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Splines on the 3-dimensional Ball and their Application to Seismic Body Wave Tomography

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Abstract. In this paper we construct spline functions based on a reproducing kernel Hilbert space to interpolate/approximate the velocity field of earthquake waves inside the Earth based on traveltime data for an inhomogeneous grid of sources (hypocenters) and receivers (seismic stations). Theoretical aspects including error estimates and convergence results as well as numerical results are demonstrated.

Key Words: inverse problem, splines, Sobolev spaces, body wave tomography, integral equation, localizing basis

AMS(2000) Classification: 45Q05, 41A15, 86A15.

1. Introduction

Although major earthquakes cause serious harm at irregular intervals, fortunately the big majority of all earthquakes does not cause relevant damage or even is not felt by any human being. Independent of the consequences of such an event, seismic data reveal secrets of the Earth's interior. One typical data source of this kind are traveltimes of earthquake waves. The time a wave needs from the source, i.e. the hypocenter, to the receiver, i.e. a seismic station, depends on the spatially varying speed of the wave on its journey through the Earth or across the Earth's surface. This paper is focussed on the first case, i.e. on body waves, which travel through the Earth's interior. The analysis of the traveltimes allows the approximate determination of the velocity field. This is a typical example of an inverse problem. It is related to the Radon transform of computerized tomography, where, however, in seismic traveltime tomography the rays of the waves are essentially curvilinear. Numerous authors have studied this problem from theoretical as well as numerical point of view. A brief and certainly incomplete list of examples is [1], [3], [6], [7], [9], [19], [20], [37], [38], [40]. However, it should be mentioned that surface wave tomography can be used to study the deeper structure of the Earth as well (see e.g. [32], [33]). In contrast to surface wave tomography a parametrization for the radial dependence of the unknown velocity field, or more general a basis system for functions on the 3D ball, has to be chosen for numerically solving the body wave tomography problem. We will demonstrate here that reproducing kernel based splines are an appropriate tool for this purpose.

This paper consists of two parts: First, we derive the theoretical details of a spline interpolation/approximation method for functions on a three–dimensional ball. The given data are here represented as linear and continuous functionals that are applied to the unknown function. This approach is motivated by the spherical spline approach based on spherical harmonics as introduced in [14], [15], [16], [17]. This way of constructing approximating structures out of a reproducing kernel has been applied to different constellations in the meantime, see e.g. [13], [18], [21], [24], [25], [36]. Following this line the present paper discusses the specific aspects of the transfer of the approach to the 3-dimensional ball (see also [2]). The name "spline" refers here to the fact that the interpolating spline minimizes a certain Sobolev–like norm among all interpolating functions, where this norm can be regarded as a kind of measure for the oscillatory behavior of the function. The characteristic minimum properties as well as an error estimate and a convergence result are proved. In the second part we prove that the theoretical conditions for the application of the spline method to the traveltime tomography are satisfied. Then we show the results of some numerical tests. For this purpose we use certain given functions for the velocity distribution and calculate the corresponding traveltimes as data for the spline approximation. These computations include the recovery of a radially symmetric velocity model as well as a laterally heterogeneous function. We obtain very good approximations in comparison to the used reference velocity fields.

Note that there also exist different approaches to construct approximating tools out of reproducing kernel Hilbert spaces such as in [4], [30], [31]. Furthermore, we remark that in [10], [11] the idea of using reproducing kernels for regularizing ill-posed problems is established in detail for Fredholm integral equations of first kind on $\mathrm{L}^2[0,1]$, where the probable extendability to further reproducing kernel Hilbert spaces is stated. A generalization of this to arbitrary real Hilbert spaces can be found at [12] which partially provides a basis for further results and applications presented here. In [27], [28] reproducing kernel based approaches are used to approximate the generalized inverse of an operator from an arbitrary Hilbert space into a set of real-valued functions on an interval or in between two such sets. Those approaches are related to this one presented here in the idea of using reproducing kernels for regularizing ill-posed problems, but are clearly different in the technical details and realization.

2. Jacobi Polynomials

In this paper $\mathbb{N}$ represents the set of all positive integers, where $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, and $\mathbb{R}$ denotes the set of all real numbers.
Let $b > 0$ and $a > b-1$ be given real numbers. The *Jacobi polynomials* are defined by the following Rodriguez's formula ([23])

$$G_n(a,b;x) := \frac{(-1)^n\Gamma(n+a)}{\Gamma(2n+a)} x^{1-b}(1-x)^{b-a}\left(\frac{d}{dx}\right)^n \left(x^{n+b-1}(1-x)^{n+a-b}\right)$$

for $n \in \mathbb{N}_0$ and $x \in [0,1]$, where Γ is the Gamma function.
The Jacobi polynomials $\{G_n(a,b;x)\}_{n\in\mathbb{N}_0}$ are the only polynomials to satisfy the following properties for all $n \in \mathbb{N}_0$:

(*i*) $G_n(a,b;\cdot)$ is a polynomial of degree n, defined on $[0,1]$.

(*ii*) $G_n(a,b;0) = 1$.

