Pareto Navigation – interactive multiobjective optimisation and its application in radiotherapy planning

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Dedicated to my parents for
their support throughout my life.

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Abstract

In this work a novel interactive multiobjective optimisation method is introduced. Distinct features of the method are the use of very few parameters to steer the exploration and the explicit manipulation of the underlying partial order during decision making to control the partial tradeoffs.

The thesis starts with an extensive introduction of the topic that sketches the main results of the work.

Then, a framework for reference point based scalarisation functions compatible with a partial order given by an ordering cone is introduced. The framework is then analysed, so that valuable properties of the resulting scalarisation function are linked to properties of the so called *cone scalarising function*. Among others, efficiency of the outcomes, reachability of efficient points, convexity of the scalarisation function and semi-continuity with respect to the reference point are investigated.

Then, *Pareto navigation* the novel interactive multiobjective optimisation method is proposed and analysed. It features three mechanisms that manipulate the upper bounds, the current solution and the bounds on the partial tradeoffs, respectively. The first two mechanisms just need one objective and one parameter and the third mechanism just two objectives and one parameter as input.

Mathematical models for the different mechanisms are introduced and discussed in detail. It is shown that among the set of possible solutions – which depends on the chosen cone scalarising function – every efficient outcome can be reached in a fixed number of steps and that the change of the current solution is upper semicontinuous with respect to the reference point. The potential non-efficiency of the outcomes is analysed and demonstrated on a critical example.

Furthermore, the application of the method as a second phase in a two phase approach is described. Here, the focus is on the efficient use of the pre-computed data, so as to turn the mechanisms into real-time procedures. Finally, the extension of the method to nonconvex cases is presented.

The last major topic of the thesis is the application of Pareto navigation to intensity modulated radiotherapy (IMRT). First, the IMRT planning problem and its inherent multiobjective character are described. Then, some modelling options are presented and algorithms and heuristic approaches to the calculation of the phase one approximation discussed. Finally, the clinical prototype is presented and its graphical user interface visualising the solution and outcome set information and offering direct, graphical access to the mechanisms is described.

The thesis ends with a outlook that lists interesting aspects or possible extensions that would deserve further attention.

Contents

Chapter 1

Introduction

Multiobjective optimisation is the art of detecting and making good compromises. It bases upon the fact that most real-world decisions are compromises between partially conflicting objectives that cannot easily be offset against each other. Thus, one is forced to look for possible compromises and finally decide which one to implement. So, the final decision in multiobjective optimisation is always with a person – the decision maker.

This lack of commitment through the optimisation routines has its roots in the lack of a suitable total order in the so-called decision space – the space where each alternative has its objective values as coordinates. In absence of a total order there is no definite way to decide for two alternatives which of the them is the better choice. Therefore, there are pairs of alternatives, where one neither is better nor worse than the other.

The first notion of optimality in this setting goes back to Edgeworth in 1881 ([34]) and Pareto in 1896 ([90]) and is still the most widely used. In (Edgeworth-)Pareto optimality every feasible alternative that is not dominated by any other in terms of the componentwise partial order is considered to be optimal. Hence each solution is considered optimal that is not definitely worse than another. Thus, multiobjective optimisation does not yield a single or a set of equally good answers, but rather suggests a range of potentially *very* different answers.

Our initial definition of multiobjective optimisation addresses two main activities – detecting and making compromises. Although the progress in the former is impressive over the last half of a century the focus of this work is on *supporting* the making of a compromise.

The speed of dissemination of multiobjective optimisation to the different fields of application seems to still increase ([123]). Thus, multiobjective optimisation is considered for problems that previously were processed with scalar optimisation at best.

Due to the increasing computing power available and the improved algorithms, problems that were considered not to be tractable for multiobjective optimisation are now within reach. So, large-scale problems as well as problems with a large number of objectives are realistic challenges for multiobjective optimisation nowadays. Moreover, the improved graphical capabilities together with the modern interaction devices allow for new forms of communicating the decision maker's input to the computer system and the system's responses to the decision maker.

Dealing with a large number of objectives clearly has an influence on the decision process. While it is relatively easy to control a decision process and specify the input parameters for two and three objectives, the set of parameters becomes increasingly unhandy, if the dimension grows. Moreover, the input parameters usually *all* influence the outcome, so that it gets more and more complicated to estimate the influence of the input on the outcome. Therefore, for multiobjective optimisation problems with a high-dimensional decision space a direct control on the outcome and simple interactions are highly important for a successful decision process.

Besides, high-dimensional multiobjective problems usually also have a large set of alternatives. So, it is unlikely that a decision maker comes past some solution by chance. Therefore, the complementary information needs to convey the set of efficient alternatives as good as possible to the decision maker. (S)he will then be able to enjoy the options the efficient alternatives offer, instead of getting lost in them. Good complementary information also offers the opportunity to break new grounds instead of sticking to the conventional choice.

The decision maker should be able to organise the decision process as freely as possible to avoid unnecessary restrictions as to how the (s)he has to work. Instead of enforcing a strict workflow the system should invite the decision maker to experiment and develop his or her own style of making a decision. This most likely also increases the confidence in the result.

A last important point is the speed of interaction. If the response times are lengthy, the decision maker will try to cut the decision process as early as possible. This deprives him or her of the chance of detecting a much more suitable compromise. Furthermore, the intuitive connection between the last result, the decision maker's action and the reaction of the system to it is not as clearly perceived, if the response times get too long.

In this work we introduce a new interactive multiobjective optimisation method. It tries to meet the described properties needed to do successful decision making for high-dimensional multiobjective problems.

It offers three mechanisms to manipulate the current solution and to change the set of feasible outcomes. The three mechanisms can be applied in arbi-

trary order and every decision can be reversed, if the decision maker changes his or her mind. In particular, the system does not expect or even enforce consistent input from the decision maker, but willingly executes the requested changes. So, the decision maker can organise the decision process to his or her own liking.

The three mechanisms *restriction*, *selection* and *discounting* are designed to be as simple as possible. With the restriction mechanism the upper bound for a chosen objective is changed to a new admissible value. The selection mechanism is the main tool to change the current solution. Here again, the input is just the selected objective and the feasible new value it should attain. The discounting mechanism allows the introduction of bounds on the so-called partial tradeoffs. Thereby, the decision maker can specify a maximum price (s)he is willing to pay in terms of one objective for a unit gain in the other objective. Thus, for the discounting two objectives and a value need to be specified. Overall, the number of parameters for the mechanisms is small and in particular independent of the number of objectives. Besides, with the possible exception of the input for the discounting, the parameters needed by the mechanisms are directly related to the model and easy to understand.

The complementary information provided to the decision maker includes estimates for the ideal and nadir point – the points composed of the individual minima and maxima for the different objectives – for the currently set bounds with and without considering the bounds on the partial tradeoffs. This information is updated, whenever there is a change in the upper bounds for the objectives due to the restriction mechanism or a change in the bounds for the partial tradeoffs due to the discounting mechanism. So, the decision maker always has an up-to-date view on the ranges still accessible. (S)he will in particular see whether there are still very promising alternatives available that substantially differ from the current solution or if further exploration will not yield significantly different results. The estimates give a kind of global picture of the set of efficient solutions that helps to range in the current solution. Nevertheless, more complementary information especially more 'global' information would presumably lead to a faster and more straight decision process.

The approach regarding control is somewhat different to the existent methods. First, the control of the partial tradeoffs is a new feature for methods that primarily work with reference points. Second, the way the selection mechanism works differs from the typical approach. Usually, in reference point based methods input is given regarding the desired or accepted change in *every* objective. The methods then try to achieve a solution that is *overall* close to these goals. In contrast to this, the solution produced by the selection mechanism attains *one value exactly*, while the control of the change in

the other objectives is second to the achievement of the specified value in the chosen objective.

Thus, the decision maker's wish regarding the chosen objective is command to the system and the decision maker has to care for the consequences of this wish him-/herself. Nevertheless, the system tries to keep the consequences at a minimum, i.e. it tries to find the solution with the most favourable distance to the previous solution given the constraint that the chosen objective is fixed to the specified value. Combined with the manipulation of the upper bounds by the restriction mechanism and the adjustment of the bounds on the partial tradeoffs this in our opinion offers very good control on the outcome of the decision process.

Achievement scalarising functions introduced by Wierzbicki ([124, 125]) are a popular choice for reference point based interactive optimisation methods ([65], [69], [71], [86], [124]), because they likewise work with feasible and infeasible reference points. We pursue this direction, but we need to generalise the concept to be able to include the control on the partial tradeoffs.

The corresponding mathematical theory is developed in chapter 2. The bounds on the partial tradeoffs are included through the choice of a so-called ordering cone, that defines the partial order in the outcome space. Thus, we build up a framework for scalarising functions similar to the achievement scalarising functions, that directly incorporate ordering cones.

The *cone scalarising functions* – the functions belonging to that framework – are then analysed and found to behave very similar to the achievement scalarising functions ([125]):
One can either choose to use a cone scalarising function that is able to reach all efficient points for the given ordering, but only guarantees weak efficiency for its outcomes or a function that yields efficient solutions, but is not able to reach them all.

The result that each outcome is associated to infinitely many reference points, proven in the original paper ([125]) for a specific achievement scalarising function is generalised to certain classes of functions.

Additionally, the continuous dependence of the outcomes on the reference points is investigated and the well-known Pascoletti-Serafini scalarisation approach ([91]) is shown to belong to the framework.

Chapter 3 introduces Pareto navigation and analyses it. Here, the consequences of the two distinct features of Pareto navigation become apparent.

On the one hand side, the enforced achievement of the specified value of the chosen objective results in problem formulations that can in certain cases yield dominated solutions. Fortunately, the decision maker can be made aware of this. On the other hand, the combination of restriction

and selection mechanism enables the decision maker to reach any efficient solution that can be reached in at most K selection steps, where K is the dimension of the decision space.

Thus, there is a kind of tradeoff between improved control and guaranteed efficiency. This cannot easily by circumvented. A critical example demonstrates that the problem of finding the efficient point of most favourable distance under the equality constraint on the chosen objective has global character, i.e. distinct and separated local optima. So, the selection mechanism essentially would have to *choose* between a priori equally good but distinct alternatives.

Aside from the above mentioned problems the selection mechanism performs its work smoothly. The optimal value of the selection mechanism problems is Lipschitz-continuous and the optimal set mapping upper semicontinuous and closed for values in the interior of the range of admissible values. Nevertheless, the result can jump at the start of a selection step. This happens, e.g. when the previous solution was not efficient and will in any case always improve the objective values.

The inclusion of the bounds on the partial tradeoffs is similarly two-edged. While it offers an improved control for the decision maker on the set of considered alternatives, it increases the complexity of the update mechanisms for the ideal and nadir point estimates. In particular, the estimate for the ideal point of the efficient set with regard to the ordering cone becomes much more difficult compared to the case for the Pareto cone.

The execution of the discounting mechanism in turn poses no particular difficulties. We demonstrate that some specific homogeneous inequalities on the elements of the dual ordering cone imply the bounds on the tradeoffs. Every such inequality restricting the dual ordering cone corresponds to one generator for the ordering cone. More specifically, the ordering cone is a positive linear combination of the rows of the constraint matrix for the dual ordering cone. Thus, we have a polyhedral ordering cone that can easily be numerically represented. Moreover, the number of generators is limited to at most K^2 – a relatively moderate number. A further nice property is that we can easily check the compatibility of the imposed constraints and we can precompute the admissible range prior to a change, so that we can enforce the compatibility.

Large scale problems represent a problem for interactive multiobjective optimisation. Lengthy computations lead to long response times and if the response times become too long, the advantage of incorporating the decision maker in the solution process diminishes more and more. First of all the decision maker might become frustrated by the slow progress and stop the search prematurely. Second, the intuitive feeling for action and reaction is lost.

Therefore, we demonstrate how Pareto navigation can be used as a second phase of a two-phase approach. Here, the first phase creates an approximation to the efficient set without interacting with the decision maker. Then, the decision maker uses Pareto navigation to find a good alternative among the convex hull of the precomputed solutions. This solution can then be projected to the efficient set of the original problem by one more scalar optimisation problem. The advantage of primarily working in the convex hull of the pre-computed solutions only is that for appropriate choices of the cone scalarising function – say the function of the Pascoletti-Serafini approach – the mechanisms can be implemented by small to moderate sized linear programs that can be executed in real-time. Hence, the decision maker gets an approximate answer immediately and only for a few promising alternatives the original problem needs to be solved again.

The results described so far for the Pareto navigation all assume convexity of the original problem. This is a necessary requirement, since for nonconvex situations it is a priori unknown whether the efficient set is connected or not. Thus, there could be gaps in the ranges of the objective functions for the efficient solutions. Therefore, Pareto navigation needs to be adapted for nonconvex problems by slightly changing the selection mechanism.

The equality constraint is relaxed so that the control is not as good as in the convex case. The compensation is that the resulting solutions are (weakly) efficient. The relaxation of the equality constraint enables the selection mechanism to jump over the gaps, but then its results will clearly not (always) depend continuously on the change to the parameters inflicted by the decision maker.

Chapter 4 describes the application of Pareto navigation to intensity modulated radiotherapy (IMRT) planning.

According to Bortfeld ([12]) "IMRT is a radiation treatment technique with multiple beams in which at least some of the beams are intensity-modulated and intentionally deliver a non-uniform intensity to the target. The desired dose distribution in the target is achieved after superimposing such beams from different directions. The additional degrees of freedom are utilised to achieve a better target dose conformality and / or better sparing of critical structures."

The IMRT planning problem is the determination of the directions from which the patient is irradiated, i.e. the 'beams' and the non-uniform intensities for the different beams. The full planning problem is a global optimisation problem that is immensely difficult to tackle, which is mainly due to the beam directions. If one considers the optimisation of the intensities only, there are good convex models for the planning problem. So, we are considering the beam directions as given and only consider the optimisation of the intensities.

The objectives of the planning are to simultaneously deliver a high dose to the tumour (target) and to spare the surrounding organs (critical structures) and the body tissue. Since the X-ray used for the treatment inevitably passes through the part of the body in front of and behind the tumour, these different objectives are conflicting – a classical multiobjective optimisation situation.

When the first of our multiobjective radiotherapy planning projects started many people thought the multiobjective modelling of the problem to be computationally intractable, since the single objective IMRT planning problem already is a large-scale problem. Since then much progress has been achieved in speeding up the scalar problem (e.g. [105, 106]), so that multiobjective IMRT planning is tractable now, even though it still is very computationally involved. This is now mainly due to the high dimension of the decision space and the resulting high-dimensional set of efficient solutions. Dimensions of 6-8 are not uncommon and as usually the curse of dimension applies to approximations in high dimensions ([89]).

In these dimensions the creation and especially storage of the auxiliary data for deterministic methods becomes troublesome. Therefore, we discuss a heuristic – the so-called extreme compromises – that is constructed to fathom the variety of solutions the efficient set offers. The approximation is then improved by intermediate points computed using stochastically chosen scalarisation parameters.

An alternative to this approach is the use of bilevel optimisation. Here, part of the description of the feasible domain for an optimisation problem – the so-called upper level problem – is given as the set of optima for a second optimisation problem – the so-called lower level problem. Leyffer ([78]) recently proposed to formulate the detection of the next point to insert into the approximation of the efficient set as a bilevel problem. Thereby, the auxiliary data is only implicitly present in the lower level problem. Although this seems promising, so far no numerical experience exists on how the approach practically works.

Independent from the way it is created the approximation is then fed into the *navigator*, where the decision making takes place.

The navigator's user interface is split in two halves. The right half is used to display the current treatment plan using standard dose visualisations. The left half is used to display the variety of treatment plans at hand and embedded into the visualisation the controls used to communicate the decision maker's actions.

The main element is the navigation star – an *interactive* spider web diagram, where the ranges of the approximation, the currently accessible part of the ranges and the current solution is visualised. Its values on the different axes are connected by lines and there are sliders, where these lines meet the axes.

The decision maker can now grab a slider with the mouse and drag it and thereby change the value of the current objective. The navigator solves selection mechanism problems for the different values and updates the values for the current solution and the visualisations on the right-hand side several times a second. Thus, the feedback to a move of the decision maker is (almost) immediate and (s)he can adapt the change based on that feedback.

The navigation star furthermore features some brackets at the upper end of the spider web's axes. Grabbing a bracket and moving it towards the inner part of the spider web activates the restriction mechanism and initiates an update of the ranges on the axes. The part of the approximation's range behind the bracket is now out of bounds for the selection mechanism.

The discounting mechanism is not yet implemented and so are features like the post-optimisation. The latter will most likely be combined with the so-called sequencing – the decomposition of the optimal intensities into a sequence of hardware configurations ([120]).

The two-phase multiobjective optimisation approach for IMRT planning is currently clinically evaluated ([115]) and will become part of the radiotherapy planning system of Siemens Oncology Care Systems. Since IMRT planning is currently done using trial and error approaches, we think that Pareto navigation will simultaneously make treatment planning more systematic and easier.

Chapter 5 presents some of the questions whose answers would have impact on the presented work or seem promising. For some of them potential lines of attack are described.

Chapter 2

Achievement scalarising with cones

A multiobjective optimisation problem is considered to be solved, when the decision maker has found his or her preferred compromise among the variety of efficient solutions. This is either done by computing a few efficient solutions and choosing among them, approximating the set of efficient solutions (more on that in chapter 4) and choosing within this approximation, or by directly computing one or several points according to the decision maker's preferences.

Either way, the most commonly used technique for carrying out the computations is *scalarisation*. Scalarisation turns the multiobjective optimisation problem into a parametrised family of scalar optimisation problems. Thus, the set of efficient, properly efficient, or weakly efficient solutions or reasonable subsets thereof are parametrised.
Executing an effective approximation of the efficient solutions, computing some representative solutions or executing an interactive search is transformed into the problem of choosing appropriate parameters. The huge advantage of scalarising multiobjective problems is that the numerical methods for processing the scalar optimisation problems are well developed and a sound theory is available for them.

Only very few methods do not use scalarisation at all. The most well known such methods are probably the ones developed by Dellnitz et al. ([32, 108]) and Hillermeier ([57, 58]).

Most scalarisation methods are contained in one or more of the following classes

1. weighted sum based methods
2. weighted metric based methods
3. directional search methods

4. ε-constraint based methods

5. achievement scalarising functions

Weighted sum based methods minimise a linear functional $\langle \mathbf{w}, \cdot \rangle$, where the nontrivial weight is chosen from an appropriate set. The weighted metric based methods minimise the distance to an utopia point and thereby scale the underlying metric with different positive scaling or weight vectors. Directional search methods push a given point as long into one direction as the result stays feasible. The ε-constraint method on the other hand, sets upper bounds for all but one objective and determines an efficient point in the remaining outcome set with the best possible value in the unrestricted objective. Achievement scalarising functions assign a quality value to all the points in the outcome space relative to a given reference point. Then, a point of the outcome space with optimal quality value is found.

The basic versions and most common derivations of these methods can be found in the book of Miettinen [83] that also contains a large list of references for further reading.

Most methods are devised for the case, where the componentwise partial order is used to decide domination. The efficient solutions in this case are usually called *(Edgeworth-)Pareto optimal* solutions.
For partial orders that are reflexive, transitive and antisymmetric there is a convex, closed and pointed cone that can be used to decide dominance: if one point lies in the convex cone attached to the other point, it is dominated by that point.

Using more general partial orders has the advantage that one can incorporate bounds on the *partial tradeoffs* or *prices* (see [56, 66, 67]). Hence, being able to work with ordering cones allows to incorporate more preference information into the method and thus narrows down the set of alternatives to consider.

The incorporation of a more general partial order varies in difficulty for the different classes of methods mentioned above. For the weighted sum based methods it is very easy, because the main parameters are the prices. Therefore, the price bounds directly translate into bounds on the weights being used in the method.
For weighted metric based approaches Jahn ([64]) describes how the metric can be chosen to reflect the partial order.
The approach of Pascoletti and Serafini ([91]) is a very general directional search method that generalises different other approaches (see [51] and references therein) and allows the direct incorporation of ordering cones.
For the ε-constraint based methods no article incorporating bounds on the partial tradeoffs is known to the author.

Some specific achievement scalarisation functions ([125]) allow the integration of ordering cones – the Pascoletti-Serafini approach fits into this framework – but no general theory for their incorporation exists.

While most of the above classes are composed of some standard methods and their derivations the achievement scalarising approach formulates properties that scalarising functions need to have in order to be an achievement scalarising function. We will follow this approach and formulate a general framework of scalarising functions that directly incorporates ordering cones.

In section 2.1 we will introduce the basic definitions and terminology and state the abstract multiobjective optimisation problem under consideration. Then we will have a closer look at ordering cones and order intervals in section 2.2.
In section 2.3 we will state the reference point based cone scalarising problem with a not yet specified cone scalarising function. We will define the distance evaluation function associated to the cone scalarising function and will study how certain properties of the cone scalarising function yield corresponding properties of the distance evaluation function.
After that we will relate properties of the scalarising with properties of the resulting minima in section 2.4. Within the section we will also investigate the range of reachable efficient solutions and state an incompatibility result. Furthermore, we will see that under reasonable assumptions the set of optima for the different reference points is connected.
Section 2.5 is used to analyse the dependency of the minimisation results on the reference points and to formulate conditions under which the optimum is unique.
Then we will identify situations where optima of the scalarising can be reached from infinitely many reference points in section 2.6.
In section 2.7 we will demonstrate that the Pascoletti-Serafini scalarisation approach is contained in our framework and conclude the chapter with a summary of the results in section 2.8.

2.1 The multiobjective optimisation problem

In this section we will state the abstract multiobjective problem that is considered in the remainder of the thesis. We will not try for the most general results, but use a setting that fits for the mathematical application of the framework described later. For more general approaches see e.g. Jahn ([64]), Tammer and Göpfert ([114]) or Tammer and Weidner ([46]).

For the ease of reading we use specific fonts to express the affiliation to certain categories. Vector spaces and predefined number sets are expressed in capital black board bold letters, e.g. $\mathbb{R}, \mathbb{N}$ and $\mathbb{Y}$. Sets and set-valued mappings are denoted in calligraphic capital letters, e.g. $\mathcal{Y}$, $\mathcal{O}_{\mathcal{Y},s}$. Matrices

are denoted by capital bold font letters, whereas vectors and vector-valued mappings use bold lower case letters. Specifically, we denote by $\mathbf{0}$ and $\mathbf{1}$ the vectors composed of only zeros, respectively only ones. Furthermore, $\mathbf{e}_k$ denotes the k^{th} Cartesian unit vector. Mappings that operate on sets like bd, cl are written as a series of lower case letters.

Usually members of a set are denoted by the same letter than the sets themselves, e.g. $\mathbf{y} \in \mathcal{Y} \subseteq \mathbb{Y}$. Differences between elements of a space or directions in a space are denoted by Δ followed by a letter suitable for that space, e.g. $\Delta\mathbf{y} := \mathbf{y} - \mathbf{y}'$. The asterisk "$*$" as a superscript marks something as being dual to the same object without it.

In the following we consider the *decision space* $\mathbb{X}$ to be the m-dimensional Euclidean space $\mathbb{R}^m$ and denote the set of feasible decisions by $\mathcal{X}$. $\mathcal{X}$ is assumed to be nonempty, closed and convex. The *outcome space* $\mathbb{Y}$ is the K-dimensional Euclidean space $\mathbb{R}^K$. We will use $\|\cdot\|_2$ to denote the Euclidean norm and $\langle\cdot,\cdot\rangle$ for the inner product. If one side of the inner product is fixed, we sometimes speak of a linear functional.

The vector $\mathbf{f} : \mathbb{X} \to \mathbb{Y}$ of objective functions $\mathbf{f} := (f_k, k \in \mathcal{K})$, $\mathcal{K} := \{1, ..., K\}$ is assumed to be continuous over the set $\mathcal{X}$. The feasible outcomes are denoted by $\mathcal{Y} := \mathbf{f}(\mathcal{X})$. We will use the terms solution, outcome and point in the outcome set interchangeably.

By $\mathfrak{P}(\mathcal{S})$ we denote the power set of a given set $\mathcal{S}$ and by $argmin$ the set of minimisers and analogously by $argmax$ the set of maximisers. These sets are empty, if the minimum or maximum is not attained. $lexmin$ states that we will minimise a list of objectives lexicographically, i.e. we will first minimise the first objective, fix it to the optimal value and then minimise the second objective subject to that constraint and so on. The symbol mo-min signals that the abstract problem of solving a multiobjective problem is considered. Throughout the text we will interchange 'minimal' with 'optimal' or 'maximal' with 'optimal', where the meaning is unambiguous.

The characteristic making multiobjective optimisation different from scalar optimisation is the lack of a canonical total order in $\mathbb{Y}$. So, there is no standard way to judge which of two arbitrary points in $\mathbb{Y}$ is smaller or bigger than the other. There are total orders on $\mathbb{Y}$ like the lexicographic order for example, but there is no numerically stable one ([41]). Hence, one usually works with a partial order. Let us now formally define partial orders ([36]).

Definition 2.1.1 *A partial order is a binary relation $\leq_o$ on $\mathbb{R}^n$ that is*

(i) reflexive: $\mathbf{x} \leq_o \mathbf{x}$

(ii) transitive: $\mathbf{x} \leq_o \mathbf{y}$, $\mathbf{y} \leq_o \mathbf{z} \implies \mathbf{x} \leq_o \mathbf{z}$

(iii) antisymmetric: $\mathbf{x} \leq_o \mathbf{y}$ *and* $\mathbf{y} \leq_o \mathbf{x} \implies \mathbf{x} = \mathbf{y}$

for arbitrary $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^n$.
The partial order is called compatible with scalar multiplication if

$$\mathbf{x} \leq_o \mathbf{y}, \alpha \geq 0 \implies \alpha\,\mathbf{x} \leq_o \alpha\,\mathbf{y}$$

and compatible with addition if

$$\mathbf{w} \leq_o \mathbf{x}, \mathbf{y} \leq_o \mathbf{z} \implies \mathbf{w} + \mathbf{y} \leq_o \mathbf{x} + \mathbf{z}$$

for arbitrary $\mathbf{w}, ..., \mathbf{z} \in \mathbb{R}^n$.

If for two arbitrary points $\mathbf{y}, \mathbf{y}' \in \mathbb{R}^n$ either $\mathbf{y} \leq_o \mathbf{y}'$ or $\mathbf{y}' \leq_o \mathbf{y}$ then the order "$\leq_o$" is a total order.

The abstract *multiobjective* or *multicriteria* or *vector optimisation problem* we will consider is the minimisation of the vector-valued objective function $\mathbf{f}$ over the set $\mathcal{X}$

$$\text{mo-min}\left\{\mathbf{f}(\mathbf{x}) \mid \mathbf{x} \in \mathcal{X}\right\}, \tag{2.1}$$

where the partial order $\leq_o$ is used to compare different outcomes. For ease of exposition we often work exclusively in the objective space $\mathbb{Y}$, i.e. we are considering the problem of minimising the points in the outcome set $\mathcal{Y}$

$$\text{mo-min}\left\{\mathbf{y} \mid \mathbf{y} \in \mathcal{Y}\right\}, \tag{2.2}$$

where the partial order $\leq_o$ is used to compare different elements of $\mathcal{Y}$.

In the next section we will show that the partial order can be represented by a *domination* or *ordering cone* and investigate some of its geometrical properties.

2.2 Ordering cones

As we will see in this section one can associate to a reflexive, antisymmetric and transitive partial order relation compatible with scalar multiplication and addition a convex pointed and closed cone, the so-called ordering cone. Since ordering cones play an important role in the remainder of the work we devote this section to their derivation and analysis.

Let us start with the definition of the Minkowski sum of two sets. For two nonempty subsets $\mathcal{S}_1, \mathcal{S}_2 \subseteq \mathbb{R}^n$ of the Euclidean space, we denote by $\mathcal{S}_1 + \mathcal{S}_2$ the algebraic or Minkowski sum of $\mathcal{S}_1$ and $\mathcal{S}_2$,

$$\mathcal{S}_1 + \mathcal{S}_2 = \left\{s \in \mathbb{R}^n \mid s = s_1 + s_2, s_1 \in \mathcal{S}_1, s_2 \in \mathcal{S}_2\right\}.$$

If $\mathcal{S}_1 = \{s_1\}$ is a singleton, we write $s_1 + \mathcal{S}_2$ to represent $\{s_1\} + \mathcal{S}_2$. The following proposition collects some of the properties of the Minkowski sum (see [95]) for convex sets.

Proposition 2.2.1 *For convex sets $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3$ and real numbers $\lambda_1, \lambda_2 \in \mathbb{R}$ the following equalities hold:*

$$\begin{aligned} \mathcal{S}_1 + \mathcal{S}_2 &= \mathcal{S}_2 + \mathcal{S}_1 \\ (\mathcal{S}_1 + \mathcal{S}_2) + \mathcal{S}_3 &= \mathcal{S}_1 + (\mathcal{S}_2 + \mathcal{S}_3) \\ \lambda_1 (\lambda_2 \mathcal{S}_1) &= (\lambda_1 \lambda_2) \mathcal{S}_1 \\ \lambda_1 (\mathcal{S}_1 + \mathcal{S}_2) &= \lambda_1 \mathcal{S}_1 + \lambda_1 \mathcal{S}_2 \end{aligned}$$

For arbitrary $\mathcal{S} \subseteq \mathbb{R}^n$ we will denote by $cl(\mathcal{S})$, $int(\mathcal{S})$, $bd(\mathcal{S})$, and $ri(\mathcal{S})$, the closure, the interior, the boundary, and the relative interior of $\mathcal{S}$, respectively (see [95]). We will now define the notion of a cone.

Definition 2.2.2 *A cone $\mathcal{C} \subseteq \mathbb{Y}$ is a set with the property that*

$$\alpha \mathbf{c} \in \mathcal{C}, \text{ for all } \mathbf{c} \in \mathcal{C} \text{ and all } 0 < \alpha \in \mathbb{R}.$$

The cone $\mathcal{C}$ is called convex, if it is convex as a subset of $\mathbb{Y}$. It is called closed, if $cl(\mathcal{C}) = \mathcal{C}$. The cone is called pointed, if it does not contain any linear subspaces of $\mathbb{Y}$, i.e. if $-\mathcal{C} \cap \mathcal{C} \subseteq \{\mathbf{0}\}$.

Note that by this definition the interior of a cone is a cone as well. In the case of a pointed and closed cone $\mathcal{C}$ the intersection $-\mathcal{C} \cap \mathcal{C} = \{\mathbf{0}\}$ contains just the origin.

Throughout the work we will use a antisymmetric partial order in the outcome space $\mathbb{Y}$. The following theorem (see [36], [40], [64]) relates properties of the partial order with properties of the ordering cone $\mathcal{C}$.

Theorem 2.2.3 *Let $\mathcal{C} \subseteq \mathbb{Y}$ be a set and let the binary relation $\leq_{\mathcal{C}}$ be defined by*

$$y \leq_{\mathcal{C}} y' \Longleftrightarrow y' - y \in \mathcal{C}.$$

If $\leq_{\mathcal{C}}$ is a partial order and compatible with scalar multiplication and addition, the set $\mathcal{C}$ is a cone. Furthermore, the following statements hold:

1. *$\mathbf{0} \in \mathcal{C}$ if and only if $\leq_{\mathcal{C}}$ is reflexive.*
2. *$\mathcal{C} + \mathcal{C} \subseteq \mathcal{C}$ if and only if $\leq_{\mathcal{C}}$ is transitive.*
3. *$\mathcal{C}$ contains no lines, i.e. is pointed if and only if $\leq_{\mathcal{C}}$ is antisymmetric.*

Note that $\mathcal{C} + \mathcal{C} \subseteq \mathcal{C}$ holds for a cone if and only if it is convex. Hence, working with a partial antisymmetric order implies that the corresponding ordering cone is pointed and convex.

For convenience we introduce the following derived order relations. For arbitrary $\mathbf{x}$ and $\mathbf{y} \in \mathbb{R}^n$ let

- $\mathbf{x} \lneq_{\mathcal{C}} \mathbf{y} \iff \mathbf{x} \leq_{\mathcal{C}} \mathbf{y}$ and $\mathbf{y} \not\leq_{\mathcal{C}} \mathbf{x} \iff \mathbf{x} \leq_{\mathcal{C} \setminus \{\mathbf{0}\}} \mathbf{y}$

- $\mathbf{x} <_{\mathcal{C}} \mathbf{y} \iff \mathbf{x} \leq_{int(\mathcal{C})} \mathbf{y}$
- $\mathbf{x} \geq_{\mathcal{C}} \mathbf{y} \iff \mathbf{y} \leq_{\mathcal{C}} \mathbf{x}$ and analogously for '$\gneq_{\mathcal{C}}$' and '$>_{\mathcal{C}}$'

The order relations can now be used to define the notion of dominance. A point $\mathbf{y} \in \mathbb{Y}$ is called

- weakly dominated by $\mathbf{y}' \in \mathbb{Y}$, iff $\mathbf{y}' \leq_{\mathcal{C}} \mathbf{y}$.
- dominated by $\mathbf{y}' \in \mathbb{Y}$, iff $\mathbf{y}' \lneq_{\mathcal{C}} \mathbf{y}$.
- strictly dominated by $\mathbf{y}' \in \mathbb{Y}$, iff $\mathbf{y}' <_{\mathcal{C}} \mathbf{y}$.

For a given cone $\mathcal{C}$ the *dual cone* $\mathcal{C}^*$ is defined as

$$\mathcal{C}^* := \left\{ y \in \mathbb{Y} \mid \forall\, \mathbf{y}' \in \mathcal{C} : \langle \mathbf{y}, \mathbf{y}' \rangle \geq 0 \right\}$$

as the set of vectors that has nonnegative inner product with all elements of $\mathcal{C}$.

The cone that corresponds to the componentwise order $\leq$ on $\mathbb{Y}$ is the positive orthant and is sometimes called the *Pareto cone*. Working with a cone $\mathcal{C}$ that is not closed is problematic with respect to numerical algorithms ([41]). Furthermore, we will only need supersets of the Pareto cone for Pareto navigation. Thus, unless stated otherwise, we will use cones $\mathcal{C}$ that fulfil the following assumption.

Assumption 2.2.4 *The cone $\mathcal{C}$ is closed, pointed and convex and contains*

$$\mathcal{C} \supseteq \mathbb{Y}_+ := \mathbb{R}^K_+ := \{\mathbf{y} \in \mathbb{R}^K \mid \mathbf{y} \geq \mathbf{0}\}$$

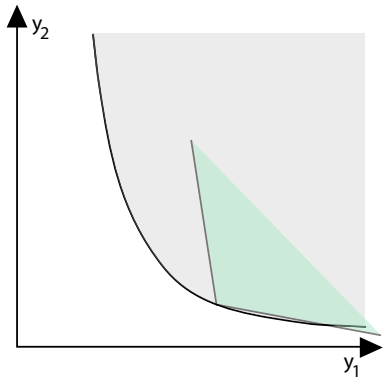
the Pareto cone.

Thus, the dual cone $\mathcal{C}^* \subseteq \mathbb{Y}_+$ is closed, convex, pointed and a subset of the Pareto cone.

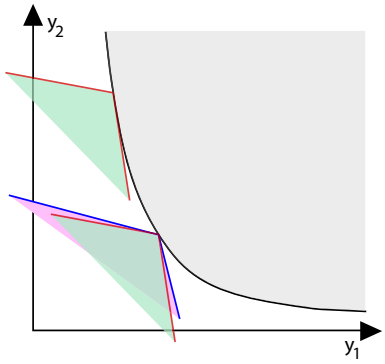
As the partial order used does in general not allow for a single best solution, we are interested in solutions being best possible.

Definition 2.2.5 *Let $\mathcal{Y} \subseteq \mathbb{Y}$ and let $\mathcal{C}$ fulfil 2.2.4. Then a point $\mathbf{y} \in \mathcal{Y}$ is*

- *efficient, if there is no other point $\mathbf{y}' \in \mathcal{Y}$ with $\mathbf{y}' \lneq_{\mathcal{C}} \mathbf{y}$, i.e. $(\mathbf{y} - \mathcal{C}) \cap \mathcal{Y} = \{\mathbf{y}\}$.*
- *weakly efficient, if there is no other point $\mathbf{y}' \in \mathcal{Y}$ with $\mathbf{y}' <_{\mathcal{C}} \mathbf{y}$, i.e. $\big(\mathbf{y} - int(\mathcal{C})\big) \cap \mathcal{Y} = \emptyset$.*



(a) Points inside the green area are strictly dominated by the tip of the cone, points on the grey lines are dominated except the tip itself.



(b) Points where the interior of the attached cone does not intersect $\mathcal{Y}$ are weakly efficient, points where just the tip intersects are efficient and points where just the tip of a slightly larger cone intersects are proper efficient.

Figure 2.1:

- *proper efficient in the sense of Henig ([54, 55]), if there exists a cone* $\mathcal{C}'$ *with* $\mathcal{C} \subseteq int(\mathcal{C}')$ *such that there is no point* $\mathbf{y}' \in \mathcal{Y}$ *with* $\mathbf{y}' \leq_{\mathcal{C}'} \mathbf{y}$, *i.e.* $(\mathbf{y} - \mathcal{C}') \cap \mathcal{Y} = \{\mathbf{y}\}$.

We denote by $eff_{\mathcal{C}}(\mathcal{Y})$, $eff_{w-\mathcal{C}}(\mathcal{Y})$, and $eff_{p-\mathcal{C}}(\mathcal{Y})$ the set of efficient, weakly efficient and properly efficient solutions in $\mathcal{Y}$ with respect to the ordering cone $\mathcal{C}$.

We assume that the problem to be solved is nontrivial and thus there is more than just one (weakly / properly) $\mathcal{C}$-efficient point in $\mathcal{Y}$. Hence, solving the multiobjective optimisation problem (2.2) means finding the decision maker's most preferred solution among the efficient ones. The non-efficient solutions can be ruled out since they leave room for improvement in some objectives without compromising the others.

Since we are interested in the different efficiency sets only, we can change the set $\mathcal{Y}$ without changing the multiobjective problem as long as the efficiency sets stay the same.

Proposition 2.2.6 *Let* $\mathcal{C}$ *be an ordering cone that fulfils 2.2.4 and let* $\mathcal{Y} \subseteq \mathbb{Y}$ *be a nonempty set. Then*

(i) $eff_{\mathcal{C}}(\mathcal{Y}) = eff_{\mathcal{C}}(\mathcal{Y} + \mathcal{C})$

(ii) $eff_{w-\mathcal{C}}(\mathcal{Y}) = eff_{w-\mathcal{C}}(\mathcal{Y} + \mathcal{C}) \cap \mathcal{Y}$

(iii) $eff_{p-\mathcal{C}}(\mathcal{Y}) = eff_{p-\mathcal{C}}(\mathcal{Y}+\mathcal{C})$

Proof:

For (i) and (ii) see e.g. lemmata 4.7 and 4.13 in the book of Jahn ([64]).

We will prove (iii). For an arbitrary $\mathbf{c} \in \mathcal{C} \setminus \{\mathbf{0}\}$ *and any* $\mathbf{y} \in \mathcal{Y}$ *the sum of the two*

$$\mathbf{y} \ <_{\bar{\mathcal{C}}} \ \mathbf{y}+\mathbf{c}$$

is strictly $\bar{\mathcal{C}}$*-dominated for any closed, convex and pointed cone* $\bar{\mathcal{C}}$ *with its interior* $int(\bar{\mathcal{C}}) \supseteq \mathcal{C}$ *containing* $\mathcal{C}$*. Thus,*

$$\Big(\mathcal{Y} + \big(\mathcal{C} \setminus \{\mathbf{0}\}\big)\Big) \cap eff_{p-\mathcal{C}}(\mathcal{Y}+\mathcal{C}) \ = \ \emptyset$$

and hence

$$eff_{p-\mathcal{C}}(\mathcal{Y}+\mathcal{C}) \ \subseteq \ \mathcal{Y}.$$

So take any $\mathbf{y} \in eff_{p-\mathcal{C}}(\mathcal{Y}+\mathcal{C})$*. As* $\mathbf{y} \in \mathcal{Y}$ *by the previous argument, the following inclusions hold:*

$$\mathbf{y} \ \in \ (\mathbf{y}-\bar{\mathcal{C}}) \cap \mathcal{Y} \ \subseteq \ (\mathbf{y}-\bar{\mathcal{C}}) \cap (\mathcal{Y}+\mathcal{C}) \ = \ \{\mathbf{y}\},$$

where we used the efficiency definition in the last step. Therefore, $\mathbf{y} \in eff_{p-\mathcal{C}}(\mathcal{Y})$ *and consequently*

$$eff_{p-\mathcal{C}}(\mathcal{Y}+\mathcal{C}) \ \subseteq \ eff_{p-\mathcal{C}}(\mathcal{Y}).$$

Now let $\mathbf{y} \notin eff_{p-\mathcal{C}}(\mathcal{Y}+\mathcal{C})$*. Hence, for every cone* $\bar{\mathcal{C}}$ *with* $int(\bar{\mathcal{C}}) \supseteq \mathcal{C}$ *the intersection* $(\mathbf{y}-\bar{\mathcal{C}}) \cap \mathcal{Y} \supsetneq \{\mathbf{y}\}$ *contains more than one element. So let* $\mathbf{y} \neq \tilde{\mathbf{y}} \in \mathcal{Y}+\mathcal{C}$ *be such an element. Consequentially, there exists a* $\tilde{\mathbf{c}} \in \mathcal{C} \setminus \{\mathbf{0}\}$ *with* $\tilde{\mathbf{y}} = \mathbf{y} - \tilde{\mathbf{c}}$*. Since* $\tilde{\mathbf{y}} \in \mathcal{Y}+\mathcal{C}$ *there exist* $\hat{\mathbf{y}} \in \mathcal{Y}$ *and* $\hat{\mathbf{c}} \in \mathcal{C}$ *with* $\tilde{\mathbf{y}} = \hat{\mathbf{y}} + \hat{\mathbf{c}}$*. Putting the two equalities together, we can represent*

$$\mathbf{y} \ = \ \hat{\mathbf{y}} + \underbrace{\hat{\mathbf{c}} + \tilde{\mathbf{c}}}_{\in \mathcal{C} \setminus \{\mathbf{0}\}}$$

by a point of the outcome set $\mathcal{Y}$ *plus a nonzero element of the ordering cone. Therefore,* $\mathbf{y} \notin eff_{p-\mathcal{C}}(\mathcal{Y})$ *and thus*

$$eff_{p-\mathcal{C}}(\mathcal{Y}+\mathcal{C}) \ \supseteq \ eff_{p-\mathcal{C}}(\mathcal{Y})$$

□

Since the sets of efficient and properly efficient outcomes do not change when we add the ordering cone to the outcome set, scalarisations that yield (properly) $\mathcal{C}$-efficient outcomes, cannot discriminate between $\mathcal{Y}$ and $\mathcal{Y}+\mathcal{C}$. If one represents the elements of the sum $\mathcal{Y}+\mathcal{C}$ canonically, then the '$\mathcal{Y}$-portion' of any weakly efficient element of $\mathcal{Y}+\mathcal{C}$ is weakly efficient in $\mathcal{Y}$. Thus, $\mathcal{Y}$ itself does not need to be convex or closed as long as $\mathcal{Y}+\mathcal{C}$ is closed or convex.

Definition 2.2.7 *A nonempty set $\mathcal{Y} \subseteq \mathbb{Y}$ is called*

(i) $\mathcal{C}$-convex, *if $\mathcal{Y} + \mathcal{C}$ is convex*

(ii) $\mathcal{C}$-closed, *if $\mathcal{Y} + \mathcal{C}$ is closed*

For outcome sets with these properties, there is an alternative description available for the different types of efficiency. We will bundle the corresponding results in the book of Jahn ([64]) in the following proposition. Here, we exploit the fact that the definitions of properly efficient points of Jahn and Henig ([54]) coincide for the considered situation.

Proposition 2.2.8 *Let the set of outcomes $\mathcal{Y}$ be $\mathcal{C}$-closed and $\mathcal{C}$-convex. Then*

i) A point $\mathbf{y} \in \mathcal{Y}$ is weakly $\mathcal{C}$-efficient if and only if there exists a nonzero element of the dual cone $\mathbf{c}^* \in \mathcal{C}^* \setminus \{\mathbf{0}\}$ such that

$$\langle \mathbf{c}^*, \mathbf{y} \rangle \;\leq\; \langle \mathbf{c}^*, \mathbf{y}' \rangle \quad \text{for all } \; \mathbf{y}' \in \mathcal{Y},$$

i.e. $\mathbf{y}$ minimises the linear functional $\langle \mathbf{c}^*, \cdot \rangle$.

ii) A point $\mathbf{y} \in \mathcal{Y}$ is properly $\mathcal{C}$-efficient if and only if there exists a element in the interior of the dual cone $\mathbf{c}^* \in int(\mathcal{C}^*)$ such that

$$\langle \mathbf{c}^*, \mathbf{y} \rangle \;\leq\; \langle \mathbf{c}^*, \mathbf{y}' \rangle \quad \text{for all } \; \mathbf{y}' \in \mathcal{Y},$$

i.e. $\mathbf{y}$ minimises the linear functional $\langle \mathbf{c}^*, \cdot \rangle$.

iii) A point $\mathbf{y} \in \mathcal{Y}$ is $\mathcal{C}$-efficient, if it is properly $\mathcal{C}$-efficient or if there exists a nonzero element of the dual cone $\mathbf{c}^* \in \mathcal{C}^* \setminus \{\mathbf{0}\}$ such that

$$\langle \mathbf{c}^*, \mathbf{y} \rangle \;<\; \langle \mathbf{c}^*, \mathbf{y}' \rangle \quad \text{for all } \; \mathbf{y}' \in \mathcal{Y} \setminus \{\mathbf{y}\},$$

i.e. $\mathbf{y}$ is the unique minimum of the linear functional $\langle \mathbf{c}^*, \cdot \rangle$.

With the preceding proposition we have a full characterisation of weakly and properly $\mathcal{C}$-efficient points for $\mathcal{C}$-closed and $\mathcal{C}$-convex outcome sets $\mathcal{Y}$. For $\mathcal{C}$-efficient points, however, we do not have a sufficient condition, which was the original reason to introduce the notion of *proper* efficiency ([44]).

We will now have a closer look at a construction frequently used in the remainder of the chapter.

Definition 2.2.9 *For a pair of points $\mathbf{y}, \mathbf{y}' \in \mathbb{Y}$ with $\mathbf{y} \leq_{\mathcal{C}} \mathbf{y}'$ the set*

$$\big(\mathbf{y} + \mathcal{C}\big) \cap \big(\mathbf{y}' - \mathcal{C}\big)$$

is called order interval *([64]).*

The properties that we imposed on the ordering cone $\mathcal{C}$ imply useful properties for the order intervals.

Proposition 2.2.10 *For a convex, closed, and pointed ordering cone $\mathcal{C}$ the interior $int(\mathcal{C}) \neq \emptyset$ of which is nonempty, the order interval for two points $\mathbf{y} \leq_{\mathcal{C}} \mathbf{y}' \in \mathbb{Y}$ is convex and compact.*

Proof:

By lemma 1.22 in the book of Jahn ([64]) the order interval is convex, closed and bounded. Thus, being subset of the Euclidean space $\mathbb{Y}$ the order interval is convex and compact.

□

Next, we will see how to construct an order interval containing two arbitrary given points. For a given $\Delta\mathbf{y} \in \mathbb{Y}$ let us consider the problem

$$\min \left\{ \left\| \mathbf{c}^+ + \mathbf{c}^- \right\|_2 \;\middle|\; \Delta\mathbf{y} = \mathbf{c}^+ - \mathbf{c}^-,\ \mathbf{c}^+,\ \mathbf{c}^- \in \mathcal{C} \right\}. \tag{2.3}$$

Proposition 2.2.11 *The solution of (2.3) is uniquely determined.*

Proof:

For cones $\mathcal{C}$ with nonempty interior every element $\mathbf{y}' \in \mathbb{Y}$ can be written as the difference of two elements from $\mathcal{C}$ (see e.g. [64, Lemma 1.13 plus Lemma 1.32]), a property that is called reproducing. *Hence, the feasible set of problem 2.3 is nonempty.*

Assume there is a second solution $\bar{\mathbf{c}}^+ \neq \mathbf{c}^+$, $\bar{\mathbf{c}}^- \neq \mathbf{c}^-$ with $\bar{\mathbf{c}}^+ - \bar{\mathbf{c}}^- = \Delta\mathbf{y}$ and the same value for $\|\bar{\mathbf{c}}^+ + \bar{\mathbf{c}}^-\|_2$. Clearly, for all $\lambda \in [0,1]$

$$\begin{aligned} \Delta\mathbf{y} &= (1-\lambda)\,\Delta\mathbf{y} + \lambda\,\Delta\mathbf{y} \\ &= (1-\lambda)\left(\mathbf{c}^+ - \mathbf{c}^-\right) + \lambda\left(\bar{\mathbf{c}}^+ - \bar{\mathbf{c}}^-\right) \\ &= \left((1-\lambda)\,\mathbf{c}^+ + \lambda\,\bar{\mathbf{c}}^+\right) - \left((1-\lambda)\,\mathbf{c}^- + \lambda\,\bar{\mathbf{c}}^-\right). \end{aligned}$$

If the differences and the sums of the two pairs $\mathbf{c}^+, \mathbf{c}^-$ and $\bar{\mathbf{c}}^+, \bar{\mathbf{c}}^-$ would both be equal, then we would have equality for $\bar{\mathbf{c}}^+ = \mathbf{c}^+$ and $\bar{\mathbf{c}}^- = \mathbf{c}^-$. Thus, the assumption implies that the sums

$$\mathbf{c}^+ + \mathbf{c}^- \;\neq \bar{\mathbf{c}}^+ + \bar{\mathbf{c}}^- \tag{2.4}$$

of the pairs are different. As the cone $\mathcal{C}$ is convex $\mathbf{c}_\lambda^+ := (1-\lambda)\,\mathbf{c}^+ + \lambda\,\bar{\mathbf{c}}^+$ and $\mathbf{c}_\lambda^- := (1-\lambda)\,\mathbf{c}^- + \lambda\,\bar{\mathbf{c}}^-$ are in $\mathcal{C}$ for all $\lambda \in [0,1]$. But for $\lambda \in (0,1)$

$$\begin{aligned} \|\,\mathbf{c}_\lambda^+ + \mathbf{c}_\lambda^-\|_2 &= \left\|\,(1-\lambda)\,\mathbf{c}^+ + \lambda\,\bar{\mathbf{c}}^+ + (1-\lambda)\,\mathbf{c}^- + \lambda\,\bar{\mathbf{c}}^-\right\|_2 \\ &= \left\|\,(1-\lambda)\big(\mathbf{c}^+ + \mathbf{c}^-\big) + \lambda\left(\bar{\mathbf{c}}^+ + \bar{\mathbf{c}}^-\right)\right\|_2 \\ &\overset{(2.4)}{<} (1-\lambda)\left\|\,\mathbf{c}^+ + \mathbf{c}^-\right\|_2 + \lambda\left\|\,\bar{\mathbf{c}}^+ + \bar{\mathbf{c}}^-\right\|_2 \\ &= \left\|\,\mathbf{c}^+ + \mathbf{c}^-\right\|_2 \end{aligned}$$

due to the strict convexity of $\|\cdot\|_2$. *But this is a contradiction to the minimality of* $\mathbf{c}^+$, $\mathbf{c}^-$.

□

The functions

$$\mathbf{c}^{\oplus} : \mathbb{Y} \to \mathcal{C} \quad \text{and} \quad \mathbf{c}^{\ominus} : \mathbb{Y} \to \mathcal{C} \tag{2.5}$$

mapping a $\Delta\mathbf{y} \in \mathbb{Y}$ to the optimal argument of (2.3) are therefore well defined. Let us define the projection onto the positive part

$$(\cdot)_+ : \mathbb{Y} \to \mathbb{Y}_+, \quad (\mathbf{y})_+ := \sum_{k\in\mathcal{K}} \left\{ \begin{array}{ll} \mathbf{y}_k & \text{for } \mathbf{y}_k > 0 \\ 0 & \text{for } \mathbf{y}_k \leq 0 \end{array} \right\} \mathbf{e}_k.$$

We will show now that the norm of the minimal result is bounded by the norm of the argument.

$$\begin{aligned} &\min\left\{ \left\| \mathbf{c}^+ + \mathbf{c}^- \right\|_2 \;\middle|\; \Delta\mathbf{y} = \mathbf{c}^+ - \mathbf{c}^-,\ \mathbf{c}^+, \mathbf{c}^- \in \mathcal{C} \right\} \\ &\qquad \overset{\mathbb{Y}_+\subseteq\mathcal{C}}{\leq} \quad \min\left\{ \left\| \mathbf{c}^+ + \mathbf{c}^- \right\|_2 \;\middle|\; \Delta\mathbf{y} = \mathbf{c}^+ - \mathbf{c}^-,\ \mathbf{c}^+, \mathbf{c}^- \in \mathbb{Y}_+ \right\} \\ &\qquad = \quad \left\| (\Delta\mathbf{y})_+ + (-\Delta\mathbf{y})_+ \right\|_2, \end{aligned}$$

because the decomposition into positive and negative entries is optimal for the ordering cone $\mathbb{Y}_+$, since any other decomposition simultaneously increases the corresponding components in $\mathbf{c}^+$ *and* $\mathbf{c}^-$. We continue the argument with the square of the norm.

$$\begin{aligned} &\left\| (\Delta\mathbf{y})_+ + (-\Delta\mathbf{y})_+ \right\|_2^2 \\ &\qquad = \quad \left\langle (\Delta\mathbf{y})_+ + (-\Delta\mathbf{y})_+,\ (\Delta\mathbf{y})_+ + (-\Delta\mathbf{y})_+ \right\rangle \\ &\qquad = \quad \left\| (\Delta\mathbf{y})_+ \right\|_2^2 \ + \ 2\left\langle (\Delta\mathbf{y})_+,\ (-\Delta\mathbf{y})_+ \right\rangle \ + \ \left\| (-\Delta\mathbf{y})_+ \right\|_2^2 \\ &\qquad = \quad \left\| (\Delta\mathbf{y})_+ \right\|_2^2 \ - \ 2\left\langle (\Delta\mathbf{y})_+,\ -(-\Delta\mathbf{y})_+ \right\rangle \ + \ \left\| -(-\Delta\mathbf{y})_+ \right\|_2^2 \\ &\qquad = \quad \left\| (\Delta\mathbf{y})_+ - (-\Delta\mathbf{y})_+ \right\|_2^2 \\ &\qquad = \quad \left\| \Delta\mathbf{y} \right\|_2^2, \end{aligned}$$

where we exploit that the positive $(\Delta\mathbf{y})_+$ and negative part $(-\Delta\mathbf{y})_+$ of a vector $\Delta\mathbf{y}$ are perpendicular and consequentially their inner product is zero. Thus, the norm of the sum of $\mathbf{c}^{\oplus}$ and $\mathbf{c}^{\ominus}$

$$\left\| \mathbf{c}^{\oplus}(\Delta\mathbf{y}) + \mathbf{c}^{\ominus}(\Delta\mathbf{y}) \right\|_2 \quad \leq \quad \left\| \Delta\mathbf{y} \right\|_2 \tag{2.6}$$

is bounded by the norm of the argument. With a little extra effort we can derive the Lipschitz-continuity of $\mathbf{c}^{\oplus}$ and $\mathbf{c}^{\ominus}$ from this result.

