper modeling of inflation forecast and dynamics can be done using multi-factor models that calibrate easily market instruments, are consistent with inflation forecast, and are linked against some general pricing engine like a Monte Carlo or a PDE engine to allow pricing of complex structures.

In this chapter two models for inflation are suggested: geometric Brownian motion for consumer price index and extended Vasicek model for inflation rate. In addition, their characteristics are discussed in general in the same way as it was done in Korn and Kruse (2004) [17].

4.1 CPI as Geometric Brownian Motion

We consider a continuous trading economy, where the uncertainty in the economy is characterized by the complete probability space $(\Omega, \mathcal{F}, Q)$, where Ω is the state space, $\mathcal{F}$ is the σ-algebra representing measurable events and Q is a risk neutral probability measure. Information evolves over the trading interval $[0,T]$, $T < \infty$, according to the augmented, right continuous, complete filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$ generated by the standard Brownian motion $\{W_I(t)\}_{t\in[0,T]}$, under the risk neutral probability measure Q, in $\mathbb{R}$ initialized at zero.

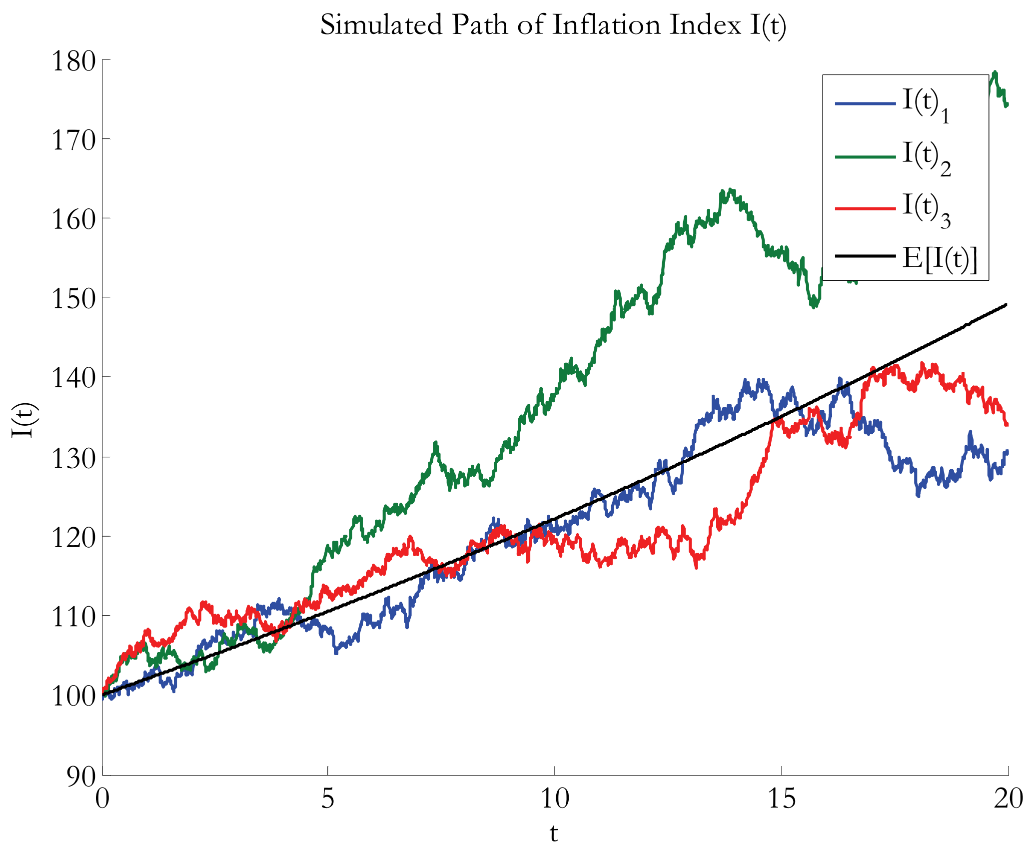

Figure 4.1 Consumer price index as a geometric Brownian motion under the risk neutral probability measure.

Under these probabilistic assumptions and taking into considerations Fisher's equation (3.2.1) we assume the dynamic of consumer price index $I(t)$ under the risk neutral probability measure Q can be described by the geometric Brownian motion according to the following equation

$$dI(t) = I(t)((r_N(t) - r_R(t))dt + \sigma_I dW_I(t)), \quad I(0) = i\,, \tag{4.1.1}$$

where the coefficients $r_N(t)$ and $r_R(t)$, nominal and real interest rates respectively, are assumed to be deterministic functions and σ_I is a constant volatility of the process.

Applying the Itô-calculus one can easily get the solution of the stochastic differential equation above, i.e.

$$I(t) = i\exp\Big(\int_0^t (r_N(s) - r_R(s))ds - \frac{1}{2}\sigma_I^2 t + \sigma_I W_I(t)\Big). \tag{4.1.2}$$

Let $E[.]$ denote the expectation with respect to the risk neutral probability measure Q, then the expectation of consumer price index $I(t)$ is

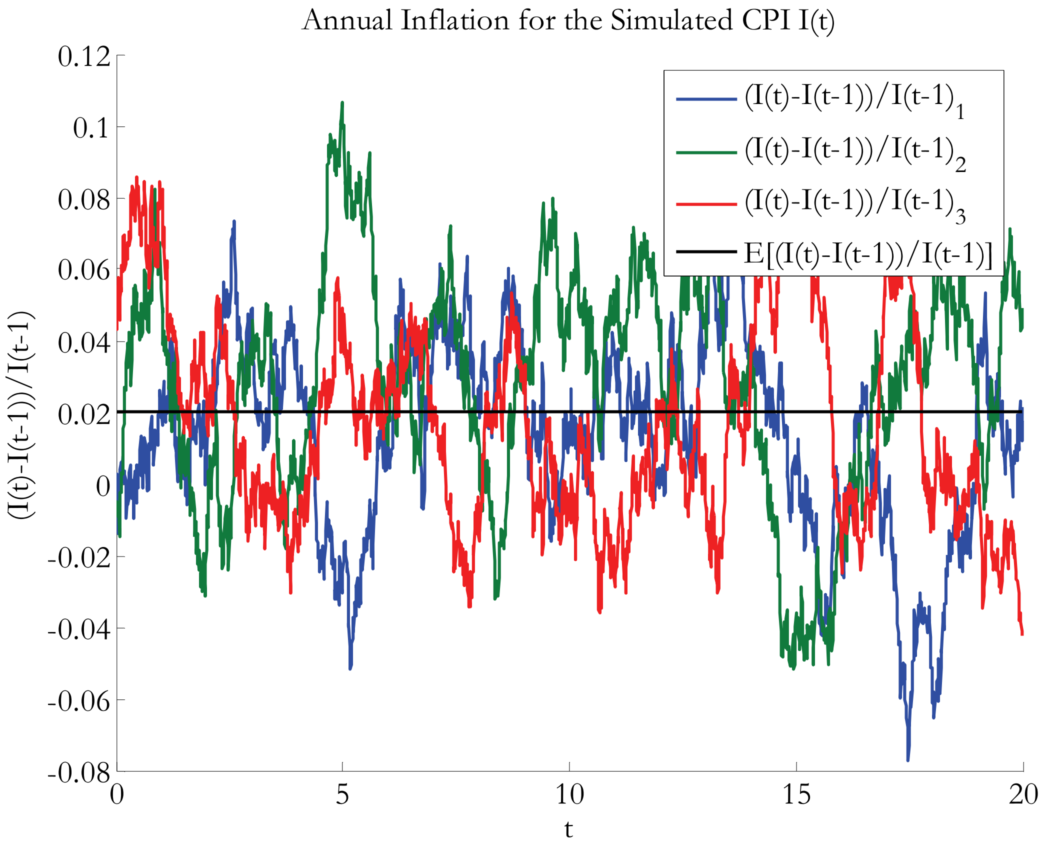

Figure 4.2 Annual inflation for the simulated inflation indexes in Figure 4.1 under the risk neutral probability measure.

$$E[I(t)] = i \exp\left(\int_0^t (r_N(s) - r_R(s))ds\right). \tag{4.1.3}$$

Figure 4.1 visualizes the geometric Brownian motion model for CPI under the risk neutral probability measure Q, where three realizations of simulated consumer price index processes are shown for $t \in [0, T]$, where maturity date $T = 20$ years, as well as the expectation of the index process is plotted. For the simulation the following set of parameter values was used on the yearly basis

$$\begin{aligned} r_N(t) &\equiv 0.05, \\ r_R(t) &\equiv 0.03, \\ \sigma_I &= 0.03, \\ i &= I(0) = 100. \end{aligned}$$

In addition, in Figure 4.2 annual inflation is plotted basing on the simulated paths of inflation index $I(t)$ in Figure 4.1.

In fact, the choice of the drift term in the stochastic equation for consumer price index (4.1.1) affected by the form of Fisher's equation (3.2.2). The incorporation of Fisher's equation could be visualized in the following way. According to (4.1.3) and using the notation of simple inflation rate $i_S(0,t)$ of (3.1.1), we can write down

$$E[1 + i_S(0,t)] = \frac{\exp\left(\int_0^t r_N(s)ds\right)}{\exp\left(\int_0^t r_R(s)ds\right)}. \tag{4.1.4}$$

Substituting the exponents with the corresponding effective interest rates

$$\tilde{r}_x = \exp\left(\int_0^t r_x(s)ds\right) - 1, \quad x \in \{N, R\} \tag{4.1.5}$$

we get Fisher's equation (3.2.2) in the form

$$1 + E[i_S(0,t)] = \frac{1 + \tilde{r}_N(t)}{1 + \tilde{r}_R(t)}. \tag{4.1.6}$$

4.2 Inflation Rate as Extended Vasicek Process

Another way to model the movements of price indexes is heavily based on the interest rate modeling. In this case, instead of modeling consumer price index, directly the inflation rate is modeled as stochastic process analogically to the short rate in the interest rate theory. Also in this model we assume the same probabilistic assumptions as in the previous section 4.1, i.e. the uncertainty in the continuous trading economy is characterized by the complete probability space $(\Omega, \mathcal{F}, Q)$, where Ω is the state space, $\mathcal{F}$ is the σ-algebra representing measurable events and Q is a risk neutral probability measure. Information evolves over the trading interval $[0,T]$, $T < \infty$, according to the augmented, right continuous, complete filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$ generated by the standard Brownian motion $\{W_i(t)\}_{t\in[0,T]}$, under the risk neutral probability measure Q, in $\mathbb{R}$ initialized at zero. Having these probabilistic assumptions, the dynamics of inflation rate $i(t)$ under the risk neutral probability measure Q is assumed to follow the extended Vasicek process

$$di(t) = (\theta(t) - \alpha i(t))dt + \sigma_i dW_i(t), \quad i(0) = i_0, \tag{4.2.1}$$

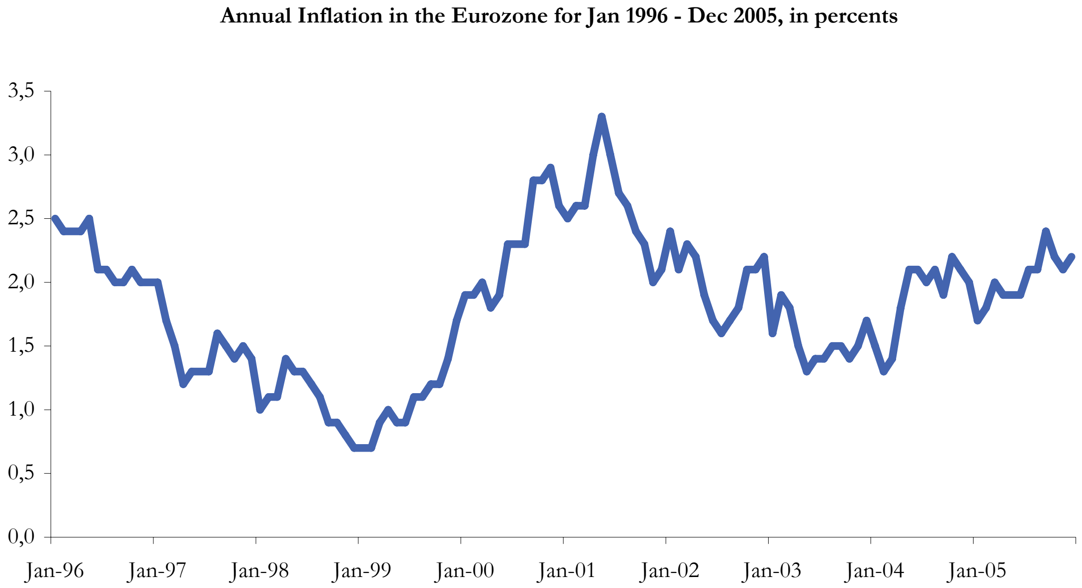

Figure 4.3 Annual inflation in the Eurozone during the period of January 1996 to December 2005.

where α is a positive constant, $\theta(t)$ is a deterministic function and σ_i is a constant volatility of the process. Empirical results do not support the hypothesis that inflation rate is constant over time or that there exists a long run mean towards which inflation rate regresses, which however does not rule out the possibility of a time-dependent reversion level $\theta(t)$ used in equation (4.2.1). As a visualization of this fact the annual inflation in the Eurozone during the period of January 1996 to December 2005 is shown in Figure 4.3.

The solution of the stochastic differential equation (4.2.1) is given by

$$i(t) = i_0 e^{-\alpha t} + \int_0^t e^{-\alpha(t-s)}\theta(s)ds + \sigma_i \int_0^t e^{-\alpha(t-s)} dW_i(s)\,. \tag{4.2.2}$$

From the model settings (4.2.1) we have expected inflation rate of the form

$$E[i(t)] = i_0 e^{-\alpha t} + \int_0^t e^{-\alpha(t-s)}\theta(s)ds \tag{4.2.3}$$

and variance of inflation rate

$$Var[i(t)] = \frac{\sigma_i^2}{2\alpha}(1 - e^{-2\alpha t})\,. \tag{4.2.4}$$

In the special case, when $\theta(t)$ is a constant function, i.e. $\theta(t) \equiv \theta$, the model in (4.2.1) is called Vasicek model with mean-reversion level θ / α and mean-reversion speed α. When the value of the process is bigger than the mean-reversion level then the dt-term in (4.2.1) is negative and the process has a tendency to decrease. In the case when the value is below the mean-reversion level, the trend of the process is up. This means among other things that despite the fact that a process has a relatively high or low initial value, in the long run it will approach the mean-reversion level θ / α requiring that volatility of the process σ_i is small enough. The same fact one can see, when one integrates the expected value of the process (4.2.3) having $\theta(t) \equiv \theta$ as a constant function

$$E[i(t)] = \frac{\theta}{\alpha} + e^{-\alpha t}\left(i_0 - \frac{\theta}{\alpha}\right), \tag{4.2.5}$$

where we notice that the expected value of the Vasicek process in the infinity is the mean-reversion level θ / α, i.e.

$$\lim_{t\to\infty} E[i(t)] = \frac{\theta}{\alpha}. \tag{4.2.6}$$

The incorporation of Fisher's equation into our model setup can be done via specific choice of parameters. By choosing the model parameters in the following way

$$\theta(t) = r_N(t) - r_R(t), \tag{4.2.7}$$

$$\alpha = 1 \tag{4.2.8}$$

Fisher's equation (3.2.1) is reflected in the model settings asymptotically due to

$$E[i(t)] = i_0 e^{-t} + \int_0^t e^{-(t-s)}(r_N(s) - r_R(s))ds. \tag{4.2.9}$$

In the case of constant nominal and real interest rates, r_N and r_R, we have even

$$E[i(t)] = r_N - r_R + (i_0 - (r_N - r_R))e^{-t}, \tag{4.2.10}$$

where for a large time horizon $t \gg 0$ we get an approximated Fisher's equation

$$E[i(t)] \approx r_N - r_R. \tag{4.2.11}$$

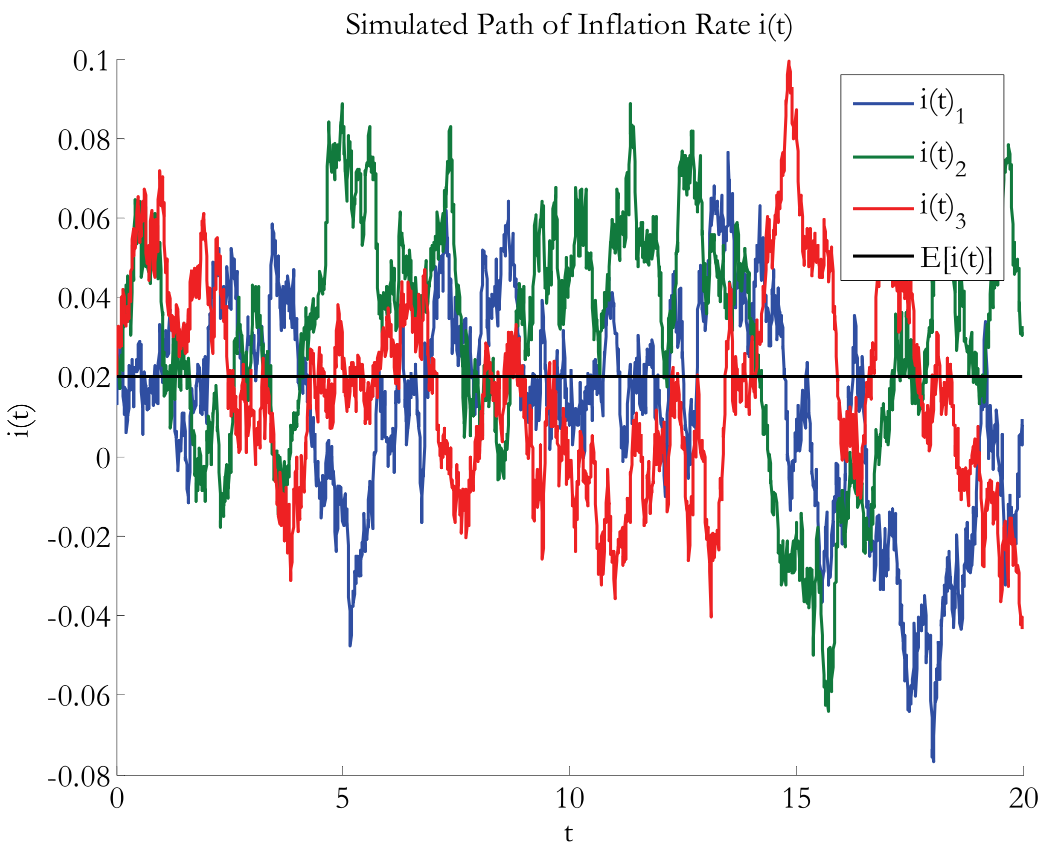

Figure 4.4 Inflation rate as a Vasicek process under the risk neutral probability measure.

In Figure 4.4 three different simulated Vasicek processes under the risk neutral probability measure Q are plotted for $t \in [0, T]$, where maturity date $T = 20$ years, as well as the expected inflation rate. For the simulation the following set of parameter values was used on the yearly basis

$$\theta(t) \equiv r_N - r_R = 0.02,$$
$$\alpha = 1,$$
$$\sigma_i = 0.04,$$
$$i(0) = 0.02.$$

For the simulation of the process in Figure 4.4 the same innovations of the Brownian motion $W_i(t)$ are used as for $W_I(t)$ in Figure 4.1 and Figure 4.2.

5 Pricing of Inflation-linked Derivatives

In this chapter we will derive the price of an inflation-linked bond for two different model setups presented in the previous chapter. We will take the dynamics of inflation for granted by assuming that inflation follows one of two option pricing based models from sections 4.1 and 4.2 and aim at providing derivatives' prices based on the assumed dynamics of inflation. Due to the way how an inflation-linked bond is constructed the process of price derivation for an inflation-linked bond is heavily based on the price of an inflation European call option. In the case of geometric Brownian motion model call option on consumer price index is used whereas in the case of extended Vasicek model call option on inflation evolution is applied in the calculation. The results of this chapter are in line with Korn and Kruse (2004) [17].

In addition to the European call option on inflation and an inflation-linked bond, for the both two models for inflation introduced in the previous chapter, the geometric Brownian motion for inflation index and the extended Vasicek process for inflation rate, we will present the closed form solutions for a European put option on inflation, an inflation cap and floor, an inflation swap and an inflation swaption. Only for the extended Vasicek process for inflation rate, we are unable to present a closed form solution for the inflation swaption due to the high complexity of the involved functions.

5.1 GBM Model

In this section we will derive the fair prices of inflation-linked financial products under the assumption of a geometric Brownian motion for inflation index.

5.1.1 European Call Option on Inflation Index

To start with, we will derive the price of a European call option on consumer price index $I(T)$ with strike price K, which is a simple contingent claim $\mathcal{X} \in \mathcal{F}_T$ with date of maturity T and a contract function Φ, i.e.

$$\mathcal{X} = \Phi(I(T)), \tag{5.1.1}$$

where the contract function Φ has the form

$$\Phi(x) = \max[x - K, 0]. \tag{5.1.2}$$

Then the arbitrage-free price $C_I(t, I(t))$ of the European call option on consumer price index $I(T)$ is given by the formula

$$C_I(t, I(t)) = E\left[\exp\left(-\int_t^T r_N(s)ds\right)\Phi(I(T))\middle|\mathcal{F}_t\right], \tag{5.1.3}$$

where $E[.|\mathcal{F}_t]$ denotes the conditional expectation with respect to the risk neutral probability measure Q.

Proposition 1. Under the assumptions that consumer price index $I(t)$ follows the geometric Brownian motion (4.1.1) and nominal and real interest rates, $r_N(t)$ and $r_R(t)$, are deterministic functions, the fair price of the European call option on consumer price index $I(T)$ at time $t \in [0,T]$ with strike price K and date of maturity T is given by

$$\begin{aligned} C_I(t,I(t)) = I(t)\exp\Big(-\int_t^T r_R(s)ds\Big)N(d(t)) - \\ K\exp\Big(-\int_t^T r_N(s)ds\Big)N(d(t)-\sigma_I\sqrt{T-t}), \end{aligned} \tag{5.1.4}$$

where N is the cumulative distribution function for the standard normal distribution $\mathcal{N}(0,1)$ and

$$d(t) = \frac{\ln\Big(\frac{I(t)}{K}\Big) + \int_t^T (r_N(s) - r_R(s))ds + \frac{1}{2}\sigma_I^2(T-t)}{\sigma_I\sqrt{T-t}}. \tag{5.1.5}$$

Proof of Proposition 1. The proof of the option pricing formula (5.1.4) and (5.1.5) is given in the Appendix 1. □

5.1.2 Inflation-linked Bond

Proposition 2. Under the assumptions that consumer price index $I(t)$ follows the geometric Brownian motion (4.1.1) and nominal and real interest rates, $r_N(t)$ and $r_R(t)$, are deterministic functions, the fair price of an inflation-linked T-bond at time $t \in [0,T]$ with a reference date $t_0 \le 0$, face value F and coupon payments before adjustment to the inflation, C_i, at times t_i is given by

$$\begin{aligned} B_{IL}(t,I(t)) = \sum_{i:t_i \ge t} C_i \frac{I(t)}{I(t_0)}\exp\Big(-\int_t^{t_i} r_R(s)ds\Big) + \\ F\bigg(\exp\Big(-\int_t^T r_N(s)ds\Big) + \frac{C_I(t,I(t))}{I(t_0)}\bigg), \end{aligned} \tag{5.1.6}$$

where $0 < t_1 \le \ldots \le t_n = T$ and $C_I(t,I(t))$ is a fair price of the European call option on consumer price index $I(T)$ at time t with strike price $K = I(t_0)$ and date of maturity T.

Proof of Proposition 2. In analogy to (5.1.3), under the assumption of arbitrage-free market the fair price of the inflation-linked T-bond at time t is given by

$$\begin{aligned} B_{IL}(t,I(t)) = E\Bigg[\sum_{i:t_i \geq t} \exp\Big(-\int_t^{t_i} r_N(s)ds\Big) C_i \frac{I(t_i)}{I(t_0)} + \\ \exp\Big(-\int_t^T r_N(s)ds\Big) F \max\Big[\frac{I(T)}{I(t_0)}, 1\Big] \Bigg| \mathcal{F}_t \Bigg], \end{aligned} \tag{5.1.7}$$

where $E[.|\mathcal{F}_t]$ denotes the conditional expectation with respect to the risk neutral probability measure Q.

Using the fact that nominal and real interest rates, $r_N(t)$ and $r_R(t)$, are deterministic functions and that the dynamics of consumer price index $I(t)$ can be described by the stochastic differential equation (4.1.1) with expectation function given by (4.1.3) we get

$$\begin{aligned} B_{IL}(t,I(t)) = \sum_{i:t_i \geq t} \exp\Big(-\int_t^{t_i} r_N(s)ds\Big) C_i \frac{I(t)}{I(t_0)} \exp\Big(\int_t^{t_i} (r_N(s) - r_R(s))ds\Big) + \\ E\Bigg[\exp\Big(-\int_t^T r_N(s)ds\Big) F \Big(1 + \frac{\max[I(T) - I(t_0), 0]}{I(t_0)}\Big) \Bigg| \mathcal{F}_t \Bigg], \end{aligned} \tag{5.1.8}$$

where we recognize the fair price of the European call option on consumer price index $I(T)$ at time t with strike price $K = I(t_0)$ and date of maturity T, equation (5.1.3), i.e.

$$C_I(t,I(t)) = E\Big[\exp\Big(-\int_t^T r_N(s)ds\Big) \max[I(T) - I(t_0), 0] \Big| \mathcal{F}_t\Big], \tag{5.1.9}$$

which gives us

$$\begin{aligned} B_{IL}(t,I(t)) = \sum_{i:t_i \geq t} C_i \frac{I(t)}{I(t_0)} \exp\Big(-\int_t^{t_i} r_R(s)ds\Big) + \\ F\Big(\exp\Big(-\int_t^T r_N(s)ds\Big) + \frac{C_I(t,I(t))}{I(t_0)}\Big) \end{aligned} \tag{5.1.10}$$

that completely coincides with Proposition 2 equation (5.1.6). □

Remark 1 on Proposition 2 *(Zero-coupon Inflation-linked Bond)***.** In the case of a zero-coupon inflation-linked bond (the one that does not pay coupon payments during its life), i.e.

$$C_i = 0, \quad \forall i = 1,\ldots,n\,, \tag{5.1.11}$$

according to the bond pricing formula (5.1.6) the fair price of a zero-coupon inflation-linked bond would be

$$B_{IL}(t,I(t)) = F\left(\exp\left(-\int_t^T r_N(s)ds\right) + \frac{C_I(t,I(t))}{I(t_0)}\right), \tag{5.1.12}$$

whereas the price of a regular bond with the same face value F is given by

$$P_0(t) = F\exp\left(-\int_t^T r_N(s)ds\right). \tag{5.1.13}$$

This means that one has to pay a higher price for getting the protection against purchasing power erosion, which is in itself natural. In the case of zero-coupon bonds this price difference is

$$B_{IL}(t,I(t)) - P_0(t) = F\frac{C_I(t,I(t))}{I(t_0)}. \tag{5.1.14}$$

Remark 2 on Proposition 2 *(Deflation-unprotected Principal Payment)***.** Concerning different structures of inflation-linked bonds, there exist a group of inflation-linked bonds, whose inflation-protected principal is not protected against deflation. For this type of structure of inflation-linked bonds we do not have maximum in (3.3.1), i.e. the principal payment at the maturity date T is

$$F\frac{I(T)}{I(t_0)}. \tag{5.1.15}$$

For this type of inflation-linked bonds the fair price would be

$$B_{IL}(t,I(t)) = \frac{I(t)}{I(t_0)}\left(\sum_{i:t_i \geq t} C_i \exp\left(-\int_t^{t_i} r_R(s)ds\right) + F\exp\left(-\int_t^T r_R(s)ds\right)\right), \tag{5.1.16}$$

which actually is a linear combination of the inflation index $I(t)$.

Remark 3 on Proposition 2 *(Asymptotic Behavior)***.** The other interesting aspect about Proposition 2 is the following fact. The bond-pricing formula (5.1.6) can be rewritten as

$$B_{IL}(t,I(t)) = \psi(t)I(t) + F\exp\left(-\int_t^T r_N(s)ds\right)(1 - N(d(t) - \sigma_I\sqrt{T-t})), \quad (5.1.17)$$

where $\psi(t)$ is a replicating strategy defined as

$$\psi(t) := \frac{\partial B_{IL}(t,I(t))}{\partial I(t)}, \quad (5.1.18)$$

i.e.

$$\psi(t) = \sum_{i:t_i \geq t} \frac{C_i}{I(t_0)} \exp\left(-\int_t^{t_i} r_R(s)ds\right) + \frac{F\exp\left(-\int_t^T r_R(s)ds\right)N(d(t))}{I(t_0)}. \quad (5.1.19)$$

From (5.1.17) one can easily see that for large $I(t) \gg I(t_0)$ the term $\psi(t)I(t)$ becomes dominating in the bond-pricing formula (5.1.17) for $B_{IL}(t,I(t))$, in other words for large $I(t) \gg 0$ the price of inflation-linked bond $B_{IL}(t,I(t))$ behaves asymptotically as the part replicated by inflation index I, $\psi(t)I(t)$, i.e.

$$B_{IL}(t,I(t)) \sim \psi(t)I(t), \quad I(t) \to \infty, \quad (5.1.20)$$

which means that

$$\lim_{I(t)\to\infty} \frac{B_{IL}(t,I(t))}{\psi(t)I(t)} = 1. \quad (5.1.21)$$

This can be explained in the following way. When the current inflation index is far over the inflation index reference, $I(t) \gg I(t_0)$, then it is highly probable that inflation index at the maturity date $I(T)$ is also higher than $I(t_0)$, which means that we have $FI(T)/I(t_0)$ for the principal payment. The final payment of this form and also inflation protected coupon payments can be perfectly replicated by the inflation index with $\psi(t)I(t)$.

When inflation index $I(t) \ll I(t_0)$ is reasonably close to zero, then inflation index at the maturity date T is also close to zero and lower than $I(t_0)$ with a high probability. In such a case the principal payment is redeemed at par, i.e. equal to F. Thus, for this situation the price of inflation-linked bond $B_{IL}(t,I(t))$ is equal to the coupon payments replicated by the inflation

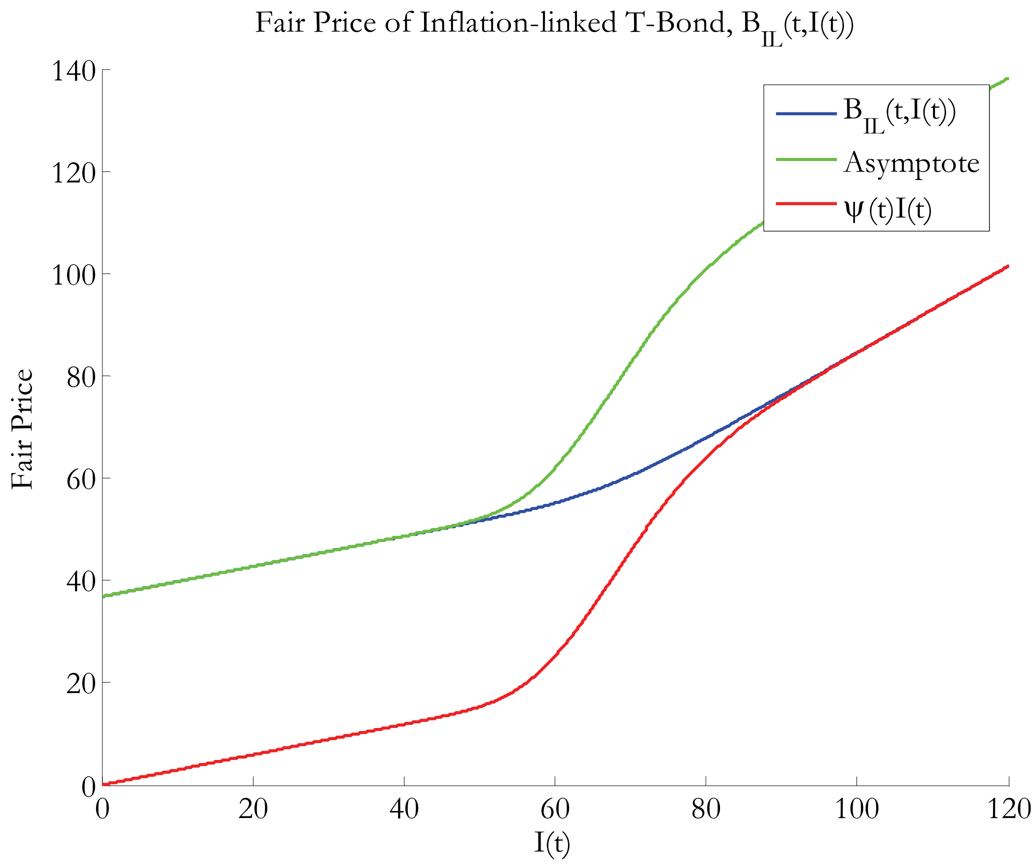

Figure 5.1 Fair price of inflation-linked bond as a function of consumer price index.

index with $\psi(t)I(t)$ plus the price of a normal bond with face value F. The same can be seen from (5.1.17) as an asymptotic behavior for $B_{IL}(t,I(t))$ when $I(t)$ approaches zero, i.e.

$$B_{IL}(t,I(t)) \sim \psi(t)I(t) + F\exp\left(-\int_t^T r_N(s)ds\right), \quad I(t) \to 0. \tag{5.1.22}$$

This asymptotic behavior of the price of inflation-linked bond $B_{IL}(t,I(t))$ is also visualized in Figure 5.1, where the fair price of the inflation-linked T-bond at time $t=0$, $B_{IL}(t,I(t))$, the asymptote function (5.1.22) as well as the term $\psi(t)I(t)$ are plotted. The following set of parameters was used for the simulation

$$\begin{aligned} r_N(t) &\equiv 0.05, \\ r_R(t) &\equiv 0.03, \\ \sigma_I &= 0.03, \\ T &= 20, \qquad F = 100, \\ C_i &= 2, \qquad t_i = 1,2,\ldots,20, \end{aligned}$$

$$I(t_0) = 100.$$

In Figure 5.1 is clearly seen the asymptotic behavior of the inflation-linked bond price $B_{IL}(t, I(t))$ at large inflation indexes $I(t) \gg 0$ as the term $\psi(t)I(t)$. Whereas for small inflation indexes $I(t) \to 0$ the difference between $B_{IL}(t, I(t))$ and $\psi(t)I(t)$ is asymptotically

$$F \exp\left(-\int_t^T r_N(s)ds\right) \approx 36.7879.$$

The other more interesting asymptotic behavior of the fair price of the inflation-linked bond $B_{IL}(t, I(t))$ is the situation, when the time t approaches the maturity date T. Depending on the level of inflation index $I(t)$ the price of inflation-lined bond $B_{IL}(t, I(t))$ approaches three different asymptotical functions. In the case of inflation, i.e. when $I(t) > I(t_0)$, the asymptote is equal to $\psi(t)I(t)$. For $I(t) = I(t_0)$ we have $\psi(t)I(t) + F/2$ as an asymptote and in the case of deflation, i.e. $I(t) < I(t_0)$, the asymptotic function for $B_{IL}(t, I(t))$ is $\psi(t)I(t) + F$. In all the asymptotic behavior of the fair price of the inflation-linked bond $B_{IL}(t, I(t))$ has the following form

$$B_{IL}(t, I(t)) \sim \psi(t)I(t), \quad t \to T, \quad I(t) > I(t_0), \tag{5.1.23}$$

$$B_{IL}(t, I(t)) \sim \psi(t)I(t) + \frac{1}{2}F, \quad t \to T, \quad I(t) = I(t_0), \tag{5.1.24}$$

$$B_{IL}(t, I(t)) \sim \psi(t)I(t) + F, \quad t \to T, \quad I(t) < I(t_0). \tag{5.1.25}$$

For $I(t) = I(t_0)$ it is equally probable that inflation index at the maturity date $I(T)$ is bigger or smaller than base index $I(t_0)$. That is why we have for this case the asymptotic behavior of the form (5.1.24). The other exciting feature of the asymptotic behavior of inflation-linked bond price $B_{IL}(t, I(t))$ at time t approaching the maturity date T can be noticed when we realize that at the maturity date T the price $B_{IL}(T, I(T))$ is equal to the sum of inflation-linked bond's inflation adjusted face value $F \max[I(T)/I(t_0), 1]$ and the last inflation adjusted coupon payment $C_i I(T)/I(t_0)$. That is why we can make the following conclusions about the limiting behavior of the term $\psi(t)I(t)$

Figure 5.2 Fair price of inflation-linked bond as a function of time, when inflation index is higher than the base index.

$$\lim_{t\to T}\psi(t)I(t) = (F + C_i)\frac{I(T)}{I(t_0)}, \quad I(t) > I(t_0), \tag{5.1.26}$$

$$\lim_{t\to T}\psi(t)I(t) = \frac{1}{2}F + C_i, \quad I(t) = I(t_0), \tag{5.1.27}$$

$$\lim_{t\to T}\psi(t)I(t) = C_i\frac{I(T)}{I(t_0)}, \quad I(t) < I(t_0). \tag{5.1.28}$$

The asymptotic behavior of the inflation-linked bond price $B_{IL}(t, I(t))$ at the maturity date T (5.1.23)-(5.1.25) as well as the limiting behavior (5.1.26)-(5.1.28) are depicted in Figure 5.2 – Figure 5.4 for $t \in [0, T]$, where maturity date $T = 20$ years and we have a different levels of inflation index $I(t)$ compared to the base index $I(t_0)$. For all three cases we use the usual set of model parameters

$$r_N(t) \equiv 0.05,$$
$$r_R(t) \equiv 0.03,$$

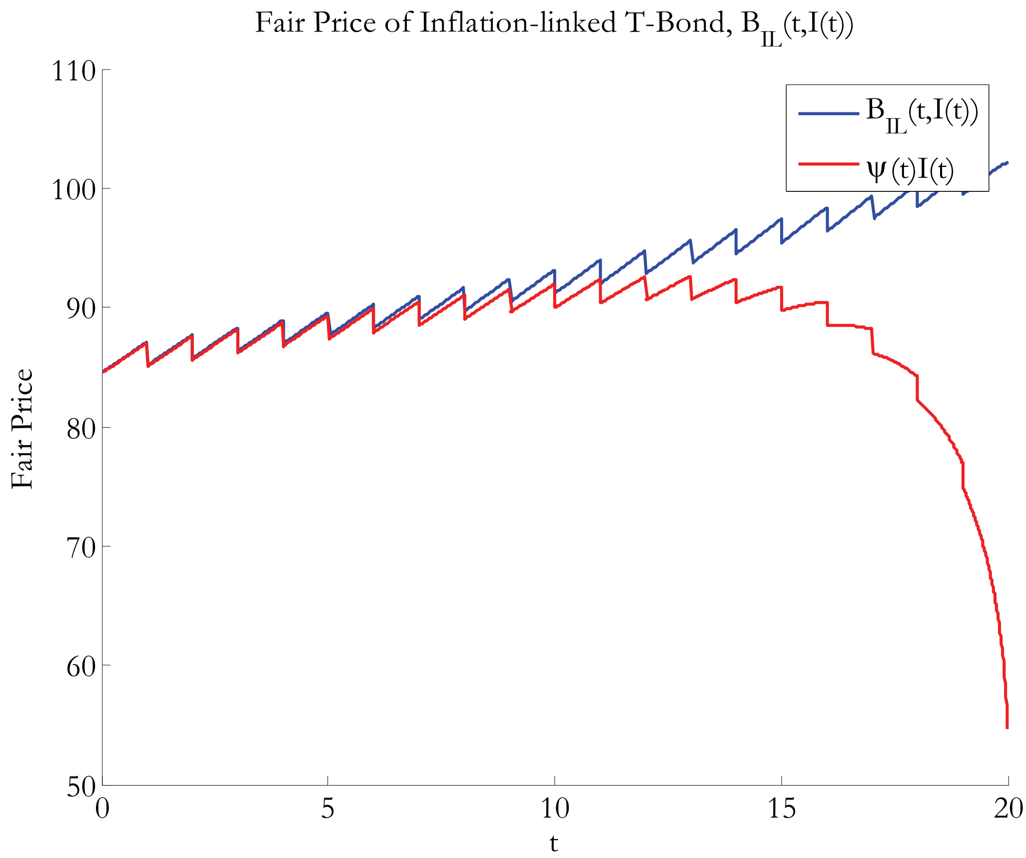

Figure 5.3 Fair price of inflation-linked bond as a function of time, when inflation index is equal to the base index.

$$\begin{aligned} \sigma_I &= 0.03, \\ T &= 20, \qquad F = 100, \\ C_i &= 2, \qquad t_i = 1,2,\ldots,20, \\ I(t_0) &= 100. \end{aligned}$$

For the first case, Figure 5.2, we assume a constant inflation index $I(t) \equiv 110$ for the whole time period $t \in [0,T]$, which corresponds to the case of inflation at the maturity date T,(5.1.23) and (5.1.26). In this case the term $\psi(t)I(t)$ almost totally corresponds to the inflation-linked bond price $B_{IL}(t,I(t))$ on the whole interval $t \in [0,T]$ and especially at time t approaching the maturity date T. In addition it is seen in the figure that the limit for the term $\psi(t)I(t)$ is in line with (5.1.26), i.e.

$$\lim_{t\to T} \psi(t)I(t) = (F + C_i)\frac{I(T)}{I(t_0)} = 112.2\,.$$

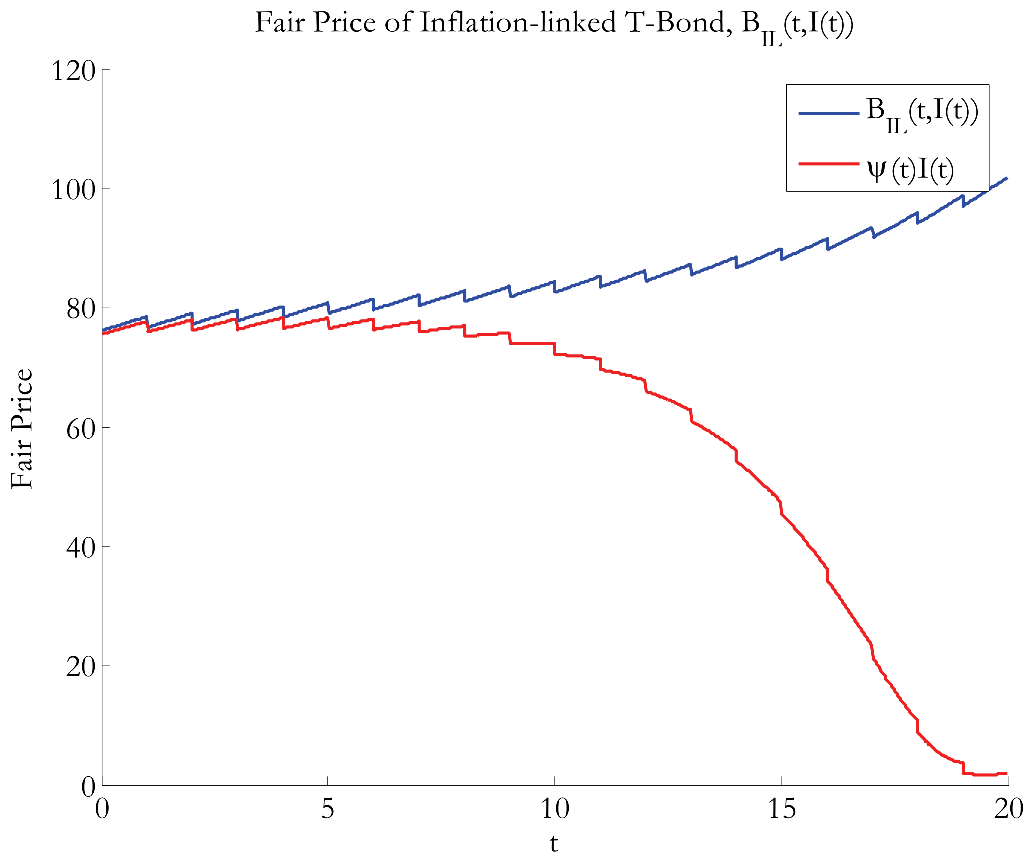

Figure 5.4 Fair price of inflation-linked bond as a function of time, when inflation index is lower than the base index.

