Diploma Thesis

Polyhedral Analysis of Hub Center Problems

by

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Contents

Chapter 1

Introduction

1.1 Hub Location Problems

The task of a logistics network is to ship goods from supply to demand nodes using certain transportation paths. Generally, when constructing such a network, the central aim is to reduce the overall costs for transportation. If the distance between the nodes is seen as only determinant of the transportation costs, one would try to establish a logistics network with goods being shipped on the shortest distance paths – a solution which results in a high number of supply-demand pairs that are directly connected (*non–stop connection*). However, besides the travelled distance on a path from supply to demand node, we often also have to consider fixed costs such as investments for establishing certain node-connections, the payment of the employees, the maintenance of a fleet of vehicles, etc. Now, in a network with lots of different non-stop connections between the nodes, certain connections might be travelled through with only small flow value; this results in high fixed costs compared to the number of items transported. Consequently, such "linear networks" [3] do not yield optimal overall transportation costs for a logistics problem of the type described above.

In modern logistics, the concept of *hub networks* has been introduced as a compromise between fast connection of supply and demand nodes and low investment costs [15]. The idea is to establish certain centralized nodes, called *hubs*, which collect flow from the supply nodes (*origin nodes*), sort it, eventually transfer it to another hub, and finally distribute it to the demand nodes (*destination nodes*)[1]. Hence, we are dealing with a *two-level network* (see [10], [8]): The first level is a distributed *backbone network* consisting of hubs that are fully interconnected (*hub level*), the second level consists of centralized *local access networks* for each hub, where each non-hub node (*spoke node*) is connected to a hub. Throughout this thesis, a *strict hubbing policy* will be assumed, meaning that each flow between two nodes is forced to pass either one or two hub nodes[2]. Note that, if there exist flows from one spoke to itself, strict hubbing even enforces this flow to pass through a hub node; for an example of such flows occurring see subsection 1.2.3.

An example for a hub network[3] with three hubs is given in figure 1.1. Square nodes correspond to hub locations, circle nodes to spokes. The doubled lines represent inter-hub connections, whereas single lines stand for *spoke arcs*. Note that in this example, every spoke is assigned to a single hub, whereas there are models in which a spoke can be assigned to more than just one hub (see section 2.2.2).

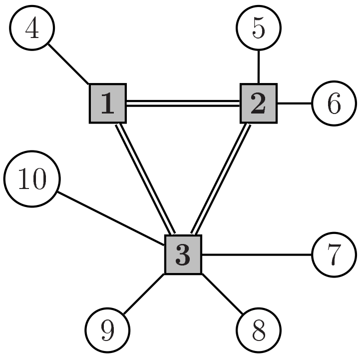


Figure 1.1: Example of a hub network with three hubs

At the hub nodes, we can centralize operations [3]: On inter-hub connections, we are now able to transport huge amounts of flow together. Consequently, we can use larger, faster and more efficient means of transport on these arcs; as a result, the transportation time reduces and the cost per item decreases compared to

[1] A pair of origin and destination node will in the following be denoted by *o-d pair*.

[2] In our model, transportation paths with more than two hub-stops are inefficient; see remark 2.1.1.

[3] Note that real-world examples, in contrast to the examples presented in this introductory part, often consist of several hundred nodes to be considered. Hence, any heuristic or exact solution approach to the problem has to face high computational challenges.

other node connections. Furthermore, since only few node connections are used compared to a linear network, a smaller fleet of vehicles and fewer employees are needed, which again makes transportation via hubs more attractive.

Having a closer look at the establishment of a hub network, one can distinguish between two subproblems: *Hub location* usually refers to the determination of the location of the hub sites among a set of graphs, whereas the *spoke allocation* subproblem is the problem of assigning the spoke nodes to the hub nodes. Though spoke allocation can be seen as an independent subproblem (once hub locations are fixed), the hub location problem cannot be solved to optimality without including the spoke allocation part. For the types of hub location problems that are considered in this thesis, both hub location and spoke allocation are known to be hard computational problems [16] [17].

1.2 Real-World Applications

Hub networking has been elected as one of the best ideas in the "American Marketing Association series of 'Great Ideas in the Decade of Marketing'" (Marketing News, June 20, 1986; see [29]); this hints at the fact that hub networks play a central role in many fields of modern logistics. Important applications of the theory can be found in airline traffic systems, telecommunications and postal delivery services. In the following, we will have a closer look at each of these fields.

1.2.1 Airline Networks

When in 1978 the Airline Deregulation Act[4] took effect in the United States, passenger and cargo airlines were, for the first time, allowed to make their own decisions on flight routes to cover and fares for their service. Many passenger airlines figured out that they could take advantage of economies of scale when introducing certain hub airports as traversal point for airline passengers: Firstly, passengers departing from one city, though having different destinations, could

[4]A similar law for bus traffic, the "Bus Regulatory Reform Act", took effect in 1982, thus enabling the development of bus hub networks as well (see [4]).

be grouped on a flight to a hub. Secondly, once arrived at the hub airport, these passengers could be recombined with passengers from other cities to fly to a second hub, or directly to common destinations. This hubbing concept among airline services has become especially popular for international traffic, where an airline might introduce hubs for any larger region to be covered by its flights (e. g. a certain country or even continent); these hubs then serve as key entrance from international (hub-to-hub) flights to local (hub-to-spoke) flights and vice versa.

For more information on airline hub networks, see [4] and [8].

1.2.2 Telecommunication Networks

As another field of application, hubs are widely used within telecommunication systems. Since communication via computers and voice has increased rapidly in the last years, the need for efficient networks to handle these data transactions became an urgent question. Similar to airline networks, the idea in this field is to design backbone networks, i. e. to locate certain facilities to concentrate communication flow [10]. Economies of scale between these established hub nodes might result from the usage of high-capacity fiber optic links between hubs. The notion of a hub in this context is often used for a special hardware product for computer networks, namely a device which connects several personal computers. However, there are a lot more fields in data transaction where hub networks are used. One special case is telecommunication via satellites, where the satellites themselves can be considered as hubs.

1.2.3 Postal Delivery Networks

In postal transportation, it has been the express parcel delivery sector which pushed forward new networking ideas. The challenge consisted of establishing– with a limited budget– a transportation network that could give a guarantee that parcels are delivered within a given time. One of the earliest companies to introduce a hub network approach was Federal Express. Realizing the benefits of shipping their freight in larger aircrafts between several hub nodes, they started running their business in the 1970s including one hub airport. Due to the growth of the company's transportation network on the one hand and increased mathematical understanding of hub location concepts on the other hand, the Federal Express hub network has been revised several times since then. Recently, the network possesses a hierarchical hub structure with a "super-hub" in Memphis and several additional "overlay hubs"[5]. Additionally, several "linear elements" have been introduced, including the use of stopover points during a transportation path to a hub node [23]. Altogether, the success of Federal Express has been highly linked to the use of hub networks [24] [23].

Stepping back from the special case of Federal Express, where most transportation paths are exclusively operated via airplanes, we can generally identify a hub node of a postal delivery network with a mail sorting center; those centers are meant to collect, sort and distribute incoming mail [19]. Note that in such a postal delivery network, it might be possible that there are flows from one postcode district to itself; as long as sorting is only possible in hub facilities, such mail, too, has to be sent to a hub and back.

A related issue to postal hub networks is the general use of hub networks among shipping agencies. In this context, savings by flow concentration between hub nodes are especially important for so-called *less-than-truckload trucking*, where a transporting vehicle carries less than actually possible; see [8].

1.3 Chapter Outline

This diploma thesis will deal with hub center problems, that is, hub location problems with a min max objective function. The rest of this thesis is divided into two main parts:

Part I will provide an overview on the field of hub location theory and introduce the main problem this thesis will focus on: In **chapter 2**, some basic notions are introduced and different problem types are listed. Finally, an overview on existing literature concerning hub location problems is given. **Chapter 3** focuses

[5] For this concept of hierarchical hub networks, refer to chapter 7.

on the hub center problem and summarizes known formulations for the *uncapacitated single allocation p-hub center problem*, which will be dealt with in the following chapters, and which is known to be an $\mathcal{NP}$-hard problem. A comparison between the different formulations will be given, and the recently developed *radius formulation* will turn out to be suitable for further analysis.

The following **part II** focuses on the polyhedron defined by the radius formulation above: **Chapter 4** determines the dimension of this polyhedron and examines its relationship to a special kind of uncapacitated facility location polyhedron. All given constraints will be evaluated concerning the dimension of the induced faces; finally, as core part of this thesis, three new classes of facet-defining inequalities will be derived. In **chapter 5**, the corresponding separation problems of the new facet-classes will be discussed. Based on these results, a branch-and-cut algorithm is suggested, and first numerical experiments are described in **chapter 6**.

The final **chapter 7** summarizes the main results of this thesis and lists some further research topics in the field of hub location theory.

Part I

Basic Definitions and Problem Formulations

Chapter 2

Basic Definitions

This chapter is meant to introduce the mathematical background for hub location theory. Section 2.1 will give a general mathematical description of a hub location problem instance. Section 2.2 provides an overview on different problem types occurring in the context of hub location; finally, section 2.3 presents a summary on recent research concerning hub networks.

2.1 A Hub Location Problem Instance

Given an undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V}$ is the set of demand nodes and potential hub sites, $|\mathcal{V}| = n$, and $\mathcal{E}$ is the set of inter-node connections. Denote an edge connecting nodes i and j by $e_{i,\,j}$. Imagine that a flow of unit volume one has to be shipped between any pair of nodes in $\mathcal{V}$ along edges in $\mathcal{E}$.

Assume that a cost factor $d_{i,\,j}$ is associated with every pair $i, j \in \mathcal{V}$, satisfying the following conditions [21]:

- $d_{i,\,j} \geq 0 \;\forall\, i, j \in \mathcal{V}$ (non-negativity)
- $d_{i,\,j} = 0 \;\Leftrightarrow\; i = j$
- $d_{i,\,j} = d_{j,\,i} \;\forall\, i, j \in \mathcal{V}$ (symmetry)
- $d_{i,\,j} \leq d_{i,\,k} + d_{k,\,j} \;\forall\, i, j, k \in \mathcal{V}$ (triangle inequality holds)

Furthermore, assume that $d_{i,j}$ is set to a large positive number if $e_{i,j} \notin \mathcal{E}$. In general, the factors $d_{i,j}$ represent the travel distance or travel time on the direct connection between two nodes.

Definition:

- A **hub node** is a centralized facility among the nodes in $\mathcal{V}$ that is used to collect, transfer and distribute flow within the network. Every flow between a pair of nodes has to pass by one or two hub nodes. An arc connecting two hub nodes is called a **hub arc**.

- A node in $\mathcal{V}$ that is not chosen as hub is called a **spoke node**. Every spoke node has to be assigned to (at least) one hub node, and all flow leaving a spoke node has to pass through this (these) particular hub(s). A **spoke arc** is an arc connecting a spoke node to a hub node.

- The **discount factor** is a factor $0 \leq \alpha \leq 1$ that represents the percentile cost savings on inter-hub connections, resulting from economies of scale.

□

If the cost coefficients $d_{i,j}$ are interpreted as travel time, the factor α might reflect time savings due to the usage of faster means of transportation on the hub arcs. The smaller the discount factor α, the more efficient a hub network becomes compared to a linear network.

Once the discount factor is defined, we can calculate the transportation cost $c_{i,j}$ between origin i and destination j as follows: If i is assigned to hub k and j is assigned to hub l, the cost for shipping one unit of goods from i to j is $c_{i,j} = d_{i,k} + \alpha d_{k,l} + d_{l,j}$.

Now, the following special cases might occur:

- $k = l$ (allocation to the same hub). Then, $c_{i,j} = d_{i,k} + d_{k,j}$, and no hub arc is used.

- $i = k$. Then, the origin node of the flow is itself a hub, that is, node i is "assigned to itself". Consequently, $c_{i,j} = \alpha d_{i,l} + d_{l,j}$ (similar if $j = l$).

- $i = j$. If there occurs flow from a spoke node to itself, it also has to be shipped via a hub node[1]: $c_{i,\,i} = d_{i,\,k} + d_{k,\,i}$. If the considered node i is a hub node itself ($i = k$), no costs occur: $c_{i,\,i} = 0$.

Remark 2.1.1
Note that, due to the triangle inequality for the $d_{i,\,j}$, a connection of an o-d pair via more than two nodes is not taken into account in the above: Assume that origin i and destination j are linked by three hubs k, l, m, inducing a transportation cost of $c_{i,\,j} = d_{i,\,k} + \alpha d_{k,\,l} + \alpha d_{l,\,m} + d_{m,\,j}$, By triangle inequality, this cost is larger than $d_{i,\,k} + \alpha d_{k,\,m} + d_{m,\,j}$, i. e. the transportation cost when linking i and j only by hubs k and m.

Definition: Given the graph $\mathcal{G}$ with cost coefficients $d_{i,\,j}$ as defined above, the **discrete hub location problem** consists of choosing a set of hubs among the nodes in $\mathcal{V}$ and allocating all non-hub nodes to those hub nodes such that a given function based on the arc costs $d_{i,\,j}$ is minimized. The subproblem of allocating the non-hub nodes to *given* hub nodes is referred to as the **hub allocation subproblem**. □

In literature on hub networks, the term "hub location problem" sometimes refers to a problem where the number of hubs to be located is not given a priori, but has to be determined as part of the solution (see [8], for instance). In this thesis, however, a "hub location problem" will denote both the case that the number of hubs *is* given a priori and the case that it is *not*; if needed, it will be stated separately which special kind of problem is referred to.

2.2 Hub Location Problem Types

When dealing with hub location problems, one will soon figure out that there does not exist anything like *the* hub location problem, but that the notion of "hub location" summarizes a huge amount of different problem types. Here, a main outline on the most important distinctions concerning hub location will be

[1]We obviously choose $k = l$, since $d_{i,\,k} + \alpha d_{k,\,l} + d_{l,\,i} \geq 2\,min\{d_{i,\,k}, d_{i,\,l}\}$.

given; chapter 3 will then introduce the main problem type that will be dealt with throughout this thesis.

2.2.1 Median and Center Problems

The definition of a discrete hub location problem in section 2.1 left quite unclear which objective function is considered. There are actually two main objective functions dealt with in the literature of hub networks. The straightforward one is the median problem:

Definition: The **hub median problem** consists of solving a hub location problem with median objective function; that is, the objective of the optimization problem is to minimize the sum of the transportation costs for all origin-destination pairs. □

However, it may make sense to consider different objective functions as well. Imagine a postal delivery network, where arc costs represent transportation times, and parcels should be delivered within a certain time. In order not to exceed this time limit, one might be interested in the maximum transportation time for an o-d pair, rather than in the sum of all transportation times. Consequently, we define the center problem:

Definition: In a **hub center problem**, the optimization aim is to locate the hub facilities such that the maximum transportation cost for the o-d pairs is minimized. □

The notions of hub median and hub center problems are directly lent from location theory, where a p–median problem, for instance, consists of locating p facilities in the plane or among a set of sites, such that the sum of distances of some given demand nodes to the nearest facility is minimized. Clearly, hub location represents a similar problem. But since we have interaction between the located facilities as well, the problem gets more complex; a greedy approach to allocate each demand point simply to the nearest hub, for instance, does in general not solve the problem to optimality (see e. g. [26]). Nevertheless, if, in the hub location problem, we are only asked to locate a single hub, no interaction between several hubs has to be taken into account, and the hub location problem consequently reduces to the 1-median (resp. 1-center) problem in location theory.

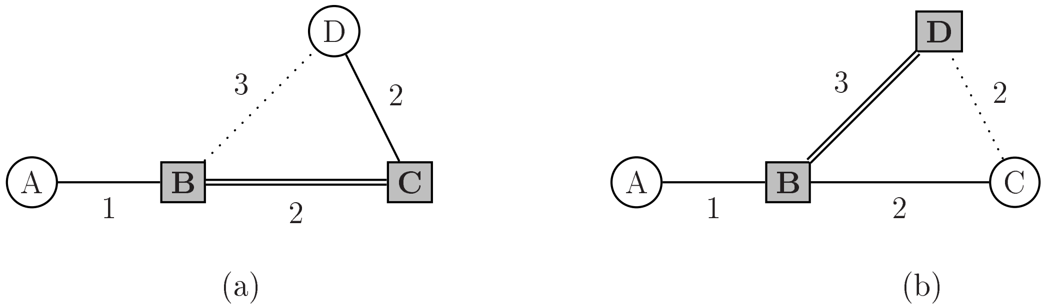


Figure 2.1: Comparison of median and center objective function

Example: Given a graph with 4 nodes A, B, C, D, where node A is only connected to node B, and $d_{AB} = 1$, $d_{BC} = d_{CD} = 2$, $d_{BD} = 3$. Assume that we are given $\alpha = \frac{2}{3}$ and that we have to locate two hubs in this graph. Figure 2.1 shows two possible ways of locating the hubs and allocating the spokes:
In part (a), nodes B and C have been chosen as hubs; node A is allocated to hub B and node D to hub C. The path from node A to D is of maximum length in the network, that is, the center objective function value is $d_{AB} + \alpha\, d_{BC} + d_{CD} = 4\,\frac{1}{3}$. If we add up the lengths of all node connections in the hub network, we obtain a median objective function value of $20\,\frac{1}{3}$ (recall that we also have to take into account the connection of a node to itself).
Now, consider the hub network given in part (b), where nodes B and D are chosen as hub nodes and both spokes are allocated to hub B. Here, the center objective function value is $d_{CB} + \alpha d_{BD} = 4 < 4\,\frac{1}{3}$. But the median objective function value in this case is $21 > 20\,\frac{1}{3}$.

2.2.2 Single and Multiple Allocation

Regarding the allocation of spokes to hubs, up to now we only stated that each spoke has to be allocated "in some way". Now, a second main field to distinguish between hub location problems is the way the spokes are allowed to be allocated. One possible idea is to allow allocation of a spoke node to only one hub:

Definition: In a **single allocation** hub location problem, each spoke node has to be allocated to a single hub facility. That is, all flow with a particular node as origin has to be shipped through the same hub node, and all flow with that node as destination has to be transferred to it via the same hub node. □

Intuitively, the single allocation variation makes spoke allocation a hard problem: A spoke allocation that produces little costs for the flow of one particular o-d pair might result in high costs for transporting items between another origin and destination. In fact, the spoke allocation subproblem with single allocation restriction is $\mathcal{NP}$- hard [16] [17].

Considering the single allocation restriction, one might argue that there are real-world problems where this constraint does not hold: In an airline hub network with several hubs, for instance, it might be quite sensible to connect a non-hub airport to several hub airports to achieve overall transportation costs that are as low as possible. This variation is called *multiple allocation*:

Definition: In the **multiple allocation** hub location problem, spokes are allowed to be allocated to several hub facilities. Consequently, for each origin-destination pair, the cheapest hub allocations can be chosen. □

In the airline example, a multiple allocation assumption makes it possible for flight passengers with different final destinations to travel to different hub airports. When looking at the postal delivery example, multiple allocation allows mail to be routed via different sorting centers according to its destination. Note that in the postal delivery case, multiple allocation requires that mail can be pre-sorted at every node, such that a distribution to different hubs is possible (see [19]).

It is quite obvious that, leaving the single allocation restriction out of consideration, the general hub allocation problem becomes easier:

If we are allowed to choose new allocations for each o-d pair, the problem simply reduces to a shortest paths problem and can therefore be solved in polynomial time [16]. Note that, once the single allocation restriction is skipped, the optimal objective function value decreases; this, as well, is due to the fact that the shortest-path connection via hubs can be chosen for any o-d pair.

To summarize, single allocation hub networking is a special case of multiple allocation, just including one more restriction. Thus, an optimal multiple allocation solution will always have an objective value at least as good as the corresponding single allocation optimal solution. Furthermore, if each spoke is connected to only one hub in the optimal multiple allocation solution, then the solution is also optimal for the single allocation case.

2.2.3 Fixed and Variable Number of Hubs

When we assume that a certain number of hubs has to be established in a hub location problem, this often does not reflect the whole real-world problem: It is more likely that the number of hubs to be located is *not* known a priori. Now, if we are free to choose the number of hubs, it would be the best choice to establish a hub at every single node: Then, every arc could be used, and, if $\alpha < 1$, all arc costs would additionally be reduced[2].
However, this again would not be an appropriate solution to the real-world problem. To mirror the situation of variable number of hubs correctly, we have to introduce some sort of "punishment" for opening hubs: On the one hand, an additional hub enables us to use more arcs and eventually reduces the costs on some arcs, but on the other hand, establishing a hub should result in paying some fixed establishing costs. These costs can vary depending on the potential hub location.

2.2.4 Capacitated and Uncapacitated Hub Location

In section 2.2.2, we claimed that the multiple allocation variant of the hub allocation problem can be solved by simply calculating the shortest paths between all o–d pairs. Here, we made use of the fact that the arcs of our network had no capacity restriction. Now, in an airline hub network, flights between certain cities might be overbooked, such that some passengers have to travel on an alternative, more expensive path from origin to destination. In our model, this can be reflected by imposing capacities on the arcs of the network. Similarly, in the postal

[2]Remember that the arc costs are supposed to be nonnegative.

delivery example, a mail sorting center can only handle a limited amount of mail per day, such that we have to include capacities on the nodes of the network.

2.2.5 Summary of Problem Types

objective function		**median**	minimize overall transportation costs
		center	minimize maximum transportation costs
constraints	overall restrictions	**capacitated**	capacity restrictions on nodes or arcs
		uncapacitated	unbounded capacity on nodes and arcs
	hub location restrictions	**p fixed**	number of hubs given a priori
		p variable	determining the number of hubs is part of the problem
	spoke allocation restrictions	**single allocation**	only one hub per spoke allowed
		multiple allocation	several hubs per spoke allowed

Table 2.1: Hub location problem types

In the subsections above, four main design decisions for a hub location problem have been presented. While the terms *median* and *center problem* (section 2.2.1) refer to the choice of the objective function, the other three design decisions (sections 2.2.2, 2.2.3, and 2.2.4) are related with the problem constraints: They either describe special problem restrictions regarding hub location (*fixed* and *variable number of hubs*), spoke allocation (*single* and *multiple allocation*) or they give overall restrictions on the design of the hub network (*capacitated* and *uncapaci-*

tated problems). The different design decisions are represented schematically in table 2.1.

2.3 Overview on Hub Location Literature

There exists a vast variety of literature on hub median location problems. The hub center problem, however, has only been considered in recent years. Concerning polyhedral analysis, Sonneborn [32] presented several facet classes for the multiple allocation hub median problem, which were derived by lifting valid inequalities of the two-level uncapacitated facility location problem. However, the polyhedral properties of hub center problems have –to the best of our knowledge– not been analyzed before.

A thorough review on hub location research was given by Campbell, Ernst and Krishnamoorthy in [8]; the authors covered problem formulations, variations and solution approaches and provided a detailed overview on the application of hub networks for both transportation and telecommunication backgrounds.

2.3.1 Literature on Hub Median Problems

Single Allocation Hub Median Problems

One of the first to deal with a mathematical formulation of hub median location was O'Kelly in 1987 [26]. He introduced a quadratic integer program formulation for the uncapacitated single allocation hub median problem with fixed number of hubs. Furthermore, O'Kelly examined two greedy heuristics to tackle the problem, where each spoke node is assigned to either the nearest or the second nearest hub node. A transfer of O'Kelly's formulation to the problem with fixed hub costs and variable number of hubs was given by Chung, Myung and Tcha [10] in 1992; in this context, an exact solution algorithm based on branch-and-cut was considered.

Skorin-Kapov and Skorin-Kapov [30] presented a tabu-search heuristic for the uncapacitated single allocation hub median problem in 1994. In 1998, Abdinnour-Helm and Venkataramanan ([1], [2]) combined the tabu-search algorithm of Skorin-

Kapov and Skorin-Kapov with a genetic algorithm to a new hybrid heuristic for solving single allocation hub median problems. Furthermore, they presented a special branch-and-bound strategy as exact solution algorithm.

O'Kelly, Skorin-Kapov and Skorin-Kapov [27] were able to derive lower bounds on the optimal objective function value of a single allocation hub median problem in 1995, and thus provided a means of checking the goodness of existing heuristic solutions.

In 1998, Pirkul and Schilling [29] presented a skilful Lagrangian Relaxation approach to solve single allocation hub median problems. Their procedure allowed them to split the problem into two subproblems that were easily solved. In the course of their solution algorithm, the authors then made use of subgradient optimization to improve the solution they found.

Ernst and Krishnamoorthy [18] considered the capacitated version of single allocation hub median location in 1999. They introduced a simulated annealing and a random descent heuristic and provided several preprocessing steps for the problem.

Multiple Allocation Hub Median Problems

Campbell [7] was among the first researchers to consider the multiple allocation hub median problem in 1995. He proposed a four-index formulation for the uncapacitated multiple allocation hub median problem with fixed number of hubs and investigated a greedy-interchange heuristic for this problem. In addition, Campbell examined possibilities of using multiple allocation solutions to generate solutions to the single allocation case. In 1996, Skorin-Kapov, Skorin-Kapov and O'Kelly [31] were able to tighten the multiple allocation formulation given by Campbell; furthermore, they discussed the general idea of applying a branch-and-cut procedure for solving both single and multiple allocation hub location problems.

A variation of the multiple allocation problem, assuming capacities for the hub nodes, was considered by Aykin [3] in 1994. Aykin applied Lagrangian Relaxation and arrived at a greedy-interchange heuristic and a branch-and-bound algorithm that uses a subgradient subroutine. Another variation of multiple allocation hub problems was given by Aykin in 1995 [4]: Here, the author considered nonstrict hubbing policies, where it is allowed to establish nonstop connections between spoke nodes under certain conditions. Furthermore, Aykin discussed the impact of variable flow volumes that depend on the type of service offered between origin and destination.

Ernst and Krishnamoorthy [19] gave a multicommodity-flow formulation of the multiple allocation hub median problem. Besides a branch-and-cut approach for exact solving, they also considered two heuristics, based on explicit enumeration and the construction of shortest paths, respectively.

2.3.2 Literature on Hub Center Problems

The hub location problem with center objective was not considered unless, in 1994, Campbell [6] presented a quadratic formulation for single allocation hub center problems. Campbell also gave a straightforward linearization of his formulation, but with the grave disadvantage of creating a huge amount of additional variables. In 2000, Kara and Tansel [21] were able to provide a linearization to Campbell's quadratic formulation that does not need any additional variables; the performance of the new formulation with respect to exact solution approaches was shown to improve substantially with this step.

Ernst, Hamacher, Jiang, Krishnamoorthy and Woeginger ([16], [17]) concentrated both on single and multiple allocation hub center problems. Concerning multiple allocation, they introduced a new three-index formulation that needed clearly less variables than the four-index formulation, which can be transferred from the median case in a straightforward manner. With regard to the single allocation problem, the authors presented a substantially new formulation, based on the concept of a "hub radius": This new formulation exploited the special properties of a center objective function, instead of just transferring results from the median case. Besides a branch-and-bound approach for exact solving, several heuristic methods have been considered by the authors, among those special heuristics for solving the allocational subproblem in the single allocation case [15].

Another recent heuristic approach to hub center problems was given by Pamuk and Sepil [28]; the authors examined a 1-exchange heuristic featuring tabu search and several possible spoke allocation strategies. Kozlova [22] gave a summary on known heuristic solution algorithms for hub center problems.

The hub center problem formulations of Campbell, Kara and Tansel and Ernst et al. will be presented in detail in chapter 3.

Chapter 3

Hub Center Problem Formulations

As stated in section 2.2, there is a huge amount of different problem types all summarized under the notion of "hub location problems". This thesis will deal with hub location problems that are equipped with a center objective function. Now, the concept of fixed costs for establishing hubs represents some kind of global approach on hub location: Establishing *any* hub is penalized. In comparison, center location can be seen as a local approach: Only *one* origin-destination pair determines the objective function value). Consequently, the introduction of fixed costs does not make sense for center hub location. This enforces us to fix the number of hubs to be located a priori: If not, the optimal solution would always consist of opening a hub in every single node (see the argumentation of section 2.2.3).

Since we want to derive results for a basic problem in hub location theory, we restrict ourselves to the uncapacitated case. Finally, the allocation type dealt with throughout this thesis will be single allocation, since recently, promising advances have been made in formulating single allocation hub center problems (see section 3.3).

So, the hub location problem type dealt with in this thesis is the *uncapacitated single-allocation p-hub center problem*, where the parameter p reflects the (fixed) number of hubs to be located. Ernst et al. [16] have proven that this problem is

$\mathcal{NP}$-hard. They use the notation of **USApHCP**, which will be adopted in the following.

Note that if $p = 1$, we are dealing with the 1-center problem, which can be solved in polynomial time, e. g. by complete enumeration of the possible center nodes. Similarly, $p = n - 1$ can be solved polynomially by complete enumeration, and $p = n$ is the trivial case where all nodes are opened as hubs. In the following, it will thus be assumed that $p \in \{2, \ldots, n-2\}$ (which implies that $n \geq 4$).

However, there will be facet proofs in chapter 4 where it is stated explicitly that $p \leq \lfloor \frac{n}{2} \rfloor$ is required. Note that for most real-world problems, the number of hubs is very small compared to the number of nodes, such that this additional requirement does not restrict the problem too much.

The following three sections give an overview on known formulations of the **USApHCP**: Section 3.1 presents Campbell's quadratic integer problem formulation. Several linearizations are dealt with in section 3.2, and the recently developed radius formulation of Ernst et al. is given in 3.3.

3.1 Quadratic Formulation

A straightforward formulation for **USApHCP** was given in 1994 by Campbell [6], who used the original median formulation of O'Kelly [26] and equipped it with a center objective function:

Let n be the number of nodes in the problem instance and p the number of hubs to be located among them. For $i, k \in \{1, \ldots, n\}$, define the variables $X_{i,\,k}$ as follows:

For $i = k$:

$$X_{k,\,k} = \begin{cases} 1 & , \quad \text{node } k \text{ is chosen as hub} \\ 0 & , \quad \text{node } k \text{ is a spoke} \end{cases}$$

For $i \neq k$:

$$X_{i,\,k} = \begin{cases} 1 & , \quad \text{node } i \text{ is allocated to hub } k \\ 0 & , \quad \text{else} \end{cases}$$

With the help of the $X_{i,\,k}$, the **USApHCP** is then formulated as follows:

(**HCP − Q**)

$$\min\max \quad \sum_{i,j,k,m=1}^{n} d_{i,\,k}X_{i,\,k} + \alpha\, d_{k,\,m}X_{i,\,k}X_{j,\,m} + d_{m,\,j}X_{j,\,m}$$

$$s.t. \quad \sum_{k=1}^{n} X_{i,\,k} = 1 \quad \forall\, i = 1,\ldots,n \tag{3.1}$$

$$X_{i,\,k} \leq X_{k,\,k} \quad \forall\, i,\, k = 1,\ldots,n \tag{3.2}$$

$$\sum_{k=1}^{n} X_{k,\,k} = p \tag{3.3}$$

$$X_{i,\,k} \in \{0,\ 1\} \quad \forall\, i,\, k = 1,\ldots,n \tag{3.4}$$

Constraint 3.1 guarantees that each spoke node is allocated to a hub node. With 3.2, we make sure that node i can only be allocated to k if k is a hub node. The number of hubs to be located is fixed to p by constraint 3.3, and finally, all variables are assumed to be binary (3.4).