Proposition 2.2.12 *The functions* $\mathbf{c}^{\oplus}$ *and* $\mathbf{c}^{\ominus}$ *are Lipschitz-continuous with a Lipschitz constant of* 1.

Proof:

Note that by their definition

$$\mathbf{c}^{\oplus}(\Delta\mathbf{y}) = \mathbf{c}^{\ominus}(-\Delta\mathbf{y}) \quad \textit{and} \quad \mathbf{c}^{\ominus}(\Delta\mathbf{y}) = \mathbf{c}^{\oplus}(-\Delta\mathbf{y}).$$

Therefore, we just need to prove Lipschitz-continuity for one of the functions. So

$$\begin{aligned} \left\| \mathbf{c}^{\oplus}(\Delta\mathbf{y}) \right\|_2 &= \left\| \tfrac{1}{2}\left(\mathbf{c}^{\oplus}(\Delta\mathbf{y}) + \mathbf{c}^{\ominus}(\Delta\mathbf{y})\right) + \tfrac{1}{2}\left(\mathbf{c}^{\oplus}(\Delta\mathbf{y}) - \mathbf{c}^{\ominus}(\Delta\mathbf{y})\right) \right\|_2 \\ &\leq \tfrac{1}{2}\left\| \mathbf{c}^{\oplus}(\Delta\mathbf{y}) + \mathbf{c}^{\ominus}(\Delta\mathbf{y}) \right\|_2 + \tfrac{1}{2}\left\| \mathbf{c}^{\oplus}(\Delta\mathbf{y}) - \mathbf{c}^{\ominus}(\Delta\mathbf{y}) \right\|_2. \end{aligned}$$

Now, using (2.6) the first part

$$\tfrac{1}{2}\left\| \mathbf{c}^{\oplus}(\Delta\mathbf{y}) + \mathbf{c}^{\ominus}(\Delta\mathbf{y}) \right\|_2 \leq \tfrac{1}{2}\left\| \Delta\mathbf{y} \right\|_2$$

can be bounded by half the norm of the argument. Applying the defining equality $\mathbf{c}^{\oplus}(\Delta\mathbf{y}) - \mathbf{c}^{\ominus}(\Delta\mathbf{y}) = \Delta\mathbf{y}$ *to the second part we get the overall inequality*

$$\begin{aligned} \left\| \mathbf{c}^{\oplus}(\Delta\mathbf{y}) \right\|_2 &\leq \tfrac{1}{2}\left\| \mathbf{c}^{\oplus}(\Delta\mathbf{y}) + \mathbf{c}^{\ominus}(\Delta\mathbf{y}) \right\|_2 + \tfrac{1}{2}\left\| \mathbf{c}^{\oplus}(\Delta\mathbf{y}) - \mathbf{c}^{\ominus}(\Delta\mathbf{y}) \right\|_2 \\ &\leq \tfrac{1}{2}\left\|\Delta\mathbf{y}\right\|_2 + \tfrac{1}{2}\left\|\Delta\mathbf{y}\right\|_2 \\ &= \left\| \Delta\mathbf{y} \right\|_2, \end{aligned}$$

proving the claim.

□

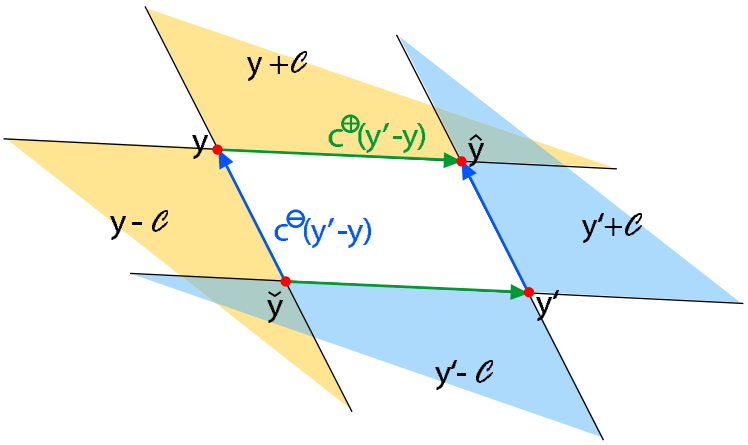


Figure 2.2: The points $\check{\mathbf{y}}$ and $\hat{\mathbf{y}}$ define the order interval for $\mathbf{y}$ and $\mathbf{y}'$ – the inner white part.

The importance of the functions $\mathbf{c}^{\oplus}$ and $\mathbf{c}^{\ominus}$ lies in the fact that they can be used to construct an order interval containing the two points. The two interval anchor points are a common upper and lower element in the lattice $\big(\mathbb{Y}, \leq_{\mathcal{C}}\big)$ for the two given points.
Suppose we have two arbitrary elements $\mathbf{y}, \mathbf{y}' \in \mathbb{Y}$. Setting

$$\mathbf{c}^{+} := \mathbf{c}^{\oplus}\big(\mathbf{y}' - \mathbf{y}\big) \quad \text{and} \quad \mathbf{c}^{-} := \mathbf{c}^{\ominus}\big(\mathbf{y}' - \mathbf{y}\big)$$

the difference between the two chosen points can be expressed as

$$\mathbf{y}' - \mathbf{y} = \mathbf{c}^{+} - \mathbf{c}^{-}.$$

Therefore, a common maximum element for the chosen points is given by

$$\mathbf{y}' \;\leq_{\mathcal{C}}\; \mathbf{y}' + \mathbf{c}^{-} \;=\; \mathbf{y} + \mathbf{c}^{+} \;\geq_{\mathcal{C}}\; \mathbf{y}$$

and a common minimum element by

$$\mathbf{y}' \;\geq_{\mathcal{C}}\; \mathbf{y}' - \mathbf{c}^{+} \;=\; \mathbf{y} - \mathbf{c}^{-} \;\leq_{\mathcal{C}}\; \mathbf{y}$$

Hence, the two points $\mathbf{y}, \mathbf{y}'$ are contained

$$\mathbf{y}, \mathbf{y}' \in \big(\mathbf{y} - \mathbf{c}^{-} + \mathcal{C}\big) \cap \big(\mathbf{y} + \mathbf{c}^{+} - \mathcal{C}\big),$$

in the order interval for $\mathbf{y} - \mathbf{c}^{-}$ and $\mathbf{y} + \mathbf{c}^{+}$. By the definition of $\mathbf{c}^{\oplus}$ and $\mathbf{c}^{\ominus}$ the norm $\big\| \mathbf{c}^{+} + \mathbf{c}^{-} \big\|_2$ is minimal and therefore the construction produces the order interval of minimal length.

Let

$$\mathcal{B}_{\rho}(\hat{\mathbf{y}}) := \{\mathbf{y} \in \mathbb{Y} \mid \|\mathbf{y} - \hat{\mathbf{y}}\|_2 \leq \rho\}$$

be the ball of radius ρ around the point $\hat{\mathbf{y}}$. We will now extend the above construction to include not just two points, but a ball of a given radius around a given centre.

Proposition 2.2.13 *For a ball $\mathcal{B}_{\rho}(\hat{\mathbf{y}})$ with radius ρ around the centre $\hat{\mathbf{y}}$ and the ordering cone $\mathcal{C}$*

$$\mathcal{B}_{\rho}(\hat{\mathbf{y}}) \quad \subseteq \quad \big(\hat{\mathbf{y}} - \rho\,\mathbf{1} + \mathcal{C}\big) \cap \big(\hat{\mathbf{y}} + \rho\,\mathbf{1} - \mathcal{C}\big).$$

Proof:
Since the ball is contained in a cube having the diameter of the ball as side length,

$$\begin{aligned} \mathcal{B}_{\rho}(\hat{\mathbf{y}}) \;&\subseteq\; \big(\hat{\mathbf{y}} - \rho\,\mathbf{1} + \mathbb{Y}_{+}\big) \cap \big(\hat{\mathbf{y}} + \rho\,\mathbf{1} - \mathbb{Y}_{+}\big) \\ &\subseteq\; \big(\hat{\mathbf{y}} - \rho\,\mathbf{1} + \mathcal{C}\big) \cap \big(\hat{\mathbf{y}} + \rho\,\mathbf{1} - \mathcal{C}\big) \end{aligned}$$

it is also contained in the potentially larger order interval.

□

For ordering cones $\mathcal{C}$ with $int(\mathcal{C}) \supset \mathbb{Y}_+$ the anchor points could be chosen lying closer to each other, but the anchor points used in the proposition suffice for our purposes. Using the fact that

$$\mathcal{B}_{\hat{\rho}}(\hat{\mathbf{y}}) + \mathcal{B}_{\bar{\rho}}(\bar{\mathbf{y}}) = \mathcal{B}_{\hat{\rho}+\bar{\rho}}(\hat{\mathbf{y}} + \bar{\mathbf{y}})$$

the above result can easily be extended to a finite number of balls.

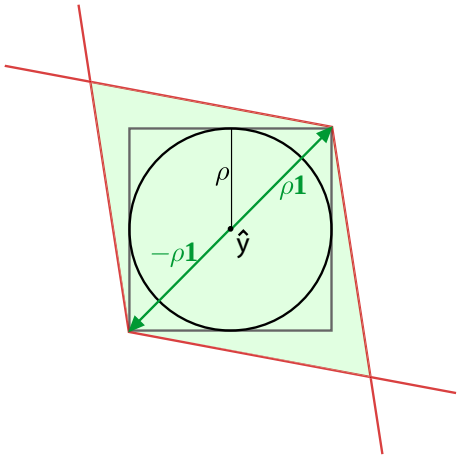


Figure 2.3: The ball can be packed into an order interval of appropriate size.

We will use the representation of arbitrary elements by differences of cone elements to formulate our scalarisation framework.

2.3 Cone scalarising functions

In this section we will express the difference between the reference point $\mathbf{y}^R$ and the point $\mathbf{y} \in \mathcal{Y}$ to be optimised by the difference of two elements of the ordering cone. We will then apply achievement scalarising-like functions called *cone scalarising functions* to these cone elements.

The cone scalarising functions will then be translated into so-called *distance evaluation functions*. They operate on distances between elements of $\mathbb{Y}$ and are thus much easier to grasp. They are a main tool for the forthcoming theory and therefore we use this section to derive relations between properties of the cone scalarising functions and their corresponding distance evaluation functions.

We will start by imposing conditions that ensure a well defined distance evaluation function. These properties suffice to show continuity and different degrees of monotonicity as well. We will then see that Lipschitz-continuity transfers from a cone scalarising function to its distance evaluation function. The same holds true for quasiconvexity and strict quasiconvexity. Finally,

we will prove that convexity and strict convexity of the cone scalarising function transfers to the distance evaluation too.

The problem that we will consider in the remainder of the chapter – called the *cone scalarising problem* – reads as follows:

$$\inf\left\{s\big(\mathbf{c}^+,\,\mathbf{c}^-\big)\;\middle|\;\mathbf{y}-\mathbf{y}^R=\mathbf{c}^+-\mathbf{c}^-,\;\mathbf{y}\in\mathcal{Y},\;\mathbf{c}^+,\,\mathbf{c}^-\in\mathcal{C}\right\},\qquad(2.7)$$

where the function s

$$s\colon\mathcal{C}\times\mathcal{C}\to\mathbb{R}$$

maps a pair of ordering cone elements to the real line. We will assume henceforth that the function s is defined for *all* $(\mathbf{c}^+,\,\mathbf{c}^-)\in\mathcal{C}\times\mathcal{C}$.

Since our ordering cone $\mathcal{C}$ has nonempty interior and is thus reproducing, the difference $\mathbf{y}-\mathbf{y}^R$ of an arbitrary pair $\mathbf{y}\in\mathcal{Y}$, $\mathbf{y}^R\in\mathbb{Y}$ can be expressed by the difference of some $\mathbf{c}^+,\,\mathbf{c}^-\in\mathcal{C}$ ([64]). The following proposition gives an explicit description of the domain of minimisation in (2.7).

Proposition 2.3.1 *For any* $\Delta\mathbf{y}\in\mathbb{Y}$

$$\mathcal{C}^+(\Delta\mathbf{y}):=\left\{\mathbf{c}^+\in\mathcal{C}\;\middle|\;\Delta\mathbf{y}=\mathbf{c}^+-\mathbf{c}^-,\;\mathbf{c}^-\in\mathcal{C}\right\}=\big(\Delta\mathbf{y}+\mathcal{C}\big)\cap\mathcal{C}.$$

Proof:

$$\begin{aligned}\mathcal{C}^+(\Delta\mathbf{y}) &= \left\{\mathbf{c}^+\in\mathcal{C}\;\middle|\;\Delta\mathbf{y}=\mathbf{c}^+-\mathbf{c}^-,\;\mathbf{c}^-\in\mathcal{C}\right\}\\ &= \left\{\mathbf{c}^+\in\mathcal{C}\;\middle|\;\Delta\mathbf{y}+\mathbf{c}^-=\mathbf{c}^+,\;\mathbf{c}^-\in\mathcal{C}\right\}\\ &= \left\{\Delta\mathbf{y}+\mathbf{c}^-\;\middle|\;\mathbf{c}^-\in\mathcal{C}\right\}\cap\mathcal{C}\\ &= \big(\Delta\mathbf{y}+\mathcal{C}\big)\cap\mathcal{C}.\end{aligned}$$

□

The set $\mathcal{C}^+(\Delta\mathbf{y})$ can be written as the intersection of two convex sets and is therefore convex itself. Besides, it can be written as the algebraic sum of its efficient set and the ordering cone.

Proposition 2.3.2 *For any* $\Delta\mathbf{y}\in\mathbb{Y}$

$$\mathcal{C}^+(\Delta\mathbf{y}) \quad = \quad eff_{\mathcal{C}}\big(\mathcal{C}^+(\Delta\mathbf{y})\big)+\mathcal{C}.$$

Proof:

The "$\subseteq$" inclusion is given by the definition of $\mathcal{C}$-efficiency. Thus, only the "$\supseteq$" inclusion remains to be shown. Let $\mathbf{c}^+\in\mathcal{C}^+(\Delta\mathbf{y})$. *Then, for an arbitrary* $\mathbf{c}\in\mathcal{C}$

$$\begin{aligned}\mathbf{c}^+ + \mathbf{c} &\in \mathbf{c}+\mathcal{C}^+(\Delta\mathbf{y})\\ &\subseteq \mathbf{c}+\big(\Delta\mathbf{y}+\mathcal{C}\big)\\ &= \Delta\mathbf{y}+\big(\mathbf{c}+\mathcal{C}\big)\\ &\subseteq \Delta\mathbf{y}+\mathcal{C}.\end{aligned}$$

With an analogous argument the sum $\mathbf{c}^+ + \mathbf{c} \subseteq \mathcal{C}$ *is also contained in the cone. Since the choice of* $\mathbf{c}$ *was arbitrary*

$$\mathbf{c}^+ + \mathcal{C} \;\subseteq\; \mathcal{C} \cap \big(\Delta\mathbf{y} + \mathcal{C}\big) \;=\; \mathcal{C}^+(\Delta\mathbf{y}) \quad \textit{for every} \quad \mathbf{c}^+ \in \mathcal{C}^+(\Delta\mathbf{y}).$$

Consequently, the efficient set plus the ordering cone

$$eff\big(\mathcal{C}^+(\Delta\mathbf{y})\big) + \mathcal{C} \quad \subseteq \quad \mathcal{C}^+(\Delta\mathbf{y})$$

form a subset of the original set.

□

The set

$$\mathcal{C}^-(\Delta\mathbf{y}) := \big\{\mathbf{c}^- \in \mathcal{C} \;\big|\; \Delta\mathbf{y} = \mathbf{c}^+ - \mathbf{c}^-,\ \mathbf{c}^+ \in \mathcal{C}\big\}$$

by definition fulfils

$$\mathcal{C}^-(\Delta\mathbf{y}) = \mathcal{C}^+(-\Delta\mathbf{y})$$

and thus is convex as well.

For the special cases of $\Delta\mathbf{y} \leq_{\mathcal{C}} \mathbf{0}$ and $\mathbf{0} \leq_{\mathcal{C}} \Delta\mathbf{y}$ the result of $\big(\Delta\mathbf{y} + \mathcal{C}\big) \cap \mathcal{C}$ reduces to $\mathcal{C}$ and $\Delta\mathbf{y} + \mathcal{C}$ respectively.

A main tool in the remainder of the chapter will be the distance evaluation function associated with the cone scalarising function. For a given distance between two points of $\mathbb{Y}$ it finds the s-minimal representation for this distance in terms of a pair of elements from $\mathcal{C}$.

Definition 2.3.3 *For a cone scalarising function* s *the function*

$$\begin{aligned} t_s: \quad & \mathbb{Y} \to \mathbb{R} \\ t_s(\Delta\mathbf{y}) \;:=\; & \inf\big\{s\big(\mathbf{c}^+, \mathbf{c}^-\big) \;\big|\; \Delta\mathbf{y} = \mathbf{c}^+ - \mathbf{c}^-,\ \mathbf{c}^+, \mathbf{c}^- \in \mathcal{C}\big\} \end{aligned}$$

is called distance evaluation function.

It can be equivalently expressed as

$$\begin{aligned} t_s(\Delta\mathbf{y}) \;&=\; \inf\big\{s\big(\mathbf{c}^+, \mathbf{c}^-\big) \;\big|\; \mathbf{c}^- = \mathbf{c}^+ - \Delta\mathbf{y}, \mathbf{c}^+ \in \mathcal{C}^+(\Delta\mathbf{y})\big\} \\ &=\; \inf\big\{s\big(\mathbf{c}^+, \mathbf{c}^+ - \Delta\mathbf{y}\big) \;\big|\; \mathbf{c}^+ \in \mathcal{C}^+(\Delta\mathbf{y})\big\}. \end{aligned}$$

Now we can decompose the cone scalarising problem (2.7) into the search for the $\mathbf{y} \in \mathcal{Y}$ with t_s-minimal distance to the reference point $\mathbf{y}^R$

$$\inf\big\{t_s\big(\mathbf{y} - \mathbf{y}^R\big) \;\mid\; \mathbf{y} \in \mathcal{Y}\big\}, \tag{2.8}$$

while by its definition t_s searches for the s-minimal representation of the distance.

Thus, choosing s such that t_s is suitable for solving (2.8) implies the appropriateness of s for solving (2.7). The following two properties of s are important for this purpose.

Definition 2.3.4 *A function $s : \mathcal{C} \times \mathcal{C} \to \mathbb{R}$ is called $\mathcal{C}$-centred, if for every $(\mathbf{c}^+, \mathbf{c}^-) \in \mathcal{C} \times \mathcal{C}$ and for every $\mathbf{c} \in \mathcal{C}$*

$$s(\mathbf{c}^+, \mathbf{c}^-) \leq s(\mathbf{c}^+ + \mathbf{c}, \mathbf{c}^- + \mathbf{c}).$$

The centredness of s ensures that we cannot improve the infimum in the definition of t_s by adding the same cone element to both sides of the difference.

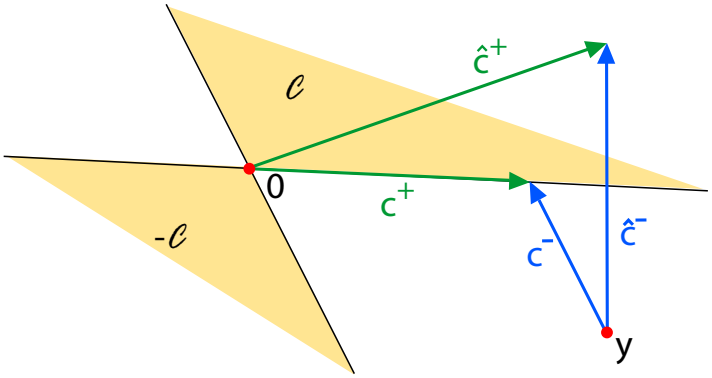


Figure 2.4: The point $\mathbf{y}$ represented by two different pairs of cone elements.

The second property that we need is consistency.

Definition 2.3.5 *A continuous cone scalarising function $s : \mathcal{C} \times \mathcal{C} \to \mathbb{R}$ is called $\mathcal{C}$-consistent, if for every $(\mathbf{c}^+, \mathbf{c}^-) \in \mathcal{C} \times \mathcal{C}$ and for every $\mathbf{c} \in \mathcal{C}$*

$$s(\mathbf{c}^+, \mathbf{c}^-) \leq s(\mathbf{c}^+ + \mathbf{c}, \mathbf{c}^-) \quad \text{and} \quad s(\mathbf{c}^+, \mathbf{c}^-) \geq s(\mathbf{c}^+, \mathbf{c}^- + \mathbf{c}).$$

It is called strictly $\mathcal{C}$-consistent, if for every $(\mathbf{c}^+, \mathbf{c}^-) \in \mathcal{C} \times \mathcal{C}$ and for every $\mathbf{c} \in int(\mathcal{C})$

$$s(\mathbf{c}^+, \mathbf{c}^-) < s(\mathbf{c}^+ + \mathbf{c}, \mathbf{c}^-) \quad \text{and} \quad s(\mathbf{c}^+, \mathbf{c}^-) > s(\mathbf{c}^+, \mathbf{c}^- + \mathbf{c}).$$

It is called strongly $\mathcal{C}$-consistent, if for every $(\mathbf{c}^+, \mathbf{c}^-) \in \mathcal{C} \times \mathcal{C}$ and for every $\mathbf{c} \in \mathcal{C} \setminus \{\mathbf{0}\}$

$$s(\mathbf{c}^+, \mathbf{c}^-) < s(\mathbf{c}^+ + \mathbf{c}, \mathbf{c}^-) \quad \text{and} \quad s(\mathbf{c}^+, \mathbf{c}^-) > s(\mathbf{c}^+, \mathbf{c}^- + \mathbf{c}).$$

We will now verify that the infimum in the definition of t_s is finite for continuous, $\mathcal{C}$-centred and $\mathcal{C}$-consistent cone scalarising functions s.

Lemma 2.3.6 *Suppose s is a $\mathcal{C}$-consistent $\mathcal{C}$-centred and continuous cone scalarising function. Then the infimum in the definition of the function t_s is finite for all $\Delta\mathbf{y} \in \mathbb{Y}$.*

Proof:
For every $\mathbf{c}^+ \in \mathcal{C}^+(\Delta\mathbf{y})$

$$\begin{aligned} s\big(\mathbf{c}^+, \mathbf{c}^+ - \Delta\mathbf{y}\big) &= s\big(\mathbf{c}^+, \mathbf{c}^+ + \mathbf{c}^{\ominus}(\Delta\mathbf{y}) - \mathbf{c}^{\oplus}(\Delta\mathbf{y})\big) \\ &\overset{\mathcal{C}\text{-consistent}}{\geq} s\big(\mathbf{c}^+, \mathbf{c}^+ + \mathbf{c}^{\ominus}(\Delta\mathbf{y}) - \mathbf{c}^{\oplus}(\Delta\mathbf{y}) + \mathbf{c}^{\oplus}(\Delta\mathbf{y})\big) \\ &= s\big(\mathbf{c}^+, \mathbf{c}^+ + \mathbf{c}^{\ominus}(\Delta\mathbf{y})\big) \\ &\overset{\mathcal{C}\text{-centred}}{\geq} s\big(\mathbf{0}, \mathbf{c}^{\ominus}(\Delta\mathbf{y})\big) \end{aligned} \tag{2.9}$$

Thus, we have a lower bound for

$$\begin{aligned} t_s(\Delta\mathbf{y}) &= \inf\big\{ s\big(\mathbf{c}^+, \mathbf{c}^+ - \Delta\mathbf{y}\big) \;\big|\; \mathbf{c}^+ \in \mathcal{C}^+(\Delta\mathbf{y}) \big\} \\ &\geq s\big(\mathbf{0}, \mathbf{c}^{\ominus}(\Delta\mathbf{y})\big) \\ &> -\infty, \end{aligned} \tag{2.10}$$

which is finite, because s *is continuous and defined on the whole of* $\mathcal{C} \times \mathcal{C}$.
□

Note that we so far only know that the infimum is bounded. This does not imply that the infimum is attained for some pair $\mathbf{c}^+, \mathbf{c}^- \in \mathcal{C}$. Therefore, we will work with the following assumption.

Assumption 2.3.7 *The cone scalarising function* s *is chosen such that for every* $\Delta\mathbf{y} \in \mathbb{Y}$ *a pair* $\mathbf{c}^+, \mathbf{c}^- \in \mathcal{C}$ *exists such that*

$$s\big(\mathbf{c}^+, \mathbf{c}^-\big) = \inf\big\{ s\big(\mathbf{c}^+, \mathbf{c}^+ - \Delta\mathbf{y}\big) \;\big|\; \mathbf{c}^+ \in \mathcal{C}^+(\Delta\mathbf{y}) \big\}$$

the infimum is attained.

This allows us to write 'min' instead of 'inf' in (2.3.3). A pair of cone elements $\mathbf{c}^+, \mathbf{c}^- \in \mathcal{C}$ will be called minimal or optimal representation for $\Delta\mathbf{y} \in \mathbb{Y}$, if $t_s(\Delta\mathbf{y}) = s(\mathbf{c}^+, \mathbf{c}^-)$.

We will therefore establish a sufficient condition for the assumption (2.3.7) to be fulfilled. For that, we will first derive a reduction result.

Proposition 2.3.8 *Let* s *be a continuous,* $\mathcal{C}$*-consistent and* $\mathcal{C}$*-centred cone scalarising function and let* $\Delta\mathbf{y} \in \mathbb{Y}$ *be given. Then for every pair* $\mathbf{c}^+, \mathbf{c}^- \in \mathcal{C}$ *with* $\mathbf{c}^+ - \mathbf{c}^- = \Delta\mathbf{y}$ *there exists a* $\bar{\mathbf{c}}^+ \in eff_{\mathcal{C}}\big(\mathcal{C}^+(\Delta\mathbf{y})\big)$ *such that*

$$s\big(\bar{\mathbf{c}}^+, \bar{\mathbf{c}}^+ - \Delta\mathbf{y}\big) \leq s\big(\mathbf{c}^+, \mathbf{c}^-\big).$$

Proof:
The assumptions imply that $\mathbf{c}^+ \in \mathcal{C}^+(\Delta\mathbf{y})$. *By proposition 2.3.2 for every* $\mathbf{c}^+ \in \mathcal{C}^+(\Delta\mathbf{y})$ *there exist* $\bar{\mathbf{c}}^+ \in eff_{\mathcal{C}}\big(\mathcal{C}^+(\Delta\mathbf{y})\big)$ *and* $\mathbf{c} \in \mathcal{C}$ *with* $\mathbf{c}^+ = \bar{\mathbf{c}}^+ + \mathbf{c}$. *Consequentially,*

$$\mathbf{c}^- = \mathbf{c}^+ - \Delta\mathbf{y} = \bar{\mathbf{c}}^+ + \mathbf{c} - \Delta\mathbf{y}.$$

Putting this representation into the cone scalarising function s and using its $\mathcal{C}$-centredness we get

$$\begin{aligned} s\big(\mathbf{c}^+, \mathbf{c}^-\big) &= s\big(\bar{\mathbf{c}}^+ + \mathbf{c}, \bar{\mathbf{c}}^+ + \mathbf{c} - \Delta\mathbf{y}\big) \\ &\geq s\big(\bar{\mathbf{c}}^+, \bar{\mathbf{c}}^+ - \Delta\mathbf{y}\big) \end{aligned}$$

□

Thus, the set of pairs with minimal t_s values always intersects with the set of $\mathcal{C}$-efficient elements of $\mathcal{C}^+(\Delta\mathbf{y})$. Therefore, a sufficient condition for the existence of a t_s-minimal pair is the boundedness of $eff_{\mathcal{C}}\big(\mathcal{C}^+(\Delta\mathbf{y})\big)$.

Lemma 2.3.9 *For a polyhedral cone $\mathcal{C}$, i.e. a cone*

$$\mathcal{C} := \Big\{ \sum_{i=1}^{n} \alpha_i \mathbf{v}^{(i)} \mid \alpha_i \in \mathbb{R}_+ \Big\}$$

given by the positive linear combination of a finite number of generators the set $eff_{\mathcal{C}}\big(\mathcal{C}^+(\Delta\mathbf{y})\big)$ is bounded.

Proof:

$\mathcal{C}^+(\Delta\mathbf{y})$ is polyhedral as an intersection of two polyhedral sets. Thus, it can be written as the algebraic sum

$$\mathcal{C}^+(\Delta\mathbf{y}) \;=\; conv\big\{\mathbf{y}^{(j)} \mid j = 1, ..., m\big\} + \Big\{ \sum_{i=1}^{k} \alpha_i \mathbf{w}^{(i)} \mid \alpha_i \in \mathbb{R}_+ \Big\} \;=:\; \mathcal{S} + \tilde{\mathcal{C}}$$

of a polytope and a polyhedral cone ([95, 19.1.1]). We first show that the representation cone $\tilde{\mathcal{C}}$ coincides with the ordering cone $\mathcal{C}$.

In proposition 2.3.2 we have seen that

$$\mathcal{C}^+(\Delta\mathbf{y}) + \mathcal{C} \;=\; \mathcal{C}^+(\Delta\mathbf{y}).$$

Therefore, the ordering cone $\mathcal{C} \subseteq \tilde{\mathcal{C}}$ is contained in the representation cone. Now assume $\mathbf{c} \in \tilde{\mathcal{C}} \setminus \mathcal{C}$ to be an element of the representation cone that is not contained in the ordering cone. Since $\mathcal{C}$ is reproducing there exist $\mathbf{c}^+, \mathbf{c}^- \in \mathcal{C}$ with

$$\mathbf{c} \;=\; \mathbf{c}^+ - \mathbf{c}^- \;=\; \sum_{i=1}^{n} \big(\alpha_i^+ - \alpha_i^-\big)\mathbf{v}^{(i)}.$$

Some of the $\alpha_i^+ - \alpha_i^- < 0$ must be smaller than zero, because the element $\mathbf{c} \in \mathcal{C}$ would otherwise be contained in the ordering cone. Now we take an arbitrary element $\mathbf{y} \in \mathcal{S}$ from the polytope. By the definition of the set $\mathcal{S}$ the element $\mathbf{y}$ is contained in the ordering cone. Hence, it can be written as

$$\mathbf{y} \;=\; \sum_{i=1}^{n} \bar{\alpha}_i \mathbf{v}^{(i)}$$

for $\bar{\alpha}_i \in \mathbb{R}_+$. *Due to the representation of* $\mathcal{C}^+(\Delta\mathbf{y})$

$$\mathbf{y} + \beta\mathbf{c} \;=\; \sum_{i=1}^{n} \big(\bar{\alpha} + \beta\big(\alpha_i^+ - \alpha_i^-\big)\big)\mathbf{v}^{(i)} \;\in\; \mathcal{C}^+(\Delta\mathbf{y}) \tag{2.11}$$

is contained in the set for arbitrary $\beta \in \mathbb{R}_+$. *But however the representations of* $\mathbf{y}$ *and* $\mathbf{c}$ *are chosen, one of the coefficients in (2.11) will become negative for an appropriately large* $\tilde{\beta}$. *Hence, the resulting point*

$$\mathbf{y} + \tilde{\beta}\mathbf{c} \;\notin\; \mathcal{C} \;\supseteq\; \mathcal{C}^+(\Delta\mathbf{y})$$

is not contained in the set. Therefore, the cone difference $\tilde{\mathcal{C}} \setminus \mathcal{C} = \emptyset$ *is empty and thus the representation cone* $\tilde{\mathcal{C}} \subseteq \mathcal{C}$ *contained in the ordering cone.*

Let $\mathbf{y} \in \mathcal{C}^+(\Delta\mathbf{y})$ *be an arbitrary element of the set. By the above arguments it can be represented as*

$$\mathbf{y} \;=\; \mathbf{s} + \mathbf{c}, \quad \mathbf{s} \in \mathcal{S}, \mathbf{c} \in \mathcal{C}.$$

But $\mathbf{s}$ *dominates* $\mathbf{y}$ *whenever the cone element* $\mathbf{c} \neq \mathbf{0}$ *is nonzero. Thus, the efficient set*

$$eff_{\mathcal{C}}\big(\mathcal{C}^+(\Delta\mathbf{y})\big) \;\subseteq\; \mathcal{S}$$

is a subset of the polytope and therefore bounded.

□

Hence, for a polyhedral ordering cone $\mathcal{C}$ the assumptions of lemma 2.3.6 are sufficient for the infimum in (2.3.3) to be attained. But polyhedral ordering cones are exactly the class that we will consider in chapter 3.

As a next step we will derive that the $\mathcal{C}$-consistency of s in its different occurrences implies the corresponding $\mathcal{C}$-monotonicity of t_s.

Definition 2.3.10 *A function* $f : \mathbb{Y} \to \mathbb{R}$ *is called* $\mathcal{C}$*-monotone, if*

$$f(\Delta\mathbf{y}) \leq f(\Delta\mathbf{y} + \mathbf{c}) \quad \textit{for every } \mathbf{c} \in \mathcal{C}.$$

It is called strictly $\mathcal{C}$*-monotone, if*

$$f(\Delta\mathbf{y}) < f(\Delta\mathbf{y} + \mathbf{c}) \quad \textit{for every } \mathbf{c} \in int(\mathcal{C}).$$

It is called strongly $\mathcal{C}$*-monotone, if*

$$f(\Delta\mathbf{y}) < f(\Delta\mathbf{y} + \mathbf{c}) \quad \textit{for every } \mathbf{c} \in \mathcal{C} \setminus \{\mathbf{0}\}.$$

Note that continuous, strongly $\mathcal{C}$-monotone functions f are also strictly $\mathcal{C}$-monotone. Analogously, continuous, strictly $\mathcal{C}$-monotone functions are also $\mathcal{C}$-monotone.

Lemma 2.3.11 *For a continuous, $\mathcal{C}$-consistent, and $\mathcal{C}$-centred s the function t_s is $\mathcal{C}$-monotone. If s is even strictly or strongly $\mathcal{C}$-consistent, the function t_s is strictly or strongly $\mathcal{C}$-monotone respectively.*

Proof:

Let $\Delta\bar{\mathbf{y}}, \Delta\tilde{\mathbf{y}} \in \mathbb{Y}$ and $\tilde{\mathbf{c}} \in \mathcal{C}$ such that $\Delta\bar{\mathbf{y}} = \Delta\tilde{\mathbf{y}} + \tilde{\mathbf{c}}$. Then

$$\begin{aligned} t_s(\Delta\bar{\mathbf{y}}) &= \min\left\{ s\big(\mathbf{c}^+, \mathbf{c}^+ - \Delta\bar{\mathbf{y}}\big) \;\middle|\; \mathbf{c}^+ \in \mathcal{C}^+(\Delta\bar{\mathbf{y}}) \right\} \\ &= \min\left\{ s\big(\mathbf{c}^+, \mathbf{c}^+ - \Delta\tilde{\mathbf{y}} - \tilde{\mathbf{c}}\big) \;\middle|\; \mathbf{c}^+ \in \mathcal{C}^+(\Delta\bar{\mathbf{y}}) \right\} \\ &\geq \min\left\{ s\big(\mathbf{c}^+, \mathbf{c}^+ - \Delta\tilde{\mathbf{y}} - \tilde{\mathbf{c}} + \tilde{\mathbf{c}}\big) \;\middle|\; \mathbf{c}^+ \in \mathcal{C}^+(\Delta\bar{\mathbf{y}}) \right\}. \end{aligned} \tag{2.12}$$

So far we exploited the $\mathcal{C}$-consistency of s. Now we will use the representation of $\mathcal{C}^+(\Delta\bar{\mathbf{y}})$ to continue the chain of inequalities.

$$\begin{aligned} t_s(\Delta\bar{\mathbf{y}}) &\geq \min\left\{ s\big(\mathbf{c}^+, \mathbf{c}^+ - \Delta\tilde{\mathbf{y}}\big) \;\middle|\; \mathbf{c}^+ \in \big(\Delta\bar{\mathbf{y}} + \mathcal{C}\big) \cap \mathcal{C} \right\} \\ &= \min\left\{ s\big(\mathbf{c}^+, \mathbf{c}^+ - \Delta\tilde{\mathbf{y}}\big) \;\middle|\; \mathbf{c}^+ \in \big(\Delta\tilde{\mathbf{y}} + \tilde{\mathbf{c}} + \mathcal{C}\big) \cap \mathcal{C} \right\} \\ &\geq \min\left\{ s\big(\mathbf{c}^+, \mathbf{c}^+ - \Delta\tilde{\mathbf{y}}\big) \;\middle|\; \mathbf{c}^+ \in \big(\Delta\tilde{\mathbf{y}} + \mathcal{C}\big) \cap \mathcal{C} \right\} \\ &= t_s(\Delta\tilde{\mathbf{y}}), \end{aligned}$$

because $\Delta\tilde{\mathbf{y}} + \tilde{\mathbf{c}} + \mathcal{C} \subseteq \Delta\tilde{\mathbf{y}} + \mathcal{C}$.

Note that for strictly $\mathcal{C}$-consistent s and $\tilde{\mathbf{c}} \in int(\mathcal{C})$ the inequality (2.12) is strict and t_s therefore strictly $\mathcal{C}$-monotone. For strongly $\mathcal{C}$-consistent s and $\tilde{\mathbf{c}} \in \mathcal{C} \setminus \{\mathbf{0}\}$ the inequality (2.12) is again strict and thus t_s is strongly $\mathcal{C}$-monotone in that case.

□

$\mathcal{C}$-monotonicity is an essential property for the function t_s, since t_s is consistent with the partial order in this case, i.e. dominated points have larger values than the ones dominating them.

We can now combine the $\mathcal{C}$-monotonicity with the order interval construction (2.5) to construct lower and upper bounds. Applying a $\mathcal{C}$-monotone function to the anchor points of an order interval for two given points, we get a common lower and upper bound for the function values of the original points. We will utilise this construction in the subsequent lemma to prove the continuity of t_s.

Lemma 2.3.12 *For a continuous, $\mathcal{C}$-consistent and $\mathcal{C}$-centred s the distance evaluation function t_s is continuous.*

Proof:

Let $\left(\Delta\mathbf{y}^{(i)}\right)_{i\in\mathbb{N}}$ be a sequence with $\lim\limits_{i\to\infty} \Delta\mathbf{y}^{(i)} = \Delta\mathbf{y}$ and let

$$\Delta\check{\mathbf{y}}^{(i)} := \Delta\mathbf{y} - \mathbf{c}^{\oplus}\big(\Delta\mathbf{y} - \Delta\mathbf{y}^{(i)}\big) = \Delta\mathbf{y}^{(i)} - \mathbf{c}^{\ominus}\big(\Delta\mathbf{y} - \Delta\mathbf{y}^{(i)}\big) \in \mathbb{Y}$$

and

$$\Delta\hat{\mathbf{y}}^{(i)} := \Delta\mathbf{y} + \mathbf{c}^{\ominus}\big(\Delta\mathbf{y} - \Delta\mathbf{y}^{(i)}\big) = \Delta\mathbf{y}^{(i)} + \mathbf{c}^{\oplus}\big(\Delta\mathbf{y} - \Delta\mathbf{y}^{(i)}\big) \in \mathbb{Y}$$

be the upper and lower anchor point of the order interval enclosing $\Delta\mathbf{y}$ *and* $\Delta\mathbf{y}^{(i)}$*. The anchor points form a common upper and lower bound*

$$\begin{array}{lllll} \Delta\check{\mathbf{y}}^{(i)} & \leq_{\mathcal{C}} & \Delta\mathbf{y} & \leq_{\mathcal{C}} & \Delta\hat{\mathbf{y}}^{(i)} \quad \textit{and} \\ \Delta\check{\mathbf{y}}^{(i)} & \leq_{\mathcal{C}} & \Delta\mathbf{y}^{(i)} & \leq_{\mathcal{C}} & \Delta\hat{\mathbf{y}}^{(i)}. \end{array}$$

for the original points in the lattice $\big(\mathbb{Y}, \leq_{\mathcal{C}}\big)$*. By the* $\mathcal{C}$*-monotonicity of* t_s *it follows that their* t_s*-values*

$$\begin{array}{lllll} t_s\big(\Delta\check{\mathbf{y}}^{(i)}\big) & \leq & t_s\big(\Delta\mathbf{y}\big) & \leq & t_s\big(\Delta\hat{\mathbf{y}}^{(i)}\big) \quad \textit{and} \\ t_s\big(\Delta\check{\mathbf{y}}^{(i)}\big) & \leq & t_s\big(\Delta\mathbf{y}^{(i)}\big) & \leq & t_s\big(\Delta\hat{\mathbf{y}}^{(i)}\big) \end{array}$$

are an upper and lower bound for the t_s*-values of the two given points. Choosing some minimal representation* $\check{\mathbf{c}}^{+}(i), \check{\mathbf{c}}^{-}(i) \in \mathcal{C}$ *for the lower anchor point*

$$\Delta\check{\mathbf{y}}^{(i)} = \check{\mathbf{c}}^{+}(i) - \check{\mathbf{c}}^{-}(i) \quad \textit{and} \quad s\big(\check{\mathbf{c}}^{+}(i), \check{\mathbf{c}}^{-}(i)\big) = t_s\big(\Delta\check{\mathbf{y}}^{(i)}\big).$$

and abbreviating the $\mathbf{c}^{\oplus}$ *and* $\mathbf{c}^{\ominus}$ *function values by*

$$\bar{\mathbf{c}}^{+}(i) := \mathbf{c}^{\oplus}\big(\Delta\mathbf{y} - \Delta\mathbf{y}^{(i)}\big) \quad \textit{and} \quad \bar{\mathbf{c}}^{-}(i) := \mathbf{c}^{\ominus}\big(\Delta\mathbf{y} - \Delta\mathbf{y}^{(i)}\big)$$

we can write the upper anchor point as

$$\Delta\hat{\mathbf{y}}^{(i)} = \Delta\check{\mathbf{y}}^{(i)} + \bar{\mathbf{c}}^{+}(i) + \bar{\mathbf{c}}^{-}(i).$$

The minimality of $t_s\big(\Delta\hat{\mathbf{y}}^{(i)}\big)$ *now yields the following inequality:*

$$\begin{array}{rcl} t_s\big(\Delta\hat{\mathbf{y}}^{(i)}\big) & = & \min\big\{s\big(\mathbf{c}^{+},\, \mathbf{c}^{+} - \Delta\hat{\mathbf{y}}^{(i)}\big) \;\big|\; \mathbf{c}^{+} \in \mathcal{C}^{+}\big(\Delta\hat{\mathbf{y}}^{(i)}\big)\big\} \\ & \leq & s\big(\check{\mathbf{c}}^{+}(i) + \bar{\mathbf{c}}^{+}(i) + \bar{\mathbf{c}}^{-}(i),\, \check{\mathbf{c}}^{-}(i)\big). \end{array}$$

Thus, $t_s\big(\Delta\mathbf{y}\big)$ *and* $t_s\big(\Delta\mathbf{y}^{(i)}\big)$ *are bounded by*

$$s\big(\check{\mathbf{c}}^{+}(i),\, \check{\mathbf{c}}^{-}(i)\big) \;\leq\; \min\big\{t_s\big(\Delta\mathbf{y}\big), t_s\big(\Delta\mathbf{y}^{(i)}\big)\big\} \tag{2.13}$$

from below and by

$$\max\big\{t_s\big(\Delta\mathbf{y}\big), t_s\big(\Delta\mathbf{y}^{(i)}\big)\big\} \;\leq\; s\big(\check{\mathbf{c}}^{+}(i) + \bar{\mathbf{c}}^{+}(i) + \bar{\mathbf{c}}^{-}(i),\, \check{\mathbf{c}}^{-}(i)\big) \tag{2.14}$$

from above. Just the first argument differs for the bounds $s\big(\check{\mathbf{c}}^{+}(i),\, \check{\mathbf{c}}^{-}(i)\big)$ *and* $s\big(\check{\mathbf{c}}^{+}(i) + \bar{\mathbf{c}}^{+}(i) + \bar{\mathbf{c}}^{-}(i),\, \check{\mathbf{c}}^{-}(i)\big)$*. This difference is bounded*

$$\begin{array}{rcl} \big\|\check{\mathbf{c}}^{+}(i) + \bar{\mathbf{c}}^{+}(i) + \bar{\mathbf{c}}^{-}(i) - \check{\mathbf{c}}^{+}(i)\big\|_2 & = & \big\|\bar{\mathbf{c}}^{+}(i) + \bar{\mathbf{c}}^{-}(i)\big\|_2 \\ & \overset{(2.6)}{\leq} & \big\|\Delta\mathbf{y} - \Delta\mathbf{y}^{(i)}\big\|_2. \end{array}$$

by the distance of the i^{th} member of the sequence to its limit. Thus, $t_s\big(\Delta\mathbf{y}\big)$ and $t_s\big(\Delta\mathbf{y}^{(i)}\big)$ are bounded by s-values for two points, the distance of which is 0 in one component and smaller than or equal to $\big\|\Delta\mathbf{y}-\Delta\mathbf{y}^{(i)}\big\|_2$ in the other component. Because the function s is continuous in both arguments and $\lim\limits_{i\to\infty}\big\|\Delta\mathbf{y}-\Delta\mathbf{y}^{(i)}\big\|_2=0$ the upper and lower bound in (2.13) and (2.14) converge to the same value and thus

$$\lim_{i\to\infty} t_s\big(\Delta\mathbf{y}^{(i)}\big) = t_s(\Delta\mathbf{y}).$$

□

For Lipschitz-continuous s we can directly extend the above result. Here, the arguments will be similar, because the above reasoning is already centred around distances.

Lemma 2.3.13 *For a Lipschitz-continuous, $\mathcal{C}$-consistent and $\mathcal{C}$-centred s the function t_s is Lipschitz-continuous.*

Proof:

Let two distances $\Delta\mathbf{y},\Delta\bar{\mathbf{y}}\in\mathbb{Y}$ be given. Then set

$$\bar{\mathbf{c}}^+ := \mathbf{c}^{\oplus}\big(\Delta\mathbf{y}-\Delta\bar{\mathbf{y}}\big) \quad \textit{and} \quad \bar{\mathbf{c}}^- := \mathbf{c}^{\ominus}\big(\Delta\mathbf{y}-\Delta\bar{\mathbf{y}}\big).$$

Furthermore, let $\tilde{\mathbf{c}}^+,\,\tilde{\mathbf{c}}^-\in\mathcal{C}$ be an optimal representation

$$s\big(\tilde{\mathbf{c}}^+,\,\tilde{\mathbf{c}}^-\big) = t_s\big(\tilde{\mathbf{c}}^+-\tilde{\mathbf{c}}^-\big) \quad \textit{and} \quad \tilde{\mathbf{c}}^+-\tilde{\mathbf{c}}^- = \Delta\mathbf{y}-\bar{\mathbf{c}}^+ = \Delta\bar{\mathbf{y}}-\bar{\mathbf{c}}^-$$

for the lower end of the order interval. From the proof of lemma 2.3.12 we know that the two values can be bounded from both sides

$$\begin{aligned} t_s\big(\Delta\mathbf{y}-\bar{\mathbf{c}}^+\big) &\leq \min\big\{t_s(\Delta\mathbf{y}),t_s(\Delta\bar{\mathbf{y}})\big\} \\ &\leq \max\big\{t_s(\Delta\mathbf{y}),t_s(\Delta\bar{\mathbf{y}})\big\} \;\leq\; t_s\big(\Delta\mathbf{y}+\bar{\mathbf{c}}^-\big) \end{aligned}$$

by the t_s values of the order interval anchor points. Thus, we can rewrite their difference as

$$\begin{aligned} &\big|\,t_s(\Delta\mathbf{y})-t_s(\Delta\bar{\mathbf{y}})\,\big| \\ &\quad= \max\big\{t_s(\Delta\mathbf{y}),t_s(\Delta\bar{\mathbf{y}})\big\}-\min\big\{t_s(\Delta\mathbf{y}),t_s(\Delta\bar{\mathbf{y}})\big\} \\ &\quad\leq t_s\big(\Delta\mathbf{y}+\bar{\mathbf{c}}^-\big)-t_s\big(\Delta\mathbf{y}-\bar{\mathbf{c}}^+\big) \\ &\quad= \min\big\{s\big(\mathbf{c}^+,\mathbf{c}^-\big)\;\big|\;\Delta\mathbf{y}+\bar{\mathbf{c}}^-=\mathbf{c}^+-\mathbf{c}^-,\;\mathbf{c}^+,\mathbf{c}^-\in\mathcal{C}\big\}-s\big(\tilde{\mathbf{c}}^+,\tilde{\mathbf{c}}^-\big) \\ &\quad= \min\big\{s\big(\mathbf{c}^+,\mathbf{c}^-\big)\;\big|\;\tilde{\mathbf{c}}^+-\tilde{\mathbf{c}}^-+\bar{\mathbf{c}}^++\bar{\mathbf{c}}^-=\mathbf{c}^+-\mathbf{c}^-,\;\mathbf{c}^+,\mathbf{c}^-\in\mathcal{C}\big\} \\ &\qquad -s\big(\tilde{\mathbf{c}}^+,\tilde{\mathbf{c}}^-\big). \end{aligned}$$

The s-value of any specific realisation of the upper anchor point

$$\begin{aligned} &\min\big\{s\big(\mathbf{c}^+,\mathbf{c}^-\big)\;\big|\;\tilde{\mathbf{c}}^+-\tilde{\mathbf{c}}^-+\bar{\mathbf{c}}^++\bar{\mathbf{c}}^-=\mathbf{c}^+-\mathbf{c}^-,\;\mathbf{c}^+,\mathbf{c}^-\in\mathcal{C}\big\} \\ &\qquad\qquad\leq\; s\big(\tilde{\mathbf{c}}^++\bar{\mathbf{c}}^++\bar{\mathbf{c}}^-,\;\tilde{\mathbf{c}}^-\big) \end{aligned}$$

is bigger than or equal to the minimal one. Hence, for L_s being the Lipschitz constant of the function s we can estimate the difference by

$$\begin{aligned} \big|\, t_s(\Delta\mathbf{y}) - t_s(\Delta\bar{\mathbf{y}}) \,\big| &\leq s\big(\check{\mathbf{c}}^+ + \bar{\mathbf{c}}^+ + \bar{\mathbf{c}}^-,\ \check{\mathbf{c}}^-\big) - s\big(\check{\mathbf{c}}^+,\ \check{\mathbf{c}}^-\big) \\ &\leq L_s \,\big\|\, \bar{\mathbf{c}}^+ + \bar{\mathbf{c}}^- \,\big|_2 \\ &\overset{2.6}{\leq} L_s \,\big\|\, \Delta\mathbf{y} - \Delta\bar{\mathbf{y}} \,\big\|_2, \end{aligned}$$

where we use the bound on the sum of $\mathbf{c}^\oplus$ and $\mathbf{c}^\ominus$ in the last step.

□

The next step in our investigation deals with the transfer of different variants of convexity from s to t_s. So let us first recall the definition of quasiconvexity.

Definition 2.3.14 *A function $f\colon \mathcal{Y} \subseteq \mathbb{R}^n \to \mathbb{R}$ is called quasiconvex, if for $\check{\mathbf{y}}, \hat{\mathbf{y}} \in \mathcal{Y}$*

$$f\big((1-\lambda)\,\check{\mathbf{y}} + \lambda\,\hat{\mathbf{y}}\big) \ \leq\ \max\{f(\check{\mathbf{y}}),\ f(\hat{\mathbf{y}})\} \quad \textit{for all}\ \ \lambda \in (0,1)$$

or equivalently, if the lower level sets

$$\{\mathbf{y} \in \mathcal{Y} \mid f(\mathbf{y}) \leq \tau\}$$

are convex for every $\tau \in \mathbb{R}$ (see [5]). The function is called strictly quasiconvex, when the inequality is strict for all $\lambda \in (0,1)$.