Also, the downwards jumps directly after the yearly coupon payments are clearly seen in the plots.

Figure 5.3 shows the case, when inflation index $I(t)$ does not deviate from the base level $I(t_0)$ at all, i.e. $I(t) \equiv 100$. Here the term $\psi(t)I(t)$ can only partially replicate the inflation-linked bond price $B_{IL}(t, I(t))$ at the maturity date T. From the plot we straightforwardly see that the part not replicated by the term $\psi(t)I(t)$ is asymptotically equal to the half face value of the inflation-linked bond $F/2 = 50$ at the maturity date T. We also observe that the limiting behavior of the term $\psi(t)I(t)$ as given by (5.1.27)

$$\lim_{t \to T} \psi(t)I(t) = \frac{1}{2}F + C_i = 52.$$

In the third Figure 5.4 we consider the case of deflation, where we assume a constant inflation index $I(t) \equiv 90$ for the whole time period $t \in [0, T]$. In this setup the term $\psi(t)I(t)$ can hardly replicate the price of inflation-linked bond $B_{IL}(t, I(t))$ at time t close to bond's maturity date T. In line with (5.1.25), from the plot we observe that the different between the inflation-linked bond price $B_{IL}(t, I(t))$ and the term $\psi(t)I(t)$ is actually asymptotically equal to the face value of the inflation-linked bond $F = 100$ at the maturity date T. Also the limit of the term $\psi(t)I(t)$ at the maturity date T, (5.1.28), can be easily detected in the figure

$$\lim_{t \to T} \psi(t)I(t) = C_i \frac{I(T)}{I(t_0)} = 1.8 .$$

5.1.3 European Put Option on Inflation Index

A European put option on consumer price index $I(T)$ with strike price K is a simple contingent claim $\mathcal{X} \in \mathcal{F}_T$ with date of maturity T and a contract function Φ of the form

$$\mathcal{X} = \Phi(I(T)) = \max[K - I(T), 0]. \tag{5.1.29}$$

Proposition 3. Under the assumptions that consumer price index $I(t)$ follows the geometric Brownian motion (4.1.1) and nominal and real interest rates, $r_N(t)$ and $r_R(t)$, are deterministic functions, the fair price of the European put option on consumer price index $I(T)$ at time $t \in [0, T]$ with strike price K and date of maturity T is given by

$$\begin{aligned} P_I(t, I(t)) = K \exp\Big(-\int_t^T r_N(s)ds\Big) N(-d(t) + \sigma_I \sqrt{T-t}) - \\ I(t) \exp\Big(-\int_t^T r_R(s)ds\Big) N(-d(t)), \end{aligned} \tag{5.1.30}$$

where N is the cumulative distribution function for the standard normal distribution $\mathcal{N}(0,1)$ and

$$d(t) = \frac{\ln\Big(\frac{I(t)}{K}\Big) + \int_t^T (r_N(s) - r_R(s))ds + \frac{1}{2}\sigma_I^2 (T-t)}{\sigma_I \sqrt{T-t}}. \tag{5.1.31}$$

Proof of Proposition 3. As usual, the arbitrage-free price $P_I(t, I(t))$ of the European put option on consumer price index $I(T)$ is given by the formula

$$P_I(t, I(t)) = E\Big[\exp\Big(-\int_t^T r_N(s)ds\Big)\Phi(I(T))\Big| \mathcal{F}_t\Big], \tag{5.1.32}$$

where $E[.|\mathcal{F}_t]$ denotes the conditional expectation with respect to the risk neutral probability measure Q. The contract function Φ (5.1.29) we can rewrite as

$$\Phi(I(T)) = K - I(T) + \max[I(T) - K, 0], \tag{5.1.33}$$

where the last term is obviously a contract function of a European call option on inflation index $I(T)$. Substituting this form of the contract function Φ (5.1.33) into the expectation in (5.1.32) we get for the fair price $P_I(t, I(t))$ of a European put option on consumer price index $I(T)$

$$P_I(t, I(t)) = K \exp\Big(-\int_t^T r_N(s)ds\Big) - I(t)\exp\Big(-\int_t^T r_R(s)ds\Big) + C_I(t, I(t)). \tag{5.1.34}$$

Substituting into the above equation (5.1.34) the fair price $C_I(t, I(t))$ of a European call option on consumer price index $I(T)$, (5.1.4), and rearranging the terms we directly get (5.1.30). □

5.1.4 Inflation Cap and Floor

An inflation cap/floor pays out if inflation evolution exceeds/drops below a certain threshold K_i over a given period $[t_{i-1}, t_i]$. At time t an inflation cap/floor with notional F has the following future payoffs

$$F \sum_{i:t_i \geq t} \max\Big[\omega\Big(\frac{I(t_i)}{I(t_{i-1})} - K_i\Big), 0\Big], \tag{5.1.35}$$

where $\omega = 1$ corresponds to a cap and $\omega = -1$ to a floor.

Proposition 4. Under the assumptions that consumer price index $I(t)$ follows the geometric Brownian motion (4.1.1) and nominal and real interest rates, $r_N(t)$ and $r_R(t)$, are deterministic functions, the fair price of an inflation cap/floor with notional F at time $t \in [0, T]$ is given by

$$\begin{aligned} CF_I(t, I(t)) = \omega F \sum_{i:t_{i-1} \geq t} & \exp\Big(-\int_t^{t_{i-1}} r_N(s)ds - \int_{t_{i-1}}^{t_i} r_R(s)ds\Big) N(\omega d_i(t)) - \\ & K_i \exp\Big(-\int_t^{t_i} r_N(s)ds\Big) N(\omega(d_i(t) - \sigma_I\sqrt{t_i - t_{i-1}})) + \\ & \omega F \sum_{i:t_i \geq t \wedge t_{i-1} < t} \frac{I(t)}{I(t_{i-1})} \exp\Big(-\int_t^{t_i} r_R(s)ds\Big) N(\omega d(t)) - \\ & K_i \exp\Big(-\int_t^{t_i} r_N(s)ds\Big) N(\omega(d(t) - \sigma_I\sqrt{t_i - t})), \end{aligned} \tag{5.1.36}$$

where $0 \leq t_0 \leq \ldots \leq t_n = T$, N is the cumulative distribution function for the standard normal distribution $\mathcal{N}(0,1)$ and

$$d_i(t) = \frac{\ln\left(\frac{1}{K_i}\right) + \int_{t_{i-1}}^{t_i} (r_N(s) - r_R(s))ds + \frac{1}{2}\sigma_I^2(t_i - t_{i-1})}{\sigma_I \sqrt{t_i - t_{i-1}}}, \tag{5.1.37}$$

$$d(t) = \frac{\ln\left(\frac{I(t)}{K_i I(t_{i-1})}\right) + \int_{t}^{t_i} (r_N(s) - r_R(s))ds + \frac{1}{2}\sigma_I^2(t_i - t)}{\sigma_I \sqrt{t_i - t}}. \tag{5.1.38}$$

Proof of Proposition 4. As usual, the arbitrage-free price $CF_I(t, I(t))$ of inflation cap/floor is given by the formula

$$CF_I(t, I(t)) = E\left[F \sum_{i:t_i \geq t} \exp\left(-\int_t^{t_i} r_N(s)ds\right) \max\left[\omega\left(\frac{I(t_i)}{I(t_{i-1})} - K_i\right), 0\right] \middle| \mathcal{F}_t\right], \tag{5.1.39}$$

where $E[.|\mathcal{F}_t]$ denotes the conditional expectation with respect to the risk neutral probability measure Q. The expectation in (5.1.39) can be divided in two parts, where in the first one the sum goes over all time intervals $[t_{i-1}, t_i]$ that lie after time t and the second one contains the intervals $[t_{i-1}, t_i]$ that include time t

$$\begin{aligned} CF_I(t, I(t)) = {} & F \sum_{i:t_{i-1} \geq t} \exp\left(-\int_t^{t_{i-1}} r_N(s)ds\right) \\ & E\left[\exp\left(-\int_{t_{i-1}}^{t_i} r_N(s)ds\right) \max\left[\omega\left(\frac{I(t_i)}{I(t_{i-1})} - K_i\right), 0\right] \middle| \mathcal{F}_t\right] + \\ & F \sum_{i:t_i \geq t \wedge t_{i-1} < t} \frac{I(t)}{I(t_{i-1})} \\ & E\left[\exp\left(-\int_t^{t_i} r_N(s)ds\right) \max\left[\omega\left(\frac{I(t_i)}{I(t)} - K_i \frac{I(t_{i-1})}{I(t)}\right), 0\right] \middle| \mathcal{F}_t\right]. \end{aligned} \tag{5.1.40}$$

The inflation evolutions in the expectation above can be represented as

$$\frac{I(t_i)}{I(t_{i-1})} = \exp(\tilde{\mu}_I(t_{i-1}, t_i) + \tilde{\sigma}_I(t_{i-1}, t_i)Z), \tag{5.1.41}$$

$$\frac{I(t_i)}{I(t)} = \exp(\tilde{\mu}_I(t, t_i) + \tilde{\sigma}_I(t, t_i)Z), \tag{5.1.42}$$

where

$$\tilde{\mu}_I(t_{i-1},t_i) = \int_{t_{i-1}}^{t_i} (r_N(s) - r_R(s))ds - \frac{1}{2}\sigma_I^2(t_i - t_{i-1}), \tag{5.1.43}$$

$$\tilde{\mu}_I(t,t_i) = \int_{t}^{t_i} (r_N(s) - r_R(s))ds - \frac{1}{2}\sigma_I^2(t_i - t), \tag{5.1.44}$$

$$\tilde{\sigma}_I(t_{i-1},t_i) = \sigma_I\sqrt{t_i - t_{i-1}}, \tag{5.1.45}$$

$$\tilde{\sigma}_I(t,t_i) = \sigma_I\sqrt{t_i - t} \tag{5.1.46}$$

and $Z \sim \mathcal{N}(0,1)$ is a standard normal variable. Consequently, applying the same techniques as in Appendix 1 we can easily get a Black and Scholes type pricing formula (5.1.36). □

5.1.5 Inflation Swap

In a year-on-year payer inflation swap one exchanges a fixed leg K_i for a floating leg $I(t_i)/I(t_{i-1})$ at time t_i. At time t a year-on-year payer inflation swap with notional F has the following future payoffs

$$F\sum_{i:t_i\geq t} \frac{I(t_i)}{I(t_{i-1})} - K_i\,. \tag{5.1.47}$$

Proposition 5. Under the assumptions that consumer price index $I(t)$ follows the geometric Brownian motion (4.1.1) and nominal and real interest rates, $r_N(t)$ and $r_R(t)$, are deterministic functions, the fair price of a year-on-year payer inflation swap with notional F at time $t \in [0,T]$ is given by

$$\begin{aligned} S_I(t,I(t)) = F\sum_{i:t_{i-1}\geq t} & \exp\Big(-\int_t^{t_{i-1}} r_N(s)ds - \int_{t_{i-1}}^{t_i} r_R(s)ds\Big) - \\ & K_i\exp\Big(-\int_t^{t_i} r_N(s)ds\Big) + \\ & F\sum_{i:t_i\geq t\wedge t_{i-1}<t} \frac{I(t)}{I(t_{i-1})}\exp\Big(-\int_t^{t_i} r_R(s)ds\Big) - K_i\exp\Big(-\int_t^{t_i} r_N(s)ds\Big), \end{aligned} \tag{5.1.48}$$

where $0 \leq t_0 \leq \ldots \leq t_n = T$.

Proof of Proposition 5. As usual, the arbitrage-free price $S_I(t, I(t))$ of a year-on-year payer inflation swap is given by the formula

$$S_I(t, I(t)) = E\left[F \sum_{i:t_i \geq t} \exp\left(-\int_t^{t_i} r_N(s)ds\right)\left(\frac{I(t_i)}{I(t_{i-1})} - K_i\right) \middle| \mathcal{F}_t\right], \tag{5.1.49}$$

where $E[.|\mathcal{F}_t]$ denotes the conditional expectation with respect to the risk neutral probability measure Q. The expectation in (5.1.49) can be divided in two parts, where in the first one the sum goes over all time intervals $[t_{i-1}, t_i]$ that lie after time t and the second one contains the intervals $[t_{i-1}, t_i]$ that include time t

$$\begin{aligned} S_I(t, I(t)) = {} & F \sum_{i:t_{i-1} \geq t} \exp\left(-\int_t^{t_{i-1}} r_N(s)ds\right) \\ & E\left[\exp\left(-\int_{t_{i-1}}^{t_i} r_N(s)ds\right)\left(\frac{I(t_i)}{I(t_{i-1})} - K_i\right) \middle| \mathcal{F}_t\right] + \\ & F \sum_{i:t_i \geq t \wedge t_{i-1} < t} \frac{I(t)}{I(t_{i-1})} \\ & E\left[\exp\left(-\int_t^{t_i} r_N(s)ds\right)\left(\frac{I(t_i)}{I(t)} - K_i \frac{I(t_{i-1})}{I(t)}\right) \middle| \mathcal{F}_t\right]. \end{aligned} \tag{5.1.50}$$

Evaluating the expectations in (5.1.50) we directly get (5.1.48). □

5.1.6 Inflation Swaption

A swaption gives a right to enter a year-on-year payer inflation swap contract at time T. A swaption with notional F has the following future payoff at the maturity T

$$\max[S_I(T, I(T)), 0], \tag{5.1.51}$$

where $S_I(T, I(T))$ is a fair price of a year-on-year payer inflation swap with notional F at time T. We assume that the swaption maturity T coincides with the first reset date of the underlying inflation swap, i.e. $T = t_0$.

Proposition 6. Under the assumptions that consumer price index $I(t)$ follows the geometric Brownian motion (4.1.1) and nominal and real interest rates, $r_N(t)$ and $r_R(t)$, are deterministic functions, the fair price of a swaption with notional F at time $t \in [0, T]$ is given by

$$SO_I(t,I(t)) = \exp\Big(-\int_t^T r_N(s)ds\Big)$$
$$\max\Big[F\sum_{i=1}^{n}\exp\Big(-\int_T^{t_{i-1}} r_N(s)ds - \int_{t_{i-1}}^{t_i} r_R(s)ds\Big) - K_i\exp\Big(-\int_T^{t_i} r_N(s)ds\Big), 0\Big], \tag{5.1.52}$$

where $T = t_0 \leq \ldots \leq t_n$.

Proof of Proposition 6. As usual, the arbitrage-free price $SO_I(t,I(t))$ of an inflation swaption is given by the formula

$$SO_I(t,I(t)) = E\Big[\exp\Big(-\int_t^T r_N(s)ds\Big)\max[S_I(T,I(T)),0]\Big|\mathcal{F}_t\Big], \tag{5.1.53}$$

where $E[.|\mathcal{F}_t]$ denotes the conditional expectation with respect to the risk neutral probability measure Q. According to (5.1.48) and taking into consideration the swap reset sate structure $T = t_0 \leq \ldots \leq t_n$, the fair price of a year-on-year payer inflation swap with notional F at time T is given by

$$S_I(T,T(t)) = F\sum_{i=1}^{n}\exp\Big(-\int_T^{t_{i-1}} r_N(s)ds - \int_{t_{i-1}}^{t_i} r_R(s)ds\Big) - K_i\exp\Big(-\int_T^{t_i} r_N(s)ds\Big). \tag{5.1.54}$$

Substituting this fair price of an inflation swap (5.1.54) into (5.1.53) we directly get (5.1.52). □

5.2 Extended Vasicek Model

Contrary to the previous section, in this one we will assume an extended Vasicek model for inflation rate and basing on this market model we will derive fair prices of inflation-linked financial products.

5.2.1 European Call Option on Inflation Evolution

In this section we proceed in the same way as we did in the previous section 5.1 for geometric Brownian motion model, i.e. we will first derive the price of a European call option on inflation, in this case on inflation evolution $I(T)/I(t) = \exp\Big(\int_t^T i(s)ds\Big)$ with strike price K, which is a simple contingent claim $\mathcal{X} \in \mathcal{F}_T$ with date of maturity T and a contract function Φ of the form

$$\mathcal{X} = \Phi(I(T)/I(t)) = \max[I(T)/I(T) - K, 0]. \tag{5.2.1}$$

Then the arbitrage-free price $C_I(t,i(t))$ of the European call option on inflation evolution $\exp\left(\int_t^T i(s)ds\right)$ is given by the formula

$$C_I(t,i(t)) = E\left[\exp\left(-\int_t^T r_N(s)ds\right)\Phi(I(T)/I(t))\middle|\mathcal{F}_t\right], \tag{5.2.2}$$

where $E[.|\mathcal{F}_t]$ denotes the conditional expectation with respect to the risk neutral probability measure Q.

Proposition 7. Under the assumptions that inflation rate $i(t)$ follows the extended Vasicek model (4.2.1) and nominal and real interest rates, $r_N(t)$ and $r_R(t)$, are deterministic functions, the fair price of the European call option on inflation evolution $\exp\left(\int_t^T i(s)ds\right)$ at time $t \in [0,T]$ with strike price K and date of maturity T is given by

$$\begin{aligned} C_I(t,i(t)) = \exp\left(-\int_t^T r_N(s)ds + \tilde{\mu}_i(t,T) + \frac{1}{2}\tilde{\sigma}_i^2(t,T)\right)N(d(t)) - \\ K\exp\left(-\int_t^T r_N(s)ds\right)N(d(t) - \tilde{\sigma}_i(t,T)), \end{aligned} \tag{5.2.3}$$

where N is the cumulative distribution function for the standard normal distribution $\mathcal{N}(0,1)$,

$$d(t) = \frac{\ln\left(\frac{1}{K}\right) + \tilde{\mu}_i(t,T) + \tilde{\sigma}_i^2(t,T)}{\tilde{\sigma}_i(t,T)} \tag{5.2.4}$$

and parameters $\tilde{\mu}_i(t,T)$ and $\tilde{\sigma}_i^2(t,T)$ are

$$\tilde{\mu}_i(t,T) = i(t)\frac{1-e^{-\alpha(T-t)}}{\alpha} + \int_t^T \frac{1-e^{-\alpha(T-s)}}{\alpha}\theta(s)ds, \tag{5.2.5}$$

$$\tilde{\sigma}_i^2(t,T) = \frac{\sigma_i^2}{\alpha^2}\left(\frac{1-\left(e^{-\alpha(T-t)}-2\right)^2}{2\alpha} + (T-t)\right). \tag{5.2.6}$$

Proof of Proposition 7. The proof of the option pricing formula (5.2.3)-(5.2.6) is given in the Appendix 2. □

The pricing formulas of the European call option on consumer price index $I(T)$, (5.1.4) and (5.1.5), as well as on inflation evolution $\exp\left(\int_t^T i(s)ds\right)$, (5.2.3)-(5.2.6), are both of the so-called Black and Scholes type.

5.2.2 Inflation-linked Bond

Proposition 8. Under the assumptions that inflation rate $i(t)$ follows the extended Vasicek model (4.2.1) and nominal and real interest rates, $r_N(t)$ and $r_R(t)$, are deterministic functions, the fair price of an inflation-linked T-bond at time $t \in [0,T]$ with a reference date $t_0 \leq 0$, face value F and actual coupon payments before adjustment to the inflation, C_k, at times t_k is given by

$$\begin{aligned} B_{IL}(t,i(t)) = \sum_{k:t_k \geq t} C_k \frac{I(t)}{I(t_0)} \exp\left(-\int_t^{t_k} r_N(s)ds + \tilde{\mu}_i(t,t_k) + \frac{1}{2}\tilde{\sigma}_i^2(t,t_k)\right) + \\ F\left(\exp\left(-\int_t^T r_N(s)ds\right) + \frac{I(t)}{I(t_0)} C_I(t,i(t))\right), \end{aligned} \tag{5.2.7}$$

where $0 < t_1 \leq \ldots \leq t_n = T$, $C_I(t,i(t))$ is a fair price of the European call option on inflation evolution $\exp\left(\int_t^T i(s)ds\right)$ at time t with strike price $K = I(t_0)/I(t)$ and date of maturity T and parameters $\tilde{\mu}_i(t,t_k)$ and $\tilde{\sigma}_i^2(t,t_k)$ are the similar ones as in (5.2.5) and (5.2.6), i.e.

$$\tilde{\mu}_i(t,t_k) = i(t)\frac{1-e^{-\alpha(t_k-t)}}{\alpha} + \int_t^{t_k} \frac{1-e^{-\alpha(t_k-s)}}{\alpha}\theta(s)ds\,, \tag{5.2.8}$$

$$\tilde{\sigma}_i^2(t,t_k) = \frac{\sigma_i^2}{\alpha^2}\left(\frac{1-(e^{-\alpha(t_k-t)}-2)^2}{2\alpha} + (t_k - t)\right). \tag{5.2.9}$$

Proof of Proposition 8. In analogy to (5.2.2), under the assumption of arbitrage-free market the fair price of the inflation-linked T-bond at time t is given by

$$\begin{aligned} B_{IL}(t,i(t)) = E\Bigg[\sum_{k:t_k \geq t} \exp\left(-\int_t^{t_k} r_N(s)ds\right) C_k \frac{I(t_k)}{I(t_0)} + \\ \exp\left(-\int_t^T r_N(s)ds\right) F \max\left[\frac{I(T)}{I(t_0)},1\right] \Bigg| \mathcal{F}_t\Bigg], \end{aligned} \tag{5.2.10}$$

where $E[.|\mathcal{F}_t]$ denotes the conditional expectation with respect to the risk neutral probability measure Q.

Using the fact that nominal and real interest rates, $r_N(t)$ and $r_R(t)$, are deterministic functions and rearranging the terms we can write the equation (5.2.10) as

$$\begin{aligned} B_{IL}(t,i(t)) = \sum_{k:t_k \geq t} \exp\Big(-\int_t^{t_k} r_N(s)ds\Big) C_k \frac{I(t)}{I(t_0)} E\Big[\frac{I(t_k)}{I(t)}\Big|\mathcal{F}_t\Big] + \\ E\Big[\exp\Big(-\int_t^T r_N(s)ds\Big) F\Big(1 + \frac{I(t)}{I(t_0)} \max\Big[\frac{I(T)}{I(t)} - \frac{I(t_0)}{I(t)}, 0\Big]\Big)\Big|\mathcal{F}_t\Big]. \end{aligned} \tag{5.2.11}$$

Here we recognize the fair price of the European call option on inflation evolution $\exp\Big(\int_t^T i(s)ds\Big)$ at time t with strike price $K = I(t_0)/I(t)$ and date of maturity T, equation (5.2.2), i.e.

$$C_I(t,i(t)) = E\Big[\exp\Big(-\int_t^T r_N(s)ds\Big)\max\Big[\frac{I(T)}{I(t)} - \frac{I(t_0)}{I(t)}, 0\Big]\Big|\mathcal{F}_t\Big]. \tag{5.2.12}$$

The expectation of inflation evolution in equation (5.2.11) can be calculated with the help of the following equation

$$E[\mathrm{e}^X] = e^{E[X] + \frac{1}{2}Var[X]}, \tag{5.2.13}$$

where X is a random variable. We define $\tilde{\mu}_i(t,t_k)$ and $\tilde{\sigma}_i^2(t,t_k)$ as expectation and variance of the integral over inflation rate $\int_t^{t_k} i(s)ds$, i.e.

$$\tilde{\mu}_i(t,t_k) := E\Big[\int_t^{t_k} i(s)ds\Big] = i(t)\frac{1 - e^{-\alpha(t_k - t)}}{\alpha} + \int_t^{t_k} \frac{1 - e^{-\alpha(t_k - s)}}{\alpha}\theta(s)ds\,, \tag{5.2.14}$$

$$\tilde{\sigma}_i^2(t,t_k) := Var\Big[\int_t^{t_k} i(s)ds\Big] = \frac{\sigma_i^2}{\alpha^2}\Big(\frac{1 - (e^{-\alpha(t_k - t)} - 2)^2}{2\alpha} + (t_k - t)\Big), \tag{5.2.15}$$

for the proof see the Appendix 2 equations (A.2.3) and (A.2.4). Then we can express the expectation of inflation evolution as

$$E\left[\frac{I(t_k)}{I(t)}\middle|\mathcal{F}_t\right] = E\left[\frac{I(t_k)}{I(t)}\right] = E\left[\exp\left(\int_t^{t_k} i(s)ds\right)\right] = \exp\left(\tilde{\mu}_i(t,t_k) + \frac{1}{2}\tilde{\sigma}_i^2(t,t_k)\right). \quad (5.2.16)$$

Using equations (5.2.12) and (5.2.16) we get for the fair price of the inflation-linked T-bond at time t

$$\begin{aligned} B_{IL}(t,i(t)) = &\sum_{k:t_k\geq t} C_k \frac{I(t)}{I(t_0)}\exp\left(-\int_t^{t_k} r_N(s)ds + \tilde{\mu}_i(t,t_k) + \frac{1}{2}\tilde{\sigma}_i^2(t,t_k)\right) + \\ &F\left(\exp\left(-\int_t^T r_N(s)ds\right) + \frac{I(t)}{I(t_0)}C_I(t,i(t))\right) \end{aligned} \quad (5.2.17)$$

that completely coincides with Proposition 8 equations (5.2.7)-(5.2.9). □

Remark 1 on Proposition 8 *(Zero-coupon Inflation-linked Bond)***.** Also in this model setup, extended Vasicek model, the same as in the geometric Brownian motion model, the price of a zero-coupon inflation-linked bond that is given by

$$B_{IL}(t,i(t)) = F\left(\exp\left(-\int_t^T r_N(s)ds\right) + \frac{I(t)}{I(t_0)}C_I(t,i(t))\right) \quad (5.2.18)$$

is higher than the price of a regular zero-coupon bond with the same face value F by the amount

$$B_{IL}(t,i(t)) - P_0(t) = F\frac{I(t)}{I(t_0)}C_I(t,i(t)). \quad (5.2.19)$$

Remark 2 on Proposition 8 *(Deflation-unprotected Principal Payment)***.** Analogically to the previous section we can derive the following result for a deflation-unprotected principal payment. Assuming the extended Vasicek model (4.2.1) for inflation rate $i(t)$ the fair price of an inflation-linked bond whose principal is not protected against deflation is given by

$$\begin{aligned} B_{IL}(t,i(t)) = \frac{I(t)}{I(t_0)}\Bigg(&\sum_{k:t_k\geq t} C_k \exp\left(-\int_t^{t_k} r_N(s)ds + \tilde{\mu}_i(t,t_k) + \frac{1}{2}\tilde{\sigma}_i^2(t,t_k)\right) + \\ &F\exp\left(-\int_t^T r_N(s)ds + \tilde{\mu}_i(t,T) + \frac{1}{2}\tilde{\sigma}_i^2(t,T)\right)\Bigg). \end{aligned} \quad (5.2.20)$$

5.2.3 European Put Option on Inflation Evolution

A European put option on inflation evolution $I(T)/I(t) = \exp\left(\int_t^T i(s)ds\right)$ with strike price K is a simple contingent claim $\mathcal{X} \in \mathcal{F}_T$ with date of maturity T and a contract function Φ of the form

$$\mathcal{X} = \Phi(I(T)/I(t)) = \max[K - I(T)/I(T), 0]. \tag{5.2.21}$$

Proposition 9. Under the assumptions that inflation rate $i(t)$ follows the extended Vasicek model (4.2.1) and nominal and real interest rates, $r_N(t)$ and $r_R(t)$, are deterministic functions, the fair price of the European put option on inflation evolution $\exp\left(\int_t^T i(s)ds\right)$ at time $t \in [0,T]$ with strike price K and date of maturity T is given by

$$\begin{aligned} P_I(t,i(t)) = {} & K\exp\left(-\int_t^T r_N(s)ds\right)N(-d(t)+\tilde{\sigma}_i(t,T)) - \\ & \exp\left(-\int_t^T r_N(s)ds + \tilde{\mu}_i(t,T) + \frac{1}{2}\tilde{\sigma}_i^2(t,T)\right)N(-d(t)), \end{aligned} \tag{5.2.22}$$

where N is the cumulative distribution function for the standard normal distribution $\mathcal{N}(0,1)$,

$$d(t) = \frac{\ln\left(\frac{1}{K}\right) + \tilde{\mu}_i(t,T) + \tilde{\sigma}_i^2(t,T)}{\tilde{\sigma}_i(t,T)} \tag{5.2.23}$$

and parameters $\tilde{\mu}_i(t,T)$ and $\tilde{\sigma}_i^2(t,T)$ are

$$\tilde{\mu}_i(t,T) = i(t)\frac{1-e^{-\alpha(T-t)}}{\alpha} + \int_t^T \frac{1-e^{-\alpha(T-s)}}{\alpha}\theta(s)ds, \tag{5.2.24}$$

$$\tilde{\sigma}_i^2(t,T) = \frac{\sigma_i^2}{\alpha^2}\left(\frac{1-\left(e^{-\alpha(T-t)}-2\right)^2}{2\alpha} + (T-t)\right). \tag{5.2.25}$$

Proof of Proposition 9. As usual, the arbitrage-free price $P_I(t,i(t))$ of the European put option on inflation evolution $\exp\left(\int_t^T i(s)ds\right)$ is given by the formula

$$P_I(t,i(t)) = E\left[\exp\left(-\int_t^T r_N(s)ds\right)\Phi(I(T)/I(t))\middle|\mathcal{F}_t\right], \tag{5.2.26}$$

where $E[.|\mathcal{F}_t]$ denotes the conditional expectation with respect to the risk neutral probability measure Q. The contract function Φ (5.2.21) we can rewrite as

$$\Phi(I(T)/I(t)) = K - \exp\Big(\int_t^T i(s)ds\Big) + \max[I(T)/I(t) - K, 0], \tag{5.2.27}$$

where the last term is obviously a contract function of a European call option on inflation evolution $\exp\Big(\int_t^T i(s)ds\Big)$. Substituting this form of the contract function Φ (5.2.27) into the expectation in (5.2.26) we get for the fair price $P_I(t, i(t))$ of a European put option on inflation evolution $\exp\Big(\int_t^T i(s)ds\Big)$

$$\begin{aligned} P_I(t, i(t)) = {} & K\exp\Big(-\int_t^T r_N(s)ds\Big) - \exp\Big(-\int_t^T r_N(s)ds + \tilde{\mu}_i(t,T) + \frac{1}{2}\tilde{\sigma}_i^2(t,T)\Big) \\ & + C_I(t, i(t)). \end{aligned} \tag{5.2.28}$$

Substituting into the above equation (5.2.28) the fair price $C_I(t, i(t))$ of a European call option on inflation evolution $\exp\Big(\int_t^T i(s)ds\Big)$, (5.2.3), and rearranging the terms we directly get (5.2.22). □

5.2.4 Inflation Cap and Floor

An inflation cap/floor pays out if inflation evolution exceeds/drops below a certain threshold K_i over a given period $[t_{i-1}, t_i]$. At time t an inflation cap/floor with notional F has the following future payoffs

$$F\sum_{i:t_i \geq t} \max\Big[\omega\Big(\frac{I(t_i)}{I(t_{i-1})} - K_i\Big), 0\Big], \tag{5.2.29}$$

where $\omega = 1$ corresponds to a cap and $\omega = -1$ to a floor.