Now, the objective function of the above formulation contains the quadratic terms $\alpha\ d_{k,\,m}X_{i,\,k}X_{j,\,m}$, which are not suitable if we want to apply techniques of linear programming. The next section deals with two approaches to reformulate (**HCP − Q**) as an integer linear program.

3.2 Linearizations

One first approach to linearize the quadratic formulation from section 3.1 above was given by Campbell himself [6]: In order to avoid the quadratic terms $\alpha\, d_{k,\,m}X_{i,\,k}X_{j,\,m}$, he introduced new variables Y_{ijkm} which represent the product $X_{i,\,k}X_{j,\,m}$. If additionally, the min max objective function is resolved by applying a standard procedure for center problems, the problem formulation looks like follows:[1]

[1] In his linearization, Campbell worked with different flow values for different o-d pairs; Ernst et al. [16] simplified this to the above form where a flow of one unit each is assumed for any pair of nodes.

(**HCP – Lin1**)

$$
\begin{aligned}
& min \quad z \\
& s.t. \quad z \geq \sum_{k=1}^{n}\sum_{m=1}^{n}(d_{i,\,k} + \alpha\, d_{k,\,m} + d_{m,\,j})Y_{ijkm} \quad \forall\, i, j = 1, \ldots, n && (3.5)\\
& \sum_{k=1}^{n}\sum_{m=1}^{n} Y_{ijkm} = 1 \quad \forall\, i, j = 1, \ldots, n && (3.6)\\
& \sum_{j=1}^{n}\sum_{m=1}^{n}(Y_{ijkm} + Y_{jimk}) = 2\sum_{j=1}^{n} X_{i,\,k} \quad \forall\, i, k = 1, \ldots, n && (3.7)\\
& X_{i,\,k} \leq X_{k,\,k} \quad \forall\, i, k = 1, \ldots, n && (3.8)\\
& \sum_{k=1}^{n} X_{k,\,k} = p && (3.9)\\
& X_{i,\,k}, Y_{ijkm} \in \{0,\ 1\} \quad \forall\, i, j, k, m = 1, \ldots && (3.10)
\end{aligned}
$$

Note that constraints 3.5 have been summarized in comparison to the original constraints given in [6].

It is easy to deduce from 3.7 that $Y_{ijkm} = 1 \;\Leftrightarrow\; X_{i,\,k} = X_{j,\,m} = 1$ (which is equivalent to $Y_{ijkm} = X_{i,\,k}\, X_{j,\,m}$ since all variables are binary):[2]

Lemma 3.2.1
$Y_{ijkm} = X_{i,\,k}\, X_{j,\,m} \quad \forall\, i, j, k, m \in \{1, \ldots, n\}$.

Proof:

1. Assume $X_{i,\,k}\, X_{j,\,m} = 0$. W.l.o.g., $X_{i,\,k} = 0$. Then, by constraint 3.7:
   $0 = 2\sum_j X_{i,\,k} = \sum_j \sum_m (Y_{ijkm} + Y_{jimk})$.
   From 3.10, we know that $Y_{ijkm} \geq 0$ and $Y_{jimk} \geq 0$,
   and thus $Y_{ijkm} = Y_{jimk} = 0$.

[2] Pay attention to the fact that the summing index at the right hand side of 3.7 is j.

2. Now, assume $X_{i,\,k}\,X_{j,\,m} = 1$.
   From $X_{i,\,k} = 1$, we can deduce

$$\begin{aligned} 2\,n &= 2\sum_{\hat{j}=1}^{n} X_{i,\,k} = \sum_{\hat{j}=1}^{n}\sum_{\hat{m}=1}^{n}(Y_{i\hat{j}k\hat{m}} + Y_{\hat{j}i\hat{m}k}) \\ &= \sum_{\hat{j}=1}^{n}\underbrace{\sum_{\hat{m}=1}^{n} Y_{i\hat{j}k\hat{m}}}_{\leq 1} + \sum_{\hat{j}=1}^{n}\underbrace{\sum_{\hat{m}=1}^{n} Y_{\hat{j}i\hat{m}k}}_{\leq 1} \end{aligned}$$

   and consequently, $\sum_{\hat{m}} Y_{i\hat{j}k\hat{m}} = 1\ \forall\ \hat{j} \in \{1, \ldots, n\}$,
   hence especially $\sum_{\hat{m}} Y_{ijk\hat{m}} = 1$.
   Analogously, $X_{j,\,m} = 1$ leads to $\sum_{\hat{k}} Y_{ij\hat{k}m} = 1$.
   To conclude, $\sum_{\hat{m}} Y_{ijk\hat{m}} = \sum_{\hat{k}} Y_{ij\hat{k}m} = 1$ and $\sum_{\hat{k}}\sum_{\hat{m}} Y_{ij\hat{k}\hat{m}} = 1$ (constraint 3.6) imply that $Y_{ij\hat{k}m} = 0 \quad \forall \hat{k} \neq k$ and $Y_{ijk\hat{m}} = 0 \quad \forall \hat{m} \neq m$, such that $Y_{ijkm} = 1$ follows.

□

Thus, the new constraints 3.6 can substitute constraints 3.1 of the quadratic formulation.

Obviously, the main disadvantage of Campbell's linearization is the high number of new variables that have been introduced.

In 2000, Kara and Tansel [21] were able to show that it is possible to resolve the quadratic term in the objective function without adding any new variables to the formulation. The new linearization suggested by Kara and Tansel is:

(HCP – Lin2)

$$\begin{aligned} &min \quad z \\ &s.t. \quad z \geq \sum_{k=1}^{n}(d_{i,\,k} + \alpha\, d_{k,\,m})X_{i,\,k} + d_{m,\,j}X_{j,\,m} \quad \forall\, i, j, m = 1, \ldots, n \end{aligned} \tag{3.11}$$

$$\sum_{k=1}^{n} X_{i,\,k} = 1 \quad \forall\, i = 1, \ldots, n \tag{3.12}$$

$$X_{i,\,k} \leq X_{k,\,k} \quad \forall\, i, k = 1, \ldots, n \tag{3.13}$$

$$\sum_{k=1}^{n} X_{k,\,k} = p \tag{3.14}$$

$$X_{i,\,k} \in \{0,\ 1\} \quad \forall\, i, k = 1, \ldots, n \tag{3.15}$$

The central point in the argumentation of Kara and Tansel is that the addend $\alpha\, d_{k,\,m} X_{i,\,k} X_{j,\,m}$ can be replaced by $\alpha\, d_{k,\,m} X_{i,\,k}$:

Lemma 3.2.2 (Kara and Tansel, 2000 [21])
The constraint $z \geq \sum_{k=1}^{n} (d_{i,\,k} + \alpha\, d_{k,\,m}) X_{i,\,k} + d_{m,\,j} X_{j,\,m} \quad \forall\, i, j, m = 1, \ldots, n$ *is valid for the quadratic problem formulation and satisfies to describe the objective function value* z*.*

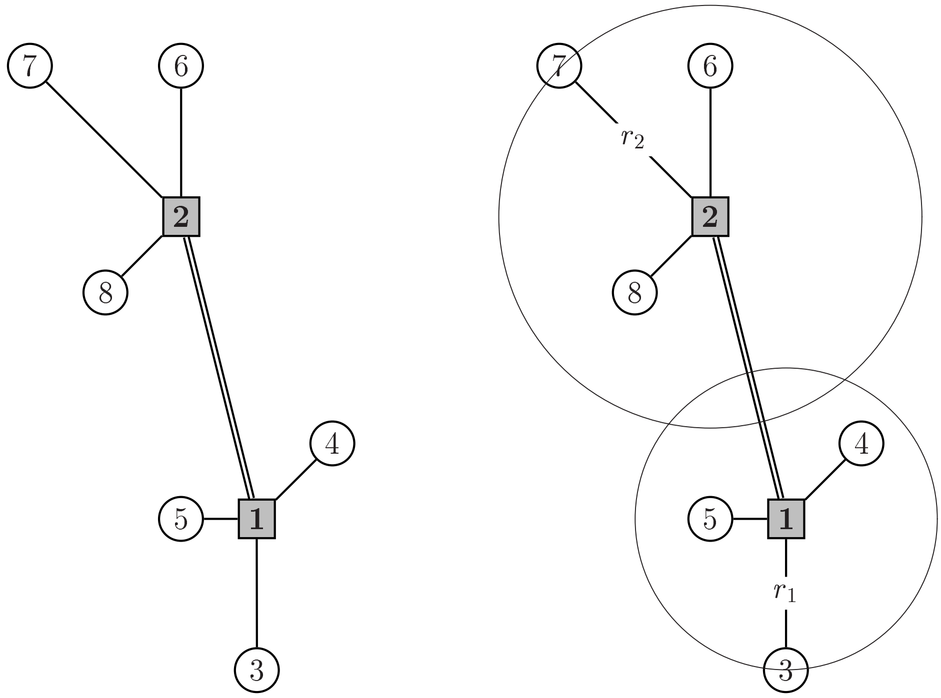
The Kara and Tansel linearization requires only few variables and has been shown to be more efficient in CPU time than the Campbell linearization [21].

3.3 Radius Formulation

So far, the approaches to give formulations for the **USApHCP** were a straightforward extension from known formulations of the median case: The standard hub median constraints were equipped with a center objective function, and the formulation was rewritten with a linear objective function and additional constraints.

Recently, Ernst et al. [16] came up with a new formulation for the **USApHCP** that, in contrast to all formulations given so far, exploits the special structure of center problems. Their central idea was to introduce the notion of a *hub radius*:

Example: Consider the hub network given in figure 3.1 a), where nodes 1 and 2 are hub nodes, and three spoke nodes are assigned to each hub. Imagine the arc distances $d_{i,\,j}$ are proportional to their lengths and we want to determine the maximum travel costs between two nodes in the graph.
Now, it is easy to see that the path from node 8 to 5, for instance, cannot have maximum cost, since spoke 7 is even further away from hub 2 than spoke 8 is, and consequently, the path from node 7 to 5 is more expensive. But once again, this is not the maximum cost path, since the travel cost can be increased by substituting node 5 by node 3.
Consequently, to determine the maximum travel cost of an o-d pair using two



a) Sample hub network b) Network with hub radii

Figure 3.1: Introduction of hub radii

fixed hubs, we only have to consider the spokes with maximum distance from the corresponding hubs.

Definition: The **radius** r_k of a hub node k is the maximum distance to node k of all spoke nodes that are allocated to hub k. □

Now, the idea for the radius formulation of **USApHCP** is the following: Firstly, given a pair of hubs, determine (via the hub radii) the path with maximum length that uses these hubs. Secondly, compare those maximum path lengths for all given pairs of hubs.

If $M \geq \max_{k,m} (\max_j d_{j,\,k} + \max_j d_{j,\,m} + \alpha d_{k,\,m})$ is a large positive number[3], the radius formulation of **USApHCP** is given as follows:

[3]For the determination of the lower bound on M, see section 4.2.

(**HCP – Rad**)

$$
\begin{aligned}
\min \quad & z \\
s.t. \quad z &\geq r_k + r_m + \alpha\, d_{k,\,m} \;\forall\; k, m = 1, \ldots, n && (3.16) \\
r_k &\geq d_{i,\,k} X_{i,\,k} \;\forall\; i, k = 1, \ldots, n && (3.17) \\
\Big(r_k &\leq M X_{k,\,k} \;\forall\; k = 1, \ldots, n \Big) && (3.18) \\
\sum_k X_{i,\,k} &= 1 \;\forall\; i = 1, \ldots, n && (3.19) \\
X_{i,\,k} &\leq X_{k,\,k} \;\forall\; i, k = 1, \ldots, n && (3.20) \\
\sum_k X_{k,\,k} &= p && (3.21) \\
X_{i,\,k} &\in \{0, 1\} \;\forall\; i, k = 1, \ldots, n && (3.22) \\
r_k &\geq 0 \;\forall\; k = 1, \ldots, n && (3.23)
\end{aligned}
$$

Constraints 3.17 set the radius variable r_k to (at least) the maximum distance to k of a spoke node that is allocated to k.

Note that the constraints 3.18 are not included in the formulation given in [16]; however, since they only restrict the radius of a non-hub node to zero (and there is always an optimal solution with $r_k = 0$ if $X_{k,\,k} = 0$), they do not change the optimal objective function value. Constraints 3.18 have been included in the formulation since they will be of help when examining the facets of the radius formulation polyhedron (see chapter 4).

Inequalities 3.16 define the objective function value z as the maximum path length between any two hub nodes by using the respective hub radii. Ernst et al. [16] shortly stated that these inequalities 3.16 stay valid for all possible values of $X_{i,\,k}$ and $X_{j,\,m}$:

Lemma 3.3.1
The inequalities $z \geq r_k + r_m + \alpha\, d_{k,\,m} \;\forall\; k, m = 1, \ldots, n$ for the objective function value z of **USApHCP** *stay valid even if k and/or m are non-hub nodes.*

Proof: If $X_{k,\,k} = 0$, then $r_k = 0$ due to 3.18. Thus

$$
\begin{aligned}
r_k + r_m + \alpha\, d_{k,\,m} &= r_m + \alpha\, d_{k,\,m} \overset{\text{triangle ineq}}{\leq} r_m + \alpha\,(d_{k,\,i_k} + d_{i_k,\,m}) \\
&\overset{\text{def. radius; } \alpha \leq 1}{\leq} r_m + \alpha\, d_{i_k,\,m} + r_{i_k} \leq z,
\end{aligned}
$$

where i_k is chosen such that $X_{k,\, i_k} = 1$. An analogous procedure can be used if $X_{m,\, m} = 0$, and a similar one if both $X_{k,\, k}$ and $X_{m,\, m}$ are zero. □

At first sight, the radius formulation for **USApHCP** has the disadvantage that it requires more variables than the Kara and Tansel linearization. However, the radius formulation considers only those spokes which are at maximum distance from their hubs; hence, a lot of origin-destination pairs can be neglected when determining the overall maximum transportation cost. This fact is mirrored in a fewer number of constraints compared to Kara and Tansel, a first hint that the radius formulation might behave computationally better. Indeed, Ernst et al. could show that for the standard test sets of hub location theory (see chapter 6), the radius formulation is on average ten times more efficient in CPU time than the Kara and Tansel linearization.

Part II

Polyhedral Analysis

Chapter 4

Facet Derivation

As shortly stated in section 2.3, various approaches have been made to try and solve hub location problems to optimality. The corresponding algorithms included several branch-and-bound and branch-and-cut techniques. But, due to the complexity of the problem, so far, for hub center problems, only instances with up to 100 nodes could be solved to optimality within a reasonable amount of time[1].

One way to improve those exact solution procedures is to concentrate on the problem formulation the algorithms are based on: If one is able to derive more valid inequalities for the polyhedron induced by the problem formulation, this new knowledge can be exploited to design better branch-and-cut algorithms, where a cut consists of introducing one or more of those further valid inequalities. Certainly, we are especially interested in facet-defining inequalities, since these provide a tight description of the integer polyhedron.

In the research on hub location problems, polyhedral analysis has played a minor role so far. The only results in this field that have been achieved so far are those of Sonneborn [32] and Hamacher et al. [20], concerning the median case. The authors gave a multi-commodity problem formulation for the uncapacitated multiple allocation hub median problem and showed that this formulation can be considered as a relaxation of the uncapacitated facility location problem (UFL).

[1] See [17] for a variety of numerical experiments based on the Kara and Tansel and radius formulation.

They presented a method to lift facet defining inequalities from UFL to the hub location problem and applied this to derive facets for the hub location problem[2].

Now, at first sight, the only difference between hub median and hub center problems is a change in the objective function of the problem. Consequently, the results in [32] and [20] could be directly applied to the center case. However, in the problem that is dealt with by Sonneborn and Hamacher et al., it is assumed that the number of hubs to be located is *not* given a priori; but, as already stated in chapter 3, this does not make sense for hub center problems. Thus, to investigate the polyhedral properties of the hub center case, we either have to transform given approaches to the case that $p = \mathit{fix}$, or proceed in a different way to derive facets. Since with the radius formulation of Ernst et al. [16] (see section 3.3), a promising new approach has been made for hub center problems, the second alternative will be chosen in this thesis, i. e. the polyhedral analysis will be based on this new problem formulation.

The following section 4.1 will provide a summary of basic terms and results that are needed throughout this chapter. A connection between **USApHCP** and a special kind of uncapacitated facility location problem is presented in section 4.2 and used to derive the dimension of the radius formulation polyhedron in section 4.3. In section 4.4, the faces that are induced by the problem constraints are examined. The final section 4.5, which represents the core of the polyhedral analysis, provides three new groups of facet-defining inequalities for the radius formulation polyhedron.

4.1 Basic Definitions of Polyhedral Theory

This section is meant to shortly introduce main definitions and results of polyhedral theory that will be used throughout this chapter. A thorough introduction to polyhedral theory can be found in [25].

[2] Furthermore, a projection theorem was given to proceed the other way round and derive facets for the UFL problem from facets of the uncapacitated multiple allocation hub median problem.

Definition: A **polyhedron** $\mathcal{P}$ in $\mathbb{R}^n$ is a set of points in $\mathbb{R}^n$ that satisfy a finite number of linear inequalities: $\mathcal{P} = \{\underline{x} \in \mathbb{R}^n : A\underline{x} \leq \underline{b}\}$, where A is an $m \times n$ matrix and b an m-dimensional vector. □

Definition:

- A set of points $\underline{x}^1, \ldots, \underline{x}^k \in \mathbb{R}^n$ is said to be **affinely independent** if the following is valid: If $\sum_i \alpha_i \underline{x}^i = 0$ and $\sum_i \alpha_i = 0$, then $\alpha_i = 0 \; \forall \; i$.
- Let $\mathcal{P} \subset \mathbb{R}^n$ be a polyhedron. The **dimension** of P is equal to k ($\dim \mathcal{P} = k$) if and only if there are exactly $k + 1$ affinely independent points in $\mathcal{P}$.

□

Proposition 4.1.1
Let $\mathcal{P}$ be a k-dimensional polyhedron in $\mathbb{R}^n$, and let m be the maximum number of linear independent equations that are satisfied by all points in $\mathcal{P}$.
Then, $k + m = n$.

Consequently, one way to determine the dimension of a polyhedron is to consider the equations that are satisfied by all points in $\mathcal{P}$. Every such equation, if linear independent from the ones considered so far, reduces the dimension of the polyhedron by one.

Definition: Let $\mathcal{P} \subset \mathbb{R}^n$ be a polyhedron.

- If the inequality $\underline{\pi}\,\underline{x} \leq \pi_0$ is satisfied by all $\underline{x} \in \mathcal{P}$, it is called a **valid inequality** of $\mathcal{P}$.
- Let $\underline{\pi}\,\underline{x} \leq \pi_0$ be a valid inequality of $\mathcal{P}$. The set $\mathcal{F} := \{\underline{x} \in P : \underline{\pi}\,\underline{x} = \pi_0\}$ is called the corresponding **face** of $\mathcal{P}$.

□

It can be shown that a face of a polyhedron is once again a polyhedron. Consequently, the term "dimension" can be applied to a face, and we can define a facet of the polyhedron:

Definition: Let $\mathcal{P} \subset \mathbb{R}^n$ be a polyhedron and $\mathcal{F} \subset \mathcal{P}$ a face. $\mathcal{F}$ is called **facet** of $\mathcal{P}$ if $\dim \mathcal{F} = \dim \mathcal{P} - 1$. □

Hence, a facet of a polyhedron is a face of maximum dimension. There are two main approaches to show that a given valid inequality $\underline{\pi}\,\underline{x} \leq \pi_0$ of a polyhedron $\mathcal{P}$ defines a facet:

1. Identify $\dim \mathcal{P}$ affinely independent points in $\mathcal{F} := \{\underline{x} \in \mathcal{P} : \underline{\pi}\,\underline{x} = \pi_0\}$ that satisfy $\underline{\pi}\,\underline{x} \leq \pi_0$ with equality.

2. Let $\underline{a_1}\,\underline{x} = b_1, \ldots, \underline{a_m}\,\underline{x} = b_m$ be the equations that are satisfied for all points in $\mathcal{P}$. Assume that there exists a further equation that is valid for all points in $\mathcal{F}$, show that it has to be a linear combination of the equations $\underline{a_1}\,\underline{x} = b_1, \ldots, \underline{a_m}\,\underline{x} = b_m$ and $\underline{\pi}\,\underline{x} = \pi_0$.

In mixed integer programming, we deal with optimization problems where the set of feasible solutions is of the following form:
$X := \{\underline{x} \in \mathbb{R}^n_+ : A\underline{x} \leq \underline{b}\} \cap \left(\mathbb{Z}^k \times \mathbb{R}^{n-k}\right)$
It is known that the convex hull of X, $conv(X)$, is a polyhedron;

hence, it can be decribed by a finite number of linear inequalities. Furthermore, a well-known result from polyhedral theory states that it suffices to restrict oneself to facet-defining inequalities. If one was able to derive all those facet-defining inequalities for the description of $conv(X)$, the original mixed integer problem would reduce to a linear problem, and standard linear techniques could be applied. However, the complete derivation of all facets of a polyhedron is very unlikely if the corresponding mixed integer problem is $\mathcal{NP}$-hard. Nevertheless, if one can obtain at least some facets, the polyhedron can be approximized, which hopefully results in time savings when applying mixed integer programming solution approaches.

4.2 (pUHL) and (pUFL) Problems

Recall the radius formulation of **USApHCP** given in [16]:

(**pUHL**)[3]

$$\min z \tag{4.1}$$

$$\text{s.t.} \quad z \geq r_k + r_m + \alpha\, d_{k,\,m} \quad \forall\, k, m \in \{1, \ldots, n\} \tag{4.2}$$

$$r_k \geq d_{i,\,k} X_{i,\,k} \quad \forall\, i, k \in \{1, \ldots, n\} \tag{4.3}$$

$$r_k \leq M X_{k,\,k} \quad \forall\, k \in \{1, \ldots, n\} \tag{4.4}$$

$$\sum_{k=1}^{n} X_{i,\,k} = 1 \quad \forall\, i \in \{1, \ldots, n\} \tag{4.5}$$

$$X_{i,\,k} \leq X_{k,\,k} \quad \forall\, i, k \in \{1, \ldots, n\} \tag{4.6}$$

$$\sum_{k=1}^{k} X_{k,\,k} = p \tag{4.7}$$

$$r_k \geq 0 \quad \forall\, k \in \{1, \ldots n\} \tag{4.8}$$

$$X_{i,\,k} \in \{0, 1\} \quad \forall\, i, k \in \{1, \ldots n\} \tag{4.9}$$

For the following analysis, we define the integer polyhedron $\mathcal{P}_{pUHL}$ of the problem:

Definition:

- $\mathcal{X}_{pUHL} := \{P = (X_{1,\,1}, \ldots, X_{n,\,n}) \in \mathbb{R}^{n^2+n+1} : \text{P feasible for } (\mathbf{pUHL})\}$
  is the set of feasible solutions to (**pUHL**).
- $\mathcal{Z}_{pUHL} := \mathcal{X}_{pUHL} \cap \left(\{0, 1\}^{n^2} \times \mathbb{R}^{n+1}\right)$
  is the set of feasible solutions to (**pUHL**) with binary $X_{i,\,k}$.
- $\mathcal{P}_{pUHL} := conv\left(\mathcal{Z}_{pUHL}\right)$
  is the polyhedron defined by the integer solutions to (**pUHL**).

□

Throughout this chapter, let $X^P_{i,\,k}$ denote the value of variable $X_{i,\,k}$ for the point P; analogously for r^P_k and z^P.

[3]In chapter 4 and 5, **USApHCP** will be the only hub location problem to be considered. To underline the parallels to the UFL problem (see below), the radius formulation of **USApHCP** will henceforth be referred to by (**pUHL**) (uncapacitated p-hub location) rather than (**HCP − Rad**).

As already stated in section 3.3, in contrast to the formulation of Ernst et al. [16] we included another constraint 4.4 ensuring that, whenever a hub is *not* opened, the radius of the corresponding node is set to zero. Though this is not a valid inequality for the original radius formulation, Ernst et al. [16] state that there always exists an optimal solution to their problem with $r_k = 0$ whenever $X_{k,\,k} = 0$. In the following, constraint 4.4 will be included since it makes no restriction on the optimal solution (see construction of M below), but nevertheless restricts the set of solutions to be considered.

To ensure that constraint 4.4 does not give any other restriction apart from "$r_k = 0$ whenever $X_{k,\,k} = 0$", we choose M large enough; a convenient choice for the following constructions[4] is

$$M \geq \max_{k,m} \left(\max_j d_{j,\,k} + \max_j d_{j,\,m} + \alpha\, d_{k,\,m}\right). \tag{4.10}$$

Inequalities 4.2, 4.3 and 4.4 reflect the radius idea, as described in section 3.3. All other inequalities only include $X_{i,\,k}$- variables. Going even further, inequalities 4.5 - 4.7 and 4.9 can be seen as the constraints of a special kind of uncapacitated facility location problem (see [12] for a general discussion of this problem and its polyhedron):

In the uncapacitated facility location problem, the demand of a given set of customers has to be met by locating one or more facilities. Now, assume the set $\mathcal{K} = \{1, \ldots, n\}$ of customers is equal to the set $\mathcal{H}$ of possible facilities, that is: Every customer can become a facility himself. If so for customer k, this customer is seen as allocated to himself, which means that k must not be allocated to any other facility. Furthermore, different to the general notion of the uncapacitated facility location problem, we assume that the number of facilities to be opened is fixed to a number p, and we impose a center objective function rather than a median one. Then, this variation of the uncapacitated facility location problem can be formulated as follows:

[4] e. g. needed in the proof of theorem 4.3.5

(**pUFL**)

$$\min \max_{i,k} d_{i,\,k}\, x_{i,\,k} \tag{4.11}$$

$$\text{s.t.} \sum_{k=1}^{n} x_{i,\,k} \;=\; 1 \quad \forall\, i \in \{1, \ldots, n\} \tag{4.12}$$

$$x_{i,\,k} \;\leq\; x_{k,\,k} \quad \forall\, i, k \in \{1, \ldots, n\} \tag{4.13}$$

$$\sum_{k=1}^{k} x_{k,\,k} \;=\; p \tag{4.14}$$

$$x_{i,\,k} \;\in\; \{0, 1\} \quad \forall\, i, k \in \{1, \ldots n\} \tag{4.15}$$

Similar to (**pUHL**), let $\mathcal{X}_{pUFL}$ denote the set of feasible solutions to (**pUFL**), $\mathcal{Z}_{pUFL}$ be the set of feasible solutions with all $x_{i,\,k}$ integer, and $\mathcal{P}_{pUFL}$ the polyhedron obtained by the convex hull of these integer solutions.

Analogously[5] to (**pUHL**), (**pUFL**) will only be considered for $p \in \{2, \ldots, n-2\}$.

In the next subsections, (**pUFL**) will be used to derive the dimension of $\mathcal{P}_{pUHL}$ and to facilitate the search for facets of $\mathcal{P}_{pUHL}$ that involve only $X_{i,\,k}$-variables.

4.3 Dimension of the Polyhedron

The formulation of (**pUFL**) above is nothing else than constraints 4.5 - 4.7 and 4.9, but equipped with the center objective function for (**pUFL**)[6]. However, as we are now looking at the polyhedron, the difference in the objective functions can be neglected.

Obviously, the following result holds:

Proposition 4.3.1
Let $P = (x_{1,\,1},\, x_{1,\,2},\, \ldots,\, x_{n,\,n}) \in \mathcal{P}_{pUFL}$ *be a feasible solution to* (**pUFL**).
Then, $P^* := (X_{1,\,1},\, X_{1,\,2},\, \ldots, X_{n,\,n},\, r_1,\, \ldots,\, r_n,\, z)$ *with*
$X_{i,\,k} := x_{i,\,k}\ \forall\, i, k,$
$r_k := \max_i d_{i,\,k} X_{i,\,k}\ \forall\, k,$

[5] see introductory remarks in chapter 3

[6] Remember that in the hub location case, interaction between the hub nodes has to be taken into account, but that the facility nodes of (**pUHL**) do not interact.

$z := max_{k,m}\, r_k + r_m + \alpha\, d_{k,\,m}$,
*is a feasible solution to (**pUHL**), and thus, $P^* \in \mathcal{P}_{pUHL}$.*

Furthermore, if we have affinely independent points $P_1, \ldots, P_m \in \mathcal{P}_{pUFL}$, they obviously correspond to affinely independent points $P_1^*, \ldots, P_m^* \in \mathcal{P}_{pUHL}$:

Proposition 4.3.2
Let $P_1, \ldots, P_m \in \mathcal{P}_{pUFL}$ be affinely independent points in $\mathcal{P}_{pUFL}$.
Then, $P_1^*, \ldots, P_m^* \in \mathcal{P}_{pUHL}$ are affinely independent points in $\mathcal{P}_{pUHL}$,
where P_i^* is constructed out of P_i as shown in Proposition 4.3.1.

Corollary 4.3.3
$\dim \mathcal{P}_{pUHL} \geq \dim \mathcal{P}_{pUFL}$.

Next, the dimension of $\mathcal{P}_{pUHL}$ will be derived. For this sake, we first compute the dimension of the polyhedron $\mathcal{P}_{pUFL}$:

Lemma 4.3.4
$\dim \mathcal{P}_{pUFL} = n^2 - n - 1$.

Proof:

- $\dim \mathcal{P}_{pUFL} \overset{!}{\leq} n^2 - n - 1$:

  There are n^2 variables and $n + 1$ equations in the formulation of (**pUFL**). Furthermore, it is easy to see that these equations are linearly independent. Thus, $\dim \mathcal{P}_{pUFL} \leq n^2 - n - 1$.

- $\dim \mathcal{P}_{pUFL} \overset{!}{\geq} n^2 - n - 1$:

  We show: Any equation that is satisfied by all points in $\mathcal{P}_{pUFL}$ is a linear combination of equations 4.12 and 4.14.

  For this sake, let
  $$\sum_i \sum_k a_{i,\,k}\, x_{i,\,k} = d \tag{4.16}$$
  be an equation that is met by all $P \in \mathcal{P}_{pUFL}$. We will proceed in two substeps:

  1. For all j: $a_{j,\,k} \overset{!}{=} a_{j,\,m} =: a_j \quad \forall k, m \neq j$.

2. For all k, l: $a_{k,\,k} - a_k \stackrel{!}{=} a_{l,\,l} - a_l =: a$.

Once steps 1 and 2 have been proven, we can conclude

$$\begin{aligned} d &= \sum_i \sum_k a_{i,\,k}\, x_{i,\,k} \\ \overset{step\ 1,\ \sum_k x_{i,\,k}=1}{\Leftrightarrow} \quad d - \sum_j a_j &= \sum_j (a_{j,\,j} - a_j) x_{j,\,j} \\ \overset{step\ 2,\ \sum_j x_{j,\,j}=p}{\Leftrightarrow} \quad d - \sum_j a_j - a\,p &= 0, \end{aligned}$$

which shows that equation 4.16 is a linear combination of the given inequalities.