Note that for continuous functions strict quasiconvexity implies quasiconvexity. Quasiconvexity or strict quasiconvexity of the cone scalarising function s devolves to the distance evaluation function t_s.

Proposition 2.3.15 *For a continuous, $\mathcal{C}$-consistent, $\mathcal{C}$-centred and quasiconvex s the function t_s is quasiconvex. If, additionally, s is strictly quasiconvex, the distance evaluation function t_s is strictly quasiconvex, too.*

Proof:

The requirements on the cone scalarising function s imply the continuity and $\mathcal{C}$-monotonicity of the distance evaluation function t_s. Take two points $\check{\mathbf{y}}, \hat{\mathbf{y}} \in \mathbb{Y}$ in the outcome space and a $\lambda \in (0,1)$. Let us denote the convex combination of two elements by $conv_\lambda(\mathbf{z}, \bar{\mathbf{z}}) := (1-\lambda)\,\mathbf{z} + \lambda\,\bar{\mathbf{z}}$. Furthermore, let $\check{\mathbf{c}}^+, \check{\mathbf{c}}^-$ and $\hat{\mathbf{c}}^+, \hat{\mathbf{c}}^-$ be s minimal representations of the points $\check{\mathbf{y}}$ and $\hat{\mathbf{y}}$, respectively. Then

$$\begin{aligned} & t_s\big(conv_\lambda(\check{\mathbf{y}}, \hat{\mathbf{y}})\big) \\ &= \min\big\{s\big(\mathbf{c}^+, \mathbf{c}^-\big) \,\big|\, conv_\lambda(\check{\mathbf{y}}, \hat{\mathbf{y}}) = \mathbf{c}^+ - \mathbf{c}^-,\ \mathbf{c}^+, \mathbf{c}^- \in \mathcal{C}\big\} \\ &= \min\big\{s\big(\mathbf{c}^+, \mathbf{c}^-\big) \,\big|\, conv_\lambda\big(\check{\mathbf{c}}^+, \hat{\mathbf{c}}^+\big) - conv_\lambda\big(\check{\mathbf{c}}^-, \hat{\mathbf{c}}^-\big) = \mathbf{c}^+ - \mathbf{c}^-, \\ &\qquad\qquad \mathbf{c}^+, \mathbf{c}^- \in \mathcal{C}\big\} \\ &\leq s\big(conv_\lambda\big(\check{\mathbf{c}}^+, \hat{\mathbf{c}}^+\big),\ conv_\lambda\big(\check{\mathbf{c}}^-, \hat{\mathbf{c}}^-\big)\big). \end{aligned}$$

Due to the quasiconvexity of s, we can continue the inequality with

$$\begin{aligned} t_s\big(conv_\lambda(\check{\mathbf{y}}, \hat{\mathbf{y}})\big) &\leq \max\big\{s\big(\check{\mathbf{c}}^+, \check{\mathbf{c}}^-\big),\ s\big(\hat{\mathbf{c}}^+, \hat{\mathbf{c}}^-\big)\big\} \\ &= \max\big\{t_s\big(\check{\mathbf{y}}\big),\ t_s\big(\hat{\mathbf{y}}\big)\big\}. \end{aligned} \tag{2.15}$$

Thus, t_s is quasiconvex. For a strictly quasiconvex s the inequality (2.15) is strict, if the points are different. Hence, t_s is strictly quasiconvex.
Note that the continuity of t_s rules out the possibility that t_s is strictly quasiconvex, but not quasiconvex ([5]).

□

The last two properties we are going to look into in this section are convexity and strict convexity. We will show that (strict) convexity of s implies (strict) convexity of t_s.

Lemma 2.3.16 *If s is continuous, $\mathcal{C}$-consistent, $\mathcal{C}$-centred and convex, then t_s is convex as well. If s is even strictly convex, then t_s is strictly convex as well.*

Proof:

For two arbitrary points $\Delta\hat{\mathbf{y}}, \Delta\bar{\mathbf{y}} \in \mathbb{Y}$ let $\Delta\mathbf{y}_\lambda := conv_\lambda(\Delta\hat{\mathbf{y}}, \Delta\bar{\mathbf{y}})$. Then

$$\begin{aligned} t_s(\Delta\mathbf{y}_\lambda) &= \min\big\{\, s\big(\mathbf{c}^+, \mathbf{c}^-\big) \;\big|\; \Delta\mathbf{y}_\lambda = \mathbf{c}^+ - \mathbf{c}^-,\ \mathbf{c}^+, \mathbf{c}^- \in \mathcal{C} \,\big\} \\ &= \min\big\{\, s\big(\mathbf{c}^+, \mathbf{c}^-\big) \;\big|\; conv_\lambda(\Delta\hat{\mathbf{y}}, \Delta\bar{\mathbf{y}}) = \mathbf{c}^+ - \mathbf{c}^-,\ \mathbf{c}^+, \mathbf{c}^- \in \mathcal{C}\big\} \\ &= \min\big\{\, s\big(conv_\lambda\big(\hat{\mathbf{c}}^+, \bar{\mathbf{c}}^+\big), conv_\lambda\big(\hat{\mathbf{c}}^-, \bar{\mathbf{c}}^-\big)\big) \;\big| \\ &\qquad conv_\lambda(\Delta\hat{\mathbf{y}}, \Delta\bar{\mathbf{y}}) = conv_\lambda\big(\hat{\mathbf{c}}^+, \bar{\mathbf{c}}^+\big) - conv_\lambda\big(\hat{\mathbf{c}}^-, \bar{\mathbf{c}}^-\big), \\ &\qquad \hat{\mathbf{c}}^+, \bar{\mathbf{c}}^+, \hat{\mathbf{c}}^-, \bar{\mathbf{c}}^- \in \mathcal{C}\big\} \end{aligned}$$

Here, we use the convexity of the ordering cone, that implies the equality $\mathcal{C} = \mathcal{C} + \mathcal{C}$. Thus, the effective domain of minimisation is unchanged by the splitting of variables. If we now split the equation in the minimisation into two

$$\begin{aligned} &\Delta\hat{\mathbf{y}} = \hat{\mathbf{c}}^+ - \hat{\mathbf{c}}^- \ \textit{ and } \ \Delta\bar{\mathbf{y}} = \bar{\mathbf{c}}^+ - \bar{\mathbf{c}}^- \\ &\Longrightarrow conv_\lambda(\Delta\hat{\mathbf{y}}, \Delta\bar{\mathbf{y}}) = conv_\lambda\big(\hat{\mathbf{c}}^+, \bar{\mathbf{c}}^+\big) - conv_\lambda\big(\hat{\mathbf{c}}^-, \bar{\mathbf{c}}^-\big) \end{aligned}$$

the two equations imply the single one, but not vice versa. Thus, we are reducing the feasible area by this split. Therefore, we can continue the estimate

with

$$\begin{aligned} t_s(\Delta\mathbf{y}_\lambda) \quad &\leq \quad \min\{\, s\big(conv_\lambda(\hat{\mathbf{c}}^+,\, \bar{\mathbf{c}}^+), conv_\lambda(\hat{\mathbf{c}}^-,\, \bar{\mathbf{c}}^-)\big) \mid \\ &\qquad\qquad \Delta\hat{\mathbf{y}} = \hat{\mathbf{c}}^+ - \hat{\mathbf{c}}^-,\ \Delta\bar{\mathbf{y}} = \bar{\mathbf{c}}^+ - \bar{\mathbf{c}}^-, \\ &\qquad\qquad \hat{\mathbf{c}}^+,\, \bar{\mathbf{c}}^+,\, \hat{\mathbf{c}}^-,\, \bar{\mathbf{c}}^- \in \mathcal{C}\} \\ &\overset{s\ convex}{\leq} \quad \min\{\, conv_\lambda\big(s(\hat{\mathbf{c}}^+,\, \hat{\mathbf{c}}^-), s(\bar{\mathbf{c}}^+,\, \bar{\mathbf{c}}^-)\big) \mid\ \Delta\hat{\mathbf{y}} = \hat{\mathbf{c}}^+ - \hat{\mathbf{c}}^-, \\ &\qquad\qquad \Delta\bar{\mathbf{y}} = \bar{\mathbf{c}}^+ - \bar{\mathbf{c}}^-,\ \hat{\mathbf{c}}^+,\, \bar{\mathbf{c}}^+,\, \hat{\mathbf{c}}^-,\, \bar{\mathbf{c}}^- \in \mathcal{C}\} \\ &= \quad (1-\lambda)\min\big\{s(\hat{\mathbf{c}}^+,\, \hat{\mathbf{c}}^-) \mid\ \Delta\hat{\mathbf{y}} = \hat{\mathbf{c}}^+ - \hat{\mathbf{c}}^-,\ \hat{\mathbf{c}}^+,\, \hat{\mathbf{c}}^- \in \mathcal{C}\big\} \\ &\qquad + \lambda\ \ \min\big\{s(\bar{\mathbf{c}}^+,\, \bar{\mathbf{c}}^-) \mid\ \Delta\bar{\mathbf{y}} = \bar{\mathbf{c}}^+ - \bar{\mathbf{c}}^-,\ \bar{\mathbf{c}}^+,\, \bar{\mathbf{c}}^- \in \mathcal{C}\big\} \\ &= \quad (1-\lambda)\, t_s(\Delta\hat{\mathbf{y}}) + \lambda\, t_s(\Delta\bar{\mathbf{y}}), \end{aligned}$$

using in essence the convexity of s and the independence of the minima after the split of the equation.

For strictly convex s the second inequality in the last estimate is strict for either $\hat{\mathbf{c}}^+ \neq \bar{\mathbf{c}}^+$ or $\hat{\mathbf{c}}^- \neq \bar{\mathbf{c}}^-$ and $\lambda \in (0,1)$. But if $\Delta\hat{\mathbf{y}} \neq \Delta\bar{\mathbf{y}}$ the two representations by cone elements have to differ, too. Thus, the second inequality is strict, if $\Delta\hat{\mathbf{y}} \neq \Delta\bar{\mathbf{y}}$ and $\lambda \in (0,1)$. Therefore, t_s is strictly convex.

□

We have investigated how properties of the cone scalarising function s transfer to the distance evaluation function t_s. Even though the distance evaluation function is the tool that is employed in most of the forthcoming proofs there is a benefit of defining the cone scalarising problem as the minimum of s: If s is defined such that the definition works for all closed, convex and pointed ordering cones $\mathcal{C} \supseteq \mathbb{Y}_+$, then changing the cone most likely changes t_s, whereas s remains stable.

The next section analyses the outcomes of the optimisation problems (2.7) and (2.8). Here, we will build upon the foundation laid in the current section.

2.4 Cone scalarising outcomes

The aim of setting up the cone scalarising function framework is to have a class of functions for calculating (weakly / properly) $\mathcal{C}$-efficient solutions. Therefore, we will analyse the results of the cone scalarising problem and investigate the potential range of outcomes.

We will show that for appropriate choices of s the cone scalarising problem (2.7) yields weakly $\mathcal{C}$-efficient solutions.

Theorem 2.4.1 *Let s be a continuous, strongly (strictly) $\mathcal{C}$-consistent, and $\mathcal{C}$-centred function. If $\hat{\mathbf{y}}$ is a minimum of*

$$\inf\left\{s(\mathbf{c}^+,\, \mathbf{c}^-) \ \mid\ \mathbf{y} - \mathbf{y}^R = \mathbf{c}^+ - \mathbf{c}^-,\ \mathbf{y} \in \mathcal{Y},\ \mathbf{c}^+,\, \mathbf{c}^- \in \mathcal{C}\right\},$$

then it is (weakly) $\mathcal{C}$-efficient.

Proof:

We will first prove the strictly $\mathcal{C}$-consistent case. Recall that

$$\begin{aligned} & \inf\{s(\mathbf{c}^+, \mathbf{c}^-) \mid \mathbf{y} - \mathbf{y}^R = \mathbf{c}^+ - \mathbf{c}^-,\ \mathbf{y} \in \mathcal{Y},\ \mathbf{c}^+, \mathbf{c}^- \in \mathcal{C}\} \\ = & \inf\{t_s(\mathbf{y} - \mathbf{y}^R) \mid \mathbf{y} \in \mathcal{Y}\} \end{aligned}$$

Assume that there is a $\bar{\mathbf{y}} \in \mathcal{Y}$ and a $\bar{\mathbf{c}} \in int(\mathcal{C})$ with $\bar{\mathbf{y}} + \bar{\mathbf{c}} = \hat{\mathbf{y}}$. Since t_s is strictly $\mathcal{C}$-monotone,

$$t_s(\hat{\mathbf{y}}) = t_s(\bar{\mathbf{y}} + \bar{\mathbf{c}}) \ > \ t_s(\bar{\mathbf{y}})$$

contradicting the minimality of $\hat{\mathbf{y}}$.

Substituting $int(\mathcal{C})$ by $\mathcal{C} \setminus \{\mathbf{0}\}$ and using strong $\mathcal{C}$-monotonicity yields the proof for the strongly $\mathcal{C}$-consistent case.

□

For the efficiency proof we neither need convexity of s or t_s nor the convexity of the outcome set $\mathcal{Y}$. So to guarantee (weak) efficiency, strict or strong $\mathcal{C}$-monotonicity is the key ingredient. However, to ensure that we can *reach* the different (weakly) $\mathcal{C}$-efficient outcomes we need stronger requirements. Let us denote the level sets of the function t_s for the given order relation by

$$\mathcal{L}_\diamond(t_s, \tau) := \{\mathbf{y} \in \mathbb{Y} \mid t_s(\mathbf{y}) \diamond \tau\} \quad \text{for} \quad \diamond \in \{<, \leq, =\}.$$

We will now define two classes of cone scalarising functions the level sets of which (locally) approximate or represent the ordering cone. By this they (approximately) encode it into a function. In analogy to Wierzbicki ([125]) we will call them *$\mathcal{C}$-representing* and *$\mathcal{C}$-approximating* functions.

Definition 2.4.2 *The function s is called strictly $\mathcal{C}$-representing at $\mathbf{y} \in \mathbb{Y}$ and $\tau \in \mathbb{R}$, if it is strictly $\mathcal{C}$-consistent, $\mathcal{C}$-centred, continuous, and*

$$\mathbf{y} - int(\mathcal{C}) \quad = \quad \mathcal{L}_<(t_s, \tau).$$

s is called strictly $\mathcal{C}$-approximating at $\mathbf{y} \in \mathbb{Y}$ and $\tau \in \mathbb{R}$, if it is strongly $\mathcal{C}$-consistent, $\mathcal{C}$-centred, continuous, and

$$\mathbf{y} - \big(\mathcal{C} \setminus \{\mathbf{0}\}\big) \quad \subseteq \quad \mathcal{L}_<(t_s, \tau) \quad \subseteq \quad \mathbf{y} - int\big(\mathcal{C}(\mathbf{c}^*, \varepsilon)\big)$$

for $0 \leq \varepsilon < 1$ and $\mathbf{c}^* \in int(\mathcal{C}^*)$, where

$$\begin{aligned} \mathcal{C} \quad &: \quad int(\mathcal{C}^*) \times \mathbb{R}_+ \to \mathfrak{P}(\mathbb{Y}) \\ \mathcal{C}(\mathbf{c}^*, \varepsilon) \quad &:= \quad \big\{\mathbf{c} \in \mathbb{Y} \mid \mathbf{c} = \mathbf{c}^+ - \mathbf{c}^-,\ \textit{for some}\ \ \mathbf{c}^+, \mathbf{c}^- \in \mathcal{C}, \\ & \qquad\qquad\qquad\qquad \textit{with}\ \ \langle \mathbf{c}^*, \varepsilon\, \mathbf{c}^+ - \mathbf{c}^- \rangle \geq 0\big\}. \end{aligned}$$

If we want to stress that the representation or approximation property is only guaranteed for one point we add the attribute locally*.*

The definition ensures that at least one level set of the distance evaluation function t_s looks like the shifted negative ordering cone or a similar cone. We will see soon that by shifting the reference point such that the level set touches the feasible region in a certain point, we can make that point the optimum of the optimisation for the shifted reference point.

Note that by the continuity of t_s in $\mathbb{Y}$ the interior of the level set for the weak inequality $int(\mathcal{L}_{\leq}(t_s, \tau)) = \mathcal{L}_{<}(t_s, \tau)$ equals the level set for the strict inequality. Furthermore, the closure of the level set for the strict inequality $cl(\mathcal{L}_{<}(t_s, \tau)) = \mathcal{L}_{\leq}(t_s, \tau)$ equals the level set for the weak inequality if in addition t_s is convex.

Before we use the definition to establish the reachability results, we will first show that the set $\mathcal{C}(\mathbf{c}^*, \varepsilon)$ is a closed, convex and pointed cone.

Proposition 2.4.3 *The set $\mathcal{C}(\mathbf{c}^*, \varepsilon)$ is a closed, convex and pointed cone, with $\mathcal{C} \subseteq \mathcal{C}(\mathbf{c}^*, \varepsilon)$, if $0 \leq \varepsilon < 1$.*

Proof:

Note that for $\mathbf{c} \in \mathbb{Y}$ and $\mathbf{c} = \mathbf{c}^+ - \mathbf{c}^-$

$$\langle \mathbf{c}^*, \varepsilon\, \mathbf{c}^+ - \mathbf{c}^- \rangle \geq 0 \quad \Longleftrightarrow \quad \langle \mathbf{c}^*, \varepsilon\, \alpha\, \mathbf{c}^+ - \alpha\, \mathbf{c}^- \rangle \geq 0$$

for every $\alpha \in \mathbb{R}_+$. Since $\mathcal{C}$ is a cone, the positive multiples $\alpha\, \mathbf{c}^+, \alpha\, \mathbf{c}^- \in \mathcal{C}$ are contained in the cone, so that we have a representation of $\alpha\, \mathbf{c}$ as a difference of two elements of $\mathcal{C}$. By the above considerations, this representation fulfils the inequality. Thus, for every element $\mathbf{c} \in \mathcal{C}(\mathbf{c}^*, \varepsilon)$ all the positive multiples $\alpha\, \mathbf{c} \in \mathcal{C}(\mathbf{c}^*, \varepsilon)$ are contained in the set and it is therefore a cone.

Because the ordering cone $\mathcal{C}$ is convex, the sum of two of its elements is again contained in the cone. Therefore, the sum of two elements $\mathbf{c}, \bar{\mathbf{c}} \in \mathcal{C}(\mathbf{c}^*, \varepsilon)$ can be represented by the sum of their individual representations

$$\mathbf{c} + \bar{\mathbf{c}} = \left(\mathbf{c}^+ + \bar{\mathbf{c}}^+\right) - \left(\mathbf{c}^- + \bar{\mathbf{c}}^-\right).$$

Since adding up the two inequalities for $\mathbf{c}$ and $\bar{\mathbf{c}}$ yields the necessary inequality, the sum of two elements of $\mathcal{C}(\mathbf{c}^*, \varepsilon)$ lies in $\mathcal{C}(\mathbf{c}^*, \varepsilon)$. Hence, the cone $\mathcal{C}(\mathbf{c}^*, \varepsilon)$ is convex.

Let $(\mathbf{c}(i))_{i \in \mathbb{N}}$ be a convergent sequence of elements in $\mathcal{C}(\mathbf{c}^*, \varepsilon)$ represented by $\mathbf{c}^+(i)$, $\mathbf{c}^-(i)$, $i \in \mathbb{N}$. From the equality $\mathbf{c}(i) = \mathbf{c}^+(i) - \mathbf{c}^-(i)$ and the defining constraint

$$\langle \mathbf{c}^*, \varepsilon\, \mathbf{c}^+(i) - \mathbf{c}^-(i) \rangle \geq 0 \quad \Longrightarrow \quad \varepsilon\, \langle \mathbf{c}^*, \mathbf{c}(i) \rangle \geq \langle \mathbf{c}^*, (1-\varepsilon)\mathbf{c}^-(i) \rangle \geq 0$$

we can derive a bound for the negative component. Taking the maximum of the inner products $0 \leq \max_{i \in \mathbb{N}} \{\langle \mathbf{c}^*, \mathbf{c}(i) \rangle\} =: M < \infty$, we can restrict $(\mathbf{c}^-(i))_{i \in \mathbb{N}}$ to the bounded and closed set

$$(\mathbf{c}^-(i))_{i \in \mathbb{N}} \subseteq \mathcal{C} \cap \{\mathbf{y} \mid \langle \mathbf{c}^*, (1-\varepsilon)\mathbf{y} \rangle \leq \varepsilon\, M\}.$$

Thus, there exist convergent subsequences $\bar{\mathbf{c}}^+(i)$, $\bar{\mathbf{c}}^-(i), i \in \mathbb{N}$. *As* $\mathcal{C}$ *is closed, the limits of the sequences lie in* $\mathcal{C}$ *and fulfil the defining inequality. Therefore, the limit of their difference*

$$\begin{aligned}\lim_{i\to\infty} \mathbf{c}(i) \;=\; \lim_{i\to\infty} \bar{\mathbf{c}}(i) \;&=\; \lim_{i\to\infty} \left(\bar{\mathbf{c}}^+(i) - \bar{\mathbf{c}}^-(i)\right) \\ &=\; \lim_{i\to\infty} \bar{\mathbf{c}}^+(i) - \lim_{i\to\infty} \bar{\mathbf{c}}^-(i) \;\in\; \mathcal{C}(\mathbf{c}^*, \varepsilon)\end{aligned}$$

can be represented by the difference of their limits and is by the above arguments contained in $\mathcal{C}(\mathbf{c}^*, \varepsilon)$. *Hence,* $\mathcal{C}(\mathbf{c}^*, \varepsilon)$ *is closed.*

Let $\mathbf{c} \in -\mathcal{C}(\mathbf{c}^*, \varepsilon) \cap \mathcal{C}(\mathbf{c}^*, \varepsilon)$ *be in the intersection of the cone and its negative. Then, by definition there exist representations* $\hat{\mathbf{c}}^+$, $\hat{\mathbf{c}}^- \in \mathcal{C}$ *for the negative and* $\bar{\mathbf{c}}^+$, $\bar{\mathbf{c}}^- \in \mathcal{C}$ *for the positive cone*

$$\mathbf{c} \;=\; \hat{\mathbf{c}}^+ - \hat{\mathbf{c}}^- \;=\; \bar{\mathbf{c}}^+ - \bar{\mathbf{c}}^-,$$

which fulfil

$$\langle \mathbf{c}^*, \varepsilon\, \hat{\mathbf{c}}^- - \hat{\mathbf{c}}^+ \rangle \geq 0 \quad \textit{and} \quad \langle \mathbf{c}^*, \varepsilon\, \bar{\mathbf{c}}^+ - \bar{\mathbf{c}}^- \rangle \geq 0.$$

Now we can rewrite the second arguments of the inner products as

$$\varepsilon\, \hat{\mathbf{c}}^- - \hat{\mathbf{c}}^+ = -(1-\varepsilon)\, \hat{\mathbf{c}}^+ - \varepsilon\, \mathbf{c} \quad \textit{and} \quad \varepsilon\, \bar{\mathbf{c}}^+ - \bar{\mathbf{c}}^- = -(1-\varepsilon)\, \bar{\mathbf{c}}^- + \varepsilon\, \mathbf{c},$$

where the representation of $\mathbf{c}$ *by the respective pair was used. Applying this reformulation and adding the above inequalities we get*

$$-(1-\varepsilon)\, \langle \mathbf{c}^*, \underbrace{\hat{\mathbf{c}}^+ + \bar{\mathbf{c}}^-}_{\in \mathcal{C}} \rangle \geq 0.$$

But by the choice of $\mathbf{c}^*$ *the inner product is greater than zero for every second argument taken from* $\mathcal{C} \setminus \{\mathbf{0}\}$. *Thus, for* $0 \leq \varepsilon < 1$ *the second argument* $\hat{\mathbf{c}}^+ + \bar{\mathbf{c}}^- = \mathbf{0}$ *and since* $\mathcal{C}$ *is pointed* $\hat{\mathbf{c}}^+ = \bar{\mathbf{c}}^- = \mathbf{0}$. *Substituting this into the representation equalities yields*

$$\mathbf{c} = -\hat{\mathbf{c}}^- = \bar{\mathbf{c}}^+.$$

Hence, $\mathbf{c} \in -\mathcal{C} \cap \mathcal{C} = \{\mathbf{0}\}$ *and since* $\mathbf{c}$ *was arbitrary, the intersection* $-\mathcal{C}(\mathbf{c}^*, \varepsilon) \cap \mathcal{C}(\mathbf{c}^*, \varepsilon) = \{\mathbf{0}\}$ *contains just the zero element.*

□

The cone $\mathcal{C}(\mathbf{c}^*, \varepsilon)$ is an enlarged version of the ordering cone $\mathcal{C}$. One may think of it as a mapping from $[0, 1]$ to the powerset of $\mathbb{Y}$, where ε is mapped to $\mathcal{C}(\mathbf{c}^*, \varepsilon)$. Then it describes a homotopy of the ordering cone $\mathcal{C}$ to the halfspace with inner normal vector $\mathbf{c}^*$ as ε goes from 0 to 1.

By their definition strictly $\mathcal{C}$-representing and strictly $\mathcal{C}$-approximating cone scalarising functions s are strictly respectively strongly $\mathcal{C}$-consistent. The

consequential strict or strong $\mathcal{C}$-monotonicity of the distance evaluation function t_s implies that the minimisations (2.7) or (2.8) yield weakly $\mathcal{C}$-efficient, respectively $\mathcal{C}$-efficient solutions as shown in theorem (2.4.1). We will now work out the range of solutions obtained by locally strictly $\mathcal{C}$-representing and strictly $\mathcal{C}$-approximating functions.

Let $\bar{\mathcal{C}}$ denote $\mathcal{C}$ in case of the $\mathcal{C}$-representing and $\mathcal{C}(\mathbf{c}^*, \varepsilon)$ in case of the $\mathcal{C}$-approximating functions.

Theorem 2.4.4 *Every weakly $\bar{\mathcal{C}}$-efficient point $\hat{\mathbf{y}} \in \mathcal{Y}$ is an optimum of the cone scalarising problem for some reference point $\mathbf{y}^R$ with a function t_s that is strictly $\mathcal{C}$-representing respectively $\mathcal{C}$-approximating at $\mathbf{y}^A$ and $b\tau = t_s(\mathbf{y}^A)$.*

Proof:

Choose $\mathbf{y}^R = \hat{\mathbf{y}} - \mathbf{y}^A$. The strict $\mathcal{C}$-representation or $\mathcal{C}$-approximation ensures that $-int(\bar{\mathcal{C}}) \supseteq -\mathbf{y}^A + \mathcal{L}_<(t_s, \tau)$. By the weak $\bar{\mathcal{C}}$-efficiency of $\hat{\mathbf{y}}$ the attached cone $-int(\bar{\mathcal{C}})$ has empty intersection with $\mathcal{Y}$. Combining both yields the follow equalities:

$$\begin{aligned} \emptyset &= \left(\hat{\mathbf{y}} - int(\bar{\mathcal{C}})\right) \cap \mathcal{Y} \\ &\supseteq \left(\hat{\mathbf{y}} - \mathbf{y}^A + \mathcal{L}_<(t_s, \tau)\right) \cap \mathcal{Y} \\ &= \left(\hat{\mathbf{y}} - \mathbf{y}^A + \{\Delta\mathbf{y} \in \mathbb{Y} \mid t_s(\Delta\mathbf{y}) < \tau\}\right) \cap \mathcal{Y}. \end{aligned}$$

Now we can employ the definition of the Minkowski sum to continue the chain of equations.

$$\begin{aligned} \emptyset &= \left\{\hat{\mathbf{y}} - \mathbf{y}^A + \Delta\mathbf{y} \;\middle|\; \Delta\mathbf{y} \in \mathbb{Y},\, t_s(\Delta\mathbf{y}) < \tau\right\} \cap \mathcal{Y} \\ &= \left\{\hat{\mathbf{y}} - \mathbf{y}^A + \Delta\mathbf{y} \in \mathcal{Y} \;\middle|\; \Delta\mathbf{y} \in \mathbb{Y},\, t_s(\Delta\mathbf{y}) < \tau\right\} \\ &= \left\{\mathbf{y} \in \mathcal{Y} \mid t_s(\mathbf{y} - \hat{\mathbf{y}} + \mathbf{y}^A) < \tau\right\} \\ &= \left\{\mathbf{y} \in \mathcal{Y} \mid t_s(\mathbf{y} - \mathbf{y}^R) < \tau\right\}. \end{aligned}$$

Hence, there is no point that improves upon $\hat{\mathbf{y}}$ and thus $\hat{\mathbf{y}}$ is an optimum for the cone scalarising problem and the chosen reference point.

□

The choice of the reference point above is sometimes called tautological, because one has to know a weakly $\bar{\mathcal{C}}$-efficient point to choose a fitting reference point so that the optimisation yields the known optimal point again. Nonetheless, the result shows that the strict $\mathcal{C}$-representation and strict $\mathcal{C}$-approximation property for a single point is enough to potentially reach all weakly $\bar{\mathcal{C}}$-efficient solutions.

The next step is to establish a simple sufficient condition for s that renders t_s strictly $\mathcal{C}$-representing at $\mathbf{y} = \mathbf{0}$ and $\tau = 0$.

Lemma 2.4.5 *If the cone scalarising function s is $\mathcal{C}$-centred, strictly $\mathcal{C}$-consistent, continuous and if*

$$s(\mathbf{0}, \mathbf{c}^-) = 0 \quad \textit{for all } \mathbf{c}^- \in bd(\mathcal{C}),$$

then t_s is strictly $\mathcal{C}$-representing at $\mathbf{y} = \mathbf{0}$ and $\tau = 0$.

Proof:

The properties of s imply that t_s is continuous and strictly $\mathcal{C}$-monotone. Due to the $\mathcal{C}$-centredness of s the inequality

$$s\big(\mathbf{0}, \hat{\mathbf{c}}^-\big) \leq s\big(\mathbf{c}^+, \mathbf{c}^-\big) \quad \textit{for all } \mathbf{c}^+, \mathbf{c}^- \in \mathcal{C} \textit{ with } \mathbf{c}^+ - \mathbf{c}^- = \hat{\mathbf{c}}^-$$

holds for arbitrary representations of $\hat{\mathbf{c}}^-$. Thus,

$$t_s\big(\hat{\mathbf{c}}^-\big) \;=\; \min\big\{s\big(\mathbf{c}^+, \mathbf{c}^-\big) \;\big|\; \mathbf{c}^+ - \mathbf{c}^- = \hat{\mathbf{c}}^-,\; \mathbf{c}^+, \mathbf{c}^- \in \mathcal{C}\big\} \;=\; s\big(\mathbf{0}, \hat{\mathbf{c}}^-\big) \;=\; 0$$

is equal to 0 at the boundary of the negative ordering cone. Every element $\mathbf{c} \in -int(\mathcal{C})$ is strictly dominated by $\mathbf{0}$ and thus

$$t_s\big(\mathbf{c}\big) < t_s(\mathbf{0}) = 0,$$

because t_s is strictly $\mathcal{C}$-monotone. Therefore, the interior of the negative cone $-int(\mathcal{C}) \subseteq \mathcal{L}_<(t_s, 0)$ is completely contained in the level set.

For an arbitrary element $\mathbf{y} \in \mathbb{Y} \setminus (-\mathcal{C})$ let $\bar{\mathbf{c}}^+, \bar{\mathbf{c}}^- \in \mathcal{C}$ be a minimal representation. The strict $\mathcal{C}$-consistency of s then implies

$$t_s\big(\mathbf{y}\big) \;=\; s\big(\bar{\mathbf{c}}^+, \bar{\mathbf{c}}^-\big) \;\geq\; s\big(\mathbf{0}, \bar{\mathbf{c}}^-\big) \;=\; 0$$

the nonnegativity of t_s. Therefore, no element of $\mathbb{Y} \setminus (-\mathcal{C})$ is contained in $\mathcal{L}_<(t_s, 0)$. As t_s is nonnegative for $-bd(\mathcal{C})$ and $\mathbb{Y} \setminus (-\mathcal{C})$ and

$$-int(\mathcal{C}) \,\dot\cup\, -bd(\mathcal{C}) \,\dot\cup\, \mathbb{Y} \setminus -\mathcal{C} \;=\; \mathbb{Y}$$

the three sets partition the space $\mathbb{Y}$, the level set $\mathcal{L}_<(t_s, 0) = -int(\mathcal{C})$ equals the interior of the negative cone.

□

We will now derive a sufficient condition for t_s so that it is strictly $\mathcal{C}$-approximating at $\mathbf{y} = \mathbf{0}$ and $\tau = 0$.

Lemma 2.4.6 *Let the cone scalarising function s be $\mathcal{C}$-centred, strongly $\mathcal{C}$-consistent, continuous and $s(\mathbf{0}, \mathbf{0}) = 0$ and let for all $\mathbf{c}^+, \mathbf{c}^- \in \mathcal{C}$*

$$s\big(\mathbf{c}^+, \mathbf{c}^-\big) \geq \langle \mathbf{c}^*, \mathbf{c}^+ - \varepsilon\, \mathbf{c}^- \rangle,$$

where $\mathbf{c}^* \in int\big(\mathcal{C}^*\big)$. Then t_s is strictly $\mathcal{C}$-approximating at $\mathbf{y} = \mathbf{0}$ and $\tau = 0$ for the outer cone $\mathcal{C}(\mathbf{c}^*, \varepsilon)$.

Proof:
Under the given conditions the distance evaluation function t_s is continuous and strongly $\mathcal{C}$-monotone. The strong consistency of s implies for arbitrary $\mathbf{c}^- \in \mathcal{C} \setminus \{\mathbf{0}\}$

$$t_s(\mathbf{c}^-) \;\leq\; s\big(\mathbf{0}, \mathbf{c}^-\big) \;<\; s\big(\mathbf{0}, \mathbf{0}\big) \;=\; 0.$$

Therefore, the negative cone $\mathcal{C} \setminus \{\mathbf{0}\} \subseteq \mathcal{L}_<(t_s, 0)$ is contained in the level set for $\mathbf{y} = \mathbf{0}$ and $\tau = 0$.

It remains to be shown that there are no nonpositive t_s values outside the negative enlarged cone $-\mathcal{C}(\mathbf{c}^*, \varepsilon)$. So let $\mathbf{y} \in \mathbb{Y} \setminus \big(-\mathcal{C}(\mathbf{c}^*, \varepsilon)\big)$ be outside the negative enlarged cone. $-\mathbf{y}$ thus violates the inequality in the definition of the enlarged cone

$$\mathcal{C}(\mathbf{c}^*, \varepsilon) := \big\{\mathbf{c} \in \mathbb{Y} \;\big|\; \mathbf{c} = \mathbf{c}^+ - \mathbf{c}^-,\; \langle \mathbf{c}^*, \varepsilon\, \mathbf{c}^+ - \mathbf{c}^- \rangle \geq 0,\; \mathbf{c}^+,\, \mathbf{c}^- \in \mathcal{C}\big\},$$

which is exactly the case if $\langle \mathbf{c}^*, \varepsilon\, \mathbf{c}^+ - \mathbf{c}^- \rangle < 0$ the restriction in the definition is violated for all $\mathbf{c}^+, \mathbf{c}^- \in \mathcal{C}$ satisfying $-\mathbf{y} = \mathbf{c}^+ - \mathbf{c}^-$. Changing the sign of $-\mathbf{y}$ and thus changing the roles of $\mathbf{c}^+$ and $\mathbf{c}^-$, the above is equivalent to

$$\langle \mathbf{c}^*, \varepsilon\, \mathbf{c}^- - \mathbf{c}^+ \rangle < 0 \quad \Longleftrightarrow \quad \langle \mathbf{c}^*, \mathbf{c}^+ - \varepsilon\, \mathbf{c}^- \rangle > 0$$

for all $\mathbf{c}^+, \mathbf{c}^- \in \mathcal{C}$ satisfying $\mathbf{y} = \mathbf{c}^+ - \mathbf{c}^-$.
For any element $\mathbf{y} \notin -\mathcal{C}(\mathbf{c}^*, \varepsilon)$ the distance evaluation function can hence be estimated by

$$t_s(\mathbf{y}) \;=\; s\big(\bar{\mathbf{c}}^+, \bar{\mathbf{c}}^-\big) \;\geq\; \langle \mathbf{c}^*, \bar{\mathbf{c}}^+ - \varepsilon\, \bar{\mathbf{c}}^- \rangle \;>\; 0,$$

where $\bar{\mathbf{c}}^+, \bar{\mathbf{c}}^- \in \mathcal{C}$ are a t_s-minimal pair. So $t_s(\mathbf{y}) > 0$ for every $\mathbf{y} \notin -\mathcal{C}(\mathbf{c}^*, \varepsilon)$ and due to the continuity of t_s the distance evaluation function has nonnegative values $t_s(\mathbf{y}) \geq 0$ for every $\mathbf{y} \in -bd\big(\mathcal{C}(\mathbf{c}^*, \varepsilon)\big)$. Hence, the level set $\mathcal{L}_<(t_s, 0) \subseteq -int\big(\mathcal{C}(\mathbf{c}^*, \varepsilon)\big)$ is contained in the interior of the negative cone.
□

The condition that s has to be larger than the linear functional ensures that the value of s increases appropriately faster in $\mathbf{c}^+$-direction than it decreases in $\mathbf{c}^-$-direction.

The results obtained so far suggest that a strongly $\mathcal{C}$-consistent, strictly locally $\mathcal{C}$-representing and continuous s would be the perfect cone scalarising function. It would yield $\mathcal{C}$-efficient solutions due to the strong $\mathcal{C}$-consistency and would allow any $\mathcal{C}$-efficient solution to be reached, because of the exact representation of the negative ordering cone $-\mathcal{C}$. Unfortunately, there is no such cone scalarising function.

Theorem 2.4.7 *There exists no function $s : \mathcal{C} \times \mathcal{C} \to \mathbb{R}$ that is simultaneously continuous, strongly $\mathcal{C}$-consistent with its distance evaluation function being strictly $\mathcal{C}$-representing at $\mathbf{y}^A \in \mathbb{Y}$ and $\tau = t_s\big(\mathbf{y}^A\big)$.*

Proof:
Under the given conditions the distance evaluation function t_s is continuous and strongly $\mathcal{C}$-monotone. By the strict $\mathcal{C}$-representation $t_s(\mathbf{y}') < 0$ if and only if $\mathbf{y}' \in \mathbf{y}^A - int(\mathcal{C})$. Due to its continuity the value of t_s on $\mathbf{y}^A - bd(\mathcal{C})$ is 0. Thus,

$$t_s\big(\mathbf{y}^A - \mathbf{c}^-\big) = 0 \quad \textit{for all} \quad \mathbf{c}^- \in bd(\mathcal{C}).$$

But this contradicts the strong $\mathcal{C}$-monotonicity.

□

Therefore, we can either opt for $\mathcal{C}$-efficient solutions, knowing that we are not able to reach every such solution or we go for weakly $\mathcal{C}$-efficient solutions and reach them all. A combination of both is not possible in one step in the framework of cone scalarising. However, the next result shows that it is possible with a two step procedure.

Lemma 2.4.8 *Let $\mathcal{Y}$ be compact. Then every $\mathcal{C}$-efficient point $\hat{\mathbf{y}} \in \mathcal{Y}$ is a minimum of*

$$lexmin\left\{s\big(\mathbf{c}^+, \mathbf{c}^-\big),\ -\langle \mathbf{c}^-, w\rangle \ \middle|\ \mathbf{y} - \mathbf{y}^R = \mathbf{c}^+ - \mathbf{c}^-,\ \mathbf{y} \in \mathcal{Y},\ \mathbf{c}^+, \mathbf{c}^- \in \mathcal{C}\right\}$$

for a strictly $\mathcal{C}$-consistent, $\mathcal{C}$-centred and continuous function s, some $\mathbf{w} \in int(\mathcal{C}^*)$ and a suitably chosen reference point $\mathbf{y}^R$.

Proof:
In the first minimisation can be substituted with $s(\mathbf{c}^+, \mathbf{c}^-)\ t_s(\mathbf{y} - \mathbf{y}^R)$. Due to the compactness of $\mathcal{Y}$ and the continuity of t_s the minimisation attains its minimum. Theorem 2.4.1 showed that any optimum $\bar{\mathbf{y}}$ of the first step is a weakly $\mathcal{C}$-efficient solution. Thus, fixing the value of s in the second step, we are left with a domain

$$\mathcal{L}_{\leq}\big(t_s, t_s(\bar{\mathbf{y}} - \mathbf{y}^R)\big) \cap \mathcal{Y} \ \subseteq\ eff_{w-\mathcal{C}}(\mathcal{Y})$$

that only consists of weakly $\mathcal{C}$-efficient solutions.

Let $\hat{\mathbf{c}}^+ - \hat{\mathbf{c}}^- = \hat{\mathbf{y}}$ be a representation of the optimum of the second step. Assume that there is a $\tilde{\mathbf{y}} \in \mathcal{Y}$ that dominates $\hat{\mathbf{y}}$. Then, there exists a $\tilde{\mathbf{c}} \in \mathcal{C} \setminus \{\mathbf{0}\}$ with $\tilde{\mathbf{y}} + \tilde{\mathbf{c}} = \hat{\mathbf{y}}$. Due to the minimality of $\hat{\mathbf{y}}$ for the first step the t_s-value

$$\begin{aligned} t_s\big(\hat{\mathbf{y}} - \mathbf{y}^R\big) &= \min\left\{t_s\big(\mathbf{y} - \mathbf{y}^R\big) \ \middle|\ \mathbf{y} \in \mathcal{Y}\right\} \\ &\leq t_s\big(\tilde{\mathbf{y}} - \mathbf{y}^R\big) \end{aligned}$$

for $\tilde{\mathbf{y}} - \mathbf{y}^R$ cannot be smaller than the one for $\hat{\mathbf{y}} - \mathbf{y}^R$. Employing strict $\mathcal{C}$-monotonicity of t_s we can continue the chain of inequalities by

$$\begin{aligned} t_s\big(\tilde{\mathbf{y}} - \mathbf{y}^R\big) &= t_s\big(\hat{\mathbf{y}} + \tilde{\mathbf{c}} - \mathbf{y}^R\big) \\ &\leq t_s\big(\hat{\mathbf{y}} - \mathbf{y}^R\big). \end{aligned}$$

Thus, the function values for the two points are equal. Therefore, $\tilde{\mathbf{y}}$ *is contained in the feasible set for the second step. But decomposing* $\tilde{\mathbf{y}}$

$$\tilde{\mathbf{y}} \;=\; \hat{\mathbf{y}} - \tilde{\mathbf{c}} \;=\; \hat{\mathbf{c}}^{+} - \hat{\mathbf{c}}^{-} - \tilde{\mathbf{c}}$$

according to the assumption, we see that $\tilde{\mathbf{y}}$

$$-\langle \hat{\mathbf{c}}^{-} + \tilde{\mathbf{c}},\, w\rangle \;=\; -\langle \hat{\mathbf{c}}^{-},\, w\rangle \;-\; \langle \tilde{\mathbf{c}},\, w\rangle \;<\; -\langle \hat{\mathbf{c}}^{-},\, w\rangle$$

has smaller objective value than $\hat{\mathbf{y}}$*. But since* $\tilde{\mathbf{y}}$ *is feasible for the second optimisation step, this contradicts the lexicographic minimality of* $\hat{\mathbf{y}}$*.*

□

The preceding lemma shows that we can make a weakly $\mathcal{C}$-efficient solution $\mathcal{C}$-efficient by applying a 'post-optimisation' step. Combining this with a distance evaluation function t_s that is strictly $\mathcal{C}$-representing at some point and for some level, we can construct a two step method, that is able to reach every $\mathcal{C}$-efficient solution and only yields $\mathcal{C}$-efficient solutions.

The last property that we will address is the connectivity of the set of optimal outcomes for a chosen cone scalarising function s. We will use a theorem of Warburton ([118]) that he used to show the connectivity of the set of Pareto optimal outcomes for convex multiobjective problems.

Theorem 2.4.9 (Warburton) *Let* $\mathcal{V} \subseteq \mathbb{R}^n, \mathcal{W} \subseteq \mathbb{R}^m$*, and assume that* $\mathcal{V}$ *is compact and* $\mathcal{W}$ *is connected. Furthermore, let* $g : \mathcal{V} \times \mathcal{W} \longrightarrow \mathbb{R}$ *be continuous. Denote by* $\mathcal{Y}(\mathbf{w}) = argmin\{g(\mathbf{v}, \mathbf{w}) \mid \mathbf{v} \in \mathcal{V}\}$*. If* $\mathcal{Y}(\mathbf{w})$ *is connected for all* $\mathbf{w} \in \mathcal{W}$ *then* $\bigcup_{\mathbf{w}\in\mathcal{W}} \mathcal{Y}(\mathbf{w})$ *is connected.*

Let us now precisely define, what we mean by the set of optimal outcomes for a chosen cone scalarising function s.

Definition 2.4.10 *Let*

$$\begin{aligned} \mathcal{O}_{\mathcal{Y},s} : \quad & \mathbb{Y} \to \mathfrak{P}(\mathbb{Y}) \\ \mathcal{O}_{\mathcal{Y},s}(\mathbf{y}^R) \;:=\; & argmin\big\{t_s(\mathbf{y} - \mathbf{y}^R) \;\big|\; \mathbf{y} \in \mathcal{Y}\big\} \end{aligned}$$

map a reference point $\mathbf{y}^R$ *to the set of optimal points for that reference point and the chosen scalarising* s *over the domain* $\mathcal{Y}$*.*

We denote the set of optima for all possible reference points for the chosen cone scalarising function s *over the outcome set* $\mathcal{Y}$ *by*

$$eff_s(\mathcal{Y}) := \bigcup_{\mathbf{y}^R\in\mathbb{Y}} \mathcal{O}_{\mathcal{Y},s}(\mathbf{y}^R).$$

This set is connected, if we add the quasiconvexity of the function s and the compactness and convexity of the outcome set $\mathcal{Y}$ to our standard requirements.

Theorem 2.4.11 *Let $\mathcal{Y}$ be convex and compact. Moreover, let the cone scalarising function s be continuous, $\mathcal{C}$-centred, strictly $\mathcal{C}$-consistent and quasiconvex. Then, the set of optimal outcomes $eff_s(\mathcal{Y})$ is connected.*

Proof:

The conditions on s imply that the corresponding distance evaluation function t_s is continuous, strictly $\mathcal{C}$-monotone and quasiconvex. Defining the function g of Warburton's theorem

$$g : \mathcal{Y} \times \mathbb{Y} \to \mathbb{R}, \qquad g(\mathbf{y}, \mathbf{y}^R) := t_s(\mathbf{y} - \mathbf{y}^R)$$

to be the distance evaluation function for the difference of arguments, the minimal sets of the theorem are given by

$$\mathcal{Y}(\mathbf{y}^R) = argmin\{g(\mathbf{y}, \mathbf{y}^R) \mid \mathbf{y} \in \mathcal{Y}\}.$$

Since changing the reference point is just a translation, $g(\mathbf{y}, \mathbf{y}^R)$ is continuous and quasiconvex in $\mathbf{y}$ for a fixed $\mathbf{y}^R$. The minimum is attained and so the sets $\mathcal{Y}(\mathbf{y}^R)$ are closed and convex for all $\mathbf{y}^R \in \mathbb{Y}$. This implies that they are connected. Therefore, theorem 2.4.9 can be applied stating that the union of the sets $\mathcal{Y}(\mathbf{w})$ is connected.

$$\begin{aligned}
\bigcup_{\mathbf{y}^R \in \mathbb{Y}} \mathcal{Y}(\mathbf{y}^R) &= \bigcup_{\mathbf{y}^R \in \mathbb{Y}} argmin\{g(\mathbf{y}, \mathbf{y}^R) \mid \mathbf{y} \in \mathcal{Y}\} \\
&= \bigcup_{\mathbf{y}^R \in \mathbb{Y}} argmin\{t_s(\mathbf{y} - \mathbf{y}^R) \mid \mathbf{y} \in \mathcal{Y}\} \\
&= \bigcup_{\mathbf{y}^R \in \mathbb{Y}} \mathcal{O}_{\mathcal{Y},s}(\mathbf{y}^R) \\
&= eff_s(\mathcal{Y})
\end{aligned}$$

□

We have shown that the set of optimal outcomes for suitable cone scalarising functions s is connected, when the set of outcomes $\mathcal{Y}$ is convex and compact. The next step is to ensure, that the optima can be changed in a controllable way, i.e. that a small change in the reference point does not cause arbitrarily large changes in the solution.

2.5 Dependence on reference point

Having (weakly) $\mathcal{C}$-efficient outcomes and being able to reach the different (weakly) $\mathcal{C}$-efficient outcomes is important for a scalarisation. But for an effective workflow a further property is necessary: continuous dependence of the outcome on the control parameters, because otherwise the control of the outcome by manipulating input parameters is poor.

This section elaborates the dependence of the optimisation result on the reference point – the input parameter used for cone scalarising. We will show the continuity and under suitable conditions the Lipschitz-continuity of the minimal s-value function μ. Furthermore, we will show that the optimal set function $\mathcal{O}_{\mathcal{Y},s}$ is upper semicontinuous.

Let us start with the definition of the minimal value function μ.

Definition 2.5.1 *For a $\mathcal{C}$-centred, continuous and $\mathcal{C}$-consistent cone scalarising function s let*

$$\begin{array}{rcl} \mu\colon & & \mathbb{Y} \to \mathbb{R} \\ \mu\big(\mathbf{y}^R\big) & = & \min\big\{s\big(\mathbf{c}^+,\mathbf{c}^-\big) \;\big|\; \mathbf{y}-\mathbf{y}^R = \mathbf{c}^+ - \mathbf{c}^-,\ \mathbf{y}\in\mathcal{Y},\ \mathbf{c}^+,\mathbf{c}^-\in\mathcal{C}\big\}. \end{array}$$

The function μ is called minimal value function.

The minimal value function μ depends continuously on the reference point, if the outcome set $\mathcal{Y}$ is compact.

Theorem 2.5.2 *Let s be a continuous, $\mathcal{C}$-centred and $\mathcal{C}$-consistent cone scalarising function and let the outcome set $\mathcal{Y}$ be compact. Then, the minimal value function μ depends continuously on the reference point $\mathbf{y}^R$.*

Proof:

We will demonstrate that the minimal value function μ preserves the convergence of sequences.
Let $\big(\mathbf{y}^R(i)\big)_{i\in\mathbb{N}}$ be a sequence that converges to the point $\lim_{i\to\infty}\mathbf{y}^R(i)=\mathbf{y}^R$.
Now let $\mathbf{c}^+(i) := \mathbf{c}^{\oplus}\big(\mathbf{y}^R(i)-\mathbf{y}^R\big)$ and $\mathbf{c}^-(i) := \mathbf{c}^{\ominus}\big(\mathbf{y}^R(i)-\mathbf{y}^R\big)$ so that

$$\mathbf{y}^R(i) := \mathbf{y}^R + \mathbf{c}^+(i) - \mathbf{c}^-(i)$$

is an order interval representation of the distance $\mathbf{y}^R(i)-\mathbf{y}^R$. For convenience we define the functions

$$\begin{array}{rcl} g^{(i)}, g^{\infty}: & & \mathbb{Y}\to\mathbb{R} \\ g^{(i)}(\mathbf{y}) & := & t_s\big(\mathbf{y}-\mathbf{y}^R(i)\big) \quad \textit{and} \\ g^{\infty}(\mathbf{y}) & := & t_s\big(\mathbf{y}-\mathbf{y}^R\big) \end{array}$$

and denote by $\big(\mathbf{y}^{(i)}\big)_{i\in\mathbb{N}}$ and $\hat{\mathbf{y}}$ optima of the cone scalarising problems

$$\min\big\{g^{(i)}(\mathbf{y}) \;\big|\; \mathbf{y}\in\mathcal{Y}\big\} \quad \textit{and} \quad \min\big\{g^{\infty}(\mathbf{y}) \;\big|\; \mathbf{y}\in\mathcal{Y}\big\}.$$

Using this notation the claim is transformed to the convergence of

$$\lim_{i\to\infty}\big|\, g^{(i)}\big(\mathbf{y}^{(i)}\big) - g^{\infty}(\hat{\mathbf{y}})\,\big| = 0$$

to zero. We will start by showing that $\left| g^{(i)}\left(\mathbf{y}\right) - g^{\infty}(\mathbf{y}) \right|$ *vanishes for arbitrary* $\mathbf{y}$ *and* $i \to \infty$.

Due to the convergence of $\mathbf{y}^R(i)$ *there exists for every* $\delta > 0$ *an* $i(\delta) \in \mathbb{N}$ *such that the tail of the reference point sequence* $\mathbf{y}^R(i) \in \mathcal{B}_\delta\left(\mathbf{y}^R\right), i \geq i(\delta)$ *is contained in a ball of radius* δ *around the limit. Thus, the difference* $\mathbf{c}^+(i) - \mathbf{c}^-(i) \in \mathcal{B}_\delta(\mathbf{0})$ *is inside the corresponding ball around the origin. By the Lipschitz-continuity of* $\mathbf{c}^{\oplus}$ *and* $\mathbf{c}^{\ominus}$ *it follows that the norms of the representing cone elements* $\| \mathbf{c}^+(i) \|_2, \| \mathbf{c}^-(i) \|_2 \leq \delta$ *are smaller than or equal to* δ.