Proposition 10. Under the assumptions that inflation rate $i(t)$ follows the extended Vasicek model (4.2.1) and nominal and real interest rates, $r_N(t)$ and $r_R(t)$, are deterministic functions, the fair price of an inflation cap/floor with notional F at time $t \in [0,T]$ is given by

$$
\begin{aligned}
CF_I(t,i(t)) = \omega F \sum_{i:t_{i-1}\geq t} \exp\Big(-\int_t^{t_i} r_N(s)ds + \tilde{\mu}_i(t_{i-1},t_i) + \frac{1}{2}\tilde{\sigma}_i^2(t_{i-1},t_i)\Big) N(\omega d_i(t)) - \\
K_i \exp\Big(-\int_t^{t_i} r_N(s)ds\Big) N(\omega(d_i(t) - \tilde{\sigma}_i(t_{i-1},t_i))) + \\
\omega F \sum_{i:t_i\geq t \wedge t_{i-1}<t} \frac{I(t)}{I(t_{i-1})} \exp\Big(-\int_t^{t_i} r_N(s)ds + \tilde{\mu}_i(t,t_i) + \frac{1}{2}\tilde{\sigma}_i^2(t,t_i)\Big) N(\omega d(t)) - \\
K_i \exp\Big(-\int_t^{t_i} r_N(s)ds\Big) N(\omega(d(t) - \tilde{\sigma}_i(t,t_i))),
\end{aligned}
\tag{5.2.30}
$$

where $0 \leq t_0 \leq \ldots \leq t_n = T$, N is the cumulative distribution function for the standard normal distribution $\mathcal{N}(0,1)$,

$$
d_i(t) = \frac{\ln\Big(\dfrac{1}{K_i}\Big) + \tilde{\mu}_i(t_{i-1},t_i) + \tilde{\sigma}_i^2(t_{i-1},t_i)}{\tilde{\sigma}_i(t_{i-1},t_i)}, \tag{5.2.31}
$$

$$
d(t) = \frac{\ln\Big(\dfrac{1}{K_i}\Big) + \displaystyle\int_{t_{i-1}}^{t} i(s)ds + \tilde{\mu}_i(t,t_i) + \tilde{\sigma}_i^2(t,t_i)}{\tilde{\sigma}_i(t,t_i)} \tag{5.2.32}
$$

and parameters $\tilde{\mu}_i(t_{i-1},t_i)$, $\tilde{\mu}_i(t,t_i)$, $\tilde{\sigma}_i^2(t_{i-1},t_i)$ and $\tilde{\sigma}_i^2(t,t_i)$ are

$$
\begin{aligned}
\tilde{\mu}_i(t_{i-1},t_i) = i(t)\frac{e^{-\alpha(t_{i-1}-t)} - e^{-\alpha(t_i-t)}}{\alpha} + \\
\int_t^{t_{i-1}} \frac{e^{-\alpha(t_{i-1}-s)} - e^{-\alpha(t_i-s)}}{\alpha}\theta(s)ds + \int_{t_{i-1}}^{t_i} \frac{1-e^{-\alpha(t_i-s)}}{\alpha}\theta(s)ds,
\end{aligned}
\tag{5.2.33}
$$

$$
\tilde{\mu}_i(t,t_i) = i(t)\frac{1-e^{-\alpha(t_i-t)}}{\alpha} + \int_t^{t_i} \frac{1-e^{-\alpha(t_i-s)}}{\alpha}\theta(s)ds, \tag{5.2.34}
$$

$$
\begin{aligned}
\tilde{\sigma}_i^2(t_{i-1},t_i) = \frac{\sigma_i^2}{\alpha^2}\Bigg(\frac{(1-e^{-\alpha(t_i-t_{i-1})})^2 - (e^{-\alpha(t_i-t)} - e^{-\alpha(t_{i-1}-t)})^2}{2\alpha}\Bigg) + \\
\frac{\sigma_i^2}{\alpha^2}\Bigg(\frac{1-(e^{-\alpha(t_i-t_{i-1})}-2)^2}{2\alpha} + (t_i - t_{i-1})\Bigg),
\end{aligned}
\tag{5.2.35}
$$

$$
\tilde{\sigma}_i^2(t,t_i) = \frac{\sigma_i^2}{\alpha^2}\Bigg(\frac{1-(e^{-\alpha(t_i-t)}-2)^2}{2\alpha} + (t_i - t)\Bigg). \tag{5.2.36}
$$

Proof of Proposition 10. As usual, the arbitrage-free price $CF_I(t,i(t))$ of inflation cap/floor is given by the formula

$$CF_I(t,i(t)) = E\left[F\sum_{i:t_i\geq t}\exp\Big(-\int_t^{t_i} r_N(s)ds\Big)\max\left[\omega\left(\frac{I(t_i)}{I(t_{i-1})}-K_i\right),0\right]\middle|\mathcal{F}_t\right], \tag{5.2.37}$$

where $E[.|\mathcal{F}_t]$ denotes the conditional expectation with respect to the risk neutral probability measure Q. The expectation in (5.2.37) can be divided in two parts, where in the first one the sum goes over all time intervals $[t_{i-1},t_i]$ that lie after time t and the second one contains the intervals $[t_{i-1},t_i]$ that include time t

$$\begin{aligned}CF_I(t,i(t)) = {} & F\sum_{i:t_{i-1}\geq t}\exp\Big(-\int_t^{t_{i-1}} r_N(s)ds\Big)\\ & E\left[\exp\Big(-\int_{t_{i-1}}^{t_i} r_N(s)ds\Big)\max\left[\omega\left(\exp\Big(\int_{t_{i-1}}^{t_i} i(s)ds\Big)-K_i\right),0\right]\middle|\mathcal{F}_t\right]+\\ & F\sum_{i:t_i\geq t\wedge t_{i-1}<t}\frac{I(t)}{I(t_{i-1})}\\ & E\left[\exp\Big(-\int_t^{t_i} r_N(s)ds\Big)\max\left[\omega\left(\exp\Big(\int_t^{t_i} i(s)ds\Big)-K_i\frac{I(t_{i-1})}{I(t)}\right),0\right]\middle|\mathcal{F}_t\right].\end{aligned} \tag{5.2.38}$$

The integrals over inflation rate in the expectation above have the following explicit form

$$\begin{aligned}\int_{t_{i-1}}^{t_i} i(s)ds = {} & i(t)\frac{e^{-\alpha(t_{i-1}-t)}-e^{-\alpha(t_i-t)}}{\alpha}+\\ & \int_t^{t_{i-1}}\frac{e^{-\alpha(t_{i-1}-s)}-e^{-\alpha(t_i-s)}}{\alpha}\theta(s)ds+\int_{t_{i-1}}^{t_i}\frac{1-e^{-\alpha(t_i-s)}}{\alpha}\theta(s)ds+\\ & \frac{\sigma_i}{\alpha}\int_t^{t_{i-1}}(e^{-\alpha(t_{i-1}-s)}-e^{-\alpha(t_i-s)})dW_i(s)+\frac{\sigma_i}{\alpha}\int_{t_{i-1}}^{t_i}(1-e^{-\alpha(t_i-s)})dW_i(s),\end{aligned} \tag{5.2.39}$$

$$\begin{aligned}\int_t^{t_i} i(s)ds = {} & i(t)\frac{1-e^{-\alpha(t_i-t)}}{\alpha}+\int_t^{t_i}\frac{1-e^{-\alpha(t_i-s)}}{\alpha}\theta(s)ds+\\ & \frac{\sigma_i}{\alpha}\int_t^{t_i}(1-e^{-\alpha(t_i-s)})dW_i(s).\end{aligned} \tag{5.2.40}$$

Here we realize that the integrals over inflation rate $\int_{t_{i-1}}^{t_i} i(s)ds$ and $\int_t^{t_i} i(s)ds$ are normally distributed with the following means and variances

$$\tilde{\mu}_i(t_{i-1},t_i) := E\Big[\int_{t_{i-1}}^{t_i} i(s)ds\Big] = i(t)\frac{e^{-\alpha(t_{i-1}-t)} - e^{-\alpha(t_i-t)}}{\alpha} + \int_t^{t_{i-1}} \frac{e^{-\alpha(t_{i-1}-s)} - e^{-\alpha(t_i-s)}}{\alpha}\theta(s)ds + \int_{t_{i-1}}^{t_i} \frac{1-e^{-\alpha(t_i-s)}}{\alpha}\theta(s)ds, \tag{5.2.41}$$

$$\tilde{\mu}_i(t,t_i) := E\Big[\int_t^{t_i} i(s)ds\Big] = i(t)\frac{1-e^{-\alpha(t_i-t)}}{\alpha} + \int_t^{t_i}\frac{1-e^{-\alpha(t_i-s)}}{\alpha}\theta(s)ds\,, \tag{5.2.42}$$

$$\begin{aligned}
\tilde{\sigma}_i^2(t_{i-1},t_i) &:= Var\Big[\int_{t_{i-1}}^{t_i} i(s)ds\Big] \\
&= E\Big[\Big(\frac{\sigma_i}{\alpha}\int_t^{t_{i-1}}(e^{-\alpha(t_{i-1}-s)} - e^{-\alpha(t_i-s)})dW_i(s) + \\
&\quad \frac{\sigma_i}{\alpha}\int_{t_{i-1}}^{t_i}(1-e^{-\alpha(t_i-s)})dW_i(s)\Big)^2\Big] \\
&= \frac{\sigma_i^2}{\alpha^2}\int_t^{t_{i-1}}(e^{-\alpha(t_{i-1}-s)} - e^{-\alpha(t_i-s)})^2 ds + \\
&\quad \frac{\sigma_i^2}{\alpha^2}\int_{t_{i-1}}^{t_i}(1-e^{-\alpha(t_i-s)})^2 ds \\
&= \frac{\sigma_i^2}{\alpha^2}\Big(\frac{(1-e^{-2\alpha(t_{i-1}-t)})}{2\alpha} - \frac{2(e^{-\alpha(t_i-t_{i-1})} - e^{-\alpha(t_{i-1}+t_i-2t)})}{2\alpha} + \\
&\quad \frac{(e^{-2\alpha(t_i-t_{i-1})} - e^{-2\alpha(t_i-t)})}{2\alpha}\Big) + \\
&\quad \frac{\sigma_i^2}{\alpha^2}\Big(\frac{(1-e^{-2\alpha(t_i-t_{i-1})})}{2\alpha} - \frac{2(1-e^{-\alpha(t_i-t_{i-1})})}{\alpha} + (t_i - t_{i-1})\Big) \\
&= \frac{\sigma_i^2}{\alpha^2}\Big(\frac{(1-e^{-\alpha(t_i-t_{i-1})})^2 - (e^{-\alpha(t_i-t)} - e^{-\alpha(t_{i-1}-t)})^2}{2\alpha}\Big) + \\
&\quad \frac{\sigma_i^2}{\alpha^2}\Big(\frac{1-(e^{-\alpha(t_i-t_{i-1})}-2)^2}{2\alpha} + (t_i - t_{i-1})\Big),
\end{aligned} \tag{5.2.43}$$

$$\begin{aligned}
\tilde{\sigma}_i^2(t,t_i) &:= Var\Big[\int_t^{t_i} i(s)ds\Big] = E\Big[\Big(\frac{\sigma_i}{\alpha}\int_t^{t_i}(1-e^{-\alpha(t_i-s)})dW_i(s)\Big)^2\Big] \\
&= \frac{\sigma_i^2}{\alpha^2}\int_t^{t_i}(1-e^{-\alpha(t_i-s)})^2 ds \\
&= \frac{\sigma_i^2}{\alpha^2}\Big(\frac{(1-e^{-2\alpha(t_i-t)})}{2\alpha} - \frac{2(1-e^{-\alpha(t_i-t)})}{\alpha} + (t_i - t)\Big) \\
&= \frac{\sigma_i^2}{\alpha^2}\Big(\frac{1-(e^{-\alpha(t_i-t)}-2)^2}{2\alpha} + (t_i - t)\Big).
\end{aligned} \tag{5.2.44}$$

Thus, one can write the value of inflation evolutions $\exp\Big(\int_{t_{i-1}}^{t_i} i(s)ds\Big)$ and $\exp\Big(\int_t^{t_i} i(s)ds\Big)$ as

$$\exp\Big(\int_{t_{i-1}}^{t_i} i(s)ds\Big) = \exp(\tilde{\mu}_i(t_{i-1},t_i) + \tilde{\sigma}_i(t_{i-1},t_i)Z), \tag{5.2.45}$$

$$\exp\Big(\int_{t}^{t_i} i(s)ds\Big) = \exp(\tilde{\mu}_i(t,t_i) + \tilde{\sigma}_i(t,t_i)Z), \tag{5.2.46}$$

where $Z \sim \mathcal{N}(0,1)$ is a standard normal variable. Consequently, applying the same techniques as in Appendix 2 we can easily get a Black and Scholes type pricing formula (5.2.30). □

5.2.5 Inflation Swap

In a year-on-year payer inflation swap one exchanges a fixed leg K_i for a floating leg $I(t_i)/I(t_{i-1})$ at time t_i. At time t a year-on-year payer inflation swap with notional F has the following future payoffs

$$F\sum_{i:t_i\geq t}\frac{I(t_i)}{I(t_{i-1})} - K_i\,. \tag{5.2.47}$$

Proposition 11. Under the assumptions that inflation rate $i(t)$ follows the extended Vasicek model (4.2.1) and nominal and real interest rates, $r_N(t)$ and $r_R(t)$, are deterministic functions, the fair price of a year-on-year payer inflation swap with notional F at time $t\in[0,T]$ is given by

$$\begin{aligned} S_I(t,i(t)) = F\sum_{i:t_{i-1}\geq t}&\exp\Big(-\int_t^{t_i} r_N(s)ds + \tilde{\mu}_i(t_{i-1},t_i) + \frac{1}{2}\tilde{\sigma}_i^2(t_{i-1},t_i)\Big) - \\ &K_i\exp\Big(-\int_t^{t_i} r_N(s)ds\Big) + \\ &F\sum_{i:t_i\geq t\wedge t_{i-1}<t}\frac{I(t)}{I(t_{i-1})}\exp\Big(-\int_t^{t_i} r_N(s)ds + \tilde{\mu}_i(t,t_i) + \frac{1}{2}\tilde{\sigma}_i^2(t,t_i)\Big) - \\ &K_i\exp\Big(-\int_t^{t_i} r_N(s)ds\Big), \end{aligned} \tag{5.2.48}$$

where $0\leq t_0\leq \ldots \leq t_n = T$ and parameters $\tilde{\mu}_i(t_{i-1},t_i)$, $\tilde{\mu}_i(t,t_i)$, $\tilde{\sigma}_i^2(t_{i-1},t_i)$ and $\tilde{\sigma}_i^2(t,t_i)$ are

$$\begin{aligned} \tilde{\mu}_i(t_{i-1},t_i) = i(t)&\frac{e^{-\alpha(t_{i-1}-t)} - e^{-\alpha(t_i-t)}}{\alpha} + \\ &\int_t^{t_{i-1}}\frac{e^{-\alpha(t_{i-1}-s)} - e^{-\alpha(t_i-s)}}{\alpha}\theta(s)ds + \int_{t_{i-1}}^{t_i}\frac{1-e^{-\alpha(t_i-s)}}{\alpha}\theta(s)ds, \end{aligned} \tag{5.2.49}$$

$$\tilde{\mu}_i(t,t_i) = i(t)\frac{1-e^{-\alpha(t_i-t)}}{\alpha} + \int_t^{t_i}\frac{1-e^{-\alpha(t_i-s)}}{\alpha}\theta(s)ds\,, \tag{5.2.50}$$

$$\begin{aligned}\tilde{\sigma}_i^2(t_{i-1},t_i) = \frac{\sigma_i^2}{\alpha^2}\left(\frac{\left(1-e^{-\alpha(t_i-t_{i-1})}\right)^2 - \left(e^{-\alpha(t_i-t)} - e^{-\alpha(t_{i-1}-t)}\right)^2}{2\alpha}\right) + \\ \frac{\sigma_i^2}{\alpha^2}\left(\frac{1-\left(e^{-\alpha(t_i-t_{i-1})}-2\right)^2}{2\alpha} + (t_i - t_{i-1})\right),\end{aligned} \tag{5.2.51}$$

$$\tilde{\sigma}_i^2(t,t_i) = \frac{\sigma_i^2}{\alpha^2}\left(\frac{1-\left(e^{-\alpha(t_i-t)}-2\right)^2}{2\alpha} + (t_i - t)\right). \tag{5.2.52}$$

Proof of Proposition 11. As usual, the arbitrage-free price $S_I(t,i(t))$ of a year-on-year payer inflation swap is given by the formula

$$S_I(t,i(t)) = E\left[F\sum_{i:t_i\geq t}\exp\left(-\int_t^{t_i} r_N(s)ds\right)\left(\frac{I(t_i)}{I(t_{i-1})} - K_i\right)\middle|\mathcal{F}_t\right], \tag{5.2.53}$$

where $E[.|\mathcal{F}_t]$ denotes the conditional expectation with respect to the risk neutral probability measure Q. The expectation in (5.2.53) can be divided in two parts, where in the first one the sum goes over all time intervals $[t_{i-1},t_i]$ that lie after time t and the second one contains the intervals $[t_{i-1},t_i]$ that include time t

$$\begin{aligned}S_I(t,i(t)) = F\sum_{i:t_{i-1}\geq t}\exp\left(-\int_t^{t_i} r_N(s)ds\right)\left(E\left[\exp\left(\int_{t_{i-1}}^{t_i} i(s)ds\right)\middle|\mathcal{F}_t\right] - K_i\right) + \\ F\sum_{i:t_i\geq t\wedge t_{i-1}<t}\exp\left(-\int_t^{t_i} r_N(s)ds\right)\left(\frac{I(t)}{I(t_{i-1})}E\left[\exp\left(\int_t^{t_i} i(s)ds\right)\middle|\mathcal{F}_t\right] - K_i\right).\end{aligned} \tag{5.2.54}$$

From (5.2.41)-(5.2.44) we know the expectations and variances, $\tilde{\mu}_i(t_{i-1},t_i)$, $\tilde{\mu}_i(t,t_i)$, $\tilde{\sigma}_i^2(t_{i-1},t_i)$ and $\tilde{\sigma}_i^2(t,t_i)$, of the integrals over inflation rate $\int_{t_{i-1}}^{t_i} i(s)ds$ and $\int_t^{t_i} i(s)ds$. Using the following relation

$$E[\exp(X)] = \exp(E[X] + \frac{1}{2}Var[X]) \tag{5.2.55}$$

we directly get (5.2.48) by evaluating the expectations in (5.2.54). □

5.2.6 Inflation Swaption

A swaption gives a right to enter a year-on-year payer inflation swap contract at time T. A swaption with notional F has the following future payoff at the maturity T

$$\max[S_I(T, i(T)), 0], \tag{5.2.56}$$

where $S_I(T, i(T))$ is a fair price of a year-on-year payer inflation swap with notional F at time T. We assume that the swaption maturity T coincides with the first reset date of the underlying inflation swap, i.e. $T = t_0$.

Proposition 12. Under the assumptions that inflation rate $i(t)$ follows the extended Vasicek model (4.2.1) and nominal and real interest rates, $r_N(t)$ and $r_R(t)$, are deterministic functions, the fair price of a swaption with notional F at time $t \in [0, T]$ is given by

$$\begin{aligned} SO_I(t, i(t)) &= F \exp\Big(-\int_t^T r_N(s)ds\Big) \\ &\int_{z_0}^{\infty}\Big(\sum_{i=1}^{n} \exp\Big(-\int_T^{t_i} r_N(s)ds + \tilde{\mu}_i(t_{i-1}, t_i) + \frac{1}{2}\tilde{\sigma}_i^2(t_{i-1}, t_i)\Big) - \\ &K_i \exp\Big(-\int_T^{t_i} r_N(s)ds\Big)\Big) f(z)dz, \end{aligned} \tag{5.2.57}$$

where $T = t_0 \le \ldots \le t_n$, parameters $\tilde{\mu}_i(t_{i-1}, t_i)$ and $\tilde{\sigma}_i^2(t_{i-1}, t_i)$ are

$$\begin{aligned} \tilde{\mu}_i(t_{i-1}, t_i) &= i(T)\frac{e^{-\alpha(t_{i-1}-T)} - e^{-\alpha(t_i - T)}}{\alpha} + \\ &\int_T^{t_{i-1}} \frac{e^{-\alpha(t_{i-1}-s)} - e^{-\alpha(t_i - s)}}{\alpha}\theta(s)ds + \int_{t_{i-1}}^{t_i} \frac{1 - e^{-\alpha(t_i - s)}}{\alpha}\theta(s)ds, \end{aligned} \tag{5.2.58}$$

$$\begin{aligned} \tilde{\sigma}_i^2(t_{i-1}, t_i) &= \frac{\sigma_i^2}{\alpha^2}\Big(\frac{(1 - e^{-\alpha(t_i - t_{i-1})})^2 - (e^{-\alpha(t_i - T)} - e^{-\alpha(t_{i-1}-T)})^2}{2\alpha}\Big) + \\ &\frac{\sigma_i^2}{\alpha^2}\Big(\frac{1 - (e^{-\alpha(t_i - t_{i-1})} - 2)^2}{2\alpha} + (t_i - t_{i-1})\Big), \end{aligned} \tag{5.2.59}$$

inflation rate $i(T)$ is given by

$$i(T) = i(t)e^{-\alpha(T-t)} + \int_t^T e^{-\alpha(T-s)}\theta(s)ds + \sigma_i\sqrt{\frac{1 - e^{-2\alpha(T-t)}}{2\alpha}}z, \tag{5.2.60}$$

z_0 is the solution of the following equation for z

$$\sum_{i=1}^{n}\exp\Big(-\int_T^{t_i} r_N(s)ds + \tilde{\mu}_i(t_{i-1},t_i) + \frac{1}{2}\tilde{\sigma}_i^2(t_{i-1},t_i)\Big) = \sum_{i=1}^{n} K_i \exp\Big(-\int_T^{t_i} r_N(s)ds\Big) \quad (5.2.61)$$

and f is the density function of the $\mathcal{N}(0,1)$-distribution, i.e.

$$f(z) = \frac{1}{\sqrt{2\pi}}\exp\Big(-\frac{z^2}{2}\Big). \quad (5.2.62)$$

Proof of Proposition 12. As usual, the arbitrage-free price $SO_I(t,i(t))$ of an inflation swaption is given by the formula

$$SO_I(t,i(t)) = E\Big[\exp\Big(-\int_t^T r_N(s)ds\Big)\max[S_I(T,i(T)),0]\Big|\mathcal{F}_t\Big], \quad (5.2.63)$$

where $E[.|\mathcal{F}_t]$ denotes the conditional expectation with respect to the risk neutral probability measure Q. According to (5.2.48) and taking into consideration the swap reset sate structure $T = t_0 \leq ... \leq t_n$, the fair price of a year-on-year payer inflation swap with notional F at time T is given by

$$\begin{aligned} S_I(T,i(T)) = F\sum_{i=1}^{n}\exp\Big(-\int_T^{t_i} r_N(s)ds + \tilde{\mu}_i(t_{i-1},t_i) + \frac{1}{2}\tilde{\sigma}_i^2(t_{i-1},t_i)\Big) - \\ K_i \exp\Big(-\int_T^{t_i} r_N(s)ds\Big), \end{aligned} \quad (5.2.64)$$

where parameters $\tilde{\mu}_i(t_{i-1},t_i)$ and $\tilde{\sigma}_i^2(t_{i-1},t_i)$ are

$$\begin{aligned} \tilde{\mu}_i(t_{i-1},t_i) = i(T)\frac{e^{-\alpha(t_{i-1}-T)} - e^{-\alpha(t_i-T)}}{\alpha} + \\ \int_T^{t_{i-1}} \frac{e^{-\alpha(t_{i-1}-s)} - e^{-\alpha(t_i-s)}}{\alpha}\theta(s)ds + \int_{t_{i-1}}^{t_i} \frac{1-e^{-\alpha(t_i-s)}}{\alpha}\theta(s)ds, \end{aligned} \quad (5.2.65)$$

$$\begin{aligned} \tilde{\sigma}_i^2(t_{i-1},t_i) = \frac{\sigma_i^2}{\alpha^2}\Big(\frac{(1-e^{-\alpha(t_i-t_{i-1})})^2 - (e^{-\alpha(t_i-T)} - e^{-\alpha(t_{i-1}-T)})^2}{2\alpha}\Big) + \\ \frac{\sigma_i^2}{\alpha^2}\Big(\frac{1-(e^{-\alpha(t_i-t_{i-1})}-2)^2}{2\alpha} + (t_i - t_{i-1})\Big). \end{aligned} \quad (5.2.66)$$

Basing on (4.2.2) the inflation rate $i(T)$ can be written as

$$i(T) = i(t)e^{-\alpha(T-t)} + \int_t^T e^{-\alpha(T-s)}\theta(s)ds + \sigma_i\sqrt{\frac{1-e^{-2\alpha(T-t)}}{2\alpha}}Z\,, \tag{5.2.67}$$

where $Z \sim \mathcal{N}(0,1)$ is a standard normal variable. Then the expectation of maximum in (5.2.63) can be written as

$$\int_{-\infty}^{\infty} \max[S_I(T, i(T)), 0] f(z)dz\,, \tag{5.2.68}$$

where f is the density function of the $\mathcal{N}(0,1)$-distribution, i.e.

$$f(z) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right). \tag{5.2.69}$$

The integrand in the integral above (5.2.68) vanishes when $z < z_0$, where z_0 is the solution of the following equation for z

$$\sum_{i=1}^{n} \exp\left(-\int_T^{t_i} r_N(s)ds + \tilde{\mu}_i(t_{i-1}, t_i) + \frac{1}{2}\tilde{\sigma}_i^2(t_{i-1}, t_i)\right) = \sum_{i=1}^{n} K_i \exp\left(-\int_T^{t_i} r_N(s)ds\right). \tag{5.2.70}$$

The integral (5.2.68) can thus be written as

$$\int_{z_0}^{\infty} S_I(T, i(T)) f(z)dz\,. \tag{5.2.71}$$

Collecting all results we directly get (5.2.57)-(5.2.62). To present a closed form solution in this case is impossible due to the complexity of the involving functions. □

C h a p t e r 4

MAIN STUDIES AND RESULTS

6 Continuous-time Portfolio Problem with Wealth Process

So far in the continuous-time market model we have been only looking at the problem of pricing contingent claims, i.a. inflation-linked bonds. Now we look at the so-called portfolio problem, where we are given a fixed initial capital and search for an admissible self-financing portfolio process, which yields a payment stream as lucrative as possible. In other words, for a given initial capital of $x > 0$, the continuous-time portfolio problem in the given continuous-time financial market consists of the determination of an optimal investment strategy at each time instant $t \in [0, T]$ to maximize the utility from the terminal wealth or the terminal purchasing power at the time horizon $t = T$. In the following chapters we will consider the financial market, where the source of randomness is a consumer price index modeled as a geometric Brownian motion except for the second case, where we look at the financial market driven by extended Vasicek process of inflation rate. In all, four different optimization problems for the terminal wealth optimization are analyzed. In the first two we assume that financial market consists of a regular bond and an inflation-linked bond, where we distinguish two problems for two different models for inflation. In the third problem we extend the scope of financial products by introducing also the opportunity for an investor to invest into a regular stock. The last fourth case is similar to the first one but there we have a constraint on the terminal wealth, so the methods of constraint optimization are applied.

6.1 Problem 1: Conventional Bond and Inflation-linked Bond

In the first case we assume that the investor has access to the financial market where a regular bond $P_0(t)$ (risk-free asset) and an inflation-linked bond $B_{IL}(t, I(t))$ (risky asset) are traded. The dynamic of the price of a regular bond $P_0(t)$ under the physical (empirical) probability measure P (equivalent to the risk neutral probability measure Q) that describes the behavior of financial instruments' prices in the "real world", is given by

$$dP_0(t) = P_0(t) r_N(t) dt, \quad P_0(0) = 1. \tag{6.1.1}$$

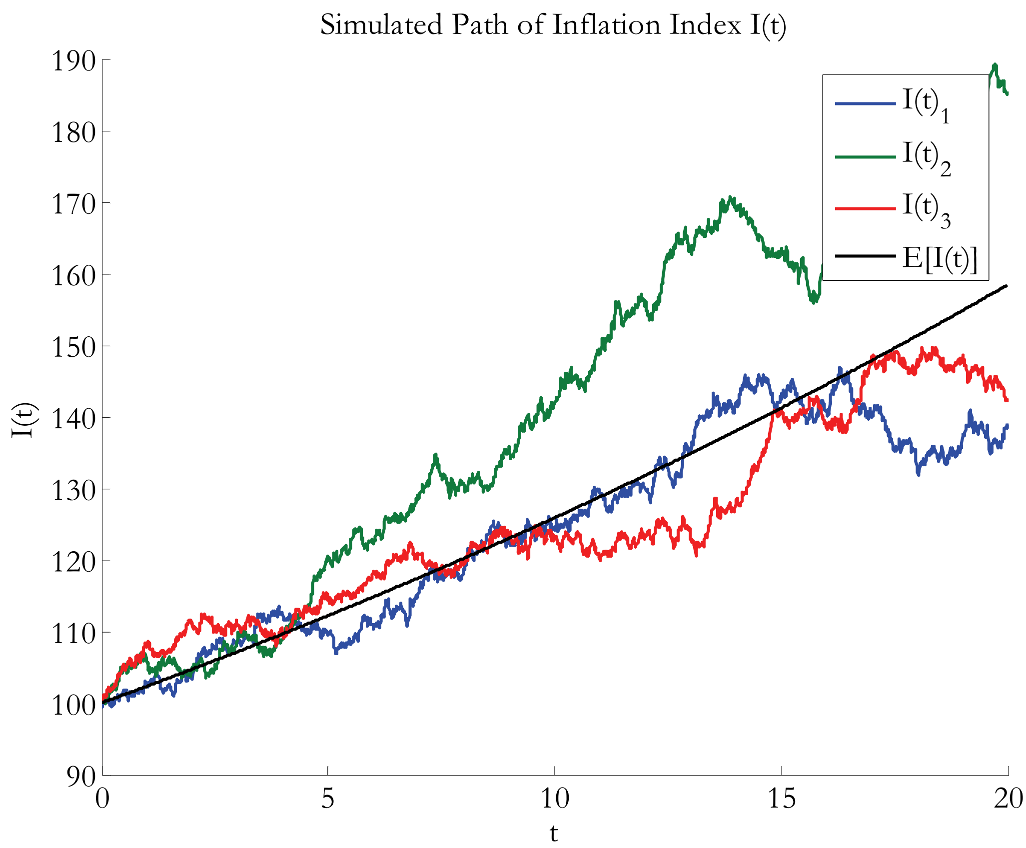

Figure 6.1 Consumer price index as a geometric Brownian motion under the physical probability measure.

The dynamic of the inflation process under the physical probability measure P is modeled by the geometric Brownian motion model for consumer price index $I(t)$, which represents the source of randomness in the financial market model, i.e.

$$dI(t) = I(t)((r_N(t) - r_R(t) + \lambda(t)\sigma_I)dt + \sigma_I d\tilde{W}_I(t)), \quad I(0) = i, \tag{6.1.2}$$

where the coefficients $r_N(t)$ and $r_R(t)$, nominal and real interest rates respectively, are assumed to be deterministic functions, σ_I is a non-negative constant, $\tilde{W}_I(t)$ is a standard Brownian motion under the physical probability measure P and $\lambda(t)$ is assumed to be a deterministic function. Naturally, the dynamics of consumer price index $I(t)$ under the risk neutral probability measure Q is given by (4.1.1).

Figure 6.1 shows the geometrical Brownian motion model for inflation index under the physical probability measure P, where three realizations of simulated consumer price index processes are shown for $t \in [0, T]$, where maturity date $T = 20$ years, as well as the expectation of the

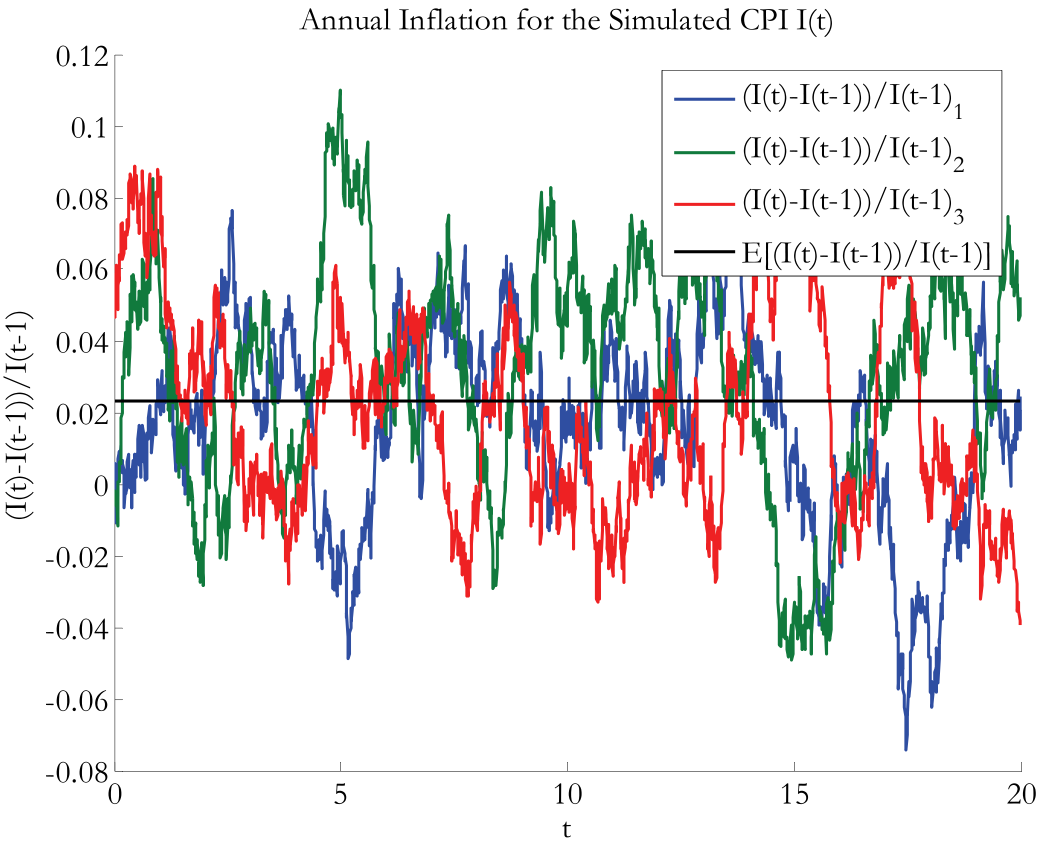

Figure 6.2 Annual inflation for the simulated inflation indexes in Figure 6.1 under the physical probability measure.

index process is plotted. For the simulation the following set of parameter values was used on the yearly basis

$$
\begin{aligned}
r_N(t) &\equiv 0.05, \\
r_R(t) &\equiv 0.03, \\
\sigma_I &= 0.03, \\
i &= I(0) = 100, \\
\lambda(t) &\equiv 0.10.
\end{aligned}
$$

For the simulation the same innovations for the Brownian motion were used as the ones for the simulation of the geometric Brownian motion under the risk neutral probability measure Q in Figure 4.1. Comparing these two figures it is naturally to see a more rapid increase in inflation index under the P-measure due to the bigger drift term because of the positive price of risk $\lambda(t) \equiv 0.10$. In addition, in Figure 6.2 annual inflation is plotted basing on the simulated paths of inflation index $I(t)$ in Figure 6.1. Also for the annual inflation we observe a higher level of inflation under the physical probability measure P compared to the level of inflation under the risk neutral probability measure Q in Figure 4.2 due to the positive price of risk $\lambda(t) \equiv 0.10$.

From section 5.1 Proposition 2 we know that under the assumption that consumer price index $I(t)$ follows the geometric Brownian motion the fair price of an inflation-linked T-bond $B_{IL}(t, I(t))$ is given by

$$\begin{aligned} B_{IL}(t, I(t)) = \sum_{i:t_i \geq t} C_i \frac{I(t)}{I(t_0)} \exp\Big(-\int_t^{t_i} r_R(s)ds\Big) + \\ F\Bigg(\exp\Big(-\int_t^T r_N(s)ds\Big) + \frac{C_I(t, I(t))}{I(t_0)}\Bigg), \end{aligned} \tag{6.1.3}$$

where $C_I(t, I(t))$ is a European call on consumer price index $I(T)$ with strike price $K = I(t_0)$, i.e.

$$\begin{aligned} C_I(t, I(t)) = I(t) \exp\Big(-\int_t^T r_R(s)ds\Big) N(d(t)) - \\ I(t_0) \exp\Big(-\int_t^T r_N(s)ds\Big) N(d(t) - \sigma_I \sqrt{T-t}), \end{aligned} \tag{6.1.4}$$

where N is the cumulative distribution function for the standard normal distribution $\mathcal{N}(0,1)$ and

$$d(t) = \frac{\ln\Big(\frac{I(t)}{I(t_0)}\Big) + \int_t^T (r_N(s) - r_R(s))ds + \frac{1}{2}\sigma_I^2(T-t)}{\sigma_I \sqrt{T-t}}. \tag{6.1.5}$$

Assuming that an investor has an opportunity to invest his initial capital $x > 0$ into a normal bond of price $P_0(t)$ and an inflation-linked bond of price $B_{IL}(t, I(t))$ investor's wealth $X(t)$ at time t can be expressed as

$$X(t) = \varphi_0(t)P_0(t) + \varphi_1(t)B_{IL}(t, I(t)), \quad X(0) = x, \tag{6.1.6}$$

where trading strategy $\varphi(t) = (\varphi_0(t), \varphi_1(t))'$ is $\mathbb{R}^2$-valued progressively measurable processes with respect to the filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$ generated by the standard Brownian motion $\{\tilde{W}_I(t)\}_{t\in[0,T]}$ satisfying

$$\int_0^T |\varphi_0(t)|\,dt < \infty \text{ a.s. } P\,, \tag{6.1.7}$$

$$\int_0^T (\varphi_1(t)B_{IL}(t,I(t)))^2\,dt < \infty \text{ a.s. } P\,. \tag{6.1.8}$$

The trading strategy $\varphi(t)$ denotes the number of shares of normal bond and inflation-linked bond held by the investor at time t respectively. The $\mathbb{R}^2$-valued process $\pi(t) = (\pi_0(t), \pi_1(t))'$ with

$$\pi_1(t) := \frac{\varphi_1(t)B_{IL}(t,I(t))}{X(t)}\,, \tag{6.1.9}$$

$$\pi_0(t) := 1 - \pi_1(t) = \frac{\varphi_0(t)P_0(t)}{X(t)} \tag{6.1.10}$$

is called portfolio process that represent the fractions of total wealth invested in the different financial instruments. We assume that the trading strategy $\varphi(t)$ is self-financing (then the corresponding portfolio process $\pi(t)$ is also self-financing) that is the following relation is satisfied

$$X(t) = x + \int_0^t \varphi_0(s)dP_0(s) + \int_0^t \varphi_1(s)dB_{IL}(s,I(s))\,. \tag{6.1.11}$$

The portfolio process $\pi(t)$ is called admissible for the initial wealth $x > 0$, if the corresponding wealth process $X(t)$ satisfies

$$X(t) \geq 0 \text{ a.s. } P \ \forall t \in [0,T]. \tag{6.1.12}$$

The set of admissible portfolio processes $\pi(t)$ is denoted by $\mathcal{A}(x)$. In our case we do not have consumption in the model, so the set of admissible portfolio processes $\mathcal{A}(x)$ is defined by

$$\mathcal{A}(x) := \left\{ (1-\pi_1(.), \pi_1(.))' \middle| X(0) = x > 0,\ \int_0^T \pi_1^2(t)dt < \infty \text{ a.s. } P \right\}. \tag{6.1.13}$$

The continuous-time portfolio problem consists of maximizing expected utility of terminal wealth $X(T)$ of the investor, i.e. the continuous-time portfolio problem looks like

$$\max_{\pi(.)\in\mathcal{A}'(x)} E[U(X(T))], \tag{6.1.14}$$

where the subset of admissible portfolio processes $\mathcal{A}'(x)$ given by

$$\mathcal{A}'(x) := \{\pi(.) \in \mathcal{A}(x) | E[U(X(T))^{-}] < \infty\} \tag{6.1.15}$$

insures the existence of the expected value in (6.1.14). The utility function $U(.) : (0,\infty) \to \mathbb{R}$ is a strictly concave C^1-function that satisfies

$$U'(0) := \lim_{x\to 0} U'(x) = \infty, \tag{6.1.16}$$

$$U'(\infty) := \lim_{x\to\infty} U'(x) = 0. \tag{6.1.17}$$

To solve the option portfolio problem (6.1.14) directly with classical stochastic control methods seems to be highly complicated due to the fact that the price of an inflation-linked bond $B_{IL}(t, I(t))$ is obviously given by the non-linear stochastic differential equation. To overcome this difficulty we will apply the results of Korn and Trautmann (1999) [18] that are summarized in the theorem below.