Steps 1 and 2 can be proven by constructing points in $\mathcal{P}_{pUFL}$ and inserting them into 4.16:

ad 1: For all j: $a_{j,\,k} \stackrel{!}{=} a_{j,\,m}\ (=: a_j) \quad \forall k, m \neq j$:

W.l.o.g., $k, m \in \{1, \ldots, p\}$ and $j \geq p{+}1$ (eventually relabel the nodes). Consider the following two points:

P_1: $x_{1,\,1} = x_{2,\,2} = \ldots = x_{p,\,p} = 1$,
$x_{i,\,k} = 1 \quad \forall\, i \in \{p+1, \ldots, n\}$,
all other values $= 0$.

P_2: all values as in P_1, except $x_{j,\,k} = 0$ and $x_{j,\,m} = 1$.

Obviously, both P_1 and P_2 are feasible for (**pUFL**). Now, inserting the values of P_1 into equation 4.16 yields

$$\sum_{l=1}^{p} a_{l,\,l} + \sum_{\substack{i \geq p+1 \\ i \neq j}} a_{i,\,k} + a_{j,\,k} = d$$

Analogously, we obtain from P_2:

$$\sum_{l=1}^{p} a_{l,\,l} + \sum_{\substack{i \geq p+1 \\ i \neq j}} a_{i,\,k} + a_{j,\,m} = d$$

Thus, $a_{j,\,k} = a_{j,\,m}$.

ad 2: For all k, l: $a_{k,\,k} - a_k \stackrel{!}{=} a_{l,\,l} - a_l =: a$:

We have shown above that, once step 1 has been proven, 4.16 can be reformulated as

$$d - \sum_j a_j = \sum_j (a_{j,\,j} - a_j)\, x_{j,\,j} \tag{4.17}$$

W.l.o.g., $k = p$, $l = p+1$ (eventually relabel the nodes). Consider the following two points:

P_1: $x_{1,\,1} = x_{2,\,2} = \ldots = x_{p,\,p} = 1$,
$x_{i1} = 1 \quad \forall i \in \{p+1, \ldots, n\}$,
all other values $= 0$.

P_2: all values as in P_1, except
$x_{p,\,p} = 0$, $x_{p+1,\,p+1} = 1$, $x_{p,\,1} = 1$, $x_{p+1,\,1} = 0$.

It is easy to see that both P_1 and P_2 are feasible for (**pUFL**). Inserting the values of P_1 into equation 4.17 yields

$$d - \sum_j a_j = \sum_{j=1}^{p-1} (a_{j,\,j} - a_j) + a_{p,\,p} - a_p$$

Analogously, we obtain from P_2:

$$d - \sum_j a_j = \sum_{j=1}^{p-1} (a_{j,\,j} - a_j) + a_{p+1,\,p+1} - a_{p+1}$$

Thus, $a_{p,\,p} - a_p = a_{p+1,\,p+1} - a_{p+1}$, i. e. $a_{k,\,k} - a_k = a_{l,\,l} - a_l$.

□

Now, we can determine the dimension of $\mathcal{P}_{pUHL}$:

Theorem 4.3.5
$\dim \mathcal{P}_{pUHL} = n^2$.

Remark 4.3.6
To prove that $dim(\mathcal{P}_{pUHL}) = n^2$, one can of course proceed in a straightforward manner as in lemma 4.3.4:

$\mathcal{P}_{pUHL}$ is a polyhedron in n^2+n+1 variables. There are $n+1$ equations fulfilled by all points in the polyhedron (see constraints 4.5 and 4.7), and it can be shown, similar to the proof of lemma 4.3.4, that any further equation satisfied by all points has to be a linear combination of the given equations.
Nevertheless, the dimension of $\mathcal{P}_{pUHL}$ will be derived in a more complicated way below: Starting from known affinely independent points in $\mathcal{P}_{pUFL}$, new points in $\mathcal{P}_{pUHL}$ will be constructed to obtain a final set of n^2+1 affinely independent points in $\mathcal{P}_{pUHL}$. The reason for this proceeding is that, by starting from the $\mathcal{P}_{pUFL}$ polyhedron, we can show the interdependencies between the (**pUFL**) and (**pUHL**) problem. This will be made use of when deriving facets of $\mathcal{P}_{pUHL}$ that only contain the variables $X_{i,\,k}$ (see proposition 4.4.1).

Proof: [of theorem 4.3.5] The radius formulation of (**pUHL**) contains n^2+n+1 variables and $n+1$ equations (constraints 4.5 and 4.7), which, as already seen in the proof of 4.3.4, are linearly independent. Consequently, $\dim \mathcal{P}_{pUHL} \leq n^2$. It remains to show that there exist n^2+1 affinely independent points in $\mathcal{P}_{pUHL}$ (and thus $\dim \mathcal{P}_{pUHL} \geq n^2$). This will be proven in two steps:

1. By lemma 4.3.4, we know that there exist n^2-n affinely independent points $P_1, \ldots, P_{n^2-n} \in \mathcal{P}_{pUFL}$. Thus, by proposition 4.3.2, we can construct n^2-n affinely independent points in $\mathcal{P}_{pUHL}$. Construct those points $P_1^*, \ldots, P_{n^2-n}^*$ as proposed in proposition 4.3.1, but set $z := 2\, max_{k,m} r_k + r_m + \alpha\, d_{k,\,m}$ for each P_i^* (obviously, the points stay feasible and affinely independent).

2. Now, we use the constructed points $P_1^*, \ldots, P_{n^2-n}^*$ to construct another $n+1$ points, such that the set of all constructed points is still affinely independent:

   First, note that for every $k \in \{1, \ldots, n\}$ there is $i \in \{1, \ldots, n^2-n\}$ such that $r_k^{P_i^*} > 0$. To see this, assume that $r_k = 0$ for all points $P_1^*, \ldots, P_{n^2-n}^*$. Then, by construction of the P_i^*, we have that $X_{m,\,k} = 0$ for all $m \neq k$ (recall that $d_{i,\,j} = 0 \Leftrightarrow i = j$). But then, these $n-1$ additional equations, which are linearly independent from constraints 4.12 and 4.14, are satisfied by the points $P_1, \ldots, P_{n^2-n}$ as well. Contradiction.

For every k, choose $i(k) \in \{1, \ldots, n^2 - n\}$ *minimal* with $r_k^{P^*_{i(k)}} > 0$ (and thus, $r_k^{P^*_l} = 0 \; \forall \, l < i(k)$). Let $\mathcal{L} = \{L_1, \ldots, L_s\}$ be the index set of chosen points $P^*_{L_j}$ (ordered such that $L_1 < L_2 < \ldots < L_s$), and let $\mathcal{L}_j$ be the set of k-values for which $P^*_{L_j}$ has been chosen, i. e. $\mathcal{L}_j = \{k : i(k) = L_j\}$. Construct new points $\hat{P}_1, \ldots, \hat{P}_n$ with $\hat{P}_k = P^*_{i(k)}$, except that $r_k^{\hat{P}_k} = 2\, r_k^{P^*_{i(k)}}$. Due to the choice of M, the new points $\hat{P}_1, \ldots, \hat{P}_n$ stay feasible:

Constraint 4.4:

$$\begin{aligned} r_k^{\hat{P}_k} &= 2\, r_k^{P^*_{i(k)}} \\ &\leq 2\, \max_j d_{j,\,k} \\ &= \max_j d_{j,\,k} + \max_j d_{j,\,k} + \alpha\, d_{k,\,k} \\ &\leq \max_{l,m}(\max_j d_{j,\,l} + \max_j d_{j,\,m} + \alpha d_{l,\,m}) \\ &\leq M \end{aligned}$$

Constraint 4.2:

- First, note that

$$\begin{aligned} r_k^{\hat{P}_k} + r_k^{\hat{P}_k} + \alpha d_{k,\,k} &= 2\, r_k^{P^*_{i(k)}} + 2\, r_k^{P^*_{i(k)}} \\ &= 2\, (r_k^{P^*_{i(k)}} + r_k^{P^*_{i(k)}}) \\ &\leq 2\, \max_{l,m}(r_l^{P^*_{i(k)}} + r_m^{P^*_{i(k)}} + \alpha\, d_{l,\,m}) \\ &\overset{step\ 1}{=} z^{P^*_{i(k)}} \\ &= z^{\hat{P}_k} \end{aligned}$$

- Now, for all $m \neq k$:

$$\begin{aligned} r_k^{\hat{P}_k} + r_m^{\hat{P}_k} + \alpha d_{k,\,m} &= 2\, r_k^{P^*_{i(k)}} + r_m^{P^*_{i(k)}} + \alpha d_{k,\,m} \\ &\leq 2\, (r_k^{P^*_{i(k)}} + r_m^{P^*_{i(k)}} + \alpha d_{k,\,m}) \\ &\leq 2\, \max_{l,t}(r_l^{P^*_{i(k)}} + r_t^{P^*_{i(k)}} + \alpha d_{l,\,t}) \\ &\overset{step\ 1}{=} z^{P^*_{i(k)}} \\ &= z^{\hat{P}_k} \end{aligned}$$

- Finally, for all $l, m \neq k$,
  $z^{\hat{P}_k} \geq r_l^{\hat{P}_k} + r_m^{\hat{P}_k} + \alpha\, d_{l,\,m}$, since this was already valid for $P^*_{i(k)}$.

Thus, we have shown that the constructed points all lie in $\mathcal{P}_{pUHL}$. Next, we show that the points $P_1^*, \ldots, P_{n^2-n}^*, \hat{P}_1, \ldots, \hat{P}_n$ are affinely independent. For this sake, assume that

$$\sum_{i=1}^{n^2-n} \alpha_i^* P_i^* + \sum_{k=1}^{n} \hat{\alpha}_k \hat{P}_k = 0 \tag{4.18}$$

$$\text{with} \quad \sum_{i=1}^{n^2-n} \alpha_i^* + \sum_{k=1}^{n} \hat{\alpha}_k = 0. \tag{4.19}$$

Equation 4.18 and the construction of the points $\hat{P}_k$ induce the following equations for the variables $X_{m,\,r} \quad (m, r \in \{1, \ldots, n\})$:

$$\begin{aligned} \sum_{i=1}^{n^2-n} \alpha_i^* X_{m,\,r}^{P_i^*} + \sum_{k=1}^{n} \hat{\alpha}_k X_{m,\,r}^{P_{i(k)}^*} &= 0 \\ \overset{def.\ of\ L_j\ and\ \mathcal{L}_j}{\Leftrightarrow} \sum_{\substack{i=1 \\ i \notin \mathcal{L}}}^{n^2-n} \alpha_i^* X_{m,\,r}^{P_i^*} + \sum_{j=1}^{s} \Big(\alpha_{L_j}^* + \sum_{l \in \mathcal{L}_j} \hat{\alpha}_l \Big) X_{m,\,r}^{P_{L_j}^*} &= 0 \end{aligned} \tag{4.20}$$

Due to the construction of the points P_i^* $(i = 1, \ldots, n^2 - n,\ i \notin \mathcal{L})$ and $P_{L_j}^*$ $(j = 1, \ldots, s)$ out of $P_1, \ldots, P_{n^2-n} \in \mathcal{P}_{pUFL}$, we can conclude from 4.20 that

$$\sum_{\substack{i=1 \\ i \notin \mathcal{L}}}^{n^2-n} \alpha_i^* P_i + \sum_{j=1}^{s} \Big(\alpha_{L_j}^* + \sum_{l \in \mathcal{L}_j} \hat{\alpha}_l \Big) P_{L_j} = 0.$$

Since

$$\sum_{\substack{i=1 \\ i \notin \mathcal{L}}}^{n^2-n} \alpha_i^* + \sum_{j=1}^{s} \Big(\alpha_{L_j}^* + \sum_{l \in \mathcal{L}_j} \hat{\alpha}_l \Big) = \sum_{i=1}^{n^2-n} \alpha_i^* + \sum_{k=1}^{n} \hat{\alpha}_k = 0,$$

(by 4.19) and $P_1, \ldots, P_{n^2-n}$ have been chosen affinely independent, we can conclude that

$$\begin{aligned} \alpha_i^* &= 0 \quad \forall\, i \in \{1, \ldots, n^2 - n\},\ i \notin \mathcal{L} \\ \text{and} \quad \alpha_{L_j}^* + \sum_{l \in \mathcal{L}_j} \hat{\alpha}_l &= 0 \quad \forall\, j \in \{1, \ldots, s\}. \end{aligned} \tag{4.21}$$

Thus, equation 4.18 reduces to

$$\sum_{j=1}^{s}\Big(\alpha^*_{L_j}P^*_{L_j}+\sum_{l\in\mathcal{L}_j}\hat{\alpha}_l\hat{P}_l\Big)=0 \tag{4.22}$$

Now, consider the equations for the variables r_k ($k\in\{1,\dots,n\}$) induced by 4.22:

$$\sum_{j=1}^{s}\Big(\alpha^*_{L_j}r_k^{P^*_{L_j}}+\sum_{l\in\mathcal{L}_j}\hat{\alpha}_l r_k^{\hat{P}_l}\Big)=0$$

Let $k\in\mathcal{L}_s$. Then, by construction,
$r_k^{P^*_{L_s}}>0$ and $r_k^{\hat{P}_l}=2\,r_k^{P^*_{L_s}}>0\quad\forall\, l\in\mathcal{L}_s$;
furthermore $r_k^{P^*_{L_j}}=0$ and $r_k^{\hat{P}_l}=0\quad\forall\, l\in\mathcal{L}_j,\ j<s$.

Inserting this result into the equation above yields

$$\begin{aligned}\alpha^*_{L_s}r_k^{P^*_{L_s}}+\sum_{l\in\mathcal{L}_s}\hat{\alpha}_l r_k^{\hat{P}_l} &= 0\\ \overset{def.of\ r_k^{\hat{P}_l}}{\Rightarrow}\quad \alpha^*_{L_s}r_k^{P^*_{L_s}}+2\sum_{l\in\mathcal{L}_s}\hat{\alpha}_l r_k^{P^*_{L_s}} &= 0\end{aligned}$$

$$\overset{r_k^{P^*_{L_s}}>0}{\Leftrightarrow}\ \alpha^*_{L_s}+2\sum_{l\in\mathcal{L}_s}\hat{\alpha}_l=0\quad \overset{equ.4.21}{\Rightarrow}\ \sum_{l\in\mathcal{L}_s}\hat{\alpha}_l=0\quad\Rightarrow \alpha^*_{L_s}=0.$$

Next, let $k\in\mathcal{L}_{s-1}$. Then, by construction,
$r_k^{P^*_{L_{s-1}}}>0$ and $r_k^{\hat{P}_l}=2\,r_k^{P^*_{L_{s-1}}}>0\quad\forall\, l\in\mathcal{L}_{s-1}$;
furthermore $r_k^{P^*_{L_j}}=0$ and $r_k^{\hat{P}_l}=0\quad\forall\, l\in\mathcal{L}_j, j<s-1$,
and we obtain, similar to the above:

$$\begin{aligned}\alpha^*_{L_{s-1}}r_k^{P^*_{L_{s-1}}}+\sum_{l\in\mathcal{L}_{s-1}}\hat{\alpha}_l r_k^{\hat{P}_l}+\alpha^*_{L_s}r_k^{P^*_{L_s}}+\sum_{l\in\mathcal{L}_s}\hat{\alpha}_l r_k^{\hat{P}_l} &= 0\\ \overset{def.\ of\ r_k^{\hat{P}_l}}{\Longrightarrow}\ \alpha^*_{L_{s-1}}r_k^{P^*_{L_{s-1}}}+\sum_{l\in\mathcal{L}_{s-1}}\hat{\alpha}_l r_k^{\hat{P}_l}+\alpha^*_{L_s}r_k^{P^*_{L_s}}+\Big(\sum_{l\in\mathcal{L}_s}\hat{\alpha}_l\Big)2r_k^{P^*_{L_s}} &= 0\\ \overset{\sum_{l\in\mathcal{L}_s}\hat{\alpha}_l=0,\ \alpha^*_{L_s}=0}{\Longrightarrow}\ \alpha^*_{L_{s-1}}r_k^{P^*_{L_{s-1}}}+\sum_{l\in\mathcal{L}_{s-1}}\hat{\alpha}_l r_k^{\hat{P}_l} &= 0\\ \overset{as\ for\ k\in\mathcal{L}_s}{\Longrightarrow}\ \sum_{l\in\mathcal{L}_{s-1}}\hat{\alpha}_l=0\Rightarrow\alpha^*_{L_{s-1}} &= 0\end{aligned}$$

Repeatedly applying this argumentation to $\mathcal{L}_{s-2}, \ldots, \mathcal{L}_1$ gives $\alpha^*_{L_j} = 0$ and $\sum_{l \in \mathcal{L}_j} \hat{\alpha}_l = 0$ for all $j \in \{1, \ldots, s\}$. This means that we have reduced equation 4.22 to

$$\sum_{j=1}^{s} \sum_{l \in \mathcal{L}_j} \hat{\alpha}_l \hat{P}_l = 0 \tag{4.23}$$

$$\text{with} \sum_{l \in \mathcal{L}_j} \hat{\alpha}_l = 0 \quad \forall\, j \in \{1, \ldots, s\}. \tag{4.24}$$

Let $j = s$, $k \in \mathcal{L}_s$ and consider the equation above for the variable r_k: By construction, $r_k = 0$ for all $\hat{P}_l$, $l \in \mathcal{L}_j$, $j < s$, and $r_k^{P^*_{L_s}} > 0$. Consequently, we obtain:

$$0 = \sum_{l \in \mathcal{L}_s} \hat{\alpha}_l r_k^{\hat{P}_l} = r_k^{P^*_{L_s}} \Big(\sum_{\substack{l \in \mathcal{L}_s \\ l \neq k}} \hat{\alpha}_l + 2\hat{\alpha}_k \Big)$$

$$\overset{r_k^{P^*_{L_s}} > 0}{\Longrightarrow} \sum_{\substack{l \in \mathcal{L}_s \\ l \neq k}} \hat{\alpha}_l + 2\hat{\alpha}_k = 0 \overset{4.24}{\Longrightarrow} \hat{\alpha}_k = 0.$$

We can transfer this argument to $k \in \mathcal{L}_j$ with $j < s$ similar to the procedure when regarding the a^*_i - values, and finally obtain that $\hat{a}_k = 0$ for all $k \in \{1, \ldots, s\}$.

Hence, we have shown that the points $P^*_1, \ldots, P^*_{n^2-n}, \hat{P}_1, \ldots, \hat{P}_n$ are affinely independent in $\mathcal{P}_{pUHL}$. To complete this proof, consider the point $\hat{P}_{n+1}$ with $\hat{P}_{n+1} = P^*_1$ except for $z^{\hat{P}_{n+1}} = \frac{1}{2} z^{P^*_1}$. By choice of z in step 1, $\hat{P}_{n+1}$ stays feasible. Using a similar argumentation as above, it is easy to see that the points $P^*_1, \ldots, P^*_{n^2-n}, \hat{P}_1, \ldots, \hat{P}_{n+1}$ are affinely independent in $\mathcal{P}_{pUHL}$.

□

Remark 4.3.7 (Technical details of facet proofs)

Throughout section 4.4 and 4.5, several inequalities will be proven to be (non-) facet-defining. To prove that an inequality does not represent a facet, it suffices to give further equations that are fulfilled by all points lying in the corresponding face. In contrast, it requires quite a lot of technical effort to prove that a given inequality does in fact represent a facet. Assume that the set of decision variables is represented by the vector $\underline{x}$. Then, all following facet-proofs will proceed using the following scheme (which has already been applied in the proof of lemma 4.3.4):

1. (For non-elementary facets:) Show validity of the inequality.

2. Assume that a further equation $\underline{\pi}\,\underline{x} = \pi_0$ is valid for all points lying in the considered face:

   (a) With the help of a general linear combination of the given equations, derive dependencies between $\underline{\pi}$ and π_0 that have to hold.

   (b) Show that, if these dependencies hold, $\underline{\pi}\,\underline{x} = \pi_0$ reduces to a linear combination of the given equations.

   (c) Prove the dependencies between the scalars by constructing points of the face and inserting them into the equation $\underline{\pi}\,\underline{x} = \pi_0$.

The reader who is more interested in facet results than in technical details of the proofs is recommended to skip those (in the following, such proofs are marked by the symbol *). Nevertheless, to understand the meaning of an inequality, the validity proof of newly-derived inequalities is of good help; additionally, an interpretation of each class of newly-derived facets will be given in section 4.5.

4.4 Elementary Facets

In this section, we will examine the faces that are defined by constraints 4.2 - 4.9 and check which of these are facets. This knowledge gives first hints at the inequalities that still have to be tightened to obtain a better formulation of $\mathcal{P}_{pUHL}$.

4.4.1 Elementary $X_{i,\,k}$-Facets of $\mathcal{P}_{pUHL}$

First, we will consider those constraints that contain only the $X_{i,\,k}$-variables. As a first step, the idea of the proof of theorem 4.3.5 can be used in determining facets for $\mathcal{P}_{pUHL}$, as shown in the following:

Proposition 4.4.1

(i) Given $n^2 - n - 1$ affinely independent points
$P_1, \ldots, P_{n^2-n-1} \in \mathcal{P}_{pUFL}$,

one can construct n^2 affinely independent points
$P_1^*, \ldots, P_{n^2-n-1}^*, \hat{P}_1, \ldots, \hat{P}_{n+1} \in \mathcal{P}_{pUHL}$.

Consequently, every facet of $\mathcal{P}_{pUFL}$ corresponds to a facet of $\mathcal{P}_{pUHL}$.

(ii) Conversely, if an inequality in the variables $x_{i,\,k}$ does not define a facet of $\mathcal{P}_{pUFL}$, then the corresponding inequality in the variables $X_{i,\,k}$ does not define a facet of $\mathcal{P}_{pUHL}$.

Proof:

ad (i) Transfer the points $P_1, \ldots, P_{n^2-n-1}$ to n^2-n-1 affinely independent points $P_1^*, \ldots, P_{n^2-n-1}^*$ in $\mathcal{P}_{pUHL}$ as shown in proposition 4.3.1.

Construct $n+1$ additional points as shown in the proof of theorem 4.3.5. For this construction, note:
Similar to the proof of 4.3.5, we can assume that for every $k \in \{1, \ldots, n\}$, there exists $P_{i(k)} \in \{P_1, \ldots, P_{n^2+n-1}\}$ with $r_k > 0$: Otherwise, there would be $n-1 \geq 3$ additional equations[7] $X_{m,\,k} = 0 \;\forall\; m \neq k$ which are valid for all P_i, and hence, $P_1, \ldots, P_{n^2-n-1}$ cannot be affinely independent in $\mathcal{P}_{pUFL}$; contradiction.

ad (ii) If $\mathcal{F}_{pUFL}$ is a face of $\mathcal{P}_{pUFL}$ but no facet, there exists at least one additional equation (in the variables $x_{i,\,k}$) which is linearly independent from the given equations, and which is satisfied for all points in $\mathcal{F}_{pUFL}$. If $\mathcal{F}_{pUHL}$ denotes the corresponding face of $\mathcal{P}_{pUHL}$, it follows that there is at least one additional equation (in the variables $X_{i,\,k}$) which is linearly independent from the given equations and which is satisfied for all points in $\mathcal{F}_{pUHL}$. Hence, $\mathcal{F}_{pUHL}$ cannot be a facet of $\mathcal{P}_{pUHL}$.

□

Thus, when searching for facets of $\mathcal{P}_{pUHL}$ that only include $X_{i,\,k}$-variables, we can restrict ourselves to searching for facets of $\mathcal{P}_{pUFL}$.

First note that some of the elementary inequalities of (**pUHL**) do *not* represent facets:

[7] Note the general assumption that $n \geq 4$.

Proposition 4.4.2
The following valid inequalities of (**pUHL**) *do not represent facets of* $\mathcal{P}_{pUHL}$*:*

(a) $X_{kk} \geq 0 \quad \forall\, k$

(b) $X_{kk} \leq 1 \quad \forall\, k$

(c) $X_{ik} \leq 1 \quad \forall\, i \neq k$

Proof:

(a) $X_{k,\,k} = 0 \quad \Rightarrow \quad X_{i,\,k} = 0 \quad \forall\, i.$

(b) $X_{k,\,k} = 1 \quad \Rightarrow \quad X_{k,\,i} = 0 \quad \forall\, i \neq k.$

(c) $X_{i,\,k} = 1 \quad \Rightarrow \quad X_{k,\,k} = 1$ and $X_{i,\,m} = 0 \quad \forall\, m \neq k.$

□

The remaining constraints that include only $X_{i,\,k}$-variables can be shown to define facets:

Proposition 4.4.3
The following is a facet of $\mathcal{P}_{pUHL}$*:*
$\mathcal{F}_I := \{P \in \mathcal{P}_{pUHL} : X^P_{i,\,k} = 0\}$ *(for* $i, k \in \{1, \ldots, n\}$*,* $i \neq k$*)*

Proof: * (following the pattern given in remark 4.3.7) For ease of notation, $\mathcal{F}_I$ will in the following denote both the face defined in (**pUHL**) and its correspondence in (**pUFL**). It will become clear from the context which set is being referred to. W.l.o.g., $i = 1$ and $k = 2$. Assume that the equation

$$\sum_{j=1}^{n} \sum_{l=1}^{n} a_{j,\,l} x_{j,\,l} = d \tag{4.25}$$

holds for all (**pUFL**)-solutions $P \in \mathcal{F}_I$. Show:

1. For all j, l, m with $j \notin \{l, m\}$ and $(1, 2) \notin \{(j, l), (j, m)\}$: $a_{j,\,l} \stackrel{!}{=} a_{j,\,m} =: a_j$.
2. For all l, m: $a_{m,\,m} - a_m \stackrel{!}{=} a_{l,\,l} - a_l := a$.

*Technical proof; may be omitted.

If steps 1 and 2 have been shown, we can reformulate 4.25 as a linear combination of the given equations:

$$\begin{aligned}
& & d &= \sum_{j=1}^{n}\sum_{l=1}^{n} a_{j,\,l}x_{j,\,l} \\
&\overset{step\,1,\ \sum_k x_{j,\,k}=1}{\Leftrightarrow} & d-\sum_{j=1}^{n} a_j &= \sum_{j=1}^{n}(a_{j,\,j}-a_j)x_{j,\,j}+(a_{1,\,2}-a_1)x_{1,\,2} \\
&\overset{x_{1,\,2}=0\,\forall\,P\in\mathcal{F}_I}{\Rightarrow} & d-\sum_{j=1}^{n} a_j &= \sum_{j=1}^{n}(a_{j,\,j}-a_j)x_{j,\,j} \\
&\overset{step\,2,\ \sum_k x_{k,\,k}=p}{\Leftrightarrow} & d-\sum_{j=1}^{n} a_j - a\,p &= 0.
\end{aligned}$$

To conclude, we prove steps 1 and 2 by constructing appropriate points that lie in $\mathcal{F}_I$:

1. For all j, l, m with $j \notin \{l, m\}$ and $(1, 2) \notin \{(j, l), (j, m)\}$: $a_{j,\,l} \overset{!}{=} a_{j,\,m} (=: a_j)$:

   - If $l, m \neq 2$: Choose $s_1, \ldots, s_{p-2} \in \{1, \ldots, n\}\backslash\{l, m, j\}$ pairwise different (note that $p \leq n-2$) and set

     P_1: $x_{s_1,\,s_1} = \ldots = x_{s_{p-2},\,s_{p-2}} = 1$, $x_{l,\,l} = x_{m,\,m} = 1$,
     $x_{i,\,l} = 1 \quad \forall \quad i \in \{1, \ldots, n\}\backslash\{s_1, \ldots, s_{p-2}, l, m\}$,

     P_2: all values as in P_1, except $x_{j,\,l} = 0$, $x_{j,\,m} = 1$.

     Since $x_{1,\,2} = 0$ for both points, they lie in $\mathcal{F}_I$, and inserting them into 4.25 gives $a_{j,\,l} = a_{j,\,m}$.

   - If $l = 2$ (or $m = 2$): Construction as above, but choose $s_1 = 1$ to ensure that $x_{1,\,1} = 1$ (and thus $x_{1,\,2} = 0$).

2. For all l, m: $a_{m,\,m} - a_m \overset{!}{=} a_{l,\,l} - a_l (:= a)$:

   We have shown that, using step 1, we can transform 4.25 to

$$d - \sum_{j=1}^{n} a_j = \sum_{j=1}^{n} (a_{j,\,j} - a_j)x_{j,\,j} \tag{4.26}$$

   - If $l,\ m \neq 1$: Choose $s_1 = 1$, $s_2, \ldots, s_{p-1} \in \{2, \ldots, n\}\backslash\{l, m\}$ pairwise different (recall $p \leq n-2$) and use the points

P_1: $x_{s_1,\, s_1} = \ldots = x_{s_{p-1},\, s_{p-1}} = 1,\ x_{l,\, l} = 1,$
$x_{r,\, s_1} = 1 \quad \forall\, r \in \{1, \ldots, n\} \backslash \{s_1, \ldots, s_{p-1}, l\},$

P_2: all values as in P_1, except
$x_{l,\, l} = 0,\ x_{m,\, m} = 1,\ x_{l,\, s_1} = 1,\ x_{m,\, s_1} = 0.$

$P_1, P_2 \in \mathcal{F}_I$ since for both points, $x_{1,\, 2} = 0$.

- If $1 \in \{l, m\}$: W.l.o.g., $l = 1$. Construction as above, but choose $s_1, \ldots, s_{p-1} \in \{2, \ldots, n\} \backslash \{m\}$ with $s_1 \neq 2$.

□

Proposition 4.4.4
The following is a facet of $\mathcal{P}_{pUHL}$*:*
$\mathcal{F}_{II} := \{P \in \mathcal{P}_{pUHL} : X^P_{i,\, k} = X^P_{k,\, k}\}$ *(for* $i, k \in \{1, \ldots, n\},\ i \neq k$*)*

Proof: * Similar to the proof of 4.4.3, we will use $\mathcal{F}_{II}$ to denote both the face defined in (**pUHL**) and its correspondence in (**pUFL**).

W.l.o.g., $k = 1$ and $i = 2$. Assume that the equation

$$\sum_{j=1}^{n} \sum_{l=1}^{n} a_{j,\, l} x_{j,\, l} = d \tag{4.27}$$

holds for all (**pUFL**)-solutions $P \in \mathcal{F}_{II}$.

To show that 4.27 is a linear combination of equations 4.12, 4.14 and $x_{2,\, 1} = x_{1,\, 1}$, we have to prove the following three steps:

1. For all j, l, m with $j \notin \{l, m\}$ and $(2, 1) \notin \{(j, l), (j, m)\}$: $a_{j,\, l} \overset{!}{=} a_{j,\, m} =: a_j$.
2. For all $l, m \neq 1$: $a_{l,\, l} - a_l \overset{!}{=} a_{m,\, m} - a_m =: a$.
3. $a_{2,\, 1} - a_2 \overset{!}{=} -(a_{1,\, 1} - a_1 - a) =: b$.