$\mathcal{Y}$ *is compact and* g^{∞} *is continuous. Therefore, there exists for every* $\varepsilon > 0$ *independently of the point* $\mathbf{y} \in \mathbb{Y}$, *a* $\delta(\varepsilon) > 0$ *such that for every* $\bar{\mathbf{y}} \in \mathbb{Y}$ *with* $\| \mathbf{y} - \bar{\mathbf{y}} \|_2 \leq \delta(\varepsilon)$ *the difference in function value is* $| g^{\infty}(\mathbf{y}) - g^{\infty}(\bar{\mathbf{y}}) | \leq \varepsilon$. *So, for any* $i \geq i\left(\delta\left(\frac{\varepsilon}{2}\right)\right)$ *the distances*

$$\left\| \left(\mathbf{y} - \mathbf{c}^+(i)\right) - \mathbf{y} \right\|_2, \ \left\| \left(\mathbf{y} + \mathbf{c}^-(i)\right) - \mathbf{y} \right\|_2 \ \leq \delta\left(\tfrac{\varepsilon}{2}\right)$$

are small enough so that the differences in function value

$$\left| g^{\infty}(\mathbf{y} - \mathbf{c}^+(i)) - g^{\infty}(\mathbf{y}) \right|, \ \left| g^{\infty}(\mathbf{y} + \mathbf{c}^-(i)) - g^{\infty}(\mathbf{y}) \right| \ \leq \frac{\varepsilon}{2} \tag{2.16}$$

are bounded by $\frac{\varepsilon}{2}$.

The functions $g^{(i)}$ *and* g^{∞} *are* $\mathcal{C}$*-monotone which allows us to construct an upper and lower bound for the two function values. For an arbitrary point* $\mathbf{y} \in \mathbb{Y}$

$$\begin{array}{lllll} \mathbf{y} - \mathbf{y}^R + \mathbf{c}^+(i) & \geq_{\mathcal{C}} & \mathbf{y} - \mathbf{y}^R & \geq_{\mathcal{C}} & \mathbf{y} - \mathbf{y}^R - \mathbf{c}^-(i) \\ \mathbf{y} - \mathbf{y}^R + \mathbf{c}^+(i) & \geq_{\mathcal{C}} & \mathbf{y} - \mathbf{y}^R(i) & \geq_{\mathcal{C}} & \mathbf{y} - \mathbf{y}^R - \mathbf{c}^-(i) \end{array}$$

we can construct an order interval containing the differences $\mathbf{y} - \mathbf{y}^R$ *and* $\mathbf{y} - \mathbf{y}^R(i)$. *The* $\mathcal{C}$*-monotonicity of* g^{∞} *and* $g^{(i)}$ *then yields*

$$\begin{aligned} g^{\infty}\left(\mathbf{y} - \mathbf{c}^-(i)\right) &\leq \min\left\{g^{\infty}\left(\mathbf{y}\right), g^{(i)}\left(\mathbf{y}\right)\right\} \\ &\leq \max\left\{g^{\infty}\left(\mathbf{y}\right), g^{(i)}\left(\mathbf{y}\right)\right\} \\ &\leq g^{\infty}\left(\mathbf{y} + \mathbf{c}^+(i)\right). \end{aligned}$$

upper and lower bounds on the function values. Combining this with the continuity estimates (2.16) we can estimate the difference in function value by

$$\begin{aligned} \left| g^{\infty}(\mathbf{y}) - g^{(i)}(\mathbf{y}) \right| &\leq \max\left\{g^{\infty}\left(\mathbf{y}\right), g^{(i)}\left(\mathbf{y}\right)\right\} - \min\left\{g^{\infty}\left(\mathbf{y}\right), g^{(i)}\left(\mathbf{y}\right)\right\} \\ &\leq g^{\infty}\left(\mathbf{y} + \mathbf{c}^+(i)\right) - g^{\infty}\left(\mathbf{y} - \mathbf{c}^-(i)\right) \\ &\leq \varepsilon \end{aligned}$$

for all $i \geq i\left(\delta\left(\frac{\varepsilon}{2}\right)\right)$ *and arbitrary* $\mathbf{y} \in \mathbb{Y}$.

The above estimate is independent from the point $\mathbf{y}$ *and therefore we can apply it to all the different optima* $\mathbf{y}(i)$ *and* $\hat{\mathbf{y}}$*. This yields the upper bounds*

$$\begin{aligned} \left| g^{\infty}(\hat{\mathbf{y}}) - g^{(i)}(\hat{\mathbf{y}}) \right| &= g^{(i)}(\hat{\mathbf{y}}) - g^{\infty}(\hat{\mathbf{y}}) \leq \varepsilon \quad \textit{and} \\ \left| g^{\infty}(\mathbf{y}(i)) - g^{(i)}(\mathbf{y}(i)) \right| &= g^{\infty}(\mathbf{y}(i)) - g^{(i)}(\mathbf{y}(i)) \leq \varepsilon \end{aligned}$$

for the difference in function value. Here, we could drop the absolute value signs, because the sign of the differences is known. Adding the two inequalities results

$$\underbrace{g^{\infty}(\mathbf{y}(i)) - g^{\infty}(\hat{\mathbf{y}})}_{\geq 0} + \underbrace{g^{(i)}(\hat{\mathbf{y}}) - g^{(i)}(\mathbf{y}(i))}_{\geq 0} \leq 2\,\varepsilon$$

in a lower bound of 0 *and an upper bound of* $2\,\varepsilon$ *for the two differences. Therefore, for every* $i \geq i\big(\delta\big(\frac{\varepsilon}{2}\big)\big)$ *the difference in function value for the respective optima*

$$\begin{aligned} &\left| g^{\infty}(\hat{\mathbf{y}}) - g^{(i)}(\mathbf{y}(i)) \right| \\ &\qquad \leq \left| g^{\infty}(\hat{\mathbf{y}}) - g^{\infty}(\mathbf{y}(i)) \right| + \left| g^{\infty}(\mathbf{y}(i)) - g^{(i)}(\mathbf{y}(i)) \right| \\ &\qquad \leq 2\,\varepsilon + \varepsilon = 3\,\varepsilon \end{aligned}$$

is bounded by $3\,\varepsilon$*. Hence, the sequence of minimal values*

$$\lim_{i\to\infty} t_s\big(\mathbf{y}(i) - \mathbf{y}^R(i)\big) = \lim_{i\to\infty} g^{(i)}(\mathbf{y}(i)) = g^{\infty}(\hat{\mathbf{y}}) = t_s\big(\hat{\mathbf{y}} - \mathbf{y}^R\big)$$

converges as the reference points converge.

□

In the proof the compactness of $\mathcal{Y}$ plays an important role because it allows for a common bound $\delta(\varepsilon)$ that is independent of the currently considered point. But compactness is a strong requirement and we will see later how we can relax it a bit. For Lipschitz-continuous functions s we do not need the compactness, as we get the common bound from the Lipschitz-continuity.

Theorem 2.5.3 *Suppose* s *is a Lipschitz-continuous,* $\mathcal{C}$*-consistent and* $\mathcal{C}$*-centred cone scalarising function. Then, the minimum value function* μ *is Lipschitz-continuous.*

Proof:

For the given requirements the function t_s *is Lipschitz-continuous and* $\mathcal{C}$*-monotone. Now let* $\mathbf{y}^R, \bar{\mathbf{y}}^R \in \mathbb{Y}$ *be two different reference points. Then let* $\mathbf{c}^+ := \mathbf{c}^{\oplus}\big(\mathbf{y}^R - \bar{\mathbf{y}}^R\big)$ *and* $\mathbf{c}^- := \mathbf{c}^{\ominus}\big(\mathbf{y}^R - \bar{\mathbf{y}}^R\big)$ *be the representation of the difference by cone elements. The two anchor points of the order interval that we can construct using* $\mathbf{c}^+$ *and* $\mathbf{c}^-$

$$\mathbf{y}^R - \mathbf{c}^+(i) \leq_{\mathcal{C}} \mathbf{y}^R,\, \bar{\mathbf{y}}^R \leq_{\mathcal{C}} \mathbf{y}^R + \mathbf{c}^-(i)$$

form a lower and upper bound for the two points. Thus, for an arbitrary $\mathbf{y} \in \mathbb{Y}$ *the* $\mathcal{C}$*-monotonicity of* t_s *implies that*

$$\begin{aligned} t_s\big(\mathbf{y} - \mathbf{y}^R - \mathbf{c}^-\big) &\leq \min\left\{t_s\big(\mathbf{y} - \mathbf{y}^R\big), t_s\big(\mathbf{y} - \bar{\mathbf{y}}^R\big)\right\} \\ &\leq \max\left\{t_s\big(\mathbf{y} - \mathbf{y}^R\big), t_s\big(\mathbf{y} - \bar{\mathbf{y}}^R\big)\right\} \\ &\leq t_s\big(\mathbf{y} - \mathbf{y}^R + \mathbf{c}^+\big) \end{aligned}$$

the function values corresponding to the upper and lower anchor point of the order interval yield a lower and upper bound for the function value. As the inequalities are valid for an arbitrary point $\mathbf{y}$ *they imply the following inequalities for the minimum value function* μ*:*

$$\begin{aligned} \mu\big(\mathbf{y}^R + \mathbf{c}^-\big) &\leq \min\left\{\mu\big(\mathbf{y}^R\big), \mu\big(\bar{\mathbf{y}}^R\big)\right\} \\ &\leq \max\left\{\mu\big(\mathbf{y}^R\big), \mu\big(\bar{\mathbf{y}}^R\big)\right\} \quad \leq \quad \mu\big(\mathbf{y}^R - \mathbf{c}^+\big). \end{aligned}$$

Therefore, we can estimate the difference in function value for the two reference points

$$\begin{aligned} \left|\, \mu\big(\mathbf{y}^R\big) - \mu\big(\bar{\mathbf{y}}^R\big) \,\right| &= \max\left\{\mu\big(\mathbf{y}^R\big), \mu\big(\bar{\mathbf{y}}^R\big)\right\} - \min\left\{\mu\big(\mathbf{y}^R\big), \mu\big(\bar{\mathbf{y}}^R\big)\right\} \\ &\leq \mu\big(\mathbf{y}^R - \mathbf{c}^+\big) - \mu\big(\mathbf{y}^R + \mathbf{c}^-\big) \\ &= \min\left\{t_s\big(\mathbf{y} - \mathbf{y}^R + \mathbf{c}^+\big) \;\middle|\; \mathbf{y} \in \mathcal{Y}\right\} \\ &\quad - \min\left\{t_s\big(\mathbf{y} - \mathbf{y}^R - \mathbf{c}^-\big) \;\middle|\; \mathbf{y} \in \mathcal{Y}\right\} \end{aligned}$$

by the difference of μ *for the anchor points. The Lipschitz-continuity of* t_s *enables us to limit the change in function value by the change in argument times the Lipschitz constant* $L_{t_s} = L_s$*. Therefore, the upper bound can be estimated by*

$$\begin{aligned} \min\left\{t_s\big(\mathbf{y} - \mathbf{y}^R + \mathbf{c}^+\big) \;\middle|\; \mathbf{y} \in \mathcal{Y}\right\} &\leq \min\left\{t_s\big(\mathbf{y} - \mathbf{y}^R\big) \;\middle|\; \mathbf{y} \in \mathcal{Y}\right\} \\ &\quad + L_s \left\| \mathbf{c}^+ \right\|_2 \end{aligned}$$

and the lower bound by

$$\begin{aligned} \min\left\{t_s\big(\mathbf{y} - \mathbf{y}^R - \mathbf{c}^-\big) \;\middle|\; \mathbf{y} \in \mathcal{Y}\right\} &\geq \min\left\{t_s\big(\mathbf{y} - \mathbf{y}^R\big) \;\middle|\; \mathbf{y} \in \mathcal{Y}\right\} \\ &\quad - L_s \left\| \mathbf{c}^- \right\|_2. \end{aligned}$$

Assembling the different estimates we get an upper bound of

$$\begin{aligned} \left|\, \mu\big(\mathbf{y}^R\big) - \mu\big(\bar{\mathbf{y}}^R\big) \,\right| &\leq \min\left\{t_s\big(\mathbf{y} - \mathbf{y}^R\big) \;\middle|\; \mathbf{y} \in \mathcal{Y}\right\} + L_s \left\| \mathbf{c}^+ \right\|_2 \\ &\quad - \min\left\{t_s\big(\mathbf{y} - \mathbf{y}^R\big) \;\middle|\; \mathbf{y} \in \mathcal{Y}\right\} + L_s \left\| \mathbf{c}^- \right\|_2 \\ &= L_s \left(\left\| \mathbf{c}^+ \right\|_2 + \left\| \mathbf{c}^- \right\|_2 \right) \\ &\overset{(2.2.12)}{\leq} 2\, L_s \left\| \mathbf{y}^R - \bar{\mathbf{y}}^R \right\|_2, \end{aligned}$$

for the difference in function value. Thus, μ *is Lipschitz-continuous with the Lipschitz constant* $2\, L_s$*.*

□

Under the given conditions the value obtained in the minimisations (2.7) and (2.8) is continuously dependent on the reference point. But this does not mean that the resulting optima depend continuously on the reference point. Therefore, we will look into the behaviour of the set of optimal points for changing reference points. Since the mapping $\mathcal{O}_{\mathcal{Y},s}$ is set-valued, we need a set-valued continuity definition. We will work with the definitions of Aubin and Frankowska ([3]).

Definition 2.5.4 *Let $\mathbb{X}, \mathbb{Y}$ be metric spaces and let*

$$\mathcal{F}\colon \mathbb{X} \to \mathfrak{P}\big(\mathbb{Y}\big)$$

be a set-valued mapping. Then

a) The graph $\mathcal{G}(\mathcal{F})$ *of the set-valued map is defined by* $\mathcal{G}(\mathcal{F}) := \{(\mathbf{x}, \mathbf{y}) \in \mathbb{X} \times \mathbb{Y} \mid \mathbf{y} \in \mathcal{F}(\mathbf{x})\}$.

b) The domain $\mathcal{D}(\mathcal{F})$ *of the set-valued map $\mathcal{F}$ is the set of $\mathbf{x} \in \mathbb{X}$, where the image $\mathcal{F}(\mathbf{x}) \neq \emptyset$ is nonempty.*

c) The set-valued map $\mathcal{F}$ is called closed*, if its graph $\mathcal{G}(\mathcal{F})$ is closed.*

d) The set-valued map $\mathcal{F}$ is called upper semicontinuous *at $\mathbf{x} \in \mathcal{D}(\mathcal{F})$, if and only if for any neighbourhood $\mathcal{U}$ of $\mathcal{F}(\mathbf{x})$, there exists $\delta > 0$ such that for each $\bar{\mathbf{x}} \in \mathcal{B}_\delta(\mathbf{x})$ the image $\mathcal{F}(\bar{\mathbf{x}}) \subseteq \mathcal{U}$ of $\bar{\mathbf{x}}$ is contained in the neighbourhood $\mathcal{U}$. $\mathcal{F}$ is said to be upper semicontinuous if and only if it is upper semicontinuous at any point $\mathbf{x} \in \mathcal{D}(\mathcal{F})$.*

e) The set-valued map $\mathcal{F}$ is called lower semicontinuous *at $\mathbf{x} \in \mathcal{D}(\mathcal{F})$, if and only if for any $\mathbf{y} \in \mathcal{F}(\mathbf{x})$ and for any sequence $\big(\mathbf{x}(i)\big)_{i\in\mathbb{N}}$ converging to $\mathbf{x}$, there exists a sequence of elements $\big(\mathbf{y}(i)\big)_{i\in\mathbb{N}}$ with $\mathbf{y}(i) \in \mathcal{F}(\mathbf{x}(i))$ converging to $\mathbf{y}$. $\mathcal{F}$ is said to be lower semicontinuous if and only if it is lower semicontinuous at any point $\mathbf{x} \in \mathcal{D}(\mathcal{F})$.*

So, for an $\mathcal{F}$ with compact graph upper semicontinuity is equivalent to the ε-δ-continuity-criterion. Meanwhile, lower semicontinuity ensures that convergent sequences will be mapped to convergent sequences. Unfortunately, the two criteria do not in general coincide for set-valued maps ([3]).

As a next step we will derive the upper semicontinuity of the optimal set mapping with respect to the reference point. Shifting the reference point slightly will not move the new set of optima out of an open neighbourhood of the old set of optima.

Theorem 2.5.5 *Let s be a continuous, $\mathcal{C}$-centred and $\mathcal{C}$-monotone cone scalarising function and let the set of outcomes $\mathcal{Y}$ be compact. Then, the optimal set mapping $\mathcal{O}_{\mathcal{Y},s}$ is upper semicontinuous.*

Proof:
Let $\mathcal{U} \supset \mathcal{O}_{\mathcal{Y},s}(\hat{\mathbf{y}}^R)$ be an open neighbourhood of the optimal set for a given reference point $\hat{\mathbf{y}}^R$. Due to compactness of $\mathcal{Y}$ there exists a radius $\varepsilon > 0$ for which $\mathcal{O}_{\mathcal{Y},s}(\hat{\mathbf{y}}^R) + \mathcal{B}_{2\,\varepsilon}(\mathbf{0}) \subseteq \mathcal{U}$. Now consider the family of open sets

$$\{int(\mathcal{B}_\varepsilon(\mathbf{y})) \mid \mathbf{y} \in \mathcal{Y}\},$$

which constitutes a covering of $\mathcal{Y}$. As $\mathcal{Y}$ is compact we can pick a finite number of points $\check{\mathbf{y}}(i) \in \mathcal{Y}$, $i = 1, ..., n$ such that

$$\mathcal{Y} \quad \subset \quad \bigcup_{i=1}^{n} int(\mathcal{B}_\varepsilon(\check{\mathbf{y}}(i))).$$

Let $\big(\mathbf{y}^R(i)\big)_{i\in\mathbb{N}}$ be a sequence of reference points converging to $\hat{\mathbf{y}}^R$ and let

$$\big(\mathbf{y}(i)\big)_{i\in\mathbb{N}} \in \mathcal{O}_{\mathcal{Y},s}\big(\mathbf{y}^R(i)\big)$$

be optimal for the corresponding reference points. Since we have a finite covering of $\mathcal{Y}$ there must be sets $int\big(\mathcal{B}_\varepsilon(\check{\mathbf{y}}(i))\big)$ with infinitely many points of $\big(\mathbf{y}(i)\big)_{i\in\mathbb{N}}$ in them.

Choose an arbitrary set $\mathcal{V} := int\big(\mathcal{B}_\varepsilon(\check{\mathbf{y}}(i'))\big)$ with infinitely many points of the sequence in it. Then the subsequence of $\big(\mathbf{y}(i)\big)_{i\in\mathbb{N}}$ with elements just in $\mathcal{V}$ is bounded and thus has a convergent subsequence $\big(\bar{\mathbf{y}}(i)\big)_{i\in\mathbb{N}}$. Let $\bar{\mathbf{y}} := \lim_{i\to\infty} \bar{\mathbf{y}}(i)$ be its limit and $\big(\bar{\mathbf{y}}^R(i)\big)_{i\in\mathbb{N}}$ be the corresponding sequence of reference points. The continuity of μ

$$\begin{aligned} \lim_{i\to\infty} \mu\big(\bar{\mathbf{y}}^R(i)\big) &= \mu\big(\lim_{i\to\infty} \bar{\mathbf{y}}^R(i)\big) \\ &= \mu\big(\hat{\mathbf{y}}^R\big) \end{aligned}$$

ensures that the convergence is preserved. As t_s is continuous as well we get the following result:

$$\begin{aligned} \lim_{i\to\infty} t_s\big(\bar{\mathbf{y}}(i) - \bar{\mathbf{y}}^R(i)\big) &= t_s\Big(\lim_{i\to\infty} \bar{\mathbf{y}}(i) - \lim_{i\to\infty} \bar{\mathbf{y}}^R(i)\Big) \\ &= t_s\big(\bar{\mathbf{y}} - \hat{\mathbf{y}}^R\big). \end{aligned}$$

Putting the two equations together,

$$\begin{aligned} \mu\big(\hat{\mathbf{y}}^R\big) &= \lim_{i\to\infty} \mu\big(\bar{\mathbf{y}}^R(i)\big) \\ &= \lim_{i\to\infty} t_s\big(\bar{\mathbf{y}}(i) - \bar{\mathbf{y}}^R(i)\big) \\ &= t_s\big(\bar{\mathbf{y}} - \hat{\mathbf{y}}^R\big), \end{aligned}$$

we see that the limit of the sequence $\bar{\mathbf{y}} \in \mathcal{O}_{\mathcal{Y},s}(\hat{\mathbf{y}}^R)$ is in the set of optima for the limit $\bar{\mathbf{y}}^R$ of the reference points.

Therefore, the set $cl(\mathcal{V})$ contains at least one point of $\mathcal{O}_{\mathcal{Y},s}\big(\hat{\mathbf{y}}^R\big)$. Since the choice of $\mathcal{V}$ was arbitrary among the sets containing infinitely many points of $\big(\mathbf{y}(i)\big)_{i\in\mathbb{N}}$ the closures of all these sets intersect with $\mathcal{O}_{\mathcal{Y},s}\big(\hat{\mathbf{y}}^R\big)$. As the diameters of the sets $int\big(\mathcal{B}_\varepsilon\big(\check{\mathbf{y}}(i)\big)\big)$ forming the coverage is $2\,\varepsilon$ there are only finitely many points of $\big(\mathbf{y}(i)\big)_{i\in\mathbb{N}}$ that lie outside

$$\mathcal{O}_{\mathcal{Y},s}\big(\hat{\mathbf{y}}^R\big) + \mathcal{B}_{2\,\varepsilon}(\mathbf{0}) \subseteq \mathcal{U}.$$

Hence, choosing $i_0 \in \mathbb{N}$ such that all these points are excluded, the remaining outcomes $\mathbf{y}(i) \in \mathcal{U}, i \geq i_0$ are contained in the neighbourhood. Thus, the optimal sets for a convergent sequence of reference points are eventually contained in an open neighbourhood of the optimal set for the limit of the reference points.

□

Besides the compactness of $\mathcal{Y}$ the assumptions for the preceding theorem are only just enough to make the distance evaluation function well-defined and $\mathcal{C}$-monotone. Thus, if we want to relax the condition of compactness of $\mathcal{Y}$ we will have to place some restrictions on the cone scalarising function s and the distance evaluation function t_s.

Definition 2.5.6 *If for a continuous, $\mathcal{C}$-centred and strictly $\mathcal{C}$-consistent cone scalarising function s there exist continuous $\rho : \mathbb{R} \to \mathbb{R}$ and $\mathbf{y}^A : \mathbb{R} \to \mathbb{Y}$ with $t_s\big(\mathbf{y}^A(\tau)\big) = \tau$ for every $\tau \in \mathbb{R}$ such that*

$$\mathbf{y}^A(\tau) - int(\mathcal{C}) \quad \subseteq \quad \mathcal{L}_<(t_s, \tau) \quad \subseteq \quad \mathcal{B}_\rho\big(\mathbf{y}^A(\tau) - \sqrt[K]{\rho(\tau)}\,\mathbf{1}\big) - int(\mathcal{C})$$

and the level sets are convex for every $\tau \in \mathbb{R}$, the function is called $\mathcal{C}$-representing.

If for a continuous, $\mathcal{C}$-centred and strongly $\mathcal{C}$-consistent cone scalarising function s there exist $\mathbf{c}^* \in int(\mathcal{C}^*)$ and an $0 < \varepsilon < 1$ such that the corresponding distance evaluation function t_s fulfils

$$\mathbf{y} - \big(\mathcal{C} \setminus \{\mathbf{0}\}\big) \quad \subseteq \quad \mathcal{L}_<(t_s, \tau) \quad \subseteq \quad \mathcal{B}_{\rho(\tau)}\big(\mathbf{y}^A(\tau) - \sqrt[K]{\rho(\tau)}\,\mathbf{1}\big) - \mathcal{C}(\mathbf{c}^*, \varepsilon)$$

and the level sets are convex for every $\tau \in \mathbb{R}$, the function is called $\mathcal{C}$-approximating.

If the radius is $\rho \equiv 0$, the cone scalarising function s is called strictly $\mathcal{C}$-approximating and strictly $\mathcal{C}$-representing.

Before we move on with the dependence of optimisation outcomes on the reference point we have a closer look on some aspects of the previous definition.

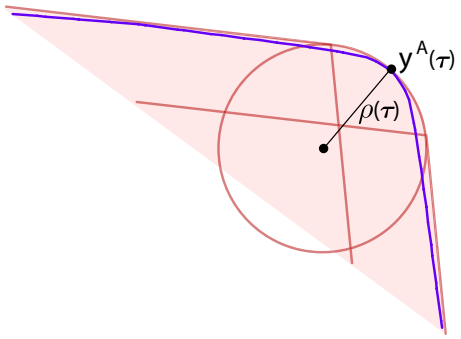


Figure 2.5: The level set is approximated from the outside by a cone plus a ball of appropriate radius.

Remark 2.5.7

- *The definition 2.5.6 extends the previous definition 2.4.2, since strictly $\mathcal{C}$-representing or strictly $\mathcal{C}$-approximating functions are strictly locally $\mathcal{C}$-representing or strictly locally $\mathcal{C}$-approximating at every $\mathbf{y}^A(\tau)$ and $\tau \in \mathbb{R}$.*

- *The choice of ε is independent from the level τ whereas the anchor point $\mathbf{y}^A$ and the radius ρ explicitely depend on the level τ.*

- *The convexity condition for the level sets directly implies that the distance evaluation function t_s is quasiconvex.*

- *Working with the composition $g \circ s$ of a strictly monotone function $g\colon \mathbb{R} \to \mathbb{R}$ with the cone scalarising function s does not affect the properties required in the above definition.*

- *A strictly $\mathcal{C}$-approximating or $\mathcal{C}$-representing function cannot be continuously differentiable at the anchor point $\mathbf{y}^A(\tau)$ for any $\tau \in \mathbb{R}$. Otherwise the level sets would not have the appropriate kink to fit into a cone. This is the reason why the non-strict concepts allow a certain rounding of the level sets at the anchor point so that continuously differentiable functions t_s fit into the discussed framework.*

The $\mathcal{C}$-representation or $\mathcal{C}$-approximation property is a strong enough regularity condition to enable us to drop the compactness of $\mathcal{Y}$ for a much weaker condition. This weaker condition together with the $\mathcal{C}$-representation or $\mathcal{C}$-approximation property allows us to concentrate on a compact subset of the original domain.

Theorem 2.5.8 *Suppose there exists an $\hat{\mathbf{a}} \in \mathbb{Y}$ with $\mathcal{Y} \subseteq \hat{\mathbf{a}} + \mathcal{C}$ for the $\mathcal{C}$-closed outcome set $\mathcal{Y}$ and let s be a $\mathcal{C}$-representing or $\mathcal{C}$-approximating*

function. Then there exists for every converging sequence $\left(\mathbf{y}^R(i)\right)_{i\in\mathbb{N}}$ *of reference points an* $i' \in \mathbb{N}$ *such that the set of optima* $\mathcal{O}_{\mathcal{Y},s}\left(\mathbf{y}^R(i)\right)$ *is contained in a fixed order interval for* $i \geq i'$*.*

Proof:

Let $\bar{\mathcal{C}}$ *denote* $\mathcal{C}$ *in case of the* $\mathcal{C}$*-representing and* $\mathcal{C}(\mathbf{c}^*, \varepsilon)$ *in case of the* $\mathcal{C}$*-approximating functions. Furthermore, let* $\hat{\mathbf{y}}^R = \lim_{i\to\infty} \mathbf{y}^R(i)$ *be the limit of the reference points and let* $\hat{\mathbf{y}} \in \mathcal{O}_{\mathcal{Y},s}\left(\hat{\mathbf{y}}^R\right)$ *be optimal for it.*

Since t_s *is continuous, there exists for every* $\varepsilon > 0$ *a* $\delta > 0$ *such that for* $\mathbf{y}^R \in \mathbb{Y}$ *with sufficiently small distance*

$$\left\| \left(\hat{\mathbf{y}} - \mathbf{y}^R\right) - \left(\hat{\mathbf{y}} - \hat{\mathbf{y}}^R\right) \right\|_2 \;=\; \left\| \mathbf{y}^R - \hat{\mathbf{y}}^R \right\|_2 \;\leq\; \delta$$

the difference in function value is smaller than $\left| t_s\left(\hat{\mathbf{y}} - \mathbf{y}^R\right) - t_s\left(\hat{\mathbf{y}} - \hat{\mathbf{y}}^R\right) \right| \leq \varepsilon$*.*

Due to the convergence of the sequence, there is for every $\delta > 0$ *an* $i(\delta)$ *such that the remainder of the sequence*

$$\left\| \mathbf{y}^R(i) - \hat{\mathbf{y}}^R \right\|_2 \leq \delta \quad \textit{i.e.} \quad \mathbf{y}^R(i) \in \mathcal{B}_\delta\left(\hat{\mathbf{y}}^R\right) \quad \textit{for all} \;\; i \geq i(\delta)$$

is contained in a δ*-ball around the limit. We can therefore apply the above continuity bound and estimate the biggest occurring minimal function value*

$$\begin{aligned} \max_{i\geq i(\delta)} \left\{\mu\left(\mathbf{y}^R(i)\right)\right\} \;&\leq\; \max_{i\geq i(\delta)} \left\{t_s\left(\hat{\mathbf{y}} - \mathbf{y}^R(i)\right)\right\} \\ &\leq\; t_s\left(\hat{\mathbf{y}} - \hat{\mathbf{y}}^R\right) + \varepsilon \;=:\; \hat{\tau} \end{aligned}$$

by the one for the limit reference point plus ε*. Therefore, the set of optima*

$$\begin{aligned} \mathcal{O}_{\mathcal{Y},s}\left(\mathbf{y}^R(i)\right) \;&\subseteq\; \hat{\mathbf{y}}^R + \mathcal{L}_{\leq}(t_s, \hat{\tau}) \\ &\subseteq\; \mathcal{B}_{\rho(\hat{\tau})}\left(\hat{\mathbf{y}}^R + \mathbf{y}^A(\hat{\tau}) - \sqrt[K]{\rho(\hat{\tau})}\,\mathbf{1}\right) - \bar{\mathcal{C}} \end{aligned}$$

is contained in the level set for $\hat{\tau}$ *for* $i \geq i(\delta)$*. By proposition 2.2.13 the ball*

$$\begin{aligned} &\mathcal{B}_{\rho(\hat{\tau})}\left(\hat{\mathbf{y}}^R + \mathbf{y}^A(\hat{\tau}) - \sqrt[K]{\rho(\hat{\tau})}\,\mathbf{1}\right) \\ &\qquad\subseteq\; \left(\left(\hat{\mathbf{y}}^R + \hat{\mathbf{y}}^A(\hat{\tau}) - \left(\rho(\hat{\tau}) + \sqrt[K]{\rho(\hat{\tau})}\right)\mathbf{1}\right) + \bar{\mathcal{C}}\right) \\ &\qquad\quad\cap \left(\left(\hat{\mathbf{y}}^R + \hat{\mathbf{y}}^A(\hat{\tau}) + \left(\rho(\hat{\tau}) - \sqrt[K]{\rho(\hat{\tau})}\right)\mathbf{1}\right) - \bar{\mathcal{C}}\right) \\ &\qquad\subseteq\; \Big(\underbrace{\hat{\mathbf{y}}^R + \hat{\mathbf{y}}^A(\hat{\tau}) + \left(\rho(\hat{\tau}) - \sqrt[K]{\rho(\hat{\tau})}\right)\mathbf{1}}_{=:\hat{\mathbf{b}}}\Big) - \bar{\mathcal{C}} \end{aligned}$$

can be embedded into an appropriate negative cone.

If we combine the inclusion derived so far we get

$$\begin{aligned}\mathcal{O}_{\mathcal{Y},s}\big(\mathbf{y}^R(i)\big) &\subseteq \mathcal{Y} \cap \big(\hat{\mathbf{y}}^R + \mathcal{L}_{\leq}(t_s, \hat{\tau})\big)\\ &\subseteq \mathcal{Y} \cap \Big(\mathcal{B}_{\rho(\hat{\tau})}\big(\mathbf{y}^A(\hat{\tau}) - \sqrt[K]{\rho(\hat{\tau})}\,\mathbf{1}\big) - \bar{\mathcal{C}}\Big)\\ &\subseteq \mathcal{Y} \cap \Big(\big(\hat{\mathbf{b}} - \bar{\mathcal{C}}\big) - \bar{\mathcal{C}}\Big),\end{aligned}$$

which holds for all $i \geq i(\delta)$. The convexity of $\bar{\mathcal{C}}$ implies that the difference $-\bar{\mathcal{C}} - \bar{\mathcal{C}} = -\bar{\mathcal{C}}$ is the negative cone itself. Using the assumed lower bound and the fact that $\mathcal{C} \subseteq \bar{\mathcal{C}}$ the outcome set $\mathcal{Y} \subseteq \hat{\mathbf{a}} + \mathcal{C} \subseteq \hat{\mathbf{a}} + \bar{\mathcal{C}}$ and thus

$$\mathcal{Y} \cap \Big(\big(\hat{\mathbf{b}} - \bar{\mathcal{C}}\big) - \bar{\mathcal{C}}\Big) \subseteq \big(\hat{\mathbf{a}} + \bar{\mathcal{C}}\big) \cap \big(\hat{\mathbf{b}} - \bar{\mathcal{C}}\big).$$

Since $\bar{\mathcal{C}}$ is by proposition 2.4.3 convex, closed and pointed and has non-empty interior, the order interval is compact by proposition 2.2.10. Thus, the set of optima $\mathcal{O}_{\mathcal{Y},s}\big(\mathbf{y}^R(i)\big)$ is contained in a compact set for every $i \geq i' := i(\delta)$. □

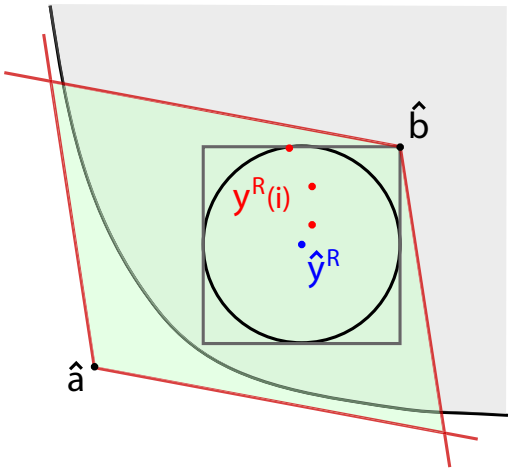


Figure 2.6: If the outcome set is contained in a translated cone, we can restrict the convergence considerations to an order interval.

The theorem now allows us to narrow the situation to a compact subset, if a lower bound $\hat{\mathbf{a}}$ exists. The following corollary uses this localisation to extend the continuity result for the minimal value function μ to the new situation.

Corollary 2.5.9 *If there exists an $\hat{\mathbf{a}} \in \mathbb{Y}$ with $\mathcal{Y} \subseteq \hat{\mathbf{a}} + \mathcal{C}$ for the $\mathcal{C}$-closed outcome set $\mathcal{Y}$ then for a $\mathcal{C}$-representing or $\mathcal{C}$-approximating cone scalarising function s, the minimal value function μ is continuous.*

Proof:
Let $\left(\mathbf{y}^R(i)\right)_{i\in\mathbb{N}}$ *be an arbitrary sequence that converges to* $\lim_{i\to\infty}\mathbf{y}^R(i)=\hat{\mathbf{y}}^R$. *By theorem 2.5.8 it suffices for every* $i\geq i'\in\mathbb{N}$ *to consider a compact order interval to find the minimum. In this order interval we can follow the reasoning of theorem 2.5.2 to prove the continuous dependence of* μ *on the reference point.*

□

The same localisation argument enables us to drop the compactness of the outcome set $\mathcal{Y}$ in the proof of the upper semicontinuity of the optimal set mapping $\mathcal{O}_{\mathcal{Y},s}$.

Corollary 2.5.10 *If there exists an* $\hat{\mathbf{a}}\in\mathbb{Y}$ *with* $\mathcal{Y}\subseteq\hat{\mathbf{a}}+\mathcal{C}$ *for the* $\mathcal{C}$*-closed outcome set* $\mathcal{Y}$ *then for a* $\mathcal{C}$*-representing or* $\mathcal{C}$*-approximating function* s*, the optimal set mapping* $\mathcal{O}_{\mathcal{Y},s}$ *is upper semicontinuous.*

Proof:
Let $\left(\mathbf{y}^R(i)\right)_{i\in\mathbb{N}}$ *be an arbitrary sequence that converges to* $\lim_{i\to\infty}\mathbf{y}^R(i)=:\hat{\mathbf{y}}^R$. *By theorem 2.5.8 it suffices to consider a compact order interval that contains* $\mathcal{O}_{\mathcal{Y},s}\left(\mathbf{y}^R(i)\right)$ *for every* $i\geq i'\in\mathbb{N}$. *In this order interval we can follow the argumentation of theorem 2.5.5 to prove the upper semicontinuity.*

□

We will now use the following proposition from the book of Aubin and Frankowska ([3]) to derive the closedness of $\mathcal{O}_{\mathcal{Y},s}$.

Proposition 2.5.11 *The graph of an upper semicontinuous set-valued map* $\mathcal{F}:\mathcal{X}\to\mathfrak{P}\left(\mathcal{Y}\right)$ *with closed domain and closed values is closed.*

Since we have all the necessary prerequisites, we can directly derive the closedness.

Corollary 2.5.12 *The optimal set mapping* $\mathcal{O}_{\mathcal{Y},s}$ *is closed.*

Proof:
The sets of minimisers of μ *are closed sets due to the closedness of* $\mathcal{Y}$ *and the continuity of* t_s. *Furthermore, the set of reference points* $\mathbb{Y}$ *is closed and therefore the set-valued mapping* $\mathcal{O}_{\mathcal{Y},s}$ *is closed, whenever it is upper semicontinuous.*

□

Lower semicontinuity is not as easy to achieve, since it states that the optimal set cannot suddenly become larger when the limit of a sequence of reference points is reached. So, it clearly does not hold for nonconvex outcome sets, where touching a new part of the outcome set can suddenly happen (figure 2.7). But even for convex cases, there can be a 'sudden' meeting of

common faces of the optimal level set with the outcome set, thus suddenly enlarging the optimal set. But there is a way to ensure lower semicontinuity in our case: if the optimal set is always a singleton.

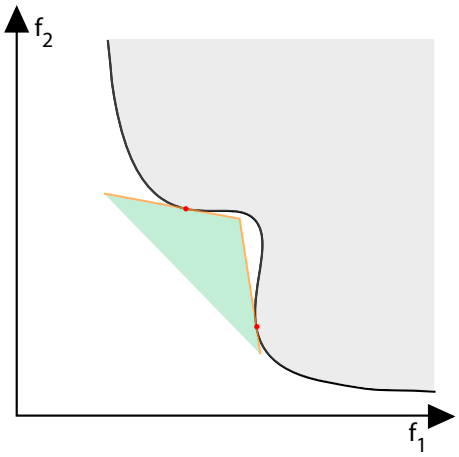


Figure 2.7: The red points depict the optimal set in this case. Thus, uniqueness is not given for general $\mathcal{Y}$.

Therefore, we will dedicate the remainder of the section to the question of uniqueness of the optimum. For that we need a few more concepts from convex analysis ([95]).

Definition 2.5.13

a) A vector $\mathbf{y}^*$ *is said to be normal to a convex set* $\mathcal{S}$ *at the point* $\mathbf{y}$*, if for every* $\mathbf{y}' \in \mathcal{S}$ *the inner product* $\langle \mathbf{y}^*, \mathbf{y}' - \mathbf{y} \rangle \leq 0$ *is nonpositive.*

b) A vector $\mathbf{y}^*$ *is said to be a subgradient of a convex function* $f : \mathbb{X} \to \mathbb{Y}$ *at a point* $\mathbf{x}$ *if* $f(\mathbf{x}') \geq f(\mathbf{x}) + \langle \mathbf{x}^*, \mathbf{x}' - \mathbf{x} \rangle$ *for all* $\mathbf{x}' \in \mathbb{X}$*. The set of all subgradients of* f *at* $\mathbf{x}$ *is called subdifferential of* f *at* $\mathbf{x}$ *and is denoted by* $\partial f(\mathbf{x})$*.*

c) A face of a convex set $\mathcal{S}$ *is a convex subset* $\mathcal{S}'$ *of* $\mathcal{S}$ *such that every (closed) line segment in* $\mathcal{S}$ *with a relative interior point in* $\mathcal{S}'$ *has both endpoints in* $\mathcal{S}'$*.*

The definition of a normal vector ensures that it does not have a acute angle with any straight line contained in the set. The subgradient inequality is a linear approximation of the function f and thus the subdifferential is a generalisation of the gradient. In particular, the gradient is the unique subgradient at a given point, if the function f is continuously differentiable at that point. The definition of a face formalises the concept of a flat part of

the boundary. Note that faces can have arbitrary dimension. In particular, the extreme points of a convex set are faces.

Additionally, we will need the following two theorems from the book of Rockafellar ([95]). The first is a separation theorem.

Theorem 2.5.14 *Let $\emptyset \neq S_1, S_2 \subseteq \mathbb{R}^n$ be convex. There is a $\mathbf{x}^* \in \mathbb{R}^n$ such that*

$$\inf_{\mathbf{x}\in S_1} \langle \mathbf{x}, \mathbf{x}^* \rangle \geq \sup_{\mathbf{x}\in S_2} \langle \mathbf{x}, \mathbf{x}^* \rangle \quad \text{and} \quad \sup_{\mathbf{x}\in S_1} \langle \mathbf{x}, \mathbf{x}^* \rangle > \inf_{\mathbf{x}\in S_2} \langle \mathbf{x}, \mathbf{x}^* \rangle$$

if and only if $ri(S_1) \cap ri(S_2) = \emptyset$.

The second one describes the relation between normal vectors to the feasible set and the subdifferential of the function to be minimised over that set.

Theorem 2.5.15 *Let h be a proper convex function and let S be a nonempty convex set. In order that $\mathbf{x}$ be a point where the infimum of h relative to S is attained, it is sufficient that there is a vector $\mathbf{x}^* \in \partial h(\mathbf{x})$ such that $-\mathbf{x}^*$ is normal to S at $\mathbf{x}$. This condition is necessary, as well as sufficient, if $ri(dom(h))$ intersects $ri(S)$, where $dom(h) := \{\mathbf{x} \in \mathbb{X} \mid h(\mathbf{x}) < \infty\}$ denotes the domain of definition.*

As can be seen in figure 2.7 uniqueness cannot be expected in general cases. Convexity of s and $\mathcal{Y}$ is enough to infer the connectedness of the set of optima, but for the uniqueness of the optima we need further assumptions.

Lemma 2.5.16 *If the set of outcomes $\mathcal{Y}$ is $\mathcal{C}$-convex, the cone scalarising function s continuous, $\mathcal{C}$-centred, strictly $\mathcal{C}$-consistent and convex and if no level set of t_s has a face of dimension 1 and higher parallel to a face of dimension 1 and higher of $eff_{w-\mathcal{C}}(\mathcal{Y})$, then the set of optima $\mathcal{O}_{\mathcal{Y},s}(\mathbf{y}^R)$ is a singleton for every reference point $\mathbf{y}^R \in \mathbb{Y}$.*

Proof:

Since $\hat{\mathbf{y}}$ is optimal for $t_s(\cdot - \mathbf{y}^R)$ and the domain of $t_s(\cdot - \mathbf{y}^R)$ is the whole space $\mathbb{Y}$, there is a subgradient $\mathbf{y}^* \in \partial t_s(\hat{\mathbf{y}} - \mathbf{y}^R)$ the negative of which is normal to $\mathcal{Y}$. By the definition of subgradient and normal it follows that the hyperplane

$$\{\mathbf{y} \in \mathbb{Y} \mid \langle \mathbf{y}^*, \mathbf{y} \rangle = \langle \mathbf{y}^*, \hat{\mathbf{y}} \rangle\} =: \mathcal{H}$$

properly separates the level set $\mathcal{L}_{\leq}(t_s, t_s(\hat{\mathbf{y}}))$ from $\mathcal{Y}$. But the optimal set

$$\mathcal{O}_{\mathcal{Y},s}(\mathbf{y}^R) \quad = \quad (\mathbf{y}^R + \mathcal{L}_{\leq}(t_s, t_s(\hat{\mathbf{y}}))) \cap \mathcal{Y}$$

is the intersection of the two sets. Thus, $\mathcal{O}_{\mathcal{Y},s}(\mathbf{y}^R) \subseteq \mathcal{H}$. If the optimal set would not be a singleton, the intersections of the level set as well as the outcome set with the hyperplane would contain more than one point and hence the two sets would have a common face of dimension 1 and higher.

□

The lemma is quite general, but unfortunately the conditions cannot be verified in practise, since they involve (almost) full knowledge about the outcome set. Therefore, we will look at some special cases, where we can derive the conditions from structural knowledge about the problem. The simplest such condition is strict convexity.

Proposition 2.5.17 *If the cone scalarising function s is continuous, $\mathcal{C}$-centred, strictly $\mathcal{C}$-consistent and convex and $\mathcal{Y}$ is strictly $\mathcal{C}$-convex with respect to $eff_s(\mathcal{Y})$, i.e. for two arbitrary points $\mathbf{y} \neq \mathbf{y}' \in eff_s(\mathcal{Y})$*

$$(1-\lambda)\,\mathbf{y} \;+\; \lambda\,\mathbf{y}' \;\in\; int\big(\mathcal{Y}+\mathcal{C}\big) \quad \textit{for all} \;\; \lambda \in (0,1),$$

then the optimal set $\mathcal{O}_{\mathcal{Y},s}\big(\mathbf{y}^R\big)$ is a singleton for all reference points $\mathbf{y}^R \in \mathbb{Y}$.

Proof:

By its definition $eff_s(\mathcal{Y})$ is the union of optimal sets $\mathcal{O}_{\mathcal{Y},s}\big(\mathbf{y}^R\big)$ for all reference points $\mathbf{y}^R \in \mathbb{Y}$. But the above strict convexity condition ensures that these sets do not contain any faces of dimension 1 and higher. Applying lemma 2.5.16 proves the claim.

□

In the preceding proposition an appropriate part of the outcome set was strictly convex. But the condition of lemma 2.5.16 also applies, if the function t_s is strictly convex in an appropriate part of its domain.

Proposition 2.5.18 *Suppose s is a continuous, $\mathcal{C}$-centred and strictly $\mathcal{C}$-consistent cone scalarising function. Let $\mathbf{y}, \mathbf{y}'$ be two arbitrary points in the level set*

$$\mathbf{y}, \mathbf{y}' \;\in\; eff_{w-(-\mathcal{C})}\big(\mathcal{L}_{\leq}(t_s, \tau)\big)$$

that are efficient for the multiobjective maximisation problem over the level set for a given $\tau \in \mathbb{R}$. If for the distance evaluation function t_s associated with s the condition

$$t_s\big((1-\lambda)\,\mathbf{y} \;+\; \lambda\,\mathbf{y}'\big) < (1-\lambda)\,t_s(\mathbf{y}) \;+\; \lambda\,t_s(\mathbf{y}') \quad \textit{for all} \;\; \lambda \in (0,1)$$

holds and the set $\mathcal{Y}$ is $\mathcal{C}$-convex, then the optimal set $\mathcal{O}_{\mathcal{Y},s}\big(\mathbf{y}^R\big)$ is a singleton for all reference points $\mathbf{y}^R \in \mathbb{Y}$.

Proof:

By the conditions on the cone scalarising function s the associated t_s is continuous and strictly $\mathcal{C}$-monotone. Hence, the optimal set $\mathcal{O}_{\mathcal{Y},s}\big(\mathbf{y}^R\big) \subseteq eff_{w-\mathcal{C}}(\mathcal{Y})$ is contained in the set of weakly $\mathcal{C}$-efficient solutions for all reference points $\mathbf{y}^R \in \mathbb{Y}$. Therefore, any face contained in $\mathcal{O}_{\mathcal{Y},s}\big(\mathbf{y}^R\big)$ lies in a hyperplane with a normal vector contained in $\mathcal{C}^* \setminus \{\mathbf{0}\}$. The points from that face would be weakly $(-\mathcal{C})$-efficient in $\mathcal{L}_{\leq}(t_s, \tau)$. But by the strict convexity condition for t_s the face can just contain one minimal point. Thus, we can again apply the lemma 2.5.16.

□

This proposition applies in particular to the case, where the function s and therefore t_s is strictly convex.

The conditions used in the different uniqueness results also imply upper semicontinuity and closedness of $\mathcal{O}_{\mathcal{Y},s}$, so that $\mathcal{O}_{\mathcal{Y},s}$ is an ordinary *continuous* function in these cases. Changing the reference point thus continuously changes the optimal solution – a very favourable property.

2.6 Reachability of outcomes

In section 2.4 we have shown that strictly locally $\mathcal{C}$-representing and strictly locally $\mathcal{C}$-approximating cone scalarising functions s can reach every $\bar{\mathcal{C}}$-efficient point, where $\bar{\mathcal{C}}$ is the cone containing the level set – $\mathcal{C}$ for the strictly locally $\mathcal{C}$-representing and $\mathcal{C}(\mathbf{c}^*, \varepsilon)$ for the strictly locally $\mathcal{C}$-approximating functions.

Nonetheless, the results were unsatisfactory, since one had to know the position of the optimum to place the reference point correspondingly only to get the optimum back. Wierzbicki has called such results "tautological" ([125]).

Thus, we want to show stronger reachability results now using 'globally' $\mathcal{C}$-representing and $\mathcal{C}$-approximating cone scalarising functions as defined in 2.5.6. We will start with the easiest case – the strictly $\mathcal{C}$-representing functions.

Theorem 2.6.1 *For a strictly $\mathcal{C}$-representing cone scalarising function s and for each of its optimal outcomes $\hat{\mathbf{y}} \in eff_s(\mathcal{Y})$ there is a continuous path of reference points $\mathbf{y}^R(\tau), \tau \in \mathbb{R}$ for which $\hat{\mathbf{y}}$ is optimal.*

Proof:

Assume that the reference point corresponding to $\hat{\mathbf{y}}$ is $\hat{\mathbf{y}}^R$ and denote the optimal value by $\hat{\tau} = t_s(\hat{\mathbf{y}} - \hat{\mathbf{y}}^R)$. By the optimality of the point $\hat{\mathbf{y}}$ we know that the difference between the point and the reference point

$$\hat{\mathbf{y}} - \hat{\mathbf{y}}^R \ \in \ \mathcal{L}_{=}(t_s, \hat{\tau}) \ \cap \ \mathcal{Y}$$

is contained in the intersection of the equality level set $\mathcal{L}_{=}(t_s, \hat{\tau})$ with the set of outcomes $\mathcal{Y}$. The strict $\mathcal{C}$-representation enables us to rewrite the level set as

$$\mathcal{L}_{=}(t_s, \hat{\tau}) \ = \ \mathbf{y}^A(\hat{\tau}) - bd(\mathcal{C}) \ = \ \big(\mathbf{y}^A(\tau) + \mathbf{y}^A(\hat{\tau}) - \mathbf{y}^A(\tau)\big) - bd(\mathcal{C}).$$

Here, we used the fact that the distance evaluation function t_s is – due to its continuity and strict $\mathcal{C}$-monotonicity – equal to τ exactly on the boundary of the negative cone attached to the corresponding anchor point $\mathbf{y}^A(\tau)$ for every $\tau \in \mathbb{R}$.

If we now define $\mathbf{y}^R(\tau)$ *to move in the opposite direction of the anchor points*

$$\mathbf{y}^R(\tau) \;:=\; \hat{\mathbf{y}}^R - \mathbf{y}^A(\tau) + \mathbf{y}^A(\hat{\tau}),$$

then the distance between $\hat{\mathbf{y}}$ *and the reference point* $\mathbf{y}^R(\tau)$

$$\hat{\mathbf{y}} - \mathbf{y}^R(\tau) \;=\; \hat{\mathbf{y}} - \hat{\mathbf{y}}^R + \mathbf{y}^A(\tau) - \mathbf{y}^A(\hat{\tau})$$

can be expressed by a term that only depends on the position of the anchor point for the level set of level τ*. The representation of the level set derived above transforms into the following equality:*

$$\begin{aligned} \hat{\mathbf{y}} \;&\in\; \big(\hat{\mathbf{y}}^R + \mathcal{L}_{=}(t_s, \hat{\tau})\big) \cap \mathcal{Y} \\ &=\; \big(\hat{\mathbf{y}}^R + \mathbf{y}^A(\hat{\tau}) - bd(\mathcal{C})\big) \cap \mathcal{Y}. \end{aligned}$$

Inserting the definition of $\mathbf{y}^R(\tau)$ *into it, we can continue with*

$$\begin{aligned} \hat{\mathbf{y}} \;&\in\; \big(\mathbf{y}^R(\tau) + \mathbf{y}^A(\tau) - \mathbf{y}^A(\hat{\tau}) + \mathbf{y}^A(\hat{\tau}) - bd(\mathcal{C})\big) \cap \mathcal{Y} \\ &=\; \big(\mathbf{y}^R(\tau) + \mathbf{y}^A(\tau) - bd(\mathcal{C})\big) \cap \mathcal{Y} \\ &=\; \big(\mathbf{y}^R(\tau) + \mathcal{L}_{=}(t_s, \tau)\big) \cap \mathcal{Y}. \end{aligned}$$

The difference between the optimal point $\hat{\mathbf{y}}$ *and the reference point* $\hat{\mathbf{y}}^R(\tau)$ *is contained in the intersection of the level set* $\mathcal{L}_{=}(t_s, \tau)$ *with the set of outcomes. Hence, the point* $\hat{\mathbf{y}}$ *is optimal for the reference point* $\hat{\mathbf{y}}^R(\tau)$*. The claim follows from the definition of* $\hat{\mathbf{y}}^R(\tau)$ *as an offset to the negative of the continuous path of anchor points* $\mathbf{y}(\tau)$*.*

□

Here, the explicit description of the level sets for the different levels enables us to directly place the reference point so that the minimum is still optimal for a given new value of the difference evaluation function t_s. Note that the same construction works whenever the distance evaluation function t_s has congruent level sets.