Theorem 1. Under the assumption that the replicating strategy $\psi(t)$ of the derivative prices $f_i(t, P_1(t),\dots, P_n(t))$ by risky assets $P_i(t)$, $i = 1,\dots,n$, given by

$$\psi(t) := \begin{bmatrix} \frac{\partial f_1(t, P_1(t),\dots,P_n(t))}{\partial P_1(t)} & \cdots & \frac{\partial f_1(t, P_1(t),\dots,P_n(t))}{\partial P_n(t)} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n(t, P_1(t),\dots,P_n(t))}{\partial P_1(t)} & \cdots & \frac{\partial f_n(t, P_1(t),\dots,P_n(t))}{\partial P_n(t)} \end{bmatrix} \tag{6.1.18}$$

is regular for $\forall t \in [0,T]$ then the option portfolio problem

$$\max_{\pi(.)\in\mathcal{A}'(x)} E[U(X(T))] \tag{6.1.19}$$

with

$$X(t) = \varphi_0(t)P_0(t) + \varphi_1(t)f_1(t, P_1(t),\dots,P_n(t)) + \dots + \varphi_n(t)f_n(t, P_1(t),\dots,P_n(t)), \tag{6.1.20}$$

$$X(0) = x \tag{6.1.21}$$

admits the following optimal solution:

a) The optimal terminal wealth B^* coincides with the optimal terminal wealth of the corresponding basic portfolio problem given by

$$\max_{\eta(.)\in\mathcal{A}'(x)} E[U(X(T))], \tag{6.1.22}$$

$$X(t) = \xi_0(t)P_0(t) + \xi_1(t)P_1(t) + \ldots + \xi_n(t)P_n(t), \quad X(0) = x. \tag{6.1.23}$$

b) Having the optimal trading strategy $\overline{\xi}(t) = (\xi_1(t),\ldots,\xi_n(t))'$ for the basic portfolio problem (6.1.22)-(6.1.23), the optimal trading strategy $\overline{\varphi}(t) = (\varphi_1(t),\ldots,\varphi_n(t))'$ and $\varphi_0(t)$ for the option portfolio problem (6.1.19)-(6.1.21) is given by

$$\overline{\varphi}(t) = (\psi(t)')^{-1}\overline{\xi}(t), \tag{6.1.24}$$

$$\varphi_0(t) = \frac{X(t) - \sum_{i=1}^{n}\varphi_i(t)f_i(t,P_1(t),\ldots,P_n(t))}{P_0(t)}. \tag{6.1.25}$$

Remark 1 on Theorem 1 *(Nature of the Continuous-time Portfolio Problem)***.** For the application to our inflation market setting one should note that the price of the inflation-linked bond is constructed in a way as if the inflation-linked bond accumulates interest at the real rate $r_R(t)$ which – as in the Garman Kohlhagen model – is interpreted as a continuous dividend yield. Thus, the trading strategy in the inflation-linked bond and the cash account (i.e. the nominal account) given indeed leads to the same wealth process as the strategy in the index and in the cash account if the real interest accumulated by the inflation-linked bond is directly reinvested in it.

Due to the fact that an inflation-linked bond $B_{IL}(t,I(t))$ is a derivative on inflation index $I(t)$, according to Korn and Trautmann (1999) [18], Theorem 1, the optimal final wealth of the option portfolio problem (6.1.14) coincides with the optimal final wealth of the basic portfolio problem, where the investor is assumed to be able to invest into regular bond $P_0(t)$ and inflation index $I(t)$. The basic portfolio problem is given by

$$\max_{\eta(.)\in\mathcal{A}'(x)} E[U(X(T))], \tag{6.1.26}$$

where the wealth process $X(t)$ can with the help of trading strategy $\xi(t) = (\xi_0(t), \xi_1(t))'$ be written as

$$X(t) = \xi_0(t)P_0(t) + \xi_1(t)I(t), \quad X(0) = x \tag{6.1.27}$$

with portfolio process $\eta(t) = (\eta_0(t), \eta_1(t))'$ having the form

$$\eta_1(t) := \frac{\xi_1(t)I(t)}{X(t)}, \tag{6.1.28}$$

$$\eta_0(t) := 1 - \eta_1(t) = \frac{\xi_0(t)P_0(t)}{X(t)}. \tag{6.1.29}$$

For the wealth process $X(t)$ of (6.1.27) we can write down the following stochastic differential equation

$$dX(t) = X(t)((r_N(t) + \eta_1(t)(\lambda(t)\sigma_I - r_R(t)))dt + \eta_1(t)\sigma_I d\tilde{W}_I(t)), \quad X(0) = x\,. \tag{6.1.30}$$

The set of admissible portfolio processes $\mathcal{A}(x)$ is then given by

$$\mathcal{A}(x) := \left\{(1-\eta_1(.), \eta_1(.))' \middle| X(0) = x > 0,\ \int_0^T \eta_1^2(t)dt < \infty \text{ a.s. } P\right\}. \tag{6.1.31}$$

In particular, the square-integrability condition on $\eta_1(.)$ in (6.1.31) ensures the uniqueness and existence of the solution of stochastic differential equation (6.1.30) as well as implies strict positivity of the wealth process $X(t)$, i.e.

$$X(0) = x > 0,\ X(t) > 0 \text{ a.s. } P\ \forall t \in [0,T]. \tag{6.1.32}$$

For the logarithmic and power utility functions

$$U(x) \in \left\{\ln(x), \frac{1}{\gamma}x^\gamma\right\}, \quad \gamma \in (0,1) \tag{6.1.33}$$

the solution of the basic portfolio problem (6.1.26) is a well-known result and the optimal portfolio process $\eta_1(t)$ is given by

$$\eta_1(t) = \frac{(r_N(t) - r_R(t) + \lambda(t)\sigma_I) - r_N(t)}{(1-\gamma)\sigma_I^2} = \frac{\lambda(t)\sigma_I - r_R(t)}{(1-\gamma)\sigma_I^2}, \tag{6.1.34}$$

where the case of $\gamma = 0$ corresponds to the optimal portfolio process for the logarithmic utility function. Having the optimal portfolio process $\eta_1(t)$ (6.1.34) for the basic portfolio problem (6.1.26) we can now express the optimal trading strategy $\xi_1(t)$ as

$$\xi_1(t) = \frac{\lambda(t)\sigma_I - r_R(t)}{(1-\gamma)\sigma_I^2} \cdot \frac{X(t)}{I(t)}. \tag{6.1.35}$$

Applying the results of Korn and Trautmann (1999) [18], Theorem 1, and having the optimal trading strategy $\xi_1(t)$ for the basic portfolio problem (6.1.26) we are able to express the optimal trading strategy $\varphi(t)$ of the option portfolio problem (6.1.14) as

$$\varphi_1(t) = \frac{\xi_1(t)}{\psi(t)} = \frac{\lambda(t)\sigma_I - r_R(t)}{(1-\gamma)\sigma_I^2} \cdot \frac{X(t)}{\psi(t)I(t)}, \tag{6.1.36}$$

$$\varphi_0(t) = \frac{X(t) - \varphi_1(t)B_{IL}(t, I(t))}{P_0(t)} = \left(1 - \frac{(\lambda(t)\sigma_I - r_R(t))B_{IL}(t, I(t))}{(1-\gamma)\sigma_I^2\psi(t)I(t)}\right) \cdot \frac{X(t)}{P_0(t)}, \tag{6.1.37}$$

where $\psi(t)$ is a replicating strategy of an inflation-linked bond $B_{IL}(t, I(t))$ by inflation index $I(t)$ defined as

$$\psi(t) := \frac{\partial B_{IL}(t, I(t))}{\partial I(t)}, \tag{6.1.38}$$

i.e.

$$\psi(t) = \sum_{i:t_i \geq t} \frac{C_i}{I(t_0)} \exp\left(-\int_t^{t_i} r_R(s)ds\right) + \frac{F \exp\left(-\int_t^T r_R(s)ds\right) N(d(t))}{I(t_0)}. \tag{6.1.39}$$

Theorem 2. The optimal portfolio process $\pi_1(t)$ of an inflation-linked bond $B_{IL}(t, I(t))$ for the continuous-time portfolio problem (6.1.14) for the HARA utility functions (logarithmic and power utility (6.1.33)) is given by

$$\pi_1(t) = \frac{\lambda(t)\sigma_I - r_R(t)}{(1-\gamma)\sigma_I^2} \cdot \frac{B_{IL}(t, I(t))}{\psi(t)I(t)}, \tag{6.1.40}$$

where $\psi(t)$ is a replicating strategy given by (6.1.39); the case of $\gamma = 0$ corresponds to the logarithmic utility function.

Proof of Theorem 2. From (6.1.9) we know that the optimal portfolio process $\pi_1(t)$ can be represented as

$$\pi_1(t) = \frac{\varphi_1(t)B_{IL}(t, I(t))}{X(t)}. \tag{6.1.41}$$

Substituting (6.1.36) into this relation (6.1.41) we directly get (6.1.40). □

Remark 1 on Theorem 2 *("Negativity" of the Optimal Portfolio Process)***.** The sign of the optimal portfolio process $\pi_1(t)$ is determined by the sign of the term

$$\lambda(t)\sigma_I - r_R(t) \tag{6.1.42}$$

due to the fact that

$$\frac{B_{IL}(t, I(t))}{\psi(t)I(t)} > 1. \tag{6.1.43}$$

This means that for a small enough term $\lambda(t)$ the optimal portfolio process $\pi_1(t)$ could be negative. In order to have the positive optimal portfolio process $\pi_1(t)$ for an inflation-linked bond $B_{IL}(t, I(t))$ the subjective excess return $\lambda(t)\sigma_I$ has to be bigger than the real interest rate $r_R(t)$ – an assumption that rarely seems to hold. So in this imaginary portfolio problem a risk-averse investor typically sells inflation-linked products when behaving optimally.

This fact is natural, due to the method of indexing payments of the inflation-linked bond on the inflation. In the basic portfolio problem, where we assume inflation to be a tradable asset, it is a risky investment whereas the drift term hardly outperforms the one of the conventional bond (see (6.1.1)-(6.1.2)). In conversion to the original option portfolio problem we just replicate the optimal trading strategies via replicating strategies. Overall the dynamics of the price of the inflation-linked bond is given by

$$dB_{IL}(t,I(t)) = \Delta(t)dI(t) + \frac{1}{2}\Gamma(t)dI^2(t), \tag{6.1.44}$$

where the greeks are

$$\Delta(t) := \frac{\partial B_{IL}(t,I(t))}{\partial I(t)} = \psi(t), \tag{6.1.45}$$

$$\Gamma(t) := \frac{\partial^2 B_{IL}(t,I(t))}{(\partial I(t))^2}. \tag{6.1.46}$$

Calculating the greeks (6.1.45)-(6.1.46) and substituting them into (6.1.44) we get

$$dB_{IL}(t,I(t)) = \psi(t)dI(t) + \frac{f(d(t))}{2\sqrt{T-t}}I(t)\sigma_I dt\,. \tag{6.1.47}$$

From this relation we can see how the dynamics of the inflation index $I(t)$ affects the form of the stochastic differential equation for $B_{IL}(t,I(t))$. Furthermore, for the special case of the inflation-linked bond without deflation protection due to the linear form of the option on the underlying we have

$$dB_{IL}(t,I(t)) = B_{IL}(t,I(t))((r_N(t) - r_R(t) + \lambda(t)\sigma_I)dt + \sigma_I d\tilde{W}_I(t)). \tag{6.1.48}$$

In general, the sign of the optimal portfolio process $\pi_1(t)$ for the regular HARA utility functions (exponential $-e^{-ax}$, logarithmic $\ln(x)$, power $\frac{1}{\gamma}x^\gamma$, quadratic $x - bx^2$) is always determined by the sign of the term in (6.1.42), which is actually negative for an investor with a normal opinion on the subjective excess return $\lambda(t)\sigma_I$ that is not higher that the real interest rate $r_R(t)$. Comparing the properties of inflation-indexed bonds with those of conventional bonds and indeed with those of all other asset classes, their major benefit is clear: a much reduced exposure to unexpected changes in the level of prices.

The form of the optimal portfolio process for an inflation-linked bond $B_{IL}(t,I(t))$ in (6.1.40), to be more precise, the negative sign of the optimal portfolio process $\pi_1(t)$, is in fact in line with the empirical observations and heuristic considerations of Deacon, Derry and Mirfendereski (2004) [9]. Since the purchase of inflation-indexed bonds reduces the real risk of a portfolio so it must increase the nominal risk, because inflation-indexed bonds offer real value certainty while conventional bonds provide nominal value certainty, and for this reason indexed bonds appeal most to those investors for whom real value certainty is the more important. Since financial assets are often acquired as a means to hedge future financial commitments, preferences here are likely to be determined by the nature of such liabilities. For instance, since banks generally have nominal liabilities they might be expected to have a preference for assets providing nominal value certainty, whereas pension funds may attach greater value to real assets to offset any inflation-indexed annuity commitment. Later in the thesis we will actually study the hedging problem of inflation-related payment in detail and will show the usefulness of having inflation-linked bonds in a hedging portfolio.

Remark 2 on Theorem 2 *(Asymptotic Behavior)***.** Analyzing the term (6.1.43) we can get some asymptotic results. For large $I(t) \gg I(t_0)$ as well as for small $I(t) \ll I(t_0)$ from (5.1.20) and (5.1.22) respectively we have for the asymptotic behavior of the optimal portfolio process $\pi_1(t)$

$$\pi_1(t) \sim \frac{\lambda(t)\sigma_I - r_R(t)}{(1-\gamma)\sigma_I^2}, \quad I(t) \to \infty, \tag{6.1.49}$$

$$\pi_1(t) \sim \frac{\lambda(t)\sigma_I - r_R(t)}{(1-\gamma)\sigma_I^2}\left(1 + \frac{F\exp\left(-\int_t^T r_N(s)ds\right)}{\psi(t)I(t)}\right), \quad I(t) \to 0. \tag{6.1.50}$$

In addition, this asymptotic behavior of the optimal portfolio process $\pi_1(t)$ of the inflation-linked bond $B_{IL}(t,I(t))$ is visualized in Figure 6.3, where the optimal portfolio process of the inflation-linked T-bond at time $t=0$, $\pi_1(t)$, as well as the asymptote functions (6.1.49)-(6.1.50) are plotted. The following set of parameters was used for the simulation

$$\begin{aligned} r_N(t) &\equiv 0.05, \\ r_R(t) &\equiv 0.03, \\ \sigma_I &= 0.03, \\ \lambda(t) &\equiv 0.10, \end{aligned}$$

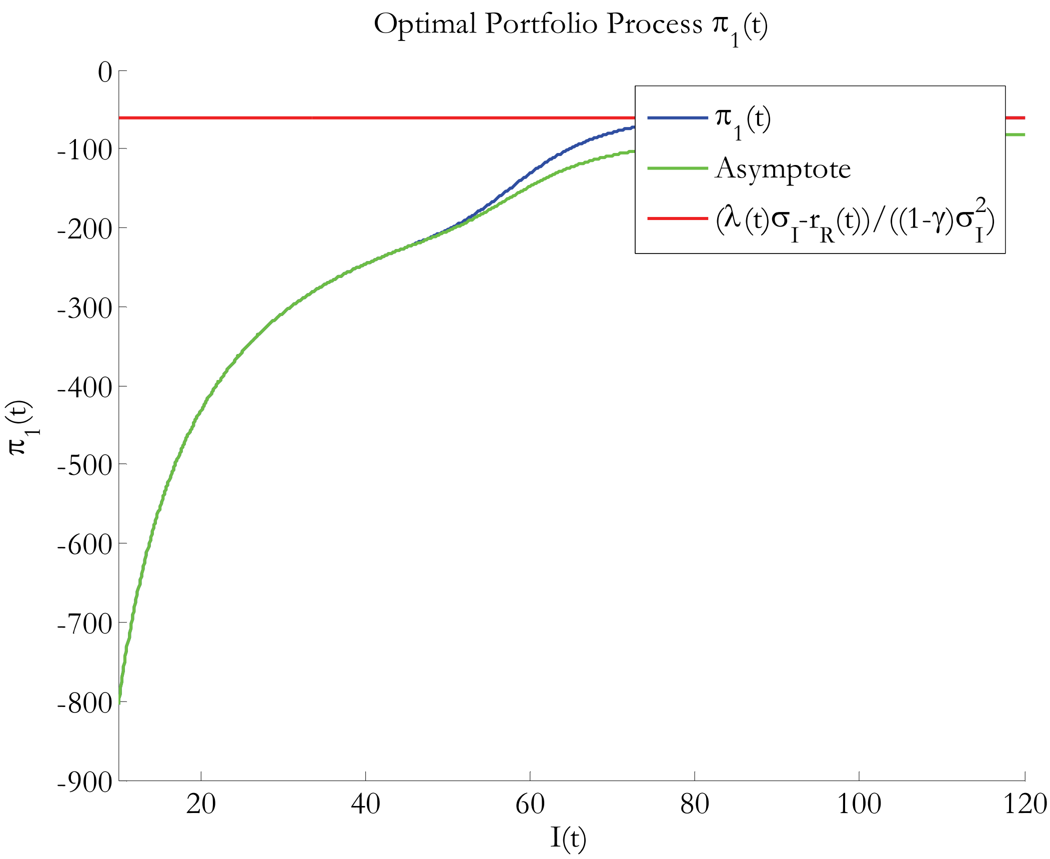

Figure 6.3 Optimal portfolio process of inflation-linked bond as a function of consumer price index.

$$
\begin{aligned}
T &= 20, \quad & F &= 100, \\
C_i &= 2, \quad & t_i &= 1,2,\ldots,20, \\
I(t_0) &= 100, \\
\gamma &= 0.5.
\end{aligned}
$$

In Figure 6.3 is clearly seen the asymptotic behavior of the optimal portfolio process $\pi_1(t)$ of the inflation-linked bond $B_{IL}(t, I(t))$ at large inflation indexes $I(t) \gg 0$ in line with (6.1.49), where for our model parameters

$$\frac{\lambda(t)\sigma_I - r_R(t)}{(1-\gamma)\sigma_I^2} = -60\,.$$

Whereas for small inflation indexes $I(t) \to 0$ it is seen from the plot that the optimal portfolio process $\pi_1(t)$ approaches its asymptote of (6.1.50).

In the same way as it was done for the fair price of inflation-linked bond $B_{IL}(t, I(t))$, from (5.1.23)-(5.1.25) we derive three cases depending from the level of inflation index $I(t)$ for the asymptotic behavior of the optimal portfolio process $\pi_1(t)$ of the inflation-linked bond $B_{IL}(t, I(t))$, when the time t approaches the maturity date T. These three cases of the level of the inflation index $I(t)$ are the case of inflation $I(t) > I(t_0)$, $I(t) = I(t_0)$ and the case of deflation $I(t) < I(t_0)$

$$\pi_1(t) \sim \frac{\lambda(t)\sigma_I - r_R(t)}{(1-\gamma)\sigma_I{}^2}, \quad t \to T, \quad I(t) > I(t_0), \tag{6.1.51}$$

$$\pi_1(t) \sim \frac{\lambda(t)\sigma_I - r_R(t)}{(1-\gamma)\sigma_I{}^2}\left(1 + \frac{F}{2\psi(t)I(t)}\right), \quad t \to T, \quad I(t) = I(t_0), \tag{6.1.52}$$

$$\pi_1(t) \sim \frac{\lambda(t)\sigma_I - r_R(t)}{(1-\gamma)\sigma_I{}^2}\left(1 + \frac{F}{\psi(t)I(t)}\right), \quad t \to T, \quad I(t) < I(t_0). \tag{6.1.53}$$

This asymptotic behavior of the optimal portfolio process $\pi_1(t)$ indirectly shows the feature of the inverse dependence between the optimal portfolio process and inflation index $I(t)$. Having higher probability for deflation, i.e. $I(t) < I(t_0)$, the relation fraction of inflation-linked bond, i.e. optimal portfolio process $\pi_1(t)$, has a higher value compared to the situation with a lower deflation probability, i.e. $I(t) > I(t_0)$.

In the same way basing on (5.1.26)-(5.1.28) we can conclude the following limiting behavior for the optimal portfolio process $\pi_1(t)$

$$\lim_{t \to T} \pi_1(t) = \frac{\lambda(t)\sigma_I - r_R(t)}{(1-\gamma)\sigma_I{}^2}, \quad I(t) > I(t_0), \tag{6.1.54}$$

$$\lim_{t \to T} \pi_1(t) = \frac{\lambda(t)\sigma_I - r_R(t)}{(1-\gamma)\sigma_I{}^2}\left(1 + \frac{F}{F + 2C_i}\right), \quad I(t) = I(t_0), \tag{6.1.55}$$

$$\lim_{t \to T} \pi_1(t) = \frac{\lambda(t)\sigma_I - r_R(t)}{(1-\gamma)\sigma_I{}^2}\left(1 + \frac{FI(t_0)}{C_i I(T)}\right), \quad I(t) < I(t_0). \tag{6.1.56}$$

The asymptotic behavior of the optimal portfolio process $\pi_1(t)$ of the inflation-linked bond price $B_{IL}(t, I(t))$ at the maturity date T (6.1.51)-(6.1.53) as well as the limiting behavior

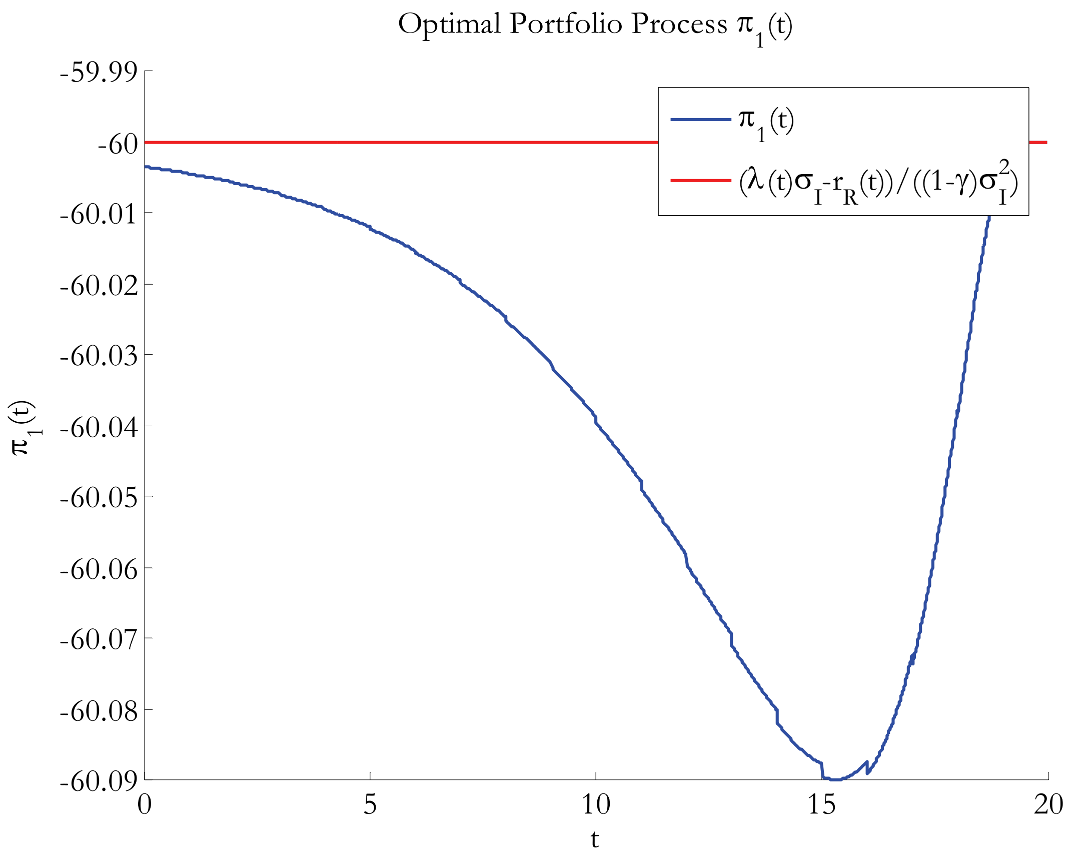

Figure 6.4 Optimal portfolio process of inflation-linked bond as a function of time, when inflation index is higher than the base index.

(6.1.54)-(6.1.56) are presented in Figure 6.4 – Figure 6.6 for $t \in [0,T]$, where maturity date $T = 20$ years and we have a different levels of inflation index $I(t)$ compared to the base index $I(t_0)$. For all three cases we use the usual set of model parameters

$$
\begin{aligned}
r_N(t) &\equiv 0.05, \\
r_R(t) &\equiv 0.03, \\
\sigma_I &= 0.03, \\
\lambda(t) &\equiv 0.10, \\
T &= 20, \qquad F = 100, \\
C_i &= 2, \qquad t_i = 1,2,\ldots,20, \\
I(t_0) &= 100, \\
\gamma &= 0.5.
\end{aligned}
$$

For the first case, Figure 6.4, we assume a constant inflation index $I(t) \equiv 110$ for the whole time period $t \in [0,T]$, in the same way as we did for the price of inflation-linked bond

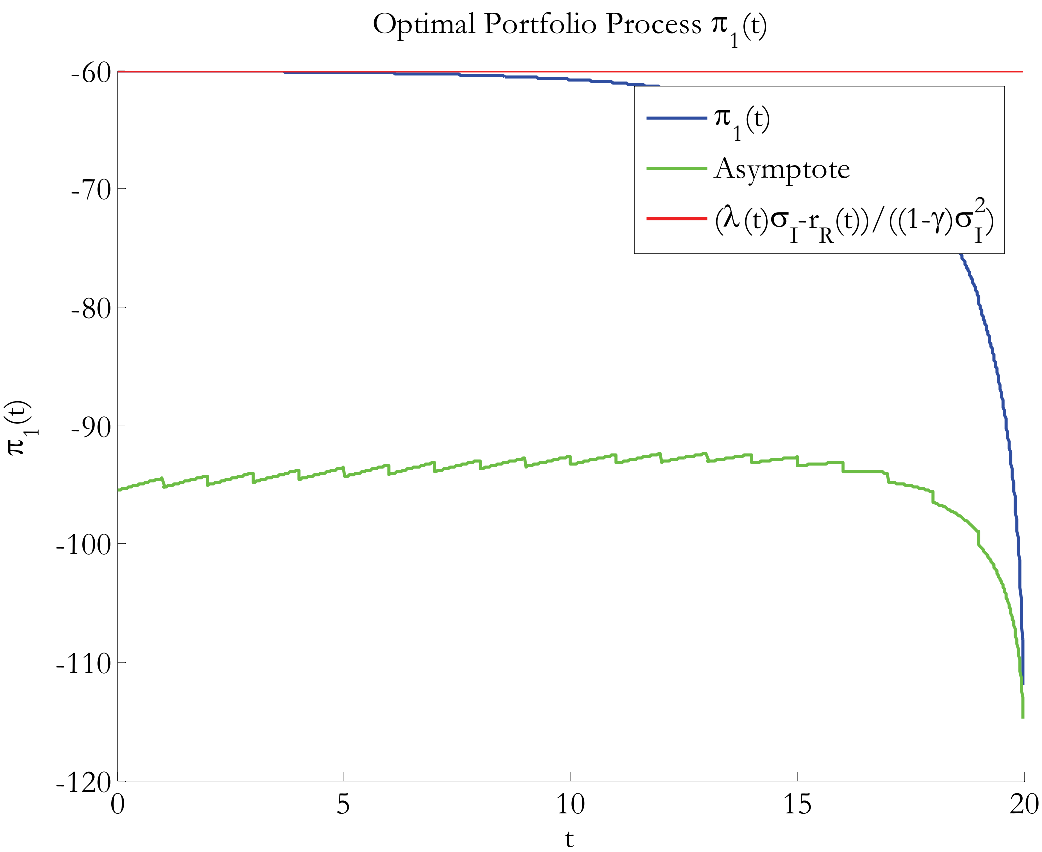

Figure 6.5 Optimal portfolio process of inflation-linked bond as a function of time, when inflation index is equal to the base index.

$B_{IL}(t, I(t))$ in Figure 5.2, which corresponds to the case of inflation at the maturity date T,(6.1.51) and (6.1.54). In this case the optimal portfolio process $\pi_1(t)$ almost totally corresponds to its limit (6.1.54) on the whole interval $t \in [0,T]$ and especially at time t approaching the maturity date T it coincides with its asymptote (6.1.51). In addition it is seen in the figure that the limit for the optimal portfolio process $\pi_1(t)$ is (6.1.54), i.e.

$$\lim_{t \to T} \pi_1(t) = \frac{\lambda(t)\sigma_I - r_R(t)}{(1-\gamma)\sigma_I{}^2} = -60.$$

Figure 6.5 shows the case, when inflation index $I(t)$ does not deviate from the base level $I(t_0)$ at all, i.e. $I(t) \equiv 100$; that is the same situation as in Figure 5.3, where we analyze the behavior of the fair price of inflation-linked bond $B_{IL}(t, I(t))$. Here we see that at time t relatively close to zero the optimal portfolio process $\pi_1(t)$ coincides with the optimal portfolio process of the basic portfolio problem (6.1.26), i.e.

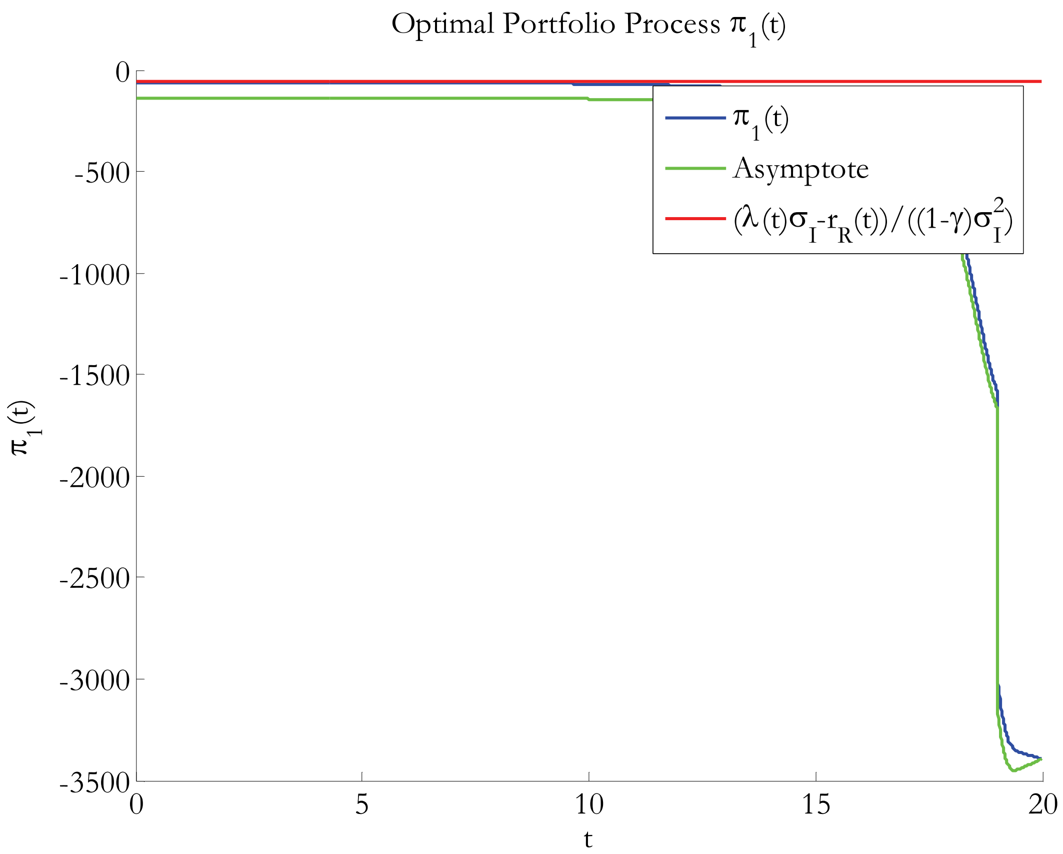

Figure 6.6 Optimal portfolio process of inflation-linked bond as a function of time, when inflation index is lower than the base index.

$$\frac{\lambda(t)\sigma_I - r_R(t)}{(1-\gamma)\sigma_I^2} = -60\,.$$

With the increasing time t the optimal portfolio process $\pi_1(t)$ deviates from its initial value and approaches its asymptote (6.1.52). At the maturity date T its limit is equal to (6.1.55), i.e.

$$\lim_{t\to T}\pi_1(t) = \frac{\lambda(t)\sigma_I - r_R(t)}{(1-\gamma)\sigma_I{}^2}\left(1+\frac{F}{F+2C_i}\right) = \text{-}117.6923\,.$$

In the third Figure 6.6 we consider the case of deflation, where we assume a constant inflation index $I(t)\equiv 90$ for the whole time period $t\in[0,T]$ that is the same as we did in Figure 5.4 for the price of the inflation-linked bond $B_{IL}(t,I(t))$. In this setup the optimal portfolio process $\pi_1(t)$ also equal to the optimal portfolio process of the basic portfolio problem (6.1.26) at time t close to zero. When time t is approaching bond's maturity date T, the optimal portfolio process $\pi_1(t)$ starts drastically to decrease to its asymptotic function (6.1.53) and approaches its limit of (6.1.56)

$$\lim_{t\to T}\pi_1(t) = \frac{\lambda(t)\sigma_I - r_R(t)}{(1-\gamma)\sigma_I{}^2}\left(1+\frac{FI(t_0)}{C_i I(T)}\right) = -3393.333\,.$$

Remark 3 on Theorem 2 *(Zero-coupon Inflation-linked Bond and Deflation-unprotected Principal Payment)***.** Having (6.1.43) we obtain that the absolute value of the optimal portfolio process $\pi_1(t)$ of the inflation-linked bond $B_{IL}(t,I(t))$ for the continuous-time portfolio problem (6.1.14) is always bigger than the one of the corresponding basic portfolio problem (6.1.26). In order to give an interpretation to this fact we first look at the optimal portfolio processes of two inflation-linked bonds with special structures studied in Remark 1 and 2 on Proposition 2: a zero-coupon inflation-linked bond and an inflation-linked bond without deflation protection.

For a zero-coupon inflation-linked bond (5.1.12), i.e.

$$C_i = 0, \quad \forall i = 1,\dots,n\,, \tag{6.1.57}$$

$$B_{IL}(t,I(t)) = F\left(\exp\left(-\int_t^T r_N(s)ds\right) + \frac{C_I(t,I(t))}{I(t_0)}\right) \tag{6.1.58}$$

we obtain the replicating strategy (6.1.38) of the form

$$\psi(t) = \frac{F\exp\left(-\int_t^T r_R(s)ds\right)N(d(t))}{I(t_0)}\,, \tag{6.1.59}$$

which would lead to the same behavior of the quotient (6.1.43).

In the case of an inflation-linked bond without deflation protection, i.e. the principal payment at the maturity date T is

$$F\frac{I(T)}{I(t_0)} \tag{6.1.60}$$

and the fair price is given by

$$B_{IL}(t,I(t)) = \frac{I(t)}{I(t_0)}\left(\sum_{i:t_i\geq t} C_i\exp\left(-\int_t^{t_i} r_R(s)ds\right) + F\exp\left(-\int_t^T r_R(s)ds\right)\right), \tag{6.1.61}$$

we obtain for the replicating strategy (6.1.38)

$$\psi(t) = \frac{1}{I(t_0)} \left(\sum_{i:t_i \geq t} C_i \exp\Big(-\int_t^{t_i} r_R(s)ds\Big) + F \exp\Big(-\int_t^{T} r_R(s)ds\Big) \right), \tag{6.1.62}$$

leading to the optimal portfolio process $\pi_1(t)$

$$\pi_1(t) = \frac{\lambda(t)\sigma_I - r_R(t)}{(1-\gamma)\sigma_I^2} \cdot \frac{B_{IL}(t, I(t))}{\psi(t)I(t)} = \frac{\lambda(t)\sigma_I - r_R(t)}{(1-\gamma)\sigma_I^2}, \tag{6.1.63}$$

i.e. we have the same optimal portfolio process as in the basic problem (6.1.26). This is not surprising as the inflation-linked bond without deflation protection is simply a linear product with regard to the inflation index which therefore can be identified as a tradeable good.

If we now put the insights from the two special cases together then we note that the higher absolute value of the optimal portfolio process for (6.1.14) compared to the corresponding basic problem (6.1.26) has its reason in the protection against deflation. As for small values of the inflation index the total payment of the inflation-linked bond is typically dominated by the final payment, the price of the inflation-linked bond then behaves more like a nominal bond. To mimic the optimal inflation position (6.1.34) of the basic problem (6.1.26), therefore more and more units of the inflation-linked bond have to be sold short.

The following numerical example illustrates the above discussion. We use the first simulated path of the inflation index $I(t)_1$ under the physical probability measure P as presented in Figure 6.1 for the time interval $t = [0, 20]$, where the time unit represents one year. The inflation index is assumed to follow the geometric Brownian motion of (6.1.2) with the same parameters

$$\begin{aligned} r_N(t) &\equiv 0.05, \\ r_R(t) &\equiv 0.03, \\ \sigma_I &= 0.03, \\ i &= I(0) = 100, \\ \lambda(t) &\equiv 0.10. \end{aligned}$$

In Figure 6.7 we present the optimal portfolio processes for three different portfolio problems, each characterized by the structure of the inflation-linked bond available as investment opportunity. These different structures are an inflation-linked bond with deflation protection (6.1.3), a zero-coupon inflation-linked bond with deflation protection (6.1.58) and an inflation-

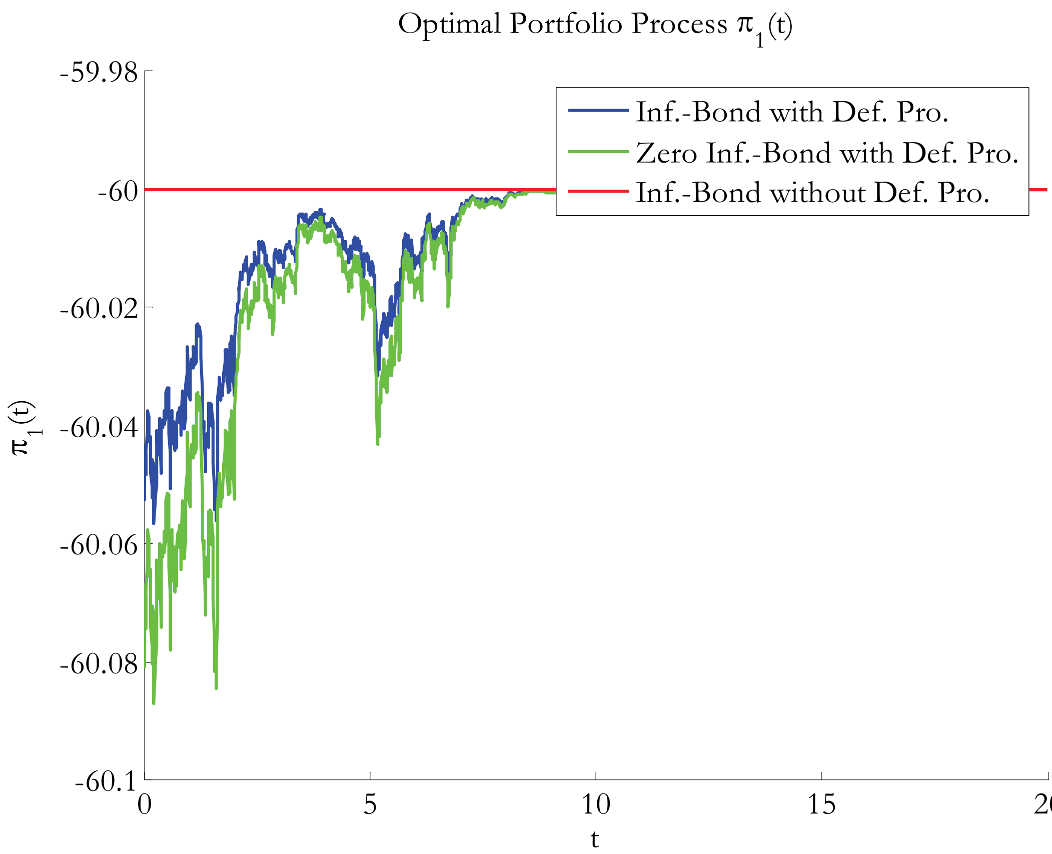

Figure 6.7 Optimal portfolio processes of inflation-linked bonds with different structure characteristics.

linked bond without deflation protection (6.1.61). The inflation-linked bonds are assumed to have the usual characteristics

$$\begin{aligned} T &= 20, \qquad F = 100, \\ C_i &= 2, \qquad t_i = 1,2,\ldots,20, \\ I(t_0) &= 100. \end{aligned}$$

The optimization is done for the HARA utility function with $\gamma = 0.5$. The deviations between optimal portfolios of different inflation-linked bonds are relatively small due to the chosen model parameters and are in the range of 10^{-1}.