Then, equation 4.27 can be transformed as follows:

$$\begin{aligned} d &= \sum_{j=1}^{n} \sum_{l=1}^{n} a_{j,\, l}\, x_{j,\, l} \\ \overset{step\,1}{\Longleftrightarrow} \quad d - \sum_{j=1}^{n} a_j &= \sum_{j=1}^{n} (a_{j,\, j} - a_j)\, x_{j,\, j} + (a_{2,\, 1} - a_2)\, x_{2,\, 1} \end{aligned}$$

*Technical proof; may be omitted.

$$\overset{step\,2}{\Longleftrightarrow} \quad d - \sum_{j=1}^{n} a_j - a\,p \;=\; (a_{1,1} - a_1 - a)\,x_{1,1} + (a_{2,1} - a_2)\,x_{2,1}$$

$$\overset{step\,3}{\Longleftrightarrow} \quad d - \sum_{j=1}^{n} a_j - a\,p \;=\; -b\,x_{1,1} + b\,x_{2,1},$$

showing that, indeed, 4.27 is a linear combination of the given equations.

Consider the steps that have to be proven:

1. For all j, l, m with $j \notin \{l, m\}$ and $(2,1) \notin \{(j,l),(j,m)\}$: $a_{j,l} \overset{!}{=} a_{j,m} (=: a_j)$.

   - If $l, m \neq 1$: Choose $s_1, \ldots, s_{p-2} \in \{2, \ldots, n\} \backslash \{l, m, j\}$ pairwise different (note $p \leq n-2$) and use the points

     P_1: $x_{s_1, s_1} = \ldots = x_{s_{p-2}, s_{p-2}} = 1$, $x_{l,l} = x_{m,m} = 1$,
     $x_{i,l} = 1 \quad \forall\, i \in \{1, \ldots, n\} \backslash \{s_1, \ldots, s_{p-2}, l, m\}$,
     P_2: all values as in P_1, except $x_{j,l} = 0$, $x_{j,m} = 1$,

     which lie in $\mathcal{F}_{II}$ since $x_{1,1} = x_{2,1} = 0$ for both points.

   - If $l = 1$ (or $m = 1$): Construction as above, but choose $s_1, \ldots, s_{p-2} \in \{1, \ldots, n\} \backslash \{l, m, j, 2\}$ to ensure that $x_{1,1} = x_{2,1} = 1$.

2. For all $l, m \neq 1$: $a_{l,l} - a_l \overset{!}{=} a_{m,m} - a_m (=: a)$:

   With step 1 being correct, we have arrived at

$$d - \sum_{j} a_j = \sum_{j=1}^{n} (a_{j,j} - a_j)\,x_{j,j} + (a_{2,1} - a_2)\,x_{2,1} \tag{4.28}$$

   Let $l, m \neq 1$. Then, $a_{l,l} - a_l \overset{!}{=} a_{m,m} - a_m$ can be shown by choosing $s_1, \ldots, s_{p-1} \in \{1, \ldots, n\} \backslash \{l, m, 1\}$ (note $p \leq n-2$) and using the points

   P_1: $x_{s_1, s_1} = \ldots = x_{s_{p-1}, s_{p-1}} = 1$, $x_{l,l} = 1$,
   $x_{r, s_1} = 1 \quad \forall\, r \in \{1, \ldots, n\} \backslash \{s_1, \ldots, s_{p-1}, l\}$,
   P_2: all values as in P_1, except
   $x_{l,l} = 0$, $x_{m,m} = 1$, $x_{l, s_1} = 1$, $x_{m, s_1} = 0$.

3. $a_{2,1} - a_2 \overset{!}{=} -(a_{1,1} - a_1 - a)$:

With step 2, we have arrived at

$$d - \sum_j a_j - a\,p = (a_{2,\,1} - a_2)\,x_{2,\,1} + (a_{1,\,1} - a_1 - a)\,x_{1,\,1} \tag{4.29}$$

With P_1 from step 1 (first case) inserted into this equation, we get that $d - \sum_j a_j - a\,p = 0$. If we then plug in a point of $\mathcal{F}_{II}$ with $x_{1,\,1} = x_{1,\,2} = 1$ (e. g. P_1 from step 1, second case), we obtain that
$(a_{2,\,1} - a_2) + (a_{1,\,1} - a_1 - a) = 0$.

□

4.4.2 Elementary Facets Involving r_k and z

In subsection 4.4.1, we have examined all inequality constraints of the radius formulation that contain only $X_{i,\,k}$-variables. Now, we focus on those constraints that include the variables r_k and z.

To start off, it is easy to see that constraint 4.8 does not represent a facet of $\mathcal{P}_{pUHL}$:

Proposition 4.4.5
The valid inequality $r_k \geq 0$ for (**pUHL**) *does not represent a facet of $\mathcal{P}_{pUHL}$.*

Proof: $r_k = 0 \overset{d_{i,\,k}>0\ \forall\, i\neq k}{\Rightarrow} X_{i,\,k} = 0 \quad \forall\, i \neq k.$ □

Next, we will show that constraint 4.2 does not define a facet either. A first hint can be the following result, which is stated in [16]:

Lemma 4.4.6 (Ernst et al., 2001 [16])
The following constraint is valid for (**pUHL**) *for all k, m:*

$$z \geq r_k + r_m + \alpha\, d_{k,\,m} + (1-\alpha)(1-X_{k,\,k}) \min_i d_{i,\,k} + (1-\alpha)(1-X_{m,\,m}) \min_i d_{i,\,m} \tag{4.30}$$

Ernst et al. [16] gave no proof of this statement; but since the proof can provide a deeper insight into the relation between inequalities 4.2 and 4.30, it is given here:

Proof:

1. Assume $X_{k,\,k} = X_{m,\,m} = 1$.
   Then, constraint 4.30 corresponds to constraint 4.2 and is thus valid for (**pUHL**).

2. Now, assume $X_{k,\,k} = 1$, $X_{m,\,m} = 0$.
   Then, by constraint 4.4, we have that $r_m = 0$. If $rhs(4.30)$ denotes the right hand side of inequality 4.30, we have that

$$\begin{aligned}
rhs(4.30) \quad &= \quad r_k + \alpha\, d_{k,\,m} + (1-\alpha)\min_i d_{i,\,m} \\
&\leq \quad r_k + \alpha\, d_{k,\,m} + (1-\alpha) d_{i_m,\,m} \quad (\text{where } X_{m,\,i_m} = 1) \\
&= \quad r_k + \alpha(d_{k,\,m} - d_{i_m,\,m}) + d_{i_m,\,m} \\
&\overset{\Delta-ineq.}{\leq} \quad r_k + \alpha\, d_{k,\,i_m} + d_{i_m,\,m} \\
&\overset{4.3}{\leq} \quad r_k + \alpha\, d_{k,\,i_m} + r_{i_m} \overset{4.2}{\leq} z.
\end{aligned}$$

   An analogous argumentation holds if $X_{m,\,m} = 1$, $X_{k,\,k} = 0$.

3. Finally, assume $X_{k,\,k} = X_{m,\,m} = 0$.
   Then, with i_m and i_k such that $X_{m,\,i_m} = X_{k,\,i_k} = 1$, we obtain, similar to the calculations above:

$$\begin{aligned}
rhs(4.30) \quad &= \quad \alpha\, d_{k,\,m} + (1-\alpha)\min_i d_{i,\,k} + (1-\alpha)\min_i d_{i,\,m} \\
&\leq \quad \alpha\, d_{k,\,m} + (1-\alpha) d_{i_k,\,k} + (1-\alpha) d_{i_m,\,m} \\
&= \quad \alpha(d_{k,\,m} - d_{i_k,\,k} - d_{i_m,\,m}) + d_{i_k,\,k} + d_{i_m,\,m} \\
&\overset{\Delta-ineq.}{\leq} \quad \alpha(d_{m,\,i_k} - d_{i_m,\,m}) + d_{i_k,\,k} + d_{i_m,\,m} \\
&\overset{\Delta-ineq.}{\leq} \quad \alpha\, d_{i_k,\,i_m} + d_{i_k,\,k} + d_{i_m,\,m} \overset{4.3}{\leq} r_{i_k} + \alpha\, d_{i_k,\,i_m} + r_{i_m} \overset{4.2}{\leq} z.
\end{aligned}$$

□

The proof above shows that, whenever hub k or hub m is not opened, there are other hubs l and/or o such that $z \geq r_k + r_m + \alpha\, d_{k,\,m}$ is dominated by $z \geq r_l + r_o + \alpha\, d_{l,\,o}$. This means that inequality 4.2 cannot represent a facet of $\mathcal{P}_{pUHL}$:

Proposition 4.4.7

Assume that $\alpha \in]\,0,\ 1[$. *Then, the valid inequality* $z \geq r_k + r_m + \alpha\, d_{k,\,m}$ *for* (**pUHL**) *does not represent a facet of* $\mathcal{P}_{pUHL}$.

Lemma 4.4.8 (Nemhauser and Wolsey, 1988 [25])
Let $\mathcal{P} = \{\underline{x} \in \mathbb{R}^n_+ : A\underline{x} \leq b\}$ *be a rational polyhedron and* $\mathcal{S} = \mathcal{P} \cap \mathbb{Z}^n \neq \emptyset$ *the set of integer points in* $\mathcal{P}$*. If* $\underline{\pi}\,\underline{x} \leq \pi_0$ *defines a face of dimension* $k-1$ *of* $conv(\mathcal{S})$*, there are* k *affinely independent points* $\underline{x}^1, \ldots, \underline{x}^k \in \mathcal{S}$ *such that* $\underline{\pi}\,\underline{x}^i = \pi_0$ *for* $i = 1, \ldots, k$*.*

Remark 4.4.9
The decisive point of lemma 4.4.8 is the fact that the affinely independent points $\underline{x}^1, \ldots, \underline{x}^k$ lie not only in $conv(\mathcal{S})$, but in $\mathcal{S}$. For our purpose, this means that the search for affinely independent points can be restricted to integral points.

Proof: [of proposition 4.4.7]
Assume that $\mathcal{F} := \{P \in \mathcal{P}_{pUHL} : z = r_k + r_m + \alpha\, d_{k,\,m}\}$ defines a facet. Lemma 4.4.8 can easily be adapted to the case that $\mathcal{P} = \mathcal{X}_{pUHL}$ and $\mathcal{S} = \mathcal{Z}_{pUHL}$; thus, there exist n^2 affinely independent points $P_1, \ldots, P_{n^2} \in \mathcal{F} \cap \mathcal{Z}_{pUHL}$. Assuming that $X_{k,\,k} = 1$ for all P_i, $i = 1, \ldots, n^2$, we would obtain that $\dim \mathcal{F} < n^2 - 1$, which means that $\mathcal{F}$ is not a facet. Contradiction. Thus, there exists P_i with $X_{k,\,k} = 0$. But then:

$$
\begin{aligned}
r_k + r_m + \alpha\, d_{k,\,m} &\overset{4.4}{=} r_m + \alpha\, d_{k,\,m} \\
&\overset{\alpha\in]0;1[,\ d_{i_k,\,k}>0}{<} r_m + \alpha\, d_{k,\,m} + (1-\alpha) d_{i_k,\,k} \\
&\qquad \text{(where } i_k \neq k \text{ such that } X_{k,\,i_k} = 1) \\
&= r_m + \alpha(d_{k,\,m} - d_{i_k,\,k}) + d_{i_k,\,k} \\
&\overset{\Delta-ineq.}{\leq} r_m + \alpha\, d_{m,\,i_k} + d_{i_k,\,k} \\
&\overset{4.3}{\leq} r_m + \alpha\, d_{m,\,i_k} + r_{i_k} \overset{4.2}{\leq} z.
\end{aligned}
$$

That is, $z > r_k + r_m + \alpha\, d_{k,\,m}$ for P_i, and thus, $P_i \notin \mathcal{F}$. Contradiction. □

However, inequality 4.30 still does not represent a facet of $\mathcal{P}_{pUHL}$; see section 4.5.2.

Next, we examine constraints 4.3:

Proposition 4.4.10
For fixed $k \in \{1, \ldots, n\}$*, let* $j \in \{1, \ldots, n\}$ *such that* $d_{j,\,k} := \max_i d_{i,\,k}$*.*
Then, $\mathcal{F}_{III} := \{P \in \mathcal{P}_{pUHL} : r_k = d_{j,\,k} X_{j,\,k}\}$ *is a facet of* $\mathcal{P}_{pUHL}$*.*

Proof: * W.l.o.g., $k = 1$ and $j = 2$, i.e. we deal with $\mathcal{F}_{III} = \{P \in \mathcal{P}_{pUHL} : r_1 = d_{2,\,1} X_{2,\,1}\}$. Assume that the equation

$$\sum_{i=1}^{n} \sum_{m=1}^{n} a_{i,\,m} X_{i,\,m} + \sum_{m=1}^{n} b_m r_m + c\,z = d \tag{4.31}$$

holds for all (**pUHL**)-solutions $P \in \mathcal{F}_{III}$. If we can show that

1. $c \overset{!}{=} 0$ and $b_m \overset{!}{=} 0 \quad \forall\, m \neq 1$
2. For all l, m, i with $i \notin \{l, m\}$ and $(2,1) \notin \{(i,l),(i,m)\}$: $a_{i,\,l} \overset{!}{=} a_{i,\,m} =: a_i$
3. For all l, m: $a_{l,\,l} - a_l \overset{!}{=} a_{m,\,m} - a_m =: a$
4. $a_{2,\,1} \overset{!}{=} a_2 - b_1 d_{2,\,1}$,

equation 4.31 can be written as a linear combination of the given equations for $\mathcal{F}_{III}$:

$$\begin{aligned}
d &= \sum_{i=1}^{n} \sum_{m=1}^{n} a_{i,\,m} X_{i,\,m} + \sum_{m=1}^{n} b_m r_m + c\,z \\
&\overset{\text{step 1}}{=} \sum_{i=1}^{n} \sum_{m=1}^{n} a_{i,\,m} X_{i,\,m} + b_1 r_1 \\
\overset{\text{step 2}}{\Longleftrightarrow}\; d - \sum_{i=1}^{n} a_i &= \sum_{m=1}^{n} (a_{m,\,m} - a_m) X_{m,\,m} + (a_{2,\,1} - a_2)\, X_{2,\,1} + b_1 r_1 \\
\overset{\text{step 3}}{\Longleftrightarrow}\; d - \sum_{i=1}^{n} a_i - a\,p &= (a_{2,\,1} - a_2)\, X_{2,\,1} + b_1 r_1 \\
&\overset{\text{step 4}}{=} b_1 (r_1 - d_{2,\,1} X_{2,\,1}).
\end{aligned}$$

ad 1: $c \overset{!}{=} 0$ and $b_m \overset{!}{=} 0 \quad \forall\, m \neq 1$:

- It is easy to construct $P_1, P_2 \in \mathcal{F}_{III}$ with $z^{P_2} = 2 z^{P_1}$, all other values equal. Comparing equation 4.31 for points P_1 and P_2 gives $c = 0$.
- Furthermore, for $m \geq 2$ fixed, one can easily construct $P_1, P_2 \in \mathcal{F}_{III}$ with $X^{P_1}_{m,\,m} = X^{P_2}_{m,\,m} = 1$, $r^{P_1}_m = \max_i d_{i,\,m}$, $r^{P_2}_m = 2 r^{P_1}_m$, all other values equal (choose z large enough). Comparing equation 4.31 for points P_1 and P_2 then gives $b_m = 0$.

*Technical proof; may be omitted.

ad 2: For all l, m, i with $i \notin \{l, m\}$ and $(2,1) \notin \{(i,l),(i,m)\}$: $a_{i,\,l} \overset{!}{=} a_{i,\,m} (=: a_i)$:

Note that there is nothing to prove unless $l \neq m$. Thus, we can assume w.l.o.g. that $l \neq 1$.

By step 1, 4.31 can be reformulated as

$$d = \sum_{i=1}^{n} \sum_{m=1}^{n} a_{i,\,m} X_{i,\,m} + b_1 r_1. \tag{4.32}$$

Choose $s_1, \ldots, s_{p-2} \in \{2, \ldots, n\} \backslash \{l, m, i\}$ pairwise different (note $p \leq n-2$) and compare equation 4.32 for the points

P_1: $X_{s_1,\,s_1} = \ldots = X_{s_{p-2},\,s_{p-2}} = 1$, $X_{l,\,l} = X_{m,\,m} = 1$,
$X_{j,\,l} = 1 \quad \forall\, j \in \{1, \ldots, n\} \backslash \{s_1, \ldots, s_{p-2}, l, m\}$,
$r_1 = 0$, $r_l = \max_j d_{j,\,l}$, $z = \max_{s,o}(r_s + r_o + \alpha\, d_{s,\,o})$,
all other values equal to zero,

P_2: all values equal to those of P_1, except $X_{i,\,m} = 1$, $X_{i,\,l} = 0$, $r_m = d_{i,\,m}$.

(Both points satisfy $r_1 = X_{2,\,1} = 0$ and thus lie in $\mathcal{F}_{III}$.)

ad 3: For all l, m: $a_{l,\,l} - a_l \overset{!}{=} a_{m,\,m} - a_m (=: a)$:

By steps 1 and 2,

$$d - \sum_i a_i = \sum_m (a_{m,\,m} - a_m) X_{m,\,m} + (a_{2,\,1} - a_2)\, X_{2,\,1} + b_1 r_1. \tag{4.33}$$

Choose $s_1, \ldots, s_{p-1} \in \{2, \ldots, n\} \backslash \{l, m\}$ pairwise different (note $p \leq n-2$) and insert the following two points into 4.33:

P_1: $X_{s_1,\,s_1} = \ldots = X_{s_{p-1},\,s_{p-1}} = 1$, $X_{l,\,l} = 1$,
$X_{i,\,s_1} = 1 \quad \forall\, i \in \{1, \ldots, n\} \backslash \{s_1, \ldots, s_{p-1}, l\}$,
$r_{s_1} = \max_i d_{i,\,s_1}$, $r_1 = 0$, $z = \max_{s,o}(r_s + r_o + \alpha\, d_{s,\,o})$,
all other values equal to zero,

P_2: all values equal to those of P_1, except
$X_{l,\,l} = 0$, $X_{m,\,m} = 1$, $X_{m,\,s_1} = 0$, $X_{l,\,s_1} = 1$.

ad 4: $a_{2,\,1} \overset{!}{=} a_2 - b_1 d_{2,\,1}$:

Using all results from above, we can reformulate 4.33 as

$$d - \sum_i a_i - ap = (a_{2,\,1} - a_2) X_{2,\,1} + b_1 r_1 \tag{4.34}$$

Inserting point P_1 from step 2 yields $lhs(4.34) = 0$. If we finally plug in $P \in \mathcal{F}_{III}$ to 4.34, with

$$\begin{aligned} P\colon \quad & X_{1,\,1} = X_{3,\,3} = \ldots = X_{p+1,\,p+1} = 1, \\ & X_{i,\,1} = 1 \;\forall\, i \in \{2, p+2, \ldots, n\},\; r_1 = d_{2,\,1}, \end{aligned}$$

we obtain that $a_{2,\,1} - a_2 + b_1\, d_{2,1} = 0$.

□

Remark 4.4.11

If $d_{j,\,k} < \max_i d_{i,\,k}$, then $r_k = d_{j,\,k} X_{j,\,k}$ implies that $X_{i,\,k} = 0$ for all i with $d_{i,\,k} > d_{j,\,k}$. Consequently, we cannot find n^2 affinely independent points satisfying $r_k \geq d_{j,\,k} X_{j,\,k}$ with equality, and thus, this inequality does not represent a facet.

Finally, note that the newly-introduced constraint 4.4 defines a facet of (**pUHL**):

Proposition 4.4.12

For any k, $\mathcal{F}_{IV} := \{P \in \mathcal{P}_{pUHL} : r_k = MX_{k,\,k}\}$ is a facet of $\mathcal{P}_{pUHL}$.

Proof: * W.l.o.g., $k = 1$, i. e. we deal with $\mathcal{F}_{IV} = \{P \in \mathcal{P}_{pUHL} : r_1 = MX_{1,\,1}\}$. Assume that the equation

$$\sum_{i=1}^{n} \sum_{m=1}^{n} a_{i,\,m} X_{i,\,m} + \sum_{m=1}^{n} b_m r_m + cz = d \tag{4.35}$$

holds for all (**pUHL**)-solutions $P \in \mathcal{F}_{IV}$.

It suffices to show:

1. $c \overset{!}{=} 0$ and $b_m \overset{!}{=} 0 \;\forall\, m \geq 2$
2. For all l, m, i with $i \notin \{l, m\}$: $a_{i,\,l} \overset{!}{=} a_{i,\,m} =: a_i$

*Technical proof; may be omitted.

3. For all $l, m \geq 2$: $a_{l,\,l} - a_l \stackrel{!}{=} a_{m,\,m} - a_m =: a$

4. $a_{1,\,1} \stackrel{!}{=} a_1 + a - b_1 M$.

Then we can write

$$\begin{aligned}
d &= \sum_{i=1}^{n}\sum_{m=1}^{n} a_{i,\,m} X_{i,\,m} + \sum_{m=1}^{n} b_m r_m + cz \\
&\stackrel{step\,1}{=} \sum_{i=1}^{n}\sum_{m=1}^{n} a_{i,\,m} X_{i,\,m} + b_1 r_1 \\
\stackrel{step\,2}{\Longleftrightarrow} d - \sum_i a_i &= \sum_m (a_{m,\,m} - a_m) X_{m,\,m} + b_1 r_1 \\
\stackrel{step\,3}{\Longleftrightarrow} d - \sum_i a_i - a\,p &= (a_{1,\,1} - a_1 - a) X_{1,\,1} + b_1 r_1 \stackrel{step\,4}{=} b_1 (r_1 - M\,X_{1,\,1}).
\end{aligned}$$

ad 1: $c \stackrel{!}{=} 0$ and $b_m \stackrel{!}{=} 0\ \forall\ m \geq 2$:

Similar to step 1 in the proof of 4.4.10.

ad 2: For all l, m, i with $i \notin \{l, m\}$: $a_{i,\,l} \stackrel{!}{=} a_{i,\,m} (=: a_i)$:

By step 1, equation 4.35 reduces to

$$d = \sum_{i=1}^{n}\sum_{m=1}^{n} a_{i,\,m} X_{i,\,m} + b_1 r_1. \tag{4.36}$$

- If $l, m \neq 1$: Choose $s_1, \ldots, s_{p-2} \in \{2, \ldots, n\} \backslash \{i, l, m\}$ pairwise different and use the points

  P_1: $X_{s_1,\,s_1} = \ldots = X_{s_{p-2},\,s_{p-2}} = 1$, $X_{l,\,l} = X_{m,\,m} = 1$,
  $X_{j,\,l} = 1\ \forall\ j \in \{1, \ldots, n\} \backslash \{s_1, \ldots, s_{p-2}, l, m\}$,
  $r_l = M$, $r_1 = 0$, $z = \max_{s,o} r_s + r_o + \alpha d_{s,\,o}$,
  all other values equal to zero,

  P_2: all values equal to those of P_1, except
  $X_{1,\,l} = 0$, $X_{1,\,m} = 1$, $r_m = d_{1,\,m}$.

- If $l = 1$: Similar construction as above, but set $r_l (= r_1) = M$ to ensure that $r_1 = M X_{1,\,1}$. Analogous for $m = 1$.

ad 3: For all $l, m \geq 2$: $a_{l,\,l} - a_l \stackrel{!}{=} a_{m,\,m} - a_m (=: a)$:

With step 2, equation 4.36 can be further reduced to

$$d - \sum_i a_i = \sum_m (a_{m,\,m} - a_m) X_{m,\,m} + b_1 r_1. \tag{4.37}$$

Choose $s_1, \ldots s_{p-1} \in \{1, \ldots, n\} \backslash \{l, m, 1\}$ pairwise different and use the points

P_1: $X_{s_1,\,s_1} = \ldots = X_{s_{p-1},\,s_{p-1}} = 1$, $X_{l,\,l} = 1$
$X_{i,\,s_1} = 1 \;\forall\, i \in \{1, \ldots, n\} \backslash \{s_1, \ldots, s_{p-1}, l\}$,
$r_{s_1} = \max_i d_{i,\,s_1}$, $r_1 = 0$, $z = \max_{s,o} r_s + r_o + \alpha\, d_{s,\,o}$,
all other values equal to zero,

P_2: all values equal to those of P_1, except
$X_{l,\,l} = 0$, $X_{m,\,m} = 1$, $X_{m,\,s_1} = 0$, $X_{l,\,s_1} = 1$.

ad 4: $a_{1,\,1} \stackrel{!}{=} a_1 + a - b_1 M$:

Using all results from above, we can reformulate 4.37 as

$$d - \sum_i a_i - a\,p = (a_{1,\,1} - a_1 - a) X_{1,\,1} + b_1 r_1 \tag{4.38}$$

Plugging in P_1 from step 3 yields $lhs(4.38) = 0$. If we finally insert $P \in \mathcal{F}_{IV}$ into 4.38, with

P: $X_{1,\,1} = \ldots = X_{p,\,p} = 1$, $X_{i,\,1} = 1 \;\forall\, i \in \{p+1, \ldots, n\}$, $r_1 = M$,
we get that $(a_{1,\,1} - a_1 - a) + b_1 M = 0$.

□

4.5 Non-elementary Facets of $\mathcal{P}_{pUHL}$

In this section, which can be seen as the core of the polyhedral analysis, several classes of non-elementary facets of $\mathcal{P}_{pUHL}$ will be presented. To deduce new valid inequalities for our problem, the PORTA tool (**Po**lyhedron **R**epresentation **T**ransformation **A**lgorithm, see [9]) has been used: This tool calculates all facet-defining inequalities for given small problem instances. Once such calculations have been carried out for different small hub location examples, patterns among the results have been identified and facet-defining inequalities have been deduced.

4.5.1 Facets in the Variables $X_{i,\,k}$

For the derivation of non-elementary facets in the variables $X_{i,\,k}$, one can once again make use of proposition 4.4.1, which allows to restrict to facets of the (**pUFL**)-polyhedron.

Spoke-concentration Facets

Theorem 4.5.1 (Spoke-concentration Facets)
Let $k \in \{1, \ldots, n\}$. The inequality

$$(n-p)X_{k,\,k} \geq \sum_{\substack{i=1 \\ i\neq k}}^{n} X_{i,\,k} \tag{4.39}$$

is valid for $\mathcal{P}_{pUHL}$ and represents a facet.

Proof: [*] Due to proposition 4.4.1, it suffices to show both validity and the facet-defining property of the inequality for the polyhedron $\mathcal{P}_{pUFL}$.

Validity:

1. Assume $x_{k,\,k} = 0$. Then, trivially, inequality 4.39 holds (with equality).

2. Now, assume $x_{k,\,k} = 1$. Since the number of hubs is fixed to p, only $n-1-(p-1) = n-p$ of the nodes $1, \ldots, k-1, k+1, \ldots, n$ are spokes and could thus be allocated to k. Thus,
   $\sum_{i=1, i\neq k}^{n} x_{i,\,k} \leq n-p = (n-p)\, x_{k,\,k}$.

Facet-defining:
For ease of notation, assume without loss of generality that $k = 1$. Set

$$\mathcal{F} = \{P \in \mathcal{P}_{pUFL} : (n-p)x_{1,\,1} = \sum_{i=2}^{n} x_{i,\,1}\}.$$

Assume that there is a further equation

$$\sum_{i=1}^{n}\sum_{k=1}^{n} a_{i,\,k}x_{i,\,k} = d \tag{4.40}$$

[*]Technical proof; may be omitted.

that is satisfied by all points in $\mathcal{F}$. We will show that this is a linear combination of the given equations. For this sake, we prove the following:

1. For all i, k, l with $k, l \notin \{1, i\}$: $a_{i,\,k} \overset{!}{=} a_{i,\,l} =: a_i$.
2. For all $k, l \geq 2$: $a_{k,\,k} + a_k \overset{!}{=} a_{l,\,l} + a_l =: a$.
3. For all $i, j \geq 2$: $a_{i,\,1} - a_i \overset{!}{=} a_{j,\,1} - a_j =: b$.
4. $a_{1,\,1} \overset{!}{=} a + a_1 - b\,(n-p)$.

Once we have proven the above, we can reformulate 4.40 as follows:

$$
\begin{aligned}
d &= \sum_{i=1}^{n}\sum_{k=1}^{n} a_{i,\,k} x_{i,\,k} \\
&\overset{1}{=} \sum_{i=1}^{n} a_i \sum_{\substack{k=2\\ k\neq i}}^{n} x_{i,\,k} + \sum_{k=1}^{n} a_{k,\,k} x_{k,\,k} + \sum_{i=2}^{n} a_{i,\,1} x_{i,\,1} \\
\overset{\sum_k x_{i,\,k}=1}{\Leftrightarrow} \quad d - \sum_{i=1}^{n} a_i &= \sum_{k=1}^{n} (a_{k,\,k} - a_k) x_{k,\,k} + \sum_{i=2}^{n} (a_{i,\,1} - a_i) x_{i,\,1} \\
&\overset{2}{=} a \sum_{k=2}^{n} x_{k,\,k} + (a_{1,\,1} - a_1) x_{1,\,1} + \sum_{i=2}^{n} (a_{i,\,1} - a_i) x_{i,\,1} \\
\overset{\sum_k x_{k,\,k}=p}{\Leftrightarrow} \quad d - \sum_{i=1}^{n} a_i - a\,p &= (a_{1,\,1} - a_1 - a) x_{1,\,1} + \sum_{i=2}^{n} (a_{i,\,1} - a_i) x_{i,\,1} \\
&\overset{3,\,4}{=} -b\,(n-p)\,x_{1,\,1} + b \sum_{i=2}^{n} x_{i,\,1} \\
&= b\big(rhs(4.39) - lhs(4.39)\big)
\end{aligned}
$$

which shows that equation 4.40 is a linear combination of the given equations. Steps 1 to 4 will now be proven one by one:

ad 1: For all i, k, l with $k, l \notin \{1, i\}$: $a_{i,\,k} \overset{!}{=} a_{i,\,l} (=: a_i)$:

Choose $s_1, \ldots, s_{p-2} \in \{2, \ldots, n\} \backslash \{i, l, k\}$ pairwise different (note $p \leq n-2$) and consider

P_1 : $x_{s_1,\, s_1} = \ldots = x_{s_{p-2},\, s_{p-2}} = x_{k,\, k} = x_{l,\, l} = 1,$
$x_{j,\, k} = 1$ for all $j \notin \{s_1, \ldots, s_{p-2}, k, l\},$
P_2 : all values as in P_1, except $x_{i,\, k} = 0$, $x_{i,\, l} = 1$.