In the slightly more general case of strictly $\mathcal{C}$-approximating cone scalarising functions the argument is very similar, but it is guaranteed to work for the $\mathcal{C}(\mathbf{c}^*, \varepsilon)$-efficient points only. This is due to the fact, that for the $\mathcal{C}$-approximating functions, we only know the shape of the level set sufficiently well at the anchor point.

Theorem 2.6.2 *For a strictly* $\mathcal{C}$*-approximating cone scalarising function* s *there is for each* $\mathcal{C}(\mathbf{c}^*, \varepsilon)$*-efficient outcome* $\hat{\mathbf{y}}$ *a continuous path of reference points* $\mathbf{y}^R(\tau), \tau \in \mathbb{R}$ *for which* $\hat{\mathbf{y}}$ *is optimal.*

Proof:

Define the path of reference points

$$\mathbf{y}^R(\tau) \;:=\; \hat{\mathbf{y}} - \mathbf{y}^A(\tau)$$

to be the optimal point minus the current anchor point of the level set. The $\mathcal{C}(\mathbf{c}^*, \varepsilon)$*-efficiency of* $\hat{\mathbf{y}}$ *guarantees that the cone* $-\mathcal{C}(\mathbf{c}^*, \varepsilon)$ *attached to the point* $\hat{\mathbf{y}}$

$$\big(\hat{\mathbf{y}} - \mathcal{C}(\mathbf{c}^*, \varepsilon)\big)\big) \cap \mathcal{Y} \;=\; \{\hat{\mathbf{y}}\}$$

intersects the outcome set $\mathcal{Y}$ *with its tip only. The cone scalarising function* s *being strictly* $\mathcal{C}$*-approximating each level set*

$$\mathcal{L}_{\leq}(t_s, \tau) \;\supset\; \mathbf{y}^A(\tau) - \mathcal{C}$$

contains the appropriately shifted negative ordering cone. Hence the shifted level set

$$\hat{\mathbf{y}} \;\in\; \mathbf{y}^R(\tau) + \mathbf{y}^A(\tau) - \mathcal{C} \;\subset\; \mathbf{y}^R(\tau) + \mathcal{L}_{\leq}(t_s, \tau)$$

contains the point $\hat{\mathbf{y}}$*. Furthermore, each level set*

$$\mathcal{L}_{\leq}(t_s, \tau) \;\subseteq\; \mathbf{y}^A(\tau) - \mathcal{C}(\mathbf{c}^*, \varepsilon)$$

is known to be contained in the appropriately shifted negative enlarged cone. But this cone

$$\begin{aligned}\big(\mathbf{y}^R(\tau) + \mathcal{L}_{\leq}(t_s, \tau)\big) \cap \mathcal{Y} \;&\subseteq\; \big(\mathbf{y}^R(\tau) + \mathbf{y}^A(\tau) - \mathcal{C}(\mathbf{c}^*, \varepsilon)\big) \cap \mathcal{Y} \\ &=\; \big(\hat{\mathbf{y}} - \mathcal{C}(\mathbf{c}^*, \varepsilon)\big) \cap \mathcal{Y} \;=\; \{\hat{\mathbf{y}}\}\end{aligned}$$

intersects the outcome set with its tip only. Combining the results, we get the following inclusions:

$$\hat{\mathbf{y}} \;\in\; \big(\mathbf{y}^R(\tau) + \mathcal{L}_{\leq}(t_s, \tau)\big) \cap \mathcal{Y} \;\subseteq\; \{\hat{\mathbf{y}}\}$$

Thus, there is no other point in the intersection of the shifted level set with the outcome set. Hence, $\hat{\mathbf{y}}$ *is optimal for the cone scalarising problem for all reference points* $\mathbf{y}^R(\tau)$*.*

□

Although the result for the $\mathcal{C}$-approximating and the $\mathcal{C}$-representing, but not strictly $\mathcal{C}$-representing cone scalarising functions is not as simple and explicit, the flavour stays the same: we show that we can change the reference point such that we get a new reference point for each level τ with the old minimal point still being optimal. But since we do not have an explicit description of the level sets for the considered cone scalarising functions, the result will be more of a qualitative nature.

Before we can derive the wanted result, we first need some more convex analysis. So, let us recall some definitions (see e.g. [95]).

Definition 2.6.3 *The recession cone of a convex set* $\mathcal{Y}$

$$\mathcal{R}(\mathcal{Y}) \;:=\; \{\Delta\mathbf{y} \in \mathbb{Y} \mid \mathbf{y} + \lambda\,\Delta\mathbf{y} \in \mathcal{Y} \;\; \textit{for all} \;\; \lambda \geq 0 \;\textit{and}\; \mathbf{y} \in \mathcal{Y}\}$$

is the cone of directions for which the set extends to infinity in every point.

The barrier cone of a convex set $\mathcal{Y}$

$$\mathcal{R}^\circ(\mathcal{Y}) \;:=\; \left\{\mathbf{y}^* \in \mathbb{Y} \;\middle|\; \langle \mathbf{y}^*, \mathbf{y}\rangle \;\leq\; \beta, \;\; \textit{for all } \mathbf{y} \in \mathcal{Y} \;\; \textit{and some } \;\beta \in \mathbb{R}\right\}$$

is the set of directions for which a finite maximum point to the corresponding linear functional exists.

For a nonempty convex cone $\mathcal{C} \subseteq \mathbb{Y}$

$$\mathcal{C}^\circ \;:=\; \{\mathbf{y}^* \in \mathbb{Y} \mid \langle \mathbf{y}^*, \mathbf{y}\rangle \leq 0 \;\; \textit{for all} \;\; \mathbf{y} \in \mathcal{C}\}$$

is called the polar cone of $\mathcal{C}$*.*

We will summerise the properties of the three defined objects in the following proposition the content of which is mostly extracted from the book of Rockafellar ([95]).

Proposition 2.6.4 *For all following claims, we will assume the set* $\mathcal{Y}$ *to be nonempty and convex.*

- *The recession cone* $\mathcal{R}(\mathcal{Y})$ *is a convex cone containing the origin. It is the same as the set of vectors* $\Delta\mathbf{y} \in \mathbb{Y}$ *such that* $\Delta\mathbf{y} + \mathcal{Y} \subseteq \mathcal{Y}$*.*
- *If there is one point* $\mathbf{y} \in \mathcal{Y}$ *for which the direction* $\Delta\mathbf{y} \in \mathbb{Y}$ *extends to infinity, i.e.* $\mathbf{y} + \lambda\Delta\mathbf{y} \in \mathcal{Y}$ *for all* $\lambda \geq 0$*, then the direction* $\Delta\mathbf{y} \in \mathcal{R}(\mathcal{Y})$ *belongs to the recession cone and hence extends to infinity from every point in* $\mathcal{Y}$*.*
- *For closed* $\mathcal{Y}$ *the recession cone* $\mathcal{R}(\mathcal{Y})$ *is the polar of the barrier cone* $\mathcal{R}^\circ(\mathcal{Y})$ *and vice versa.*
- *The polar of a cone* $\mathcal{C}$ *is the negative* $\mathcal{C}^\circ = -\mathcal{C}^*$ *of the dual cone.*
- *The polar of a cone* $\mathcal{C} \subseteq \tilde{\mathcal{C}}$ *being included in another cone, is the superset* $\mathcal{C}^\circ \supseteq \tilde{\mathcal{C}}^\circ$ *of the other cone's polar.*

Equipped with the above definitions and relations, we are now able to show the existence of infinitely many reference points for each optimal point, specifically one for each level of the distance evaluation function.

For ease of exposition we will denote by $\bar{\mathcal{C}}$ the cone used in the outer approximation of the level sets, i.e. it will stand for $\mathcal{C}$ in case of the $\mathcal{C}$-representing functions and for $\mathcal{C}(\mathbf{c}^*, \varepsilon)$ in case of the $\mathcal{C}$-approximating functions.

Theorem 2.6.5 *If the cone scalarising function* s *is* $\mathcal{C}$*-approximating or* $\mathcal{C}$*-representing and the outcome set* $\mathcal{Y}$ *is* $\mathcal{C}$*-convex, there is for each weakly* $\bar{\mathcal{C}}$*-efficient outcome* $\hat{\mathbf{y}} \in eff_{w-\bar{\mathcal{C}}}(\mathcal{Y})$ *and for each level* $\tau \in \mathbb{R}$ *a reference point* $\mathbf{y}^R(\tau)$ *for which* $\hat{\mathbf{y}}$ *is optimal.*

Proof:

Due to the restrictions on the structure of the level sets the recession cone of an arbitrary level set

$$\mathcal{R}\big(\mathcal{L}_{\leq}(t_s,\tau)\big) \;\subseteq\; -\bar{\mathcal{C}}$$

is a subset of the cone $-\bar{\mathcal{C}}$ *for all* $\tau \in \mathbb{R}$. *Therefore, the barrier cone as the polar cone of the recession cone is a superset*

$$\mathcal{R}^{\circ}\big(\mathcal{L}_{\leq}(t_s,\tau)\big) \;\supseteq\; \big(-\bar{\mathcal{C}}\big)^{\circ} \;=\; \bar{\mathcal{C}}^{*}$$

of the dual cone $\bar{\mathcal{C}}^{*}$. *Thus, by the definition of the barrier cone there exists for every* $\mathbf{c}^{*} \in \bar{\mathcal{C}}^{*}$ *a point* $\tilde{\mathbf{y}} \in \mathcal{L}_{\leq}(t_s,\tau)$

$$\langle \mathbf{c}^{*}, \mathbf{y}\rangle \;\leq\; \langle \mathbf{c}^{*}, \tilde{\mathbf{y}}\rangle \quad \textit{for all} \;\; \mathbf{y} \in \mathcal{L}_{\leq}(t_s,\tau)$$

which maximises the linear functional associated with $\mathbf{c}^{*}$. *This is true for* every *level set* $\mathcal{L}_{\leq}(t_s,\tau), \tau \in \mathbb{R}$.

Suppose we are given a minimum $\hat{\mathbf{y}}$ *of the cone scalarising problem for some reference point* $\hat{\mathbf{y}}^{R}$. *Then we know that* $\hat{\mathbf{y}}$ *is weakly* $\bar{\mathcal{C}}$*-efficient. For convex* $\mathcal{Y} + \mathcal{C}$ *it therefore minimises a linear functional* $\langle \hat{\mathbf{c}}^{*}, \cdot\rangle$ *with the nontrivial direction* $\hat{\mathbf{c}}^{*} \in \bar{\mathcal{C}}^{*} \setminus \{\mathbf{0}\}$ *taken from the dual cone (2.2.8).*

By the above discussion there exists a point $\tilde{\mathbf{y}}(\tau) \in \mathcal{L}_{\leq}(t_s,\tau), \tau \in \mathbb{R}$ *in every level set that maximises the linear functional* $\langle \hat{\mathbf{c}}^{*}, \cdot\rangle$ *for which* $\hat{\mathbf{y}}$ *is a minimum over* $\mathcal{Y}$. *Defining the reference points by*

$$\mathbf{y}^{R}(\tau) \;:=\; \hat{\mathbf{y}} - \tilde{\mathbf{y}}(\tau),$$

the original optimal point $\hat{\mathbf{y}}$

$$\hat{\mathbf{y}} \;=\; \mathbf{y}^{R}(\tau) + \tilde{\mathbf{y}}(\tau) \;\in\; \mathbf{y}^{R}(\tau) + \mathcal{L}_{\leq}(t_s,\tau)$$

is contained in the level set translated by the newly defined reference point. Consider the maximisation of the linear functional over the translated level set:

$$\begin{aligned}
\max\big\{\langle \hat{\mathbf{c}}^{*}, \mathbf{y} + \mathbf{y}^{R}(\tau)\rangle \;\big|\; \mathbf{y} \in \mathcal{L}_{\leq}(t_s,\tau)\big\} & \\
&= \max\big\{\langle \hat{\mathbf{c}}^{*}, \mathbf{y}\rangle \;\big|\; \mathbf{y} \in \mathcal{L}_{\leq}(t_s,\tau)\big\} + \langle \hat{\mathbf{c}}^{*}, \mathbf{y}^{R}(\tau)\rangle \\
&= \langle \hat{\mathbf{c}}^{*}, \tilde{\mathbf{y}}(\tau)\rangle + \langle \hat{\mathbf{c}}^{*}, \mathbf{y}^{R}(\tau)\rangle \\
&= \langle \hat{\mathbf{c}}^{*}, \tilde{\mathbf{y}}(\tau) + \mathbf{y}^{R}(\tau)\rangle \\
&= \langle \hat{\mathbf{c}}^{*}, \hat{\mathbf{y}}\rangle.
\end{aligned}$$

But at the same time $\hat{\mathbf{y}}$ *minimises the functional over the set* $\mathcal{Y}$. *Since the negative direction* $-\hat{\mathbf{c}}^{*} \in \mathcal{R}(\mathcal{L}_{\leq}(t_s,\tau)$ *is contained in the recession cone, the infimum*

$$\inf\{\langle \mathbf{c}^{*}, \mathbf{y}\rangle \mid \mathbf{y} \in \mathcal{L}_{\leq}(t_s,\tau)\} \;=\; -\infty$$

is unbounded. By the separation theorem 2.5.14 the relative interiors of the two sets

$$ri\big(\mathbf{y}^R(\tau) + \mathcal{L}_{\leq}(t_s, \tau)\big) \cap ri(\mathcal{Y} + \mathcal{C}) = \emptyset$$

have empty intersection. So there is no point of $\mathcal{Y} + \mathcal{C}$ *in the interior*

$$\begin{aligned} \emptyset &= \Big(int\big(\mathbf{y}^R(\tau) + \mathcal{L}_{\leq}(t_s, \tau)\big)\Big) \cap \big(\mathcal{Y} + \mathcal{C}\big) \\ &= \big(\mathbf{y}^R(\tau) + \mathcal{L}_{<}(t_s, \tau)\big) \cap \big(\mathcal{Y} + \mathcal{C}\big) \end{aligned}$$

of the translated level set. Thus, there is no point in $\mathcal{Y}+\mathcal{C}$ *that has a smaller distance evaluation function value than* $\hat{\mathbf{y}}$

$$t_s\big(\hat{\mathbf{y}} - \mathbf{y}^R(\tau)\big) = \min\big\{t_s\big(\mathbf{y} - \mathbf{y}^R(\tau)\big) \mid \mathbf{y} \in \mathcal{Y}\big\}$$

for the specified reference point $\mathbf{y}^R(\tau)$.

□

The convexity assumption is necessary since the radius of the "rounding ball" in the definition 2.5.6 can change, which might render a point unreachable due to a larger radius in nonconvex cases (see figure 2.8).

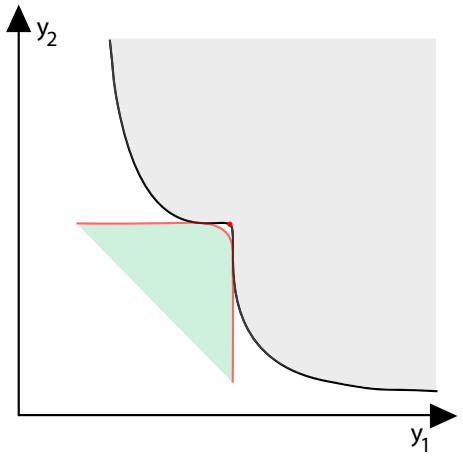


Figure 2.8: The red point cannot be reached even though it is efficient.

Note that this is in analogy to the situation in weighted metric scalarisation for the Pareto cone. For nonconvex outcome sets one needs to use the Chebycheff norm to be able to reach every efficient point, since smoother norms might not be able to reach 'hidden' efficient points ([83]).

2.7 Example – the Pascoletti-Serafini approach

In this section we will show that the well-known scalarisation approach of Pascoletti and Serafini is contained in our framework. Consider the following

cone scalarising function:

$$s\big(\mathbf{c}^+, \mathbf{c}^-\big) := \min\left\{z \in \mathbb{R} \mid \mathbf{c}^+ - \mathbf{c}^- + \mathbf{c} = z\,\mathbf{1},\ \mathbf{c} \in \mathcal{C}\right\}. \tag{2.17}$$

Note that the distance evaluation function t_s corresponding to s is constant for every given $\Delta\mathbf{y} \in \mathbb{Y}$. Therefore, assumption 2.3.7 is fulfilled for every ordering cone under consideration.

We will first prove the $\mathcal{C}$-centredness and the strict $\mathcal{C}$-consistency of this function.

Proposition 2.7.1 *The cone scalarising function (2.17) is $\mathcal{C}$-centred and strictly $\mathcal{C}$-consistent.*

Proof:

The $\mathcal{C}$-centredness is trivial as for all $\mathbf{c}^+$, $\mathbf{c}^-$ and $\mathbf{c}' \in \mathcal{C}$ the function values

$$s\big(\mathbf{c}^+,\, \mathbf{c}^-\big) = s\big(\mathbf{c}^+ + \mathbf{c}',\, \mathbf{c}^- + \mathbf{c}'\big)$$

agree. For the strict $\mathcal{C}$-consistency, we will first show that the optimal result is always associated with a $\mathbf{c} \in bd(\mathcal{C})$.

Let us assume that the $\hat{\mathbf{c}} \in int(\mathcal{C})$ for the optimum lies in the interior of the cone. Hence, for an arbitrary direction $\Delta\mathbf{c} \in \mathbb{Y}$ there is an $\bar{\varepsilon} > 0$ such that $\mathbf{c} + \varepsilon\,\Delta\mathbf{c} \in \mathcal{C}$ for all $0 \leq \varepsilon \leq \bar{\varepsilon}$. Choose $\Delta\mathbf{c}$ to be $-\mathbf{1}$ and let $\hat{\varepsilon}$ be the corresponding bound. Then for the optimal $\hat{z}$

$$\hat{z}\,\mathbf{1} = \mathbf{c}^+ - \mathbf{c}^- + \hat{\mathbf{c}} \quad\Longleftrightarrow\quad \hat{z}\,\mathbf{1} - \varepsilon\,\mathbf{1} = \mathbf{c}^+ - \mathbf{c}^- + \hat{\mathbf{c}} - \varepsilon\,\mathbf{1}$$

for all $0 < \varepsilon \leq \hat{\varepsilon}$. But this contradicts the optimality of $\hat{z}$, because the right hand side is feasible for (2.17). Therefore, the optimal $\hat{\mathbf{c}} \in bd(\mathcal{C})$ is contained in the boundary of the cone.

Now let $\tilde{\mathbf{c}} \in int(\mathcal{C})$. This implies that $(\tilde{\mathbf{c}} + \mathcal{C}) \cap bd(\mathcal{C}) = \emptyset$. The optimal elements lie on the boundary of the cone and are therefore not contained in $\tilde{\mathbf{c}} + \mathcal{C}$. Hence, adding $\tilde{\mathbf{c}}$ to $\mathbf{c}^+$

$$\begin{aligned} s\big(\mathbf{c}^+ + \tilde{\mathbf{c}}, \mathbf{c}^-\big) &= \min\left\{z \in \mathbb{R} \mid \mathbf{c}^+ - \mathbf{c}^- + \mathbf{c} + \tilde{\mathbf{c}} = z\,\mathbf{1},\ \mathbf{c} \in \mathcal{C}\right\} \\ &= \min\left\{z \in \mathbb{R} \mid \mathbf{c}^+ - \mathbf{c}^- + \mathbf{c} = z\,\mathbf{1},\ \mathbf{c} \in \tilde{\mathbf{c}} + \mathcal{C}\right\} \\ &> \min\left\{z \in \mathbb{R} \mid \mathbf{c}^+ - \mathbf{c}^- + \mathbf{c} = z\,\mathbf{1},\ \mathbf{c} \in \mathcal{C}\right\} \\ &= s\big(\mathbf{c}^+, \mathbf{c}^-\big). \end{aligned}$$

increases the value of s. For the chosen function adding a cone element to the positive part

$$s\big(\mathbf{c}^+ + \tilde{\mathbf{c}}, \mathbf{c}^-\big) = s\big(\mathbf{c}^+, \mathbf{c}^- - \tilde{\mathbf{c}}\big),$$

is equivalent to subtracting it from the negative part. Thus, the preceding inequality also applies for $s\big(\mathbf{c}^+, \mathbf{c}^- - \tilde{\mathbf{c}}\big)$ and hence the cone scalarising function (2.17) is strictly $\mathcal{C}$-consistent.

□

The function is not strongly $\mathcal{C}$-consistent. Assume the optimal $\hat{\mathbf{c}} \in \mathcal{C} \setminus \{\mathbf{0}\}$ to be nonzero. Then the function value for the parameter pair $\mathbf{c}^+ + \hat{\mathbf{c}}$, $\mathbf{c}^-$

$$s\big(\mathbf{c}^+,\, \mathbf{c}^-\big) \;=\; s\big(\mathbf{c}^+ + \hat{\mathbf{c}}, \mathbf{c}^-\big)$$

yields the same optimal function value. Thus, s is not strongly $\mathcal{C}$-consistent.

Inserting the function (2.17) into the cone scalarising problem (2.7) yields

$$\begin{aligned}
&\min\big\{\, s\big(\mathbf{c}^+,\, \mathbf{c}^-\big) \;\big|\; \mathbf{y} - \mathbf{y}^R = \mathbf{c}^+ - \mathbf{c}^-,\; \mathbf{y} \in \mathcal{Y},\; \mathbf{c}^+,\, \mathbf{c}^- \in \mathcal{C}\big\} \\
&= \min\Big\{ \min\big\{z \in \mathbb{R} \mid \mathbf{c}^+ - \mathbf{c}^- + \mathbf{c} = z\,\mathbf{1},\; \mathbf{c} \in \mathcal{C}\big\} \;\Big|\; \\
&\qquad\qquad \mathbf{y} - \mathbf{y}^R = \mathbf{c}^+ - \mathbf{c}^-,\; \mathbf{y} \in \mathcal{Y},\; \mathbf{c}^+,\, \mathbf{c}^- \in \mathcal{C}\Big\} \\
&= \min\big\{\, z \in \mathbb{R} \;\big|\; \mathbf{y} - \mathbf{y}^R + \mathbf{c} = z\,\mathbf{1},\; \mathbf{c} \in \mathcal{C},\; \mathbf{y} \in \mathcal{Y}\big\}.
\end{aligned}$$

So, for the function (2.17) the cone scalarising problem amounts to the Pascoletti-Serafini scalarisation. In the function definition (2.17) we have used the search direction $\mathbf{1}$, but we could have used any other $\mathbf{q} \in int(\mathcal{C})$ as well. We will show now, that the Pascoletti-Serafini scalarisation is $\mathcal{C}$-representing. Here we use again $\mathbf{1}$ as search direction, but the transfer to a general $\mathbf{q}$ is fairly obvious.

Proposition 2.7.2 *The cone scalarising function (2.17) associated to the Pascoletti-Serafini approach is strictly $\mathcal{C}$-representing for $\mathbf{y}^A(\tau) := \mathbf{y}^R + \tau\,\mathbf{1}$.*

Proof:

Let an arbitrary level $\tau \in \mathbb{R}$ be given. Clearly, the function value for a pair $\mathbf{c}^+,\, \mathbf{c}^- \in \mathcal{C}$ with $\mathbf{c}^+ - \mathbf{c}^- = \mathbf{y}^A(\tau)$ is

$$\begin{aligned}
s\big(\mathbf{c}^+, \mathbf{c}^-\big) \;&=\; \min\big\{z \in \mathbb{R} \;\big|\; \mathbf{y}^A(\tau) - \mathbf{y}^R + \mathbf{c} = z\,\mathbf{1},\; \mathbf{c} \in \mathcal{C},\; \mathbf{y} \in \mathcal{Y}\big\} \\
&=\; \min\big\{z \in \mathbb{R} \;\big|\; \tau\,\mathbf{1} + \mathbf{c} = z\,\mathbf{1},\; \mathbf{c} \in \mathcal{C},\; \mathbf{y} \in \mathcal{Y}\big\} \\
&=\; \tau,
\end{aligned}$$

with the optimal cone element $\mathbf{c} = \mathbf{0}$ being trivial. Due to the strict $\mathcal{C}$-consistency of s, the corresponding distance evaluation function t_s is strictly $\mathcal{C}$-monotone. Hence, for every element $\bar{\mathbf{c}} \in int(\mathcal{C})$ the value of

$$t_s\big(\mathbf{y}^A(\tau) - \bar{\mathbf{c}}\big) \;<\; t_s\big(\mathbf{y}^A(\tau)\big)$$

becomes smaller. Thus, the interior of the negative ordering cone attached to the anchor point $\mathbf{y}^A(\tau) - int(\mathcal{C}) \;\subseteq\; \mathcal{L}_<(t_s, \tau)$ is part of the level set.

Now take an offset $\bar{\mathbf{c}} \in bd(\mathcal{C})$ from the boundary of the ordering cone. As $z = \tau$ and $\mathbf{c} = \bar{\mathbf{c}}$ is a feasible representation of $\mathbf{y}^A(\tau) - \bar{\mathbf{c}}$

$$\mathbf{y}^A(\tau) - \bar{\mathbf{c}} - \mathbf{y}^R + \bar{\mathbf{c}} \;=\; \tau\,\mathbf{1} - \bar{\mathbf{c}} + \bar{\mathbf{c}} \;=\; \tau\,\mathbf{1}$$

the minimal value for the cone scalarising problem is bounded by τ*. Suppose the minimal representation for* $\mathbf{y}^A(\tau) - \bar{\mathbf{c}}$ *would have a* $\tilde{z} < \tau$ *for some cone element* $\tilde{\mathbf{c}} \in \mathcal{C}$*. Then, the difference of the cone elements*

$$\bar{\mathbf{c}} - \tilde{\mathbf{c}} \;=\; \mathbf{y}^A(\tau) + \bar{\mathbf{c}} - \mathbf{y}^A(\tau) - \tilde{\mathbf{c}} \;=\; (z - \tilde{z})\,\mathbf{1} \;\in\; int(\mathcal{C})$$

would lie in the interior of the cone. But then the original cone element

$$\bar{\mathbf{c}} = \tilde{\mathbf{c}} + (z - \tilde{z})\,\mathbf{1} \;\in\; int(\mathcal{C})$$

would be in the interior of the cone, which contradicts the assumption. Thus, the boundary of the negative cone attached to the anchor point

$$\mathbf{y}^A(\tau) - bd(\mathcal{C}) \;\subseteq\; \mathcal{L}_{=}(t_s, \tau)$$

is part of the equality level set.

It remains to show that t_s *is positive outside* $\mathbf{y}^A(\tau) - \mathcal{C}$*. But with the strict* $\mathcal{C}$*-monotonicity of* t_s *every point* $\mathbf{c}^+ - \mathbf{c}^- = \mathbf{y} \in \mathbb{Y}$ *with* $\mathbf{c}^+ \in int(\mathcal{C})$ *and* $\mathbf{c}^- \in -\mathbf{y}^A(\tau) + bd(\mathcal{C})$ *evaluates to a positive distance evaluation function value, because it is strictly dominated by a point lying on the negative boundary attached to the anchor point. Thus, the distance evaluation function*

$$t_s(\mathbf{y}) > 0 \quad \textit{for all} \;\; \mathbf{y} \in \big(\mathbf{y}^A(\tau) - bd(\mathcal{C})\big) + int(\mathcal{C})$$

has positive values for all elements composed of the anchor point plus a inner cone element minus a boundary element.

We will now show that all configurations can be reduced to the discussed cases. For the chosen s *the values* $s\big(\mathbf{c}^+ + \mathbf{c}, \mathbf{c}^- + \mathbf{c}\big) = s\big(\mathbf{c}^+, \mathbf{c}^-\big)$ *agree for any* $\mathbf{c} \in \mathcal{C}$*.*

If the considered distance $\mathbf{y} - \mathbf{y}^A(\tau) =: \Delta\mathbf{y} = \mathbf{c}^+ - \mathbf{c}^- \in -\mathcal{C}$ *between point and anchor point would lie in the negative cone, we could reduce* $\mathbf{c}^+$ *to* $\mathbf{0}$ *without changing the value of the cone scalarising function* s*. For* $\mathbf{y} \notin -\mathcal{C}$ *the corresponding* $\mathbf{c}^+ \neq \mathbf{0}$ *obviously needs to be nonzero. Hence, we can reduce* $\mathbf{c}^+$ *to* $\mathbf{0}$ *if and only if the considered distance* $\Delta\mathbf{y} \in -\mathcal{C}$ *lies in the negative cone.*

If $\mathbf{c}^- \in int(\mathcal{C})$ *is in the interior of the ordering cone for a* $\Delta\mathbf{y} \notin -\mathcal{C}$*, there is by definition a maximum amount* $\bar{\varepsilon} > 0$ *of* $-\mathbf{c}^+$ *for which* $\mathbf{c}^- - \bar{\varepsilon}\,\mathbf{c}^+ \in \mathcal{C}$ *is still in the cone. For this maximum amount* $\mathbf{c}^- - \bar{\varepsilon}\,\mathbf{c}^+ \in bd(\mathcal{C})$*. Thus, we can change the representation to*

$$\bar{\mathbf{c}}^+ \;=\; (1 - \bar{\varepsilon})\,\mathbf{c}^+ \quad \textit{and} \quad \bar{\mathbf{c}}^- \;=\; \mathbf{c}^- - \bar{\varepsilon}\,\mathbf{c}^+,$$

so that $\bar{\mathbf{c}}^- \in bd(\mathcal{C})$ *lies on he boundary of the ordering cone. Note, that* $\bar{\mathbf{c}}^+$ *cannot become* $\mathbf{0}$ *in this construction, since this would imply that the distance* $\Delta\mathbf{y} \in -\mathcal{C}$ *lies in the negative cone.*

If after the above reduction $\bar{\mathbf{c}}^+ \in bd(\mathcal{C})$ *is a boundary element, a small amount* $\Delta\mathbf{c} \in int(\mathcal{C})$ *can be added to both cone elements. Then the above reduction is repeated. Since* $\bar{\mathbf{c}}^+$ *is just rescaled, it will stay in the interior of the cone.*

Therefore, for an arbitrary distance $\Delta\mathbf{y}$ *one of the three discussed scenarios applies. Consequentially, the level sets of the distance evaluation function*

$$\mathcal{L}_<(t_s, \tau) \;=\; \mathbf{y}^A(\tau) - int(\mathcal{C})$$

equal the interior of the negative cube attached to the anchor points.

□

We have seen that the Pascoletti-Serafini scalarisation approach fits into our framework and translates to a strictly $\mathcal{C}$-representing cone scalarising function with anchor points that spread along the search direction.

2.8 Summary

In this chapter we have build a framework for scalarising functions that directly incorporate partial orders specified through closed convex ordering cones. The so-called cone scalarising functions have the property that they yield – depending whether they are strictly or strongly $\mathcal{C}$-consistent – weakly $\mathcal{C}$-efficient or $\mathcal{C}$-efficient solutions independent of the reference point's position. So, the reference point can be feasible or infeasible without impairing the result in contrast to the situation for weighted metric scalarisation, where we need an utopia point to ensure (weak) efficiency.
Furthermore, we showed that, when level sets of the distance evaluation function corresponding to a cone scalarising function have a certain geometry we are able to reach all weakly $\mathcal{C}$-efficient or a certain subset of the $\mathcal{C}$-efficient solutions. To be able to reach these solutions just one of the level sets needs to fulfil the requirements. But if a certain subset of the level sets or every level set fulfils them, we are able to get to a reachable solution from infinitely many reference points. This is important, as the reference points are specified without knowing how they *should* be chosen. If a solution can be reached from just one reference point the probability to really reach it therefore is practically 0.
Then we have demonstrated that strongly $\mathcal{C}$-consistent cone scalarising functions cannot be simultaneously strictly $\mathcal{C}$-representing, thus ruling out the possibility of a cone scalarising function that produces only $\mathcal{C}$-efficient solutions and reaches all of them in one step. We have then described a two-step lexicographic optimisation problem that combines the two properties. Unfortunately, lexicographic optimisation involves constraints that are troublesome in nonlinear optimisation, so this approach seems most appropriate for linear optimisation.

For a convex and compact outcome set we have proven that the solutions resulting from cone scalarising problems with the union of all potential reference points form a connected subset of the set of outcomes. We have then proceeded to investigate the dependence of the minima of cone scalarising problems on the chosen reference points. Here, we demonstrated that the optimal value depends continuously on the reference point for compact outcome sets. We have then proven the upper semicontinuity and closedness of the set of optimal solutions as a set-valued mapping from the reference points to the powerset of the outcome set. Again we needed a compact outcome set to derive the upper semicontinuity. Since compactness is a strong requirement, we have shown that the existence of a certain lower bound for the outcome set combined with a $\mathcal{C}$-representing or $\mathcal{C}$-approximating function is enough to restore the compact situation. This criterion was then used to prove the two continuity results without the requirement that the outcome set is compact. As a last property we have shown the uniqueness of the solutions to the cone scalarising problem for suitable strict convexity assumptions.

The last section of this chapter showed that the well-known Pascoletti-Serafini approach fits into the framework. Moreover, we demonstrated that it is a strictly $\mathcal{C}$-representing cone scalarising function.

In the next chapter we will exploit the assembled framework to construct a new interactive multiobjective optimisation method – Pareto navigation.

Chapter 3

Pareto navigation

In this chapter we will propose a new interactive multiobjective optimisation method called *Pareto navigation*. In section 3.1 we will embed Pareto navigation into the existing interactive multiobjective optimisation methods. As a basis for this embedding we use the overview provided by [83]. In section 3.2 we will then present the mechanisms being available to the decision maker. Here, we will also discuss the information provided by Pareto navigation to ease the decision making. This is followed by a in-depth discussion of the functioning and mathematical properties of the restriction mechanism in section 3.3, of the selection mechanism in section 3.4 and the discounting mechanism in section 3.5. We will then demonstrate how Pareto navigation can be sped up to turn it into a real-time procedure in section 3.6. In section 3.7 we will adapt Pareto navigation to the case of nonconvex outcome sets. The chapter will be concluded by a summary in section 3.8.

3.1 Interactive multiobjective optimisation

According to the classification presented by Hwang and Masud ([62]) and adopted by Miettinen ([83]) there are four types of multiobjective methods. Methods that do not use any articulation of preference (*no preference methods*), methods where a posteriori articulation of preference is used (*a posteriori methods*), methods where a priori articulation of preference information is used (*a priori methods*), and methods where progressive articulation of preference information is used (*interactive methods*).

The basic work cycle for interactive multiobjective optimisation methods consists of finding an initial solution, asking the decision maker for preference information and presenting one (or several) new solution in response. The latter two steps are repeated until the solution is satisfactory, some convergence criterion is met or the decision maker runs out of time.

The different interactive multiobjective optimisation methods differ in the way they generate new solutions, in the interpretation of the decision maker's

input and in the type of input used to express preference. Usually, some of the following information needs to be specified by the decision maker to guide the chosen interactive multiobjective optimisation method:

- marginal rates of substitution or tradeoffs ([20, 45, 100])
- classification of objective values ([6, 85, 84])
- reference points or aspiration levels ([16, 65, 69, 71, 84, 85, 86, 124])
- reservation levels or upper bounds ([6, 16, 20, 84])

Besides the inspection of a solution in the normal workflow, some of the methods use the comparison of solutions to find some step size ([20, 69, 71, 84, 124]) or use it to guide the method ([65, 86, 112]).

The early methods often tried to approximate a so-called *value-function* – a function implicitly used by the decision maker to judge preference ([20, 45, 100]) or just tried to find some satisfactory result ([6]).

In recent years the reference point or aspiration level based methods possibly combined with some classification have become more and more popular ([16, 65, 69, 71, 84, 85, 86, 124]). This is due to the fact, that information that is directly related to the objective function values is easier to specify than information that is meant to reveal the structure of some unknown function.

We are in line with that and present here another approach based on reference points, upper bounds and bounds on the partial tradeoffs. Since our construction is along the lines of Wierzbicki's achievement scalarising function approach ([124, 125]), they are naturally similar. From the above mentioned approaches the Pareto Race of Korhonen et al. and its predecessors ([69, 71]) is close to our approach too and inspired the name for our method.

In chapter 1 we already discussed the advantages and drawbacks of our approach, so that we now concentrate on discussing its difference to existing methods with regard to the interaction and the mathematical properties. For that we will contrast our method against its closest relatives – the Reference Point Method and the Pareto Race.

Our method deviates from them in two distinct ways: it allows the introduction of bounds on the partial tradeoffs. Furthermore, for changing the current solution, it expects just one feasible value and a chosen objective as an input and guarantees to achieve this value for the objective.

The additional feature of being able to enforce restrictions on the partial tradeoffs is reflected in the cone scalarising framework that we needed to setup in chapter 2 to be able to implement the bounds on the partial tradeoffs (see section 3.5). It allows to rule out solutions, whose quality in one

objective is too dearly bought in terms of another objective. Even though this feature will most likely not be extensively used, it helps avoiding situations that cannot easily be avoided with reference points and upper bounds only.

Enforcing the achievement of the specified value for the chosen objective also changes the nature of the mathematical setting as we will see in section 3.4. On the one hand, this might render the result dominated – a drawback that the two methods do not suffer from. On the other hand, it offers the possibility to reach a pre-specified solution in K algorithmically given steps, where K is the dimension of the decision space. This also sets our method apart from the above methods.

Furthermore, the number of parameters to be specified for the other methods is equal to the dimension of the decision space. In contrast to this, our method takes only two parameters for changing the current solution – the objective to be changed and the new value – and the same number of parameters for changing the upper bounds. For modifying the bounds on the partial tradeoffs two objectives and the new value must be specified. Thus, generally the number of parameters is independent from the number of objectives for our method.

Altogether, our method shares some properties with the methods that influenced its development, but deviated from them in some important points. Specifically, our method differs in the way it generates new solutions, interprets the decision maker's input and to some extent in the type of input used to express preference and thus constitutes a new interactive multiobjective optimisation method.

3.2 Mechanisms of Pareto navigation

An interactive multiobjective optimisation method is determined by the supported mechanisms that can be used for decision making, the input needed from the decision maker, the information the method returns in response to the input and the internal workings of the method. In this section we will present the supported mechanisms, the input needed for them and the information generated in response from the perspective of the user.

Pareto navigation offers two mechanisms to change the feasible set. The *restriction* mechanism allows the introduction, removal or change of feasible upper bounds on individual objectives. So the decision maker can introduce constraints that (s)he would have imposed, if (s)he had known possible outcomes. Furthermore, they can be used as temporary bounds to guide the change of the current solution. Therefore, it is important that they can be changed any time and even be completely removed.

The second mechanism that influences the feasible domain is the *discounting*. Here, the decision maker can specify bounds on pairwise prices. (S)he can for example state, that improving objective k by one unit should not worsen objective k' by more than three units. This allows to focus the investigation to the set of alternatives with reasonable tradeoffs. However, the discounting does not just influence the set of feasible alternatives, but adapts the current solution, if it does not meet the new restrictions.

In response to changes through any of the two mechanisms the so called *planning horizon* is updated. It is an estimate of the intervals covered by the remaining set of efficient points for the different objectives. Therefore, it is the multidimensional interval between the ideal point – the vector of minimal values for the different objectives – and the nadir point – the vector of maximal values for the different objectives – over the remaining set of efficient solutions. This interval is important information for the decision maker, since *"(t)he ideal criterion values are the most optimistic aspiration levels which are possible to set for criteria, and the nadir criterion values are the most pessimistic reservation values that are necessary to accept for the criteria"* (see [70]).

Due to the response the decision maker can get a feeling for the impact a certain constraint has on the variety of the solutions at hand. In particular, (s)he can observe strong changes for the interval in one objective, when a certain bound was introduced for another one and deduce from it that the two objectives strongly influence each other. Note that the restriction is not allowed to exclude the current solution.

Changing the current solution is done using the *selection* mechanism. The decision maker sets a value in one of the objective and the Pareto navigation determines a solution with that value, that has the most favourable distance to the previous solution with regard to the other objectives. Thus, it keeps the distance as small as possible, if the value of an objective has to be worsened and tries to improve it as much as possible, if it can be improved. Again infeasible decisions are not supported in the selection mechanism, so the decision maker cannot choose an objective value that contradicts the chosen bounds on the objective values.

The objective vector of the current solution is embedded into the planning horizon so that the decision maker has an impression of the possibilities the current problem offers and the limitations it underlies. Along with the embedding one will usually have a visualisation of the current solution in a format familiar to the decision maker.

A typical session could look like the following. The system starts by presenting the decision maker the ranges for the different objectives and a starting solution that was computed by a no preference or an a priori multiobjective

method. The decision maker then starts by setting some bounds on objectives (s)he thinks are necessary for a reasonable solution. Then (s)he changes a crucial objective until the effect on the others becomes too strong. (S)he might now want to limit the tradeoff price between the currently changed objective and several of the others.

Then (s)he might continue to change the crucial objective. (S)he could introduce a bound on that objective, so that the achieved level is not lost in the forthcoming navigation steps. Then (s)he could start to manipulate the next objective, or to narrow down the solution set by introducing tighter constraints on the objectives or their tradeoffs. (S)he could store several different promising settings for a later in-depth analysis or directly decide on the preferred solution.

The specific strength of Pareto navigation is that it requires very little knowledge on decision making and the underlying model to enable someone to use it. In particular, neither the specification of marginal rates of substitution nor the direct specification of a reference point or similar information is needed to change the current solution.

On the other hand, it consequently does not offer a wide variety of parameters to manipulate and direct the decision process, but we believe that its interaction options are wide enough to enable a fast and successful decision making.

Pareto navigation is designed to work for convex, bounded multiobjective optimisation problems. Therefore, we will make the following assumptions:

Assumption 3.2.1

- *The functions* $\mathbf{f}_k, k \in \mathcal{K}$ *are assumed to be convex with the explicit exception of section 3.7.*
- *We will assume that the outcome set is contained*

$$\mathcal{Y} \subseteq \left(\hat{\mathbf{a}} + \mathbb{Y}_+\right) \cap \left(\hat{\mathbf{b}} + \mathbb{Y}_-\right)$$

  in a suitably chosen cuboid, i.e. that $\mathcal{Y}$ *is compact. We do not assume the bounds to be given, though.*
- *We assume the problem to be nontrivial, so the set of efficient solutions* $eff(\mathcal{Y})$ *is supposed to contain more than one alternative*

The assumptions of upper and lower bounds is not a severe restriction from the viewpoint of applications, because in most real-world models unbounded values are unrealistic anyway and thus due to shortcomings in the model. The convexity assumption is a definite restriction of the applicability of Pareto navigation and therefore we devote section 3.7 to elaborate on the

nonconvex case. The third assumption is a regularity assumption that is met in virtually all applications.

We will now have a closer look at each of the three mechanisms in turn and and see how the described black box behaviour is implemented.

3.3 The restriction mechanism

The restriction mechanism allows the decision maker to change the upper bounds $\mathbf{b}$ of the multiobjective optimisation problem. Thus, it restricts the further considerations to

$$\mathcal{Y}(\mathbf{b}) \;:=\; \mathcal{Y} \cap \big(\mathbf{b} - \mathbb{Y}_+\big),$$

where the components of $\mathbf{b}$ could have the symbolic value of ∞ to indicate that no restriction is set for the objective by the decision maker.

One could argue that if good bounds exist, they should be included in the description of the problem's feasibility domain. But this only accounts for the situation where the bounds are known exactly. There is often a certain *uncertainty* about the exact value the decision maker should enforce with the bound (s)he sets. If ineptly chosen it could even render the problem infeasible.

Thus, the restriction mechanism allows the decision maker to enforce or loosen bounds *after* (s)he has examined feasible solutions and gained some insight on the effect of choosing a certain bound. Besides, it can be used to temporarily set a bound that has no specific meaning in terms of the modelling, but serves as a means of steering the selection mechanism (see section 3.4). The restriction mechanism is not allowed to exclude the current solution, so the upper bounds set by it can never render the problem infeasible. At most, the current solution is the only feasible solution left.

An important tool to judge the effects of a certain choice of bounds is the so-called *planning horizon* – the multidimensional interval between the ideal and nadir point estimate. Changing one of the bounds triggers an update of the planning horizon and thus provides the decision maker with feedback on the effect of his or her decision. If sharpening a certain upper bound hardly has an effect on the remaining objectives, one could make it even tighter than originally intended. If on the other hand the effect is more dramatic than expected, one could directly revert the change partially.

The planning horizon will later even be divided into two parts for the case of an ordering cone $\mathcal{C}$ that is a proper superset of the Pareto cone $\mathbb{Y}_+$.

Even though the use of the restriction mechanism is fairly straight forward, the mathematical implementation of the planning horizon update is not.

Therefore, we will have a closer look at it. We will start with the estimate of the ideal point and then consider the nadir point estimate.

3.3.1 Estimating the ideal point

Determining the ideal point is a relatively simple problem as long as the ordering cone is the Pareto cone $\mathbb{Y}_+$. One solves the K optimisation problems

$$\mathbf{y}_k^I(\mathbf{b}) \;:=\; \min\{\mathbf{y}_k \mid \mathbf{y} \in \mathcal{Y}(\mathbf{b})\} \quad \text{for} \;\; k \in \mathcal{K}, \tag{3.1}$$

each yielding one component of the ideal point. The situation changes, if we include the pairwise bounds on the tradeoffs into the consideration.

Each of the minimisation problems (3.1) can be seen as a weighted scalarisation problem with weight vector $\mathbf{e}_k$. If the ordering cone $\mathcal{C} \supsetneq \mathbb{Y}_+$ is a proper superset of the Pareto cone, the dual ordering cone $\mathcal{C}^* \subsetneq \mathbb{Y}_+$ is a proper subset of the dual Pareto cone – again the positive orthant. But then some or all of the unit vectors $\mathbf{e}_k$ are no longer in the dual cone and therefore minimising the corresponding functional does not necessarily yield a weakly $\mathcal{C}$-efficient solution. Moreover, weakly $\mathcal{C}$-efficient and $\mathcal{C}$-efficient solutions can now severely differ in their minimal $\mathbf{y}_k$-value, so it is no longer irrelevant whether we are dealing with weakly $\mathcal{C}$-efficient or $\mathcal{C}$-efficient solutions.

Thus, we have to explicitly restrict the minimisation to the $\mathcal{C}$-efficient outcomes in $\mathcal{Y}(\mathbf{b})$. In other words, we have to solve

$$\min\left\{\mathbf{y}_k \mid \mathbf{y} \in eff_{\mathcal{C}}(\mathcal{Y}(\mathbf{b}))\right\} \tag{3.2}$$

Unfortunately, problem (3.2) is nonconvex because the domain of minimisation is nonconvex in general. One approach to model this problem is the formulation of a *bilevel programming problem* or *Stackelberg game*.

Bilevel programming problems incorporate an optimisation problem in the description of their feasible domain. So, the set of points considered for the optimisation itself is optimal for the so-called lower level problem. The lower level problem shares some variables with the upper level problem thus making the two problems interdependent. Bilevel problems belong to the larger class of problems called mathematical programs with equilibrium constraints – an area of very active research (see e.g. [33], [80], [111] and references therein).

In our case we will encode the $\mathbf{y} \in eff_{\mathcal{C}}(\mathcal{Y}(\mathbf{b}))$ efficiency condition into the lower level problem and minimise the k^{th} component of the remaining points. So, we solve

$$\min\left\{\mathbf{y}_k \;\middle|\; \mathbf{y}^R \in \mathbb{Y},\, \min\left\{s(\mathbf{c}^+,\mathbf{c}^-) \;\middle|\; \mathbf{y}-\mathbf{y}^R = \mathbf{c}^+ - \mathbf{c}^-,\, \mathbf{y} \in \mathcal{Y},\, \mathbf{c}^+,\mathbf{c}^- \in \mathcal{C}\right\}\right\} \tag{3.3}$$

for every $k \in \mathcal{K}$ to get the K components of the ideal point. We have chosen to use a cone scalarising problem in the lower level, but any problem that can assure (near) $\mathcal{C}$-efficient outcomes is suitable. The lower level problem optimises the cone scalarising function for the given reference point to ensure the $\mathcal{C}$-efficiency. The $\mathbf{y}^R$ variables are used by the upper level to manipulate the optimum of the lower level in search for the minimum. If the global optimum for the problem is found, it will consequentially be reached at a $\mathcal{C}$-efficient point with minimal k^{th} component.

Unfortunately, bilevel problems are global optimisation problems and guarantee global optimality of their results only in very special cases. Besides, they are computationally complex and might therefore require an effort that is not justified by their use for Pareto navigation.

Thus, we will work with some estimates for the ideal point that are computationally much easier to obtain. Let $\mathbf{c}^{(k)}$ be the projection of the k^{th} unit vector $\mathbf{e}_k$ onto the dual ordering cone $\mathcal{C}^*$. This projection is uniquely defined and is nonzero, if the dual ordering cone has nonempty interior. Then we solve the K minimisation problems

$$\min \left\{ \langle \mathbf{c}^{(k)}, \mathbf{y} \rangle \mid \mathbf{y} \in \mathcal{Y}(\mathbf{b}) \right\} \quad \text{for} \quad k \in \mathcal{K} \tag{3.4}$$

that yield the corresponding components of the ideal point estimate. For the Pareto cone this problem equals problem (3.1). Hence, it is exact in this case. For all other cases it needs some further modifications. The optima are so far weakly $\mathcal{C}$-efficient and thus the value of the estimate can be smaller than the true value. On the other hand, it is not guaranteed to produce the solution of minimal $\mathbf{y}_k$-value. So the value could be too large.

To get a reliable estimate, we have to make the outcome of problem (3.4) $\mathcal{C}$-efficient. For that purpose we can either add a small amount of a vector $\mathbf{c}^* \in int(\mathcal{C}^*)$ from the interior of the dual ordering cone to the weight vector or we can lexicographically optimise a second objective like in lemma 2.4.8. Either way we will obtain a $\mathcal{C}$-efficient solution $\mathbf{y}^I_{\mathcal{C},k}(\mathbf{b})$ whose k^{th} component is an upper bound for the true minimum value.

By the optimality of $\mathbf{y}^I_{\mathcal{C},k}(\mathbf{b})$ and the convexity of $\mathcal{Y} + \mathbb{Y}_+$ ([64][theorem 2.11]) the inequality

$$\langle \mathbf{c}^{(k)}, \mathbf{y}^I_{\mathcal{C},k}(\mathbf{b}) \rangle \;\leq\; \langle \mathbf{c}^{(k)}, \mathbf{y} \rangle$$

holds for all $\mathbf{y} \in \mathcal{Y}$. Thus, we can construct a lower bound for the minimum value by calculating the minimum value of $\alpha \in \mathbb{R}$

$$\langle \mathbf{c}^{(k)}, \mathbf{y}^I_{\mathcal{C},k}(\mathbf{b}) \rangle \;\leq\; \langle \mathbf{c}^{(k)}, \mathbf{b} + \alpha\, \mathbf{e}_k \rangle,$$

which is given by

$$\alpha \;=\; -\frac{\langle \mathbf{c}^{(k)}, \mathbf{b} - \mathbf{y}^I_{\mathcal{C},k}(\mathbf{b}) \rangle}{\mathbf{c}^{(k)}_k}.$$

The lower bound on the minimal value is then given by

$$\min\left\{\mathbf{y}_k \mid \mathbf{y} \in eff_{\mathcal{C}}(\mathcal{Y}(\mathbf{b}))\right\} \;\geq\; \max\left\{\mathbf{y}_k^I(\mathbf{b}), \mathbf{b}_k + \alpha\right\}$$

the maximum of the minimal value for the Pareto cone and the newly derived estimate. The latter is probably fairly good, when the ordering cone is not too different from the Pareto cone, but can be rather pessimistic, if the two cones differ significantly.

3.3.2 Estimating the nadir point

The problem of finding the components of the nadir point

$$\max\left\{\mathbf{y}_k(\mathbf{x}) \;\middle|\; \mathbf{y} \in eff_{\mathcal{C}}(\mathcal{Y}(\mathbf{b}))\right\} \tag{3.5}$$

is even more involved. Even in the case of the Pareto ordering cone it is a convex *maximisation* problem over the nonconvex domain $eff_{\mathcal{C}}(\mathcal{Y}(\mathbf{b}))$ and thus a global optimisation problem, which is difficult to solve in three or more dimensions (see e.g. the abstract of [10]).

In [127] an overview of methods for optimisation over the efficient set is given – a class of algorithms that is more general, but can be used for the nadir point detection. More such algorithms are proposed in [30, 60, 61, 79], that all involve global optimisation subroutines and at best converge in finitely many iterations, but are inappropriate for an interactive method. An exception is the algorithm proposed in [37] that is less computationally involved, but since it heavily relies on two dimensional subproblems it only works for up to three objectives. Most of these algorithms could also be used to compute the ideal point coordinates for general ordering cones, but their computation time is unacceptable in that case as well.