As already stated, the graph illustrates the fact that if an inflation-linked bond without deflation protection is used for trading, the optimal portfolio process $\pi_1(t)$, (6.1.63), is constant. For the other two inflation-bonds that are deflation-protected at par one can observe that the optimal portfolio of the zero-coupon inflation-bond is always higher than the one of the inflation-bond containing coupon payments, which are not deflation-protected, due to the fact that the zero-coupon inflation-bond's structure is totally option-like, where as including coupon payments creates a mixture of the inflation index and an option-like inflation-bond.

Further, the conclusion on the opposite movements of the inflation index $I(t)$ and the absolute value of the optimal portfolio processes $\pi_1(t)$ of inflation-linked bonds with deflation protection can be made basing on the different asymptotical forms (6.1.51)-(6.1.53) and the limits (6.1.54)-(6.1.56). The optimal "pure" fraction of the bond and inflation index in the portfolio is given by the solution of the basic portfolio problem (6.1.26) and coincides with the optimal portfolio of the inflation-linked bond without deflation protection (6.1.63). An inflation-linked bond with deflation protection $B_{IL}(t, I(t))$ can be seen as the combination of the inflation index $I(t)$ (the part corresponding to the term $\psi(t)I(t)$, where $\psi(t)$ is a replicating strategy (6.1.39)) and the conventional bond $P_0(t)$ (the remaining part of (5.1.17)), i.e. can be replicated by the inflation index $I(t)$ and this conventional bond $P_0(t)$. In order to maintain this optimal "pure" fraction of the bond $P_0(t)$ and inflation index $I(t)$ in the portfolio, one has to increase the relative fraction of the inflation-bond in the portfolio, when the inflation index $I(t)$ is getting lower, because the replicating strategy (6.1.39) becomes smaller, too.

6.2 Problem 2: Conventional Bond and Inflation-linked Bond (Extended Vasicek Model)

In this section we will consider the same financial market as in the previous section consisting of a regular conventional bond $P_0(t)$ and an inflation-linked bond $B_{IL}(t, i(t))$. The dynamics of the conventional bond $P_0(t)$ under the physical probability measure P is given by (6.1.1), whereas the dynamics of inflation rate $i(t)$ under the physical subjective probability measure P is assumed to be of the following form

$$di(t) = (\theta(t) - (\alpha + \lambda(t)\sigma_i)i(t))dt + \sigma_i d\tilde{W}_i(t), \quad i(0) = i_0\,, \tag{6.2.1}$$

where α is a positive constant, $\theta(t)$ is a deterministic function, σ_i is a non-negative constant, $\tilde{W}_i(t)$ is a standard Brownian motion under the physical probability measure P and $\lambda(t)$ is assumed to be a deterministic function. Naturally, the dynamics of inflation rate $i(t)$ under the risk neutral probability measure Q is given by (4.2.1).

Figure 6.8 shows the extended Vasicek model for inflation rate under the physical probability measure P, where three realizations of simulated consumer price index processes are shown for

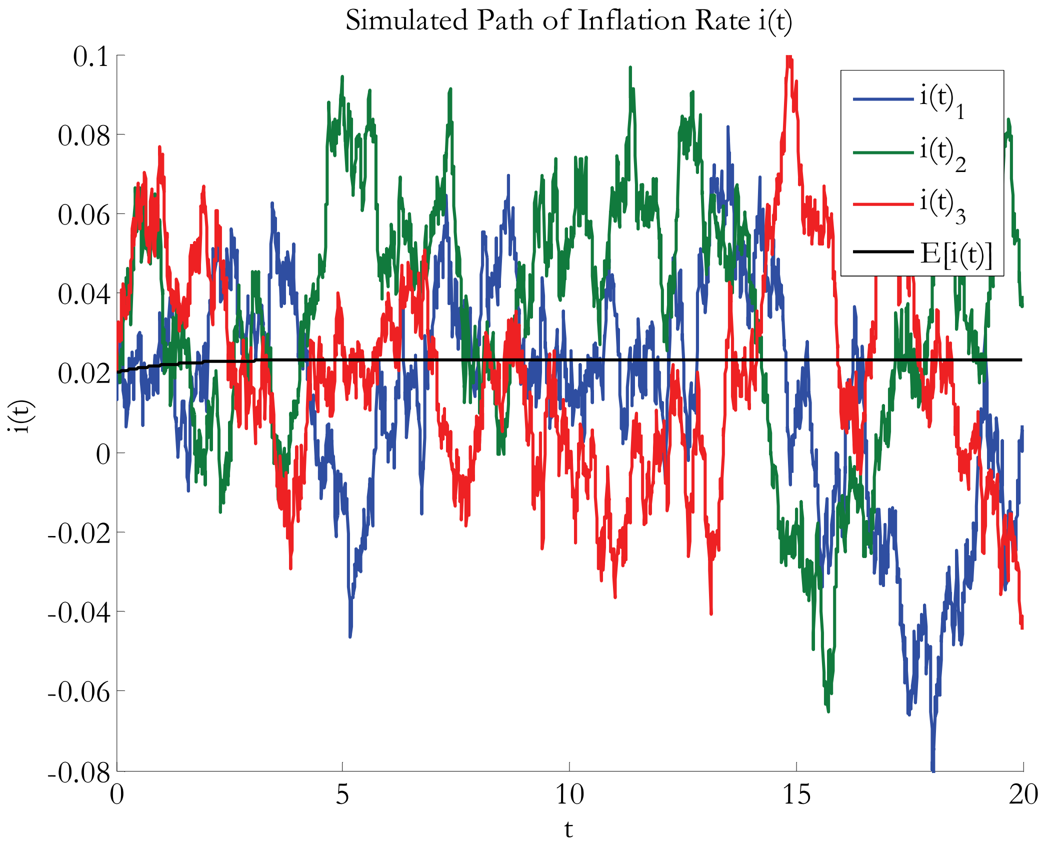

Figure 6.8 Inflation rate as a Vasicek process under the physical probability measure.

$t \in [0,T]$, where maturity date $T = 20$ years, as well as the expectation of the inflation rate process is plotted. For the simulation the following set of parameter values was used on the yearly basis

$$\begin{aligned}\theta(t) &\equiv r_N - r_R = 0.02,\\ \alpha &= 1,\\ \sigma_i &= 0.04,\\ i(0) &= 0.02,\\ \lambda(t) &\equiv -3.5.\end{aligned}$$

For the simulation the same innovations for the Brownian motion were used as the ones for the simulation of the extended Vasicek process under the risk neutral probability measure Q in Figure 4.4. Comparing these two figures it is naturally to see a higher level of inflation rate under the physical probability measure P compared to the level of inflation under the risk neutral probability measure Q in Figure 4.4 due to the negative $\lambda(t) \equiv -3.5$ with a higher mean-reversion level $\theta/(\alpha + \lambda(t)\sigma_i)$ and a smaller mean-reversion speed $\alpha + \lambda(t)\sigma_i$.

From section 5.1.3 Proposition 8 we know that under the assumption that inflation rate $i(t)$ follows an extended Vasicek process the fair price of an inflation-linked T-bond $B_{IL}(t,i(t))$ is given by

$$\begin{aligned} B_{IL}(t,i(t)) = \sum_{k:t_k \ge t} C_k \frac{I(t)}{I(t_0)} \exp\Big(-\int_t^{t_k} r_N(s)ds + \tilde{\mu}_i(t,t_k) + \frac{1}{2}\tilde{\sigma}_i^2(t,t_k)\Big) + \\ F\bigg(\exp\Big(-\int_t^T r_N(s)ds\Big) + \frac{I(t)}{I(t_0)} C_I(t,i(t))\bigg), \end{aligned} \tag{6.2.2}$$

where $C_I(t,i(t))$ is a European call option on inflation evolution $\exp\Big(\int_t^T i(s)ds\Big)$ with strike price $K = I(t_0)/I(t)$, i.e.

$$\begin{aligned} C_I(t,i(t)) = \exp\Big(-\int_t^T r_N(s)ds + \tilde{\mu}_i(t,T) + \frac{1}{2}\tilde{\sigma}_i^2(t,T)\Big) N(d(t)) - \\ \frac{I(t_0)}{I(t)} \exp\Big(-\int_t^T r_N(s)ds\Big) N(d(t) - \tilde{\sigma}_i(t,T)), \end{aligned} \tag{6.2.3}$$

where N is the cumulative distribution function for the standard normal distribution $\mathcal{N}(0,1)$,

$$d(t) = \frac{\ln\Big(\frac{I(t)}{I(t_0)}\Big) + \tilde{\mu}_i(t,T) + \tilde{\sigma}_i^2(t,T)}{\tilde{\sigma}_i(t,T)} \tag{6.2.4}$$

and parameters $\tilde{\mu}_i(t,t_k)$, $\tilde{\mu}_i(t,T)$, $\tilde{\sigma}_i^2(t,t_k)$ and $\tilde{\sigma}_i^2(t,T)$ are

$$\tilde{\mu}_i(t,t_k) = i(t)\frac{1-e^{-\alpha(t_k-t)}}{\alpha} + \int_t^{t_k} \frac{1-e^{-\alpha(t_k-s)}}{\alpha}\theta(s)ds\,, \tag{6.2.5}$$

$$\tilde{\mu}_i(t,T) = i(t)\frac{1-e^{-\alpha(T-t)}}{\alpha} + \int_t^{T} \frac{1-e^{-\alpha(T-s)}}{\alpha}\theta(s)ds\,, \tag{6.2.6}$$

$$\tilde{\sigma}_i^2(t,t_k) = \frac{\sigma_i^2}{\alpha^2}\bigg(\frac{1-(e^{-\alpha(t_k-t)}-2)^2}{2\alpha} + (t_k - t)\bigg), \tag{6.2.7}$$

$$\tilde{\sigma}_i^2(t,T) = \frac{\sigma_i^2}{\alpha^2}\bigg(\frac{1-(e^{-\alpha(T-t)}-2)^2}{2\alpha} + (T - t)\bigg). \tag{6.2.8}$$

As usual, assuming that an investor has an opportunity to invest his initial capital $x > 0$ into a normal bond of price $P_0(t)$ and an inflation-linked bond of price $B_{IL}(t, i(t))$ investor's wealth $X(t)$ at time t can be expressed as

$$X(t) = \varphi_0(t)P_0(t) + \varphi_1(t)B_{IL}(t, i(t)), \quad X(0) = x\,. \tag{6.2.9}$$

Analogically to the previous section, to the trading strategy $\varphi(t) = (\varphi_0(t), \varphi_1(t))'$ corresponding portfolio process $\pi(t) = (\pi_0(t), \pi_1(t))'$ is defined as

$$\pi_1(t) := \frac{\varphi_1(t)B_{IL}(t, i(t))}{X(t)}, \tag{6.2.10}$$

$$\pi_0(t) := 1 - \pi_1(t) = \frac{\varphi_0(t)P_0(t)}{X(t)}. \tag{6.2.11}$$

Again, the set of admissible portfolio processes $\pi(t)$ is denoted by $\mathcal{A}(x)$ and is defined by

$$\mathcal{A}(x) := \left\{(1 - \pi_1(.), \pi_1(.))' \middle| X(0) = x > 0,\ \int_0^T \pi_1^2(t)dt < \infty \text{ a.s. } P\right\}, \tag{6.2.12}$$

which consists of portfolio processes $\pi(t)$ that insure the non-negativity of the corresponding wealth processes $X(t)$ asymptotically during the whole time $t \in [0, T]$, (6.1.12).

In this section we consider the following continuous-time portfolio problem consisting of maximizing expected utility of terminal wealth $X(T)$ of the investor, i.e. the continuous-time portfolio problem looks like

$$\max_{\pi(.)\in\mathcal{A}'(x)} E[U(X(T))], \tag{6.2.13}$$

where the subset of admissible portfolio processes $\mathcal{A}'(x)$ given by

$$\mathcal{A}'(x) := \{\pi(.) \in \mathcal{A}(x) | E[U(X(T))^-] < \infty\} \tag{6.2.14}$$

insures the existence of the expected value in (6.2.13). The utility function $U(.) : (0, \infty) \to \mathbb{R}$ is a strictly concave C^1-function that satisfies (6.1.16)-(6.1.17).

Naturally, due to the fact that an inflation-linked bond $B_{IL}(t, i(t))$ is a derivative on inflation rate $i(t)$, according to Korn and Trautmann (1999) [18], Theorem 1, the optimal final wealth of the option portfolio problem (6.2.13) coincides with the optimal final wealth of the basic portfolio problem, where the investor is assumed to be able to invest into regular bond $P_0(t)$ and inflation rate $i(t)$ itself. The basic portfolio problem is given by

$$\max_{\eta(.)\in\mathcal{A}'(x)} E[U(X(T))], \tag{6.2.15}$$

where the wealth process $X(t)$ can with the help of trading strategy $\xi(t) = (\xi_0(t), \xi_1(t))'$ be written as

$$X(t) = \xi_0(t)P_0(t) + \xi_1(t)i(t), \quad X(0) = x \tag{6.2.16}$$

with portfolio process $\eta(t) = (\eta_0(t), \eta_1(t))'$ having the form

$$\eta_1(t) := \frac{\xi_1(t)i(t)}{X(t)}, \tag{6.2.17}$$

$$\eta_0(t) := 1 - \eta_1(t) = \frac{\xi_0(t)P_0(t)}{X(t)}. \tag{6.2.18}$$

For the wealth process $X(t)$ of (6.2.16) we can formally write down the following stochastic differential equation

$$dX(t) = X(t)\left(\left(r_N(t) + \eta_1(t)\left(\frac{\theta(t)}{i(t)} - \alpha - \lambda(t)\sigma_i - r_N(t)\right)\right)dt + \eta_1(t)\frac{\sigma_i}{i(t)}d\tilde{W}_i(t)\right) \tag{6.2.19}$$

with initial condition $X(0) = x$, if we assume to consider only those controls $\eta_1(.)$ with $\left|\frac{\eta_1(t)}{i(t)}\right| < \infty$ for $i(t) \to 0$. The form of the optimal portfolio strategy computed later will justify this.

In line with Björk (1998) [5] let us define the optimal value function $V(t, x)$ for the basic portfolio problem (6.2.15) as

$$V(t,x) = \sup_{\eta(.)\in\mathcal{A}'(t,x)} E[U(X(T))], \tag{6.2.20}$$

where the set of admissible portfolio processes is defines as

$$\mathcal{A}'(t,x) := \{\eta(.) \in \mathcal{A}(t,x) \mid E[U(X(T))^-] < \infty\} \tag{6.2.21}$$

with

$$\mathcal{A}(t,x) := \left\{(1-\eta_1(.),\eta_1(.))' \middle| X(t) = x > 0,\ \int_t^T \eta_1^2(s)ds < \infty \text{ a.s. } P\right\}. \tag{6.2.22}$$

Now we can write the Hamilton-Jacobi-Bellman equation for the logarithmic and power utility functions (6.1.33)

$$\begin{cases} V_t(t,x) + \sup_{\eta(.)\in\mathcal{A}'(x)} \left(\mu(t)xV_x(t,x) + \frac{1}{2}\sigma^2(t)x^2V_{xx}(t,x)\right) = 0, & \forall t \in (0,T), \forall x > 0 \\ V(T,x) = \left\{\ln(x), \frac{1}{\gamma}x^\gamma\right\}, & \gamma \in (0,1), \forall x > 0 \end{cases} \tag{6.2.23}$$

where we define $\mu(t)$ and $\sigma(t)$ as

$$\mu(t) := r_N(t) + \eta_1(t)\left(\frac{\theta(t)}{i(t)} - \alpha - \lambda(t)\sigma_i - r_N(t)\right), \tag{6.2.24}$$

$$\sigma(t) := \eta_1(t)\frac{\sigma_i}{i(t)} \tag{6.2.25}$$

and the subscripts in the optimal value function $V(t,x)$ denote partial derivatives. For an arbitrary point (t,x) the supremum is attained, when

$$\eta_1(t) = -\frac{(\theta(t) - i(t)(\alpha + \lambda(t)\sigma_i + r_N(t)))i(t)V_x(t,x)}{\sigma_i^2 xV_{xx}(t,x)}. \tag{6.2.26}$$

After substituting this candidate for the optimal control law (6.2.26) into the Hamilton-Jacobi-Bellman equation (6.2.23), we get the following partial differential equation for the optimal value function $V(t,x)$

$$V_t(t,x) + r_N(t)xV_x(t,x) - \frac{1}{2}\frac{(\theta(t) - i(t)(\alpha + \lambda(t)\sigma_i + r_N(t)))^2 V_x^2(t,x)}{\sigma_i^2 V_{xx}(t,x)} = 0\,. \quad (6.2.27)$$

Making a guess about the form of the optimal value function $V(t,x)$ as

$$V(t,x) = \ln(x) + f(t), \quad f(T) = 0 \quad (6.2.28)$$

and

$$V(t,x) = f(t)\frac{1}{\gamma}x^\gamma, \quad f(T) = 1 \quad (6.2.29)$$

for the logarithmic and power utility functions (6.1.33) respectively, we substitute our ansatz (6.2.28)-(6.2.29) into the partial differential equation (6.2.27). As a result we get a differential equation for the function $f(t)$

$$f'(t) = -r_N(t) - \frac{1}{2}\frac{(\theta(t) - i(t)(\alpha + \lambda(t)\sigma_i + r_N(t)))^2}{\sigma_i^2}, \quad f(T) = 0 \quad (6.2.30)$$

and

$$f'(t) = \left(\frac{1}{2}\frac{(\theta(t) - i(t)(\alpha + \lambda(t)\sigma_i + r_N(t)))^2}{\sigma_i^2(\gamma - 1)} - r_N(t)\right)\gamma f(t), \quad f(T) = 1 \quad (6.2.31)$$

for the logarithmic and power utility functions (6.1.33) respectively. The solutions for these differential equations (6.2.30)-(6.2.31) are given by

$$f(t) = \left(r_N(t) + \frac{1}{2}\frac{(\theta(t) - i(t)(\alpha + \lambda(t)\sigma_i + r_N(t)))^2}{\sigma_i^2}\right)(T - t) \quad (6.2.32)$$

and

$$f(t) = \exp\left(\left(r_N(t) - \frac{1}{2}\frac{(\theta(t) - i(t)(\alpha + \lambda(t)\sigma_i + r_N(t)))^2}{\sigma_i^2(\gamma - 1)}\right)\gamma(T - t)\right) \quad (6.2.33)$$

for the logarithmic and power utility functions (6.1.33) respectively. Knowing the form of the optimal value function $V(t,x)$, (6.2.28)-(6.2.29) and (6.2.32)-(6.2.33), from (6.2.26) we get the solution for the optimal portfolio process $\eta(t) = (\eta_0(t), \eta_1(t))'$ of the basic portfolio problem (6.2.15) in form of

$$\eta_0(t) = 1 - \eta_1(t) = \frac{(1-\gamma)\sigma_i^2 - (\theta(t) - i(t)(\alpha + \lambda(t)\sigma_i + r_N(t)))i(t)}{(1-\gamma)\sigma_i^2}, \tag{6.2.34}$$

$$\eta_1(t) = \frac{(\theta(t) - i(t)(\alpha + \lambda(t)\sigma_i + r_N(t)))i(t)}{(1-\gamma)\sigma_i^2}, \tag{6.2.35}$$

where the case of $\gamma = 0$ corresponds to the logarithmic utility function. From the form of the optimal portfolio strategy $\eta_1(t)$ in (6.2.35) we see that the assumption $\left|\frac{\eta_1(t)}{i(t)}\right| < \infty$ for $i(t) \to 0$ used in deriving the formal representation of the stochastic differential equation for the wealth process $X(t)$ in (6.2.19) is fulfilled.

Theorem 3. The optimal portfolio process $\pi_1(t)$ of an inflation-linked bond $B_{IL}(t, i(t))$ for the continuous-time portfolio problem (6.2.13) for the HARA utility functions (logarithmic and power utility (6.1.33)) is given by

$$\pi_1(t) = \frac{\theta(t) - i(t)(\alpha + \lambda(t)\sigma_i + r_N(t))}{(1-\gamma)\sigma_i^2} \cdot \frac{B_{IL}(t, i(t))}{\psi(t)}, \tag{6.2.36}$$

where $\psi(t)$ is a replicating strategy given by

$$\begin{aligned}\psi(t) = \sum_{k:t_k \geq t} C_k \frac{I(t)}{I(t_0)} \exp\Big(-\int_t^{t_k} r_N(s)ds + \tilde{\mu}_i(t,t_k) + \frac{1}{2}\tilde{\sigma}_i^2(t,t_k)\Big)\frac{1-e^{-\alpha(t_k - t)}}{\alpha} + \\ F\frac{I(t)}{I(t_0)} \exp\Big(-\int_t^{T} r_N(s)ds + \tilde{\mu}_i(t,T) + \frac{1}{2}\tilde{\sigma}_i^2(t,T)\Big) N(d(t))\frac{1-e^{-\alpha(T-t)}}{\alpha};\end{aligned} \tag{6.2.37}$$

the case of $\gamma = 0$ corresponds to the logarithmic utility function.

Proof of Theorem 3. Having the optimal portfolio process $\eta_1(t)$ (6.2.35) for the basic portfolio problem (6.2.15) we can now express the optimal trading strategy $\xi_1(t)$ as

$$\xi_1(t) = \frac{\theta(t) - i(t)(\alpha + \lambda(t)\sigma_i + r_N(t))}{(1-\gamma)\sigma_i^2} X(t). \tag{6.2.38}$$

Applying the results of Korn and Trautmann (1999) [18], Theorem 1, and having the optimal trading strategy $\xi_1(t)$ for the basic portfolio problem (6.2.15) we are able to express the optimal trading strategy $\varphi(t)$ of the option portfolio problem (6.2.13) as

$$\varphi_1(t) = \frac{\xi_1(t)}{\psi(t)} = \frac{\theta(t) - i(t)(\alpha + \lambda(t)\sigma_i + r_N(t))}{(1-\gamma)\sigma_i^2} \cdot \frac{X(t)}{\psi(t)}, \tag{6.2.39}$$

$$\begin{aligned} \varphi_0(t) &= \frac{X(t) - \varphi_1(t) B_{IL}(t, i(t))}{P_0(t)} \\ &= \left(1 - \frac{(\theta(t) - i(t)(\alpha + \lambda(t)\sigma_i + r_N(t)))B_{IL}(t, i(t))}{(1-\gamma)\sigma_i^2 \psi(t)}\right) \frac{X(t)}{P_0(t)}, \end{aligned} \tag{6.2.40}$$

where $\psi(t)$ is a replicating strategy of an inflation-linked bond $B_{IL}(t, i(t))$ by inflation rate $i(t)$ defined in an analogical way to (6.1.38) as

$$\psi(t) := \frac{\partial B_{IL}(t, i(t))}{\partial i(t)} \tag{6.2.41}$$

i.e.

$$\begin{aligned} \psi(t) = \sum_{k: t_k \geq t} C_k \frac{I(t)}{I(t_0)} \exp\left(-\int_t^{t_k} r_N(s) ds + \tilde{\mu}_i(t, t_k) + \frac{1}{2}\tilde{\sigma}_i^2(t, t_k)\right) \frac{1 - e^{-\alpha(t_k - t)}}{\alpha} + \\ F \frac{I(t)}{I(t_0)} \exp\left(-\int_t^{T} r_N(s) ds + \tilde{\mu}_i(t, T) + \frac{1}{2}\tilde{\sigma}_i^2(t, T)\right) N(d(t)) \frac{1 - e^{-\alpha(T - t)}}{\alpha}. \end{aligned} \tag{6.2.42}$$

From (6.2.10) we know that the optimal portfolio process $\pi_1(t)$ can be represented as

$$\pi_1(t) = \frac{\varphi_1(t) B_{IL}(t, i(t))}{X(t)}. \tag{6.2.43}$$

Substituting (6.2.39) into this relation (6.2.43) we directly get (6.2.36). □

Remark 1 on Theorem 3 *("Negativity" of the Optimal Portfolio Process)***.** The sign of the optimal portfolio process $\pi_1(t)$ is determined by the sign of the term

$$\theta(t) - i(t)(\alpha + \lambda(t)\sigma_i + r_N(t)), \tag{6.2.44}$$

due to the fact that

$$\frac{B_{IL}(t, i(t))}{\psi(t)} > 1\,. \tag{6.2.45}$$

This means that also in this situation, when we have an extended Vasicek model for inflation rate, the optimal portfolio process $\pi_1(t)$ could be negative with a high probability. In order to have the positive optimal portfolio process $\pi_1(t)$ for an inflation-linked bond $B_{IL}(t, i(t))$ in a case of instantaneous inflation $i(t) > 0$ the subjective excess return $-\lambda(t)\sigma_i$ has to be bigger than the term $\theta(t) / i(t) - \alpha - r_N(t)$, i.e. $\lambda(t)$ should be highly negative – an assumption that rarely seems to hold. So, also for this model for inflation process, an extended Vasicek process, in this imaginary portfolio problem a risk-averse investor typically sells inflation-linked products when behaving optimally. This is actually is not surprising due to the same facts presented in Remark 1 on Theorem 2. As was already mentioned, in a normal situation the drift term of the risk-free asset, i.e. a conventional bond, beats the drift term of a risky asset, in our case an inflation-linked bond. That is the reason why in an imaginary situations studied here the optimal portfolio process for an inflation-linked security is negative for a relatively small subjective excess return $-\lambda(t)\sigma_i$.

Remark 2 on Theorem 3 *(Nature of the Continuous-time Portfolio Problem)***.** Looking at the dynamics of investor's wealth process $X(t)$, (6.2.19), we realize that the basic portfolio problem (6.2.15) studied in this section is equivalent to the classical continuous-time portfolio problem, where we have a conventional bond $P_0(t)$ as a risk-free asset with an expected return of $r_N(t)$ and an inflation rate $i(t)$ as a risky asset with a drift equal to $\theta(t) / i(t) - \alpha - \lambda(t)\sigma_i$ and volatility $\sigma_i / i(t)$. The optimal portfolio process for a risky asset, in this case an inflation rate $i(t)$, is then given by the excess returns over the return of the risk-free asset divided by squared volatility multiplied with $1 - \gamma$, which is exactly the same as in (6.2.35).

6.3 Problem 3: Conventional Bond, Stock and Inflation-linked Bond

In the third case we assume that an investor has access to the market, where in addition to the settings in the previous section (regular conventional bonds $P_0(t)$ and inflation-linked bonds $B_{IL}(t, I(t))$) also stocks $P_1(t)$ are traded. The dynamics of the conventional bond $P_0(t)$ and inflation index $I(t)$ under the physical probability measure P are given by (6.1.1) and (6.1.2) respectively. The price of inflation-linked bond $B_{IL}(t, I(t))$ is given by (6.1.3)-(6.1.5). The dynamics of the stock price $P_1(t)$ we define to be correlated with the inflation index $I(t)$ as

$$dP_1(t) = P_1(t)(b(t)dt + \sigma_1 d\tilde{W}(t) + \sigma_2 d\tilde{W}_I(t)), \quad P_1(0) = p_1\,, \tag{6.3.1}$$

where $(\tilde{W}(t), \tilde{W}_I(t))'$ is an independent two-dimensional standard Brownian motion under the physical probability measure P. Additionally we define the volatility of our model σ as

$$\sigma := \begin{bmatrix} \sigma_1 & \sigma_2 \\ 0 & \sigma_I \end{bmatrix}. \tag{6.3.2}$$

Now investor's wealth process $X(t)$ can expressed as

$$X(t) = \varphi_0(t)P_0(t) + \varphi_1(t)P_1(t) + \varphi_2(t)B_{IL}(t, I(t)), \quad X(0) = x\,, \tag{6.3.3}$$

where investor's initial capital is strictly positive, i.e. $x > 0$, and trading strategy $\varphi(t) = (\varphi_0(t), \varphi_1(t), \varphi_2(t))'$ is $\mathbb{R}^3$-valued progressively measurable processes with respect to the filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$ generated by the independent standard Brownian motion $\{(\tilde{W}(t), \tilde{W}_I(t))'\}_{t\in[0,T]}$ satisfying similar technical conditions as (6.1.7)-(6.1.8). Naturally the corresponding $\mathbb{R}^3$-valued portfolio process $\pi(t) = (\pi_0(t), \pi_1(t), \pi_2(t))'$ is defined as

$$\pi_1(t) := \frac{\varphi_1(t)P_1(t)}{X(t)}, \tag{6.3.4}$$

$$\pi_2(t) := \frac{\varphi_2(t)B_{IL}(t, I(t))}{X(t)}, \tag{6.3.5}$$

$$\pi_0(t) := 1 - \overline{\pi}(t)'\underline{1} = \frac{\varphi_0(t)P_0(t)}{X(t)}, \tag{6.3.6}$$

where $\underline{1} := (1,\ldots,1)' \in \mathbb{R}^d$ and $\overline{\pi}(t)$ corresponds to the fractions of the wealth $X(t)$ invested into the risky assets, i.e.

$$\overline{\pi}(t) := (\pi_1(t), \pi_2(t))'. \tag{6.3.7}$$

We define the set of admissible portfolio processes $\mathcal{A}(x)$ in analogy to (6.1.13)

$$\mathcal{A}(x) := \left\{ (1 - \overline{\pi}(.)'\underline{1}, \overline{\pi}(.)')' \,\middle|\, X(0) = x > 0,\ \int_0^T \pi_i^2(t)dt < \infty \text{ a.s. } P,\ i = 1,2 \right\}. \tag{6.3.8}$$

Having the wealth process $X(t)$ given by (6.3.3), in this section we look at the continuous-time portfolio problem

$$\max_{\pi(.)\in\mathcal{A}'(x)} E[U(X(T))], \tag{6.3.9}$$

where the subset of admissible portfolio processes $\mathcal{A}'(x)$ is characterized as

$$\mathcal{A}'(x) := \{\pi(.) \in \mathcal{A}(x) | E[U(X(T))^{-}] < \infty\}. \tag{6.3.10}$$

Having the same arguments as in the previous section that is the fact that an inflation-linked bond $B_{IL}(t, I(t))$ is a derivative on inflation index $I(t)$, according to Korn and Trautmann (1999) [18], Theorem 1, the optimal final wealth of the option portfolio problem (6.3.9) coincides with the optimal final wealth of the basic portfolio problem defined as

$$\max_{\eta(.)\in\mathcal{A}'(x)} E[U(X(T))], \tag{6.3.11}$$

where the wealth process $X(t)$ can with the help of trading strategy $\xi(t) = (\xi_0(t), \xi_1(t), \xi_2(t))'$ be written as

$$X(t) = \xi_0(t)P_0(t) + \xi_1(t)P_1(t) + \xi_2(t)I(t), \quad X(0) = x \tag{6.3.12}$$

with the corresponding portfolio process $\eta(t) = (\eta_0(t), \eta_1(t), \eta_2(t))'$ analogous to (6.1.28)-(6.1.29).

Theorem 4. The optimal portfolio process $\overline{\pi}(t) = (\pi_1(t), \pi_2(t))'$ of a stock $P_1(t)$ and an inflation-linked bond $B_{IL}(t, I(t))$ for the continuous-time portfolio problem (6.3.9) for the HARA utility functions (logarithmic and power utility (6.1.33)) is given by

$$\overline{\pi}(t) = \frac{1}{1-\gamma}\begin{bmatrix} \dfrac{b(t) - r_N(t)}{\sigma_1^2} - \dfrac{\lambda(t)\sigma_I - r_R(t)}{\sigma_I^2} \cdot \dfrac{\sigma_2\sigma_I}{\sigma_1^2} \\ \left(\dfrac{\lambda(t)\sigma_I - r_R(t)}{\sigma_I^2}\left(1 + \dfrac{\sigma_2^2}{\sigma_1^2}\right) - \dfrac{b(t) - r_N(t)}{\sigma_1^2} \cdot \dfrac{\sigma_2\sigma_I}{\sigma_I^2}\right)\dfrac{B_{IL}(t, I(t))}{\psi_{22}(t)I(t)} \end{bmatrix}, \tag{6.3.13}$$

where $\psi_{22}(t)$ is a replicating strategy for an inflation-linked bond $B_{IL}(t, I(t))$ by inflation index $I(t)$ given by

$$\psi_{22}(t) = \sum_{i:t_i \geq t} \frac{C_i}{I(t_0)} \exp\Big(-\int_t^{t_i} r_R(s)ds\Big) + \frac{F \exp\Big(-\int_t^T r_R(s)ds\Big) N(d(t))}{I(t_0)}; \qquad (6.3.14)$$

the case of $\gamma = 0$ corresponds to the logarithmic utility function.

Proof of Theorem 4. For the logarithmic and power utility functions (6.1.33) the solution of the basic portfolio problem (6.3.11) is a well-known result and the optimal portfolio process $\overline{\eta}(t) = (\eta_1(t), \eta_2(t))'$ is given by

$$\begin{aligned}\overline{\eta}(t) &= \frac{1}{1-\gamma}(\sigma\sigma')^{-1}\begin{bmatrix} b(t) - r_N(t) \\ (r_N(t) - r_R(t) + \lambda(t)\sigma_I) - r_N(t)\end{bmatrix} \\ &= \frac{1}{1-\gamma}\begin{bmatrix} \dfrac{b(t) - r_N(t)}{\sigma_1^2} - \dfrac{\lambda(t)\sigma_I - r_R(t)}{\sigma_I^2} \cdot \dfrac{\sigma_2\sigma_I}{\sigma_1^2} \\ \dfrac{\lambda(t)\sigma_I - r_R(t)}{\sigma_I^2}\left(1 + \dfrac{\sigma_2^2}{\sigma_1^2}\right) - \dfrac{b(t) - r_N(t)}{\sigma_1^2} \cdot \dfrac{\sigma_2\sigma_I}{\sigma_I^2}\end{bmatrix},\end{aligned} \qquad (6.3.15)$$

where the case of $\gamma = 0$ corresponds to the optimal portfolio process for the logarithmic utility function. Having the optimal portfolio process $\overline{\eta}(t)$ (6.3.15) for the basic portfolio problem (6.3.11) we can now express the optimal trading strategy $\overline{\xi}(t) = (\xi_1(t), \xi_2(t))'$ as

$$\overline{\xi}(t) = \frac{X(t)}{1-\gamma}\begin{bmatrix} \dfrac{b(t) - r_N(t)}{\sigma_1^2 P_1(t)} - \dfrac{\lambda(t)\sigma_I - r_R(t)}{\sigma_I^2 P_1(t)} \cdot \dfrac{\sigma_2\sigma_I}{\sigma_1^2} \\ \dfrac{\lambda(t)\sigma_I - r_R(t)}{\sigma_I^2 I(t)}\left(1 + \dfrac{\sigma_2^2}{\sigma_1^2}\right) - \dfrac{b(t) - r_N(t)}{\sigma_1^2 I(t)} \cdot \dfrac{\sigma_2\sigma_I}{\sigma_I^2}\end{bmatrix}. \qquad (6.3.16)$$

Applying the results of Korn and Trautmann (1999) [18], Theorem 1, and having the optimal trading strategy $\overline{\xi}(t)$ for the basic portfolio problem (6.3.11) we are able to express the optimal trading strategy $\overline{\varphi}(t) = (\varphi_1(t), \varphi_2(t))'$ and $\varphi_0(t)$ of the option portfolio problem (6.3.9) as

$$\overline{\varphi}(t) = (\psi(t)')^{-1}\overline{\xi}(t) = \frac{X(t)}{1-\gamma}\begin{bmatrix} \dfrac{b(t) - r_N(t)}{\sigma_1^2 P_1(t)} - \dfrac{\lambda(t)\sigma_I - r_R(t)}{\sigma_I^2 P_1(t)} \cdot \dfrac{\sigma_2\sigma_I}{\sigma_1^2} \\ \dfrac{\lambda(t)\sigma_I - r_R(t)}{\sigma_I^2 \psi_{22}(t) I(t)}\left(1 + \dfrac{\sigma_2^2}{\sigma_1^2}\right) - \dfrac{b(t) - r_N(t)}{\sigma_1^2 \psi_{22}(t) I(t)} \cdot \dfrac{\sigma_2\sigma_I}{\sigma_I^2}\end{bmatrix}, \qquad (6.3.17)$$

$$\varphi_0(t) = \frac{X(t) - \varphi_1(t)P_1(t) - \varphi_2(t)B_{IL}(t, I(t))}{P_0(t)}, \qquad (6.3.18)$$

where $\psi(t)$ is a replicating strategy defined as

$$\psi(t) := \begin{bmatrix} \dfrac{\partial P_1(t)}{\partial P_1(t)} & \dfrac{\partial P_1(t)}{\partial I(t)} \\ \dfrac{\partial B_{IL}(t,I(t))}{\partial P_1(t)} & \dfrac{\partial B_{IL}(t,I(t))}{\partial I(t)} \end{bmatrix}, \tag{6.3.19}$$

i.e.

$$\psi(t) := \begin{bmatrix} 1 & 0 \\ 0 & \sum_{i:t_i \geq t} \dfrac{C_i}{I(t_0)} \exp\Big(-\int_t^{t_i} r_R(s)ds\Big) + \dfrac{F \exp\Big(-\int_t^{T} r_R(s)ds\Big) N(d(t))}{I(t_0)} \end{bmatrix}; \tag{6.3.20}$$

$\psi_{22}(t)$ is a 22-element of the replicating matrix $\psi(t)$, which corresponds to the replicating strategy for an inflation-linked bond $B_{IL}(t,I(t))$ by inflation index $I(t)$.

From (6.3.4)-(6.3.5) we know that the optimal portfolio process $\overline{\pi}(t) = (\pi_1(t), \pi_2(t))'$ can be represented as

$$\overline{\pi}(t) = \begin{bmatrix} \dfrac{\varphi_1(t)P_1(t)}{X(t)} \\ \dfrac{\varphi_2(t)B_{IL}(t,I(t))}{X(t)} \end{bmatrix}. \tag{6.3.21}$$

Substituting (6.3.17) into this relation (6.3.21) we directly get (6.3.13). □

Remark 1 on Theorem 4 *(Merton and Hedging Terms)***.** Looking at (6.3.13) we actually see that the optimal portfolio process $\overline{\pi}(t)$ consists of two parts: the Merton-term

$$\frac{1}{1-\gamma} \begin{bmatrix} \dfrac{b(t) - r_N(t)}{\sigma_1^2} \\ \dfrac{\lambda(t)\sigma_I - r_R(t)}{\sigma_I^2} \cdot \dfrac{B_{IL}(t,I(t))}{\psi_{22}(t)I(t)} \end{bmatrix}$$

and the hedging term

$$\frac{1}{1-\gamma}\begin{bmatrix} -\frac{\lambda(t)\sigma_I - r_R(t)}{\sigma_I^2}\cdot\frac{\sigma_2\sigma_I}{\sigma_1^2} \\ \left(\frac{\lambda(t)\sigma_I - r_R(t)}{\sigma_I^2}\cdot\frac{\sigma_2^2}{\sigma_1^2} - \frac{b(t)-r_N(t)}{\sigma_1^2}\cdot\frac{\sigma_2\sigma_I}{\sigma_I^2}\right)\frac{B_{IL}(t,I(t))}{\psi_{22}(t)I(t)} \end{bmatrix}.$$

The Merton-term corresponds to the optimal portfolio process in the situation, when an investor operates with only two securities: either a conventional bond $P_0(t)$ and a stock $P_1(t)$ or a conventional bond $P_0(t)$ and an inflation-linked bond $B_{IL}(t,I(t))$.