It is easy to see that $P_1, P_2 \in \mathcal{F}$, since for both points, we have that $x_{1,\, 1} = x_{i,\, 1} = 0 \;\forall\, i = 1, \ldots, n$. Inserting P_1 and P_2 into 4.40 yields $a_{i,\, k} = a_{i,\, l}$.

ad 2: For all $k, l \geq 2$: $a_{k,\, k} + a_k \overset{!}{=} a_{l,\, l} + a_l (=: a)$:

Using the result of 1, we can reformulate 4.40 as

$$\sum_{k=2}^{n} (a_{k,\, k} - a_k) x_{k,\, k} + (a_{1,\, 1} - a_1) x_{1,\, 1} + \sum_{i=2}^{n} (a_{i,\, 1} - a_i) x_{i,\, 1} = d - \sum_{i=1}^{n} a_i \quad (4.41)$$

(as shown above). Choose $s_1, \ldots, s_{p-1} \in \{2, \ldots, n\} \backslash \{l, k\}$ pairwise different (note $p \leq n - 2$) and set

P_1 : $x_{s_1,\, s_1} = \ldots = x_{s_{p-1},\, s_{p-1}} = x_{k,\, k} = 1,$
$x_{j,\, s_1} = 1$ for all $j \notin \{s_1, \ldots, s_{p-1}, k\},$
P_2 : all values as in P_1, except $x_{k,\, k} = 0$, $x_{l,\, l} = 1$, $x_{l,\, s_1} = 0$, $x_{k,\, s_1} = 1$.

As in 1, both P_1 and P_2 lie in $\mathcal{F}$, since for both points, node 1 is a spoke. Inserting P_1 and P_2 into equation 4.41 gives $a_{k,\, k} + a_k = a_{l,\, l} + a_l$.

ad 3: For all $i, j \geq 2$: $a_{i,\, 1} - a_i \overset{!}{=} a_{j,\, 1} - a_j (=: b)$:

As shown above, equation 4.41 can now be further reduced to

$$(a_{1,\, 1} - a_1 - a) x_{1,\, 1} + \sum_{i=2}^{n} (a_{i,\, 1} - a_i) x_{i,\, 1} = d - \sum_{i=1}^{n} a_i - a\, p. \qquad (4.42)$$

Choose $s_1, \ldots, s_{p-2} \in \{2, \ldots, n\} \backslash \{i, j\}$ pairwise different (note $p \leq n - 2$) and set

P_1 : $x_{1,\, 1} = x_{i,\, i} = x_{s_1,\, s_1} = \ldots = x_{s_{p-2},\, s_{p-2}} = 1,$
$x_{h,\, 1} = 1$ for all $h \notin \{s_1, \ldots, s_{p-2}, 1, i\},$
P_2 : all values as in P_1, except $x_{i,\, i} = 0$, $x_{j,\, j} = 1$, $x_{i,\, 1} = 1$, $x_{j,\, 1} = 0$.

For both P_1 and P_2, hub node 1 is open, and there are exactly $n - p$ spokes allocated to 1, such that both points lie in $\mathcal{F}$. Inserting P_1 and P_2 into equation 4.42 gives $a_{i,\, 1} - a_i = a_{j,\, 1} - a_j$.

ad 4: $a_{1,\,1} \stackrel{!}{=} a + a_1 - b(n-p)$:

With step 3, equation 4.42 can be reduced to

$$(a_{1,\,1} - a_1 - a)x_{1,\,1} + b\sum_{i=2}^{n} x_{i,\,1} = d - \sum_{i=1}^{n} a_i - a\,p \tag{4.43}$$

Now, for P_1 from step 1, for instance, we have that $x_{1,\,1} = x_{i,\,1} = 0\ \forall\ i$, and thus, $rhs(4.43) = 0$. Furthermore, plugging in $\sum_{i\geq 2} x_{i,\,1} = (n-p)x_{1,\,1}$ (which is valid due to the definition of $\mathcal{F}$) yields

$$\big(b\,(n-p) + (a_{1,\,1} - a_1 - a)\big)x_{1,\,1} = 0$$

Since there are points in $\mathcal{F}$ with $x_{1,\,1} = 1$ (e. g. P_1 from step 3), we finally obtain $a_{1,\,1} = a + a_1 - b\,(n-p)$.

□

Remark 4.5.2 (Interpretation of the facet)
Having a closer look at the facets in theorem 4.5.1, we can distinguish between two types of points that satisfy the facet-inequality with equality:

1. Points with $X_{k,\,k} = 0$ (and thus, of course, $X_{i,\,k} = 0\ \forall\ i$).
2. Points with $X_{k,\,k} = 1$. In this case, to fulfill the facet-inequality with equality, we are forced to assign *every* spoke to the hub in k.

Thus, the facets of theorem 4.5.1 represent all points with "trivial" spoke allocation in the sense that all spokes are allocated to one single hub. Hence, we will refer to these facets as *"spoke-concentration facets"*.

Focus-element Facets

Next, another class of facet-defining inequalities in the variables $X_{i,\,k}$ will be presented:

Theorem 4.5.3 (Focus-element Facets I)
Let $A \subset \{1,\ldots,n\}$ with $|A| = n-p$ and $a \in A$ an element of A. Then,

$$\sum_{i\in A} X_{i,\,i} \geq \sum_{j\notin A} X_{j,\,a} + \sum_{i\in A\setminus\{a\}} X_{a,\,i} \tag{4.44}$$

is a valid inequality for (**pUHL**) *and defines a facet.*

Proof: As before, we restrict ourselves to showing validity and the facet-defining property for $\mathcal{P}_{pUFL}$.

Without loss of generality, assume that $A = \{1, \ldots, n-p\}$ and $a = 1$. Then, inequality 4.44 (for $\mathcal{P}_{pUFL}$) can be written as

$$\sum_{k=1}^{n-p} x_{k,\,k} \geq \sum_{i=n-p+1}^{n} x_{i,\,1} + \sum_{k=2}^{n-p} x_{1,\,k} \tag{4.45}$$

Validity:

1. Assume $x_{1,\,1} = 0$. Then, of course, $\sum_{i=n-p+1}^{n} x_{i,\,1} = 0$. Furthermore, due to constraint 4.13, $x_{1,\,k} \leq x_{k,\,k}$, and thus:

$$rhs(4.45) = \sum_{k=2}^{n-p} x_{1,\,k} \leq \sum_{k=2}^{n-p} x_{k,\,k} = lhs(4.45).$$

2. Now, assume $x_{1,\,1} = 1$. Then obviously $\sum_{k=2}^{n-p} x_{1,\,k} = 0$. Let $\sum_{k=2}^{n-p} x_{k,\,k} = l$ be the number of hubs among nodes $2, \ldots, n-p$. $l \in \{0, \ldots, p-1\}$ obvious. Since the total number of hubs is p, there are exactly $(p-1)-l$ hubs among nodes $n-p+1, \ldots, n$, i.e. $\sum_{k=n-p+1}^{n} x_{k,\,k} = p-1-l$. Consequently,

$$\sum_{i=n-p+1}^{n} x_{i,\,1} \leq \underbrace{p}_{=|\{n-p+1,\ldots,n\}|} - \underbrace{(p-1-l)}_{\substack{\#\text{ nodes which are}\\ \text{themselves hubs}}} = l+1$$

Thus,

$$lhs(4.45) = \underbrace{1}_{x_{1,\,1}} + \underbrace{l}_{\sum_{k=2}^{n-p} x_{k,\,k}} \geq \sum_{i=n-p+1}^{n} x_{i,\,1} = rhs(4.45).$$

Facet-defining:
(Will be shown for a more general facet class in theorem 4.5.6.)

□

Remark 4.5.4 (Graphical Interpretation)

Figure 4.1 gives a graphical interpretation of the focus-element facets presented in theorem 4.5.3. The elements of A are marked by black nodes, all elements in $\bar{A} = \{1, \ldots, n\} \backslash A$ are represented as white nodes. We consider three sums:

- $\sum_{i \in A} X_{i,\,i}$ counts the hubs in A
- $\sum_{j \notin A} X_{j,\,a}$ counts the spokes in $\bar{A}$ which are allocated to a (doubled arrows)
- $\sum_{i \in A \backslash \{a\}} X_{a,\,i}$ adds an additional 1 if a is not a hub, but allocated to a hub in A (dashed arrows)

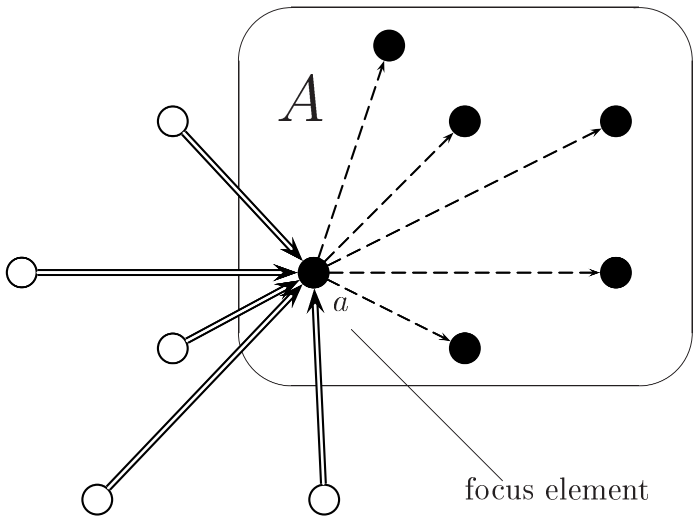


Figure 4.1: Graphical interpretation of focus-element facets I

There exist three possibilities to achieve equality in 4.44:

1. $\sum_{i \in A} X_{i,\,i} = 0$, i. e. there are no hubs in A. Then, by cardinality of A, all nodes in $\bar{A}$ have to be hubs.
2. $\sum_{i \in A} X_{i,\,i} \geq 1$, i. e. there are hubs in A.
   (a) If a is not a hub, there exists exactly one hub $b \neq a$ in A, and a is allocated to b.

(b) If a is a hub, then there is a spoke in $\bar{A}$ that is allocated to a. For any further hub in A, there is an additional spoke in $\bar{A}$ allocated to a.

Since our consideration of the node set A focusses at element a (see figure 4.1), we refer to a as *"focus element"*.

Remark 4.5.5 (Number of facets in facet class)
The above inequalities have been shown to define facets only for the representative case that $A = \{1, \ldots, n-p\}$ and $a = 1$. Note that there are actually $\binom{n}{n-p} \cdot (n-p) = \binom{n}{p} \cdot (n-p)$ different facets that are described by 4.44. Section 5.2 examines the corresponding separation problem for this facet class.

In fact, the focus-element facet class can still be further generalized, as described in the following:

1. The set A will be subdivided into two sets A^* and $\bar{A}^* = A \backslash A^* \ni a$. The addends $X_{a,\,i}$ from equation 4.44 will only be considered for $i \in \bar{A}^*$.

2. For the elements $k \in A^*$, additional addends $X_{b_k,\,k}$ will be introduced, where $b_k \in \bar{A}$.

Theorem 4.5.6 (Focus-element facets II)
Let $A \subset \{1, \ldots, n\}$ *with* $|A| = n - p$,
$a \in A$ *an element of* A,
$A^* \subset A$ *a subset of* $A \backslash \{a\}$ *with* $|A^*| \in \{0, \ldots, n-p-2\}$,
$\{b_k : k \in A^*\}$ *elements of* $\bar{A} := \{1, \ldots, n\} \backslash A$, *pairwise different. Then,*

$$\sum_{i \in A} X_{i,\,i} \geq \sum_{j \in \bar{A}} X_{j,\,a} + \sum_{i \in A \backslash (\{a\} \cup A^*)} X_{a,\,i} + \sum_{k \in A^*} X_{b_k,\,k} \tag{4.46}$$

is a valid inequality for (**pUHL**) *and defines a facet.*

Proof: * It suffices to show validity and the facet-defining property for $\mathcal{P}_{pUFL}$.

Without loss of generality, assume that $A = \{1, \ldots, n-p\}$, $a = 1$ and $A^* = \{2, \ldots, t\}$ with $t \leq n - p - 1$ ($A^* = \emptyset$ possible). Then, inequality 4.46 can be written as

$$\sum_{k=1}^{n-p} x_{k,\,k} \geq \sum_{i=n-p+1}^{n} x_{i,\,1} + \sum_{k=t+1}^{n-p} x_{1,\,k} + \sum_{k=2}^{t} x_{b_k,\,k}. \tag{4.47}$$

*Technical proof; may be omitted.

Validity:

1. Assume $x_{1,\,1} = 0$. Then,

$$\begin{aligned} lhs(4.47) &= \sum_{k=2}^{t} x_{k,\,k} + \sum_{k=t+1}^{n-p} x_{k,\,k} \\ rhs(4.47) &= \sum_{k=2}^{t} x_{b_k,\,k} + \sum_{k=t+1}^{n-p} x_{1,\,k} \end{aligned}$$

and validity follows since $x_{b_k,\,k} \le x_{k,\,k}$ and $x_{1,\,k} \le x_{k,\,k}$ for all k.

2. Now, assume $x_{1,\,1} = 1$. Set $\sum_{k=2}^{n-p} x_{k,\,k} =: s$. Then, due to $\sum_{k=1}^{n} x_{k,\,k} = p$ and $x_{1,\,1} = 1$, we have that $\sum_{k=n-p+1}^{n} x_{k,\,k} = p - s - 1$, and consequently,

$$\begin{aligned} rhs(4.47) &= \sum_{k=2}^{t} x_{b_k,\,k} + \sum_{i=n-p+1}^{n} x_{i,\,1} \\ &= \sum_{k=2}^{t} (x_{b_k,\,k} + x_{b_k,\,1}) + \sum_{\substack{i=n-p+1 \\ i \notin \{b_2,\dots,b_t\}}}^{n} x_{i,\,1} \\ &\overset{(*)}{\le} \underbrace{\sum_{k=2}^{t} (1 - x_{b_k,\,b_k})}_{\#\text{ spokes in } \{b_2,\dots,b_t\}} + \underbrace{\sum_{\substack{i=n-p+1 \\ i \notin \{b_2,\dots,b_t\}}}^{n} (1 - x_{i,\,i})}_{\substack{\#\text{ spokes in} \\ \{n-p+1,\dots,n\} \setminus \{b_2,\dots,b_t\}}} \\ &= \underbrace{p}_{\substack{\#\text{ nodes in} \\ \{n-p+1,\dots,n\}}} - \underbrace{(p - s - 1)}_{\substack{\#\text{ hubs in} \\ \{n-p+1,\dots,n\}}} = s + 1 = lhs(4.47), \end{aligned}$$

where (*) is valid since $x_{b_k,\,k} + x_{b_k,\,1} \le 1$ and $x_{b_k,\,k} = x_{b_k,\,1} = 0$ if b_k is a hub[8].

Facet-defining:
Let $\mathcal{F} := \left\{ P \in \mathcal{P}_{pUFL} : \sum_{k=1}^{n-p} x_{k,\,k} = \sum_{i=n-p+1}^{n} x_{i,\,1} + \sum_{k=t+1}^{n-p} x_{1,\,k} + \sum_{k=2}^{t} x_{b_k,\,k} \right\}$.
Assume that

$$\sum_{i=1}^{n} \sum_{k=1}^{n} a_{i,\,k} x_{i,\,k} = d \tag{4.48}$$

is a further equation that is satisfied by all points in $\mathcal{F}$; show:

[8] For the argumentation above, note futhermore that the b_k were assumed to be pairwise different.

1. (a) For $k, l = n-p+1, \ldots, n$: $a_{1,\,k} \overset{!}{=} a_{1,\,l} =: a_1$.

   (b) For $k = 2, \ldots, t$: $a_{1,\,k} \overset{!}{=} a_1$.

   (c) For $i = 2, \ldots, n-p$ and $k, l \in \{1, \ldots, n\} \backslash \{i\}$: $a_{i,\,k} \overset{!}{=} a_{i,\,l} =: a_i$.

   (d) For $i = n-p+1, \ldots, n$ and $k, l \in \{t+1, \ldots, n\} \backslash \{i\}$: $a_{i,\,k} \overset{!}{=} a_{i,\,l} =: a_i$.

   (e) For $i \in \{n-p+1, \ldots, n\} \backslash \{b_2, \ldots, b_t\}$ and $k = 2, \ldots, t$: $a_{i,\,k} \overset{!}{=} a_i$.

2. For $k, l = n-p+1, \ldots, n$: $a_{k,\,k} - a_k \overset{!}{=} a_{l,\,l} - a_l =: a$.

3. (a) For $i, j = n-p+1, \ldots, n$: $a_{i,\,1} - a_i \overset{!}{=} a_{j,\,1} - a_j =: -b$.

   (b) For $k = 2, \ldots, t$: $a_{b_k,\,k} - a_{b_k} \overset{!}{=} -b$.

   (c) For $k = t+1, \ldots, n-p$: $a_{1,\,k} - a_1 \overset{!}{=} -b$.

4. For $k = 1, \ldots, n-p$: $a_{k,\,k} - a_k - a \overset{!}{=} b$.

Once this is shown, equation 4.48 can be reformulated as a linear combination of the given equations for $\mathcal{F}$:

$$
\begin{aligned}
\sum_{i=1}^{n}\sum_{k=1}^{n} a_{i,\,k}x_{i,\,k} &= d \\
\Leftrightarrow \sum_{k=2}^{t} a_{1,\,k}x_{1,\,k} + \sum_{k=t+1}^{n-p} a_{1,\,k}x_{1,\,k} + \sum_{k=n-p+1}^{n} a_{1,\,k}x_{1,\,k} & \\
+ \sum_{i=2}^{n-p}\sum_{\substack{k=1\\k\neq i}}^{n} a_{i,\,k}x_{i,\,k} + \sum_{i=n-p+1}^{n} a_{i,\,1}x_{i,\,1} & \\
+ \sum_{\substack{i=n-p+1\\i\notin\{b_2,\ldots,b_t\}}}^{n} \sum_{k=2}^{t} a_{i,\,k}x_{i,\,k} + \sum_{k=2}^{t} a_{b_k,\,k}x_{b_k,\,k} & \\
+ \sum_{i=n-p+1}^{n} \sum_{\substack{k=t+1\\k\neq i}}^{n} a_{i,\,k}x_{i,\,k} + \sum_{k=1}^{n} a_{k,\,k}x_{k,\,k} &= d
\end{aligned}
$$

$$\begin{aligned}
\stackrel{1a-1e}{\Longleftrightarrow} \sum_{k=t+1}^{n-p} (a_{1,\,k} - a_1) x_{1,\,k} + \sum_{i=n-p+1}^{n} (a_{i,\,1} - a_i) x_{i,\,1} & \\
+ \sum_{k=2}^{t} (a_{b_k,\,k} - a_{b_k}) x_{b_k,\,k} + \sum_{k=1}^{n} (a_{k,\,k} - a_k) x_{k,\,k} &= d - \sum_{i=1}^{n} a_{i,\,k} \qquad (4.49) \\
\stackrel{2}{\Longleftrightarrow} \sum_{k=t+1}^{n-p} (a_{1,\,k} - a_1) x_{1,\,k} + \sum_{i=n-p+1}^{n} (a_{i,\,1} - a_i) x_{i,\,1} & \\
+ \sum_{k=2}^{t} (a_{b_k,\,k} - a_{b_k}) x_{b_k,\,k} + \sum_{k=1}^{n-p} (a_{k,\,k} - a_k - a) x_{k,\,k} &= d - \sum_{i=1}^{n} a_{i,\,k} - a\,p \qquad (4.50) \\
\stackrel{3a-3c}{\Leftrightarrow} -b \sum_{k=t+1}^{n-p} x_{1,\,k} - b \sum_{i=n-p+1}^{n} x_{i,\,1} - b \sum_{k=2}^{t} x_{b_k,\,k} & \\
+ \sum_{k=1}^{n-p} (a_{k,\,k} - a_k - a) x_{k,\,k} &= d - \sum_{i=1}^{n} a_{i,\,k} - a\,p \\
\stackrel{4}{\Leftrightarrow} b \Big(lhs(4.47) - rhs(4.47) \Big) &= d - \sum_{i=1}^{n} a_{i,\,k} - a\,p.
\end{aligned}$$

We will now prove steps 1a to 4. As usual, we will proceed by constructing points in $\mathcal{F}$ and inserting them into equation 4.48:

ad 1 **(a)** For $k, l = n-p+1, \dots, n$: $a_{1,\,k} \stackrel{!}{=} a_{1,\,l} (=: a_1)$: Use

P_1: $x_{n-p+1,\,n-p+1} = \ldots = x_{n,\,n} = 1$,
$x_{i,\,k} = 1 \quad \forall\, i = 1, \dots, n-p$,

P_2: all values as in P_1, except $x_{1,\,k} = 0$, $x_{1,\,l} = 1$.

($lhs(4.47) = 0 = rhs(4.47)$ for both points, $\Rightarrow P_1, P_2 \in \mathcal{F}$)

(b) For $k = 2, \dots, t$: $a_{1,\,k} \stackrel{!}{=} a_1$:

Choose $m \in \{n-p+1, \dots, n\}$ such that $b_k \neq m$ (note $p \geq 2$). Use

P_1: $x_{k,\,k} = 1, x_{s,\,s} = 1\ \forall\, s \in \{n-p+1, \dots, n\} \backslash \{b_k\}$,
$x_{i,\,k} = 1 \quad \forall\, i \in \{1, \dots, n-p,\, b_k\} \backslash \{k\}$,

P_2: all values as in P_1, except $x_{1,\,k} = 0$, $x_{1,\,m} = 1$.

($P_1, P_2 \in \mathcal{F}$ since $lhs(4.47) = x_{k,\,k} = 1 = x_{b_k,\,k} = rhs(4.47)$)

(c) For $i = 2, \dots, n-p$ and $k, l \in \{1, \dots, n\} \backslash \{i\}$: $a_{i,\,k} \stackrel{!}{=} a_{i,\,l} (=: a_i)$:

- Case "$k = 1,\ l \in \{2, \ldots, n-p\} \backslash \{i\}$":
  $a_{i,\,k} = a_{i,\,l}$ can be achieved using
  P_1: $x_{1,\,1} = x_{l,\,l} = x_{n-p+1,\,n-p+1} = \ldots = x_{n-2,\,n-2} = 1$,
  $x_{j,\,1} = 1 \quad \forall\, j \in \{2, \ldots, n-p,\ n-1,\ n\} \backslash \{l\}$,
  P_2: all values as in P_1, except $x_{i,\,1} = 0$, $x_{i,\,l} = 1$,
  which both are elements of $\mathcal{F}$, since
  $lhs(4.47) = x_{1,\,1} + x_{l,\,l} = 2 = x_{n-1,\,1} + x_{n,\,1} = rhs(4.47)$.
- Case "$k = 1,\ l \in \{n-p+1, \ldots, n\}$":
  W.l.o.g., $l \neq n-p+1$. Use the points
  P_1: $x_{1,\,1} = x_{n-p+2,\,n-p+2} = \ldots = x_{n,\,n} = 1$,
  $x_{j,\,1} = 1 \quad \forall\, j \in \{2, \ldots, n-p+1\}$,
  P_2: all values as in P_1, except $x_{i,\,1} = 0$, $x_{i,\,l} = 1$
  (with $P_1, P_2 \in \mathcal{F}$ due to
  $lhs(4.47) = x_{1,\,1} = 1 = x_{n-p+1,\,1} = rhs(4.47)$.)
- Case "$k, l \in \{2, \ldots, n\} \backslash \{i\}$":
  Can be combined from the two cases above.

(d) For $i = n-p+1, \ldots, n$ and $k, l \in \{t+1, \ldots, n\} \backslash \{i\}$: $a_{i,\,k} \stackrel{!}{=} a_{i,\,l} (=: a_i)$:

- Case "$k \in \{t+1, \ldots, n-p\},\ l \in \{n-p+1, \ldots, n\} \backslash \{i\}$":
  Use the points P_1 and P_2, with
  P_1: $x_{k,\,k} = 1, x_{s,\,s} = 1\ \forall\ s \in \{n-p+1, \ldots, n\} \backslash \{i\}$
  $x_{j,\,k} = 1 \quad \forall\, j \in \{1, \ldots, n-p,\ i\} \backslash \{k\}$,
  P_2: all values as in P_1, except $x_{i,\,k} = 0$, $x_{i,\,l} = 1$.
  $P_1, P_2 \in \mathcal{F}$ since $lhs(4.47) = x_{k,\,k} = 1 = x_{1,\,k} = rhs(4.47)$. The desired result is achieved by plugging in P_1 and P_2 to 4.48.
- Case "$k, l \in \{t+1, \ldots, n-p\}$":
  Can be shown by applying the first case twice.
- Case "$k, l \in \{n-p+1, \ldots, n\} \backslash \{i\}$":
  Can be shown by applying the first case twice.

(e) For $i \in \{n-p+1, \ldots, n\} \backslash \{b_2, \ldots, b_t\}$ and $k = 2, \ldots, t$: $a_{i,\,k} \stackrel{!}{=} a_i$:
We show: $a_{i,\,k} = a_{i,\,n-p} \stackrel{step\ 1d}{=} a_i$. (In this context, recall that by overall assumption, we have $t \leq n-p-1$.) The above can be shown using the points

P_1: $x_{k,\,k} = 1, x_{m,\,m} = 1 \quad \forall\, m \in \{n-p, \ldots, n\} \backslash \{i, b_k\},$
$x_{1,\,n-p} = 1,\; x_{j,\,k} = 1 \quad \forall\, j \in \{2, \ldots, n-p-1, i, b_k\} \backslash \{k\},$
P_2: all values as in P_1, except $x_{i,\,k} = 0$, $x_{i,\,n-p} = 1$,

which lie in $\mathcal{F}$, since they both satisfy
$lhs(4.47) = x_{k,\,k} + x_{n-p,\,n-p} = 2 = x_{b_k,\,k} + x_{1,\,n-p} = rhs(4.47).$

Once steps 1a to 1e have been proven, equation 4.48 reduces to 4.49, as shown above.

ad 2 For $k, l = n-p+1, \ldots, n$: $a_{k,\,k} - a_k \stackrel{!}{=} a_{l,\,l} - a_l (=: a)$:

Use the points

P_1: $x_{m,\,m} = 1 \quad \forall\, m \in \{n-p, \ldots, n\} \backslash \{l\},$
$x_{j,\,n-p} = 1 \quad \forall\, j \in \{1, \ldots, n-p-1, l\},$
P_2: all values as in P_1, except
$x_{k,\,k} = 0,\; x_{l,\,l} = 1,\; x_{l,\,n-p} = 0,\; x_{k,\,n-p} = 1.$

($P_1, P_2 \in \mathcal{F}$ since $lhs(4.47) = x_{n-p,\,n-p} = 1 = x_{1,\,n-p} = rhs(4.47)$) and insert them into 4.49.

With step 2, equation 4.49 can be further reduced to 4.50 (see above).

ad 3 **(a)** For $i, j = n-p+1, \ldots, n$: $a_{i,\,1} - a_i \stackrel{!}{=} a_{j,\,1} - a_j (=: -b)$:

Insert the following points $P_1, P_2 \in \mathcal{F}$ into 4.50:

P_1: $x_{1,\,1} = 1, x_{k,\,k} = 1 \;\forall\, k \in \{n-p+1, \ldots, n\} \backslash \{i\},$
$x_{h,\,1} = 1 \quad \forall\, h \in \{2, \ldots, n-p,\; i\},$
P_2: all values as in P_1, except
$x_{i,\,i} = 1, x_{j,\,j} = 0, x_{i,\,1} = 0, x_{j,\,1} = 1.$

(b) For $k = 2, \ldots, t$: $a_{b_k,\,k} - a_{b_k} \stackrel{!}{=} -b$:

Choose $s \in \{n-p+1, \ldots, n\}$ with $s \neq b_k$ (recall that $p \geq 2$). Insert the points

P_1: $x_{1,\,1} = x_{k,\,k} = 1,\; x_{m,\,m} = 1 \quad \forall\, m \in \{n-p+1, \ldots, n\} \backslash \{b_k, s\},$
$x_{b_k,\,k} = 1, x_{s,\,1} = 1,\; x_{j,\,1} = 1 \quad \forall\, j \in \{2, \ldots, n-p\} \backslash \{k\},$
P_2: all values as in P_1, except $x_{b_k,\,k} = 0, x_{b_k,\,1} = 1.$

to equation 4.50 ($P_1, P_2 \in \mathcal{F}$ is easy to see):

$$\begin{aligned} P_1: \quad & d - \sum_{i=1}^{n} a_{i,\,k} - a\,p \\ = \quad & (a_{s,\,1} - a_s) + (a_{b_k,\,k} - a_{b_k}) + (a_{1,\,1} - a_1 - a) + (a_{k,\,k} - a_k - a) \end{aligned}$$

$$\begin{aligned} P_2: \quad & d - \sum_{i=1}^{n} a_{i,\,k} - a\,p \\ = \quad & (a_{s,\,1} - a_s) + (a_{b_k,\,1} - a_{b_k}) + (a_{1,\,1} - a_1 - a) + (a_{k,\,k} - a_k - a), \end{aligned}$$

and thus, $a_{b_k,\,k} - a_{b_k} = a_{b_k,\,1} - a_{b_k} \overset{step\ 3a}{=} -b$.

(c) For $k = t+1, \ldots, n-p$: $a_{1,\,k} - a_1 \overset{!}{=} -b$:

Will be shown in context with step 4.

Now that steps 3a and 3b have been proven, we arrive at the following reformulation of 4.50:

$$\begin{aligned} d - \sum_{i=1}^{n} a_{i,\,k} - a\,p \quad = \quad & \sum_{k=t+1}^{n-p} (a_{1,\,k} - a_1) x_{1,\,k} - b \sum_{i=n-p+1}^{n} x_{i,\,1} \\ & -b \sum_{k=2}^{t} x_{b_k,\,k} + \sum_{k=1}^{n-p} (a_{k,\,k} - a_k - a) x_{k,\,k}. \end{aligned}$$

Plugging in P_1 from the proof of step 1a gives $d - \sum_{i=1}^{n} a_{i,\,k} - a\,p = 0$, and such, we have that

$$\begin{aligned} & \sum_{k=t+1}^{n-p} (a_{1,\,k} - a_1) x_{1,\,k} - b \sum_{i=n-p+1}^{n} x_{i,\,1} \\ & -b \sum_{k=2}^{t} x_{b_k,\,k} + \sum_{k=1}^{n-p} (a_{k,\,k} - a_k - a) x_{k,\,k} \quad = \quad 0. \end{aligned} \tag{4.51}$$

ad 4 For $k = 1, \ldots, n-p$: $a_{k,\,k} - a_k - a \overset{!}{=} b$:

- If $k = 1$: $a_{1,\,1} - a_1 - a = b$ can be achieved by inserting $P \in \mathcal{F}$ into 4.51, with

  P: $x_{1,\,1} = x_{n-p+1,\,n-p+1} = \ldots = x_{n-1,\,n-1} = 1$,
  $x_{j,\,1} = 1 \quad \forall\, j \in \{2, \ldots, n-p,\, n\}$.

- If $k \in \{2, \ldots, t\}$: Use point $P \in \mathcal{F}$ with

  P: $x_{k,\,k} = 1, x_{l,\,l} = 1 \;\forall\, l \in \{n-p+1, \ldots, n\} \backslash \{b_k\}$
  $x_{j,\,k} = 1 \quad \forall\, j \in \{1, \ldots, n-p,\, b_k\} \backslash \{k\}$.