Since exact methods are intractable, heuristic estimates for the nadir point have to be used. Estimates using the so called payoff table (see e.g. [83]) are problematic, because they can be too large or too small and arbitrarily far away from the true value (see [63]). But in [37], small algorithmic changes to the payoff table heuristic are proposed that make it either a lower or upper bound for the true value. In essence the algorithmic changes for the lower estimate ensure that the obtained solution is efficient. Applying these small changes to the problems solved when looking for the ideal point (3.1) and (3.4), the improved payoff table entries can be computed with almost no additional effort.

The points gained in the computation should be stored, so that when the bounds $\mathbf{b}$ are modified, the stored points are checked for feasibility and only the problems corresponding to infeasible points are computed anew. Depending on the chosen optimisation algorithm the infeasible points could even be used as starting points.

In [70], a heuristic to approximate the nadir point for linear multiobjective optimisation based on the simplex algorithm is proposed. It uses its objective function to enforce efficiency and successively changes the right hand side to maximise the currently considered objective. Furthermore, a cutting plane is used to cut off the part of the polyhedron that contains smaller values than the most current estimate. The heuristic yields a lower bound for the true nadir value, since in general it only detects local maxima. It involves no global optimisation subroutines and is thus adequate for our purposes. Additionally, it can be stopped at any time still yielding a lower bound for the nadir point, although the estimate is less accurate then. Even though the original method was proposed for the Pareto ordering cone, it can easily be adapted to the more general situation of polyhedral ordering cones, since it is based on the Pascoletti-Serafini scalarisation.

In case of a linear multiobjective problem one can either evaluate the payoff tables in conjunction with the ideal point detection or use the more sophisticated nadir point heuristic above depending on the time restrictions and the problem complexity. In nonlinear multiobjective optimisation problems payoff tables are the only feasible options for interactive methods.

Again, a more rigorous way of determining the nadir point components is the use of bilevel programming. The model is essentially the same as for the ideal point components

$$\max\left\{\mathbf{y}_k \;\middle|\; \mathbf{y}^R \in \mathbb{Y},\ \min\left\{s(\mathbf{c}^*, \mathbf{c}^-) \;\middle|\; \mathbf{y} - \mathbf{y}^R = \mathbf{c}^+ - \mathbf{c}^-,\ \mathbf{y} \in \mathcal{Y},\ \mathbf{c}^+, \mathbf{c}^- \in \mathcal{C}\right\}\right\}, \tag{3.6}$$

but naturally we are now maximising the objective function. In fact the above heuristic for the case of linear multiobjective problems ([70]) can be seen as a heuristic to solve the bilevel problem: It uses its 'explicit' objective function to ensure efficiency and has some outer routine – the counterpart of the upper level problem – that tries to enlarge the objective value by changes to the reference point in the underlying Pascoletti-Serafini approach.

We will estimate the bound for the Pareto cone $\mathbf{y}^N(\mathbf{b})$ as well as $\mathbf{y}^N_{\mathcal{C}}(\mathbf{b})$ – the one for the ordering cone $\mathcal{C}$ and split the planning horizon into an inner part – the estimated interval $\left[\mathbf{y}^I_{\mathcal{C}}(\mathbf{b}),\ \mathbf{y}^N_{\mathcal{C}}(\mathbf{b})\right]$ for the currently used ordering cone – and an outer part – the estimated interval $\left[\mathbf{y}^I(\mathbf{b}),\ \mathbf{y}^N(\mathbf{b})\right]$ for the Pareto cone. For the outer interval the ideal point is accurate while the nadir point estimate is heuristic, for the inner interval both bounds are heuristic.

Working with two intervals has the additional advantage that we can see the (approximate) effect of the pairwise price bounds on the size of the set of alternatives. Thus, we see the effect that the restriction mechanism has itself and the cumulative effect of the restriction and discounting mechanism.

3.4 The selection mechanism

The selection mechanism is the main tool of Pareto navigation to change the current solution. The decision maker chooses one objective and changes it to any feasible value that (s)he finds desirable. The solution given back in response to this action attains the specified value in the chosen objective and has the most favourable distance to the remaining objectives. In particular, it tries to improve beyond the previous level when this is possible. Furthermore, it observes the upper bounds on the objectives introduced by the restriction mechanism and mostly observes the bounds on the pairwise prices introduced by the discounting mechanism.

Internally, the selection mechanism executes a cone scalarising problem that incorporates the upper bounds directly and the price bounds by its choice of the ordering cone (see section 3.5). The choice of the cone scalarising function thereby expresses, what is considered to be the most favourable distance in a specific application context. The reference point is determined internally and strongly depends on the previous solution, thus trying to keep the changes small in the remaining objectives.

An individual selection step is determined by the choice of the objective function and its new value only, so the number of parameters is very small. So, there is not an overwhelming number of controls one has to twiddle to trigger a certain change. Besides, whenever the objective functions have a direct interpretation the control parameters have one as well. Consequentially, the decision maker does not need to be familiar with the internal working of the mechanism to be able to specify appropriate control parameters.

The approach taken here differs from the approaches taken by known interactive multiobjective methods in some important aspects. Therefore, we will thoroughly investigate its behaviour.

We will present the mathematical model of the selection mechanism in subsection 3.4.1. In particular, we will discuss the effect of fixing the function value in the chosen objective.
In subsection 3.4.2 we will present a critical example of an outcome set that is used throughout the remainder of the chapter to demonstrate undesirable behaviour of the selection mechanism. This behaviour can be seen as the downside of the improved control on the current solution.
The subsection 3.4.3 investigates the properties of the solutions returned by the selection mechanism. This comprises the efficiency of the outcomes, the connectedness of the set of outcomes, the reachability of specific outcomes, the continuous dependence on the control parameter and uniqueness of the optimal outcome.

3.4.1 Modelling the selection mechanism

The most obvious distinctiveness of the selection mechanism is that the specified value τ for the chosen objective k' is guaranteed to be attained. For that we introduce a temporary equality constraint. Geometrically we are looking for the solution with the best possible distance to the values of the remaining objectives on the intersection of the set of outcomes $\mathcal{Y}$ with the hyperplane

$$\mathcal{H}_{k'}(\tau) := \{\mathbf{y} \in \mathbb{Y} \mid \mathbf{y}_{k'} = \tau\}. \tag{3.7}$$

Thus, the dimension of the problem considered for an individual selection step is one less than the dimension of the original problem.

The upper bounds $\mathbf{b}$ from the restriction mechanism are directly put into the problem. In contrast, the bounds on the pairwise prices are incorporated using an ordering cone $\mathcal{C}$ that reflects these bounds (see section 3.5).

The distance is optimised by the cone scalarising problem

$$\min \left\{ s\big(\mathbf{c}^+, \mathbf{c}^-\big) \;\middle|\; \mathbf{y} - \mathbf{y}^R = \mathbf{c}^+ - \mathbf{c}^-,\ \mathbf{y}_{k'} = \tau,\ \mathbf{y} \leq \mathbf{b},\ \mathbf{y} \in \mathcal{Y},\ \mathbf{c}^+, \mathbf{c}^- \in \mathcal{C} \right\}, \tag{3.8}$$

where $\tau \in \left[\mathbf{y}^I_{k'}(\mathbf{b}), \mathbf{y}^N_{k'}(\mathbf{b})\right] =: \mathcal{I}_{k'}(\mathbf{b})$ is taken from the interval given by the k'-th component of the estimated ideal and nadir point.

Here, we have two choices for the interval. On the one hand side, we can take the estimates for the Pareto cone to be sure not to exclude solutions with good $\mathbf{y}_{k'}$ values due to an inexact estimate of $\mathbf{y}^I_{\mathcal{C},k'}(\mathbf{b})$. Thereby, we allow the navigation to violate the bounds on the pairwise prices for the chosen objective, if the parameter τ is chosen to be larger than the exact k'-th nadir point component for the current ordering cone $\mathcal{C}$ and upper bounds $\mathbf{b}$.

On the other hand, we could work with the estimates of the ideal and nadir point for the current ordering cone $\mathcal{C}$. Thus, we would enforce the price restrictions, but could potentially exclude some of the best solutions for individual objectives due to the lack of accuracy in the lower bound estimate.

Hence, if the estimates $\mathbf{y}^I_{\mathcal{C},k'}(\mathbf{b})$ and $\mathbf{y}^N_{\mathcal{C},k'}(\mathbf{b})$ were reliable, we would use them. Otherwise, it is a tradeoff between excluding some efficient solutions against a violation of the bounds on the prices. We will use $\mathcal{I}_k(\mathbf{b})$ for both possible choices and make the choice explicit, if there is a difference in behaviour due to it.

The reference point in (3.8) is equal to the previous solution $\mathbf{y}^S$

$$\mathbf{y}^R := \left\{ \begin{array}{ll} \mathbf{y}^S_k & \text{for } k \neq k' \\ \tau & \text{for } k = k' \end{array} \right\}$$

except for the k'-th component which is set to τ. Thus, we try to find a solution in $\mathcal{Y} \cap \mathcal{H}_{k'}(\tau)$ that has the most favourable distance to the projection of

the previous solution onto $\mathcal{H}_{k'}(\tau)$. Thereby, the distances are evaluated by the distance evaluation function t_s. Thus, the chosen cone scalarising function reflects what is considered the most favourable reaction to the change in $\mathbf{y}_{k'}$.

We benefit from working with cone scalarising functions in at least two ways here. First, the bounds on the pairwise prices can be incorporated seamlessly. Second, the position of the reference point can be arbitrary relative to the feasible set $\mathcal{Y} \cap \mathcal{H}_{k'}(\tau)$ without compromising the quality of the outcome. In particular, the selection mechanism will improve the objective values beyond the reference point, if the objective value for the chosen objective is worsened.

In order to work in conjunction with the discounting mechanism there is one further restriction regarding the choice of the cone scalarising function s. It needs to be able to work with changing ordering cones, i.e. its definition must make sense for every ordering cone $\mathcal{C}$ that is constructed by the discounting mechanism. In particular, it should therefore not exploit the structure of one specific ordering cone. We assume to be working with such a cone scalarising function s henceforth.

We will call optimal solutions to the cone scalarising problem s-efficient to avoid having to distinguish between the different efficiency notions for the different types of cone scalarising functions. To more clearly express the dependency on the chosen ordering cone, we will slightly modify the notation for the set of all optima for the cone scalarising function s over the set $\mathcal{S}$

$$eff_{s,\mathcal{C}}(\mathcal{S}) := \bigcup_{\mathbf{y}^R \in \mathbb{Y}} \mathcal{O}_{\mathcal{S},s}(\mathbf{y}^R)$$

and include the ordering cone in the subscript. If no cone or cone scalarising function is mentioned, we are considering efficiency with respect to the Pareto ordering cone.

Before we investigate the mathematical properties of the selection mechanisms, we will introduce an example outcome set that is frequently used in the remainder of the chapter. Therefore, we will present it upfront to bundle the corresponding explanations.

3.4.2 Critical example

In this subsection we introduce a simple three objective example, whose variations suffice to illustrate undesirable, but somewhat unavoidable behaviour of the selection mechanism. The construction has a nontrivial set of efficient solutions that is sub-dimensional, i.e. the set of efficient outcomes has no 'inner' part. We have chosen to construct it for the Pareto cone,

but the basic idea can be transferred to other ordering cones in a straight forward way.

The set of outcomes $\mathcal{Y}^{ex}$ (figure 3.1) is the convex combination

$$\mathcal{Y}^{ex} := conv\left\{\mathbf{y}^{ex_1}, \mathbf{y}^{ex_2}, \mathbf{y}^{ex_3}, \mathbf{y}^{ex_4}\right\},$$

of the objective vectors

$$\mathbf{y}^{ex_1} := \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix}, \mathbf{y}^{ex_2} := \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix}, \mathbf{y}^{ex_3} := \begin{pmatrix} 2 \\ 0.8 \\ 4.8 \end{pmatrix}, \mathbf{y}^{ex_4} := \begin{pmatrix} 0.8 \\ 2 \\ 4.8 \end{pmatrix}.$$

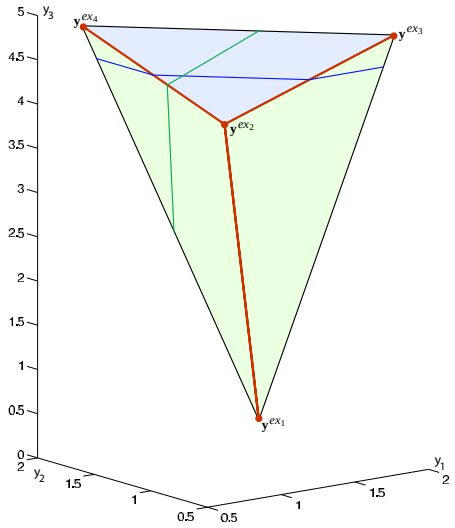
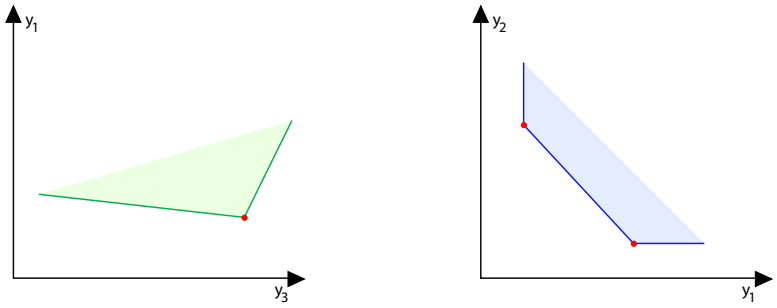


Figure 3.1: Simple yet critical example.

The greenish side areas in figure 3.1 are weakly dominated by the upright red line and the bluish upper area is dominated by the two upper red lines. Thus, just the red lines are efficient. By just slightly changing the y_1-coordinate of $\mathbf{y}^{ex_4}$ or the y_2-coordinate of $\mathbf{y}^{ex_3}$ the side areas can either be made efficient or strictly dominated. Furthermore, by making the upper right line marginally shorter an arbitrarily small slope between $\mathbf{y}^{ex_3}$ and $\mathbf{y}^{ex_4}$ regarding the y_3-coordinate can be realised.

Now let us look at intersections of $\mathcal{Y}^{ex}$ with hyperplanes normal to an axis. Due to the symmetry of the example, we only need to consider the hyperplane perpendicular to the second and third objective axis.

Looking at the result of the vertical intersection - the greenish area - we see that the parts that were weakly dominated in the three-dimensional setting are now efficient - properly efficient even. Note that the slope of this part



(a) (b)

Figure 3.2: The intersection of the hyperplanes with the 3D polyhedron.

of the boundary can be made arbitrarily small by moving $\mathbf{y}^{ex_2}$ and $\mathbf{y}^{ex_3}$ along the y_1-coordinate. The red point is the single 3D-efficient point on the intersection.

The middle part of the boundary of the bluish area - the result of the horizontal intersection - is properly efficient in the two-dimensional setting while it is strictly dominated in the three-dimensional. The weakly efficient outer parts of the boundary can either be made efficient or strictly dominated by slightly modifying $\mathbf{y}^{ex_3}$ or $\mathbf{y}^{ex_4}$. If the intersecting hyperplane is moved towards the lower y_3 values, the (two dimensional) efficient middle part will eventually become just a point. Here again the red points are the 3D-efficient points on the intersection.

3.4.3 Results of the selection mechanism

In this subsection we will analyse the properties of the selection mechanism. We will describe the set of selection outcomes and the properties of these outcomes. We will see that there are non-s-efficient outcomes and we will discuss the inevitability of their existence. Then we will demonstrate the connectedness of the set of selection outcomes and formulate an algorithm to reach any given efficient outcome in $eff_{s,\mathcal{C}}(\mathcal{Y}(\mathbf{b}))$. Within the algorithm we will use a combination of the selection and restriction mechanism that can be regarded as a possible workflow for using Pareto navigation. We will then investigate the continuity of the selection mechanism and consider uniqueness of the solutions to the selection mechanism problem (3.8).

The solutions to (3.8) are contained in the set $eff_{s,\mathcal{C}}\big(\mathcal{Y}(\mathbf{b}) \cap \mathcal{H}_k(\eta)\big)$ of optimal solutions of the intersection of the outcome set with the hyperplane defined in (3.7). But they are not necessarily s-efficient with respect to the full-dimensional outcome set $\mathcal{Y}(\mathbf{b})$. Looking at the critical example (figure 3.1), we see that any 2D-efficient point on the blue line (figure 3.2(b)) except the corners is dominated in 3D.

Thus, we need to have a closer look at the set of potential navigation outcomes. The set

$$\mathcal{Y}_{\mathcal{C}}^{Nav}(\mathbf{b}) := \bigcup_{k\in\mathcal{K}} \Big(\bigcup_{\beta\in\mathcal{I}_k(\mathbf{b})} eff_{s,\mathcal{C}}\big(\underbrace{\mathcal{Y}(\mathbf{b}) \cap \mathcal{H}_k(\beta)}_{=:\mathcal{Y}_k(\mathbf{b},\beta)} \big) \Big) \tag{3.9}$$

is the domain of outcomes for problem (3.8), if we allow arbitrary reference points $\mathbf{y}^R \in \mathbb{Y}$. In the selection mechanism the reference point is not arbitrary but determined by the previous solution $\mathbf{y}^S$. Therefore, we will in general not be able to reach every point in $\mathcal{Y}_{\mathcal{C}}^{Nav}(\mathbf{b})$. We are nevertheless able to reach the points which we are most interested in reaching: the efficient points in $\mathcal{Y}_{\mathcal{C}}^{Nav}(\mathbf{b})$.

First note that changing the bounds with the aid of the restriction mechanism

$$eff_{s,\mathcal{C}}\big(\hat{\mathcal{Y}} \cap (\bar{\mathbf{b}} - \mathbb{Y}_+)\big) = eff_{s,\mathcal{C}}\big(\hat{\mathcal{Y}}\big) \cap \big(\hat{\mathbf{b}} - \mathbb{Y}_+\big)$$

does not effectively change the set of s-efficient outcomes, but rather determines which part of it is still considered. So we can freely interchange the restriction and the selection mechanism. Since the restriction mechanism is not allowed to exclude the current solution, the problem will never be infeasible.

Suppose that we start the navigation at a point $\mathbf{y}^{(0)} \in \mathcal{Y}_{\mathcal{C}}^{Nav}(\mathbf{b})$ and want to reach the $\mathcal{C}$-efficient point $\hat{\mathbf{y}} \in eff_{s,\mathcal{C}}\big(\mathcal{Y}(\mathbf{b})\big)$. To arrive at the point we execute the following algorithm:

Algorithm 3.4.1

1. *Let* $i = 0$ *and* $\mathbf{b}^{(0)} := \mathbf{b}$.
2. *Set* $\mathcal{J}_i = \Big\{ k \in \mathcal{K} \;\Big|\; \mathbf{y}_k^{(i)} > \hat{\mathbf{y}}_k \Big\}$ *and* $\mathbf{b}^{(i+1)} = \left\{ \begin{array}{c} \mathbf{b}_k^{(i)} \;\; \textit{for } k \in \mathcal{J}_i \\ \hat{\mathbf{y}}_k \;\; \textit{for } k \notin \mathcal{J}_i \end{array} \right\}$.
3. *If* $\mathcal{J}_i = \emptyset$, *stop.*
4. *Take an arbitrary index* $k' \in \mathcal{J}_i$ *and solve the cone scalarising problem (3.8) for* $\tau = \hat{\mathbf{y}}_{k'}$, $\mathbf{b} = \mathbf{b}^{(i+1)}$ *and* $\mathbf{y}^{(i)}$ *as the starting point. Let* $\mathbf{y}^{(i+1)}$ *be the result.*
5. *Set* $i = i + 1$ *and goto 2.*

The algorithm sets the entries of the target point as upper bounds for the selection mechanism, if the current solution fulfils them. The components with too large values are successively chosen for the next selection step. This step changes the chosen component to the wanted value and tightens the bounds where possible. Thus, the changed component is now bound to not exceed the value of the corresponding target point component. This process

is continued until the target point is reached – which we will verify in the next theorem.

Theorem 3.4.2 *Let s be a $\mathcal{C}$-centred, strictly $\mathcal{C}$-consistent and continuous cone scalarising function. Then, starting from an arbitrary point $\mathbf{y}^S \in \mathcal{Y}_{\mathcal{C}}^{Nav}(\mathbf{b})$ we can reach every $\mathbb{Y}_+$-efficient point $\hat{\mathbf{y}} \in eff_{s,\mathcal{C}}\big(\mathcal{Y}(\mathbf{b})\big)$ using the algorithm 3.4.1.*

Proof:

Since the bounds $\mathbf{b}^{(i)}$ and τ are chosen such that $\hat{\mathbf{y}}$ is feasible for these restrictions, the cone scalarising problem in step 4 of the algorithm always has a nonempty feasible domain.

The starting set of indices $\mathcal{J}_0$, where $\mathbf{y}^{(0)}$ exceeds $\hat{\mathbf{y}}$ clearly is a subset of $\mathcal{K}$. In each iteration solving the cone scalarising problem (3.8) yields a solution that fulfils the previous constraints $\mathbf{y}^{(i+1)} \leq \mathbf{b}^{(i+1)}$ and additionally the constraint $\mathbf{y}_{k'}^{(i+1)} \leq \hat{\mathbf{y}}_{k'}$. Hence, the set of exceeding indices $\mathcal{J}_{i+1} \subseteq \mathcal{J}_i \setminus \{k'\}$ is reduced by at least the currently chosen index k'. So the stopping criterion of the algorithm in step 3 is met after $i' \leq K$ steps.

The bound vector $\mathbf{b}^{(i'+1)} = \hat{\mathbf{y}}$ equals the target point when the algorithm terminates. By the definition of $\mathbb{Y}_+$-efficiency

$$\big(\mathbf{b}^{(i'+1)} - \mathbb{Y}_+\big) \cap \big(\mathcal{Y}(\mathbf{b}) + \mathcal{C}\big) = \{\hat{\mathbf{y}}\}$$

only the target point $\hat{\mathbf{y}}$ remains feasible. But the current solution is always feasible and therefore the final solution of the algorithm is the target point.

□

So, we can *exactly* reach every $\mathbb{Y}_+$-efficient point in $eff_{s,\mathcal{C}}\big(\mathcal{Y}(\mathbf{b})\big)$. But the algorithm 3.4.1 is useful for more general situations.

Corollary 3.4.3 *Let s be a $\mathcal{C}$-centred, strictly $\mathcal{C}$-consistent and continuous cone scalarising function. Then, starting from an arbitrary point $\mathbf{y}^S \in \mathcal{Y}_{\mathcal{C}}^{Nav}(\mathbf{b})$ we can find a point $\tilde{\mathbf{y}} \leq \hat{\mathbf{y}}$ that is smaller or equal to an arbitrary point $\hat{\mathbf{y}} \in eff_{s,\mathcal{C}}\big(\mathcal{Y}(\mathbf{b})\big)$ using the algorithm 3.4.1.*

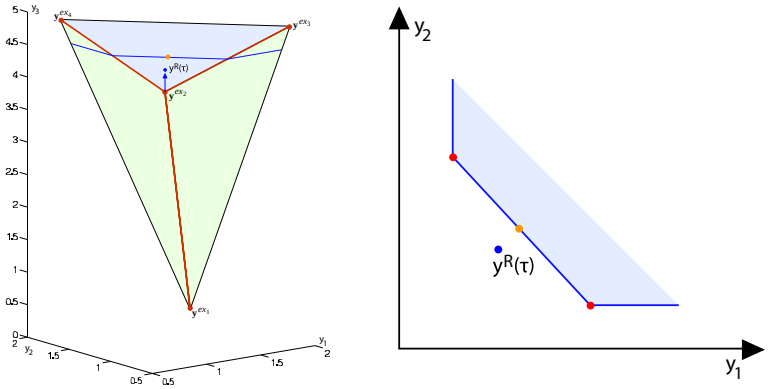
Proof:

The proof of the theorem remains valid except the last argument. The intersection

$$\big(\mathbf{b}^{(i'+1)} - \mathbb{Y}_+\big) \cap \big(\mathcal{Y}(\mathbf{b}) + \mathcal{C}\big) \supseteq \{\hat{\mathbf{y}}\}$$

can be larger than just the target point singleton. But every other point in the intersection is smaller or equal to the target point.

□



(a) (b)

Figure 3.3: The problem that results, when the decision maker moves from the centre point in positive y_3 direction.

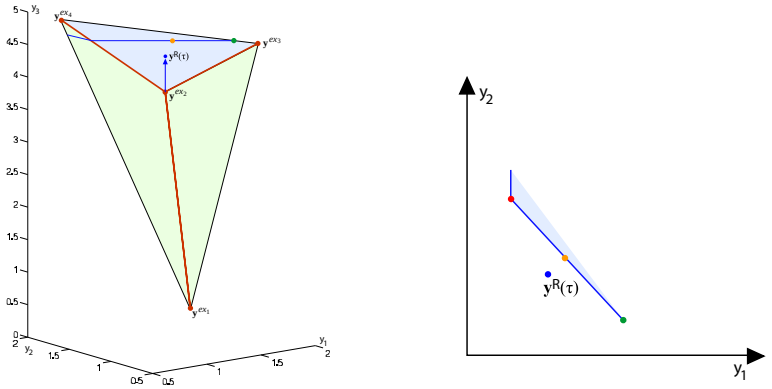
As $\mathcal{C}$-efficiency implies efficiency for the Pareto cone, the set of $\mathcal{C}$-efficient solutions is contained in the set of efficient solutions. Thus, theorem 3.4.2 and corollary 3.4.3 also apply to $\mathcal{C}$-efficient and weakly $\mathcal{C}$-efficient solutions.

But even when we use the specific reference points for the selection mechanism problem (3.8) the outcomes can be weakly or strictly dominated. If we start at the centre point in the critical example and increase the y_3 component of the current solution (see figure 3.3(a)) the optimum for the 2D situation (see figure 3.3(b)) – the orange point – is strictly dominated in 3D.

Moreover, the situation cannot easily be changed, because the next efficient point is not uniquely determined in the situation, i.e. the two red points in figure 3.3(b) have the same distance to the reference point. Such ties can be constructed for an arbitrary distance evaluation function, so this situation cannot easily be avoided.

Using lexicographic optimisation to enforce efficiency on the 2D section does not solve the problem either. Assume the upper right point in the critical example is closer to the centre point than the upper left point (see figure 3.4(a)). Furthermore, assume that we decided to lexicographically optimise the y_2-component first and the y_1-component second. Thus, the lexicographic optimisation will always push the solution to the right-most point on the 2D section in our example (see figure 3.4(b)). If we increase the y_3-component beyond the level of $\mathbf{y}_3^{ex_3}$ – an admissible level, since there are still efficient points with such a y_3-component, the right-most point will no longer be efficient (see figure 3.4(a)).

So this potential non-efficiency seems to be inherently linked to the equality constraint used in the selection mechanism. Thus, it is the 'price' for the im-



(a) (b)

Figure 3.4: The result of using lexicographic optimisation on the 2D section

proved control one has due to the equality constraint. Put in multiobjective terms we are trading improved control for guaranteed efficiency.

Nevertheless, we can display the efficient solution with most favourable distance to the current point by solving one cone scalarising problem without the equality constraint and the current point as starting point. If the current point is efficient for the full dimensional problem as well, the two points will be equal and if that is not the case it can directly be detected.

Note that the critical character of the example becomes even more pronounced, if we consider the behaviour of other interactive multiobjective optimisation methods. Assume the centre point to be the current solution and the decision maker decided to increase the y_3-component of the current solution to gain some improvement in the remaining objectives. If the method is working with marginal rates of substitution, decreasing the value for the y_3-component will not change the current solution. So the decision maker has to decide explicitely which component (s)he wants to improve to inflict a change. In other words, the strategy has to be changed.

Similar effects occur when the method works with distance based methods. To move from the centre point to any of the two upper points one of them most be favoured explicitely, otherwise the method is stuck at the centre point. Again, this asks for a change in strategy. Besides, the decision maker might give up, if (s)he does not know that the upper two points exist – say through the planning horizon information – and think that the centre point is the point of largest y_3-value among the efficient points.

Being able to set a chosen component to a feasible, specified value is a very controlled way of changing a solution. Nevertheless, the feeling of control is not given if the whole procedure is discontinuous. Therefore, we will investigate the continuity of the selection mechanism. If τ just ranges over

the *true* inner planning horizon $\tau \in [\mathbf{y}^I_{\mathcal{C}}(\mathbf{b}),\, \mathbf{y}^N_{\mathcal{C}}(\mathbf{b})]$, the navigation outcomes form a connected set.

As a preparation for the proof, we first cite a lemma ([36]) stating a connectivity criterion.

Lemma 3.4.4 *If $\{\mathcal{A}_j \mid j \in \mathcal{J}\}$ is a family of connected sets with $\bigcap_{j\in\mathcal{J}} \mathcal{A}_j \neq \emptyset$, then $\bigcup_{j\in\mathcal{J}} \mathcal{A}_j$ is connected.*

We will now apply this lemma to show the connectedness of $\mathcal{Y}^{Nav}_{\mathcal{C}}$.

The following theorem first shows that the efficient set for the full dimensional problem intersects the efficient sets for the problems on the different intersections. Thus, the union of efficient set of some intersection with the efficient set for the full dimensional problem is a connected set. But all these unions share a common 'central' set, so that their intersection is non-empty, too. Lemma 3.4.4 then yields the claim.

Theorem 3.4.5 *Let s be continuous, $\mathcal{C}$-centred, strictly $\mathcal{C}$-consistent and convex. Moreover, let the intervals*

$$\mathcal{I}_k = \big[\min\{\mathbf{y}_k \mid \mathbf{y} \in eff_{\mathcal{C}}(\mathcal{Y}(\mathbf{b})\},\max\{\mathbf{y}_k \mid \mathbf{y} \in eff_{\mathcal{C}}(\mathcal{Y}(\mathbf{b})\}\big]$$

in the definition of $\mathcal{Y}^{Nav}_{\mathcal{C}}(\mathbf{b})$ represent the true inner planning horizon. Then $\mathcal{Y}^{Nav}_{\mathcal{C}}(\mathbf{b})$ is connected and contains $eff_{s,\mathcal{C}}(\mathcal{Y}(\mathbf{b}))$.

Proof:

We show the second statement first. Let $\hat{\mathbf{y}} \in eff_{s,\mathcal{C}}(\mathcal{Y}(\mathbf{b}))$ be an arbitrary s-efficient point and let $\mathbf{y}^R$ be a reference point for which $\hat{\mathbf{y}}$ is minimal. Its k^{th} component $\hat{\mathbf{y}}_k \in \mathcal{I}_k(\mathbf{b})$ is contained in the corresponding interval of the inner planning horizon. Placing the intersecting hyperplane at just this level the point $\hat{\mathbf{y}} \in \mathcal{Y}_k(\mathbf{b}, \hat{\mathbf{y}}_k)$ is contained in the intersection. Therefore, the intersection $\mathcal{Y}_k(\mathbf{b}, \hat{\mathbf{y}}_k)$ is nonempty.

Hence, the minimisation over the intersection is well-defined and the following inequality holds:

$$\begin{aligned} t_s\big(\hat{\mathbf{y}} - \mathbf{y}^R\big) &\leq \min\big\{t_s\big(\mathbf{y} - \mathbf{y}^R\big) \;\big|\; \mathbf{y} \in \mathcal{Y}(\mathbf{b})\big\} \\ &\leq \min\big\{t_s\big(\mathbf{y} - \mathbf{y}^R\big) \;\big|\; \mathbf{y} \in \mathcal{Y}_k(\mathbf{b}, \hat{\mathbf{y}}_k)\big\} \\ &\leq t_s\big(\hat{\mathbf{y}} - \mathbf{y}^R\big). \end{aligned} \tag{3.10}$$

Thus, $\hat{\mathbf{y}}$ is an optimum for the given reference point and therefore s-efficient on the intersection

$$\hat{\mathbf{y}} \;\in\; eff_{s,\mathcal{C}}\big(\mathcal{Y}_k(\mathbf{b}, \hat{\mathbf{y}}_k)\big) \;\subseteq\; \mathcal{Y}^{Nav}_{\mathcal{C}}(\mathbf{b}).$$

Since $\hat{\mathbf{y}} \in eff_{s,\mathcal{C}}(\mathcal{Y}(\mathbf{b}))$ was chosen arbitrarily from the set of s-efficient points, every s-efficient solution $eff_{s,\mathcal{C}}(\mathcal{Y}(\mathbf{b})) \subseteq \mathcal{Y}^{Nav}_{\mathcal{C}}(\mathbf{b})$ is contained in the set of navigation outcomes.

Using the argument (3.10) again, we see that for every $\tau \in \mathcal{I}_k(\mathbf{b})$ *the intersection*

$$eff_{s,\mathcal{C}}\big(\mathcal{Y}(\mathbf{b})\big) \cap eff_{s,\mathcal{C}}\big(\mathcal{Y}_k(\mathbf{b},\tau)\big) \neq \emptyset$$

of the two efficiency sets is nonempty.

Now define the family of sets

$$\mathcal{A}_\tau^{(k)} := eff_{s,\mathcal{C}}\big(\mathcal{Y}_k(\mathbf{b},\tau)\big) \cup eff_{s,\mathcal{C}}\big(\mathcal{Y}(\mathbf{b})\big)$$

for all $k \in \mathcal{K}$ *and* $\tau \in \mathcal{I}_k$ *as the union of the 'central' set* $eff_{s,\mathcal{C}}\big(\mathcal{Y}(\mathbf{b})\big)$ *and the set of* $\mathcal{C}$*-efficient solution on the section. Then, the sets* $\mathcal{A}_\tau^{(k)}$ *are unions of connected sets with nonempty intersection and thus by lemma 3.4.4 connected sets. Furthermore, the elements of the family* $\mathcal{A}_\tau^{(k)}$ *have a nonempty intersection, because every set contains* $eff_{s,\mathcal{C}}\big(\mathcal{Y}(\mathbf{b})\big)$*. Thus,*

$$\bigcup_{k\in\mathcal{K}} \Bigg(\bigcup_{\tau\in\mathcal{I}_k\left(eff_{s,\mathcal{C}}(\mathcal{Y}(\mathbf{b}))\right)} \mathcal{A}_\tau^{(k)} \Bigg) = \mathcal{Y}_{\mathcal{C}}^{Nav}(\mathbf{b}) \cup eff_{s,\mathcal{C}}\big(\mathcal{Y}(\mathbf{b})\big) = \mathcal{Y}_{\mathcal{C}}^{Nav}(\mathbf{b})$$

is connected by lemma 3.4.4.

□

The preceding connectedness result holds for the real inner planning horizon and hence for the conservative estimates we use as well. Unfortunately, the proof cannot be easily adapted for the case of the outer planning horizon, since there is no corresponding 'central' set.

The next step in our analysis concerns the continuity of the optimal distance value. We will show that the optimal distance value changes continuously with the changes of the parameter τ. Here, the result holds for all τ-values in the outer planning horizon. For the proof we will use another result from convex analysis ([95]). We will slightly reformulate the theorem to avoid the definition of further concepts of convex analysis.

Consider the problem of minimising a proper convex function f

$$\min\{f(\mathbf{x}) \mid \mathbf{g}(\mathbf{x}) \leq \mathbf{0}, \mathbf{h}(\mathbf{x}) = \mathbf{0}, \mathbf{x} \in \mathcal{S}\} \tag{3.11}$$

over a convex domain $\mathcal{S}$, subject to convex inequality constraints $\mathbf{g}(\mathbf{x}) \leq \mathbf{0}$ and some affine linear equality constraints $\mathbf{h}(\mathbf{x}) = \mathbf{0}$. If we now look at disturbed right hand sides $\mathbf{u}, \mathbf{v}$ for the inequality and equality constraints respectively, the following theorem holds.

Theorem 3.4.6 *For the problem (3.11), the infimum function*

$$\inf\{f(\mathbf{x}) \mid \mathbf{g}(\mathbf{x}) \leq \mathbf{u},\ \mathbf{h}(\mathbf{x}) = \mathbf{v},\ \mathbf{x} \in \mathcal{S}\}$$

is convex as a function of $(\mathbf{u}, \mathbf{v})$ *and has the effective domain*

$$\big\{(\mathbf{u},\mathbf{v}) \;\big|\; \{\mathbf{x} \in \mathcal{S} \mid \mathbf{g}(\mathbf{x}) \leq \mathbf{u},\ \mathbf{h}(\mathbf{x}) = \mathbf{v}\} \neq \emptyset\big\}.$$

The effective domain is the part of the pre-image space, where the convex function has finite values.

The convex set in our case is $\mathcal{Y}(\mathbf{b}) \times \mathcal{C} \times \mathcal{C}$ and the convex function is $s\big(\cdot,\cdot\big)$. We have just affine equality constraints, namely $\mathbf{y}_k = \tau$ and $\mathbf{y} - \mathbf{y}^R(\tau) = \mathbf{c}^+ - \mathbf{c}^-$, where

$$\mathbf{y}^R(\tau) \;:=\; \left\{ \begin{array}{ll} \mathbf{y}_k^S & \text{for } \; k \neq k' \\ \tau & \text{for } \; k = k' \end{array} \right\}$$

for a fixed starting point $\mathbf{y}^S$. Thus, $\mathbf{g}$ is not present in our case and $\mathbf{h}_1(\mathbf{y}) = \mathbf{y} - \mathbf{y}^R(\tau) - \mathbf{c}^+ + \mathbf{c}^-$ and $\mathbf{h}_2(\mathbf{y}) = \mathbf{y}_k - \tau$. Applying the above theorem we get the following corollary.

Corollary 3.4.7 *Let s be a continuous, $\mathcal{C}$-centred, strictly $\mathcal{C}$-consistent and convex cone scalarising function. Then, the optimal value of*

$$\begin{array}{l} \mu(\tau) \;:= \\ \quad \min\big\{ s\big(\mathbf{c}^+,\, \mathbf{c}^-\big) \;\big|\; \mathbf{y} - \mathbf{y}^R(\tau) = \mathbf{c}^+ - \mathbf{c}^-,\; \mathbf{y}_k = \tau,\; \mathbf{y} \in \mathcal{Y}(\mathbf{b}),\; \mathbf{c}^+,\, \mathbf{c}^- \in \mathcal{C} \big\} \end{array}$$

is continuous as a function of τ for $\tau \in int(\mathcal{I}_k(\mathbf{b}))$.

Proof:

By theorem 3.4.6 the minimum is a convex function for all values of τ for which the feasible set is nonempty – by definition at least the interval $\mathcal{I}_k(\mathbf{b})$. But convex functions are continuous in the interior of their effective domain. Moreover, they are Lipschitz-continuous relative to every bounded subset of the relative interior of their effective domain. Hence, μ is Lipschitz-continuous for every closed subset of $\mathcal{I}_k$.

□

Thus, we have continuous behaviour of the optimal cone scalarising value, possibly except for $\tau \in \big\{\mathbf{y}^I(\mathbf{b}),\, \mathbf{y}^N(\mathbf{b})\big\}$. But as the boundary values can cause numerical difficulties one should slightly shorten the interval anyway. For the slightly shortened interval the function is then Lipschitz-continuous.

The next thing we will investigate is the upper semicontinuity of the set of optima for a selection step. So the set-valued mapping we will investigate is

$$\mathcal{O}(k,\tau) \;:=\; argmin\big\{ t_s\big(\mathbf{y} - \mathbf{y}^R(\tau)\big) \;\big|\; \mathbf{y}_k = \tau,\, \mathbf{y} \in \mathcal{Y}(\mathbf{b}) \big\}.$$

The proof is almost a replica of the proof of upper semicontinuity for the optimal set map in theorem 2.5.5.

Theorem 3.4.8 *Let s be a continuous, $\mathcal{C}$-centred, $\mathcal{C}$-monotone and convex cone scalarising function. Then, the corresponding optimal set mapping $\mathcal{O}$ is upper semicontinuous.*

Proof:

Let $\mathcal{U} \supset \mathcal{O}(k,\hat{\tau})$ be an open neighbourhood of the optimal set for a given $\hat{\tau}$. Due to compactness of $\mathcal{Y}(\mathbf{b})$ there exists a radius $\varepsilon > 0$ for which $\mathcal{O}(k,\hat{\tau}) + \mathcal{B}_{2\,\varepsilon}(\mathbf{0}) \subseteq \mathcal{U}$. Now consider the family of open sets

$$\big\{int\big(\mathcal{B}_{\varepsilon}(\mathbf{y})\big) \;\big|\; \mathbf{y} \in \mathcal{Y}\big\},$$

which constitutes a covering of $\mathcal{Y}(\mathbf{b})$. As $\mathcal{Y}(\mathbf{b})$ is compact we can pick a finite number of points $\check{\mathbf{y}}(i) \in \mathcal{Y}$, $i = 1, ..., n$ such that

$$\mathcal{Y}(\mathbf{b}) \quad \subset \quad \bigcup_{i=1}^{n} int\big(\mathcal{B}_{\varepsilon}(\check{\mathbf{y}}(i))\big).$$

Let $(\tau(i))_{i\in\mathbb{N}}$ be a sequence of values converging to $\hat{\tau}$ and let

$$\big(\mathbf{y}(i)\big)_{i\in\mathbb{N}} \in \mathcal{O}(k,\tau(i))$$

be optimal for the corresponding τ-value. Since we have a finite covering of $\mathcal{Y}(\mathbf{b})$ there must be sets $int\big(\mathcal{B}_{\varepsilon}(\check{\mathbf{y}}(i))\big)$ with infinitely many points of $\big(\mathbf{y}(i)\big)_{i\in\mathbb{N}}$ in them.

Choose an arbitrary set $\mathcal{V} := int\big(\mathcal{B}_{\varepsilon}(\check{\mathbf{y}}(i'))\big)$ with infinitely many points of the sequence in it. Then the subsequence of $\big(\mathbf{y}(i)\big)_{i\in\mathbb{N}}$ with elements just in $\mathcal{V}$ is bounded and thus has a convergent subsequence $\big(\bar{\mathbf{y}}(i)\big)_{i\in\mathbb{N}}$. Let $\bar{\mathbf{y}} := \lim\limits_{i\to\infty} \bar{\mathbf{y}}(i)$ be its limit and $\big(\bar{\tau}(i)\big)_{i\in\mathbb{N}}$ be the corresponding sequence of τ-values. The continuity of μ

$$\begin{aligned} \lim_{i\to\infty} \mu\big(\bar{\tau}(i)\big) &= \mu\big(\lim_{i\to\infty} \bar{\tau}(i)\big) \\ &= \mu\big(\hat{\tau}\big) \end{aligned}$$

ensures that the convergence of $\big(\bar{\tau}(i)\big)_{i\in\mathbb{N}}$ is preserved. As t_s is continuous as well we get the following result:

$$\begin{aligned} \lim_{i\to\infty} t_s\big(\bar{\mathbf{y}}(i) - \mathbf{y}^R(\bar{\tau}(i))\big) &= t_s\Big(\lim_{i\to\infty} \bar{\mathbf{y}}(i) - \lim_{i\to\infty} \mathbf{y}^R(\bar{\tau}(i))\Big) \\ &= t_s\big(\bar{\mathbf{y}} - \mathbf{y}^R(\hat{\tau})\big). \end{aligned}$$

Putting the two equations together,

$$\begin{aligned} \mu\big(\hat{\tau}\big) &= \lim_{i\to\infty} \mu\big(\bar{\tau}(i)\big) \\ &= \lim_{i\to\infty} t_s\big(\bar{\mathbf{y}}(i) - \mathbf{y}^R(\bar{\tau}(i))\big) \\ &= t_s\big(\bar{\mathbf{y}} - \mathbf{y}^R(\hat{\tau})\big), \end{aligned}$$

we see that the limit of the sequence $\bar{\mathbf{y}} \in \mathcal{O}(k,\hat{\tau})$ is in the set of optima for the limit $\bar{\tau}$-value.

Therefore, the set $cl(\mathcal{V})$ contains at least one point of $\mathcal{O}(k,\hat{\tau})$. Since the choice of $\mathcal{V}$ was arbitrary among the sets containing infinitely many points of $\big(\mathbf{y}(i)\big)_{i\in\mathbb{N}}$ the closures of all these sets intersect with $\mathcal{O}(k,\hat{\tau})$. As the diameters of the sets $int\big(\mathcal{B}_\varepsilon\big(\check{\mathbf{y}}(i)\big)\big)$ forming the coverage is $2\,\varepsilon$ there are only finitely many points of $\big(\mathbf{y}(i)\big)_{i\in\mathbb{N}}$ that lie outside

$$\mathcal{O}(k,\hat{\tau}) + \mathcal{B}_{2\,\varepsilon}(\mathbf{0}) \;\subseteq\; \mathcal{U}.$$

Hence, choosing $i_0 \in \mathbb{N}$ such that all these points are excluded, the remaining outcomes $\mathbf{y}(i) \in \mathcal{U}, i \geq i_0$ are contained in the neighbourhood. Thus, the optimal sets for a convergent sequence of τ-values are eventually contained in an open neighbourhood of the optimal set for the limit $\hat{\tau}$.

□

In contrast to the upper semicontinuity proof for the optimal set map in chapter 2 we require the convexity of s and $\mathcal{Y}(\mathbf{b})$ here to be able to use the continuity of the minimal value function μ. Again, we can use the closedness criterion of proposition 2.5.11 to prove the closedness of $\mathcal{O}$.

Corollary 3.4.9 *The optimal set mapping $\mathcal{O}$ is closed.*

Proof:

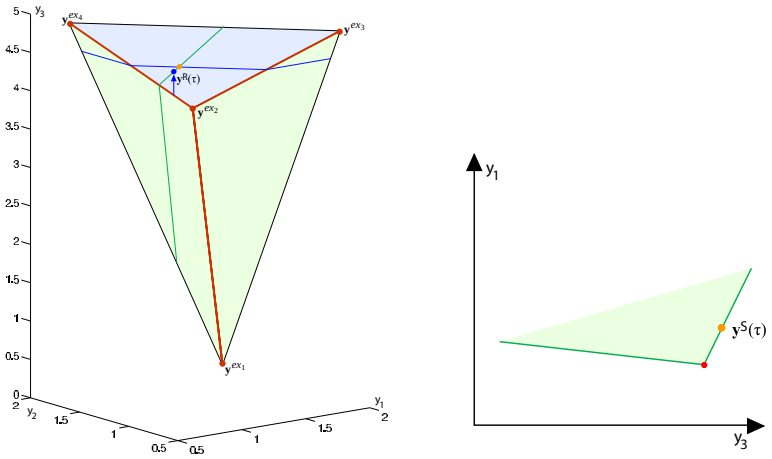
The sets of minimisers of t_s are closed sets due to $\mathcal{Y}_k(\mathbf{b},\tau)$ being closed and t_s being continuous. Furthermore, the interval $\mathcal{I}_k(\mathbf{b})$ is closed. Therefore, the upper semicontinuity of the set-valued mapping $\mathcal{O}$ implies that it is closed.

□

The uniqueness results of section 2.5 can directly be transferred to the case of the selection mechanism. If the set of outcomes $\mathcal{Y}$ does not have a face of dimension 1 or higher in common with the level sets of t_s, its restriction $\mathcal{Y}(\mathbf{b})$ clearly does not have one either. Moreover, the intersection of the restricted set of outcomes with a hyperplane $\mathcal{H}_k(\tau)$ cannot have a face of dimension 1 or higher in common with the level sets of t_s either. Thus, if the conditions hold for the unrestricted and full dimensional set of outcomes, they also hold for the case of the selection mechanism problem (3.8).

Note, that the combination of upper semicontinuity and uniqueness yields continuity. So, for a continuous $\mathcal{C}$-centred, strictly $\mathcal{C}$-consistent and convex cone scalarising function s and a outcome set $\mathcal{Y}$ that either fulfil the conditions of proposition 2.5.17 or proposition 2.5.18 the optima for a selection step form a path parametrised by τ.

Note that the continuity of the optima of the selection mechanism does neither mean that the set of navigation outcomes is connected nor that there are no jumps in the selection. It just means that during one selection step there is no jump. But there can be one at the beginning – for the parameter $\tau = \mathbf{y}_k^S$ when the k^{th} objective was chosen for manipulation. Consider the



(a) (b)

Figure 3.5: With a strictly dominated starting point a jump can happen at the start of the next selection step.

situation depicted in figure 3.5(a) for a selection step that yields a dominated solution. If now the second objective is chosen for the next movement, we start with the orange point in figure 3.5(b) that will directly be improved, leading to a jump at the start of a selection step.

We now turn our attention to the third and last mechanism, which allows the introduction of restrictions regarding the relative solution quality – the discounting.

3.5 The discounting mechanism

Cone scalarising is constructed to work with more general ordering cones than the Pareto cone. We will now use this feature to control the *partial tradeoffs* (see [20], [83]). The original definition is made in terms of pre-images and the objective functions. We will reformulate it with points of the outcome set instead.

Definition 3.5.1 *Let $\mathbf{y} \neq \mathbf{y}' \in \mathcal{Y}$ be two feasible outcomes. Then the slope*

$$\frac{\mathbf{y}_k - \mathbf{y}'_k}{\mathbf{y}_l - \mathbf{y}'_l}$$

of the k^{th} objective relative to the l^{th} objective is called partial tradeoff*, if all remaining objectives $\mathbf{y}_{k'} = \mathbf{y}'_{k'}, k' \in \mathcal{K} \setminus \{k, l\}$ for the two outcomes are equal.*

Naturally, we are not interested in the partial tradeoff for *any* two outcomes, but are interested in the partial tradeoffs for the efficient solutions. Here, the partial tradeoffs are always negative, since we cannot gain in both objectives due to the definition of efficiency. One can see the partial tradeoffs as the price one has to pay – therefore the negative sign – in the k^{th} objective for an unit gain in the l^{th} objective. Therefore, we will sometimes use prices as a synonym for partial tradeoff.

We will derive in this section a method to translate bounds on the partial tradeoffs into an ordering cone $\mathcal{C}$. The ordering cone $\mathcal{C}$ is constructed such that the $\mathcal{C}$-efficient solutions always fulfil the given bounds on the partial tradeoffs.

Assume we do not want to deteriorate the k^{th} objective by more than $\mathbf{T}_{l,k}$ the amount we gain in the l^{th} objective while the other objectives are fixed at their current values. So, for any two efficient solutions $\hat{\mathbf{y}} \neq \bar{\mathbf{y}} \in eff(\mathcal{Y}(\mathbf{b}))$ with $\hat{\mathbf{y}}_{k'} = \bar{\mathbf{y}}_{k'}$ for all $k' \in \mathcal{K} \setminus \{k, l\}$ the following inequality is supposed to hold:

$$\frac{\bar{\mathbf{y}}_k - \hat{\mathbf{y}}_k}{\hat{\mathbf{y}}_l - \bar{\mathbf{y}}_l} \leq \mathbf{T}_{l,k}.$$

Since the points are efficient, the difference in the two components is nonzero. Besides we assume, that $\bar{\mathbf{y}}_l < \hat{\mathbf{y}}_l$ and that we want to control the price for that gain. Therefore, the inequalities

$$\begin{aligned} \frac{\bar{\mathbf{y}}_k - \hat{\mathbf{y}}_k}{\hat{\mathbf{y}}_l - \bar{\mathbf{y}}_l} \leq \mathbf{T}_{l,k} \quad &\Longleftrightarrow \quad -\hat{\mathbf{y}}_k + \bar{\mathbf{y}}_k \leq \mathbf{T}_{l,k}\,(\hat{\mathbf{y}}_l - \bar{\mathbf{y}}_l) \\ &\Longleftrightarrow \quad 0 \;\leq\; \mathbf{T}_{l,k}\,(\hat{\mathbf{y}}_l - \bar{\mathbf{y}}_l) + \hat{\mathbf{y}}_k - \bar{\mathbf{y}}_k \\ &\Longleftrightarrow \quad \left\langle \mathbf{T}_{l,k}\,\mathbf{e}_l + \mathbf{e}_k,\; \hat{\mathbf{y}} - \bar{\mathbf{y}} \right\rangle \;\geq\; 0 \end{aligned} \tag{3.12}$$

are equivalent. Note that in case of $\bar{\mathbf{y}}_l > \hat{\mathbf{y}}_l$ the sign of the denominator $\bar{\mathbf{y}}_k - \hat{\mathbf{y}}_k < 0$ is negative due to the efficiency of the points $\bar{\mathbf{y}}, \hat{\mathbf{y}}$. Thus, we can just exchange the roles of the two points, so that either way the price in the k^{th} objective for the gain in the l^{th} objective is bounded.

We will use the last formulation, because we do not need a strict inequality $\bar{\mathbf{y}}_l < \hat{\mathbf{y}}_l$ and it allows the combination of several such inequalities into a matrix. Note that $\mathbf{T}_{l,k} < 0$ are infeasible for efficient points. Therefore, we will only consider nonnegative $\mathbf{T}_{l,k}$.

We will now couple the above tradeoff restriction (3.12) with a restriction on the elements of the dual ordering cone.

Theorem 3.5.2 *The inequality (3.12) is met for all $\mathcal{C}$-efficient points, if every element $\mathbf{c}^* \in \mathcal{C}^*$ of the dual ordering cone fulfils the inequality*

$$\mathbf{c}_l^* \leq \mathbf{T}_{l,k}\,\mathbf{c}_k^*.$$

Proof:

Assume the inequality for the elements of the dual cone holds, but there are two $\mathcal{C}$-efficient points $\hat{\mathbf{y}}, \bar{\mathbf{y}}$ that violate the tradeoff restriction (3.12), i.e.

$$\left\langle \mathbf{T}_{l,k}\,\mathbf{e}_l + \mathbf{e}_k,\ \hat{\mathbf{y}} - \bar{\mathbf{y}} \right\rangle \ < \ 0.$$

Without loss of generality assume that $\bar{\mathbf{y}}_k > \hat{\mathbf{y}}_k$ and hence $\bar{\mathbf{y}}_l < \hat{\mathbf{y}}_l$.