Another remarkable observation one can make is that the optimal portfolio process $\pi_1(t)$ for the stock $P_1(t)$ does not depend from the level of the inflation, inflation index $I(t)$, directly despite the fact that it is correlated with inflation index $I(t)$. Furthermore, in the case of constant model parameters the optimal portfolio process $\pi_1(t)$ is even constant.

Remark 2 on Theorem 4 *("Negativity" of the Optimal Portfolio Process)***.** Having

$$\frac{B_{IL}(t,I(t))}{\psi_{22}(t)I(t)} > 1 \tag{6.3.22}$$

and looking at the explicit form of the optimal portfolio process $\pi_2(t)$ for the inflation-linked bond $B_{IL}(t,I(t))$ in (6.3.13), we realize that unless we have a very strong opinion for a very high inflation rate (the excess return $\lambda(t)$) it would be optimal to sell short inflation-linked bonds. The conclusion one can make from this comment and previous Remark 1 on Theorem 2 is that the type of risk-averse investor typically sells short inflation-linked products.

Remark 3 on Theorem 4 *(Expected Utility from the Optimal Terminal Wealth)***.** In general, it is clear that having an opportunity to invest additionally into inflation-linked bonds $B_{IL}(t,I(t))$ is better compared to the situation, when we are investing (positively or negatively) into all other financial assets expect inflation-linked products. The form of the optimal portfolio process $\overline{\pi}(t)$ in (6.3.13) only instructs us how we should act in order to optimize our expected utility from the terminal wealth $E[U(X(T))]$. It is natural that if we were not handling an inflation-linked bond $B_{IL}(t,I(t))$ at all (in the case of a risk-averse investor we were selling this inflation-linked bond $B_{IL}(t,I(t))$) our expected utility from the terminal wealth $E[U(X(T))]$ would be smaller, which means that it is not optimal to have portfolio process $\pi_2(t)$ equal to zero. In the following we will present the mathematical justification to this concept in addition to the already stated heuristic reasoning.

The optimal terminal wealth B^* in the continuous-time portfolio problem (6.3.9) is given by (see for example Korn (1997) [14])

$$B^* = I(Y(x)H(T)), \tag{6.3.23}$$

where $I(y)$ is the inverse function of the first derivative of the logarithmic and power utility function $U(.)$, (6.1.33), that is

$$I(y) := (U')^{-1}(y) = y^{\frac{1}{\gamma-1}}, \tag{6.3.24}$$

where the case of $\gamma = 0$ again corresponds to the logarithmic utility function. The process $H(T)$ is given by

$$H(T) := \exp\Big(-\int_0^T (r_N(s) + \frac{1}{2}\theta(s)'\theta(s))ds - \int_0^T \theta(s)' d(\tilde{W}(s), \tilde{W}_I(s))'\Big), \tag{6.3.25}$$

$$\theta(s) := \sigma^{-1}(b(s) - r_N(s), \lambda(s)\sigma_I - r_R(s))'. \tag{6.3.26}$$

The process $X(y)$ is defined as

$$X(y) := E[H(T)I(yH(T))] = y^{\frac{1}{\gamma-1}} E[H(T)^{\frac{\gamma}{\gamma-1}}]. \tag{6.3.27}$$

Now, let $Y(.)$ be an inverse function of $X(.)$. From (6.3.27) one can obtain $Y(x)$ as the unique solution of the equation

$$X(Y(x)) = x. \tag{6.3.28}$$

This unique solution is given by

$$Y(x) = \frac{x^{\gamma-1}}{E[H(T)^{\frac{\gamma}{\gamma-1}}]^{\gamma-1}}. \tag{6.3.29}$$

Having the explicit form of $Y(x)$ in (6.3.29), we can rewrite the optimal terminal wealth B^* of (6.3.23) as

$$B^* = \frac{xH(T)^{\frac{1}{\gamma-1}}}{E[H(T)^{\frac{\gamma}{\gamma-1}}]}. \tag{6.3.30}$$

The expected utility from the optimal terminal wealth $E[U(B^*)]$ is given by

$$E[\ln(B^*)] = E\left[\ln\left(\frac{x}{H(T)}\right)\right] \tag{6.3.31}$$

and

$$E\left[\frac{1}{\gamma}B^{*\gamma}\right] = \frac{1}{\gamma}x^\gamma E[H(T)^{\frac{\gamma}{\gamma-1}}]^{1-\gamma} \tag{6.3.32}$$

for the logarithmic and power utility functions (6.1.33) respectively. For the logarithmic utility function we can substitute the explicit form of $H(T)$, (6.3.25)-(6.3.26), and evaluate the expected value in (6.3.31).

$$E[\ln(B^*)] = \ln(x) + \int_0^T (r_N(s) + \frac{1}{2}\theta(s)'\theta(s))ds\,, \tag{6.3.33}$$

where

$$\theta(s)'\theta(s) = \left(\frac{b(s) - r_N(s)}{\sigma_1} - \frac{\sigma_2}{\sigma_1}\cdot\frac{\lambda(s)\sigma_I - r_R(s)}{\sigma_I}\right)^2 + \left(\frac{\lambda(s)\sigma_I - r_R(s)}{\sigma_I}\right)^2. \tag{6.3.34}$$

Analogically we can proceed for the power utility function in (6.3.32)

$$\begin{aligned} E\left[\frac{1}{\gamma}B^{*\gamma}\right] &= \frac{1}{\gamma}x^\gamma E\left[\exp\left(\frac{\gamma}{1-\gamma}\int_0^T (r_N(s) + \frac{1}{2}\theta(s)'\theta(s))ds + \right.\right. \\ &\left.\left.\frac{\gamma}{1-\gamma}\int_0^T \theta(s)'d(\tilde{W}(s),\tilde{W}_I(s))'\right)\right]^{1-\gamma} \\ &= \frac{1}{\gamma}x^\gamma \exp\left(\gamma\int_0^T\left(r_N(s) + \frac{1}{2}\cdot\frac{1}{1-\gamma}\cdot\theta(s)'\theta(s)\right)ds\right), \end{aligned} \tag{6.3.35}$$

where $\theta(s)'\theta(s)$ is the same as in (6.3.34). From the explicit forms of the expected utility from the optimal terminal wealth $E[U(B^*)]$ in (6.3.33) and (6.3.35) we see that the optimal expected utility $E[U(B^*)]$ is increasing in $\theta(s)'\theta(s)$. In the situation, when we are not handling an inflation-linked bond $B_{IL}(t, I(t))$ at all, we are in the situation of a new financial market that is represented only by a conventional bond $P_0(t)$ and a stock $P_1(t)$. Therefore, the dynamics of a stock $P_1(t)$ given in (6.3.1) can be rewritten as

$$dP_1(t) = P_1(t)(b(t)dt + \sigma_{12}d\tilde{W}_{12}(t)), \quad P_1(0) = p_1, \tag{6.3.36}$$

where $\tilde{W}_{12}(t)$ is a new standard Brownian motion given by

$$\tilde{W}_{12}(t) := \frac{\sigma_1\tilde{W}(t) + \sigma_2\tilde{W}_I(t)}{\sqrt{\sigma_1^2 + \sigma_2^2}} \tag{6.3.37}$$

and σ_{12} is a new volatility of the process defined as

$$\sigma_{12} := \sqrt{\sigma_1^2 + \sigma_2^2}\,. \tag{6.3.38}$$

Consequently, we can apply the same techniques as above in order to derive the form of the expected utility from the optimal terminal wealth $E[U(B^*)]$ in this new financial market. All the equations presented above are still valid, when we define $\theta(s)$ as

$$\theta(s) := \frac{b(s) - r_N(s)}{\sigma_{12}}\begin{bmatrix}\dfrac{\sigma_1}{\sqrt{\sigma_1^2 + \sigma_2^2}} \\ \dfrac{\sigma_2}{\sqrt{\sigma_1^2 + \sigma_2^2}}\end{bmatrix} = \frac{b(s) - r_N(s)}{\sigma_1^2 + \sigma_2^2}\begin{bmatrix}\sigma_1 \\ \sigma_2\end{bmatrix}. \tag{6.3.39}$$

Therefore, the explicit forms of the expected utility from the optimal terminal wealth $E[U(B^*)]$ are still given by (6.3.33) and (6.3.35), where we substitute the new $\theta(s)'\theta(s)$ given by

$$\theta(s)'\theta(s) = \frac{(b(s) - r_N(s))^2}{\sigma_1^2 + \sigma_2^2}. \tag{6.3.40}$$

One can easily see that it is highly probable that $\theta(s)'\theta(s)$ in (6.3.34) bigger than $\theta(s)'\theta(s)$ in (6.3.40), especially in the case of a risk-averse investor, when $\lambda(s)\sigma_I - r_R(s)$ is negative, which means that the expected utility from the optimal terminal wealth $E[U(B^*)]$ is higher for the case, when we additionally trade an inflation-linked bond $B_{IL}(t,I(t))$. This proves our heuristic judgment that it is optimal to use all possible opportunities to trade different financial assets, i.e. also an inflation-linked bond $B_{IL}(t,I(t))$ that a risk-averse investor would like to sell, when behaving optimally.

Remark 4 on Theorem 4 *(Asymptotic Behavior)***.** Also in this problem setup we can without difficulties derive asymptotic and limiting behaviors for the optimal portfolio process $\pi_2(t)$ for the inflation-linked bond $B_{IL}(t,I(t))$ similar to (6.1.49)-(6.1.56) from Remark 2 on Theorem 2, where we only substitute

$$\frac{\lambda(t)\sigma_I - r_R(t)}{(1-\gamma)\sigma_I{}^2}$$

with

$$\frac{1}{1-\gamma}\left(\frac{\lambda(t)\sigma_I - r_R(t)}{\sigma_I^2}\left(1+\frac{\sigma_2^2}{\sigma_1^2}\right) - \frac{b(t)-r_N(t)}{\sigma_1^2}\cdot\frac{\sigma_2\sigma_I}{\sigma_I^2}\right).$$

Moreover, the results and observations about the optimal portfolio processes for inflation-linked bonds with different structures from Remark 3 on Theorem 2 also apply for the current problem setup of this section.

6.4 Problem 4: Conventional Bond and Inflation-linked Bond with a Positive Lower Bound on Final Wealth

In practice it is natural for an investor to desire some minimum for his future final wealth. That is why in real life portfolio optimization problems often include a positive lower bound on the final wealth. This could be for example a contract with a guaranteed lower bound on the return on the capital plus additionally possible extra returns. In this situation we consider the problem of determination of optimal portfolio processes under a lower bound constraint on the final wealth.

For simplicity we look at a complete market studied in section 6.1, which is made up of a money market account $P_0(t)$, (6.1.1), and an inflation linked-bond $B_{IL}(t,I(t))$, (6.1.3)-(6.1.5), driven by inflation index $I(t)$, (6.1.2). Consequently investor's wealth process can be expressed as

$$X(t) = \varphi_0(t)P_0(t) + \varphi_1(t)B_{IL}(t, I(t)), \quad X(0) = x\,, \tag{6.4.1}$$

where investor's initial capital is strictly positive, i.e. $x > 0$. To the trading strategy $\varphi(t) = (\varphi_0(t), \varphi_1(t))'$ corresponding portfolio process $\pi(t) = (\pi_0(t), \pi_1(t))'$ is defined as

$$\pi_1(t) := \frac{\varphi_1(t)B_{IL}(t, I(t))}{X(t)}, \tag{6.4.2}$$

$$\pi_0(t) := 1 - \pi_1(t) = \frac{\varphi_0(t)P_0(t)}{X(t)}. \tag{6.4.3}$$

Again, the set of admissible portfolio processes $\pi(t)$ is denoted by $\mathcal{A}(x)$ and is defined by

$$\mathcal{A}(x) := \left\{ (1 - \pi_1(.), \pi_1(.))' \middle| X(0) = x > 0, \int_0^T \pi_1^2(t)dt < \infty \text{ a.s. } P \right\}, \tag{6.4.4}$$

which consists of portfolio processes $\pi(t)$ that insure the non-negativity of the corresponding wealth processes $X(t)$ asymptotically during the whole time $t \in [0, T]$, (6.1.12).

In this section we consider the similar optimization problem of maximizing expected utility of terminal wealth $X(T)$ as in section 6.1, but here we have additionally the constraint on the final wealth $X(T)$ that it should be minimum $B \geq 0$. Thus, having the wealth process $X(t)$ given by (6.4.1) the constrained continuous-time portfolio problem of this section looks like

$$\max_{\pi(.) \in \mathcal{A}'(x)} E[U(X(T))], \tag{6.4.5}$$

where the subset of admissible portfolio processes $\mathcal{A}'(x)$ given by

$$\mathcal{A}'(x) := \{\pi(.) \in \mathcal{A}(x) | E[U(X(T))^-] < \infty, X(T) \geq B \text{ a.s. } P\} \tag{6.4.6}$$

insures the existence of the expected value in (6.4.5). Solutions of similar problems are presented in Korn (2005) [16], where the requirements for the utility function $U(.)$ are more general than the usual conditions assumed in (6.1.16)-(6.1.17). In Korn (2005) [16] the utility function $U(.) : (0, \infty) \to \mathbb{R}$ is assumed to be a strictly concave C^1-function that satisfies

$$U'(0) := \lim_{x \to 0} U'(x) > 0\,, \tag{6.4.7}$$

$$U'(x) > 0, \quad \forall x \in (0, z), \tag{6.4.8}$$

$$U'(z) = 0 \tag{6.4.9}$$

for a unique value $z \in (0, \infty]$. In our problem we stay to the classical assumptions on the utility function $U(.) : (0, \infty) \to \mathbb{R}$, (6.1.16)-(6.1.17).

The only difference we have in our problem (6.4.5) compared to Korn (2005) [16] is that in our case instead of a basic portfolio problem we have an option portfolio problem (6.4.5). This difficulty can be overcome with the results of Korn and Trautmann (1999) [18], Theorem 1, where we know how to transfer the solution of the basic portfolio problem into the solution of the option portfolio problem. Having the same arguments as in the previous sections that is the fact that an inflation-linked bond $B_{IL}(t, I(t))$ is a derivative on inflation index $I(t)$, according to Korn and Trautmann (1999) [18], Theorem 1, the optimal final wealth of the option portfolio problem (6.4.5) coincides with the optimal final wealth of the basic portfolio problem defined as

$$\max_{\eta(.) \in \mathcal{A}'(x)} E[U(X(T))], \tag{6.4.10}$$

where the wealth process $X(t)$ can with the help of trading strategy $\xi(t) = (\xi_0(t), \xi_1(t))'$ be written as

$$X(t) = \xi_0(t) P_0(t) + \xi_1(t) I(t), \quad X(0) = x \tag{6.4.11}$$

with the corresponding portfolio process $\eta(t) = (\eta_0(t), \eta_1(t))'$, (6.1.28)-(6.1.29).

From Korn (2005) [16] we know that in order to solve this constrained basic portfolio problem (6.4.10) we need to define the equivalent auxiliary portfolio problem of the form

$$\max_{\tilde{\eta}(.) \in \tilde{\mathcal{A}}'(\tilde{x})} E[\tilde{U}(\tilde{X}(T))], \tag{6.4.12}$$

where the auxiliary utility function $\tilde{U}(.)$ is defined as

$$\tilde{U}(x) := U(x + B) \tag{6.4.13}$$

and the wealth process $\tilde{X}(t)$ is given by

$$\tilde{X}(t) = \tilde{\xi}_0(t)P_0(t) + \tilde{\xi}_1(t)I(t), \quad \tilde{X}(0) = \tilde{x} \tag{6.4.14}$$

with the portfolio process $\tilde{\eta}(t) = (\tilde{\eta}_0(t), \tilde{\eta}_1(t))'$ corresponding to the trading strategy $\tilde{\xi}(t) = (\tilde{\xi}_0(t), \tilde{\xi}_1(t))'$. The initial wealth $\tilde{x}$ in the auxiliary portfolio problem (6.4.12) is set to

$$\tilde{x} := x - E[H(T)B], \tag{6.4.15}$$

where the process $H(t)$ is given by

$$H(t) := \exp\Big(-\int_0^t \Big(r_N(s) + \frac{1}{2}\theta^2(s)\Big)ds - \int_0^t \theta(s)d\tilde{W}_I(s)\Big), \tag{6.4.16}$$

$$\theta(s) := \frac{\lambda(s)\sigma_I - r_R(s)}{\sigma_I}. \tag{6.4.17}$$

In the case of a constant B the initial wealth $\tilde{x}$, (6.4.15), in the auxiliary portfolio problem (6.4.12) is reduced to

$$\tilde{x} = x - \exp\Big(-\int_0^T r_N(s)ds\Big)B. \tag{6.4.18}$$

The subset of admissible portfolio processes $\tilde{\mathcal{A}}'(\tilde{x})$ in the auxiliary portfolio problem (6.4.12) is given by

$$\tilde{\mathcal{A}}'(\tilde{x}) := \{\pi(.) \in \mathcal{A}(\tilde{x}) | E[U(X(T))^-] < \infty\}. \tag{6.4.19}$$

We want to notice that despite the fact that the auxiliary utility function $\tilde{U}(.)$ does not satisfy the classical assumptions on the utility function, (6.1.16)-(6.1.17), it satisfies the more general assumptions (6.4.7)-(6.4.9) of Korn (2005) [16], which provides us with the following theorem.

Theorem 5. Let $x \geq E[H(T)B]$ then the optimal portfolio process $\pi_1(t)$ of an inflation-linked bond $B_{IL}(t, I(t))$ for the constrained continuous-time portfolio problem (6.4.5) for the HARA utility functions (logarithmic and power utility (6.1.33)) is given by

$$\pi_1(t) = \frac{\tilde{\eta}_1(t)\tilde{X}(t) + \hat{\eta}_1(t)E\Big[\frac{H(T)}{H(t)}B\Big|\mathcal{F}_t\Big]}{X(t)} \cdot \frac{B_{IL}(t, I(t))}{\psi(t)I(t)}, \tag{6.4.20}$$

where $X(t)$ is the wealth process corresponding to $\pi_1(t)$ via (6.4.11) and (6.1.28)-(6.1.29), $\tilde{\eta}_1(t)$ is an optimal portfolio process for inflation index $I(t)$ in the auxiliary portfolio problem (6.4.12) with a corresponding wealth process $\tilde{X}(t)$, $\hat{\eta}_1(t)$ is an optimal portfolio process for inflation index $I(t)$ needed for the replication of the final payment B at the maturity date T with an initial capital $\hat{x} = E[H(T)B]$ and $\psi(t)$ is a replicating strategy for an inflation-linked bond $B_{IL}(t, I(t))$ by inflation index $I(t)$ given by

$$\psi(t) = \sum_{i:t_i \geq t} \frac{C_i}{I(t_0)} \exp\Big(-\int_t^{t_i} r_R(s)ds\Big) + \frac{F\exp\Big(-\int_t^T r_R(s)ds\Big)N(d(t))}{I(t_0)}. \tag{6.4.21}$$

Proof of Theorem 5. According to Korn (2005) [16] the optimal portfolio process $\eta_1(t)$ in the constrained basic portfolio problem (6.4.10) can be expressed as

$$\eta_1(t)X(t) = \tilde{\eta}_1(t)\tilde{X}(t) + \hat{\eta}_1(t)E\left[\frac{H(T)}{H(t)}B\middle|\mathcal{F}_t\right], \tag{6.4.22}$$

where $X(t)$ is the wealth process corresponding to $\eta_1(t)$ via (6.4.1)-(6.4.3), $\tilde{\eta}_1(t)$ is an optimal portfolio process for inflation index $I(t)$ in the auxiliary portfolio problem (6.4.12) with a corresponding wealth process $\tilde{X}(t)$ and $\hat{\eta}_1(t)$ is an optimal portfolio process for inflation index $I(t)$ needed for the replication of the final payment B at the maturity date T with an initial capital $\hat{x} = E[H(T)B]$.

Now it is possible to express the trading strategy $\xi_1(t)$ for inflation index $I(t)$ in the constrained basic portfolio problem (6.4.10) as

$$\xi_1(t) = \frac{\eta_1(t)X(t)}{I(t)} = \frac{\tilde{\eta}_1(t)\tilde{X}(t) + \hat{\eta}_1(t)E\left[\frac{H(T)}{H(t)}B\middle|\mathcal{F}_t\right]}{I(t)}. \tag{6.4.23}$$

Applying the results of Korn and Trautmann (1999) [18], Theorem 1, and having the optimal trading strategy $\xi_1(t)$ for the constrained basic portfolio problem (6.4.10) we are able to express the optimal trading strategy $\varphi_1(t)$ of the option portfolio problem (6.4.5) as

$$\varphi_1(t) = \frac{\xi_1(t)}{\psi(t)} = \frac{\tilde{\eta}_1(t)\tilde{X}(t) + \hat{\eta}_1(t)E\left[\frac{H(T)}{H(t)}B\middle|\mathcal{F}_t\right]}{\psi(t)I(t)}, \tag{6.4.24}$$

where $\psi(t)$ is a replicating strategy of an inflation-linked bond $B_{IL}(t, I(t))$ by inflation index $I(t)$ defined in (6.1.38), i.e.

$$\psi(t) = \sum_{i:t_i \geq t} \frac{C_i}{I(t_0)} \exp\left(-\int_t^{t_i} r_R(s)ds\right) + \frac{F\exp\left(-\int_t^T r_R(s)ds\right)N(d(t))}{I(t_0)}. \tag{6.4.25}$$

Substituting this relation (6.4.25) into (6.4.2) we directly get (6.4.20). □

6.4.1 Optimal Investment with a Performance Benchmark

In this application to Theorem 5 we consider the situation of an investor who wants to maximize the growth of his wealth while being sure that the performance of his optimized wealth is at least as good as a fraction of the initial wealth adjusted to the inflation. More precisely, we consider a lower bond B of the form

$$B = \alpha\frac{I(T)}{i}x = \beta I(T), \tag{6.4.26}$$

$$\alpha \in [0,1), \tag{6.4.27}$$

$$\beta := \frac{\alpha x}{i}. \tag{6.4.28}$$

We now consider the auxiliary portfolio problem (6.4.12) with the initial wealth $\tilde{x}$, (6.4.15), equal to

$$\tilde{x} = x - E[H(T)B] = x - E\left[H(T)\frac{\alpha x}{i}I(T)\right] = (1-\alpha)x \tag{6.4.29}$$

and the auxiliary utility function $\tilde{U}(.)$ given by

$$\tilde{U}(x) = U(x + \beta I(T)), \tag{6.4.30}$$

where $U(.)$ is a logarithmic or power utility (6.1.33).

In order to solve this auxiliary portfolio problem (6.4.12) we will apply the modified martingale approach (see for example Korn (1997) [14]). In line with Korn (2005) [16], the inverse function of the first derivative of the utility function $\tilde{U}(.)$ is given by

$$\tilde{I}(y) := (\tilde{U}')^{-1}(y) = \left(y^{\frac{1}{\gamma-1}} - \beta I(T)\right)^{+}, \quad \forall y > 0, \tag{6.4.31}$$

where the case of $\gamma = 0$ corresponds to the logarithmic utility function. The process $\tilde{X}(y)$ is defined as

$$\tilde{X}(y) := E\left[H(T)\tilde{I}(yH(T))\right] = E\left[\left(y^{\frac{1}{\gamma-1}} H(T)^{\frac{\gamma}{\gamma-1}} - \beta H(T)I(T)\right)^{+}\right], \quad \forall y > 0. \tag{6.4.32}$$

The process $\tilde{X}(y)$, (6.4.32), can be expressed as

$$\begin{aligned}\tilde{X}(y) = E\Bigg[\Bigg(& y^{\frac{1}{\gamma-1}} \exp\left(\frac{\gamma}{\gamma-1}\left(-\int_0^T \left(r_N(s) + \frac{1}{2}\theta^2(s)\right)ds - \sqrt{\int_0^T \theta^2(s)ds}Z\right)\right) - \\ & \beta i \exp\left(\int_0^T \left(\lambda(s)\sigma_I - r_R(s) - \frac{1}{2}(\theta^2(s) + \sigma_I^2)\right)ds + \right. \\ & \left.\left.\sqrt{\int_0^T (\sigma_I - \theta(s))^2 ds}Z\right)\Bigg)^{+}\right],\end{aligned} \tag{6.4.33}$$

where $Z \sim \mathcal{N}(0,1)$ is a standard normal variable. The difference in (6.4.32) is positive, when $z < z_0$, where

$$\begin{aligned} z_0 = \Bigg(\ln\left(\frac{y^{\frac{1}{\gamma-1}}}{\beta i}\right) & - \int_0^T \left(\lambda(s)\sigma_I - r_R(s) - \frac{1}{2}(\theta^2(s) + \sigma_I^2)\right)ds - \\ & \frac{\gamma}{\gamma-1}\int_0^T \left(r_N(s) + \frac{1}{2}\theta^2(s)\right)ds \Bigg) \Big/ \Bigg(\sqrt{\int_0^T (\sigma_I - \theta(s))^2 ds} + \\ & \frac{\gamma}{\gamma-1}\sqrt{\int_0^T \theta^2(s)ds}\Bigg). \end{aligned} \tag{6.4.34}$$

Further on, the expectation in (6.4.33) can be written as

$$\tilde{X}(y) = \int_{-\infty}^{z_0} \left(y^{\frac{1}{\gamma-1}} \exp\left(\frac{\gamma}{\gamma-1} \left(-\int_0^T \left(r_N(s) + \frac{1}{2}\theta^2(s) \right) ds - \sqrt{\int_0^T \theta^2(s) ds z} \right) \right) - \right.$$
$$\beta i \exp\left(\int_0^T \left(\lambda(s)\sigma_I - r_R(s) - \frac{1}{2}(\theta^2(s) + \sigma_I^2) \right) ds + \tag{6.4.35}$$
$$\left. \sqrt{\int_0^T (\sigma_I - \theta(s))^2 ds z} \right) \Bigg) f(z) dz,$$

where f is the density function of the $\mathcal{N}(0,1)$-distribution, i.e.

$$f(z) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2} \right). \tag{6.4.36}$$

Calculating the integrals in (6.4.35) we get for the process $\tilde{X}(y)$

$$\begin{aligned} \tilde{X}(y) = y^{\frac{1}{\gamma-1}} \exp\left(-\frac{\gamma}{\gamma-1} \int_0^T r_N(s) ds \right) N\left(z_0 + \frac{\gamma}{\gamma-1} \sqrt{\int_0^T \theta^2(s) ds} \right) - \\ \beta i N\left(z_0 - \sqrt{\int_0^T (\sigma_I - \theta(s))^2 ds} \right), \end{aligned} \tag{6.4.37}$$

where N is a cumulative distribution function of the $\mathcal{N}(0,1)$-distribution, i.e.

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp\left(-\frac{z^2}{2} \right) dz. \tag{6.4.38}$$

Let $\tilde{Y}(.)$ be an inverse function of $\tilde{X}(.)$. From (6.4.37) one can obtain $\tilde{Y}(\tilde{x})$ as the unique solution of the equation

$$\tilde{X}(\tilde{Y}(\tilde{x})) = \tilde{x} \tag{6.4.39}$$

via numerical methods, where we have $\tilde{x} = (1-\alpha)x$, (6.4.29). The optimal terminal wealth $\tilde{B}^*$ of the auxiliary portfolio problem (6.4.12) is given by

$$\tilde{B}^* = \tilde{I}(\tilde{Y}(\tilde{x})H(T)) = \left((\tilde{Y}(\tilde{x})H(T))^{\frac{1}{\gamma-1}} - \beta I(T) \right)^+. \tag{6.4.40}$$

The value of the optimal portfolio process $\tilde{X}(t)$ in the auxiliary portfolio problem (6.4.12) can be expressed as

$$\tilde{X}(t) = E\left[\frac{H(T)}{H(t)}\tilde{B}^*\,\middle|\,\mathcal{F}_t\right] = E\left[\left(\left(\tilde{Y}(\tilde{x})H(t)\right)^{\frac{1}{\gamma-1}}\left(\frac{H(T)}{H(t)}\right)^{\frac{\gamma}{\gamma-1}} - \beta\frac{H(T)}{H(t)}I(T)\right)^+\,\middle|\,\mathcal{F}_t\right]. \quad (6.4.41)$$

Applying the same techniques as we did in (6.4.32)-(6.4.38) we get for the optimal portfolio process $\tilde{X}(t)$ in the auxiliary portfolio problem (6.4.12)

$$\begin{aligned}\tilde{X}(t) = &\left(\tilde{Y}(\tilde{x})H(t)\right)^{\frac{1}{\gamma-1}}\exp\left(-\frac{\gamma}{\gamma-1}\int_t^T r_N(s)ds\right)N\left(z_0 + \frac{\gamma}{\gamma-1}\sqrt{\int_t^T \theta^2(s)ds}\right) - \\ &\beta I(t)N\left(z_0 - \sqrt{\int_t^T (\sigma_I - \theta(s))^2 ds}\right),\end{aligned} \quad (6.4.42)$$

where z_0 is given by

$$\begin{aligned}z_0 = &\left(\ln\left(\frac{\left(\tilde{Y}(\tilde{x})H(t)\right)^{\frac{1}{\gamma-1}}}{\beta I(t)}\right) - \int_t^T\left(\lambda(s)\sigma_I - r_R(s) - \frac{1}{2}(\theta^2(s)+\sigma_I^2)\right)ds - \right.\\ &\left.\frac{\gamma}{\gamma-1}\int_t^T\left(r_N(s)+\frac{1}{2}\theta^2(s)\right)ds\right)\Big/\left(\sqrt{\int_t^T (\sigma_I - \theta(s))^2 ds} + \right.\\ &\left.\frac{\gamma}{\gamma-1}\sqrt{\int_t^T \theta^2(s)ds}\right)\end{aligned} \quad (6.4.43)$$

and N is a cumulative distribution function of the $\mathcal{N}(0,1)$-distribution. Now the fraction of the optimal wealth $\tilde{X}(t)$ invested into inflation index $I(t)$ in the auxiliary portfolio problem (6.4.12) can be written as

$$\tilde{\eta}_1(t)\tilde{X}(t) = \tilde{\xi}_1(t)I(t), \quad (6.4.44)$$

where the trading strategy $\tilde{\xi}_1(t)$ is calculated as

$$\begin{aligned}\tilde{\xi}_1(t) = \frac{\partial\tilde{X}(t)}{\partial I(t)} = &-\left(\tilde{Y}(\tilde{x})H(t)\right)^{\frac{1}{\gamma-1}}\exp\left(-\frac{\gamma}{\gamma-1}\int_t^T r_N(s)ds\right)\cdot\\ &\frac{f\left(z_0 + \frac{\gamma}{\gamma-1}\sqrt{\int_t^T \theta^2(s)ds}\right)}{I(t)\left(\sqrt{\int_t^T (\sigma_I - \theta(s))^2 ds} + \frac{\gamma}{\gamma-1}\sqrt{\int_t^T \theta^2(s)ds}\right)} - \\ &\beta N\left(z_0 - \sqrt{\int_t^T (\sigma_I - \theta(s))^2 ds}\right) +\end{aligned}$$

$$\beta \frac{f\left(z_0 - \sqrt{\int_t^T (\sigma_I - \theta(s))^2 ds}\right)}{\sqrt{\int_t^T (\sigma_I - \theta(s))^2 ds} + \frac{\gamma}{\gamma - 1}\sqrt{\int_t^T \theta^2(s) ds}}, \tag{6.4.45}$$

where z_0 is given by (6.4.43) and f is the density function of the $\mathcal{N}(0,1)$-distribution.

The part of the final payment B, (6.4.26), replicated by inflation index $I(t)$ is equal in our application to

$$\hat{\eta}_1(t) E\left[\frac{H(T)}{H(t)} B \middle| \mathcal{F}_t\right] = \beta I(t). \tag{6.4.46}$$

Applying (6.4.44) and (6.4.46) to Theorem 5 we get the following corollary.

Corollary 1 to Theorem 5. The optimal portfolio process $\pi_1(t)$ of an inflation-linked bond $B_{IL}(t, I(t))$ for the constrained continuous-time portfolio problem (6.4.5) with a lower bound $B = \beta I(T)$ defined in (6.4.26) for the HARA utility functions (logarithmic and power utility (6.1.33)) is given by

$$\pi_1(t) = (\tilde{\xi}_1(t) + \beta) \cdot \frac{B_{IL}(t, I(t))}{\psi(t) X(t)}, \tag{6.4.47}$$

where $X(t)$ is the wealth process corresponding to $\pi_1(t)$ via (6.4.11) and (6.1.28)-(6.1.29), $\psi(t)$ is a replicating strategy for an inflation-linked bond $B_{IL}(t, I(t))$ by inflation index $I(t)$ given by

$$\psi(t) = \sum_{i:t_i \geq t} \frac{C_i}{I(t_0)} \exp\left(-\int_t^{t_i} r_R(s) ds\right) + \frac{F \exp\left(-\int_t^T r_R(s) ds\right) N(d(t))}{I(t_0)} \tag{6.4.48}$$

and the trading strategy $\tilde{\xi}_1(t)$ of the auxiliary portfolio problem (6.4.12) is given by

$$\tilde{\xi}_1(t) = \frac{\beta f\left(z_0 - \sqrt{\int_t^T (\sigma_I - \theta(s))^2 ds}\right)}{\sqrt{\int_t^T (\sigma_I - \theta(s))^2 ds} + \frac{\gamma}{\gamma - 1}\sqrt{\int_t^T \theta^2(s) ds}} -$$
$$\beta N\left(z_0 - \sqrt{\int_t^T (\sigma_I - \theta(s))^2 ds}\right) -$$

$$\frac{\left(\tilde{Y}(\tilde{x})H(t)\right)^{\frac{1}{\gamma-1}} \exp\left(-\frac{\gamma}{\gamma-1}\int_t^T r_N(s)ds\right) f\left(z_0+\frac{\gamma}{\gamma-1}\sqrt{\int_t^T \theta^2(s)ds}\right)}{I(t)\left(\sqrt{\int_t^T (\sigma_I-\theta(s))^2 ds}+\frac{\gamma}{\gamma-1}\sqrt{\int_t^T \theta^2(s)ds}\right)} \tag{6.4.49}$$

with

$$z_0 = \left(\ln\left(\frac{\left(\tilde{Y}(\tilde{x})H(t)\right)^{\frac{1}{\gamma-1}}}{\beta I(t)}\right) - \int_t^T \left(\lambda(s)\sigma_I - r_R(s) - \frac{1}{2}(\theta^2(s)+\sigma_I^2)\right)ds - \frac{\gamma}{\gamma-1}\int_t^T \left(r_N(s)+\frac{1}{2}\theta^2(s)\right)ds\right) \Big/ \left(\sqrt{\int_t^T (\sigma_I-\theta(s))^2 ds} + \frac{\gamma}{\gamma-1}\sqrt{\int_t^T \theta^2(s)ds}\right) \tag{6.4.50}$$

and $\tilde{Y}(\tilde{x})$ is a unique solution of the equation (6.4.39).

7 Continuous-time Portfolio Problem with Purchasing Power

In this section we will consider the utility maximization from the purchasing power of the investor. There we will also have two different cases with different market setups. The first one is similar to the one of wealth optimization, where the market is assumed to have only two assets: a conventional bond and an inflation-linked bond. In the second case, we will expand the market by introducing a stock.

7.1 Problem 5: Conventional Bond and Inflation-linked Bond (Purchasing Power in Utility)

So far we have been considering only utility functions on the terminal wealth directly. Now we want to look at the situation, where we are interested in maximizing the terminal purchasing power of the investor that we define in the following way

$$\hat{X}(t) := \frac{i}{I(t)} X(t). \tag{7.1.1}$$

To start with, we look at a complete market studied in section 6.1, which is made up of a money market account $P_0(t)$, (6.1.1), and an inflation linked-bond $B_{IL}(t, I(t))$, (6.1.3)-(6.1.5), driven by inflation index $I(t)$, (6.1.2). Consequently investor's wealth process can be expressed as

$$X(t) = \varphi_0(t)P_0(t) + \varphi_1(t)B_{IL}(t, I(t)), \quad X(0) = x\,, \tag{7.1.2}$$

where investor's initial capital is strictly positive, i.e. $x > 0$. To the trading strategy $\varphi(t) = (\varphi_0(t), \varphi_1(t))'$ corresponding portfolio process $\pi(t) = (\pi_0(t), \pi_1(t))'$ is defined as

$$\pi_1(t) := \frac{\varphi_1(t)B_{IL}(t, I(t))}{X(t)}\,, \tag{7.1.3}$$

$$\pi_0(t) := 1 - \pi_1(t) = \frac{\varphi_0(t)P_0(t)}{X(t)}\,. \tag{7.1.4}$$

Again, the set of admissible portfolio processes $\pi(t)$ is denoted by $\mathcal{A}(x)$ and is defined by

$$\mathcal{A}(x) := \left\{ (1 - \pi_1(.), \pi_1(.))' \middle| X(0) = x > 0,\ \int_0^T \pi_1^2(t)dt < \infty \text{ a.s. } P \right\}, \tag{7.1.5}$$

which consists of portfolio processes $\pi(t)$ that insure the non-negativity of the corresponding wealth processes $X(t)$ asymptotically during the whole time $t \in [0, T]$, (6.1.12).

We consider the continuous-time portfolio problem of maximizing expected utility of terminal purchasing power $\hat{X}(T)$ of the investor, (7.1.1), i.e. the continuous-time portfolio problem looks like

$$\max_{\pi(.) \in \mathcal{A}'(x)} E\left[U\left(\frac{i}{I(T)} X(T) \right) \right], \tag{7.1.6}$$

where the subset of admissible portfolio processes $\mathcal{A}'(x)$ given by

$$\mathcal{A}'(x) := \left\{ \pi(.) \in \mathcal{A}(x) \middle| E\left[U\left(\frac{i}{I(T)} X(T) \right)^{-} \right] < \infty \right\} \tag{7.1.7}$$

insures the existence of the expected value in (7.1.6). The utility function $U(.) : (0, \infty) \to \mathbb{R}$ is a strictly concave C^1-function that satisfies (6.1.16)-(6.1.17).