- If $k \in \{t+1, \ldots, n-p\}$: Consider

  P: $x_{k,\,k} = x_{n-p+1,\,n-p+1} = \ldots = x_{n-1,\,n-1} = 1,$
  $x_{j,\,k} = 1 \quad \forall\, j \in \{1, \ldots, n-p,\, n\} \backslash \{k\}$.

  $P \in \mathcal{F}$ since $lhs(4.47) = x_{k,\,k} = 1 = x_{1,\,k} = rhs(4.47)$. Inserting P into 4.51, we get that
  $(a_{k,\,k} - a_k - a) + (a_{1,\,k} - a_1) = 0 \Rightarrow \underbrace{(a_{k,\,k} - a_k - a)}_{=:c_k} = -(a_{1,\,k} - a_1)$.

  So far, we have reduced equation 4.51 to

$$-c_k \sum_{k=t+1}^{n-p} x_{1,\,k} - b \sum_{i=n-p+1}^{n} x_{i,\,1} - b \sum_{k=2}^{t} x_{b_k,\,k} + b \sum_{k=1}^{t} x_{k,\,k} + c_k \sum_{k=t+1}^{n-p} x_{k,\,k} = 0. \qquad (4.52)$$

  It remains to show that $c_k = b$. (Then, both step 3c and 4 are proven correct.)

  Due to definition, all points in $\mathcal{F}$ satisfy

$$b\Big(\sum_{k=1}^{n-p} x_{k,\,k} - \sum_{k=t+1}^{n-p} x_{1,\,k} - \sum_{i=n-p+1}^{n} x_{i,\,1} - \sum_{k=2}^{t} x_{b_k,\,k}\Big) = 0,$$

  and such, by setting the left hand side of the above equation equal to the left hand side of 4.52, we get that

$$\begin{aligned}\sum_{k=t+1}^{n-p} (c_k - b) x_{k,\,k} - \sum_{k=t+1}^{n-p} (c_k - b) x_{1,\,k} &= 0\\ \Leftrightarrow \sum_{k=t+1}^{n-p} (c_k - b)\big(x_{k,\,k} - x_{1,\,k}\big) &= 0\end{aligned}$$

  Now, for fixed $k \in \{t+1, \ldots, n-p\}$, we plug in $P^* \in \mathcal{F}$, with

  P^*: $x_{1,\,1} = x_{k,\,k} = x_{n-p+1,\,n-p+1} = \ldots = x_{n-2,\,n-2} = 1,$
  $x_{j,\,1} = 1 \quad \forall\, j \in \{2, \ldots, n-p,\, n-1,\, n\} \backslash \{k\}$.

and such arrive at
$c_k - b = 0 \; \forall \; k \in \{t+1, \ldots, n-p\} \Rightarrow c_k = b \; \forall \; k \in \{t+1, \ldots, n-p\}$.

□

Remark 4.5.7 (Graphical Interpretation)
Figure 4.2 gives an interpretation of the facet class presented in theorem 4.5.6, based on the interpretation of theorem 4.5.3 given in figure 4.1. Once again, black nodes denote elements of A, white nodes elements of $\bar{A}$. Additionally, the subset $A^* \subset A$ has been marked. As in figure 4.1, the arcs that contribute to $\sum_{i \in \bar{A}} X_{i,\, a}$ are marked by doubled arrows. Note that $\sum_{j \in A \setminus (A^* \cup \{a\})} X_{a,\, j}$ (dashed arrows) now considers only nodes in $A \setminus A^*$. The arcs that contribute to the new addend $\sum_{k \in A^*} X_{b_k,\, k}$ are marked by dotted arrows.

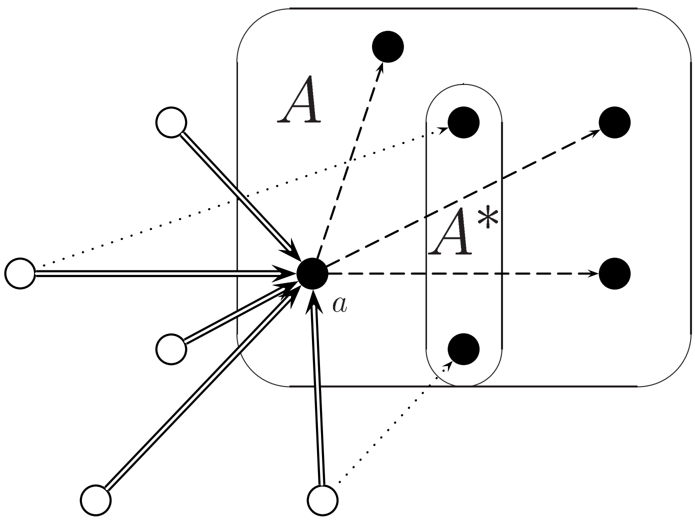
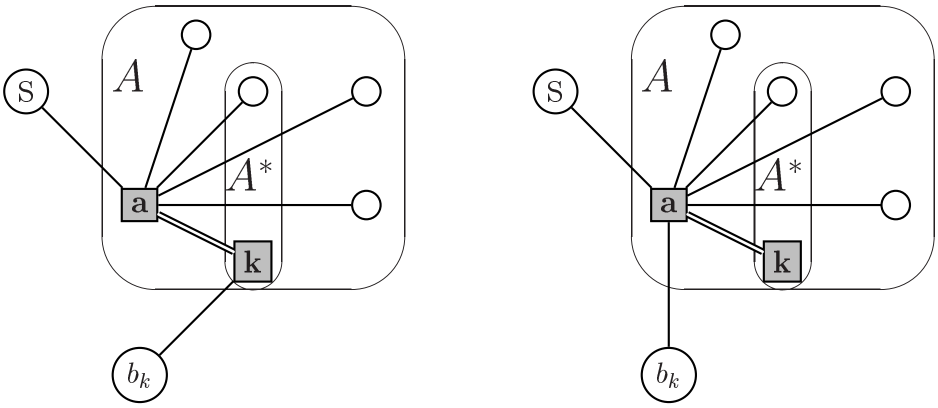


Figure 4.2: Graphical interpretation of focus-element facets II

Example: Given a problem instance with $n = 8$ and $p = 2$, assume the sets A, A^* and the node a are chosen as shown in figure 4.3 below. Consider point P_1 of step 3 (b) in the above proof (seen as an element of $\mathcal{P}_{pUHL}$):
$X_{a,\, a} = X_{k,\, k} = 1,\; X_{m,\, m} = 1 \; \forall \; m \in \bar{A} \setminus \{b_k,\; s\}$,
$X_{b_k,\, k} = 1,\; X_{s,\, a} = 1,\; X_{j,\, a} = 1 \; \forall \; j \in A \setminus \{a,\; k\}$
for some $k \in A^*$ and $s \in \bar{A} \setminus \{b_k\}$.

This point is shown in figure 4.3(a). Since $X_{a,\,a} + X_{k,\,k} = 2 = X_{b_k,\,k} + X_{s,\,a}$, it fulfills inequality 4.46 with equality; but $X_{a,\,a} + X_{k,\,k} = 2 > 1 = X_{s,\,a}$, and hence, inequality 4.44 is strict. To fulfill 4.44 with equality, we would have to set $X_{b_k,\,k} = 0$ and $X_{b_k,\,a} = 1$ instead, which corresponds to point P_2 of step 3 (b); see figure 4.3(b).



(a) Solution satisfying focus-el. inequality II with equality

(b) Solution satisfying focus-el. inequality I (and II) with equality

Figure 4.3: Comparison of focus-element facets I and II

Remark 4.5.8 (number of facets in facet class)
To calculate the total number of facets that are represented by inequality 4.46, note:

- The number of possible choices of set A is $\binom{n}{n-p} = \binom{n}{p}$.

- The number of possible elements a to choose out of A is $n - p$.

- The number of possible choices of set $A^* \subset A \backslash \{a\}$ (with $|A^*| \in \{0, \ldots, n-p-2\}$ fixed) is $\binom{n-p-1}{|A^*|}$.

- The number of possible choices of the elements b_k with $\{b_k : k \in A^*\} \subset \{1, \ldots, n\} \backslash A$ and $b_1, \ldots, b_{|A^*|}$ pairwise different is $\binom{p}{|A^*|}\,|A^*|!$ (note that the order of the chosen elements b_k is important).

Hence, the total number of facets in the given facet class is

$$\binom{n}{p}(n-p)\sum_{i=0}^{n-p-2}\left(\binom{n-p-1}{i}\binom{p}{i}i!\right).$$

4.5.2 Facets Including the Variable z

In subsection 4.4.2, we noted that inequality 4.2 ($z \geq r_k + r_m + \alpha\, d_{k,\,m}$) does not define a facet of the polyhedron $\mathcal{P}_{pUHL}$, since using inequality 4.30 instead provides a tighter description of $\mathcal{P}_{pUHL}$. However, inequality 4.30 still does not define a facet of $\mathcal{P}_{pUHL}$:

Proposition 4.5.9
For any $k, m \in \{1, \ldots, n\}$ and $\alpha \in]0;\, 1[$, the inequality

$$\begin{aligned} z \quad \geq \quad & r_k + r_m + \alpha\, d_{k,\,m} \\ & +(1-\alpha)(1-X_{k,\,k}) \min_{i \neq k} d_{i,\,k} + (1-\alpha)(1-X_{m,\,m}) \min_{i \neq m} d_{i,\,m} \end{aligned}$$

does not represent a facet of $\mathcal{P}_{pUHL}$.

Proof: Without loss of generality, $k = 1$ and $m = 2$. Let

$$\begin{aligned} \mathcal{F} := \{P \in \mathcal{P}_{pUHL} : z = r_1 + r_2 + \alpha\, d_{1,\,2} \\ +(1-\alpha)(1-X_{1,\,1}) \min_{i \neq 1} d_{i,\,1} + (1-\alpha)(1-X_{2,\,2}) \min_{i \neq 2} d_{i,\,2}\} \end{aligned}$$

1. Assume that there exist nodes $r, s \neq 1$ with different distance to node 1, i. e. $d_{1,\,r} \neq d_{1,\,s}$.
   Let the set $\mathcal{M}$ be defined as

   $$\mathcal{M} := \{k \in \{2, \ldots, n\} : d_{1,\,k} > \min_{i \neq 1} d_{1,\,i}\}$$

   $\mathcal{M} \neq \emptyset$ by assumption.

   <u>Claim:</u> For all $P \in \mathcal{F} \cap \mathcal{Z}_{pUHL}$ and all $k \in \mathcal{M}$, we have that $X^P_{1,\,k} = 0$.

   (a) If $X_{1,\,1} = 1$, then $X_{1,\,l} = 0 \quad \forall\, l \geq 2$,
   and thus also $X_{1,\,k} = 0 \quad \forall\, k \in \mathcal{M}$.

(b) Now, consider the case that $X_{1,\,1} = 0$. Then, assuming that $X_{1,\,k} = 1$ for $k \in \mathcal{M}$ requires that $X_{k,\,k} = 1$.

i. Assume $X_{2,\,2} = 1$. Due to the assumption that $X_{1,\,k} = 1$, we have $r_k \geq d_{1,\,k}$. Hence

$$\begin{aligned} z &\overset{P\in\mathcal{F}}{=} r_2 + \alpha\, d_{1,\,2} + (1-\alpha) \min_{i\neq 1} d_{i,\,1} \\ &\overset{k\in\mathcal{M},\,\alpha<1}{<} r_2 + \alpha\, d_{1,\,2} + (1-\alpha) d_{1,\,k} \\ &\overset{\Delta-ineq.}{\leq} r_2 + \alpha(d_{1,\,k} + d_{k,\,2}) + (1-\alpha) d_{1,\,k} \\ &= r_2 + \alpha\, d_{2,\,k} + d_{1,\,k} \overset{r_k \geq d_{1,\,k}}{\leq} r_2 + \alpha\, d_{2,\,k} + r_k \end{aligned}$$

that is, any such point P violates constraint 4.2; contradiction.

ii. Now, assume $X_{2,\,2} = 0$. That is, we have that $X_{1,\,1} = X_{2,\,2} = 0$ and $X_{k,\,k} = 1$. Assume that $X_{1,\,k} = 1$. Then, $r_k \geq d_{1,\,k}$. Let $l \in \{3, \ldots, n\}$ be a hub with $X_{2,\,l} = 1$. Consequently, $r_l \geq d_{2,\,l}$, and we have[9]

$$\begin{aligned} z &\overset{P\in\mathcal{F}}{=} \alpha\, d_{1,\,2} + (1-\alpha) \min_{i\neq 1} d_{i,\,1} + (1-\alpha) \min_{i\neq 2} d_{i,\,2} \\ &\overset{k\in\mathcal{M},\,\alpha<1}{<} \alpha\, d_{1,\,2} + (1-\alpha) d_{1,\,k} + (1-\alpha) d_{2,\,l} \\ &= d_{1,\,k} + \alpha(d_{1,\,2} - d_{1,\,k} - d_{2,\,l}) + d_{2,\,l} \\ &\overset{\Delta-ineq.}{\leq} d_{1,\,k} + \alpha(d_{k,\,2} - d_{2,\,l}) + d_{2,\,l} \\ &\overset{\Delta-ineq.}{\leq} d_{1,\,k} + \alpha d_{k,\,l} + d_{2,\,l} \overset{r_k \geq d_{1,\,k},\, r_l \geq d_{2,\,l}}{\leq} r_k + \alpha\, d_{k,\,l} + r_l \end{aligned}$$

that is, any such point P violates constraint 4.2; contradiction.

2. Now, assume that all nodes $r, s \neq 1$ have equal distances to node 1, i. e. $d_{1,\,r} = d_{1,\,s}$.

<u>Claim:</u> For all $P \in \mathcal{F} \cap \mathcal{Z}_{pUHL}$ and $k \geq 3$, we have that $X^P_{1,\,k} = 0$.

We prove this claim for a fixed k, w.l.o.g. $k = 3$. For the following argumentation, recall the general assumption that $d_{i,\,j} > 0$ for all $i \neq j$, and thus $d_{3,\,2} > 0$.

[9]Note that $l = k$ is possible; in this case, $r_k \geq \max\{d_{1,\,k}, d_{2,\,k}\}$.

(a) If $X_{1,\,1} = 1$, then $X_{1,\,l} = 0 \quad \forall\, l \geq 2$, and thus also $X_{1,\,3} = 0$.

(b) Now, consider the case $X_{1,\,1} = 0$. Then, assuming that $X_{1,\,3} = 1$ requires $X_{3,\,3} = 1$.

i. Assume that $X_{2,\,2} = 1$. Due to the assumption that $X_{1,\,3} = 1$, we have $r_3 \geq d_{1,\,3}$. Hence

$$\begin{aligned}
z &\overset{P\in\mathcal{F}}{=} r_2 + \alpha\, d_{1,\,2} + (1-\alpha)\min_{i\neq 1} d_{i,\,1} \\
&\overset{d_{1,\,3}=d_{3,\,1}=d_{i,\,1}\,\forall\, i\neq 1}{=} r_2 + \alpha\, d_{1,\,2} + (1-\alpha) d_{1,\,3} \\
&\overset{d_{3,\,2}>0}{<} r_2 + \alpha(d_{1,\,2} + d_{3,\,2}) + (1-\alpha) d_{1,\,3} \\
&\overset{d_{1,\,2}=d_{1,\,3}}{=} r_2 + \alpha(d_{1,\,3} + d_{3,\,2}) + (1-\alpha) d_{1,\,3} \\
&= r_2 + \alpha\, d_{3,\,2} + d_{1,\,3} \overset{r_3\geq d_{1,\,3}}{\leq} r_2 + \alpha\, d_{3,\,2} + r_3
\end{aligned}$$

that is, any such point P violates constraint 4.2; contradiction.

ii. Now, assume that $X_{2,\,2} = 0$. That is, we have that $X_{1,\,1} = X_{2,\,2} = 0$ and $X_{3,\,3} = 1$. Assume that $X_{1,\,3} = 1$. Then, $r_3 \geq d_{1,\,3}$. Let $l \in \{3, \ldots, n\}$ be a hub with $X_{2,\,l} = 1$. Consequently, $r_l \geq d_{2,\,l}$, and we have[10]

$$\begin{aligned}
z &\overset{P\in\mathcal{F}}{=} \alpha\, d_{1,\,2} + (1-\alpha)\min_{i\neq 1} d_{i,\,1} + (1-\alpha)\min_{i\neq 2} d_{i,\,2} \\
&\leq \alpha\, d_{1,\,2} + (1-\alpha) d_{1,\,3} + (1-\alpha) d_{2,\,l} \\
&= d_{1,\,3} + \alpha(d_{1,\,2} - d_{1,\,3} - d_{2,\,l}) + d_{2,\,l} \\
&\overset{d_{1,\,2}=d_{1,\,3}}{=} d_{1,\,3} - \alpha\, d_{2,\,l} + d_{2,\,l} \overset{d_{2,\,l}>0,\ \alpha>0}{<} d_{1,\,3} + d_{2,\,l} \\
&\overset{r_3\geq d_{1,\,3},\ r_l\geq d_{2,\,l}}{\leq} r_3 + r_l \overset{d_{3,\,l}\geq 0,\ \alpha>0}{\leq} r_3 + \alpha\, d_{3,\,l} + r_l
\end{aligned}$$

that is, any such point P violates constraint 4.2; contradiction.

□

Remark 4.5.10

Note that this chapter does not present any facet-defining inequalities containing the variable z. The fact that the variable z defines the objective function value makes it extremely hard to determine such facets. However, future work might have to concentrate more intensively on facets including the variable z; see chapter 7.

[10]Note that $l = 3$ is possible; in this case, $r_3 \geq \max\{d_{1,\,3}, d_{2,\,3}\}$.

4.5.3 Facets Including the Variables r_k

Increasing-distances Facets

As stated in section 4.4.2, the constraint "$r_k \geq d_{i,\,k} X_{i,\,k}$" represents a facet if $d_{i,\,k} = max_j d_{j,\,k}$ (see proposition 4.4.10). Now, new facet-defining inequalities that generalize the above constraint will be presented.

Theorem 4.5.11 (Increasing-distances Facets I)
Let $k \in \{1, \ldots, n\}$ *and let* $\{a_1, \ldots, a_{n-1}\} = \{1, \ldots, n\} \backslash \{k\}$ *such that* $d_{a_1,\,k} \leq d_{a_2,\,k} \leq \ldots \leq d_{a_{n-1},\,k}$. *The inequality*

$$r_k \geq d_{a_{n-p},\,k} X_{a_{n-p},\,k} + \sum_{i=n-p+1}^{n-1} (d_{a_i,\,k} - d_{a_{i-1},\,k}) X_{a_i,\,k} \qquad (4.53)$$

is valid for (**pUHL**)*; if* $p \leq \lfloor \frac{n}{2} \rfloor$, *it represents a facet of the polyhedron* $\mathcal{P}_{pUHL}$.

Proof: For ease of notation, assume that $k = 1$ and

$$d_{n,\,1} \geq d_{n-1,\,1} \geq d_{n-2,\,1} \geq \ldots \geq d_{2,\,1}, \qquad (4.54)$$

and thus consider the inequality

$$r_1 \geq d_{n-p+1,\,1} X_{n-p+1,\,1} + \sum_{i=n-p+2}^{n} (d_{i,\,1} - d_{i-1,\,1}) X_{i,\,1}. \qquad (4.55)$$

Validity:

1. Assume that none of the nodes in $\{n-p+2, \ldots, n\}$ is allocated to node 1. Then, inequality 4.55 reduces to constraint 4.3 and thus is valid.

2. Now, assume that at least one of the nodes in $\{n-p+2, \ldots, n\}$ is allocated to node 1, i.e. $\exists\, i_1, \ldots, i_s \in \{n-p+2, \ldots, n\}$ (with $s \geq 1$ and $i_1 < i_2 < \ldots < i_s$) such that $X_{i_1,\,1} = X_{i_2,\,1} = \ldots = X_{i_s,\,1} = 1$ and $X_{i,\,1} = 0 \quad \forall\, i \in \{n-p+2, \ldots, n\} \backslash \{i_1, \ldots, i_s\}$.

   Then, due to constraint 4.3, we have that $r_1 \geq d_{i_s,\,1}$. Now consider the right hand side of inequality 4.55:

$$
\begin{aligned}
& d_{n-p+1,\,1}X_{n-p+1,\,1} + \sum_{i=n-p+2}^{n} (d_{i,\,1} - d_{i-1,\,1})X_{i,\,1} \\
= \ & d_{n-p+1,\,1}X_{n-p+1,\,1} \\
& +(d_{i_1,\,1} - d_{i_1-1,\,1}) + \ldots + (d_{i_{s-1},\,1} - d_{i_{s-1}-1,\,1}) + (d_{i_s,\,1} - d_{i_s-1,\,1}) \\
= \ & d_{i_s,\,1} + (d_{i_{s-1},\,1} - d_{i_s-1,\,1}) + (d_{i_{s-2},\,1} - d_{i_{s-1}-1,\,1}) + \ldots \\
& +(d_{i_1,\,1} - d_{i_2-1,\,1}) - d_{i_1-1,\,1} + d_{n-p+1,\,1}X_{n-p+1,\,1}
\end{aligned}
$$

Due to the assumption that $i_1 < i_2 < \ldots < i_s$ and integer, we have that $i_1 \leq i_2 - 1, \ldots, i_{s-1} \leq i_s - 1$, and thus, using the general assumption 4.54, we get that $d_{i_{s-1},\,1} - d_{i_s-1,\,1} \leq 0, \ldots, d_{i_1,\,1} - d_{i_2-1,\,1} \leq 0$. Similarly, since $n-p+1 \leq i_1 - 1$, we have that $-d_{i_1-1,\,1} + d_{n-p+1,\,1}X_{n-p+1,\,1} \leq 0$, and such, finally, we arrive at $rhs(4.55) \leq d_{i_s,\,1} \leq r_1$.

Facet-defining:
(Will be shown for a more general facet class in theorem 4.5.14.)

□

Remark 4.5.12
In inequality 4.53, we consider the nodes $\{1, \ldots, n\}\backslash\{k\}$ in increasing order regarding their distance to node k. Hence, we refer to the corresponding facet class as *"increasing-distances facets"*.

Example: Let $n \geq 4$ and $p = 2$. Consider the facet-defining inequality from theorem 4.5.11 for $k = 1$, where we assume that node n is furthest and node $n-1$ second furthest away from node 1:

$$r_1 \geq d_{n-1,\,1}X_{n-1,\,1} + (d_{n,\,1} - d_{n-1,\,1})X_{n,\,1}. \tag{4.56}$$

There are three different types of points for which inequality 4.56 is satisfied with equality:

1. $X_{n-1,\,1} = X_{n,\,1} = 0$ and no other allocation to node 1 (trivial).

2. $X_{n-1,\,1} = 1$, $X_{n,\,1} = 0$. (Then, $r_1 = d_{n-1,\,1}$.)

3. $X_{n-1,\,1} = X_{n,\,1} = 1$. (Then, $r_1 = d_{n,\,1}$.)

Now assume a point with $X_{n,\,1} = 1$ and $X_{n-1,\,1} = 0$. In this case, we get

$$\begin{aligned} r_1 \quad & \geq \quad d_{n,\,1} \overset{d_{n-1,\,1}>0}{>} d_{n,\,1} - d_{n-1,\,1} \\ & \overset{X_{n-1,\,1}=0}{=} d_{n-1,\,1}X_{n-1,\,1} + (d_{n,\,1} - d_{n-1,\,1})X_{n,\,1}, \end{aligned}$$

which means that such a point does *not* satisfy 4.56 with equality; this fact will become important when examining the inequality of theorem 4.5.11 for $p > \lfloor \frac{n}{2} \rfloor$; see the following remark.

Remark 4.5.13 (the case $p > \lfloor \frac{n}{2} \rfloor$)
Assume that the distances to node k are pairwise different[11]; for the case $k = 1$, which was presented in the proof of 4.5.11, this means that
$d_{n,\,1} > d_{n-1,\,1} > d_{n-2,\,1} > \ldots > d_{2,\,1}$.

It is easy to see[12] that a necessary condition for a point P to satisfy 4.55 with equality is $X_{n-p+1,\,1} = \ldots = X_{n-p+i,\,1} = 1$ and $X_{n-p+i+1,\,1} = \ldots = X_{n,\,1} = 0$ for some $i \in \{0, \ldots, p\}$. Now, assume that a point P satisfies 4.55 with equality and has $X_{n,\,1} = 1$. Then, due to the above, $X_{n-p+1,\,1} = \ldots = X_{n-1,\,1} = 1$, i. e. point P contains at least p spokes. But if $p > \lfloor \frac{n}{2} \rfloor$, we have $n - p < p$, which means that for point P, there are not enough nodes left to locate the p hubs. Hence, P is not feasible for (**pUHL**). Consequently, all points that satisfy 4.55 with equality also satisfy $X_{n,\,1} = 0$; thus, 4.55 is not facet-defining for $p > \lfloor \frac{n}{2} \rfloor$.

We have shown so far that, if $d_{a_1,\,k} \leq \ldots \leq d_{a_{n-1},\,k}$ and $p \leq \lfloor \frac{n}{2} \rfloor$, the following inequalities are valid and define facets of $\mathcal{P}_{pUHL}$:

- $r_k \geq d_{a_{n-1},\,k} X_{a_{n-1},\,k}$
- $r_k \geq d_{a_{n-p},\,k} X_{a_{n-p},\,k} + \sum_{i=n-p+1}^{n-1} (d_{a_i,\,k} - d_{a_{i-1},\,k}) X_{a_i,\,k}$

[11]This is the case for most real-world problems, where two distance values are very unlikely to be exactly the same.

[12]For any point with $X_{k,\,1} = 0$ and $X_{m,\,1} = 1$ for some $k < m$, $k, m \in \{n-p+1, \ldots, n\}$, we have that $rhs(4.55) \leq d_{m,\,1} - (d_{k,\,1} - d_{k-1,\,1}) < d_{m,\,1} \leq lhs(4.55)$.

Next, we will see that the above inequalities are only special cases of a more general class of facet-defining inequalities:

Theorem 4.5.14 (Increasing-distances Facets II)
Let $p \leq \lfloor \frac{n}{2} \rfloor$, $k \in \{1, \ldots, n\}$ *and* $i_k := arg\,max_i\{d_{i,\,k} : i = 1, \ldots, n\}$.
Let $A = \{a_1, \ldots, a_t\} \subset \{1, \ldots, n\} \backslash \{k, i_k\}$ *with* $|A| = t \in \{1, \ldots, p-1\}$, *where* $d_{a_i,\,k} \leq d_{a_{i+1},\,k} \quad \forall\, i = 1, \ldots, t-1$. *Consider the inequality*

$$r_k \geq d_{a_1,\,k} X_{a_1,\,k} + \sum_{i=2}^{t} (d_{a_i,\,k} - d_{a_{i-1},\,k}) X_{a_i,\,k} + (d_{i_k,\,k} - d_{a_t,\,k}) X_{i_k,\,k}. \tag{4.57}$$

(i) Inequality 4.57 is valid for $\mathcal{P}_{pUHL}$.

(ii) If $t \leq p-2$, *then 4.57 represents a facet of* $\mathcal{P}_{pUHL}$.

(iii) If $t = p-1$ *and* ($p \leq \lfloor \frac{n-1}{2} \rfloor$ *or* $d_{i,\,k} \leq d_{a_t,\,k} \quad \forall\, i \notin A$), *then 4.57 represents a facet of* $\mathcal{P}_{pUHL}$.

Proof: * Assume without loss of generality[13] that $k = 1$, $i_k = n$ and $A = \{a_1 = n-t, a_2 = n-t+1, \ldots, a_t = n-1\}$. Then, inequality 4.57 can be written as follows:

$$r_1 \geq d_{n-t,\,1} X_{n-t,\,1} + \sum_{i=n-t+1}^{n} (d_{i,\,1} - d_{i-1,\,1}) X_{i,\,1} \tag{4.58}$$

ad (i): Validity:

(Analogous to the proof of 4.5.11.)

ad (ii): Facet-defining (Case $t \leq p-2$):

We will show that, if all points lying on the face

$$\mathcal{F} := \{P \in \mathcal{P}_{pUHL} : r_1 = d_{n-t,\,1} X_{n-t,\,1} + \sum_{i=n-t+1}^{n} (d_{i,\,1} - d_{i-1,\,1}) X_{i,\,1}\}$$

satisfy another equation

$$\sum_{i=1}^{n} \sum_{k=1}^{n} a_{i,\,k} X_{i,\,k} + \sum_{k=1}^{n} b_k r_k + c\, z = d, \tag{4.59}$$

*Technical proof; may be omitted.

[13] Note that, in contrast to the w.l.o.g. assumption in the proof of 4.5.11, it might here be possible that $d_{i,\,1} > d_{n-1,\,1}$ for some $i \in \{1, \ldots, n-t-1\}$.

then this new equation 4.59 is a linear combination of the other equations that are fullfilled.

We will proceed by showing the following:

1. (a) For $k \geq 2$: $b_k \stackrel{!}{=} 0$.
   (b) $c \stackrel{!}{=} 0$.
2. (a) For $i \leq n-t-1$ and $k, l \in \{1, \ldots, n\} \backslash \{i\}$: $a_{i,\,k} \stackrel{!}{=} a_{i,\,l} =: a_i$.
   (b) For $i \geq n-t$ and $k, l \in \{2, \ldots, n\} \backslash \{i\}$: $a_{i,\,k} \stackrel{!}{=} a_{i,\,l} =: a_i$.
3. For $k, l \in \{1, \ldots, n\}$: $a_{k,\,k} - a_k \stackrel{!}{=} a_{l,\,l} - a_l =: a$.
4. (a) $a_{n-t,\,1} - a_{n-t} \stackrel{!}{=} -b_1\, d_{n-t,\,1}$.
   (b) For $i \geq n-t+1$: $a_{i,\,1} - a_i \stackrel{!}{=} -b_1\,(d_{i,\,1} - d_{i-1,\,1})$.