For a $\mathcal{C}$-efficient point $\bar{\mathbf{y}}$ in a convex multiobjective optimisation problem there exists a $\bar{\mathbf{c}}^* \in \mathcal{C}^* \setminus \{\mathbf{0}\}$

$$\langle \bar{\mathbf{c}}^*, \bar{\mathbf{y}} \rangle = \min \left\{ \langle \bar{\mathbf{c}}^*, \mathbf{y} \rangle \ \middle|\ \mathbf{y} \in \mathcal{Y}(\mathbf{b}) \right\}$$

for which it minimises the corresponding linear functional (see proposition 2.2.8). We will choose $\bar{\mathbf{c}}^*$ such that $\bar{\mathbf{c}}^*_l > 0$. Thus, we have the following equivalent formulations

$$\begin{aligned} \left\langle \mathbf{T}_{l,k}\,\mathbf{e}_l + \mathbf{e}_k,\ \hat{\mathbf{y}} - \bar{\mathbf{y}} \right\rangle \ < \ 0 \quad &\iff \quad \bar{\mathbf{c}}^*_l \left\langle \mathbf{T}_{l,k}\,\mathbf{e}_l + \mathbf{e}_k,\ \hat{\mathbf{y}} - \bar{\mathbf{y}} \right\rangle \ < \ 0 \\ &\iff \quad \left\langle \bar{\mathbf{c}}^*_l\,\mathbf{T}_{l,k}\,\mathbf{e}_l + \bar{\mathbf{c}}^*_l\,\mathbf{e}_k,\ \hat{\mathbf{y}} - \bar{\mathbf{y}} \right\rangle \ < \ 0. \end{aligned}$$

Now using the restriction $\mathbf{c}^*_l \leq \mathbf{T}_{l,k}\,\mathbf{c}^*_k$ of the marginal rates of substitution and the assumed negativity of the difference $\hat{\mathbf{y}}_k - \bar{\mathbf{y}}_k < 0$, we can proceed with

$$\begin{aligned} &\left\langle \bar{\mathbf{c}}^*_l\,\mathbf{T}_{l,k}\,\mathbf{e}_l + \bar{\mathbf{c}}^*_l\,\mathbf{e}_k,\ \hat{\mathbf{y}} - \bar{\mathbf{y}} \right\rangle \ < \ 0 \\ &\qquad\qquad \iff \quad \left\langle \bar{\mathbf{c}}^*_l\,\mathbf{T}_{l,k}\,\mathbf{e}_l + \bar{\mathbf{c}}^*_k\,\mathbf{T}_{l,k}\,\mathbf{e}_k,\ \hat{\mathbf{y}} - \bar{\mathbf{y}} \right\rangle \ < \ 0 \\ &\qquad\qquad \overset{\mathbf{T}_{l,k}>0}{\iff} \quad \left\langle \bar{\mathbf{c}}^*_l\,\mathbf{e}_l + \bar{\mathbf{c}}^*_k\,\mathbf{e}_k,\ \hat{\mathbf{y}} - \bar{\mathbf{y}} \right\rangle \ < \ 0. \end{aligned}$$

By assumption the difference $\hat{\mathbf{y}} - \bar{\mathbf{y}}$ of the two points is nonzero in the components l, k only. Therefore, the two inner products

$$\left\langle \bar{\mathbf{c}}^*_l\,\mathbf{e}_l + \bar{\mathbf{c}}^*_k\,\mathbf{e}_k,\ \hat{\mathbf{y}} - \bar{\mathbf{y}} \right\rangle \ = \ \left\langle \bar{\mathbf{c}}^*,\ \hat{\mathbf{y}} - \bar{\mathbf{y}} \right\rangle$$

coincide. Hence, we can continue the equivalences with

$$\begin{aligned} \left\langle \bar{\mathbf{c}}^*_l\,\mathbf{e}_l + \bar{\mathbf{c}}^*_k\,\mathbf{e}_k,\ \hat{\mathbf{y}} - \bar{\mathbf{y}} \right\rangle \ < \ 0 \quad &\iff \quad \left\langle \bar{\mathbf{c}}^*,\ \hat{\mathbf{y}} - \bar{\mathbf{y}} \right\rangle \ < \ 0 \\ &\iff \quad \left\langle \bar{\mathbf{c}}^*,\ \hat{\mathbf{y}} \right\rangle \ < \ \left\langle \bar{\mathbf{c}}^*,\ \bar{\mathbf{y}} \right\rangle, \end{aligned}$$

which contradicts the minimality of $\bar{\mathbf{y}}$. The constraint on the elements of the dual ordering cone therefore implies the bound on the partial tradeoff. □

We can thus control the tradeoffs by imposing restrictions on the elements of the dual ordering cone. Assume we are looking for efficient solutions and want to impose a set of tradeoff restrictions. Let $\bar{\mathcal{T}} \subseteq \mathcal{K} \times \mathcal{K}$ be the set of

pairs for which we want to limit the tradeoffs and let $\bar{\mathcal{T}}(j)$ be the j^{th} pair in the set. To control the tradeoffs we impose the restrictions

$$\mathbf{c}^*_{\bar{\mathcal{T}}(j)_1} \leq \mathbf{T}_{\bar{\mathcal{T}}(j)}\, \mathbf{c}^*_{\bar{\mathcal{T}}(j)_2},$$

on the elements of the dual ordering cone. The dual cone for the Pareto ordering cone $\mathbb{Y}_+$ is $\mathbb{Y}_+$, so the dual ordering cone

$$\begin{aligned} \mathcal{C}^* &= \left\{\mathbf{c}^* \in \mathbb{Y}_+ \;\middle|\; \mathbf{c}^*_{\bar{\mathcal{T}}(j)_1} \leq \mathbf{T}_{\bar{\mathcal{T}}(j)}\, \mathbf{c}^*_{\bar{\mathcal{T}}(j)_2}\right\} \\ &= \left\{\mathbf{c}^* \in \mathbb{Y} \;\middle|\; \mathbf{c}^*_{\bar{\mathcal{T}}(j)_1} \leq \mathbf{T}_{\bar{\mathcal{T}}(j)}\, \mathbf{c}^*_{\bar{\mathcal{T}}(j)_2},\ \mathbf{c}^* \geq \mathbf{0}\right\} \end{aligned}$$

incorporates the tradeoff restrictions. If we rewrite the nonnegativity constraints as

$$\mathbf{c}^* \geq \mathbf{0} \quad \Longleftrightarrow \quad \mathbf{c}^* \leq 2\,\mathbf{c}^*,$$

we can integrate them into the tradeoff restrictions by setting $\mathbf{T}_{k,k} = 2$ for all $k \in \mathcal{K}$. Thus, we get a pair set

$$\mathcal{T} \;:=\; \bar{\mathcal{T}} \;\cup\; \{(k,k) \mid k \in \mathcal{K}\}$$

that is always nonempty. Using the equivalence

$$\mathbf{c}^*_{\mathcal{T}(j)_1} \leq \mathbf{T}_{\mathcal{T}(j)}\, \mathbf{c}^*_{\mathcal{T}(j)_2} \quad \Longleftrightarrow \quad \left\langle \mathbf{T}_{\mathcal{T}(j)}\, \mathbf{e}_{\mathcal{T}(j)_2} - \mathbf{e}_{\mathcal{T}(j)_1},\ \mathbf{c}^* \right\rangle \;\geq\; 0$$

we can combine the different vectors $\mathbf{T}_{\mathcal{T}(j)}\, \mathbf{e}_{\mathcal{T}(j)_2} - \mathbf{e}_{\mathcal{T}(j)_1}$, $j = 1, ..., |\mathcal{T}|$ into a matrix $\mathbf{C}$ with entries

$$\mathbf{C}_{i,j} := \left\{ \begin{array}{lll} 1 & \text{for} & \mathcal{T}(j)_1 = i = \mathcal{T}(j)_2 \\ -1 & \text{for} & \mathcal{T}(j)_1 = i \neq \mathcal{T}(j)_2 \\ \mathbf{T}_{\mathcal{T}}(j) & \text{for} & \mathcal{T}(j)_1 \neq i = \mathcal{T}(j)_2 \\ 0 & \text{for} & \mathcal{T}(j)_1, \mathcal{T}(j)_2 \neq i \end{array} \right\}.$$

Note that due to the nonnegativity constraints $\mathcal{T} \supseteq \{(k,k,) \mid k \in \mathcal{K}\}$. Therefore, the matrix contains the identity matrix as a submatrix. The matrix notation now allows us to rewrite the full set of restrictions as $\mathbf{C}^T\mathbf{c}^* \geq \mathbf{0}$. Hence, the dual ordering cone is given by

$$\mathcal{C}^* := \left\{\mathbf{c}^* \in \mathbb{Y} \;\middle|\; \mathbf{C}^T\mathbf{c}^* \;\geq\; \mathbf{0}\right\}.$$

From this we will derive a similarly easy explicit representation of the ordering cone. To derive one of the inclusions we will use the strong duality theorem of linear programming (see e.g. [25]). This correspondence is a special case of results in [95, section 14].

Lemma 3.5.3 *The ordering cone implying the above tradeoff restrictions is given by*

$$\mathcal{C} = \{\mathbf{C}\,\mathbf{w} \mid \mathbf{w} \geq \mathbf{0}\}.$$

Proof:
In the setting considered here it suffices to show that the given cone equals the dual $\{\mathbf{C}\,\mathbf{w} \mid \mathbf{w} \geq \mathbf{0}\} = (\mathcal{C}^*)^*$ *of the dual ordering cone* $\mathcal{C}^*$ *(see [64]).*
$\subseteq$*:*
For any given $\mathbf{w} \geq \mathbf{0}$ *and any* $\mathbf{c}^* \in \mathcal{C}^*$

$$\langle \mathbf{C}\,\mathbf{w}, \mathbf{c}^* \rangle \quad = \quad \langle \mathbf{w}, \mathbf{C}^T\,\mathbf{c}^* \rangle \quad \geq \quad 0.$$

Therefore, every element of $\{\mathbf{C}\,\mathbf{w} \mid \mathbf{w} \geq \mathbf{0}\}$ *is dual to every element of* $\mathcal{C}^*$ *and hence contained in the bidual. This implies* $\{\mathbf{C}\,\mathbf{w} \mid \mathbf{w} \geq \mathbf{0}\} \subseteq (\mathcal{C}^*)^*$.
$\supseteq$*:*
We will show that every $\Delta\mathbf{y} \in \mathbb{Y} \setminus \{\mathbf{C}\,\mathbf{w} \mid \mathbf{w} \geq \mathbf{0}\}$ *outside the given cone is not contained in* $(\mathcal{C}^*)^*$ *either. Since* $\mathbf{C}$ *contains the identity matrix,* $\Delta\mathbf{y}$ *can be written as a linear combination* $\Delta\mathbf{y} = \mathbf{C}\,\mathbf{w}$ *for a suitable* $\mathbf{w} \in \mathbb{R}^{|\mathcal{T}|}$. *For every* $\Delta\mathbf{y} \notin \{\mathbf{C}\,\mathbf{w} \mid \mathbf{w} \geq \mathbf{0}\}$ *the optimal value*

$$\max \min_{0 \leq j \leq |\mathcal{T}|} \{\mathbf{w}_j \mid \Delta\mathbf{y} = \mathbf{C}\,\mathbf{w}\} < 0 \tag{3.13}$$

is negative. The problem (3.13) can be reformulated as the linear program

$$\max\{z \in \mathbb{R} \mid \mathbf{w} \geq \mathbf{1}\,z,\ \mathbf{C}\,\mathbf{w} = \Delta\mathbf{y}\},$$

whose dual can be simplified to

$$\min\left\{\Delta\mathbf{y}^T\mathbf{u} \,\middle|\, \mathbf{C}^T\mathbf{u} \geq \mathbf{0}, \left(\mathbf{1}^T\,\mathbf{C}^T\right)\mathbf{u} = 1\right\}. \tag{3.14}$$

Because the primal problem (3.13) is bounded and has feasible points, its optimum is attained. Duality theory of linear programming thus yields that the optimal value of (3.14) is negative as well. Looking at the constraints of (3.14) we see that its feasible set is contained in $\mathcal{C}^* \setminus \{\mathbf{0}\}$. *But for the optimal* $\mathbf{u}$ *the scalar product with* $\Delta\mathbf{y}$ *is negative. Hence, the chosen point* $\Delta\mathbf{y} \notin (\mathcal{C}^*)^*$ *is not contained in the bidual. Since* $\Delta\mathbf{y}$ *is an arbitrary point outside* $\{\mathbf{C}\,\mathbf{w} \mid \mathbf{w} \geq \mathbf{0}\}$, *no such point can be contained in the bidual. Therefore, the proposed cone* $\{\mathbf{C}\,\mathbf{w} \mid \mathbf{w} \geq \mathbf{0}\} \supseteq (\mathcal{C}^*)^*$ *is a superset of the bidual.*

□

So the ordering cone is the positive linear combination of the columns of $\mathbf{C}$. If we have no user defined tradeoff restrictions then $\mathbf{C}$ is just the identity matrix and hence the ordering cone is just the Pareto cone $\mathbb{Y}_+$. Any nontrivial tradeoff restriction will add a generator for the ordering cone and consequentially the new cone is a superset of the old one.

The tradeoff restrictions should leave enough 'room' so that the dual ordering cone still has nonempty interior. For that the combination of inequalities must neither become sharp nor incommensurate with positive $\mathbf{c}_k^*$. Since the

restrictions are onesided, only a chain of inequalities having the same element as starting and end point combined with the nonnegativity constraints can cause such a situation. So assume there is a closed path $\mathcal{P} = (k_1, ..., k_n)$ with $(k_i, k_{i+1}) \in \mathcal{T}$ for all $i = 1, ..., n-1$ and $(k_n, k_1) \in \mathcal{T}$. Hence, we have the set of inequalities

$$\mathbf{c}_{k_1}^* \leq \mathbf{T}_{k_1,k_2}\, \mathbf{c}_{k_2}^* \ , \ \ ... \ , \ \ \mathbf{c}_{k_{n-1}}^* \leq \mathbf{T}_{k_{n-1},k_n}\, \mathbf{c}_{k_n}^* \ , \ \ \mathbf{c}_{k_n}^* \leq \mathbf{T}_{k_n,k_1}\, \mathbf{c}_{k_1}^*.$$

Setting $k_{n+1} = k_1$ and combining all the inequalities we can derive the following condition on $\mathbf{c}_{k_1}^*$:

$$\mathbf{c}_{k_1}^* \leq \prod_{i=1}^{n} \mathbf{T}_{k_i,k_{i+1}}\, \mathbf{c}_{k_1}^*$$

Thus, for allowing positive $\mathbf{c}_k^*$ the product

$$1 \leq \prod_{i=1}^{n} \mathbf{T}_{k_i,k_{i+1}}. \tag{3.15}$$

needs to be greater than or equal to 1. We capture this in the following definition.

Definition 3.5.4 *Assume the tradeoff bounds are given by $\mathcal{T}$ and $\mathbf{T}$. Then*

i) they are called compatible*, if the product $\prod_{i=1}^{n} \mathbf{T}_{k_i,k_{i+1}} \geq 1$ and*

ii) strictly compatible*, if $\prod_{i=1}^{n} \mathbf{T}_{k_i,k_{i+1}} > 1$*

for every existing chain of inequalities in $\mathcal{T}$. If some chain exists with a product $\prod_{i=1}^{n} \mathbf{T}_{k_i,k_{i+1}} < 1$ smaller than 1, we call the bounds incompatible*.*

For incompatible bounds, the objectives whose corresponding components in the dual ordering cone are restricted to 0 do not play a role in the optimisation any longer. Thus, if all of the components of the dual ordering cone are involved in incompatible inequality cycles, the problem is turned into a pure feasibility problem.

For compatible but tight bounds the relative values of the elements of the dual cone are fixed for the involved components. Hence, the dual ordering cone has empty interior, and therefore there are no properly $\mathcal{C}$-efficient solutions (see proposition 2.2.8). Moreover, we have fixed marginal rates of substitution for the corresponding objectives, which means that we are effectively considering their weighted sum as *one* objective in the multiobjective problem. So, tightening the tradeoff constraints results in an increasingly strong coupling of the corresponding objective functions.

If all tradeoff rates are fixed we effectively have a halfspace as an ordering cone, so that the optimisation result can equivalently be gained by minimising the weighted sum with one of the elements of the half-ray $\mathcal{C}^*$ used as

weight vector. Note that the $\mathcal{C}$-approximating functions are $\mathcal{C}$-representing in this case, because they are nonnegative in $\mathcal{C}$ and nonpositive in $-\mathcal{C}$ and $-\mathcal{C} \cup \mathcal{C} = \mathbb{Y}$. So independently from the chosen $\mathcal{C}$-representing or $\mathcal{C}$-approximating function the cone scalarising problem will yield the weighted sum solution in this case.

To check the compatibility of the constraint combination we need to ensure that the restriction (3.15) holds for all closed circles in the directed graph given by the vertices $k \in \mathcal{K}$ and the edges $\mathcal{T}$.

Proposition 3.5.5 *If the shortest path problem on the directed graph given by the vertices $k \in \mathcal{K}$ and the edges $\mathcal{T}$ with edge weights $\ln\left(\mathbf{T}_{k,k'}\right)$ detects no negative cycle for the paths from each vertex k to itself, the constraints $\mathbf{C}^{\mathcal{T}}\,\mathbf{c}^* \geq \mathbf{0}$ are compatible.*

Proof:

If no closed path exists, the claim holds trivially. So let $(k_1, ..., k_n, k_{n+1} = k_1)$ be a closed path with minimal length. Due to the monotonicity of the natural logarithm

$$\sum_{i=1}^{n} \ln\left(\mathbf{T}_{k_i,k_{i+1}}\right) = \ln\left(\prod_{i=1}^{n} \mathbf{T}_{k_i,k_{i+1}}\right) = \ln(1) \geq 0 \iff \prod_{i=1}^{n} \mathbf{T}_{k_i,k_{i+1}} \geq 1.$$

Therefore, the nonnegativity of the sum of logarithms is equivalent to the product being bigger than or equal to 1. Since we test each vertex in turn, we will detect negative cycles, if they exist.

□

Detecting negative cycles in shortest path problems is easy ([2]) and for the typical number of vertices ($= K$) and edges ($\leq K^2$) present in our specific network its runtime is negligible.

The situation is even simpler when the tradeoff bounds are introduced one at a time. Here, we restrict the edge length of the new constraint such that system of constraints stays compatible – which it is for the Pareto cone. If a decision maker wants to introduce a bound on the deterioration of objective k per unit gain in objective l, we can detect the shortest path from l to k, if such a path exists. If it does not exist, the decision maker can introduce a bound with an arbitrary $\mathbf{T}_{l,k} > 0$. If such a path exists we get the path length ρ of the shortest path. Now choosing $\mathbf{T}_{l,k} \geq e^{-\rho}$ we ensure that the cycle composed from the shortest path and the new edge has nonnegative length. Thus, the corresponding chain of inequalities is compatible. If we want to enforce strict compatibility the inequality must be strict, i.e. $\mathbf{T}_{l,k} > e^{-\rho}$.

To avoid numerical difficulties, one should set a minimum distance $\delta > 0$ and ensure by the above procedure, that there is always a slack of at least δ in

every chain of inequalities. This can easily be incorporated into the methods. So not just negative cycles imply incompatibility in this stronger sense, but also detecting a shortest cyclic path of length smaller than $\ln(1+\delta)$ is then judged to be incompatible.

So far we just analysed the effects the discounting mechanism has on the forthcoming selection steps and thus on the choice of the subsequent solutions. We will now have a closer look on the effects it has on the current solution.

The current solution is modified to conform to the newly introduced price bounds. For that we solve a cone scalarising problem on the full feasibility domain $\mathcal{Y}(\mathbf{b})$

$$\min\left\{s\big(\mathbf{c}^+,\,\mathbf{c}^-\big)\;\middle|\;\mathbf{y}-\mathbf{y}^S=\mathbf{c}^+-\mathbf{c}^-,\;\mathbf{y}\in\mathcal{Y}(\mathbf{b}),\;\mathbf{c}^+,\,\mathbf{c}^-\in\mathcal{C}\right\} \tag{3.16}$$

with the previous solution $\mathbf{y}^S$ as a reference point. If the point is s-efficient for the old cone and remains so for the new cone, i.e. it already fulfils the new pairwise bounds on the prices, it will not be changed. If it is not s-efficient with respect to the old ordering cone or does not fulfil the new price bound it will be changed.

This change does in general not happen continuously, as it is very similar to changing the weights in a weighted sum scalarisation. In particular, polyhedral outcome sets $\mathcal{Y}(\mathbf{b})$ are likely to exhibit discontinuous behaviour. A small enlargement of the previous ordering cone $\mathcal{C}'$ could 'remove' a whole facet of the corresponding efficient set $eff_{\mathcal{C}'}(\mathcal{Y}(\mathbf{b}))$ and thus force the cone scalarising problem to jump. For continuous behaviour the condition that the set of efficient points has just on point with each facet of $\mathcal{Y}+\mathbb{Y}_+$ in common is needed, to rule out the described situation (see [41]).

3.6 Two-phase approach to Pareto navigation

Interactive multiobjective optimisation methods are especially useful when the response times of the system to user input are short. If the decision maker has to wait noticeable fractions of a minute for the system to answer the interactivity is somewhat lost and the decision process becomes awkward. Therefore, large-scale multiobjective optimisation problems are a challenge for interactive methods.

Sometimes one is able to improve the speed of the individual calculation to overcome the problem, but if no special structure in the scalar problems can be exploited this approach is not feasible.

A more general technique is the use of a two phase approach. Here, one combines a method that generates an approximation to the set of efficient

solutions with an interactive approach. The approximation method is referred to as phase I and will usually involve lengthy computations for high-dimensional and large-scale multiobjective optimisation problems. This part can be done offline, so after the computations are started the first phase does not require any further interaction.

The approximation gained in phase I is then used for decision making in the interactive phase – phase II. Here, optimisation problems are solved *in* the given approximation ([40]).

We will describe a two-phase approach to Pareto navigation meant to be real-time compatible. Thus, we consider the *evaluation* of the objective function to be potentially time-critical. We will discuss at several points possibilities to improve the current estimate by some computationally more involved method. The implicit assumption is that the time intervals between individual decisions will vary. The corrections are meant to be executed while the system is idle, e.g. when the decision maker closely inspects the current solution or rethinks his or her strategy.

In the remainder of this section we will assume the approximation from the first phase to be given. For most of the ideas and methods put forward in this section an approximation of the efficient set by a set of efficient solutions is enough. They do not need to be ordered or linked to some parameters. Although, at times we discuss refined or auxiliary methods that include the use of normal vectors to the efficient set or equivalently subgradients of the scalarisation function at the given points.

This information is readily available for some of the approximation methods, e.g. the methods based on weighted sum scalarisation. For other methods it needs to be generated, but the basic information for its generation will most likely be present during the calculation of the efficient points. But it can sometimes even be generated afterwards (see [56]), if one judges the effort to be worthwhile.

We will not elaborate the question of how to construct the approximation here. The article of Ruzika and Wiecek ([99]) contains an overview of available methods. The exact choice depends to some extent on the characteristics of the multiobjective problem under consideration. Therefore, we will discuss this issue in more detail in conjunction with the intensity modulated radiotherapy planning problem in chapter 4.

3.6.1 Simplifying the problem formulations

We will use the precomputed solutions to considerably simplify the problem formulation. Solving the simplified problems will be amenable to fast execution. Naturally, the solutions gained by solving the reformulation will only be approximations to solutions of the original problem. If the precomputed

solutions form a good approximation of the efficient set, the results of the simplified problems will give a fairly accurate picture though.

Assume we are given a number of pre-computed $\mathbb{Y}_+$-efficient decision vectors $\mathbf{x}^{(1)}, ..., \mathbf{x}^{(m)}$ and their objective vectors $\mathbf{y}^{(1)}, ..., \mathbf{y}^{(m)}$. For convenience we group the vectors into matrices

$$\mathbf{X} := \left(\mathbf{x}^{(1)} \mid ... \mid \mathbf{x}^{(m)}\right) \quad \text{and} \quad \mathbf{Y} := \left(\mathbf{y}^{(1)} \mid ... \mid \mathbf{y}^{(m)}\right).$$

This information allows us to formulate two simplified versions of our multiobjective optimisation problem. The first reformulation substitutes the potentially complex and implicitly given feasible domain by the convex hull of the precomputed solutions and thus by an explicit representation. So we consider the multiobjective optimisation problem

$$\text{mo-min}\left\{\mathbf{y} \mid \mathbf{y} \in \mathbf{f}\left(\mathcal{X}^P\right)\right\}, \tag{3.17}$$

where $\mathcal{X}^P := conv\left\{\mathbf{x}^{(1)}, ..., \mathbf{x}^{(m)}\right\}$ is the convex hull of the precomputed solutions.

The second approximation goes one step further and substitutes the outcome set by the convex hull of the precomputed objective vectors

$$\mathcal{Y}^P := conv\left\{\mathbf{y}^{(1)}, ..., \mathbf{y}^{(m)}\right\}.$$

The corresponding multiobjective optimisation problem is

$$\text{mo-min}\left\{\mathbf{y} \mid \mathbf{y} \in \mathcal{Y}^P\right\}. \tag{3.18}$$

In both cases the convex hull is spanned by a finite number of given extreme points. Hence, it can be expressed in a constructive way by

$$\mathcal{X}^P = \{\mathbf{X}\,\mathbf{v} \mid \mathbf{v} \in \mathcal{V}_m\} \quad \text{and} \quad \mathcal{Y}^P = \{\mathbf{Y}\,\mathbf{v} \mid \mathbf{v} \in \mathcal{V}_m\}$$

respectively, where

$$\mathcal{V}_m := \left\{\mathbf{v} \in \mathbb{R}^m_+ \mid \langle \mathbf{1}, \mathbf{v}\rangle = 1\right\}$$

is the simplex of convex combination coefficients in m dimensions. Substituting these representations into the problem formulations, we get

$$\text{mo-min}\left\{\mathbf{f}\left(\mathbf{X}\mathbf{v}\right) \mid \mathbf{v} \in \mathcal{V}_m\right\} \tag{3.19}$$

for the problem (3.17) and

$$\text{mo-min}\left\{\mathbf{Y}\mathbf{v} \mid \mathbf{v} \in \mathcal{V}_m\right\} \tag{3.20}$$

for the problem (3.18).

So the first approximation replaces the potentially complicated feasibility domain $\mathcal{X}$ by the simplex $\mathcal{V}_m$ and a linear mapping and the second approximation turns the original problem (2.2) into a linear multiobjective problem with the simplex $\mathcal{V}_m$ as feasible domain. Thereby, the linear model (3.20) can be seen as a linearisation of the partially simplified model (3.19). The relation of the original, the partially simplified and the fully simplified model are depicted in figure 3.6.

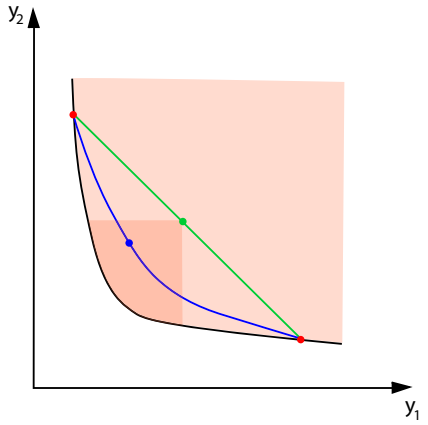


Figure 3.6: The green line is the approximation to the set of efficient points given by problem (3.20) and the blue curve the one given by problem (3.19). For the same convex combination coefficients $\mathbf{v}$, the blue point $\mathbf{f}(\mathbf{Xv})$ is by convexity contained in the darker area $\left(\mathbf{Yv} - \mathbb{Y}_+\right) \cap \mathcal{Y}$ given by the feasible outcomes contained in the negative Pareto cone attached to the green point $\mathbf{Yv}$.

3.6.2 Update of the planning horizon

The simplified structure of the approximate problems allows a simplification of the optimisation problems related to the navigation mechanisms. We will start with the update routines for the lower end of the planning horizon.

The range of solutions present in decision making is now just the set of outcomes for the partially simplified problem. Thus, we try to estimate the ideal and nadir point for this problem. The quality of the estimate with respect to the original problem can be quite bad, if the approximation computed in the first phase only covers some portion of the efficient set. Therefore, the ability of the a posteriori method used in the first phase to cover the set of efficient solutions is an important quality marker.

To determine the minima of the different objective functions K convex minimisation problems have to be solved.

$$\min \left\{ \mathbf{f}_k(\mathbf{Xv}) \;\middle|\; \mathbf{f}(\mathbf{Xv}) \leq \mathbf{b},\ \mathbf{v} \in \mathcal{V}_m \right\} \tag{3.21}$$

yields the minimum value for the k^{th} objective function. If solving the partially simplified problem K times is too time consuming, the fully simplified problem can be used to gain some estimates.

$$\min\left\{(\mathbf{Yv})_k \;\middle|\; \mathbf{Yv} \leq \mathbf{b},\, \mathbf{v} \in \mathcal{V}_m\right\} \tag{3.22}$$

yields an approximation of the minimum value for the k^{th} dimension. The linearisation overestimates the convex functions $\mathbf{f}$. Therefore, the upper bounds are more restrictive and thus the feasible region is at most as large as the one for the partially simplified model. Moreover, linearisation overestimates the value of the objective function, so that the overall estimate is an upper bound. To improve the estimate, $\mathbf{f}_k(\mathbf{Xv}^{(k)})$ can be evaluated for the optimal $\mathbf{v}^{(k)}$ of the fully simplified problem giving a sharper estimate of the minimum.

If an outer approximation of the original problem by supporting hyperplanes is given, we can solve another set of linear programs to get lower bounds on the minima:

$$\min\left\{\mathbf{y}_k \;\middle|\; \mathbf{N}^T\mathbf{y} \geq \mathbf{1},\, \mathbf{y} \leq \mathbf{b}\right\} \tag{3.23}$$

The intersection of the halfspaces $\mathbf{N}^T\mathbf{y} \geq \mathbf{1}$ corresponding to the supporting hyperplanes is a superset of $\mathcal{Y}$. Hence, the resulting objective values underestimate the ideal point's corresponding coordinate.

The linear programs (3.22) and (3.23) are very moderately sized and therefore usually can be solved in some fraction of a second. The linear program implementing (3.22) has m variables – the number of precomputed solutions – and K restrictions. The linear program for problem (3.23) has K variables and $K + m$ restrictions.

The interval given by the objective values for two linear programs (3.22) and (3.23) contains the true value and thus the interval length is a worst-case bound for the linearisation error. Since the same is true if we use $\mathbf{f}_k\big(\mathbf{Xv}^{(k)}\big)$ instead of $\mathbf{Yv}^{(k)}$ and the former is smaller or equal to the latter, evaluating the function improves the error estimate.

If solving K problems of type (3.21) is real-time feasible, then this formulation should be used for the navigation. Otherwise, the linearisation (3.22), combined, if possible, with function evaluation should be used. (3.23) should be used to determine the interval of uncertainty for the estimates, if the tangent information is present. If (3.21) is fast, but not real-time feasible, a hybrid strategy can be employed: While the bound is changed the linearisation is used and when the system is idle again, (3.21) is used to determine the true minima and correct the visualisation.

The determination of the individual minima of the objectives under tradeoff restrictions remains difficult. The exact formulations for the approximate

problems still are bilevel problems, whose application does not seem appropriate in a real-time environment.

Thus, we will work with the simplified version of the heuristic (3.4). So we will either solve the K minimisation problems

$$\min\left\{\langle \mathbf{c}^{(k)}, \mathbf{f}(\mathbf{X}\mathbf{v})\rangle \mid \mathbf{f}(\mathbf{X}\mathbf{v}) \leq \mathbf{b},\ \mathbf{v} \in \mathcal{V}_m\right\} \quad \text{for } k \in \mathcal{K} \tag{3.24}$$

or their linear counterparts

$$\min\left\{\langle \mathbf{c}^{(k)}, \mathbf{Y}\mathbf{v}\rangle \mid \mathbf{Y}\mathbf{v} \leq \mathbf{b},\ \mathbf{v} \in \mathcal{V}_m\right\} \quad \text{for } k \in \mathcal{K}. \tag{3.25}$$

Here, we cannot improve the estimate of the linear program (3.25) by computing $\mathbf{f}_k(\mathbf{X}\mathbf{v}^{(k)})$ unless we are able to check that the resulting outcome fulfils the tradeoff restrictions.

The objective function of the linear program (3.25) should be transformed to

$$\langle \mathbf{c}^{(k)}, \mathbf{Y}\mathbf{v}\rangle = \langle \mathbf{Y}^T\mathbf{c}^{(k)}, \mathbf{v}\rangle.$$

The right-hand side $\mathbf{Y}^T\mathbf{c}^{(k)}$ in the inner product can be precomputed and only needs to be updated, when the bounds on the partial tradeoffs change. The linear program (3.25) then has K constraints and m variables.

The estimates for the upper end of the planning horizon remain a difficult task. We can use the payoff-table heuristic on the results gained during the lower bound computations. Another possibility is the use of the simplex based heuristic for the nadir point approximation presented in [70]. Since it works for linear multiobjective optimisation problems and can be adapted to work with ordering cones (see 3.3.2), we could apply it for the estimate of the upper end of the inner and outer planning horizon.

To make the estimate a lower bound for the true maximum, one can simply evaluate $\mathbf{f}_k(\mathbf{X}\mathbf{v}^{(k)})$ for the optimum $\mathbf{v}^{(k)}$ on the linear approximation. So, depending on the time restrictions we could work with the payoff table estimates or the results of the heuristic. If we choose the payoff tables, we can still apply the heuristic for corrections when the system is idle.

3.6.3 Selection and discounting mechanism

Making the selection mechanism faster is maybe the most important of the described speedups. Changing the current solution is very likely the mechanism that is most often used. Besides, being able to (approximately) execute the selection in real-time allows the decision maker to directly react to the side-effects of his or her changes. The ability to modify the current solution in real-time raises the appeal to move away from standard solutions and to really explore the variety of solutions multiobjective optimisation offers.

The reformulation of the selection problem (3.8) is straight forward:

$$\min\left\{s\big(\mathbf{c}^+,\mathbf{c}^-\big) \;\middle|\; \begin{array}{l}\mathbf{f}(\mathbf{X}\mathbf{v})-\mathbf{y}^R=\mathbf{c}^+-\mathbf{c}^-,\ \mathbf{f}_{k'}(\mathbf{X}\mathbf{v})=\tau,\\ \mathbf{f}(\mathbf{X}\mathbf{v})\le\mathbf{b},\ \ \mathbf{v}\in\mathcal{V}_m,\ \ \mathbf{c}^+,\mathbf{c}^-\in\mathcal{C}\end{array}\right\}. \tag{3.26}$$

This problem is somewhat simpler compared to the original problem (3.8), but as a convex minimisation problem it will most likely not be fast enough for a real-time execution. Thus, we reformulate the selection mechanism problem using the fully simplified problem.

$$\min\left\{s\big(\mathbf{c}^+,\mathbf{c}^-\big) \;\middle|\; \begin{array}{l}\mathbf{Y}\mathbf{v}-\mathbf{y}^R=\mathbf{c}^+-\mathbf{c}^-,\ (\mathbf{Y}\mathbf{v})_{k'}=\tau,\\ \mathbf{Y}\mathbf{v}\le\mathbf{b},\ \ \mathbf{v}\in\mathcal{V}_m,\ \ \mathbf{c}^+,\mathbf{c}^-\in\mathcal{C}\end{array}\right\}. \tag{3.27}$$

The equality constraint is certainly much easier to implement in the linearised case, as it restricts the *variables* $\mathbf{v}$ to an affine linear subspace. So, one can enforce it by only allowing changes, for which the variables stay in this subspace. But the equality constraint is only an approximate. If we reevaluate the optimal convex combination $\hat{\mathbf{v}}$ with the original convex functions $\mathbf{f}(\mathbf{X}\hat{\mathbf{v}})$, the convexity of $\mathbf{f}_{k'}$ ensures that τ is an upper bound for the convex function value. In general it will be smaller than τ. Nevertheless, the deviation is bounded by the distance of the approximation to the original problem, so that for a good approximation the deviation is small.

The simplified problems can be incorporated into the solution modification problem (3.16) for the discounting in a very similar way. Evaluating the optimal convex combination in case of the fully simplified problem again improves the approximation. Since a change of the bounds on the pairwise prices is very likely not used as often as the selection mechanism, we consider it to be less critical in terms of time. Hence, using the partially simplified problem to adapt the solution to changing price bounds can be considered.

In both cases one could employ the fastest implementation while a change is induced by the decision maker and use the more complex problem formulation to correct the outcomes of the coarser approximation, when the system is idle.

The convex combinations of precomputed solutions are in general suboptimal even for a good approximation – a slightly unsatisfactory situation. Fortunately, we can estimate the degree to which the current solution is suboptimal. There are different ways to estimate the distance, but all approaches exploit the fact that we have an inner and outer approximation to the efficient set.

So one simply has to choose a sensible direction and do a line search from the current solution and compute the intersection point with the outer approximation. The distance between the current solution and the point on the

outer approximation gives an upper bound for the distance to the efficient set. To compute the intersection point one just has to solve

$$\max\left\{\alpha \mid \mathbf{N}^T\left(\mathbf{y}^S + \alpha\mathbf{q}\right) \geq \mathbf{1}\right\},$$

for the search direction $\mathbf{q}$. The distance of the current solution from the set of efficient outcomes is then bounded by $\alpha \parallel \mathbf{q} \parallel_2$.

The direction for this search process can be chosen in many different ways. We propose four different heuristics to choose it here:

1. Choose a fixed $\mathbf{q} \in -int(\mathcal{C}^*)$.
2. Take $-\mathbf{N}\mathbf{v}$.
3. Choose $\mathbf{y}^S - \mathbf{b}$.
4. Take $\mathbf{y}^I(\mathbf{b}) - \mathbf{y}^S$.

The four different ways of choosing the search direction correspond to heuristics used to build up approximations of the set of efficient points ([17], [28], [91]), where the second choice tries to approximate the normal vector of the chosen point.

If the distance is large in an area that is of interest to the decision maker, (s)he could decide to invest some time into improving the approximation. Then the line search would be executed on the real problem yielding another efficient point. This point can be added to the simplified problems and greatly improves the local approximation. This process can be seen as a column generation for the simplified problems. After the point is added, the planning horizon is updated and the newly computed point is set to be the current solution.

The decision making process could thus start with a relatively coarse approximation and be refined in regions of interest to the decision maker. This strikes a balance between executing Pareto navigation on the original problem and just working with the approximation. Again it is a tradeoff: if we choose to work with a coarse approximation, phase I will be shorter, but we need additional time in phase II.

The possibility of locally refining the approximation influences the perception of the quality measures for the approximation. Coverage now is the most important, since we cannot correct for lack of it later on, whereas too large distances can be compensated. Still, an approximation with small distances is an advantage, since the simplified problems then convey the problem characteristics correctly to the decision maker.

The simplified problem description can be used to speedup the selection and parts of the discounting mechanism through the simpler description of the

feasible domain and the simpler structure of the constraints. But to really profit from the simplification we have to choose a cone scalarising function that can be implemented by a linear program. One such function is the Pascoletti-Serafini scalarisation approach.

3.6.4 An implementation with Pascoletti-Serafini

We have shown in section 2.7 that the Pascoletti-Serafini scalarisation approach can be transformed into the cone scalarising function (2.17),

$$s\big(\mathbf{c}^+, \mathbf{c}^-\big) := \min\left\{z \in \mathbb{R} \mid \mathbf{c}^+ - \mathbf{c}^- + \mathbf{c} = z\,\mathbf{1},\ \mathbf{c} \in \mathcal{C}\right\}.$$

If we insert this function into the selection mechanism problem (3.8)

$$\min\left\{s\big(\mathbf{c}^+, \mathbf{c}^-\big) \;\middle|\; \mathbf{y} - \mathbf{y}^R = \mathbf{c}^+ - \mathbf{c}^-,\ \mathbf{y}_{k'} = \tau,\ \mathbf{y} \leq \mathbf{b},\ \mathbf{y} \in \mathcal{Y},\ \mathbf{c}^+, \mathbf{c}^- \in \mathcal{C}\right\},$$

we get the following linear optimisation problem:

$$\begin{aligned} & \min\big\{\min\{z \in \mathbb{R} \mid \mathbf{c}^+ - \mathbf{c}^- + \mathbf{c} = z\,\mathbf{1},\ \mathbf{c} \in \mathcal{C}\} \mid \\ & \qquad \mathbf{y} - \mathbf{y}^R(\tau) = \mathbf{c}^+ - \mathbf{c}^-,\ \mathbf{y}_{k'} = \tau,\ \mathbf{y} \leq \mathbf{b},\ \mathbf{y} \in \mathcal{Y}^P,\ \mathbf{c}^+, \mathbf{c}^- \in \mathcal{C}\big\}, \\ = \quad & \min\big\{z \in \mathbb{R} \;\big|\; \mathbf{Y}\mathbf{v} - \mathbf{y}^R(\tau) + \mathbf{c} = z\,\mathbf{1},\ (\mathbf{Y}\mathbf{v})_{k'} = \tau,\ \mathbf{Y}\mathbf{v} \leq \mathbf{b}, \\ & \qquad \mathbf{v} \in \mathcal{V}_m,\ \mathbf{c} \in \mathcal{C}\big\} \end{aligned} \tag{3.28}$$

The problem for the discounting mechanism is very similar to (3.28), but with $\mathbf{y}^S$ instead of $\mathbf{y}^R(\tau)$ and without the equality constraint $(\mathbf{Y}\mathbf{v})_{k'} = \tau$. Because of their similarity we will only describe the implementation of the selection mechanism in detail.

The first step is to introduce the constructive definition of the ordering cone and to replace the multiple occurrences of the matrix vector multiplication $\mathbf{Y}\mathbf{v}$ by a set of variables to make the matrix of the linear program sparser.

$$\begin{aligned} \min\big\{\ z \ \big|\ & \mathbf{y} = \mathbf{Y}\mathbf{v},\ \mathbf{y}_{k'} = \tau,\ \mathbf{y} \leq \mathbf{b},\ \mathbf{1}^T\mathbf{v} = 1, \\ & \mathbf{y}_k - \mathbf{y}_k^R(\tau) + (\mathbf{C}\mathbf{w})_k = z,\ k \in \mathcal{K} \setminus \{k'\} \\ & \mathbf{v} \geq \mathbf{0},\ \mathbf{w} \geq \mathbf{0},\ z \in \mathbb{R},\ \mathbf{y} \in \mathbb{Y}\big\} \end{aligned} \tag{3.29}$$

From the standpoint of implementation it is beneficial to use *one* linear program that can be adapted to realise the different selection mechanisms instead of setting up one for each request. Thus, we modify the linear program (3.29) so that changing τ as well as changing the chosen objective can be realised by a change of the right-hand side. The $-\infty$ constraints express that the corresponding variables are unbounded from below.

$$\begin{aligned} \min\big\{\ z \ \big|\ & \mathbf{Y}\mathbf{v} - \Delta\mathbf{y} = \mathbf{y}_k^R(\tau),\ \Delta\mathbf{y} + (\mathbf{C}\mathbf{w})_k = \mathbf{1}\,z,\ \mathbf{1}^T\mathbf{v} = 1, \\ & \Delta\mathbf{y} \leq \sum_{k \neq k'} \big(\mathbf{b}_k - \mathbf{y}_k^R(\tau)\big)\,\mathbf{e}_k,\ \Delta\mathbf{y} \geq -\infty\,(\mathbf{1} - \mathbf{e}_{k'}) \\ & \mathbf{v} \geq \mathbf{0},\ \mathbf{w} \geq \mathbf{0},\ z \in \mathbb{R},\ \mathbf{y} \in \mathbb{Y}\big\} \end{aligned} \tag{3.30}$$

Here, the difference $\Delta\mathbf{y}$ of the convex combination of the precomputed solutions $\mathbf{Yv}$ to the reference point $\mathbf{y}_k^R(\tau)$ is the main set of variables. The first part of second line of the linear program represents the $\mathbf{y} \leq \mathbf{b}$ constraint, but the first equality of the first line was used to substitute $\mathbf{y}$ by $\Delta\mathbf{y}$. Additionally, the constraint $\mathbf{y}_{k'} \leq \tau$ was incorporated into the inequalities by $\Delta\mathbf{y}_{k'} \leq 0$, which substitutes the weaker $\mathbf{y}_{k'} \leq \mathbf{b}_{k'}$. The second set of inequalities in the second line is trivial except for the k'-th component, which ensures that $\Delta\mathbf{y}_{k'} \geq 0$.

Thus, the change of τ as well as the change in the chosen objective function can be implemented by changing just the right hand side of the linear program. In particular, the solution of several problems (3.30) for the same starting point but different values of τ becomes a parametric linear programming problem, which we solve with the simplex method ([25]).

If we view the Pascoletti-Serafini approach as a line search from the reference point along a pre-specified direction, we see that the optimal solution is on the boundary of $\mathcal{Y}^P$. Since $\mathcal{Y}^P$ is the convex hull of the precomputed solutions, its boundary is given by convex hulls of subsets of the points. If no $K+1$ points lie on a common hyperplane, the boundary is composed of simplices. The linear program (3.30) chooses the simplex with the optimal point automatically, which can be seen as an implicit triangulation of the surface.

If τ changes slightly and the new optimum lies on the same boundary simplex, the simplex method does not even need to pivot, but can directly compute the new point from the new right-hand side. The inverse basis matrix is continuous as a linear mapping and therefore the change in the point is continuous until the next basis change. Then we topple over to the next boundary simplex and so on. Thus, if the change in τ is not too large, the linear program for the previous τ value or selection step can be used to almost instantly get a response.

When employing the simplex method to solve the linear program, at most $K+1$ components of the $\mathbf{v}$ vector are nonzero. This is due to a theorem due to Caratheodory (see [95]) and the property of the simplex method to use minimal representations. Thus, independent from the number of precomputed solutions the number of convex combination partners only depends on the number of objectives.

Changing the restrictions to the partial tradeoffs cannot be incorporated as easily. Here, a part of the constraint matrix of the linear program changes. This is uncritical, if the corresponding variable is not contained in the basis matrix, but otherwise requires some updating of the solution. This corresponds to the situation that the changed bound played a role in determining the optimal solution. Newly introduced bounds are uncritical though, since

their introduction can be viewed as a new column being added to the system. Thus, the previously optimal solution stays at least feasible for the dual problem.

3.7 Pareto navigation for nonconvex problems

So far we only considered the case of convex multiobjective optimisation problems. If the nonconvexity can be remedied by strictly monotone coordinate transformations say, the method could work on the transformed outcome set and the results could be transformed back to the original outcome set. Ideally, the transformation would be applied twice, so that the input of the decision maker is given for the original outcome set, is then transformed and applied on the transformed set and the transformed back to the original set. In essence this is just a nonlinear deformation of scales.

The more interesting case is the case where even the set of weakly efficient outcomes is no longer connected. Here, a selection step that moves into one of these 'holes' is bound to give a result that could be improved in *all* objectives. Therefore, such a move should be avoided.

By slightly reformulating the selection problem (3.8)

$$\min\left\{s\big(\mathbf{c}^+,\mathbf{c}^-\big) \;\middle|\; \begin{array}{l}\mathbf{y}-\mathbf{y}^R=\mathbf{c}^+-\mathbf{c}^-,\ \mathbf{y}_{k'}+\mathbf{c}_{k'}=\tau,\\ \mathbf{y}\le\mathbf{b},\ \mathbf{y}\in\mathcal{Y},\ \mathbf{c}^+,\mathbf{c}^-,\mathbf{c}\in\mathcal{C}\end{array}\right\}, \tag{3.31}$$

we can avoid the strictly dominated parts at the price of a reduced control on the manipulated objective. The only change compared to the original problem is the changed left-hand side of the equality constraint. Due to the change we are no longer searching for the point of best distance to the reference point in the intersection of the outcome set $\mathcal{Y}(\mathbf{b}) \cap \mathcal{H}_{k'}(\tau)$ with a hyperplane. Instead we search on the intersection of the hyperplane $(\mathcal{Y}(\mathbf{b}) + \mathcal{C}) \cap \mathcal{H}_{k'}(\tau)$ with the Minkowski sum of the outcome set and the ordering cone.

The weakly $\mathcal{C}$-efficient solutions of this set are again connected and the $\mathcal{Y}(\mathbf{b})$ portion of the Minkowski sum is weakly $\mathcal{C}$-efficient.

Theorem 3.7.1 *Let the cone scalarising function s be $\mathcal{C}$-centred, strictly $\mathcal{C}$-consistent and continuous. Then optimal results to the problem (3.31) are weakly $\mathcal{C}$-efficient.*

Proof:

Let $\hat{\mathbf{y}}$ be optimal for the problem (3.31) and a given reference point $\mathbf{y}^R$. Furthermore, let $\hat{\mathbf{c}}^+, \hat{\mathbf{c}}^- \in \mathcal{C}$ be an s-optimal representation of $\hat{\mathbf{y}}$. Now assume that $\hat{\mathbf{y}}$ is not weakly $\mathcal{C}$-efficient. Then the interior of the negative ordering cone

$$\big(\hat{\mathbf{y}} - int(\mathcal{C})\big) \cap \mathcal{Y}(\mathbf{b}) \neq \emptyset$$

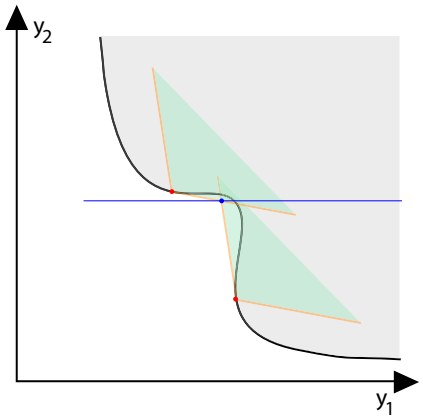


Figure 3.7: Due to the redefinition of the equality constraint we are able to bridge potential gaps in nonconvex problems.

has nontrivial intersection with the set of outcomes. Let $\tilde{\mathbf{y}} + \tilde{\mathbf{c}} = \hat{\mathbf{y}}$ *be a representation of the optimum with a feasible outcome* $\tilde{\mathbf{y}} \in \mathcal{Y}(\mathbf{b})$ *and an element of the ordering cone's interior* $\tilde{\mathbf{c}} \in int(\mathcal{C})$.

But since $\tilde{\mathbf{y}}$ *fulfils the constraints*

$$\tilde{\mathbf{y}}_{k'} + \tilde{\mathbf{c}}_{k'} + \mathbf{c}_{k'} = \tau \quad \text{and} \quad \tilde{\mathbf{y}} - \mathbf{y}^R = \hat{\mathbf{c}}^+ - \underbrace{(\hat{\mathbf{c}}^- + \tilde{\mathbf{c}})}_{\in \mathcal{C}},$$

it is feasible for problem (3.31).

Now the minimality of $\hat{\mathbf{y}}$ *implies the following inequalities:*

$$\begin{aligned} s(\hat{\mathbf{c}}^+, \hat{\mathbf{c}}^-) &= \min \left\{ s(\mathbf{c}^+, \mathbf{c}^-) \;\middle|\; \mathbf{y} - \mathbf{y}^R = \mathbf{c}^+ - \mathbf{c}^-,\ \mathbf{y}_{k'} + \mathbf{c}_{k'} = \tau, \right. \\ &\qquad \left. \mathbf{y} \in \mathcal{Y},\ \mathbf{c}^+, \mathbf{c}^- \in \mathcal{C} \right\} \\ &\leq s(\hat{\mathbf{c}}^+, \hat{\mathbf{c}}^- + \tilde{\mathbf{c}}) \\ &\underset{\tilde{\mathbf{c}} \in int(\mathcal{C})}{<} s(\hat{\mathbf{c}}^+, \hat{\mathbf{c}}^- + \tilde{\mathbf{c}} - \tilde{\mathbf{c}}) \\ &= s(\hat{\mathbf{c}}^+, \hat{\mathbf{c}}^-). \end{aligned}$$

This contradicts the assumed minimality. Hence, the interior of the negative ordering cone

$$\left(\hat{\mathbf{y}} - int(\mathcal{C})\right) \cap \mathcal{Y}(\mathbf{b}) = \emptyset$$

has empty intersection with the set of outcomes and therefore $\hat{\mathbf{y}}$ *is weakly* $\mathcal{C}$*-efficient.*

□

A further nice property of the resulting weakly $\mathcal{C}$-efficient solution is that it fulfils the price bounds by the construction of the ordering cone.

Modifications to problem (3.31) analogous to the ones in lemma 2.4.8 can be used to yield even $\mathcal{C}$-efficient outcomes. Mind though that the equality constraint necessary in the lexicographic optimisation can be troublesome for nonlinear problems.

The result of theorem (3.7.1) is further weakened by the fact that it is in general unclear how to obtain a *global* optimum of the cone scalarising function over the intersection $\mathcal{Y}(\mathbf{b}) + \mathcal{C} \cap \mathcal{H}_{k'}(\tau)$ for a given reference point $\mathbf{y}^R$. This crucially depends on the structure of the underlying scalar problems and thus on the structure of the individual multiobjective optimisation problem at hand.