Naturally, due to the fact that an inflation-linked bond $B_{IL}(t, I(t))$ is a derivative on inflation index $I(t)$, according to Korn and Trautmann (1999) [18], Theorem 1, the optimal final wealth of the option portfolio problem (7.1.6) coincides with the optimal final wealth of the basic portfolio problem, where the investor is assumed to be able to invest into regular bond $P_0(t)$ and inflation index $I(t)$. The basic portfolio problem is given by

$$\max_{\eta(.)\in\mathcal{A}'(x)} E\left[U\left(\frac{i}{I(T)}X(T)\right)\right], \tag{7.1.8}$$

where the wealth process $X(t)$ can with the help of trading strategy $\xi(t)=(\xi_0(t),\xi_1(t))'$ be written as

$$X(t)=\xi_0(t)P_0(t)+\xi_1(t)I(t), \quad X(0)=x \tag{7.1.9}$$

with portfolio process $\eta(t)=(\eta_0(t),\eta_1(t))'$ having the form

$$\eta_1(t):=\frac{\xi_1(t)I(t)}{X(t)}, \tag{7.1.10}$$

$$\eta_0(t):=1-\eta_1(t)=\frac{\xi_0(t)P_0(t)}{X(t)}. \tag{7.1.11}$$

For the wealth process $X(t)$ of (7.1.9) we can write down the following stochastic differential equation

$$dX(t)=X(t)((r_N(t)+\eta_1(t)(\lambda(t)\sigma_I-r_R(t)))dt+\eta_1(t)\sigma_I d\tilde{W}_I(t)), \quad X(0)=x. \tag{7.1.12}$$

Having (6.1.2) and (7.1.12), with a help of Itô's formula we compute the dynamics of the purchasing power process $\hat{X}(t)$, (7.1.1),

$$\begin{aligned} d\hat{X}(t) &= d\left(\frac{i}{I(t)}X(t)\right)=d\left(\frac{i}{I(t)}\right)X(t)+\frac{i}{I(t)}dX(t)+d\left(\frac{i}{I(t)}\right)dX(t) \\ &= \hat{X}(t)(\eta_0(t)(r_R(t)-\lambda(t)\sigma_I+\sigma_I^2)dt-\eta_0(t)\sigma_I d\tilde{W}_I(t)). \end{aligned} \tag{7.1.13}$$

In line with Björk (1998) [5] let us define the optimal value function $V(t,\hat{x})$ for the basic portfolio problem (7.1.8) as

$$V(t,\hat{x})=\sup_{\eta(.)\in\hat{\mathcal{A}}'(t,\hat{x})} E[U(\hat{X}(T))], \tag{7.1.14}$$

where the set of admissible portfolio processes is defines as

$$\hat{\mathcal{A}}'(t,\hat{x}) := \{\eta(.) \in \hat{\mathcal{A}}(t,\hat{x}) \mid E[U(\hat{X}(T))^{-}] < \infty\} \tag{7.1.15}$$

with

$$\hat{\mathcal{A}}(t,\hat{x}) := \left\{(\eta_0(.),1-\eta_0(.))' \middle| \hat{X}(t) = \hat{x} > 0,\ \int_t^T (1-\eta_0(s))^2 ds < \infty \text{ a.s. } P\right\}. \tag{7.1.16}$$

Now we can write the Hamilton-Jacobi-Bellman equation for the logarithmic and power utility functions (6.1.33)

$$\begin{cases} V_t(t,\hat{x}) + \sup\limits_{\eta(.)\in\mathcal{A}'(x)} \left(\hat{\mu}(t)\hat{x}V_{\hat{x}}(t,\hat{x}) + \dfrac{1}{2}\hat{\sigma}^2(t)\hat{x}^2 V_{\hat{x}\hat{x}}(t,\hat{x})\right) = 0, & \forall t \in (0,T), \forall \hat{x} > 0 \\ V(T,\hat{x}) = \left\{\ln(\hat{x}), \dfrac{1}{\gamma}\hat{x}^{\gamma}\right\}, & \gamma \in (0,1), \forall \hat{x} > 0 \end{cases} \tag{7.1.17}$$

where we define $\hat{\mu}(t)$ and $\hat{\sigma}(t)$ as

$$\hat{\mu}(t) := \eta_0(t)(r_R(t) - \lambda(t)\sigma_I + \sigma_I^2), \tag{7.1.18}$$

$$\hat{\sigma}(t) := -\eta_0(t)\sigma_I \tag{7.1.19}$$

and the subscripts in the optimal value function $V(t,\hat{x})$ denote partial derivatives. For an arbitrary point $(t,\hat{x})$ the supremum is attained, when

$$\eta_0(t) = -\frac{(r_R(t) - \lambda(t)\sigma_I + \sigma_I^2)V_{\hat{x}}(t,\hat{x})}{\sigma_I^2 \hat{x} V_{\hat{x}\hat{x}}(t,\hat{x})}. \tag{7.1.20}$$

After substituting this candidate for the optimal control law (7.1.20) into the Hamilton-Jacobi-Bellman equation (7.1.17), we get the following partial differential equation for the optimal value function $V(t,\hat{x})$

$$V_t(t,\hat{x}) - \frac{1}{2}\frac{(r_R(t) - \lambda(t)\sigma_I + \sigma_I^2)^2 V_{\hat{x}}^2(t,\hat{x})}{\sigma_I^2 V_{\hat{x}\hat{x}}(t,\hat{x})} = 0. \tag{7.1.21}$$

Making a guess about the form of the optimal value function $V(t,\hat{x})$ as

$$V(t,\hat{x}) = \ln(\hat{x}) + f(t), \quad f(T) = 0 \tag{7.1.22}$$

and

$$V(t,\hat{x}) = f(t)\frac{1}{\gamma}\hat{x}^{\gamma}, \quad f(T) = 1 \tag{7.1.23}$$

for the logarithmic and power utility functions (6.1.33) respectively, we substitute our ansatz (7.1.22)-(7.1.23) into the partial differential equation (7.1.21). As a result we get a differential equation for the function $f(t)$

$$f'(t) = -\frac{1}{2}\frac{(r_R(t) - \lambda(t)\sigma_I + \sigma_I^2)^2}{\sigma_I^2}, \quad f(T) = 0 \tag{7.1.24}$$

and

$$f'(t) = \frac{1}{2}\frac{(r_R(t) - \lambda(t)\sigma_I + \sigma_I^2)^2}{\sigma_I^2} \cdot \frac{\gamma}{\gamma - 1} f(t), \quad f(T) = 1 \tag{7.1.25}$$

for the logarithmic and power utility functions (6.1.33) respectively. The solutions for these differential equations (7.1.24)-(7.1.25) are given by

$$f(t) = \frac{1}{2}\frac{(r_R(t) - \lambda(t)\sigma_I + \sigma_I^2)^2}{\sigma_I^2}(T - t) \tag{7.1.26}$$

and

$$f(t) = \exp\left(\frac{1}{2}\frac{(r_R(t) - \lambda(t)\sigma_I + \sigma_I^2)^2}{\sigma_I^2} \cdot \frac{\gamma}{1 - \gamma}(T - t)\right) \tag{7.1.27}$$

for the logarithmic and power utility functions (6.1.33) respectively. Knowing the form of the optimal value function $V(t,\hat{x})$, (7.1.22)-(7.1.23) and (7.1.26)-(7.1.27), from (7.1.20) we get the solution for the optimal portfolio process $\eta(t) = (\eta_0(t), \eta_1(t))'$ of the basic portfolio problem (7.1.8) in form of

$$\eta_0(t) = \frac{r_R(t) - \lambda(t)\sigma_I + \sigma_I^2}{(1 - \gamma)\sigma_I^2}, \tag{7.1.28}$$

$$\eta_1(t) = 1 - \eta_0(t) = \frac{\lambda(t)\sigma_I - r_R(t) - \gamma\sigma_I^2}{(1-\gamma)\sigma_I^2}, \tag{7.1.29}$$

where the case of $\gamma = 0$ corresponds to the logarithmic utility function.

Theorem 6. The optimal portfolio process $\pi_1(t)$ of an inflation-linked bond $B_{IL}(t, I(t))$ for the continuous-time portfolio problem (7.1.6) for the HARA utility functions (logarithmic and power utility (6.1.33)) is given by

$$\pi_1(t) = \left(\frac{\lambda(t)\sigma_I - r_R(t)}{(1-\gamma)\sigma_I^2} - \frac{\gamma}{1-\gamma}\right) \cdot \frac{B_{IL}(t, I(t))}{\psi(t) I(t)}, \tag{7.1.30}$$

where $\psi(t)$ is a replicating strategy given by (6.1.39) that is

$$\psi(t) = \sum_{i: t_i \geq t} \frac{C_i}{I(t_0)} \exp\left(-\int_t^{t_i} r_R(s) ds\right) + \frac{F \exp\left(-\int_t^T r_R(s) ds\right) N(d(t))}{I(t_0)}; \tag{7.1.31}$$

the case of $\gamma = 0$ corresponds to the logarithmic utility function.

Proof of Theorem 6. Having the optimal portfolio process $\eta_1(t)$ (7.1.29) for the basic portfolio problem (7.1.8) we can now express the optimal trading strategy $\xi_1(t)$ as

$$\xi_1(t) = \frac{\lambda(t)\sigma_I - r_R(t) - \gamma\sigma_I^2}{(1-\gamma)\sigma_I^2} \cdot \frac{X(t)}{I(t)}. \tag{7.1.32}$$

Applying the results of Korn and Trautmann (1999) [18], Theorem 1, and having the optimal trading strategy $\xi_1(t)$ for the basic portfolio problem (7.1.8) we are able to express the optimal trading strategy $\varphi(t)$ of the option portfolio problem (7.1.6) as

$$\varphi_1(t) = \frac{\xi_1(t)}{\psi(t)} = \frac{\lambda(t)\sigma_I - r_R(t) - \gamma\sigma_I^2}{(1-\gamma)\sigma_I^2} \cdot \frac{X(t)}{\psi(t) I(t)}, \tag{7.1.33}$$

$$\varphi_0(t) = \frac{X(t) - \varphi_1(t) B_{IL}(t, I(t))}{P_0(t)} = \left(1 - \frac{(\lambda(t)\sigma_I - r_R(t) - \gamma\sigma_I^2) B_{IL}(t, I(t))}{(1-\gamma)\sigma_I^2 \psi(t) I(t)}\right) \frac{X(t)}{P_0(t)}, \tag{7.1.34}$$

where $\psi(t)$ is a replicating strategy of an inflation-linked bond $B_{IL}(t, I(t))$ by inflation index $I(t)$ defined in (6.1.38)-(6.1.39).

From (7.1.3) we know that the optimal portfolio process $\pi_1(t)$ can be represented as

$$\pi_1(t) = \frac{\varphi_1(t) B_{IL}(t, I(t))}{X(t)}. \tag{7.1.35}$$

Substituting (7.1.33) into this relation (7.1.35) we directly get (7.1.30). □

Remark 1 on Theorem 6 *("Negativity" of the Optimal Portfolio Process)***.** It turns out that also in this situation, when one is maximizing his terminal wealth of the purchasing power, a risk-averse investor does typically sell inflation-linked products in order to behave in an optimal way. This is actually is not surprising due to the same facts presented in Remark 1 on Theorem 2. As was already mentioned, in a normal situation the drift term of the risk-free asset, i.e. a conventional bond, beats the drift term of a risky asset, in our case an inflation-linked bond. That is the reason why in an imaginary situations studied here the optimal portfolio process for an inflation-linked security is negative for a relatively small subjective excess return $\lambda(t)\sigma_I$. Especially, this result for the case of purchasing power maximization can be in some way foreseen when knowing the results for the case of wealth maximization, due to the fact that we just discount our wealth process with a non-negative stochastic process of inflation index, which should not change the value of the optimal portfolio process a lot, when we have only two tradable assets. Actually, for the case of the logarithmic utility function the form of the optimal portfolio process stays the same. Indeed, all we do is just make a change of measure, where one have the same subordination of the portfolio processes of different financial assets.

Remark 2 on Theorem 6 *(Nature of the Continuous-time Portfolio Problem)***.** Looking at the dynamics of purchasing power process $\hat{X}(t)$, (7.1.13), we realize that the basic portfolio problem (7.1.8) studied in this section is equivalent to the classical continuous-time portfolio problem, where we have an inflation index $I(t)$ as a risk-free asset with an expected return of zero and a conventional bond $P_0(t)$ as a risky asset with a drift equal to $r_R(t) - \lambda(t)\sigma_I + \sigma_I^2$ and volatility $-\sigma_I$. The optimal portfolio process for a risky asset, in this case a conventional bond $P_0(t)$, is then given by the excess returns over the return of the risk-free asset divided by squared volatility multiplied with $1-\gamma$, which is exactly the same as in (7.1.28).

7.2 Problem 6: Conventional Bond, Stock and Inflation-linked Bond (Purchasing Power in Utility)

As an extension to the previous section, we will consider in this section a more general model, where we introduce a possibility for an investor to invest also into a stock $P_1(t)$ in addition to a conventional bond $P_0(t)$ and an inflation-linked bond $B_{IL}(t, I(t))$ driven by inflation index $I(t)$. The dynamics of the conventional bond $P_0(t)$ and inflation index $I(t)$ under the physical probability measure P are given by (6.1.1) and (6.1.2) respectively. The price of inflation-linked bond $B_{IL}(t, I(t))$ is given by (6.1.3)-(6.1.5). The dynamics of the stock price $P_1(t)$ is defined to be correlated with the inflation index $I(t)$ as

$$dP_1(t) = P_1(t)(b(t)dt + \sigma_1 d\tilde{W}(t) + \sigma_2 d\tilde{W}_I(t)), \quad P_1(0) = p_1, \tag{7.2.1}$$

where $(\tilde{W}(t), \tilde{W}_I(t))'$ is an independent two-dimensional standard Brownian motion under the physical probability measure P. Additionally we define the volatility of our model σ as

$$\sigma := \begin{bmatrix} \sigma_1 & \sigma_2 \\ 0 & \sigma_I \end{bmatrix}. \tag{7.2.2}$$

Consequently, investor's wealth process $X(t)$ can expressed as

$$X(t) = \varphi_0(t)P_0(t) + \varphi_1(t)P_1(t) + \varphi_2(t)B_{IL}(t, I(t)), \quad X(0) = x, \tag{7.2.3}$$

where investor's initial capital is strictly positive, i.e. $x > 0$, and trading strategy $\varphi(t) = (\varphi_0(t), \varphi_1(t), \varphi_2(t))'$ is $\mathbb{R}^3$-valued progressively measurable processes with respect to the filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$ generated by the independent standard Brownian motion $\{(\tilde{W}(t), \tilde{W}_I(t))'\}_{t\in[0,T]}$ satisfying similar technical conditions as (6.1.7)-(6.1.8). Naturally the corresponding $\mathbb{R}^3$-valued portfolio process $\pi(t) = (\pi_0(t), \pi_1(t), \pi_2(t))'$ is defined as

$$\pi_1(t) := \frac{\varphi_1(t)P_1(t)}{X(t)}, \tag{7.2.4}$$

$$\pi_2(t) := \frac{\varphi_2(t)B_{IL}(t, I(t))}{X(t)}, \tag{7.2.5}$$

$$\pi_0(t) := 1 - \overline{\pi}(t)'\underline{1} = \frac{\varphi_0(t)P_0(t)}{X(t)}, \tag{7.2.6}$$

where $\underline{1} := (1,\ldots,1)' \in \mathbb{R}^d$ and $\overline{\pi}(t)$ corresponds to the fractions of the wealth $X(t)$ invested into the risky assets, i.e.

$$\overline{\pi}(t) := (\pi_1(t), \pi_2(t))'. \tag{7.2.7}$$

As usual, we define the set of admissible portfolio processes $\mathcal{A}(x)$ in analogy to (7.1.5)

$$\mathcal{A}(x) := \left\{(1 - \overline{\pi}(.)'\underline{1}, \overline{\pi}(.)')' \middle| X(0) = x > 0, \int_0^T \pi_i^2(t)dt < \infty \text{ a.s. } P,\ i = 1,2\right\}, \tag{7.2.8}$$

which insures the non-negativity of the corresponding wealth process $X(t)$ asymptotically during the whole time $t \in [0,T]$, (6.1.12).

Having the wealth process $X(t)$ given by (7.2.3), in this section we look at the continuous-time portfolio problem of maximizing expected utility of terminal purchasing power $\hat{X}(T)$ of the investor, (7.1.1), i.e. the continuous-time portfolio problem looks like

$$\max_{\pi(.) \in \mathcal{A}'(x)} E\left[U\left(\frac{i}{I(T)}X(T)\right)\right], \tag{7.2.9}$$

where the subset of admissible portfolio processes $\mathcal{A}'(x)$ characterized as

$$\mathcal{A}'(x) := \left\{\pi(.) \in \mathcal{A}(x) \middle| E\left[U\left(\frac{i}{I(T)}X(T)\right)^-\right] < \infty\right\} \tag{7.2.10}$$

insures the existence of the expected value in (7.2.9). The utility function $U(.) : (0,\infty) \to \mathbb{R}$ is a strictly concave C^1-function that satisfies the usual conditions of (6.1.16)-(6.1.17).

Again, having the same arguments as in the previous section that is the fact that an inflation-linked bond $B_{IL}(t, I(t))$ is a derivative on inflation index $I(t)$, according to Korn and Trautmann (1999) [18], Theorem 1, the optimal final wealth of the option portfolio problem (7.2.9) coincides with the optimal final wealth of the basic portfolio problem, where the investor is assumed to be able to invest into regular bond $P_0(t)$, stock $P_1(t)$ and inflation index $I(t)$, i.e. we hypothetically assume an inflation index $I(t)$ to be tradable. Consequently, the basic portfolio problem is given by

$$\max_{\eta(.)\in\mathcal{A}'(x)} E\left[U\left(\frac{i}{I(T)}X(T)\right)\right], \tag{7.2.11}$$

where the wealth process $X(t)$ can with the help of trading strategy $\xi(t) = (\xi_0(t), \xi_1(t), \xi_2(t))'$ be written as

$$X(t) = \xi_0(t)P_0(t) + \xi_1(t)P_1(t) + \xi_2(t)I(t), \quad X(0) = x \tag{7.2.12}$$

with the corresponding portfolio process $\eta(t) = (\eta_0(t), \eta_1(t), \eta_2(t))'$ analogous to (7.1.10)-(7.1.11).

For the wealth process $X(t)$ of (7.2.12) we can write down the following stochastic differential equation

$$\begin{aligned} dX(t) = X(t)(((1-\eta_1(t))r_N(t) + \eta_1(t)b(t) + \eta_2(t)(\lambda(t)\sigma_I - r_R(t)))dt + \\ \eta_1(t)\sigma_1 d\tilde{W}(t) + (\eta_1(t)\sigma_2 + \eta_2(t)\sigma_I)d\tilde{W}_I(t)), \quad X(0) = x. \end{aligned} \tag{7.2.13}$$

Having (6.1.2) and (7.2.13), with a help of Itô's formula we compute the dynamics of the purchasing power process $\hat{X}(t)$, (7.1.1),

$$\begin{aligned} d\hat{X}(t) &= d\left(\frac{i}{I(t)}X(t)\right) = d\left(\frac{i}{I(t)}\right)X(t) + \frac{i}{I(t)}dX(t) + d\left(\frac{i}{I(t)}\right)dX(t) \\ &= \hat{X}(t)\left(\hat{\mu}(t)dt + \hat{\sigma}(t)\begin{bmatrix}\tilde{W}(t) \\ \tilde{W}_I(t)\end{bmatrix}\right), \end{aligned} \tag{7.2.14}$$

where we define $\hat{\mu}(t)$ and $\hat{\sigma}(t)$ as

$$\hat{\mu}(t) := (\eta_1(t), \eta_2(t) - 1)\begin{bmatrix} b(t) - r_N(t) - \sigma_2\sigma_I \\ \lambda(t)\sigma_I - r_R(t) - \sigma_I^2 \end{bmatrix}, \tag{7.2.15}$$

$$\hat{\sigma}(t) := (\eta_1(t), \eta_2(t) - 1)\sigma. \tag{7.2.16}$$

In line with Björk (1998) [5] let us define the optimal value function $V(t,\hat{x})$ for the basic portfolio problem (7.2.11) as

$$V(t,\hat{x}) = \sup_{\eta(.)\in\hat{\mathcal{A}}'(t,\hat{x})} E[U(\hat{X}(T))], \tag{7.2.17}$$

where the set of admissible portfolio processes is defines as

$$\hat{\mathcal{A}}'(t,\hat{x}) := \{\eta(.) \in \hat{\mathcal{A}}(t,\hat{x}) \mid E[U(\hat{X}(T))^{-}] < \infty\} \tag{7.2.18}$$

with

$$\hat{\mathcal{A}}(t,\hat{x}) := \left\{(1-\overline{\eta}(.)'\underline{1},\overline{\eta}(.)')' \middle| \hat{X}(t) = \hat{x} > 0,\ \int_t^T \eta_i^2(s)ds < \infty \text{ a.s. } P,\ i = 1,2\right\}, \tag{7.2.19}$$

where $\overline{\eta}(t)$ corresponds to the fractions of the wealth $X(t)$ invested into the risky assets, i.e.

$$\overline{\eta}(t) := (\eta_1(t),\eta_2(t))'. \tag{7.2.20}$$

Now we can write the Hamilton-Jacobi-Bellman equation for the logarithmic and power utility functions (6.1.33)

$$\begin{cases} V_t(t,\hat{x}) + \sup\limits_{\eta(.)\in\mathcal{A}'(x)}\left(\hat{\mu}(t)\hat{x}V_{\hat{x}}(t,\hat{x}) + \dfrac{1}{2}\hat{\sigma}^2(t)\hat{x}^2V_{\hat{x}\hat{x}}(t,\hat{x})\right) = 0, & \forall t \in (0,T), \forall \hat{x} > 0 \\ V(T,\hat{x}) = \left\{\ln(\hat{x}), \dfrac{1}{\gamma}\hat{x}^{\gamma}\right\}, & \gamma \in (0,1), \forall \hat{x} > 0 \end{cases} \tag{7.2.21}$$

where $\hat{\mu}(t)$ and $\hat{\sigma}^2(t) := \hat{\sigma}(t)\hat{\sigma}(t)'$ are given by (7.2.15) and (7.2.16) respectively and the subscripts in the optimal value function $V(t,\hat{x})$ denote partial derivatives. For an arbitrary point $(t,\hat{x})$ the supremum is attained, when

$$\begin{bmatrix} \eta_1(t) \\ \eta_2(t) - 1 \end{bmatrix} = -\frac{V_{\hat{x}}(t,\hat{x})}{\hat{x}V_{\hat{x}\hat{x}}(t,\hat{x})}(\sigma\sigma')^{-1}\begin{bmatrix} b(t) - r_N(t) - \sigma_2\sigma_I \\ \lambda(t)\sigma_I - r_R(t) - \sigma_I^2 \end{bmatrix}. \tag{7.2.22}$$

After the substituting this candidate for the optimal control law (7.2.22) into the Hamilton-Jacobi-Bellman equation (7.2.21), we get the following partial differential equation for the optimal value function $V(t,\hat{x})$

$$V_t(t,\hat{x}) - \frac{1}{2}\frac{V_{\hat{x}}^2(t,\hat{x})}{V_{\hat{x}\hat{x}}(t,\hat{x})}\begin{bmatrix} b(t) - r_N(t) - \sigma_2\sigma_I \\ \lambda(t)\sigma_I - r_R(t) - \sigma_I^2 \end{bmatrix}' (\sigma\sigma')^{-1} \begin{bmatrix} b(t) - r_N(t) - \sigma_2\sigma_I \\ \lambda(t)\sigma_I - r_R(t) - \sigma_I^2 \end{bmatrix} = 0\,. \quad (7.2.23)$$

Making a guess about the form of the optimal value function $V(t,\hat{x})$ as

$$V(t,\hat{x}) = \ln(\hat{x}) + f(t), \quad f(T) = 0 \quad (7.2.24)$$

and

$$V(t,\hat{x}) = f(t)\frac{1}{\gamma}\hat{x}^{\gamma}, \quad f(T) = 1 \quad (7.2.25)$$

for the logarithmic and power utility functions (6.1.33) respectively, we substitute our ansatz (7.2.24)-(7.2.25) into the partial differential equation (7.2.23). As a result we get a differential equation for the function $f(t)$

$$f'(t) = -\frac{1}{2}\begin{bmatrix} b(t) - r_N(t) - \sigma_2\sigma_I \\ \lambda(t)\sigma_I - r_R(t) - \sigma_I^2 \end{bmatrix}' (\sigma\sigma')^{-1} \begin{bmatrix} b(t) - r_N(t) - \sigma_2\sigma_I \\ \lambda(t)\sigma_I - r_R(t) - \sigma_I^2 \end{bmatrix}, \quad f(T) = 0 \quad (7.2.26)$$

and

$$\begin{aligned} f'(t) &= \frac{1}{2}\begin{bmatrix} b(t) - r_N(t) - \sigma_2\sigma_I \\ \lambda(t)\sigma_I - r_R(t) - \sigma_I^2 \end{bmatrix}' (\sigma\sigma')^{-1} \begin{bmatrix} b(t) - r_N(t) - \sigma_2\sigma_I \\ \lambda(t)\sigma_I - r_R(t) - \sigma_I^2 \end{bmatrix} \cdot \frac{\gamma}{\gamma - 1} f(t), \\ f(T) &= 1 \end{aligned} \quad (7.2.27)$$

for the logarithmic and power utility functions (6.1.33) respectively. The solutions for these differential equations (7.2.26)-(7.2.27) are given by

$$f(t) = \frac{1}{2}\begin{bmatrix} b(t) - r_N(t) - \sigma_2\sigma_I \\ \lambda(t)\sigma_I - r_R(t) - \sigma_I^2 \end{bmatrix}' (\sigma\sigma')^{-1} \begin{bmatrix} b(t) - r_N(t) - \sigma_2\sigma_I \\ \lambda(t)\sigma_I - r_R(t) - \sigma_I^2 \end{bmatrix} (T - t) \quad (7.2.28)$$

and

$$f(t)=\exp\left(\frac{1}{2}\begin{bmatrix}b(t)-r_N(t)-\sigma_2\sigma_I\\ \lambda(t)\sigma_I-r_R(t)-\sigma_I^2\end{bmatrix}'(\sigma\sigma')^{-1}\begin{bmatrix}b(t)-r_N(t)-\sigma_2\sigma_I\\ \lambda(t)\sigma_I-r_R(t)-\sigma_I^2\end{bmatrix}\cdot\frac{\gamma}{1-\gamma}(T-t)\right)\quad(7.2.29)$$

for the logarithmic and power utility functions (6.1.33) respectively. Knowing the form of the optimal value function $V(t,\hat{x})$, (7.2.24)-(7.2.25) and (7.2.28)-(7.2.29), from (7.2.22) we get the solution for the optimal portfolio process $\overline{\eta}(t)=(\eta_1(t),\eta_2(t))'$ of the basic portfolio problem (7.2.11) in form of

$$\begin{aligned}\overline{\eta}(t)&=\frac{1}{1-\gamma}(\sigma\sigma')^{-1}\begin{bmatrix}b(t)-r_N(t)-\sigma_2\sigma_I\\ \lambda(t)\sigma_I-r_R(t)-\sigma_I^2\end{bmatrix}+\begin{bmatrix}0\\1\end{bmatrix}\\&=\frac{1}{1-\gamma}\begin{bmatrix}\dfrac{b(t)-r_N(t)}{\sigma_1^2}-\dfrac{\lambda(t)\sigma_I-r_R(t)}{\sigma_I^2}\cdot\dfrac{\sigma_2\sigma_I}{\sigma_1^2}\\ \dfrac{\lambda(t)\sigma_I-r_R(t)}{\sigma_I^2}\left(1+\dfrac{\sigma_2^2}{\sigma_1^2}\right)-\dfrac{b(t)-r_N(t)}{\sigma_1^2}\cdot\dfrac{\sigma_2\sigma_I}{\sigma_I^2}\end{bmatrix},\end{aligned}\quad(7.2.30)$$

where the case of $\gamma=0$ corresponds to the logarithmic utility function.

Theorem 7. The optimal portfolio process $\overline{\pi}(t)=(\pi_1(t),\pi_2(t))'$ of a stock $P_1(t)$ and an inflation-linked bond $B_{IL}(t,I(t))$ for the continuous-time portfolio problem (7.2.9) for the HARA utility functions (logarithmic and power utility (6.1.33)) is given by

$$\overline{\pi}(t)=\frac{1}{1-\gamma}\begin{bmatrix}\dfrac{b(t)-r_N(t)}{\sigma_1^2}-\dfrac{\lambda(t)\sigma_I-r_R(t)}{\sigma_I^2}\cdot\dfrac{\sigma_2\sigma_I}{\sigma_1^2}\\ \left(\dfrac{\lambda(t)\sigma_I-r_R(t)}{\sigma_I^2}\left(1+\dfrac{\sigma_2^2}{\sigma_1^2}\right)-\dfrac{b(t)-r_N(t)}{\sigma_1^2}\cdot\dfrac{\sigma_2\sigma_I}{\sigma_I^2}\right)\dfrac{B_{IL}(t,I(t))}{\psi_{22}(t)I(t)}\end{bmatrix},\quad(7.2.31)$$

where $\psi_{22}(t)$ is a replicating strategy for an inflation-linked bond $B_{IL}(t,I(t))$ by inflation index $I(t)$ given by

$$\psi_{22}(t)=\sum_{i:t_i\geq t}\frac{C_i}{I(t_0)}\exp\left(-\int_t^{t_i}r_R(s)ds\right)+\frac{F\exp\left(-\int_t^T r_R(s)ds\right)N(d(t))}{I(t_0)};\quad(7.2.32)$$

the case of $\gamma=0$ corresponds to the logarithmic utility function.

Proof of Theorem 7. Having the optimal portfolio process $\overline{\eta}(t)=(\eta_1(t),\eta_2(t))'$ (7.2.30) for the basic portfolio problem (7.2.11) we can now express the optimal trading strategy $\overline{\xi}(t)=(\xi_1(t),\xi_2(t))'$ as

$$\overline{\xi}(t)=\frac{X(t)}{1-\gamma}\begin{bmatrix}\dfrac{b(t)-r_N(t)}{\sigma_1^2 P_1(t)}-\dfrac{\lambda(t)\sigma_I-r_R(t)}{\sigma_I^2 P_1(t)}\cdot\dfrac{\sigma_2\sigma_I}{\sigma_1^2}\\ \dfrac{\lambda(t)\sigma_I-r_R(t)}{\sigma_I^2 I(t)}\left(1+\dfrac{\sigma_2^2}{\sigma_1^2}\right)-\dfrac{b(t)-r_N(t)}{\sigma_1^2 I(t)}\cdot\dfrac{\sigma_2\sigma_I}{\sigma_I^2}\end{bmatrix}. \qquad (7.2.33)$$

Applying the results of Korn and Trautmann (1999) [18], Theorem 1, and having the optimal trading strategy $\overline{\xi}(t)$ for the basic portfolio problem (7.2.11) we are able to express the optimal trading strategy $\overline{\varphi}(t)=(\varphi_1(t),\varphi_2(t))'$ and $\varphi_0(t)$ of the option portfolio problem (7.2.9) as

$$\overline{\varphi}(t)=(\psi(t)')^{-1}\overline{\xi}(t)=\frac{X(t)}{1-\gamma}\begin{bmatrix}\dfrac{b(t)-r_N(t)}{\sigma_1^2 P_1(t)}-\dfrac{\lambda(t)\sigma_I-r_R(t)}{\sigma_I^2 P_1(t)}\cdot\dfrac{\sigma_2\sigma_I}{\sigma_1^2}\\ \dfrac{\lambda(t)\sigma_I-r_R(t)}{\sigma_I^2\psi_{22}(t)I(t)}\left(1+\dfrac{\sigma_2^2}{\sigma_1^2}\right)-\dfrac{b(t)-r_N(t)}{\sigma_1^2\psi_{22}(t)I(t)}\cdot\dfrac{\sigma_2\sigma_I}{\sigma_I^2}\end{bmatrix}, \qquad (7.2.34)$$

$$\varphi_0(t)=\frac{X(t)-\varphi_1(t)P_1(t)-\varphi_2(t)B_{IL}(t,I(t))}{P_0(t)}, \qquad (7.2.35)$$

where $\psi(t)$ is a replicating strategy defined as

$$\psi(t):=\begin{bmatrix}\dfrac{\partial P_1(t)}{\partial P_1(t)} & \dfrac{\partial P_1(t)}{\partial I(t)}\\ \dfrac{\partial B_{IL}(t,I(t))}{\partial P_1(t)} & \dfrac{\partial B_{IL}(t,I(t))}{\partial I(t)}\end{bmatrix}, \qquad (7.2.36)$$

i.e.

$$\psi(t):=\begin{bmatrix}1 & 0\\ 0 & \displaystyle\sum_{i:t_i\geq t}\frac{C_i}{I(t_0)}\exp\left(-\int_t^{t_i}r_R(s)ds\right)+\frac{F\exp\left(-\int_t^T r_R(s)ds\right)N(d(t))}{I(t_0)}\end{bmatrix}; \qquad (7.2.37)$$

$\psi_{22}(t)$ is a 22-element of the replicating matrix $\psi(t)$, which corresponds to the replicating strategy for an inflation-linked bond $B_{IL}(t,I(t))$ by inflation index $I(t)$.

From (7.2.4)-(7.2.5) we know that the optimal portfolio process $\overline{\pi}(t) = (\pi_1(t), \pi_2(t))'$ can be represented as

$$\overline{\pi}(t) = \begin{bmatrix} \dfrac{\varphi_1(t)P_1(t)}{X(t)} \\ \dfrac{\varphi_2(t)B_{IL}(t, I(t))}{X(t)} \end{bmatrix}. \tag{7.2.38}$$

Substituting (7.2.34) into this relation (7.2.38) we directly get (7.2.31). □

Remark 1 on Theorem 7 *("Negativity" of the Optimal Portfolio Process)***.** Using the usual arguments about very unrealistic strong opinion for a high inflation, one can conclude that a normal risk-averse investor will still want to sell short inflation-linked financial products. The form of the optimal portfolio process in (7.2.31) actually coincides with the one in (6.3.13), where we were optimizing the expected utility from investor's wealth. As already stated in Remark 1 on Theorem 6, the fact that two optimal portfolio processes of different continuous-time portfolio problems coincide is not surprising, because all we do is just make a change of measure, where one have the same subordination of the portfolio processes of different financial assets, because we scale our original price processes with a non-negative inflation index process. In addition, in the same way as we did in Remark 3 on Theorem 4 one can easily show that despite the fact of a highly probable negative optimal portfolio process for inflation-linked bond $B_{IL}(t, I(t))$ it is still optimal to trade an inflation-linked bond $B_{IL}(t, I(t))$ than rather having a zero position in it and trading all other financial products optimally. It is obvious, that the sigh of the optimal portfolio process does not affect the expected utility from the optimal terminal wealth in a sense that the sigh only tells us how we should handle, sell or buy, financial products, because the optimal trading strategy is the one that maximizes the expected utility from the terminal wealth.

Remark 2 on Theorem 7 *(Nature of the Continuous-time Portfolio Problem)***.** Looking at the dynamics of purchasing power process $\hat{X}(t)$, (7.2.14)-(7.2.16), we realize that the basic portfolio problem (7.2.11) studied in this section is equivalent to the classical continuous-time portfolio problem, where we have original financial assets, $P_0(t)$, $P_1(t)$ and $I(t)$, discounted with an inflation index $I(t)$. For the new discounted financial assets $\hat{P}_0(t)$, $\hat{P}_1(t)$ and $\hat{I}(t)$ we have the following dynamics

$$d\left(\frac{i}{I(t)}P_0(t)\right) = d\hat{P}_0(t) = \hat{P}_0(t)((r_R(t) - \lambda(t)\sigma_I + \sigma_I^2)dt - \sigma_I d\tilde{W}_I(t)), \quad \hat{P}_0(0) = 1, \tag{7.2.39}$$

$$d\left(\frac{i}{I(t)}P_1(t)\right) = d\hat{P}_1(t) = \hat{P}_1(t)((b(t) - r_N(t) + r_R(t) - \lambda(t)\sigma_I + \sigma_I^2 - \sigma_2\sigma_I)dt + \sigma_1 d\tilde{W}(t) + (\sigma_2 - \sigma_I)d\tilde{W}_I(t)), \quad \hat{P}_1(0) = p_1, \tag{7.2.40}$$

$$d\left(\frac{i}{I(t)}I(t)\right) = d\hat{I}(t) = 0, \quad \hat{I}(0) = i. \tag{7.2.41}$$

We define the portfolio process $\hat{\eta}(t) = (\hat{\eta}_0(t), \hat{\eta}_1(t), \hat{\eta}_2(t))'$ as

$$\hat{\eta}_0(t) := \frac{\xi_0(t)P_0(t)/I(t)}{X(t)/I(t)} = \eta_0(t), \tag{7.2.42}$$

$$\hat{\eta}_1(t) := \frac{\xi_1(t)P_1(t)/I(t)}{X(t)/I(t)} = \eta_1(t), \tag{7.2.43}$$

$$\hat{\eta}_2(t) := 1 - \hat{\eta}_0(t) - \hat{\eta}_1(t) = \eta_2(t), \tag{7.2.44}$$

which clearly coincides with portfolio process $\eta(t) = (\eta_0(t), \eta_1(t), \eta_2(t))'$ of the basic portfolio problem (7.2.11). According to the classical well-known results the optimal portfolio processes $(\hat{\eta}_0(t), \hat{\eta}_1(t))'$ are given by

$$\begin{bmatrix} \hat{\eta}_0(t) \\ \hat{\eta}_1(t) \end{bmatrix} = \frac{1}{1-\gamma}(\hat{\sigma}\hat{\sigma}')^{-1}\begin{bmatrix} r_R(t) - \lambda(t)\sigma_I + \sigma_I^2 \\ b(t) - r_N(t) + r_R(t) - \lambda(t)\sigma_I + \sigma_I^2 - \sigma_2\sigma_I \end{bmatrix}, \tag{7.2.45}$$

where

$$\hat{\sigma} = \begin{bmatrix} 0 & -\sigma_I \\ \sigma_1 & \sigma_2 - \sigma_I \end{bmatrix}. \tag{7.2.46}$$

Having the relations (7.2.42)-(7.2.44) between portfolio processes $\hat{\eta}(t) = (\hat{\eta}_0(t), \hat{\eta}_1(t), \hat{\eta}_2(t))'$ and $\eta(t) = (\eta_0(t), \eta_1(t), \eta_2(t))'$ as well as knowing (7.2.45)-(7.2.46) we get for the optimal portfolio process of the basic portfolio problem (7.2.11)

$$\eta_0(t) = \frac{1}{1-\gamma}\left(1 - \frac{b(t)-r_N(t)}{\sigma_1^2}\left(1-\frac{\sigma_2\sigma_I}{\sigma_I^2}\right) - \frac{\lambda(t)\sigma_I - r_R(t)}{\sigma_I^2}\left(1+\frac{\sigma_2^2}{\sigma_1^2}-\frac{\sigma_2\sigma_I}{\sigma_1^2}\right)\right), \quad (7.2.47)$$

$$\eta_1(t) = \frac{1}{1-\gamma}\left(\frac{b(t)-r_N(t)}{\sigma_1^2} - \frac{\lambda(t)\sigma_I - r_R(t)}{\sigma_I^2}\cdot\frac{\sigma_2\sigma_I}{\sigma_1^2}\right), \quad (7.2.48)$$

$$\eta_2(t) = \frac{1}{1-\gamma}\left(\frac{\lambda(t)\sigma_I - r_R(t)}{\sigma_I^2}\left(1+\frac{\sigma_2^2}{\sigma_1^2}\right) - \frac{b(t)-r_N(t)}{\sigma_1^2}\cdot\frac{\sigma_2\sigma_I}{\sigma_I^2}\right), \quad (7.2.49)$$

which is exactly the same as in (7.2.30).