Once this has been shown, we can reformulate equation 4.59 and thus show that it has to be a linear combination of the given equations:

$$
\begin{aligned}
\sum_{i=1}^{n}\sum_{k=1}^{n} a_{i,\,k}X_{i,\,k} + \sum_{k=1}^{n} b_k r_k + c\, z &= d \\
\stackrel{1a,1b}{\Longleftrightarrow} \quad \sum_{i=1}^{n}\sum_{k=1}^{n} a_{i,\,k}X_{i,\,k} + b_1\, r_1 &= d \qquad (4.60)\\
\stackrel{2a}{\Longleftrightarrow} \quad \sum_{i=1}^{n-t-1} a_i \sum_{k=1}^{n} X_{i,\,k} + \sum_{k=1}^{n-t-1} (a_{k,\,k} - a_k) X_{k,\,k} & \\
+ \sum_{i=n-t}^{n}\sum_{k=1}^{n} a_{i,\,k}X_{i,\,k} + b_1\, r_1 &= d \\
\stackrel{2b}{\Longleftrightarrow} \quad \sum_{i=1}^{n} a_i \sum_{k=1}^{n} X_{i,\,k} + \sum_{k=1}^{n} (a_{k,\,k} - a_k) X_{k,\,k} & \\
+ \sum_{i=n-t}^{n} (a_{i,\,1} - a_i) X_{i,\,1} + b_1\, r_1 &= d \\
\Leftrightarrow \quad \sum_{i=n-t}^{n} (a_{i,\,1} - a_i) X_{i,\,1} + \sum_{k=1}^{n} (a_{k,\,k} - a_k) X_{k,\,k} + b_1\, r_1 &= d - \sum_{i=1}^{n} a_i \qquad (4.61)
\end{aligned}
$$

$$\begin{aligned}
\overset{3}{\Leftrightarrow} \quad \sum_{i=n-t}^{n} (a_{i,\,1} - a_i) X_{i,\,1} + b_1\, r_1 &= d - \sum_{i=1}^{n} a_i - a\, p \\
\overset{4a,4b}{\Longleftrightarrow} \quad b_1 (r_1 - rhs(4.58)) &= d - \sum_{i=1}^{n} a_i - a\, p.
\end{aligned} \tag{4.62}$$

Now, we prove the required steps 1a to 4b:

ad 1: **(a)** For $k \geq 2$: $b_k \overset{!}{=} 0$:

Choose $h_1, \ldots, h_{p-1} \in \{1, \ldots, n\} \backslash \{k\}$ pairwise different and set

P_1: $X_{k,\,k} = 1,\ X_{h_1,\,h_1} = \ldots = X_{h_{p-1},\,h_{p-1}} = 1,$
$X_{i,\,k} = 1 \quad \forall\, i \in \{1, \ldots, n\} \backslash \{k, h_1, \ldots, h_{p-1}\},$
$r_k = max_i \{d_{i,\,k} : i \in \{1, \ldots, n\} \backslash \{k, h_1, \ldots, h_{p-1}\}\},\ r_1 = 0,$

P_2 all values as in P_1, except $r_k^{P_2} = 2\, r_k^{P_1}$.

Obviously, both points are elements of $\mathcal{F}$, and inserting them into 4.59 yields $b_k = 0$.

(b) $c \overset{!}{=} 0$:

Choose P_1 as in step 1a above, and point P_2 with all values as in P_1 except that $z^{P_2} = 2\, z^{P_1}$. Once again, it is obvious that those points lie in $\mathcal{F}$, and inserting them into 4.59 delivers the desired result.

Having proven the steps above, equation 4.59 reduces to 4.60.

ad 2: **(a)** For $i \leq n - t - 1$ and $k, l \in \{1, \ldots, n\} \backslash \{i\}$: $a_{i,\,k} \overset{!}{=} a_{i,\,l}\, (=: a_i)$:

- Case 1: $k = 1,\ l \in \{2, \ldots, n - t - 1\} \backslash \{i\}$.

  Note that $p \leq \lfloor \frac{n}{2} \rfloor$ and $t \leq p - 2$ by assumption. Hence, $n - t - 1 \geq n - p + 1 \geq p + 1$. Consequently, it is possible to choose nodes $h_1, \ldots, h_{p-2} \in \{1, \ldots, n - t - 1\} \backslash \{1, l, i\}$ pairwise different. Do so and set

  P_1: $X_{1,\,1} = X_{l,\,l} = 1,\ X_{h_1,\,h_1} = \ldots = X_{h_{p-2},\,h_{p-2}} = 1,$
  $X_{j,\,1} = 1\ \forall\, j \in \{2, \ldots, n\} \backslash \{h_1, \ldots, h_{p-2}, l\},$
  $r_1 = d_{n,\,1},\ r_l = 0,$

  P_2: all values as in P_1, except $X_{i,\,1} = 0,\ X_{i,\,l} = 1,\ r_l = d_{i,\,l}.$

P_1 and P_2 satisfy
$lhs(4.58) = d_{n,\,1} = d_{n-t,\,1} + \sum_{j=n-t+1}^{n} (d_{j,\,1} - d_{j-1,\,1}) = rhs(4.58)$,
and thus lie in $\mathcal{F}$. Inserting them into 4.60 gives $a_{i,\,1} = a_{i,\,l}$.

- Case 2: $k \in \{2, \ldots, n-t-1\} \backslash \{i\}$, $l \in \{n-t, \ldots, n\}$.
  Choose $h_1, \ldots, h_{p-2} \in \{1, \ldots, n\} \backslash \{k, l, i\}$ (possible due to the overall assumption that $p \leq n-2$), and consider

  P_1: $X_{k,\,k} = X_{l,\,l} = 1$, $X_{h_1,\,h_1} = \ldots = X_{h_{p-2},\,h_{p-2}} = 1$,
  $X_{j,\,k} = 1 \quad \forall\, j \in \{1, \ldots, n\} \backslash \{k, l, h_1, \ldots, h_{p-2}\}$, $r_1 = 0$,
  $r_k = max_j \{ d_{j,\,k} : j \in \{1, \ldots, n\} \backslash \{k, l, h_1, \ldots, h_{p-2}\} \}$,
  P_2: all values as in P_1, except $X_{i,\,k} = 0$, $X_{i,\,l} = 1$, $r_l = d_{i,\,l}$.

  For both points, we have $r_1 = X_{i,\,1} = 0 \;\forall\, i$, and hence, $P_1, P_2 \in \mathcal{F}$. $a_{i,\,k} = a_{i,\,l}$ follows by plugging these points in to 4.60.

- Case 3: $k, l \in \{1, \ldots, n\} \backslash \{i\}$ other than in cases 1 and 2.
  Then, $a_{i,\,k} = a_{i,\,l}$ can be shown by applying cases 1 and/or 2 several times.

(b) For $i \geq n-t$ and $k, l \in \{2, \ldots, n\} \backslash \{i\}$: $a_{i,\,k} \stackrel{!}{=} a_{i,\,l}\,(=: a_i)$:

Can be shown using the same construction as in case 2 of the proof of step 2a.

As shown above, once that 2a and 2b have been proven correct, equation 4.60 reduces to 4.61.

ad 3: For $k, l \in \{1, \ldots, n\}$: $a_{k,\,k} - a_k \stackrel{!}{=} a_{l,\,l} - a_l\,(=: a)$:

Can be shown using the points

P_1: $X_{k,\,k} = 1$, $X_{h_1,\,h_1} = \ldots = X_{h_{p-1},\,h_{p-1}} = 1$,
$X_{i,\,h_1} = 1 \quad \forall\, i \in \{1, \ldots, n\} \backslash \{k, h_1, \ldots, h_{p-1}\}$, $r_1 = 0$,
$r_{h_1}^{P_1} = max_i \{ d_{i,\,h_1} : i \in \{1, \ldots, n\} \backslash \{k, h_1, \ldots, h_{p-1}\} \}$,
P_2: all values as in P_1, except
$X_{k,\,k} = 0$, $X_{k,\,h_1} = 1$, $X_{l,\,l} = 1$, $X_{l,\,h_1} = 0$, $r_{h_1}^{P_2} = max\{ r_{h_1}^{P_1}, d_{k,\,h_1} \}$.

where $h_1, \ldots, h_{p-1} \in \{2, \ldots, n\} \backslash \{k, l\}$ pairwise different (note $p \leq n-2$).

With step 3, we have arrived at equation 4.62. If we insert point P_1 from step 3 into 4.62, we get that $d - \sum_{i=1}^{n} a_i - a\,p = 0$, and thus,

$$\sum_{i=n-t}^{n} (a_{i,\,1} - a_i) X_{i,\,1} + b_1\, r_1 = 0. \tag{4.63}$$

ad 4: **(a)** $a_{n-t,\,1} - a_{n-t} \stackrel{!}{=} -b_1\, d_{n-t,\,1}$:

Insert $P \in \mathcal{F}$ into equation 4.63, where

$$\begin{aligned} P\colon\quad & X_{1,\,1} = \ldots = X_{p,\,p} = 1, \\ & X_{n-t,\,1} = 1,\ X_{i,\,2} = 1 \quad \forall\, i \in \{p+1, \ldots, n\} \backslash \{n-t\}, \\ & r_1 = d_{n-t,\,1},\ r_2 = max_i \big\{ d_{i,\,2} : i \in \{p+1, \ldots, n\} \backslash \{n-t\} \big\}. \end{aligned}$$

($n-t$ is spoke since $t \leq p-1$ and $p \leq \lfloor \frac{n}{2} \rfloor$)

(b) For $i \geq n-t+1$: $a_{i,\,1} - a_i \stackrel{!}{=} -b_1\,(d_{i,\,1} - d_{i-1,\,1})$:

Will be shown via induction on i:

Start of Induction: $i = n-t+1$. Consider

$$\begin{aligned} P\colon\quad & X_{1,\,1} = \ldots = X_{p,\,p} = 1,\ X_{n-t,\,1} = X_{n-t+1,\,1} = 1, \\ & X_{j,\,2} = 1 \quad \forall\, j \in \{p+1, \ldots, n\} \backslash \{n-t, n-t+1\}, \\ & r_1 = d_{n-t+1,\,1},\ r_2 = max_i \big\{ d_{i,\,2} : i \ s.th.\ X_{i,\,2} = 1 \big\}. \end{aligned}$$

Clearly, $P \in \mathcal{F}$, and if we insert this point into 4.63, we get

$$\underbrace{(a_{n-t,\,1} - a_{n-t})}_{=-b_1 d_{n-t,\,1}\,(from\,4a)} + (a_{n-t+1,\,1} - a_{n-t+1}) + b_1 d_{n-t+1,\,1} = 0,$$

and $a_{n-t+1,\,1} - a_{n-t+1} = -b_1(d_{n-t+1,\,1} - d_{n-t,\,1})$ follows.

Step from i to $i-1$:

Assume that the claim holds up to an $i \geq n-t+1$. Set

$$\begin{aligned} P\colon\quad & X_{1,\,1} = \ldots = X_{p,\,p} = 1,\ X_{n-t,\,1} = \ldots = X_{i,\,1} = X_{i+1,\,1} = 1, \\ & X_{j,\,2} = 1 \quad \forall\, j \in \{p+1, \ldots, n\} \backslash \{n-t, \ldots, i+1\}, \\ & r_1 = d_{i+1,\,1},\ r_2 = max_i \big\{ d_{i,\,2} : i \ s.th.\ X_{i,\,2} = 1 \big\}. \end{aligned}$$

$P \in \mathcal{F}$ can easily be seen. If we insert the values above into 4.63, we obtain

$$\underbrace{(a_{n-t,\,1} - a_{n-t})}_{(*)} + \sum_{j=n-t+1}^{i} \underbrace{(a_{j,\,1} - a_j)}_{(**)} + (a_{i+1,\,1} - a_{i+1}) + b_1\, d_{i+1,\,1} = 0.$$

By step 4a, we have $(*) = -b_1 d_{n-t,\,1}$, and $(**) = -b_1(d_{i,\,1} - d_{i-1,\,1})$ follows by assumption of the induction. Hence, we can deduce $(a_{i+1,\,1} - a_{i+1}) = -b_1(d_{i+1,\,1} - d_{i,\,1})$.

ad (iii): Facet-defining (Case $t = p - 1$):

In (ii), the only time that $t \leq p - 2$ was needed was in case 1 of step 2a: We needed $t \leq p - 2$ to guarantee that $|\{1, \ldots, n - t - 1\} \backslash \{1, l, i\}| \geq p - 2$.

Now, if $t = p - 1$, the additional assumptions
"$p \leq \lfloor \frac{n-1}{2} \rfloor$" or "$d_{i,\,k} \leq d_{a_t,\,k}\ \forall\, i \notin A$"
can be used to prove case 1 of step 2a:

- Assume $t = p - 1$ and $p \leq \lfloor \frac{n-1}{2} \rfloor (\Leftrightarrow n - p - 1 \geq p)$. Then, $|\{1, \ldots, n - t - 1\} \backslash \{1, l, i\}| = n - t - 4 = n - p - 3 = n - p - 1 - 2 \geq p - 2$, and thus, step 2a can be shown exactly as in (ii).

- Assume $t = p - 1$ and $d_{i,\,1} \leq d_{n-1,\,1} \quad \forall\, i \leq n - t$.
  We have to show that
  $a_{i,\,l} = a_{i,\,1}$ for $i \leq n - t - 1$ and $l \in \{2, \ldots, n - t - 1\} \backslash \{i\}$.
  Choose $h_1, \ldots, h_{p-3} \in \{1, \ldots, n - t - 1 = n - p\} \backslash \{1, l, i\}$ pairwise different (possible since $p \leq n - p$) and consider

  P_1: $X_{1,\,1} = X_{l,\,l} = X_{n,\,n} = 1,\ X_{h_1,\,h_1} = \ldots = X_{h_{p-3},\,h_{p-3}} = 1,$
  $X_{j,\,1} = 1 \quad \forall\, j \in \{2, \ldots, n - 1\} \backslash \{l, h_1, \ldots, h_{p-3}\},$
  $r_1 = d_{n-1,\,1},\ r_l = d_{i,\,l}$
  P_2: all values as in P_1, except $X_{i,\,1} = 0,\ X_{i,\,l} = 1$.

  Note that P_1 and P_2 are feasible, since $r_1 = d_{n-1,\,1} \geq d_{i,\,1} = d_{i,\,1} X_{i,\,1}$ by assumption. $P_1, P_2 \in \mathcal{F}$ is easy to see, and inserting the points into 4.60 yields $a_{i,\,1} = a_{i,\,l}$.

□

Remark 4.5.15 (Graphical Interpretation)
The facet class given above can be interpreted quite similar to the special case presented in proposition 4.5.11: The right hand side term
$d_{a_1,\,k} X_{a_1,\,k} + \sum_{i=2}^{t} (d_{a_i,\,k} - d_{a_{i-1},\,k}) X_{a_i,\,k} + (d_{i_k,\,k} - d_{a_t,\,k} X_{i_k,\,k}$

considers, starting from the node a_1 that is nearest to k, the increase of the radius r_k when stepwise allocating new spokes to hub k, each one being further away from k than the ones considered so far. Figure 4.4 marks the distances that are added up in the increasing-distances facets (doubled lines) for an example with $t = 3$.

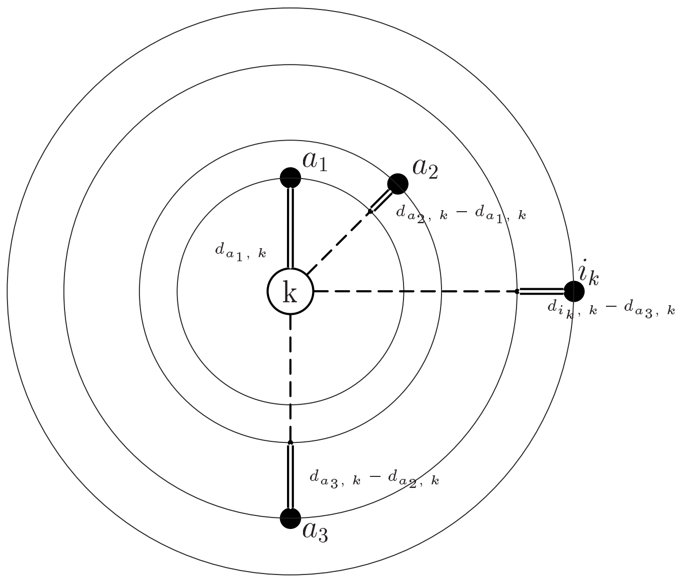


Figure 4.4: Graphical interpretation of increasing-distances facets

Remark 4.5.16 (number of facets in facet class)
Recall that the choice of $k = 1$ and $A = \{n - t, \ldots, n - 1\}$ has only been one possible way to choose a facet from the facet class described by 4.57. All in all, there are n ways to choose the node k and $\binom{n-2}{t}$ ways to choose the set A, resulting in a total sum of $n \sum_{t=1}^{p-2} \binom{n-2}{t}$ facets in the facet class (plus additional facets for the case that $t = p - 1$ and the additional conditions hold).

Remark 4.5.17 (Case $t = p - 1$ in general)
Assume that $d_{i,\,k} \neq d_{i+1,\,k}\ \forall\, i \in A$. Then, if $t = p - 1$ and neither $p \leq \lfloor \frac{n-1}{2} \rfloor$ nor $d_{i,\,k} \leq d_{a_t,\,k} \quad \forall\, i \notin A$, inequality 4.58 does *not* represent a facet:
From $p = \lfloor \frac{n}{2} \rfloor$ and p, n integer, one can deduce that $p \geq n - p$. Let $i \notin A$ be a node with $d_{i,\,k} \geq d_{a_t,\,k}$. Assume that $X_{i,\,k} = 1$ for some point $P \in \mathcal{F}$. Then, $r_k \geq d_{i,\,k} > d_{a_t,\,k}$. In this case, by construction of $\mathcal{F}$, the only way to guarantee that $P \in \mathcal{F}$ is to even out r_k on the left hand side of 4.57 by setting

$X_{a_1,\,1} = \ldots = X_{a_t,\,1} = X_{i_k,\,k} = 1$. But, if we do so, we have already chosen $t + 1 = p \geq n - p$ nodes to be spokes, meaning that i cannot be a spoke, too. Hence, $X_{i,\,k} = 0 \; \forall \; P \in \mathcal{F}$, and consequently, $\mathcal{F}$ is not a facet.

4.5.4 Summary of Non-elementary Facets

In section 4.5, the following inequalities have been shown to represent facets of $\mathcal{P}_{pUHL}$:

- **Spoke-concentration facets:**
  $$\mathbf{(n-p)X_{k,\,k} \geq \sum_{i \neq k} X_{i,\,k}}$$

  # facets in class: n
  see theorem : *4.5.1*

- **Focus-element facets:**
  $$\mathbf{\sum_{i \in A} X_{i,\,i} \geq \sum_{j \notin A} X_{j,\,a} + \sum_{i \in A \setminus (\{a\} \cup A^*)} X_{a,\,i} + \sum_{k \in A^*} X_{b_k,\,k}}$$

  prerequisites: $|A| = n-p,\ a \in A,$
  $A^* \subset A \setminus \{a\},\ |A^*| \in \{0, \ldots, n-p-2\},$
  $b_k \in \bar{A}\ \forall\ k \in A^*$ *pairwise different*
  # facets in class: $\binom{n}{p}\,(n-p)\,\sum_{i=0}^{n-p-2}[\,\binom{n-p-1}{i}\,\binom{p}{i}\,i!\,]$
  special cases: $A^* = \emptyset$:
  $\sum_{i \in A} X_{i,\,i} \geq \sum_{j \notin A} X_{j,\,a} + \sum_{i \in A \setminus \{a\}} X_{a,\,i}$
  see theorem : *4.5.3 and 4.5.6*

- **Increasing-distances facets:**
  $$\mathbf{r_k \geq d_{a_1,\,k} X_{a_1,\,k} + \sum_{i=2}^{t} (d_{a_i,\,k} - d_{a_{i-1},\,k}) X_{a_i,\,k} + (d_{i_k,\,k} - d_{a_t,\,k}) X_{i_k,\,k}}$$

  prerequisites: $p \leq \lfloor \frac{n}{2} \rfloor,\ i_k = arg\,max_i d_{i,\,k},$
  $\{k, i_k\} \not\subset A = \{a_1, \ldots, a_t\},\ t \in \{1, \ldots, p-1\},$
  $d_{a_i,\,k} \leq d_{a_{i+1},\,k}\ \forall\ i,$
  $\big(p \leq \lfloor \frac{n-1}{2} \rfloor$ *or* $d_{i,\,k} \leq d_{a_t,\,k}\ \forall\ i \notin A\big)$ *if* $t = p-1$
  # facets in class: $n\ \sum_{t=1}^{p-2} \binom{n-2}{t}$ *(for* $t \leq p-2$*)*
  special cases: $t = p-1,\ d_{a_1,\,k} \leq \ldots \leq d_{a_{n-1},\,k}$:
  $r_k \geq d_{a_{n-p},\,k} X_{a_{n-p},\,k} + \sum_{i=n-p+1}^{n-1} (d_{a_i,\,k} - d_{a_{i-1},\,k}) X_{a_i,\,k}$
  see theorem : *4.5.11 and 4.5.14*

Chapter 5

Separation

With the help of the new valid inequalities derived in section 4.5, a new exact solution algorithm to (**pUHL**) can be designed, which is expected to improve the CPU-times for solving problem instances to optimality. But, since we have identified a large number of new inequalities, it is highly inefficient to simply include all those inequalities to the original formulation and then start a simple branch-and-bound algorithm. Instead, a *branch-and-cut strategy* will be used to make sure that inequalities are only added to the original formulation if needed, i. e. if they cut off a current (non-integral) solution.

Definition: Assume that, for a problem instance of (**pUHL**), we are given a point P^* and a family of valid inequalities for (**pUHL**). The *separation problem* then consists of either finding an inequality in the above family that is violated by P^*, or proving that none exists. □

In the following, we will concentrate on the respective separation problems for the different classes of facets that were derived so far.

5.1 Separating Spoke-concentration Facets

Note that there are only n inequalities

$$(n-p)X_{k,\,k} \geq \sum_{\substack{i=1\\ i\neq k}}^{n} X_{i,\,k} \quad \forall\, k \in \{1,\ldots,n\}$$

representing the class of spoke-concentration facets (see theorem 4.5.1). Hence, a simple enumeration procedure will solve the corresponding separation problem efficiently.

5.2 Separating Focus-element Facets

Recall the inequalities representing the focus-element facet class, as given in theorem 4.5.6:

$$\sum_{i \in A} X_{i,\,i} \geq \sum_{j \notin A} X_{j,\,a} + \sum_{i \in A \setminus (\{a\} \cup A^*)} X_{a,\,i} + \sum_{k \in A^*} X_{b_k,\,k}$$

with $A \subset \{1, \ldots, n\}$, $|A| = n - p$, $a \in A$ an element of A, $A^* \subset A \setminus \{a\}$ with $|A^*| \in \{0, \ldots, n-p-2\}$, and $b_k \in \bar{A} \;\forall\, k \in A^*$ pairwise different.

In section 5.2.1, an exact solution algorithm will be presented for the special case that $A^* = \emptyset$ (see theorem 4.5.3). Section 5.2.2 then suggests a heuristic for the general separation problem, based on the exact algorithm of 5.2.1.

5.2.1 Focus-element Facets, Case $A^* = \emptyset$

Assume we are given a point $P^* = (\mathbf{X}^*, \mathbf{r}^*, z^*)$ and have to identify, if existent, an inequality of the focus-element facet class that is violated by P^*. That is, find a set A of $n - p$ nodes and a node a among those nodes such that the term $\sum_{j \notin A} X^*_{j,\,a} + \sum_{i \in A \setminus \{a\}} X^*_{a,\,i} - \sum_{i \in A} X^*_{i,\,i}$ is maximized; if the value of the term is strictly larger than zero, a violated inequality has been identified. It will turn out that this separation problem can be solved in polynomial time, using a greedy strategy[1].

In the following, the class of focus-element facets with $A^* = \emptyset$ will be split off in n subclasses regarding the element a:

For a fixed node a in $\{1, \ldots, n\}$, the separation problem reduces to finding $n-p-1$ further nodes such as to maximize $\sum_{j \notin A} X^*_{j,\,a} + \sum_{i \in A \setminus \{a\}} X^*_{a,\,i} - \sum_{i \in A} X^*_{i,\,i}$. Assume

[1]One first hint that a greedy solution might already be optimal, is the fact that we do not have any further restrictions on A despite that $|A| = n - p$.

without loss of generality that $a = n$. Introduce index variables $A_1, \ldots, A_{n-1}$ for the set A: $A_i = 1$ if node i is chosen as further element of A, $A_i = 0$ else.

With the help of the $(n-1)$-dimensional vector $\mathcal{A} = (A_1, \ldots, A_{n-1})$, the separation problem can be formulated as follows:

$$max_{\mathcal{A}} \quad \Big(\sum_{j=1}^{n-1} X^*_{j,\,a}(1 - A_j) + \sum_{i=1}^{n-1} X^*_{a,\,i}\; A_i - \sum_{i=1}^{n-1} X^*_{i,\,i}\; A_i - X^*_{a,\,a} \Big)$$

$$s.t. \quad \sum_{i=1}^{n-1} A_i \;=\; n - p - 1, \quad A_i \in \{0; 1\} \quad \forall\, i$$

The above objective function can be reformulated as follows:

$$\begin{aligned} & max_{\mathcal{A}} \Big(\sum_{j=1}^{n-1} X^*_{j,\,a}(1 - A_j) + \sum_{i=1}^{n-1} X^*_{a,\,i}\; A_i - \sum_{i=1}^{n-1} X^*_{i,\,i}\; A_i - X^*_{a,\,a} \Big) \\ = \;& max_{\mathcal{A}} \Big(\sum_{i=1}^{n-1} (X^*_{a,\,i} - X^*_{i,\,a} - X^*_{i,\,i})\, A_i + \sum_{i=1}^{n-1} X^*_{i,\,a} - X^*_{a,\,a} \Big) \\ = \;& max_{\mathcal{A}} \Big[\sum_{i=1}^{n-1} (X^*_{a,\,i} - X^*_{i,\,a} - X^*_{i,\,i})\, A_i \Big] + \underbrace{\sum_{i=1}^{n-1} X^*_{i,\,a} - X^*_{a,\,a}}_{const.} \end{aligned}$$

Thus, it suffices to solve the following integer optimization problem:

$$max_{\mathcal{A}} \Big\{ \sum_{i=1}^{n-1} (X^*_{a,\,i} - X^*_{i,\,a} - X^*_{i,\,i})\, A_i \;\Big|\; \sum_{i=1}^{n-1} A_i = n - p - 1, A_i \in \{0; 1\} \;\forall\, i \Big\}.$$

Since there is no further restriction on the set A, the above problem can be solved by calculating the terms $X^*_{a,\,i} - X^*_{i,\,a} - X^*_{i,\,i}$ for all $i = 1, \ldots, n-1$, sorting the i in decreasing order corresponding to the value of the above term and, for the first $n - p - 1$ nodes, setting $A_i = 1$.

As this greedy strategy has to be applied for all $a \in \{1, \ldots, n\}$, the complete separation algorithm has a complexity of $\mathcal{O}(n^2)$.

5.2.2 Focus-element Facets, General Case

Assume that, by the procedure described in 5.2.1, no violating inequality is found (and such, since we are using an exact solution algorithm for the separation of the facets so far, none exist). Though, it might be possible that the given point P^* violates an inequality of the focus-element facet class with $A^* \neq \emptyset$.

For a given set A with element a, it is quite easy to determine $A^* \subset A$ and nodes $\{b_k : k \in A^*\} \subset \bar{A}$ such that the term

$\sum_{j \notin A} X^*_{j,\,a} + \sum_{i \in A \setminus (\{a\} \cup A^*)} X^*_{a,\,i} + \sum_{k \in A^*} X^*_{b_k,\,k} - \sum_{i \in A} X^*_{i,\,i}$ (*) is maximized:

Start with $A^* = \emptyset$. For every node $k \in A$, determine a node $n_k \in \bar{A}$ with maximum value $X^*_{n_k,\,k}$. Starting with the node $k \in A$ with largest value $X^*_{n_k,\,k} - X^*_{a,\,k}$, check if $X^*_{n_k,\,k} > X^*_{a,\,k}$; if yes, substitute those addends in the above term[2], i. e. set $A^* := A^* \cup \{k\}$ and $b_k := n_k$. Stop if either $n - p - 2$ nodes have been substituted or the value of the term (*) exceeds zero.

However, as soon as the set A is not given any more, it does in general *not* suffice to choose A optimal for the case that $A^* = \emptyset$ and then apply the above procedure[3]. The general problem is that the sets A, A^* and the nodes b_k have to be determined simultaneously, but optimal choices of the nodes b_k can only be made once the sets A and A^* are known. Consequently, we will try to develop an approximate solution method for the problem: Start with the greedy approach described above, then apply local search to improve the given solution[4]:

Heuristic: Separation of Violated Focus-element Inequalities

For $a = 1, \ldots, n$

- Solve $\max_{\mathcal{A}} \left\{ \sum_{\substack{i=1 \\ i \neq a}}^{n} (X^*_{a,\,i} - X^*_{i,\,a} - X^*_{i,\,i}) A_i \;\middle|\; \sum_{\substack{i=1 \\ i \neq a}}^{n} A_i = n - p - 1, A_i \in \{0;1\}\ \forall\ i \right\}$ by greedy technique (as described in 5.2.1); let z be the optimal objective function value.
- Set $z := z + \sum_{\substack{i=1 \\ i \neq a}}^{n} X^*_{i,\,a} - X^*_{a,\,a}$, and $A := \{i : A_i = 1\} \cup \{a\}$.

---

[2]To guarantee that the b_k are pairwise different, a sort of "tabu list" containing all chosen b_k so far has to be kept; see the heuristic algorithm below.

[3]Intuitively, if we choose A worse for the general formula facets, we might save "better" candidates for b_k, such that the worse choice of A in the beginning could be compensated.

[4]In this context, a neighborhood of a set A^* could, for instance, be defined as the set of all 1-exchange sets to A^*.

- If $z > 0$: STOP:
  $\sum_{i \in A} X_{i,\,i} \geq \sum_{j \notin A} X_{j,\,a} + \sum_{i \in A\setminus\{a\}} X_{a,\,i}$
  is a violated inequality.

- Else

  – $nNodes := 0,\ A^* = \emptyset$

  – Calculate $n_k := arg\,max_i X^*_{i,\,k}$ for all $k \in A\setminus\{a\}$, and let $incr_k := X^*_{n_k,\,k} - X^*_{a,\,k}$ be the increase of z when adding node k to A^*. Let $A = \{a, a_1, \ldots, a_{n-p-1}\}$ with $incr_{a_i} \geq incr_{a_{i+1}}\ \forall\ i$.

  – For $i = 1, \ldots, n-p-1$:
    If $nNodes < n-p-1$ and $incr_{a_i} > 0$, then

    * $A^* := A^* \cup \{a_i\},\ A := A\setminus\{a_i\},\ b_{a_i} := n_{a_i}$
    * $z := z + incr_{a_i},\ nNodes+ = 1$
    * For all k, redefine $n_k := arg\,max_{i \notin A^*} X^*_{i,\,k}$, update $incr_k$ with these new n_k and resort $\{a_{i+1}, \ldots, a_{n-p+1}\}$ by increasing value of $incr_{a_j}$.[5]

  – Apply a local search procedure to improve the solution found.

  – If $z > 0$: STOP:
    $\sum_{i \in A} X_{i,\,i} \geq \sum_{j \notin A} X_{j,\,a} + \sum_{i \in A\setminus(\{a\}\cup A^*)} X_{a,\,i} + \sum_{k \in A^*} X_{b_k,\,k}$
    is a violated inequality.

5.3 Separating Increasing-distances Facets

We will now consider the separation problem for the increasing-distances facet class given by theorem 4.5.14:

$$r_k \geq d_{a_1,\,k} X_{a_1,\,k} + \sum_{i=2}^{t} (d_{a_i,\,k} - d_{a_{i-1},\,k}) X_{a_i,\,k} + (d_{i_k,\,k} - d_{a_t,\,k}) X_{i_k,\,k} \qquad (5.1)$$

with $i_k = arg\,max_i\ d_{i,\,k},\ \{k, i_k\} \not\subset A = \{a_1, \ldots, a_t\},\ t \leq p-1$ and $d_{a_i,\,k} \leq d_{a_{i+1}\,k}\ \ \forall\ i$. Recall that the above inequality represents a facet for $t \leq p-2$ in any case, but for $t = p-1$ only under additional conditions; but,

[5]This guarantees that the b_k are chosen pairwise different.

even if those additional conditions are not met, the inequality for $t = p - 1$ is at least valid.