The negative side effect of the problem modification is the loss in control on the value of the manipulated objective. Looking at figure 3.7 we see, that the $\mathcal{Y}(\mathbf{b})$ portion of the Minkowski sum constantly remains the same until the other connectivity component of the weakly $\mathcal{C}$-efficient set can be reached. Then a jump to the new connectivity component takes place.

One could in principle use the problem (3.31) for the convex case as well, but one would give up the ultimate control on the value of the manipulated objective, which is one of the key features of Pareto navigation.

With the above changes the selection and discounting mechanisms can be applied to nonconvex cases. The restriction mechanism can directly be transferred as well. By contrast, the updates of the planning horizon, which were already difficult to implement for the convex case cannot easily be implemented for the nonconvex case. Here, the intervals could consist of several disconnected parts, where these parts can consist of single points or be open. Thus, in the nonconvex case the planning horizon cannot be determined with the methods at hand.

3.8 Summary

In this chapter we have introduced a new interactive multiobjective optimisation method named "Pareto navigation". We have argued why we see a need for a new method and posited our main goals for the new method: control, simplicity and overview. We have then presented the three mechanisms that are at the decision maker's command – restriction, selection and discounting – and the complementary information provided to the decision maker – the planning horizon – from a user's perspective.

After that we started the in depth analysis of the mathematical realisation of the mechanisms with the restriction mechanism and the related updates of the planning horizon. We have seen that the incorporation of the bounds on the objectives set or changed by the restriction mechanism into the model is easy, but estimating their consequences is difficult. In particular, the upper end of the inner and the outer planning horizon – the estimate for the

nadir point considering respectively not considering the ordering cone – can only be exactly determined by global optimisation techniques. The same holds true for the lower end of the inner planning horizon – the estimate for the ideal point considering the ordering cone. Therefore, we proposed some heuristics to gain approximate values that are used instead of the intractable exact ones.

Then, we presented the selection mechanism arguably the most important of the three mechanisms Pareto navigation offers. We have presented the mathematical model for the selection mechanism and a critical example used to demonstrate potential problems. We then examined the set of potential results and specified an algorithm to reach every efficient point among that set. The set of results was then shown to be connected, if the objectives are allowed to change in the inner planning horizon only. For objectives that range of the full planning horizon the set of optima for an individual selection step was shown to be upper semicontinuous. Together with the uniqueness results proved in section 2.5 of chapter 2 the results of the selection mechanism are continuous, if the corresponding conditions on the set of outcomes or the distance evaluation function are met.

The last mechanism to be presented was the discounting. Here, we derived the constraints on the dual ordering cone from the restrictions on the partial tradeoffs. It was then proved that the restrictions on the dual ordering cone imply the restrictions on the partial tradeoffs. The next step was to derive a representation of the ordering cone from the representation of the dual ordering cone. Then, we defined the notion of compatible and strictly compatible tradeoff restrictions and demonstrated how the compatibility can be checked with some simple network flow problems. The discussion was completed by an examination of border cases of compatibility.

We then looked at ways to approximate the original problem formulation by simpler models employing precomputed solutions as a basis for this reformulation. The loss of accuracy and methods to keep it small and estimate it were discussed. In particular, the update of the planning horizon and the selection and discounting mechanism were considered. Finally, the implementation of the selection mechanism for the Pascoletti-Serafini scalarisation by linear programs and different formulations of it are presented.

The chapter finishes with an examination of an adapted Pareto navigation for nonconvex outcome sets. Here, we showed that dropping the equality constraint in the selection mechanism, which leads to a loss in control, enables the method to cope with the nonconvex situation. Furthermore, it is proved that the resulting outcomes are weakly efficient with respect to the given ordering cone. Thus, the discounting mechanism can be smoothly applied to that case, but we saw that the planning horizon update cannot directly transferred to that situation, so that the posited goal ‘overview’ is not achieved in this more general situation.

Chapter 4

Multiobjective IMRT Planning

In this chapter we will present the application of Pareto navigation to intensity modulated radiotherapy planning. Section 4.1 introduces intensity modulated radiotherapy (IMRT) an important cancer treatment technique. In section 4.2 the inherent multiobjective nature of the IMRT planning problem is identified. Then typical approaches to modelling the IMRT planning problem are introduced and feasible ones are identified in section 4.3.

Section 4.4 addresses the problem of computing approximations to the efficient set, which is nontrivial for the number of dimensions under consideration. We will discuss why known deterministic methods of approximation can only be used up to a certain dimension of the outcome space and propose the heuristic of the extreme compromises to treat the cases where the dimension is too large.

Then, the current prototype implementing Pareto navigation is introduced in section 4.5. Here, we will see that the user interface allows for very intuitive means of manipulating the current solution and the feasible ranges.

The chapter ends with the summary in section 4.6, where we highlight the most important points in the chapter and assess the impact of Pareto navigation on IMRT planning.

4.1 Intensity-modulated radiotherapy

Radiotherapy is, besides surgery and chemotherapy, the most important treatment option in clinical oncology. It is used with both curative and palliative intention, either solely or in combination with surgery and chemotherapy. The vast majority of all radiotherapy patients is treated with high energetic photon beams. Hereby, the radiation is produced by a linear ac-

celerator and delivered to the patient by several beams coming from different directions (see figure 4.1).

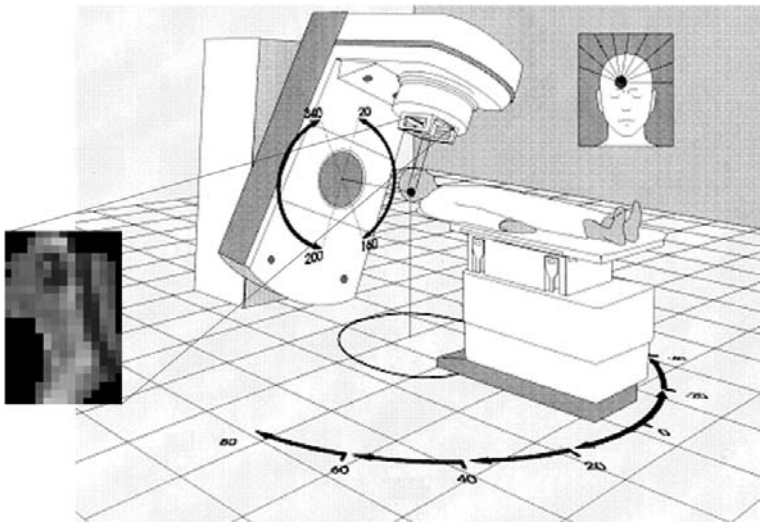


Figure 4.1: The gantry moves around the couch on which the patient lies (picture from [73]).

The goal in radiotherapy is to destroy the tumour without side effects in the surrounding critical organs or the normal tissue. Unfortunately, it is physically impossible to totally spare the surrounding body, since the photon beam deposits parts of its energy in front of and behind the tumour. To have a high enough dose in the tumour one therefore applies beams from different directions, so that the dose concentrates in the tumour and is distributed among the surrounding critical structures and normal tissue.

Using multi-leaf collimators (see figure 4.2), the intensity of a beam is modulated. This happens by uncovering specific parts of the beam and blocking the remainder of the beam opening by the collimator leafs. This setting is applied for a predetermined time. For every beam several such apertures are superimposed and add up to a two-dimensional step function for each beam.

The levels of this step function called *intensities* offer many degrees of freedom, so that many different dose distributions in the body can be achieved. Thus, there are a lot of options for designing an IMRT treatment plan for a specific tumour.

An IMRT treatment plan is fully determined by the directions from which the body is irradiated and the number of these directions along with the intensities for each direction. Although the beam geometry – the set of

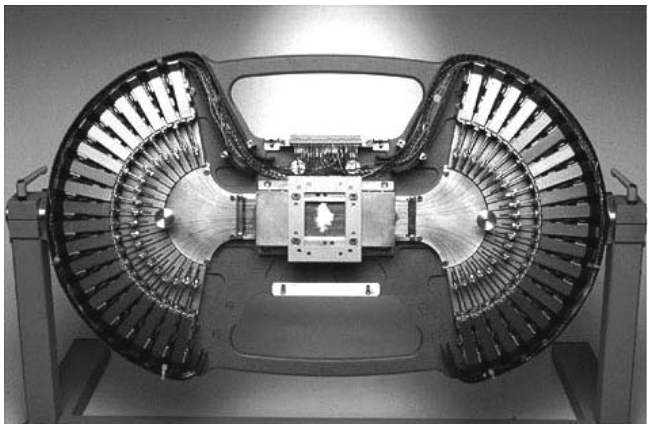


Figure 4.2: A multi-leaf collimator. The square opening in the centre of the machine is partially covered by leafs, each of which can be individually moved (picture from [107]).

chosen directions – plays an important role in the planning, it lies outside the scope of this work and we consider the number of the beams and their position as given.

The reason for this is that finding good beam geometries is a hard global optimisation problem, that has not satisfactorily been solved so far ([12]) even though there have been some attempts (see [82, 92, 93] and the references therein). Solving the full planning problem usually results in methods that are either heuristic in nature ([14, 47, 48, 93, 110]) or have potentially very long running times ([18, 77]). Practically, the beam geometry is chosen based on the experience of the planner and used as fixed input to the optimisation routines.

Thus, for our purposes the *treatment planning problem* is the detection of optimal intensities for a given number of beams irradiating from given directions. Besides, the large number of degrees of freedom allows to compensate to some extend for the beam geometry. Hence, the results will be quite similar, if the (suboptimal) choice of the beam geometry is reasonable.

4.2 The multiobjective character of IMRT treatment planning

The general goal of destroying the tumour without impairing the remaining body splits into several goals at closer inspection. To be able to destroy the tumour, first of all the dose inflicted on the tumour needs to be high enough. Moreover, there is the requirement that the dose peaks in the tumour should

not be too high. Thus, we would like to have a high and preferably uniform or *homogeneous* dose in the tumour.

Opposed to this, we would like to have low doses in the surrounding body tissue. In particular the organs lying close by should preferably not be impaired by the treatment. In case that this general goal cannot be met, the planners usually search for a good compromise of distributing the inevitable dose *among* the organs. Hence, in this case the sparing of each organ represents an individual goal.

Moreover, the tumour volume is usually split into several nested volumes, so that the region that definitely contains tumour volume, can be distinguished from the part that is likely to contain microscopic tumour. This is in turn contained in a larger volume that tries to a priori compensate for potential movement.

The multiobjective problem for the treatment planning problem naturally arises from these goals. So, a typical setting involves one or two objectives for a set of 2-3 nested target volumes and one objective for each organ – at least for the ones close to the tumour.

In a prostate case the number of nested target volumes is typically three and at least the bladder and the rectum that directly abut the tumour have to be considered. If we use constraints to ensure a curative dose in the nested target volumes and introduce an objective for the homogeneity for each of the target volumes we are left with five objectives. Additionally, the femoral heads and the unclassified normal tissue could contribute another objective each.

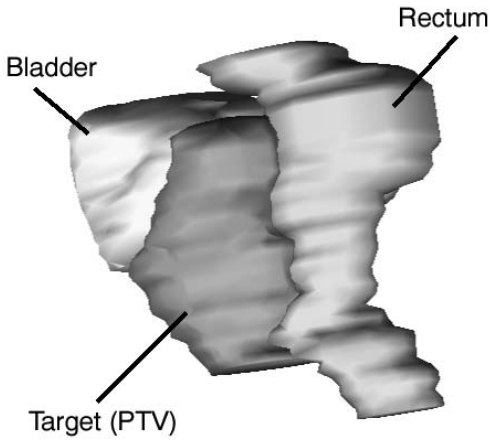


Figure 4.3: Exemplary prostate case where the target volume is situated between two critical structures (picture from [73]).

The prostate case nicely demonstrates that the objectives for the organs at risk can be opponents and cannot, therefore, be combined a priori. The rectum and the bladder abut the prostate at different ends (see figure 4.3). To achieve a high and homogeneous tumour dose the dose has to build up somewhat outside the tumour. Usually, the build up can be moved more towards the rectum or more towards the bladder, but the two organs cannot be spared simultaneously without impairing the quality of the dose distribution in the target volumes. Similar situations arise for other tumour sites.

Moreover, the patients differ in age, fitness and might already suffer from impairment of one of the organs under consideration. So, the treatment does not only depend on the tumour site, but is to differing degrees individually designed for the patient. Therefore, there is no prefixed model of how to distribute dose effects between organs at risk. The problem thus is *inherently* a multiobjective optimisation problem and should hence be treated as such.

The multiobjective character of the problem is present in almost all planning approaches as the treatment goals for the target volumes and the different critical structures are usually specified separately. However, it is currently only implicitly present in the optimisation. In recent years there have been some efforts to address the problem as multiobjective problem, though.

Yu ([128]) proposed a scheme for a systematic parameter choice based on multiobjective decision theory to substitute the trial and error process used for finding suitable parameters. Haas et al. ([47, 48]) applied multiobjective genetic algorithms to simultaneously find the beam geometry and the intensities.

The activities then shifted to deterministic approaches for the approximation of the efficient set. Linear models ([13, 49, 59, 72, 74]) as well as nonlinear ones ([23, 74, 76]) were under consideration. The two last mentioned works also included the splitting into an approximation phase and a decision making phase. However, in both articles the decision making phase only considered the precomputed solutions as discrete points in the objective space. So, no convex combination of plans was considered.

4.3 Modelling the IMRT planning problem

So far there is no commonly accepted model for the exact biological impact the irradiated dose has on the different organs and the tumour. Thus, there are many different models to capture the quality of the dose distribution in one (or a few) number(s) for each volume. Typically, the individual planner uses the model (s)he thinks appropriate for the setting encountered in the treatment or the model (s)he is most experienced in.

Many models are based on the *dose volume histogram* (DVH). It depicts the volume percentage that receives at least a certain dose as a function of the dose (see figure 4.4) for each of the considered organs and the different target volumes. Thus, there is a curve for each considered volume, which represents the dose distribution in that volume devoid of its geometrical information. So, the planner can directly read off the portion of an organ that is currently overdosed or the portion of the tumour that is underdosed.

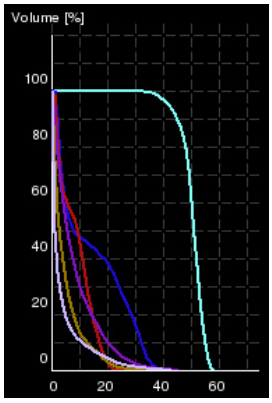


Figure 4.4: Exemplary DVH curve with a light blue tumour curve and curves for several organs at risk.

A popular choice for specifying treatment goals are so-called DVH constraints. A DVH constraint enforces one of the curves to pass above or below a specified dose volume point. So, either the percentage of volume that receives less than the specified dose or the volume that receives more than a specified dose is restricted for the chosen volume. DVH constraints are widely used, in particular some clinical protocols are formulated using DVH constraints.

Unfortunately, incorporating DVH constraints into the optimisation results in a nonconvex feasible region and thus a global optimisation problem. Hence, given a local optimum of the problem there is no guarantee for global optimality. Therefore, either an enormous computational effort has to be spent for finding all local optima or a suboptimal solution, whose deficiency in quality compared to the global optimum is unknown, has to be accepted.

Therefore, convex evaluation functions of the dose distribution in a volume have been devised that try to control the DVH. We will have a closer look at them, since they are the functions we use for the modelling.

For a numerical treatment of the planning problem, the relevant part of the patient's body is partitioned into small cubes called *voxels* $\mathcal{V}_j$. The dose dis-

tribution can then be expressed as a vector of values $\mathbf{d} := \big(d(\mathcal{V}_1), ..., d(\mathcal{V}_N)\big)$ with one dose value per voxel.

Using this notation, one such evaluation function is

$$\mathbf{f}_k(\mathbf{d}) := \sum_{\mathcal{V}_j \subseteq \mathcal{R}_k} (\max\{d(\mathcal{V}_j) - U_k, 0\})^q, \quad q \in [1, \infty), \tag{4.1}$$

where $\mathcal{R}_k$ is some organ at risk. This function penalises the parts of the volume where the dose distribution exceeds a specified threshold U_k. In terms of the DVH-curve this function penalises nonzero volume-values beyond the threshold of U_k.

In [96] Romeijn et al. propose a different type of dose-volume constraint approximation, that yields a piece-wise linear convex model analogous to the well-known conditional Value-at-Risk measure in financial engineering.

Another approach to quantify the quality of a dose distribution in a volume considers the biological impact. The biological impact is assessed using statistical data on the tumour control probability (TCP) and the normal tissue complication probability (NTCP) [119, Chapter 5]. These statistics are gained from experiences with thousands of treated patients, see e.g. [39].

The concept of equivalent uniform dose (EUD) was first introduced by Brahme in [15]. The EUD is the uniform dose that is supposed to have the same biological impact in a volume than a given non-uniform dose distribution and depends on the type of the volume.

The most well-known is Niemierko's EUD concept [88] that uses the L_a-norm to compute the EUD:

$$\mathbf{f}_k(\mathbf{d}) = \Big(\frac{1}{\#\{\mathcal{V}_j \subseteq \mathcal{R}_k\}} \cdot \sum_{\mathcal{V}_j \subseteq \mathcal{R}_k} d(\mathcal{V}_j)^a\Big)^{\frac{1}{a}}, \quad a \in (-\infty, -1) \cup (1, \infty). \tag{4.2}$$

Figure 4.5 illustrates EUD evaluations of a given DVH for two different a-parameters. The dotted and the dashed lines are EUD measures with a about 1 and a close to ∞, respectively. Organs that work in *parallel*, i.e. organs such as lungs or kidneys that are viable even after a part of them is impaired, are evaluated with an a close to 1, whereas *serial* organs, i.e. structures that depend on working as an entity like the spinal cord, are evaluated with high a values.

Romeijn et al. [98] show that for multiobjective optimisation many common evaluation functions can be expressed by convex functions yielding the same set of efficient preimages.

In our prototype we mainly employ the EUD concept of Niemierko and derivatives of it for the organs at risk. We use tail penalties or an EUD derivative as functions for the tumour volume. Here, the function that cares

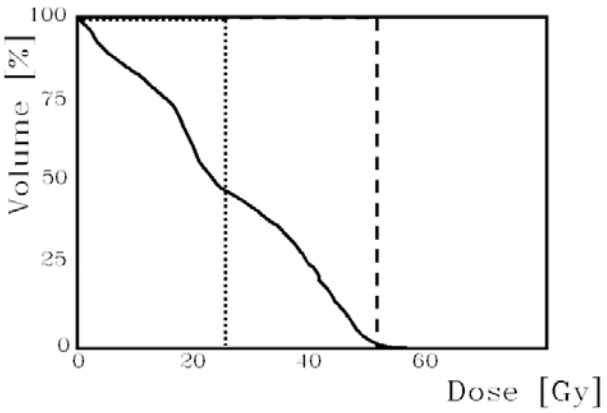


Figure 4.5: Exemplary DVH curve with the resulting EUDs for a parallel and a serial organ. (picture from [73])

for a high enough tumour dose can just be bounded and thus contributes as a convex constraint or be used as a 'real' objective. Alternatively, the mean value of the tumour can be fixed and the standard deviation is used as an objective.

We always have some upper bounds for the different functions, since in the clinical setting arbitrary values for *any* of the considered volumes are infeasible. The outcome set is thus compact, because natural lower bounds for all the objectives exist. For the organs this is the no-dose-plan and for the tumour it is the perfectly homogenous plan that exactly realises the prescribed dose. Moreover, the model is also convex due to the chosen functions.

Even the scalarised IMRT planning problem is a large-scale problem, so that the response times for any Pareto navigation action is to long for true interactivity. Therefore, we will use the two-phase approach to Pareto navigation. In the first phase the efficient set is approximated by a certain number of efficient points and in the second phase the interactive real-time decision making takes place.

4.4 Computing the approximation – phase I

A specific feature of the multiobjective setting in radiotherapy planning is its normally high dimension – usually greater than or equal to five. Therefore, computing the approximation to the efficient set in the first phase of our two phase approach is nontrivial.

Hence, we will review the different methods to approximate the efficient set listed in [99] and discuss their applicability to the IMRT planning problem.

We will see that even the deterministic approaches that work best for our case cannot currently cope with the high number of dimension in a satisfactory way. So from a certain number of dimension on alternatives are needed. We propose such an alternative – the heuristic of the *extreme compromises*.

4.4.1 Known approximation approaches

In our discussion of the different approximation methods we will follow the classification developed in [99] that distinguishes point-based approximations of $eff(\mathcal{Y})$ and $eff(\mathcal{X})$ and outer, inner and sandwich approximations to $eff(\mathcal{Y})$.

The methods for point-based approximations of $eff(\mathcal{Y})$ usually fall into one of the following categories:

1. They use fixed grids for the scalarisation parameters [17, 94, 113],
2. derive relations between approximation quality and distance of scalarisation parameters for arbitrary grids [1, 87] or
3. try to create an fine grid directly on the efficient set [42, 103].

The method of Dellnitz et al. ([32, 108]) does not approximate the efficient set with points but cubes. Still, for our purposes it falls into the third category.

The methods in (1) cover the scalarisation parameter set whose dimension is at least $|\mathcal{K}| - 1$ with regular grids of maximum distance ε. The methods in (2) and (3) in turn cover the efficient set – which is in general a $|\mathcal{K}| - 1$ dimensional manifold – with grids of maximum distance ε. In any case, the number of points needed is of order

$$O\left((1/\varepsilon)^{|\mathcal{K}|-1}\right).$$

Such a number of points is not tractable for the dimensionality of our problem. But due to the continuous approximation by convex combinations used in the decision making phase we do not need such a number of points for a good approximation.

For point-based approximations of $eff(\mathcal{X})$ the situation is potentially even worse. "Since the dimension of $eff(\mathcal{Y})$ is often significantly smaller than the dimension of $eff(\mathcal{X})$, and since the structure of $eff(\mathcal{Y})$ is often much simpler than the structure of $eff(\mathcal{X})$", a search for solutions in the outcome space $eff(\mathcal{Y})$ is more promising than in $eff(\mathcal{X})$ [11].

In essence the quality measure that underlies the point-based approximations namely having an approximation point close to every efficient point

does not match with the continuous approximation scheme used in the decision making phase. Therefore, the large number of points computed by point-based approximations is a waste of effort.

The quality measures that match with our decision making scheme are coverage of the efficient set and distance to it (see 3.6.3). Thus distance based methods for approximating the efficient set are the natural choice for the first phase of our two-phase approach. According to [99] there are three types of distance based methods:

- outer approximation methods
- inner approximation methods and
- sandwich approximation methods.

Outer approximation methods successively find new supporting hyperplanes and approximate the efficient set by the intersection of the corresponding halfspaces. The methods of Benson [8] and Voinalovich [116] for linear multiobjective problems are not directly applicable to our nonlinear case, although some of the ideas can be combined with inner approximation methods to form sandwich approximation schemes.

Inner approximation methods [22, 26, 104] create successively more efficient points and approximate the efficient set with the close-by facets of the convex hull of the computed points. Sandwich approximation methods [68, 109, 24] determine supporting hyperplanes for every computed efficient point and use the corresponding halfspaces to simultaneously update the outer approximation. Having an inner and outer approximation the sandwich approximation schemes are able to give worst case estimates for the approximation error.

All methods mentioned above try to choose the scalar subproblems such that the maximal distance between the efficient set and the approximation is systematically reduced. The construction of the inner approximation is conducted in all five methods by variations of the same basic idea:

1. create some starting approximation, that consists of one facet, i.e. a $|\mathcal{K}| - 1$ dimensional face
2. find the efficient point that is farthest from the chosen facet by solving a weighted scalarisation problem with a weight vector that is perpendicular to the facet
3. add the point to the inner approximation and update the convex hull
4. if the approximation is not yet satisfactory, choose a facet from the inner approximation and goto 2, otherwise stop

So, all of them use convex hull computations as a subroutine in step 3. But convex hull computations are known to have lengthy computation times and huge memory needs in higher dimensions. The best available algorithms for convex hull computations usually work for dimensions up to 9 [4, 22]. But the trade-off between computational and memory expense for the convex hull subroutine on the one hand and computational savings due to well-chosen scalar problems on the other hand reaches its breakeven point much earlier.

Moreover, the approximation methods use the output of the convex hull computation to construct their next problem formulation. The description of the problem thus grows with the complexity of the convex hull generated by the subroutine. Hence, not only the subroutine becomes lengthier in higher dimensions, but the problems solved to get new points are much more complex, too. That may be the reason why [24] is the only method for which numerical test for dimensions higher than three are reported.

A solution for that could be the application of bilevel programming proposed recently by Leyffer ([78]) to solve the problem of finding the next set of scalarisation parameters. Here, instead of building an explicit representation of the inner approximation by convex hull algorithms, the inner approximation is implicitly represented by the lower level problem. Thus, there is no need for time consuming convex hull computations and the updated representation after a new point has been calculated essentially only increases by that point.

On the one hand, the method has the potential to smoothly work for high dimensions, since it avoids all the problems mentioned earlier. On the other hand, bilevel programming is a global optimisation technique with potentially long running times and the problem of getting stuck in local minima. The latter is not very troublesome in this context, as local minima with similar distance values as the global optimum constitute points, that would eventually be added to the approximation anyway. Still, the lack of numerical experience with the method suggests to wait for its validation before employing it in an application context.

Therefore, the (currently) most promising scheme is the use of [24] for the comparatively low dimensional cases (up to four or maybe five) and the use of heuristic or stochastic methods for higher dimensions.

The covered range plays a crucial role in the interactive selection process and there is no guarantee that a reasonable range is achieved with a purely stochastic procedures. Hence, we propose to use a heuristic to supply the appropriate ranges and a stochastic procedure to improve the approximation with further points.

4.4.2 Extreme compromises – a heuristic approach

We will introduce the heuristic of the extreme compromises as a means to ensure reasonable ranges for the objectives. The extreme compromises probe the potential for different compromises. Therefore, it takes subsets of the objectives and finds the best compromise for them observing the upper bounds for the remaining objectives, but otherwise (almost) disregarding them completely. Thus, it employs an extreme way of compromising – the reason for its name.

The extreme compromises successively minimise the maximum values occurring in subsets of the objectives. They partition the set of objectives into the subsets of *active* and *inactive* ones. Then, the successive maxima in the active objective functions are considered (oder taken care of) first and the inactive objective functions are thereafter treated likewise.

Let $\emptyset \neq \mathcal{M} \subseteq \mathcal{K}$ be the set of indices of the active objectives. Define

$$\pi_{\mathcal{M}} : \mathbb{Y} \times \mathcal{K} \to \mathcal{K}$$

such that it sorts the objectives

$$\begin{aligned} \mathbf{y}_{\pi_{\mathcal{M}}(\mathbf{y},k)} &\geq \mathbf{y}_{\pi_{\mathcal{M}}(\mathbf{y},k')} \quad \text{for} \quad k,k' \in \mathcal{M}, k \leq k' \\ \mathbf{y}_{\pi_{\mathcal{M}}(\mathbf{y},k)} &\geq \mathbf{y}_{\pi_{\mathcal{M}}(\mathbf{y},k')} \quad \text{for} \quad k,k' \notin \mathcal{M}, k \leq k' \\ \pi_{\mathcal{M}}(\mathbf{y},k) &\leq \pi_{\mathcal{M}}(\mathbf{y},k') \quad \text{for} \quad k \in \mathcal{M}, k' \notin \mathcal{M}. \end{aligned}$$

first according to whether they belong to the active or inactive subset and second according to their current value. In analogy to [35, Chapter 6.3] let

$$sort_{\mathcal{M}}(\mathbf{y}) := \left(\mathbf{y}_{\pi_{\mathcal{M}}(\mathbf{y},k)}\right)_{k \in \mathcal{K}}$$

denote the sorted vector.

Consider the three vectors $(1,4,6,8)$, $(5,2,4,3)$, $(2,6,3,4)$ and the set of active indices $\mathcal{M} = \{1,3\}$. Then the sorted vectors are

$$\begin{aligned} sort_{\mathcal{M}}((1,4,6,8)) &= (6,1,8,4), \\ sort_{\mathcal{M}}((5,2,4,3)) &= (5,4,3,2) \quad \text{and} \\ sort_{\mathcal{M}}((2,6,3,4)) &= (3,2,6,4). \end{aligned}$$

So, the sorting first sorts the components contained in $\mathcal{M}$ in decreasing order into the first $|\mathcal{M}|$ components of the result vector followed by the remaining components of the original vector in decreasing order.

The solution of

$$lexmin\{sort_{\mathcal{M}}(\mathbf{y}) \mid \mathbf{y} \in \mathcal{Y}\} \tag{4.3}$$

is called extreme compromise for the active objectives $\mathcal{M}$. The resulting objective vector $\mathbf{y}^{\mathcal{M}}$ will consist of several groups of indices with decreasing

function value. Here, the groups lying in $\mathcal{M}$ make up the first $|\mathcal{M}|$ entries and the ones in $\mathcal{K} \setminus \mathcal{M}$ the remaining $K - |\mathcal{M}|$ components. The values for each of the parts form independent scales by construction.

Note that (4.3) is just a formal way of writing the definition of an extreme compromise. Since the sorting is supposed to change during the optimisation process whenever the definition of $\pi_{\mathcal{M}}(\mathbf{y}, k)$ demands a change, we cannot calculate the extreme compromises with lexicographic minimisation.

We will show, that the extreme compromises are efficient for every nonempty set $\mathcal{M} \neq \emptyset$. This is mainly due to the consistency of $sort_{\mathcal{M}}$ with the componentwise partial order.

Lemma 4.4.1 *For every nonempty set $\emptyset \neq \mathcal{M} \subseteq \mathcal{K}$ and two vectors $\mathbf{y}, \mathbf{y}' \in \mathbb{Y}$, where one dominates $\mathbf{y} \lneq \mathbf{y}'$ the other, the sorted vectors $sort_{\mathcal{M}}(\mathbf{y}) \lneq sort_{\mathcal{M}}(\mathbf{y}')$ have the same dominance relation.*

Proof:

Let us first look at the case $\mathcal{M} = \mathcal{K}$. The claim is equivalent to the largest remaining element of $\mathbf{y}'$ being greater than or equal to the largest remaining element of $\mathbf{y}$.

If the permutations $\pi_{\mathcal{K}}(\mathbf{y}, k) = \pi_{\mathcal{K}}(\mathbf{y}', k)$ agree for the k-th step, we just remove the index. The domination relation then ensures that the claim holds for the remaining components.

So assume that the permutations $\pi_{\mathcal{K}}(\mathbf{y}, k') \neq \pi_{\mathcal{K}}(\mathbf{y}', k')$ do not agree for the k-th index. The indices given by the permutations are by definition the maximum of the remaining components for the respective vectors. Therefore, the following chain of inequalities holds

$$\mathbf{y}'_{\pi_{\mathcal{K}}(\mathbf{y},k')} \geq \mathbf{y}_{\pi_{\mathcal{K}}(\mathbf{y},k')} \geq \mathbf{y}_{\pi_{\mathcal{K}}(\mathbf{y}',k')}.$$

Here, the centre inequality is due to the dominance relation. Thus, the counterpart $\mathbf{y}_{\pi_{\mathcal{K}}(\mathbf{y}',k')}$ in the dominating vector $\mathbf{y}$ for the maximum of the remaining entries in the dominated vector $\mathbf{y}'$ is bounded from above by the entry $\mathbf{y}'_{\pi_{\mathcal{K}}(\mathbf{y},k')}$ of $\mathbf{y}'$ – the counterpart to the maximum of the remaining entries in $\mathbf{y}$. Hence, if we consider them as a pair now, we can iteratively continue the argument.

In other words, if we remove two components differing in index, their counterparts in the other vector have the right order relation. Therefore, the maximum in the dominated vector for the remaining components is larger than the one for the dominating vector.

To prove the claim for general $\mathcal{M}$ we observe, that $sort_{\mathcal{M}}$ sorts the indices in $\mathcal{M}$ and the ones in $\mathcal{K} \setminus \mathcal{M}$ separately and puts the result of the former in the first $|\mathcal{M}|$ components of the resulting vector and the latter in the remaining components. So, applying $sort_{\mathcal{M}}$ is like applying the unpartitioned sorting

to the projections on the components belonging to $\mathcal{M}$ and to the remaining components separately. The sorting within the projections respects the partial order and thus the overall sorting as the collocation of the two respects it as well.

□

Now the proof for the efficiency of the extreme compromises combines the preceding result with the properties of the lexicographic order.

Theorem 4.4.2 *For a nonempty subset $\emptyset \neq \mathcal{M} \subseteq \mathcal{K}$ of the objective functions the corresponding extreme compromise is efficient.*

Proof:

Let $\mathbf{y}, \mathbf{y}' \in \mathbb{Y}$ be two vectors, where one dominates $\mathbf{y} \lneq \mathbf{y}'$ the other. By proposition 4.4.1 the sorted vectors $sort_{\mathcal{M}}(\mathbf{y}) \lneq sort_{\mathcal{M}}(\mathbf{y}')$ have the same relation. But by the definition of the relation $\lneq$ and the lexicographic ordering it follows that

$$sort_{\mathcal{M}}(\mathbf{y}) <_{lex} sort_{\mathcal{M}}(\mathbf{y}').$$

Thus, no dominated point can be the minimum with respect to the lexicographic ordering applied on vectors sorted by $sort_{\mathcal{M}}$. Hence, the extreme compromises are efficient.

□

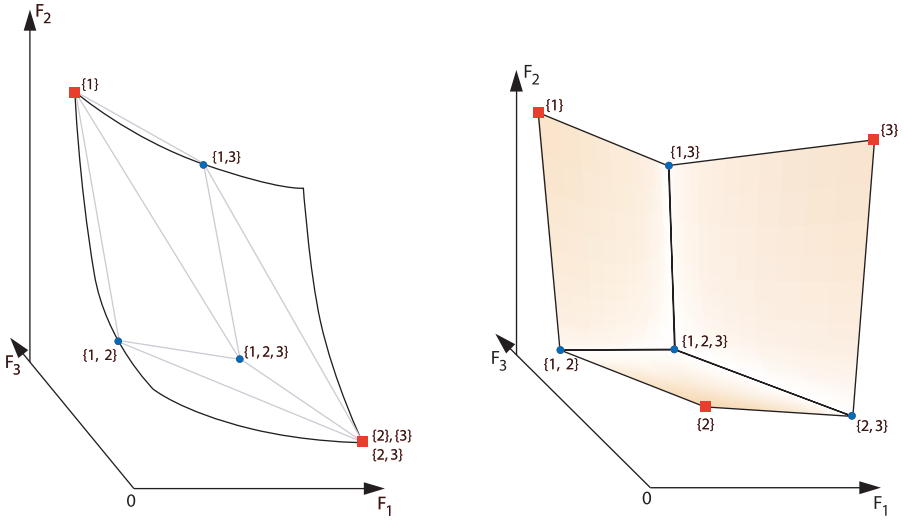
The extreme compromise with all objectives active, i.e. $\mathcal{M} = \mathcal{K}$, is known as *lexicographic max-ordering* problem [35], *variant lexicographic* optimisation problem [101] or as *nucleolar solution* [81] and *nucleolus* in game theory (see references in [81, 101]). The latter articles also describe methods for computing it. Sankaran ([101]) is able to compute it solving $|\mathcal{K}|$ optimisation problems using $|\mathcal{K}|$ additional variables and constraints.

The number of $|\mathcal{K}|$ optimisation problems sounds alarming, but looking at the algorithm, we see that in every step the $|\mathcal{K}|$ additional constraints are tightened, but leave the current optimum feasible. The other change done in each step is the adaption of the objective function, which does not hurt feasibility as well. Thus, the current optimum is not optimal for the new setting only, if the change in the objective function impairs its optimality. Typically, that is not the case for more than 2-3 steps at most. So the $|\mathcal{K}|$ optimisation problems is a worst case bound that is not met in practise.

General extreme compromises can be calculated by using Sankaran's method lexicographically for the two subsets. Alternatively, if upper bounds and lower bounds for the objective functions are known, the objectives can be scaled and shifted such that the largest value in the inactive objectives is always smaller than the smallest value in the active objectives. Sankaran's algorithm will then directly yield the corresponding extreme compromise.

The rationale behind the definition of the extreme compromises is to fathom the possibilities to simultaneously minimise a group of objectives for all possible such groups (see figure 4.6(a)). Therefore, we build up our initial range by computing the extreme compromises for all nonempty subsets $\emptyset \neq \mathcal{M} \subseteq \mathcal{K}$ of the objectives.

Note that the solutions minimising individual objective functions, the so-called *individual minima* are contained in the extreme compromises. Thus, the convex hull of individual minima (CHIM) – the starting point for the inner approximation in [26, 109] and a possible starting point in [68, 104] – is also contained in the convex hull of the extreme compromises. Figure 4.6(a) shows that the extreme compromises cover substantially more than the CHIM, which is in this case even sub-dimensional. This is due to the fact that two of the three objectives share a common minimum.



(a) (b)

Figure 4.6: Extreme compromises in 3 dimensions. The integer sets state the active set for the corresponding extreme compromise. The squares are individual minima.

Figure 4.6(b) shows the position of the extreme compromises for $\mathcal{Y}$ being a bent open cube. Again the squares depict the individual minima. As one can see, we need the full number of extreme compromises to cover the set of efficient points for this case. In the general case the "grid" given by the extreme compromises is distorted, with occasional degeneracies (see figure 4.6(a)).

The extreme compromises typically are not of high clinical relevance as the inactive objectives frequently reach their upper bounds and are thus close to the plans that were a priori characterised as clinically irrelevant.

The number of extreme compromises (4.3) to be calculated is exponential. It is equal to the number of nonempty subsets of $\mathcal{K}$ which is $2^{|\mathcal{K}|} - 1$. One can think of this number as resulting from two points for each interval in a Cartesian product.

One method to reduce the number of computations is to group related organs and treat them as one component. They are turned active or inactive as a group and hence only count as one objective in the exponential complexity term. This is necessary in cases, where the number of objectives would otherwise be more than ten. In a head and neck case – where such numbers can occur – one could for example group the eyes with their respective optical nerve since it is meaningless to spare one while applying critical doses to the other.

To improve the approximation of the efficient set we add further points to the approximation using a stochastic procedure. These points will most likely not change the range of the approximation but improve the distance between the approximation and the efficient set. This allows us to better convey the shape of the efficient set to the planner in the navigation process.

For the stochastic procedure we again use the property that cone scalarising functions can work with feasible and infeasible reference points alike. Thus, we propose two approaches to generate random reference points.

The computation of the extreme compromises included the computation of the individual minima and therefore we know the ideal point. Moreover, we can use the extreme compromises for a payoff table estimate of the nadir point. Thus, we have a bounding box, whose weakly efficient surface parts can be sampled randomly. Alternatively, we could randomly create convex combinations of the extreme compromises. On the one hand, the first approach might still enlarge the range, whereas the second will not. On the other hand, we have good starting points for the optimisation in the second approach, so that typically the running times for the corresponding scalar problems are shorter.

The number of computations to be performed in the approximation phase should be some reasonable compromise between accuracy and computational effort. The approximation should be good enough to convey the interdependencies of the objectives correctly to the decision maker, but since we can correct approximation errors by post-optimisation, the approximation phase should not be prolonged until the approximation is perfect. Typically we choose to compute roughly as many intermediate solutions as there are extreme compromises.

The computation of the extreme compromises and intermediate points are technical and done without any human interaction, so the calculations could for example run over night. So, unless the decision maker has to wait for the computation to be finished, it does not consume any of his or her time.

4.5 Interactive decision making – phase II

The real work of the planner starts after the approximation to the efficient set is computed. (S)he has to find a plan among the variety offered by the precomputed solutions. Naturally, our decision support tool – the *navigator* – uses Pareto navigation to support this search.
Since the mechanisms and internal workings were already presented in detail, we will use this section to present the user interface and graphical presentation of Pareto navigation for the case of IMRT planning. We will only present the mechanisms that are already implemented in the current prototype, i.e. selection and restriction.
The user interface of the navigator is composed of typical elements of an IMRT planning system plus the interface elements that are linked to the Pareto navigation functionality (figure 4.7).

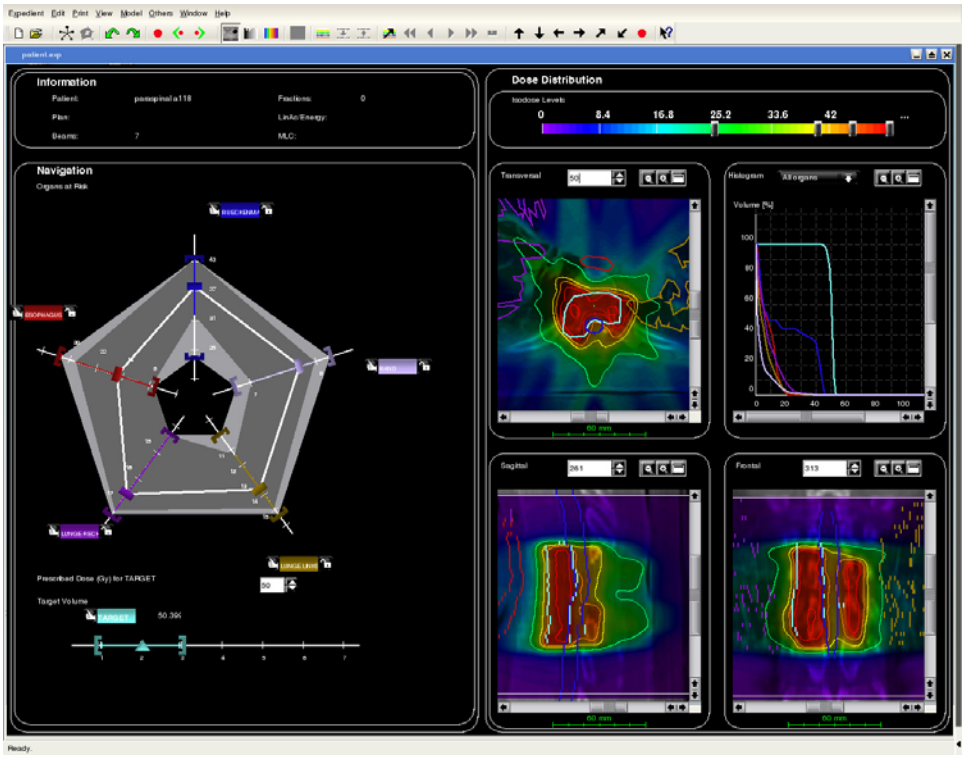


Figure 4.7: The navigation screen. The spider web chart on the left hand side shows the active and inactive planning horizon and the current solution. On the right the dose visualisation and the dose volume histogram for the current solution are shown.

The user interface is divided into two parts. The left hand side visualises the variety of plans available as a whole and embeds the current solution

into this variety. The right hand side displays dose volume histogram of the current plan and the corresponding dose distribution on transversal, frontal and sagittal slices.

The spider web diagram on the left hand side is composed of axes for the different objective functions associated to critical structures. The objectives associated with tumour volumes are shown as separate axes beneath the spider web diagram. The interval on the axes corresponds to the range of values for the respective objective function covered by the precomputed solutions. The white polygon marks the objective function values for the currently selected plan.

The shaded area in the spider web chart and the tumour axes are used to represent the planning horizon. The shaded part is divided into the (*active*) planning horizon and the part that is out of reach for the current bound called the *inactive* planning horizon. The former is visualised by a darker shade of grey in the spider web chart and the coloured parts of the tumour axes. Note that in absence of the discounting mechanism the inner and outer planning horizon are the same.

So, the decision maker can see how the current solution embeds into the potentially and the still accessible ranges for the different objectives. Moreover, (s)he can see how the imposed restrictions limit the range for the different objectives.

However, the axes are not just used to display the current situation, but also carry the controls that are used for the manipulation. The line representing the currently selected plan has handles called *selector* at each intersection with an axis and triangles in case of the tumour related axes. Furthermore, each axis carries *restrictors* represented by brackets.

The planner can now grab any of the controls with the mouse and move it to any feasible value. Meanwhile, the corresponding mechanism is executed and its result is displayed several times a second. In case of the restrictors just the spider web diagram and the tumour axes need to be updated. More time critical are selector movements, since they change the current solution and thus trigger an update of the dose visualisations, too.

Typically, the linear programs executed for estimating the upper and lower bounds in case of the restriction mechanism or the linear programs detecting the convex combination coefficients in case of the selection mechanism execute in some 10 microseconds each. The more time critical part is the execution of the convex combination and the evaluation of the objective functions. For the selection mechanism the update of the visualisation takes up further time. The time needed for the execution of the convex combination followed by the evaluation of the objective functions and the computations for the visualisation usually ranges between 100-200 microseconds. Thus,

currently the bottleneck is not the execution of the problems for the fully simplified model, but rather the surrounding technical computations. Thus, employing better visualisation algorithms, we could gain some time that we could invest in more complex approximations of the original problem.

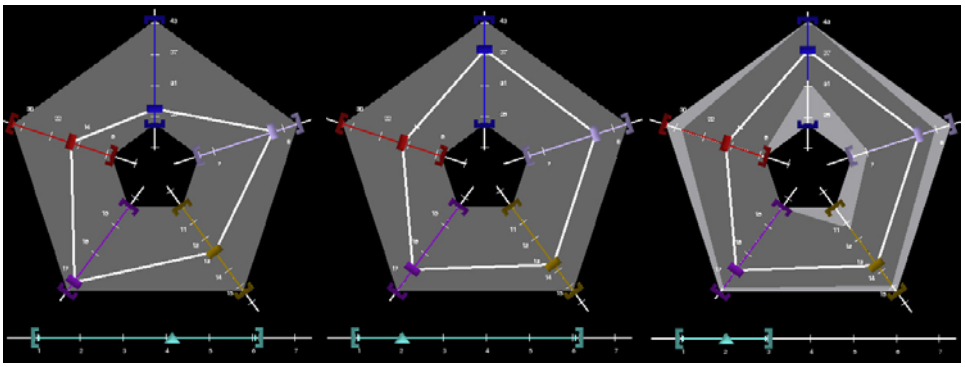


Figure 4.8: A sequence of navigation steps on the tumour axes visualised along with the corresponding changes on the spider web chart.

Figure 4.8 visualises a sequence of two navigation steps – one selector and one restrictor movement on the tumour axis – along with the corresponding reactions on the navigation star. We see that the improvement of the standard deviation in the tumour affects the organs differently. The spinal cord (the blue upright axis) that is surround by the tumour in the considered case reacts strongest and the right lung (the golden axis at the lower left) that is close by worsens too, while the oesophagus (the red axis at the upper left) is indifferent to the change and the left lung (the lilac axis at the lower left) and the unclassified tissue the (violet axes at the upper right) even improve slightly.

Then we restrict the standard deviation in the tumour and see that the loss in range is most noticeable in the organs that reacted strongest to the prior change. Note that for the unclassified tissue and the right lung the range is cut at both ends. Now that the tumour dose distribution is satisfactory the planner would probably start using the controls on the axes of the critical structures in the spider web chart.

4.6 Summary

In this chapter we described the application of Pareto navigation and in particular of the two-phase approach to Pareto navigation to the problem of intensity modulated radiotherapy planning.

We started by describing intensity modulated radiotherapy and discussed the need for a multiobjective approach to IMRT planning. Then, we introduced possible models for the IMRT planning problem and described the modelling options currently available in our prototype.

We continued by surveying different ways to implement the approximation of the efficient set, the first phase of the two-phase approach. In particular, we reviewed the different existing algorithms with respect to their ability to cope with the high-dimensional situation. It became apparent that established methods can only be used for up to 4-5 dimensional cases. Thus, we proposed a combination of the heuristic of the extreme compromises with stochastic sampling to create the approximation in higher dimensions. Our numerical experience with this approach suggests that the rationale of fathoming the possibilities put into the heuristic works for the clinical cases considered.

The next section presented the user interface implementing Pareto navigation for IMRT planning. The different elements and the mouse based input of the parameters for the Pareto navigation mechanisms were introduced. Finally, a short example for an interaction was presented.

The proposed method offers a level of interactivity that is so far unknown in radiation therapy planning. At best IMRT planning so far is an iterative search for good scalarisation parameters and at worst "it's not a science, it's an art" ([75]).

The real-time response to any changes regarding the current solution and planning horizon enables the user to get a feeling for the shape of the efficient set. Observing the changes in the objective function values implied by a modification of one of the objectives gives the planner an impression for the sensitivity and thus for the local interrelation. Observing the changes in the active planning horizon reveals the global connection between the objectives complementing the planner's mental picture of the efficient set.

The concurrent update of the visualisations of the current dose distribution on the right hand side of the navigation screen allows the planner to apply quality measures on the solutions during the navigation that were not modelled into the optimisation problem. The system thus acknowledges the existence of further clinical criteria that are relevant for the planner's final decision.

The realisation of the mechanisms in the user interface highlights, why specifying one value for one objective is also useful for realising the mechanism by a user interface: the choice and the change in value can be specified by a mouse movement, which offers a very natural way of communicating some change to the system.

We are still in search of a similarly easy way to represent the discounting information and to offer corresponding manipulation options. This is a

challenge, since it involves the development of an intuitive representation of dual problem information.

Our prototype is currently used in first comparative clinical studies. Although it lacks proper integration into the remaining clinical workflow and features a rather difficult user interface for specifying the model, the feedback is encouraging. Thus, we expect the forthcoming product to have a considerable impact on future IMRT planning.

In summary, the decision-making process for the treatment planning problem described in this thesis is a distinct improvement over the processes currently in action. Furthermore, its application is not limited to IMRT planning and could be used for other large scale and high-dimensional multiobjective problems as well.

Chapter 5

Outlook

The work presented some general theory regarding multiobjective optimisation, the interactive multiobjective optimisation method "Pareto navigation" and its application in radiotherapy planning. Although we have tried to investigate all major questions in the context, there are open questions remaining. In this chapter we will focus on the questions we think most relevant.

We have several points in the different algorithms where a very natural formulation of the problem under consideration is a bilevel programming problem. If we could solve these bilevel problems in reasonable time to (near) optimality, we would be able to have a controlled approximation without the overhead of creating, maintaining and storing convex hull data to guide the process. Furthermore, the upper end of the outer planning horizon could be detected and the upper and lower end of the inner planning horizon could be correctly determined. This would eliminate the need to work with the outer planning horizon in the selection just for lack of confidence in the estimates of the inner planning horizon.

As such bilevel problems are global problems and there are instances of hard global problems that can be formulated as bilevel problems ([111]), which indicates that they are generally hard to tackle. The advantage for the bilevel problems we have come across is, that they carry special structure that could help to make them tractable, if properly exploited.

Moreover, in all these problems we are satisfied, if the result represents a local optimum as long as the objective value is close to the one for the global optimum. For building up the approximation the insertion of a point that is located at an almost worst case distance between inner and outer approximation improves the approximation almost as much as inserting the true worst case point. For the estimates of ideal and nadir point the optimum as such is not of much interest anyway, so that it suffices, if the associated value is close to the optimum value.

The most promising start point for employing bilevel programming for the nadir and ideal point estimates would be the fully simplified model. Here, we could in particular experiment with the formulation of the lower level program, since there are many ways to enforce efficiency.

Along the way we could potentially gain insight on how to construct good heuristics to approximate the bilevel problem and thus good heuristics to estimate the ideal and nadir point components when the time restrictions do not allow for a more thorough search. The bilevel programs could then be used to correct the estimated values during idle time of the system, if we can make them fast enough.

A second complex that deserves a deeper investigation centres around the extreme compromises. First, the coverage properties should be empirically compared against deterministic methods of approximation for a suitable class of artificial examples. The second interesting point is whether payoff tables based on the extreme compromises are improvements upon the payoff tables created during ideal point component computation. For that a slight modification of the successive balancing idea could be used to avoid that the component in question is included in the balancing.

The third question around the extreme compromises asks whether any conclusions on the shape of the efficient set can be drawn from them. In high-dimensional multiobjective optimisation problems it is likely that not all objectives play an equally important role. The question is now whether the extreme compromises offer enough problem information, to tell the main players from the followers or even detect correlation that is strong enough so that one or more of the objectives could be dropped.

Many of the statements especially in chapter 2 are true in significantly more general situations. We could move to more general spaces ([64], [114]) or use more general dominance sets ([46]).

A topic that is closer to IMRT planning is the development of a better tumour objective function. So far, if we consider the function used to measure the achievement of an appropriately high tumour dose as an objective, only a comparatively small part of the resulting range of values contains clinically relevant plans. Thus, the function does not adequately represent the relaxation of the original goal a clinician would like to see.

A question that is more related to decision making is how to devise a graphical user interface to easily and intuitively set the bounds on the partial tradeoffs. The lack of such a way to express them similar to the way the other decisions are communicated is the main obstacle to incorporate them in the existing system.

Appendix A

The author's scientific career

Professional Experience

since Oct. 2005 Member of the scientific staff of the department *Optimisation* of the Fraunhofer-Institut für Techno- und Wirtschaftsmathematik (ITWM), Kaiserslautern.

Studies

Jun. 2001 - Mar. 2006 PhD study in Mathematics at the University of Kaiserslautern (funded by a scholarship of the Fraunhofer-Institut für Techno- und Wirtschaftsmathematik (ITWM)).

Oct. 1996 - Jun. 2001 Diploma study in Mathematics at the University Kaiserslautern.

Title of the diploma thesis:
Inkorrekt gestellte Probleme unter allgemeinen Quellenbedingungen

Appendix B

Publications

The following publications contain parts of this thesis or precursory work:

[13] TR Bortfeld, KH Küfer, M Monz, A Scherrer, C Thieke, and H Trinkaus. Intensity-modulated radiotherapy - a large scale multicriteria programming problem. Technical Report 43, Fraunhofer- Institut für Techno- und Wirtschaftsmathematik, 2002.

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