8 Hedging with Inflation-linked Financial Products

Taking a look at the problems studied so far, one can easily conclude that none of the problem setups delivers us an optimal portfolio solution for an inflation-linked bond $B_{IL}(t, I(t))$ that is positive for sure, unless an investor is pretty sure about relatively huge growth of an inflation index $I(t)$ in the future that is expressed in a high subjective excess return $\lambda(t)\sigma_I$. This means that almost all continuous-time portfolio problems studied so far describe only one half of the market participants, i.e. in most situations sellers of inflation-linked products. The natural buyers of inflation-linked financial products could be companies that are interested in hedging against growth of prices of materials. One typical example is a car-insurance company that obviously has to cover repairing costs of a crashed car some day in the future. To hedge against future inflation of material prices this type of insurance companies would probably find inflation-linked bonds very attractive for hedging purposes. In this section we will look at a problem of hedging an inflation-linked payment obligation in the future. First we will study a general market setup, where we have a full access to the financial instruments traded at the market; these are conventional bond, stock and inflation-linked bond. To show that it is worth handling also inflation-linked products in order to hedge an inflation-linked payment obligation, we will restrict our market to two financial assets, a conventional bond and a stock, that are available to an investor for handling.

8.1 Problem 7: Conventional Bond, Stock and Inflation-linked Bond (Hedging)

In this section we consider a financial market consisting of a risk-free conventional bond $P_0(t)$, a stock $P_1(t)$ and an inflation index $I(t)$ itself as if it would be tradable. From Remark 3 on Theorem 2 we know that assuming an inflation index $I(t)$ to be tradable is equivalent to the assumption that an inflation-linked bond $B_{IL}(t, I(t))$ without deflation protection to be traded on the market. The dynamics of these financial assets under the physical probability measure P is given by the following slightly different from the previous sections stochastic equations

$$dP_0(t) = P_0(t) r_N(t) dt, \quad P_0(0) = 1, \tag{8.1.1}$$

$$dP_1(t) = P_1(t)(b(t)dt + \sigma_{p1} d\tilde{W}_1(t)), \quad P_1(0) = p_1, \tag{8.1.2}$$

$$dI(t) = I(t)((r_N(t) - r_R(t) + \lambda(t)\sigma_I)dt + \sigma_{I1} d\tilde{W}_1(t) + \sigma_{I2} d\tilde{W}_2(t)), \quad I(0) = i, \tag{8.1.3}$$

where

$$\sigma_I = \sqrt{\sigma_{I1}^2 + \sigma_{I2}^2} \tag{8.1.4}$$

and $(\tilde{W}_1(t), \tilde{W}_2(t))'$ is an independent two-dimensional standard Brownian motion under the physical probability measure P. Additionally we define the volatility of our model σ as

$$\sigma := \begin{bmatrix} \sigma_{p1} & 0 \\ \sigma_{I1} & \sigma_{I2} \end{bmatrix}. \tag{8.1.5}$$

Consequently, investor's wealth process $X(t)$ can expressed as

$$X(t) = \xi_0(t) P_0(t) + \xi_1(t) P_1(t) + \xi_2(t) I(t), \quad X(0) = x, \tag{8.1.6}$$

where investor's initial capital is strictly positive, i.e. $x > 0$, and trading strategy $\xi(t) = (\xi_0(t), \xi_1(t), \xi_2(t))'$ is $\mathbb{R}^3$-valued progressively measurable processes with respect to the filtration $\{\mathcal{F}_t\}_{t \in [0,T]}$ generated by the independent standard Brownian motion $\{(\tilde{W}_1(t), \tilde{W}_2(t))'\}_{t \in [0,T]}$ satisfying similar technical conditions as (6.1.7)-(6.1.8). Naturally the corresponding $\mathbb{R}^3$-valued portfolio process $\eta(t) = (\eta_0(t), \eta_1(t), \eta_2(t))'$ is defined as

$$\eta_1(t) := \frac{\xi_1(t)P_1(t)}{X(t)}, \tag{8.1.7}$$

$$\eta_2(t) := \frac{\xi_2(t)I(t)}{X(t)}, \tag{8.1.8}$$

$$\eta_0(t) := 1 - \overline{\eta}(t)'\underline{1} = \frac{\xi_0(t)P_0(t)}{X(t)}, \tag{8.1.9}$$

where $\underline{1} := (1,\ldots,1)' \in \mathbb{R}^d$ and $\overline{\pi}(t)$ corresponds to the fractions of the wealth $X(t)$ invested into the risky assets, i.e.

$$\overline{\eta}(t) := (\eta_1(t), \eta_2(t))'. \tag{8.1.10}$$

As usual, we define the set of admissible portfolio processes $\mathcal{A}(x)$ as

$$\mathcal{A}(x) := \left\{ (1 - \overline{\eta}(.)'\underline{1}, \overline{\eta}(.)')' \middle| X(0) = x > 0,\ \int_0^T \eta_i^2(t)dt < \infty \text{ a.s. } P,\ i = 1,2 \right\}, \tag{8.1.11}$$

which insures the non-negativity of the corresponding wealth process $X(t)$ asymptotically during the whole time $t \in [0,T]$, (6.1.12).

Having the wealth process $X(t)$ given by (8.1.6), in this section we look at the hedging problem, where investor wants to hedge a future payment obligation that is inflation index $I(t)$ related, i.e. we have the following hedging problem

$$\min_{\eta(.)\in\mathcal{A}(x)} E\left[\left(B\frac{I(T)}{i} - X(T) \right)^2 \right], \tag{8.1.12}$$

where B is independent of the Brownian motion $(\tilde{W}_1(t), \tilde{W}_2(t))'$ and represents the value of the insurance premium that has to be paid out if the insurance case would happen today. Further, payment obligation B can be interpreted as the total outcome of the risk process of the insurance company until the time horizon T. Remembering that payment obligation B is independent of the Brownian motion $(\tilde{W}_1(t), \tilde{W}_2(t))'$ the hedging problem (8.1.12) we can rewrite as

$$E\left[\left(B\frac{I(T)}{i}\right)^2\right] + \min_{\eta(.)\in\mathcal{A}(x)}\left(E[X^2(T)] - 2E[B]E\left[\frac{I(T)}{i}X(T)\right]\right), \tag{8.1.13}$$

which means that the original hedging problem (8.1.12) is equivalent to the standard utility maximization problem

$$\max_{\eta(.)\in\mathcal{A}(x)} E[U(X(T))] \tag{8.1.14}$$

with utility function U given by

$$U(x) = 2E[B]\frac{I(T)}{i}x - x^2\,. \tag{8.1.15}$$

It is obvious that this utility function (8.1.15) is strictly concave but not strictly increasing. Despite this fact we still have an invertible derivative of the utility function and according to Korn (1997) [15] can apply the usual procedures of the martingale method of portfolio optimization (see for example Korn (1997) [14]) but cannot ensure a non-negative final wealth process $X(T)$.

We will mainly consider the situation, when an investor does not have enough initial capital x to make a perfect hedge, because the situation of a perfect hedge is trivial. Having enough initial capital x the investor will minimize the hedging error in (8.1.12) by completely replicating the inflation index $I(t)$ related payment obligation B by buying $E[B]/i$ units of inflation index $I(t)$ so that his final wealth $\widehat{X}(T)$ will satisfy

$$\widehat{X}(T) = E[B]\frac{I(T)}{i}. \tag{8.1.16}$$

For this perfect hedge investor needs an initial capital of $\widehat{x}$ equal to

$$\widehat{x} = E\left[H(T)E[B]\frac{I(T)}{i}\right] = E[B], \tag{8.1.17}$$

where process $H(T)$ is given by (6.3.25)-(6.3.26), i.e.

$$H(T) := \exp\left(-\int_0^T (r_N(s) + \frac{1}{2}\theta(s)'\theta(s))ds - \int_0^T \theta(s)'d(\tilde{W}_1(s),\tilde{W}_2(s))'\right), \tag{8.1.18}$$

$$\theta(s) := \sigma^{-1}(b(s) - r_N(s), \lambda(s)\sigma_I - r_R(s))'. \tag{8.1.19}$$

By inserting the terminal wealth $\widehat{X}(T)$ of (8.1.16) instead of $X(T)$ into (8.1.12) we get the minimal quadratic hedging error for the case of a perfect hedge that equal to

$$Var[B]\frac{E[I^2(T)]}{i^2}. \tag{8.1.20}$$

In a more interesting case, when one has an initial capital $x < E[B]$ that is not enough for a perfect hedge, one can apply the techniques of a martingale approach that is described for example in Korn (1997) [14]. For this reason we define $I(y)$ as the inverse function of the first derivative of the random utility function $U(.)$ in (8.1.15)

$$I(y) := (U')^{-1}(y) = E[B]\frac{I(T)}{i} - \frac{1}{2}y\,. \tag{8.1.21}$$

The process $X(y)$ is defined as

$$X(y) := E[H(T)I(yH(T))] = E[B] - \frac{1}{2}yE[H^2(T)]\,. \tag{8.1.22}$$

Now, let $Y(.)$ be an inverse function of $X(.)$. From (8.1.22) one can obtain $Y(x)$ as the unique solution of the equation

$$X(Y(x)) = x\,. \tag{8.1.23}$$

This unique solution is given by

$$Y(x) = 2\frac{E[B] - x}{E[H^2(T)]}. \tag{8.1.24}$$

Theorem 8. In the case of investor's initial capital being not enough for a perfect hedge in a market consisting of conventional bond $P_0(t)$, stock $P_1(t)$ and inflation index $I(t)$, i.e.

$$x < E[B], \tag{8.1.25}$$

the optimal terminal wealth B^* for the hedging problem (8.1.12) is given by

$$B^* = E[B]\frac{I(T)}{i} - \frac{E[B]-x}{E[H^2(T)]}H(T), \tag{8.1.26}$$

which implies the minimal quadratic hedging error of

$$Var[B]\frac{E[I^2(T)]}{i^2} + \frac{(E[B]-x)^2}{E[H^2(T)]}; \tag{8.1.27}$$

$H(T)$ and $\theta(s)$ are given in (8.1.18)-(8.1.19).

Proof of Theorem 8. According to Korn (1997) [15] the optimal terminal wealth B^* for the hedging problem (8.1.12) is given by

$$B^* = I(Y(x)H(T)). \tag{8.1.28}$$

Inserting the explicit forms of $I(y)$ and $Y(x)$, (8.1.21) and (8.1.24) respectively, into (8.1.28) we get for the optimal terminal wealth B^*

$$B^* = E[B]\frac{I(T)}{i} - \frac{E[B]-x}{E[H^2(T)]}H(T) \tag{8.1.29}$$

that coincides with (8.1.26). Substituting the this optimal terminal wealth B^* from (8.1.29) instead of $X(T)$ in (8.1.12) we directly get the minimal quadratic hedging error exactly equal to (8.1.27). □

Remark 1 on Theorem 8 *(Nature of the Minimal Quadratic Hedging Error)***.** The hedging error in (8.1.27) consists of two parts. The first part

$$Var[B]\frac{E[I^2(T)]}{i^2}$$

represents the uncertainty about the amount of the future payment B. This term can only vanish, when we have a deterministic B, that means that we know for sure the size of the future recovery premium happening at time T as if the insurance case would happen today, i.e. we can predict B; of course the development of the inflation index $I(t)$ in the future is not known. The second part

$$\frac{(E[B]-x)^2}{E[H^2(T)]}$$

corresponds to the quadratic hedging error caused by the lack of enough initial capital for a perfect hedge, $x < E[B]$. Also this second term can vanish; this occurs, when our initial wealth of the hedging portfolio $x = E[B]$. When investor's initial capital $x > E[B]$, he should buy $E[B]/i$ units of inflation index $I(t)$ spending $E[B]$ amount of money; the rest of his capital $x - E[B]$ he should use for the other purposes.

8.2 Problem 8: Conventional Bond and Stock (Hedging)

In general, it is clear that having an opportunity to hedge an inflation index $I(t)$ related future payment obligation B additionally with inflation index $I(t)$ itself is better compared to the situation, when we are using all other financial assets except inflation index $I(t)$. As already stated, we assume hypothetically an inflation index $I(t)$ to be traded on the market directly, which is equivalent to the assumption that an inflation-linked bond $B_{IL}(t, I(t))$ without deflation protection is traded on the market. To demonstrate this positive effect of inflation index $I(t)$ on the hedging error in problem (8.1.12) we will consider a restricted financial market, where only a conventional bond $P_0(t)$, (8.1.1), and a stock $P_1(t)$, (8.1.2), are traded. It is obvious that in this market we are not able to hedge the randomness coming from inflation index $I(t)$ completely or exactly from the Brownian motion $\tilde{W}_2(t)$. For that reason we rewrite the solution of the stochastic differential equation (8.1.3) for $I(t)$ as

$$I(t) = \breve{I}(t)\exp(-\frac{1}{2}\sigma_{I2}^2 t + \sigma_{I2}\tilde{W}_2(t)), \tag{8.2.1}$$

where $\breve{I}(t)$ is defined as

$$\breve{I}(t) := i\exp\left(\int_0^t (r_N(s) - r_R(s) + \lambda(s)\sigma_I)ds - \frac{1}{2}\sigma_{I1}^2 t + \sigma_{I1}\tilde{W}_1(t)\right). \tag{8.2.2}$$

Consequently, the expected value of $I(t)$ is equal to the one of $\breve{I}(t)$ due to

$$\begin{aligned} E[I(t)] &= E[\breve{I}(t)\exp(-\frac{1}{2}\sigma_{I2}^2 t + \sigma_{I2}\tilde{W}_2(t))] \\ &= E[\breve{I}(t)]E[\exp(-\frac{1}{2}\sigma_{I2}^2 t + \sigma_{I2}\tilde{W}_2(t))] = E[\breve{I}(t)]\cdot 1 = E[\breve{I}(t)], \end{aligned} \tag{8.2.3}$$

where we used the independence of the Brownian motions $\tilde{W}_1(t)$ and $\tilde{W}_2(t)$. Now, having this relation (8.2.3) and the independence of the wealth process $X(t)$, which in our new restricted market is made up from the conventional bond $P_0(t)$ and the stock $P_1(t)$, from the Brownian motion $\tilde{W}_2(t)$, we can simplify the hedging problem (8.1.12) in a similar way as we did in (8.1.13) to

$$E\left[\left(B\frac{I(T)}{i}\right)^2\right] + \min_{\eta(.)\in\mathcal{A}(x)}\left(E[X^2(T)] - 2E[B]E\left[\frac{\breve{I}(T)}{i}X(T)\right]\right), \tag{8.2.4}$$

which means that the original hedging problem (8.1.12) is equivalent to the standard utility maximization problem in a new complete market consisting from the conventional bond $P_0(t)$ and the stock $P_1(t)$

$$\max_{\eta(.)\in\mathcal{A}(x)} E[U(X(T))] \tag{8.2.5}$$

with utility function U given by

$$U(x) = 2E[B]\frac{\breve{I}(T)}{i}x - x^2. \tag{8.2.6}$$

Investor's wealth process $X(t)$ is given by

$$X(t) = \xi_0(t)P_0(t) + \xi_1(t)P_1(t), \quad X(0) = x, \tag{8.2.7}$$

where investor's initial capital is strictly positive, i.e. $x > 0$, and the corresponding portfolio process $\eta(t) = (\eta_0(t), \eta_1(t))'$ is defined as

$$\eta_1(t) := \frac{\xi_1(t)P_1(t)}{X(t)}, \tag{8.2.8}$$

$$\eta_0(t) := 1 - \eta_1(t) = \frac{\xi_0(t)P_0(t)}{X(t)}. \tag{8.2.9}$$

The set of admissible portfolio processes $\mathcal{A}(x)$ is defined in a usual way as

$$\mathcal{A}(x) := \left\{ (1-\eta_1(.),\eta_1(.))' \middle| X(0) = x > 0,\ \int_0^T \eta_1^2(t)dt < \infty \text{ a.s. } P \right\}, \tag{8.2.10}$$

which insures the non-negativity of the corresponding wealth process $X(t)$ asymptotically during the whole time $t \in [0,T]$, (6.1.12).

Again, having enough initial capital x to make a perfect hedge the investor will maximize the expected utility (8.2.6) by completely replicating the $\breve{I}(t)$ part of the inflation index $I(t)$ by buying $E[B]/i$ units of $\breve{I}(t)$ so that his final wealth $\widehat{X}(T)$ will satisfy

$$\widehat{X}(T) = E[B]\frac{\breve{I}(T)}{i}. \tag{8.2.11}$$

For this perfect hedge in our new market investor needs an initial capital of $\widehat{x}$ equal to

$$\widehat{x} = E\left[H(T)E[B]\frac{\breve{I}(T)}{i}\right] = E[B]\exp\left(\int_0^T (\lambda(s)\sigma_I - r_R(s) - \sigma_{I1}\theta(s))ds\right), \tag{8.2.12}$$

where process $H(T)$ is defined in an analogical way to (8.1.18)-(8.1.19) that is

$$H(T) := \exp\left(-\int_0^T (r_N(s) + \frac{1}{2}\theta^2(s))ds - \int_0^T \theta(s)d\tilde{W}_1(s)\right), \tag{8.2.13}$$

$$\theta(s) := \frac{b(s) - r_N(s)}{\sigma_{p1}}. \tag{8.2.14}$$

By inserting the terminal wealth $\widehat{X}(T)$ of (8.2.11) instead of $X(T)$ into (8.1.12) we get the minimal quadratic hedging error for the case of a perfect hedge in our new market that equal to

$$\begin{aligned} E\left[\left(B\frac{I(T)}{i} - E[B]\frac{\breve{I}(T)}{i}\right)^2\right] &= E\left[\left((B - E[B])\frac{I(T)}{i} + E[B]\left(\frac{I(T)}{i} - \frac{\breve{I}(T)}{i}\right)\right)^2\right] \\ &= E\left[\left((B - E[B])\frac{I(T)}{i}\right)^2\right] + \\ &\quad E\left[2(B - E[B])\frac{I(T)}{i}E[B]\left(\frac{I(T)}{i} - \frac{\breve{I}(T)}{i}\right)\right] + \\ &\quad E\left[\left(E[B]\left(\frac{I(T)}{i} - \frac{\breve{I}(T)}{i}\right)\right)^2\right] \end{aligned}$$

$$= Var[B]\frac{E[I^2(T)]}{i^2} + (E[B])^2\,\frac{E[(I(T)-\breve{I}(T))^2]}{i^2}. \tag{8.2.15}$$

In a more interesting case, when one has an initial capital $x < \widehat{x}$ that is not enough for a perfect hedge, one can apply the same techniques of a martingale approach that is described for example in Korn (1997) [14]. For this reason we define $I(y)$ as the inverse function of the first derivative of the random utility function $U(.)$ in (8.2.6)

$$I(y) := (U')^{-1}(y) = E[B]\frac{\breve{I}(T)}{i} - \frac{1}{2}y\,. \tag{8.2.16}$$

The process $X(y)$ is defined as

$$X(y) := E[H(T)I(yH(T))] = E[B]\frac{E[H(T)\breve{I}(T)]}{i} - \frac{1}{2}yE[H^2(T)]\,. \tag{8.2.17}$$

Now, let $Y(.)$ be an inverse function of $X(.)$. From (8.2.17) one can obtain $Y(x)$ as the unique solution of the equation

$$X(Y(x)) = x\,. \tag{8.2.18}$$

This unique solution is given by

$$Y(x) = 2\frac{E[B]\dfrac{E[H(T)\breve{I}(T)]}{i} - x}{E[H^2(T)]}\,. \tag{8.2.19}$$

Theorem 9. In the case of investor's initial capital being not enough for a perfect hedge in a restricted market consisting of conventional bond $P_0(t)$ and stock $P_1(t)$, i.e.

$$x < E[B]\exp\Big(\int_0^T (\lambda(s)\sigma_I - r_R(s) - \sigma_{I1}\theta(s))ds\Big), \tag{8.2.20}$$

the optimal terminal wealth B^* for the hedging problem (8.1.12) is given by

$$B^* = E[B]\frac{\breve{I}(T)}{i} - \frac{E[B]\dfrac{E[H(T)\breve{I}(T)]}{i} - x}{E[H^2(T)]}H(T), \tag{8.2.21}$$

which implies the minimal quadratic hedging error of

$$Var[B]\frac{E[I^2(T)]}{i^2} + (E[B])^2\,\frac{E[(I(T)-\breve{I}(T))^2]}{i^2} + \frac{\left(E[B]\frac{E[H(T)\breve{I}(T)]}{i} - x\right)^2}{E[H^2(T)]}; \quad (8.2.22)$$

$H(T)$ and $\theta(s)$ are given in (8.2.13)-(8.2.14).

Proof of Theorem 9. According to Korn (1997) [15] the optimal terminal wealth B^* for the hedging problem (8.1.12) is given by

$$B^* = I(Y(x)H(T)). \quad (8.2.23)$$

Inserting the explicit forms of $I(y)$ and $Y(x)$, (8.2.16) and (8.2.19) respectively, into (8.2.23) we get for the optimal terminal wealth B^*

$$B^* = E[B]\frac{\breve{I}(T)}{i} - \frac{E[B]\frac{E[H(T)\breve{I}(T)]}{i} - x}{E[H^2(T)]}H(T) \quad (8.2.24)$$

that coincides with (8.2.21). Substituting the this optimal terminal wealth B^* from (8.2.24) instead of $X(T)$ in (8.1.12) we easily get the minimal quadratic hedging error exactly equal to (8.2.22). □

Remark 1 on Theorem 9 *(Positive Effect of Inflation-linked Bond on the Hedging Error)***.** Comparing the two minimal quadratic hedging errors in (8.1.27) and (8.2.22) one can straightforwardly see the positive effect of using an opportunity to including inflation-linked products into portfolio in order to hedge an inflation index $I(t)$ related future payment obligation B. This effect is natural due to the nature of the calculated quadratic hedging errors. In the previous section we projected an inflation-linked payment $BI(T)/i$ onto the market, where perfect replication of inflation index $I(t)$ is possible, whereas in this section we made the succeeding projection of the results from the previous section onto the market, where only partial replication of inflation index $I(t)$ is achievable. It is apparent that the minimal hedging error in the second case, i.e. without inflation-linked products in the hedging portfolio, is bigger than the one in the first case.

9 Conclusions

This thesis has dealt with diverse problem concerning inflation-linked products. To start with, two models for inflation were first presented, including a geometric Brownian motion for consumer price index itself and an extended Vasicek model for inflation rate. For both suggested models the pricing formulas of inflation-linked products were derived using the risk-neutral valuation techniques. As a result closed form solutions for a call option on inflation index for a Brownian motion model and inflation evolution for an extended Vasicek model as well as for an inflation-linked bond were calculated. Due to the form of the pricing formulas they belong to the Black and Scholes type. In addition to these inflation-linked products, for the both inflation models the pricing formulas of a European put option on inflation, an inflation cap and floor, an inflation swap and an inflation swaption were derived.

Consequently, basing on the derived pricing formulas and assuming the geometric Brownian motion process for an inflation index, different continuous-time portfolio problems as well as hedging problems were studied using the martingale techniques as well as stochastic optimal control methods. These utility optimization problems are continuous-time portfolio problems in different financial market setups and in addition with a positive lower bound constraint on the final wealth of the investor. When one summarizes all the optimization problems studied in this work, one will have the complete picture of the inflation-linked market and both counterparts of market-participants, sellers as well as buyers of inflation-linked financial products. One of the interesting results worth mentioning here is naturally the fact that a regular risk-averse investor would like to act as a seller of inflation-linked products due to the high price of inflation-linked bonds for example and an underperformance of inflation-linked bonds compared to the conventional risk-free bonds. The relevance of this observation was proved by investigating a simple optimization problem for the extended Vasicek process, where as a result we still have an underperforming inflation-linked bond compared to the conventional bond.

This situation does not change, when one switches to an optimization of expected utility from the purchasing power, because in its nature it is only a change of measure, where we consider a different deflator. The negativity of the optimal portfolio process for a normal investor is in itself an interesting aspect, but it does not affect the optimality of handling inflation-linked products compared to the situation not including these products into investment portfolio.

In the following, hedging problems were considered as a modeling of the other half of inflation market that is inflation-linked products buyers. Natural buyers of these inflation-linked products are obviously institutions that have payment obligations in the future that are inflation connected. That is why we considered problems of hedging inflation-indexed payment obligations with different financial assets. The role of inflation-linked products in the hedging portfolio was shown to be very important by analyzing two alternative optimal hedging strategies, where in the first one an investor is allowed to trade as inflation-linked bond and in the second one he is not allowed to include an inflation-linked bond into his hedging portfolio. Technically this was done by restricting our original financial market, which was made of a conventional bond, inflation index and a stock correlated with inflation index, to the one, where an inflation index was excluded.

As a whole, this thesis presents a wide view on inflation-linked products: inflation modeling, pricing aspects of inflation-linked products, various continuous-time portfolio problems with inflation-linked products as well as hedging of inflation-related payment obligations.

APPENDIX 1: DERIVATION OF EQUATIONS (5.1.4)-(5.1.5)

Taking into consideration the fact that nominal interest rate $r_N(t)$ is a deterministic function the option pricing formula (5.1.3) will take the form

$$C_I(t, I(t)) = \exp\Big(-\int_t^T r_N(s)ds\Big) E[\max[I(T) - K, 0] | \mathcal{F}_t]. \tag{A.1.1}$$

Assuming the geometric Brownian motion model (4.1.1) for consumer price index $I(t)$ one can write the value of CPI $I(T)$ at the maturity date T as

$$I(T) = I(t)\exp(\tilde{\mu}_I(t,T) + \tilde{\sigma}_I(t,T)Z), \tag{A.1.2}$$

where

$$\tilde{\mu}_I(t,T) = \int_t^T (r_N(s) - r_R(s))ds - \frac{1}{2}\sigma_I^2(T-t), \tag{A.1.3}$$

$$\tilde{\sigma}_I(t,T) = \sigma_I\sqrt{T-t} \tag{A.1.4}$$

and $Z \sim \mathcal{N}(0,1)$ is a standard normal variable. Then the expectation in (A.1.1) can be written as

$$\int_{-\infty}^{\infty} \max[I(t)\exp(\tilde{\mu}_I(t,T) + \tilde{\sigma}_I(t,T)z) - K, 0] f(z)dz, \tag{A.1.5}$$

where f is the density function of the $\mathcal{N}(0,1)$-distribution, i.e.

$$f(z) = \frac{1}{\sqrt{2\pi}}\exp\Big(-\frac{z^2}{2}\Big). \tag{A.1.6}$$

The integrand in the integral above (A.1.5) vanishes when

$$I(t)\exp(\tilde{\mu}_I(t,T) + \tilde{\sigma}_I(t,T)z) < K, \tag{A.1.7}$$

i.e. when $z < z_0$, where

$$z_0 = \frac{\ln\left(\dfrac{K}{I(t)}\right) - \tilde{\mu}_I(t,T)}{\tilde{\sigma}_I(t,T)}. \tag{A.1.8}$$

The integral (A.1.5) can thus be written as

$$\int_{z_0}^{\infty} I(t)\exp(\tilde{\mu}_I(t,T) + \tilde{\sigma}_I(t,T)z)f(z)dz - \int_{z_0}^{\infty} Kf(z)dz = A - B\,. \tag{A.1.9}$$

The integral B can obviously be written as

$$B = K \cdot \text{Prob}(Z \geq z_0) = K \cdot N(-z_0), \tag{A.1.10}$$

where N is a cumulative distribution function of the $\mathcal{N}(0,1)$-distribution, i.e.

$$N(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} \exp\left(-\frac{z^2}{2}\right)dz\,. \tag{A.1.11}$$

In the integral A we complete the square in the exponent to obtain

$$A = \frac{I(t)\exp\left(\tilde{\mu}_I(t,T) + \frac{1}{2}\tilde{\sigma}_I^2(t,T)\right)}{\sqrt{2\pi}}\int_{z_0}^{\infty} \exp\left(-\frac{(z - \tilde{\sigma}_I(t,T))^2}{2}\right)dz\,. \tag{A.1.12}$$

Here we recognize the density of a $\mathcal{N}(\tilde{\sigma}_I(t,T),1)$-distribution, so after normalizing to a standard normal variable we have

$$A = I(t)\exp\left(\tilde{\mu}_I(t,T) + \frac{1}{2}\tilde{\sigma}_I^2(t,T)\right)N(-z_0 + \tilde{\sigma}_I(t,T))\,. \tag{A.1.13}$$

Summing up all results we have for the option price

$$\begin{aligned} C_I(t,I(t)) = \exp\left(-\int_t^T r_N(s)ds\right)(A - B) = I(t)\exp\left(-\int_t^T r_R(s)ds\right)N(d(t)) - \\ K\exp\left(-\int_t^T r_N(s)ds\right)N(d(t) - \sigma_I\sqrt{T-t}), \end{aligned} \tag{A.1.14}$$

where

$$d(t) = -z_0 + \tilde{\sigma}_I(t,T) = \frac{\ln\left(\dfrac{I(t)}{K}\right) + \int_t^T (r_N(s) - r_R(s))ds + \dfrac{1}{2}\sigma_I^2(T-t)}{\sigma_I\sqrt{T-t}}, \quad \text{(A.1.15)}$$

which actually exactly corresponds to the equations (5.1.4) and (5.1.5). □

APPENDIX 2: DERIVATION OF EQUATIONS (5.2.3)-(5.2.6)

Taking into consideration the fact that nominal interest rate $r_N(t)$ is a deterministic function the option pricing formula (5.2.2) will take the form

$$C_I(t,i(t)) = \exp\Big(-\int_t^T r_N(s)ds\Big)E\Big[\max\Big[\exp\Big(\int_t^T i(s)ds\Big) - K, 0\Big]\Big|\mathcal{F}_t\Big]. \qquad \text{(A.2.1)}$$

Assuming the extended Vasicek model (4.2.1) for inflation rate $i(t)$ we can write the integral over inflation rate $\int_t^T i(s)ds$ as

$$\begin{aligned}\int_t^T i(s)ds &= i(t)\frac{1-e^{-\alpha(T-t)}}{\alpha} + \int_t^T \frac{1-e^{-\alpha(T-s)}}{\alpha}\theta(s)ds + \\ &\frac{\sigma_i}{\alpha}\int_t^T (1-e^{-\alpha(T-s)})dW_i(s).\end{aligned} \qquad \text{(A.2.2)}$$

Here we realize that the integral over inflation rate $\int_t^T i(s)ds$ is normally distributed with the following mean and variance

$$\tilde{\mu}_i(t,T) := E\Big[\int_t^T i(s)ds\Big] = i(t)\frac{1-e^{-\alpha(T-t)}}{\alpha} + \int_t^T \frac{1-e^{-\alpha(T-s)}}{\alpha}\theta(s)ds, \qquad \text{(A.2.3)}$$

$$\begin{aligned}\tilde{\sigma}_i^2(t,T) &:= Var\Big[\int_t^T i(s)ds\Big] = E\Big[\Big(\frac{\sigma_i}{\alpha}\int_t^T (1-e^{-\alpha(T-s)})dW_i(s)\Big)^2\Big] \\ &= \frac{\sigma_i^2}{\alpha^2}\int_t^T (1-e^{-\alpha(T-s)})^2 ds \\ &= \frac{\sigma_i^2}{\alpha^2}\Big(\frac{(1-e^{-2\alpha(T-t)})}{2\alpha} - \frac{2(1-e^{-\alpha(T-t)})}{\alpha} + (T-t)\Big) \\ &= \frac{\sigma_i^2}{\alpha^2}\Big(\frac{1-(e^{-\alpha(T-t)}-2)^2}{2\alpha} + (T-t)\Big).\end{aligned} \qquad \text{(A.2.4)}$$

Thus, one can write the value of inflation evolution $\exp\Big(\int_t^T i(s)ds\Big)$ as

$$\exp\Big(\int_t^T i(s)ds\Big) = \exp(\tilde{\mu}_i(t,T) + \tilde{\sigma}_i(t,T)Z), \qquad \text{(A.2.5)}$$

where $Z \sim \mathcal{N}(0,1)$ is a standard normal variable. Then the expectation in (A.2.1) can be written as

$$\int_{-\infty}^{\infty} \max[\exp(\tilde{\mu}_i(t,T) + \tilde{\sigma}_i(t,T)z) - K, 0] f(z) dz\,, \tag{A.2.6}$$

where f is the density function of the $\mathcal{N}(0,1)$-distribution, i.e.

$$f(z) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right). \tag{A.2.7}$$

The integrand in the integral above (A.2.6) vanishes when

$$\exp(\tilde{\mu}_i(t,T) + \tilde{\sigma}_i(t,T)z) < K\,, \tag{A.2.8}$$

i.e. when $z < z_0$, where

$$z_0 = \frac{\ln(K) - \tilde{\mu}_i(t,T)}{\tilde{\sigma}_i(t,T)}. \tag{A.2.9}$$

The integral (A.2.6) can thus be written as

$$\int_{z_0}^{\infty} \exp(\tilde{\mu}_i(t,T) + \tilde{\sigma}_i(t,T)z) f(z) dz - \int_{z_0}^{\infty} K f(z) dz = A - B\,. \tag{A.2.10}$$

The integral B can obviously be written as

$$B = K \cdot \text{Prob}(Z \geq z_0) = K \cdot N(-z_0), \tag{A.2.11}$$

where N is a cumulative distribution function of the $\mathcal{N}(0,1)$-distribution, i.e.

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp\left(-\frac{z^2}{2}\right) dz\,. \tag{A.2.12}$$

In the integral A we complete the square in the exponent to obtain

$$A = \frac{\exp\left(\tilde{\mu}_i(t,T) + \frac{1}{2}\tilde{\sigma}_i^2(t,T)\right)}{\sqrt{2\pi}} \int_{z_0}^{\infty} \exp\left(-\frac{(z - \tilde{\sigma}_i(t,T))^2}{2}\right) dz\,. \tag{A.2.13}$$

Here we recognize the density of a $\mathcal{N}(\tilde{\sigma}_i(t,T),1)$-distribution, so after normalizing to a standard normal variable we have

$$A = \exp\Big(\tilde{\mu}_i(t,T) + \frac{1}{2}\tilde{\sigma}_i^2(t,T)\Big)N(-z_0 + \tilde{\sigma}_i(t,T)). \tag{A.2.14}$$

Summing up all results we have for the option price

$$\begin{aligned} C_I(t,i(t)) &= \exp\Big(-\int_t^T r_N(s)ds\Big)(A-B) \\ &= \exp\Big(-\int_t^T r_N(s)ds + \tilde{\mu}_i(t,T) + \frac{1}{2}\tilde{\sigma}_i^2(t,T)\Big)N(d(t)) - \\ &\quad K\exp\Big(-\int_t^T r_N(s)ds\Big)N(d(t) - \tilde{\sigma}_i(t,T)), \end{aligned} \tag{A.2.15}$$

where

$$d(t) = -z_0 + \tilde{\sigma}_i(t,T) = \frac{\ln\Big(\frac{1}{K}\Big) + \tilde{\mu}_i(t,T) + \tilde{\sigma}_i^2(t,T)}{\tilde{\sigma}_i(t,T)}, \tag{A.2.16}$$

which actually exactly corresponds to the equations (5.2.3)-(5.2.6). □

BIBLIOGRAPHY

[1] Agency France Trésor (2002) *Investor's Guide (2032 OAT€i).*

[2] Ang, A. and Bekaert, G. (2003) The Term Structure of Real Interest Rates and Expected Inflation. Columbia Business School, *working paper.*

[3] Atta-Mensah, J. and Yuan, M. (1998) Extraction of Expected Inflation from Canadian Forward Rates. *Proceedings of a conference held by the Bank of Canada*, May 1998.

[4] Ben-Hamou, E. (2003) *Inflation (CPI) Futures.*

[5] Björk, T. (1998) *Arbitrage Theory in Continuous Time*. Oxford University Press.

[6] Broll, U., Schweimayer, G. and Welzel, P. (2003) Managing Credit Risk with Credit and Macro Derivatives. *Discussion Paper Series* **252**, Universität Augsburg, Institute for Economics.

[7] Brown, R.H. and Schäfer, S.M. (1994) The Term Structure of Real Interest Rates and the Cox, Ingersoll, and Ross Model. *Journal of Financial Economics* **35**, p. 3-42.

[8] Cox, J.C., Ingersoll, J.E. and Ross, S.A. (1985) A Theory of the Term Structure of Interest Rates. *Econometrica* **53**, No. 2, p. 385-407.

[9] Deacon, M., Derry, A. and Mirfendereski, D. (2004) *Inflation-indexed Securities*. John Wiley & Sons, Ltd.

[10] Elhedery, G. (2003) *Inflation Toolbox: New Products and Developments*. Goldman Sachs International.

[11] Fisher, I. (1930) *The Theory of Interest*. MacMillan Press Ltd., London, Basingstoke.

[12] Gerlach, S. and Schnabel, G. (1999) The Taylor Rule and Interest Rates in the EMU Area: A Note. *Centre for Economic Policy Research Discussion Papers* **2271**.

[13] Jarrow, R. and Yildirim, Y. (2003) Pricing Treasury Inflation Protected Securities and Related Derivatives using an HJM Model. *Journal of Financial and Quantitative Analysis* **38**, No. 2.

[14] Korn, R. (1997) *Optimal Portfolios*. World Scientific, Singapore.

[15] Korn, R. (1997) Some Applications of L^2-Hedging with a Non-negative Wealth Process. *Applied Mathematical Finance* **4**, No. 1, p. 64-79.

[16] Korn, R. (2005) Optimal Portfolios with a Positive Lower Bound on Final Wealth. *Quantitative Finance.*

[17] Korn, R. and Kruse, S. (2004) Einfache Verfahren zur Bewertung von inflationsgekoppelten Finanzprodukten. *Blätter der DGVFM* **26**, No. 3, p. 351-367.

[18] Korn, R. and Trautmann, S. (1999) Optimal Control of Option Portfolios and Applications. *OR Spektrum – Quantitative Approaches in Management* **21**, No. 1-2, p. 123-146.

[19] Nielsen, H. (2003) Inflation Expectations in the EU – Results from Survey Data. Humboldt-Univesität zu Berlin, Center for Applied Statistics and Economics, *working paper*.

[20] Schweimayer, G. and Wagatha, M. (2000) Risikomanagement mit Makroderivaten am internen Markt eines Bankenverbundes. *Verbundtagung Wirtschaftsinformatik 2000*, Shaker Verlag, Aachen, p. 150-176.

[21] Schweimayer, G. (2003) *Risikomanagement mit Makroderivaten auf Basis zeitdiskreter stochastischer Prozesse*. Published Doctoral Thesis, Shaker Verlag, Aachen.

[22] Söderlind, P. (1995) Forward Interest Rates as Indicators of Inflation Expectations. *Centre for Economic Policy Research Discussion Papers* **1313**.

[23] Söderlind, P. and Svensson, L. (1997) New Techniques to Extract Market Expectations from Financial Instruments. *Centre for Economic Policy Research Discussion Papers* **1556**.

[24] Wilkie, A.D. (1986) A Stochastic Investment Model for Actuarial Use. *Transactions of the Faculty of Actuaries* **39**, p. 341-403.

[25] Woodward, G.T. (1990) The Real Thing: A Dynamic Profile of The Term Structure of Real Interest Rates and Inflation Expectations in The United Kingdom, 1982-89. *Journal of Business* **63**, No. 3, p. 373-398.

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