Before considering the facet separation problem, here is an example of how an increasing-distances inequality can be violated: At first sight, it might seem obvious that, as soon as a point P^* satisfies all the single radius constraints $r_k^* \geq d_{i,\,k} X_{i,\,k}^*$, it will also fulfill all increasing-distances inequalities. However, for points P^* with non-integral $X_{i,\,k}^*$, this might not be true, as the following example illustrates:

Example: Consider a hub center problem instance where the distances between three nodes a_1, a_2, k are given by $d_{a_1,\,k} = 1$ and $d_{a_2,\,k} = 2$, and assume that $a_2 = arg\,max_i d_{i,\,k}$. Then, given a point P^* with $X_{a_1,\,k}^* = 1$, $X_{a_2,\,k}^* = 0.5$ and $r_k^* = 1$, the radius constraints $r_k^* \geq d_{a_1,\,k} X_{a_1,\,k}^*$ and $r_k^* \geq d_{a_2,\,k} X_{a_2,\,k}^*$ are both fulfilled, whereas we have that $r_k^* = 1 < 1.5 = d_{a_1,\,k} X_{a_1,\,k}^* + (d_{a_2,\,k} - d_{a_1,\,k}) X_{a_2,\,k}^*$.

As for the focus-element facet class, the separation problem for the increasing-distances facets will be considered for each choice of node k separately. For given k and a given point P^*, we have to determine a number t and nodes $a_1, \ldots, a_t$ such that the term
$d_{a_1,\,k} X_{a_1,\,k}^* + \sum_{i=2}^{t} (d_{a_i,\,k} - d_{a_{i-1},\,k}) X_{a_i,\,k}^* + (d_{i_k,\,k} - d_{a_t,\,k}) X_{i_k,\,k}^*$
is maximized; if this value is larger than r_k^*, a violated inequality is found.

Note that the above separation problem shows similarities to a Knapsack Problem: We are allowed to choose a maximum of $p - 1$ elements (corresponding to the volume constraint for Knapsack), where each element has a certain benefit, and the total benefit that has to be maximized. However, the contribution of an element a_i to the total benefit depends on its "neighbor-element" a_{i-1}, since it is calculated by $(d_{a_i,\,k} - d_{a_{i-1},\,k}) X_{a_i,\,k}^*$. Hence, it is impossible to apply the known Knapsack strategies, e. g. by running a branch-and-bound procedure: Each time an element is added to the knapsack, its benefit has to be determined dependent on the elements that have already been chosen; even more, the benefit of already chosen elements might change by such an addition.

Nevertheless, the considerations concerning Knapsack lead to an approach that is finally capable of solving our separation problem to optimality: Since the "benefit" of each element in A depends on this particular element and its "neighbor-element", a shortest path problem on a graph can be defined: Costs occur on the edges and are thus by construction dependent on both end points of the edge.

The construction of the desired graph is done like follows:

- Each node of the shortest path graph corresponds to a node of the hub network that can be chosen as an element of A.
- For each node i, include edges to all nodes j with $d_{j,\,k} > d_{i,\,k}$.
- For an edge from $i \neq k$ to j, include costs of $-(d_{j,\,k} - d_{i,\,k})X^*_{j,\,k}$.
  For an edge from k to i, include costs of $-d_{i,\,k}X^*_{i,\,k}$.

A sample graph with $n = 5$, $k = 1$, $i_k = 5$, $d_{2,\,1} < d_{3,\,1} < d_{4,\,1} < d_{5,\,1}$ fulfilling the above construction rules is given in figure 5.1.

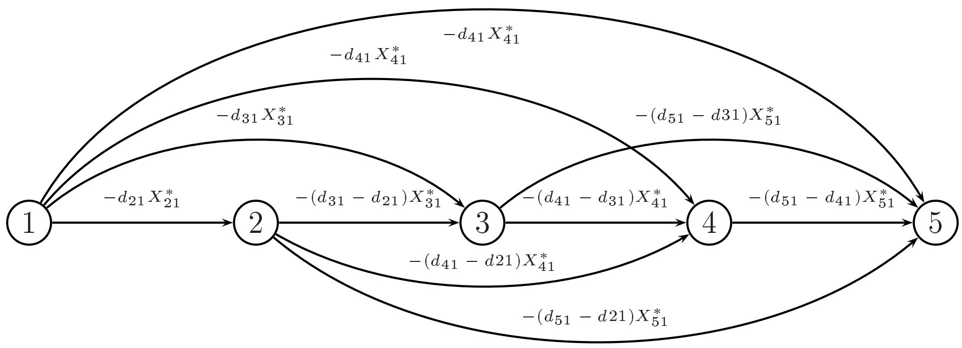


Figure 5.1: Shortest path graph for increasing-distances facet separation

For every path from k to i_k in the constructed graph, the nodes that have been traversed correspond to the choice of elements in set A. The cost of a path is equal to $-rhs(5.1)$ for the particular choice of A. Hence, the shortest path in the constructed graph yields a set A with maximum value $rhs(5.1)$. If this value is larger than $r_k = lhs(5.1)$, a violated increasing-distances inequality has been identified.

Remark 5.3.1

Some nodes and arcs in the above graph can be thrown out right away, such that we obtain a sparser graph:

- Nodes i with $X^*_{i,\,k} = 0$:

  Let $\{1, \ldots, n\} = \{k, a_1, \ldots a_{n-1}\}$, with $d_{a_1,\,k} \leq d_{a_2,\,k} \leq \ldots \leq d_{a_{n-1},\,k}$. Assume that $X^*_{a_i,\,k} = 0$ for an $i \in \{2, \ldots, n-2\}$, and that node a_i is part of a shortest path from k to a_{n-1} in the above graph. Let j be maximal such that a_j is part of the shortest path and $d_{a_j,\,k} \leq d_{a_i,\,k}$. Similar, let l be minimal such that a_l is part of the shortest path and $d_{a_l,\,k} \geq d_{a_i,\,k}$. Then,

$$\begin{aligned} -(d_{a_i,\,k} - d_{a_j,\,k})X^*_{a_i,\,k} - (d_{a_l,\,k} - d_{a_i,\,k})X^*_{a_l,\,k} &= -(d_{a_l,\,k} - d_{a_i,\,k})X^*_{a_l,\,k} \\ &\geq -(d_{a_l,\,k} - d_{a_j,\,k})X^*_{a_l,\,k}. \end{aligned}$$

  Consequently, when omitting a_i in the shortest path, the resulting new path from k to a_{n-1} is as least as short. A similar argumentation holds for $a_i = a_1$. Hence, we can right away throw out from the constructed graph all nodes $i \notin \{k,\ i_k\}$ with $X^*_{i,\,k} = 0$.

- Arcs (i, j) with $d_{i,\,k} = d_{j,\,k}$:

  Using a similar argumentation as above, we can throw out all edges (i, j) with $d_{i,\,k} = d_{j,\,k}$.

Remark 5.3.2
When reformulating the separation problem as a shortest path problem, we left out of consideration that $p \leq \lfloor \frac{n}{2} \rfloor$ and $t \leq p-1$. But, as proven in theorem 4.5.14, the increasing-distances inequalities are valid for all possible p and t. Hence, we can enlarge the search for violated inequalities to this more general case.

The constructed graph has n nodes and $\mathcal{O}(n^2)$ edges, such that a shortest path from k to i_k can be found in polynomial time with respect to n, e.g. using the Bellman-Ford algorithm, which is described in [11].

Chapter 6

Computational Results

In chapter 4 above some new classes of facet-defining inequalities for the radius formulation of **USApHCP** have been presented; the separation of violated inequalities of these types has been dealt with in chapter 5. Now, we will present a branch-and-cut solution algorithm for **USApHCP** which is based on these new results; the performance of this algorithm has been tested for different hub location problem instances.

The algorithm has been implemented in the *Mosel* programming language of Xpress-MP [13], using the Xpress-IVE (**I**ntegrated **V**isual **E**nvironment) optimization software (Version 1.14.10, Xpress Optimizer Version 14.10 [14]). Two boolean parameters *automaticcuts* and *facetcuts* have been included, which indicate whether the automatic cutting procedure of the Xpress Optimizer is enabled, and whether cuttings derived from the new facet inequalities are included. By adjusting these parameters, three different algorithms have been examined:

- $automaticcuts = facetcuts =$ false:
  pure branch-and-bound, without introducing cuttings at the different nodes. In the tables below, this algorithm will be denoted by **B&B**.

- $automaticcuts =$ true, $facetcuts =$ false:
  branch-and-cut, using the inbuilt automated cutting generation procedure of Xpress. In the tables this is denoted by **B&C-A**.

- *automaticcuts* = false, *facetcuts* = true:
  branch-and-cut, using cuts that are generated from violated facet inequalities of one of the new classes (spoke-concentration inequalities, focus-element inequalities, increasing-distances inequalities). The automated cutting generation procedure is disabled. This algorithm will be denoted by **B&C-F**.

Cuts from violated facet-inequalities are generated using the separation procedures described in chapter 5, but only for nodes up to a depth of *cuttingdepth*. This parameter can be adjusted by the user; for the tests presented, a cutting depth of both 3 and 5 has been examined – see the columns **B&C-F(3)** and **B&C-F(5)**.

Regarding focus-element inequality separation, only the greedy heuristic is applied, and further local search heuristics are left out of consideration (see section 5.2).

For the evaluation of the algorithm variations above, the common *CAB data set* has been used: It was evaluated by the Civil Aeronautics Board and contains data on passenger airline travel between 25 large US cities; see [26]. This test set is available through the OR-library [5] via internet. Subsets of the data set have been derived by variating the number of nodes, number of hubs and the discount factor[1]:

$n \in \{10, 15, 20, 25\}$, $p \in \{2, 3, 4\}$, $\alpha \in \{0.25, 0.5, 0.75\}$.

For each problem, the optimal solution with objective function value and an optimal hub set has been determined (columns **o.f.v.** and **hub set**). The CPU time (in seconds) for running each algorithm has been measured (columns **CPU**); finally, for the two algorithms **B&C-F(3)** and **B&C-F(5)** that include the new facet cuttings, the number of added cuts of each class has been determined: **1** denotes the number of spoke-concentration cuts, **2** the number of focus-element cuts and **3** the number of increasing-distances cuts.

All computations have been carried out on an AMD Athlon processor with 700 MHz and 128 MB RAM. The results are given in tables 6.1-6.3.

[1] For the CAB data, this is traditionally done by using only the first 10, 15, 20 or all 25 nodes of the example.

A first look at the CPU times (tables 6.1 and 6.2) shows that the introduction of cuttings only seems to make sense for larger-sized problems ($n \in \{20, 25\}$). For smaller problems ($n \in \{10, 15\}$), pure branch-and-bound performs best, but branch-and-cut using the new facet-cuttings (B&C-F(3) and B&C-F(5)) is at least faster as branch-and-cut with automatic cutting (B&C-A) for most problem instances.

The larger the number n of nodes gets, the more efficient a branch-and-cut strategy is, compared to pure branch-and-bound. Comparing the different branch-and-cut algorithms that have been tested, the automatic cutting algorithm B&C-A is outperformed by either B&C-F(3) or B&C-F(5) in roughly 70 % of all cases with $n \geq 15$.

Concerning the choice of the parameter *cuttingdepth*, no clear preference between a depth of 3 and 5 can be made based on the given data. The larger the cutting depth, the more time is consumed to search for violated facet-inequalities; but, if violated inequalities are found, these newly introduced cuttings give hope to reduce the total number of branch-and-bound nodes and such to accelerate the solution process. Table 6.3 lists the number of nodes in the branch-and-bound tree that are used to solve the respective problem instance. But, based on the data that has been tested, no clear statement can be made whether or not the new algorithm reduces the number of branch-and-bound nodes. A deeper investigation should include more and larger test data sets. Furthermore, different possible branching strategies should be taken into account, rather than just examining the branch-and-bound nodes in depth-first manner as done here; this, however, is beyond the scope of this thesis.

When considering the number of introduced facet-cuttings of each type, the percentage of violated increasing-distances facets (type **3** in the tables) is remarkably high. This might hint at the fact that the use of facet-cuttings of this type is highly efficient in a branch-and-cut algorithm. Future research should include a more thorough examination of this effect.

Probl.			Opt. Solution		B&B	B&C-A	B&C-F(3)				B&C-F(5)			
n	p	α	o.f.v.	hub set	CPU	CPU	CPU	cuts			CPU	cuts		
								1	2	3		1	2	3
10	2	0.25	1476.07	{6, 7}	1.2	2.6	1.5	3	3	21	1.5	2	4	31
10	2	0.5	1728.49	{6, 7}	0.8	1.9	1.3	1	2	11	1.5	1	4	16
10	2	0.75	1759,13	{5, 8}	0.7	1.8	1.2	1	2	7	1.0	2	5	9
10	3	0.25	1119.53	{6, 7, 8}	1.2	3.3	1.5	1	0	5	1.3	1	0	8
10	3	0.5	1286.03	{6, 7, 8}	1.1	2.2	1.3	0	0	16	1.8	0	0	24
10	3	0.75	1538.46	{6, 7, 8}	1.3	1.2	1.5	0	0	7	1.4	0	0	12
10	4	0.25	858.22	{3, 5, 7, 8}	1.3	3.0	1.7	0	0	11	1.8	1	0	17
10	4	0.5	1047.62	{3, 5, 7, 8}	1.0	1.9	1.3	0	1	15	1.2	0	0	19
10	4	0.75	1377.38	{3, 5, 7, 8}	0.6	1.7	1.3	0	0	12	1.2	0	1	15
15	2	0.25	2059.04	{5, 8}	5.3	11.7	8.3	0	0	15	8.6	1	0	25
15	2	0.5	2160.75	{5, 12}	4.2	6.9	3.2	0	0	8	3.5	1	0	11
15	2	0.75	2343.4	{12, 13}	2.9	3.3	3.7	0	0	7	3.8	0	0	8
15	3	0.25	1760.15	{6, 12, 13}	9.5	15.3	8.6	0	0	16	10.2	1	0	22
15	3	0.5	1760.15	{6, 12, 13}	3.5	10.3	11.0	0	0	20	10.1	0	0	30
15	3	0.75	2086.13	{3, 12, 13}	4.9	5.9	4.6	0	1	13	7.9	0	2	22
15	4	0.25	1361.42	{2, 11, 12, 14}	10.0	15.4	9.5	0	0	16	11.5	0	0	26
15	4	0.5	1530.41	{2, 11, 12, 14}	7.1	6.6	5.9	0	0	15	6.6	0	0	24
15	4	0.75	1979.01	{3, 11, 12, 14}	8.0	8.4	4.3	0	0	14	6.0	0	0	24

Table 6.1: Numerical results for CAB test sets (1): CPU times

Probl.			Opt. Solution		B&B	B&C-A	B&C-F(3)				B&C-F(5)			
n	p	α	o.f.v.	hub set	CPU	CPU	CPU	cuts			CPU	cuts		
								1	2	3		1	2	3
20	2	0.25	1933.42	{5, 19}	20.8	27.3	27.6	0	0	16	31.3	0	0	26
20	2	0.5	2224.11	{5, 19}	21.5	23.6	21.3	0	0	10	33.3	1	0	18
20	2	0.75	2444.89	{13, 19}	22.2	14.0	35.1	0	0	13	11.4	0	0	20
20	3	0.25	1635.37	{9, 16, 19}	33.9	62.3	47.0	0	0	14	52.4	0	0	19
20	3	0.5	1871.24	{6, 13, 19}	26.6	52.5	35.7	0	1	20	48.7	0	1	29
20	3	0.75	2187.63	{13, 17, 19}	52.0	30.7	51.6	0	0	10	62.2	0	0	18
20	4	0.25	1361.42	{2, 11, 14, 19}	71.6	63.4	61.4	0	0	21	38.0	0	2	32
20	4	0.5	1650.81	{11, 14, 17, 19}	123.7	63.2	49.3	0	0	16	46.4	0	0	21
20	4	0.75	2086.13	{12, 13, 17, 19}	65.0	29.9	47.1	0	0	10	57.5	0	0	17
25	2	0.25	2194.52	{21, 22}	66.9	68.5	50.7	0	0	17	63.2	0	0	24
25	2	0.5	2480.64	{8, 21}	68.6	55.5	70.4	0	0	24	66.1	0	0	35
25	2	0.75	2675.88	{8, 21}	54.5	74.8	33.6	0	0	9	30.8	0	0	13
25	3	0.25	2001.65	{5, 19, 23}	131.3	275.4	221.8	0	0	34	241.7	0	1	37
25	3	0.5	2218.32	{12, 21, 23}	344.7	289.3	160.9	0	0	16	417.5	0	0	25
25	3	0.75	2500.24	{8, 16, 20}	253.8	130.0	403.6	0	0	22	258.0	0	0	32
25	4	0.25	1703.61	{9, 16, 19, 23}	119.5	296.7	202.4	0	0	38	211.6	0	0	48
25	4	0.5	2045.65	{2, 12, 13, 23}	256.1	478.5	257.7	0	1	14	298.8	0	1	23
25	4	0.75	2372.12	{19, 21, 22, 23}	1227.9	285.2	275.6	0	0	19	237.3	0	0	29

Table 6.2: Numerical results for CAB test sets (2): CPU times

Probl.			B&B	B&C	B&C	B&C
n	p	α		A	F(3)	F(5)
10	2	0.25	61	74	78	65
10	2	0.5	65	59	85	99
10	2	0.75	24	103	72	32
10	3	0.25	84	137	105	91
10	3	0.5	180	101	148	219
10	3	0.75	166	82	158	126
10	4	0.25	209	144	180	229
10	4	0.5	162	114	151	147
10	4	0.75	22	181	275	236
15	2	0.25	182	272	301	285
15	2	0.5	127	84	104	105
15	2	0.75	55	55	64	88
15	3	0.25	810	891	474	713
15	3	0.5	56	424	825	559
15	3	0.75	107	127	110	244
15	4	0.25	1017	749	603	1028
15	4	0.5	608	571	425	440
15	4	0.75	1074	599	319	234
20	2	0.25	434	253	441	733
20	2	0.5	395	57	347	1101
20	2	0.75	680	100	705	195
20	3	0.25	2030	1501	2647	3232
20	3	0.5	875	2523	1498	2300
20	3	0.75	3388	1002	2665	4041
20	4	0.25	6196	2070	3707	1929
20	4	0.5	11091	4604	3738	3287
20	4	0.75	6368	1799	2683	4326
25	2	0.25	1396	388	456	626
25	2	0.5	1200	349	531	443
25	2	0.75	928	904	225	254
25	3	0.25	6156	1576	7787	10232
25	3	0.5	18814	13744	5874	19200
25	3	0.75	14223	5022	17494	11870
25	4	0.25	6377	11172	8873	9387
25	4	0.5	15590	17951	12140	15528
25	4	0.75	68311	14788	13656	10477

Table 6.3: Numerical results for CAB test sets (3): no. of b&b nodes

Chapter 7

Conclusions and Outlook

7.1 Summary of Main Results

This diploma thesis dealt with hub location problems with center objective. Several problem variations have been discussed, and some known formulations for the uncapacitated single-allocation p-hub center problem have been listed; the problem is known to be $\mathcal{NP}$-hard.

The polyhedron of the newly developed radius formulation by Ernst et al. [16] has been investigated: Using a connection to a special kind of uncapacitated facility location polyhedron, the dimension of the radius formulation polyhedron could be derived, and it has been examined whether the given problem constraints are facet-defining or not. In the core part of this thesis, three new classes of facet-defining inequalities have been derived, and a separation procedure has been discussed for each of these classes.

The new results are used to design a branch-and-cut algorithm where cuttings represent violated inequalities of one of the new types. First numerical evaluations show that this algorithm can compete in CPU time with an automated cutting procedure. A combination of the new branch-and-cut algorithm with an automated cutting procedure might be able to solve problem instances with even less time effort; hence, also larger problem instances –for which only heuristic solutions exist up to now– could be solved to optimality within a reasonable amount of time.

7.2 Further Research

7.2.1 Polyhedral Analysis for $\mathcal{P}_{pUHL}$

Though a number of new valid inequalities has already been derived for the radius formulation of **USApHCP**, it is worth investigating the polyhedron $\mathcal{P}_{pUHL}$ more thoroughly. A promising approach might be to search for valid inequalities that include more than just one variable r_k and thus to obtain information on the connection between the different radius variables. Furthermore, tight valid inequalities that include the variable z might be of great use. However, attempts to deduce a facet-defining inequality using z failed so far. Since this variable describes the objective function value, such inequalities are probably difficult to find at all.

Another interesting way of deriving new facets for (**pUHL**) might be to examine the (**pUFL**) problem in more detail, to find connections to the general (**UFL**) problem and to transfer known facet results for (**UFL**).

Concerning the seperation problem of the class of focus-element facets that have been derived in this thesis, a local search algorithm was suggested. Details on the design of this procedure still have to be developed, and comparisons to other heuristics such as tabu search have to me made. In fact, the complexity of the separation problem for focus-element inequalities is still an open issue.

To test the performance of the new branch-and-cut algorithm of chapter 6, further numerical examinations have to be done. These should include an examination of the optimal cutting depth to be used within the algorithm. Furthermore, the percentage of identified violated increasing-distances inequalities during the algorithm has to be investigated more thoroughly. Additionally, numerical experiments should not only consider the total running time to solve a problem instance, but also compare the CPU times that are needed to determine a solution that is ”close enough” to a known lower bound. Other branch-and-bound and branch-and-cut approaches from literature should be considered as well.

To test the algorithm’s performance for larger data sets, the *AP data set* should be taken into account; this test set consists of 200 nodes, is provided by the

Australian Post and can, as well as the CAB data set, be obtained via internet [5]. In this context, it is worth examining a combination of the new branch-and-cut algorithm with an automated cutting procedure as provided in an integer program solver such as Xpress. If, using the new procedure, optimal solutions to large problem instances are available, these can be used to test the performance of known heuristics in large scale.

7.2.2 Multiple Allocation Hub Center

This thesis concentrated on the single allocation case of hub center location. Nevertheless, the multiple allocation problem variation is of great interest, e. g. when modelling a passenger airline network, where the hub airport used depends on the final destination of the flight passenger. Concerning multiple allocation hub median problems, Sonneborn [32] presented useful work on polyhedral analysis. An attempt to transfer these results to the center case (where the main difference to Sonneborn's problem is that the number of hubs is fixed a priori) seems to be promising.

One main reason why we dealt with the single allocation hub center problem was the new radius formulation, which was developed for this problem variation. At first sight, the concept of a hub radius only makes sense for single allocation. However, if we assume that a radius depends on two hubs simultaneously, it is possible to transfer the radius idea to the multiple allocation case:

The following problem formulation of **UMApHCP** is based on a formulation for the median case by Skorin-Kapov et al. [31], reformulated by Sonneborn [32] to reflect the idea of multicommodity flows:

Let $\mathcal{K}$ be the set of all origin-destination pairs $k = (k_1,\ k_2)$, seen as the set of *commodities* that have to be shipped through the network. Let $\mathcal{H}$ be the set of potential hubs, and $Y_j = 1$ if a hub is opened at node j, $Y_j = 0$ else. Denote by the variable $X_{i,\,j,\,k}$ the allocation of a commodity k: $X_{i,\,j,\,k} = 1$ if and only if commodity k is shipped via hubs i and j (in that order).

Now, for every commodity that is shipped via hubs i and j, one can add up the lengths of the spoke arcs of this connection. The maximum such sum for two hubs i and j is then referred to as the "radius" $r_{i,\,j}$ of i and j:

(**MA – HCP – Rad**)

$$
\begin{aligned}
& \min z && && (7.1)\\
s.t. \quad & z && \geq r_{i,\,j} + \alpha\, d_{i,\,j} \ \forall\, i, j \in \mathcal{H} && (7.2)\\
& r_{i,\,j} && \geq X_{i,\,j,\,k}\,(d_{k_1,\,i} + d_{k_2,\,j}) && \\
& && \quad \forall\, i, j \in \mathcal{H},\ k = (k_2,\ k_2) \in \mathcal{K} && (7.3)\\
& \sum_{i\in\mathcal{H}}\sum_{j\in\mathcal{H}} X_{i,\,j,\,k} && = 1 \ \forall\, k \in \mathcal{K} && (7.4)\\
& \sum_{j\in\mathcal{H}} X_{i,\,j,\,k} && \leq Y_i \ \forall\, i \in \mathcal{H}, k \in \mathcal{K} && (7.5)\\
& \sum_{i\in\mathcal{H}} X_{i,\,j,\,k} && \leq Y_j \ \forall\, j \in \mathcal{H}, k \in \mathcal{K} && (7.6)\\
& \sum_{j\in\mathcal{H}} Y_j && = p && (7.7)\\
& X_{i,\,j,\,k} && \in \{0,\ 1\} \ \forall\, i, j \in \mathcal{H}, k \in \mathcal{K} && (7.8)\\
& Y_j && \in \{0,\ 1\} \ \forall\, j \in \mathcal{H} && (7.9)\\
& r_{i,\,j} && \geq 0 \ \forall\, i, j \in \mathcal{H} && (7.10)
\end{aligned}
$$

Constraints 7.4 - 7.9 are directly lent from the multicommodity formulation given by Sonneborn [32]: Constraint 7.4, together with the integrality constraints 7.8, describes that each commodity has to be allocated to exactly one pair of hubs. Constraints 7.5 and 7.6 guarantee that k can only be allocated to i and j if both i and j are hubs. Constraint 7.7 states that the number of hubs to be located is a priori fixed to p.

Now, consider the two groups of constraints 7.2 and 7.3 that contain the radius variables $r_{i,\,j}$: By constraint 7.3, the radius $r_{i,\,j}$ of two hubs is defined as the maximum sum of spoke arc lengths for any commodity that is transported via i and j. Constraints 7.2 state that the maximum transportation cost in the hub network is the maximum sum of radius and (discounted) hub arc length for any pair of hubs.

Though the notion of "radius" does not fit very well, once such "radius" depends on two different hubs, the basic concept from the single allocation case has been conserved in the above construction. This new multiple allocation radius formulation should be examined regarding its behaviour in computational tests, and it might be worth considering the corresponding polyhedron, as it is done in this thesis for the single allocation case.

7.2.3 Hub Covering

Center location problems and covering problems are tightly linked together: In a center location problem, the maximum distance from a customer to (or via) a facility is minimized; for covering problems, such maximum allowed distance is fixed a priori and facilities should be located such that all customers are covered. In [6], Campbell defined the *hub covering problem* as the covering problem corresponding to center hub location: He considers the multiple allocation case, where pairs of hubs cover a pair of origin- and destination node, and presents several problem formulations.

Now, it seems like there exists a link between hub covering and the radius concept, presented in section 3.3: If the distance between a node i and a hub k is at most as large as the hub radius r_k, we can say that i is covered by hub k. But, as already stated by Campbell, different to other location problems, we have to take the interaction between the hub facilities into account. Consequently, it does not solve the hub center problem to minimize the maximum hub radius such that all nodes are covered. A counter example is given below.

Example: Consider the hub network given in figure 7.1, and assume that distances in the network are proportional to the given arc lengths. As shown in the graph, $d_{3,\,4} - d_{2,\,4} =: \varepsilon > 0$ is very small compared to $d_{1,\,2} - d_{1,\,3} =: \delta \gg \varepsilon$. Now, if we choose nodes 1 and 2 as hub nodes (black arcs), the maximum radius is $r_2 = d_{2,\,4}$. If nodes 1 and 3 are opened as hubs instead, this maximum radius increases to $r_3 = d_{3,\,4}$; however, we have substantially decreased the length of the hub arc from $d_{1,\,2}$ to $d_{1,\,3}$. Hence, though the covering radius is not minimal for this second solution, the objective function value decreases (given that the discount factor α is large enough).

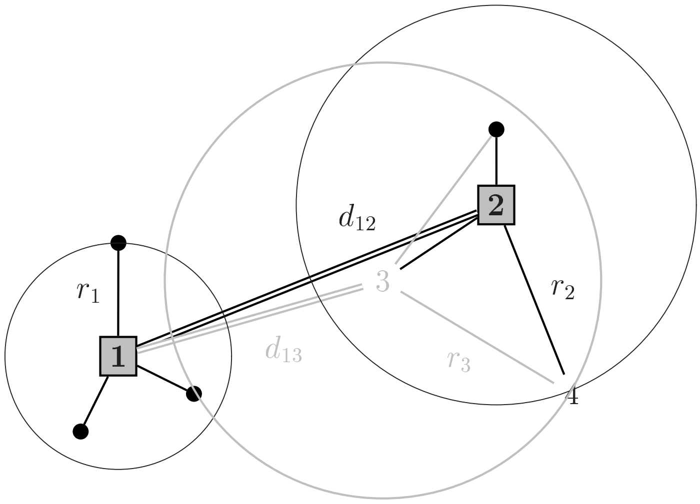


Figure 7.1: Connection between hub covering and hub radii

Future reseach should deal with the relationship between hub covering and hub radii in more detail, such that, hopefully, new solution approaches to the hub center problem can be developed.

7.2.4 Further Problem Variations

Though a couple of problem types have already been discussed in hub location literature, there is still a large potential in refining given models in order to mirror the real-world circumstances more precisely.

Concerning hub center problems, we have requested up to now that *all* flow in the network has to be transported within the given time (which has to be minimized). Now, in a postal delivery network, for instance, guaranteed transportation times are often given only for some standard routes, and transportation on other, minor routes, is still allowed to exceed these time limits. One straightforward approach to model this would be to include only those origin-destination pairs to our problem which exceed a certain flow value. A more sophisticated advance might a priori give a certain percentage of the total flow that has to be transported within the given time (e. g. 95 %). It is even imaginable to work with two objective functions simultaneously: To minimize the guaranteed transportation time and to maximize the total flow percentage that can be shipped within this time limit.

A second idea to modify given problem types can be derived from the passenger airline example: Here, the convenience of the passenger is of high interest [8] and might well determine the passenger's decision to choose an airline. For instance, if service between two major cities is only provided using a two-hub stop, passengers might change to another airline to avoid this inconvenience. Hence, the type of service provided between two nodes of a hub network might have an impact on the amount of flow in between those two nodes. Some further literature on this topic is given in [8].

Finally, future research should deal with more sophisticated designs of hub networks: In the introductory chapter 1, the notion of a *two-level network* was used to refer to a hub network system. However, a network construction with more than just two levels might be worth of consideration: Some logistics providers, e. g. Federal Express, have already put this concept into reality by introducing a couple of (regional) "mini-hubs" and one or few "major hubs" (see [23]): Hubs lying on the first level of the network are completely interconnected, whereas second-level hubs are only connected to one (or more) first-level hubs each; finally, nodes on the third level (spokes) are only connected to one (or more) second-level hubs. Some first hints at the mathematical formulation of such *hierarchical hub network problems* can be found in [8].

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