(*iii*) $\displaystyle\int_0^1 x^{b-1}(1-x)^{a-b}G_n(a,b;x)G_m(a,b;x)dx = 0 \quad \text{for all} \quad m \in \mathbb{N}_0\backslash\{n\}.$

Note that every integral in this paper is a Lebesgue integral.
In case of $m = n$, we have

$$\int_0^1 x^{b-1}(1-x)^{a-b}G_n(a,b;x)G_n(a,b;x)dx = n!\frac{\Gamma(a+n)\Gamma(b+n)\Gamma(a-b+n+1)}{(2n+a)[\Gamma(a+2n)]^2}.$$

Hence, the system $\{\tilde{G}_n(a,b;\cdot)\}_{n\in\mathbb{N}_0}$ defined by

$$\tilde{G}_n(a,b;x) := \left[\frac{(2n+a)[\Gamma(a+2n)]^2}{n!\Gamma(a+n)\Gamma(b+n)\Gamma(a-b+n+1)}\right]^{1/2} G_n(a,b;x), \tag{1}$$

will be orthonormal in $\mathrm{L}^2[0,1]$ with the weight function $w(x) = x^{b-1}(1-x)^{a-b}$. Moreover, it is known that the system $\{\tilde{G}_n(a,b;\cdot)\}_{n\in\mathbb{N}_0}$ is closed in $\mathrm{C}[0,1]$.
Note that one finds an alternative definition in the literature (see e.g. [34]), where the

functions $P_n^{(\alpha,\beta)}$, $n \in \mathbb{N}_0$, with $\alpha, \beta > -1$ fixed, are called Jacobi polynomials if they satisfy the following properties for all $n \in \mathbb{N}_0$:

(i) $P_n^{(\alpha,\beta)}$ is a polynomial of degree n, defined on $[-1, 1]$.

(ii) $\displaystyle\int_0^1 (1-x)^\alpha (1+x)^\beta P_n^{(\alpha,\beta)}(x) P_m^{(\alpha,\beta)}(x) dx = 0$ for all $m \in \mathbb{N}_0 \backslash \{n\}$.

(iii) $P_n^{(\alpha,\beta)}(1) = \dfrac{\Gamma(n+\alpha+1)}{\Gamma(n+1)\Gamma(\alpha+1)}$.

The relation between $P_n^{(\alpha,\beta)}$ and $G_n(a, b; \cdot)$ is given by

$$G_n(a, b; x) = \frac{n!\Gamma(n+a)}{\Gamma(2n+a)} P_n^{(a-b,b-1)}(2x-1),\ x \in [0, 1]. \tag{2}$$

For any $\alpha, \beta > -1$ the Jacobi polynomials $P_n^{(\alpha,\beta)}$ have the following property (see [23], p. 217):

$$\max_{x\in[-1,1]} \left| P_n^{(\alpha,\beta)}(x) \right| = \begin{cases} \mathcal{O}(n^q), & \text{if} \quad q = \max(\alpha, \beta) \geq -1/2 \\ \mathcal{O}(n^{-1/2}), & \text{if} \quad q = \max(\alpha, \beta) < -1/2 \end{cases} \tag{3}$$

as $n \to \infty$. For the calculations of the Jacobi polynomials $P_n^{(\alpha,\beta)}$ one can use the recurrence formula given in [23], p. 213.

3. Spherical Harmonics

The space $\mathrm{Harm}_n(\Omega)$ of all real-valued homogeneous harmonic polynomials restricted to the unit sphere $\Omega := \{x \in \mathbb{R}^3 | \, |x| = 1\}$, i.e. the set of all $P|_\Omega$, where $P : \mathbb{R}^3 \to \mathbb{R}$ is a homogeneous polynomial satisfying

$$\Delta_x P(x) = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} \right) P(x_1, x_2, x_3) = 0$$

for all $x \in \mathbb{R}^3$, has the dimension $2n+1$. Its elements are called *spherical harmonics*. We define $\{Y_{n,j}\}_{j=-n,\dots,n}$ as an $\mathrm{L}^2(\Omega)$-orthonormal system in $\mathrm{Harm}_n(\Omega)$, i.e.

$$(Y_{n,j}, Y_{n,k})_{\mathrm{L}^2(\Omega)} = \int_\Omega Y_{n,j}(\xi) Y_{n,k}(\xi) \, \mathrm{d}\omega(\xi) = \delta_{jk} = \begin{cases} 1, & j = k \\ 0, & j \neq k \end{cases}$$

Due to the dimension formula this system is complete in $\mathrm{Harm}_n(\Omega)$.
As it is well-known spherical harmonics of different degrees are $\mathrm{L}^2(\Omega)$-orthogonal such that we have $(Y_{n,j}, Y_{m,k})_{\mathrm{L}^2(\Omega)} = \delta_{nm}\delta_{jk}$. Moreover, it is possible to show that such an orthonormal system $\{Y_{n,j}\}_{n\in\mathbb{N}_0;\, j=-n,\dots,n}$ is always complete in $\mathrm{L}^2(\Omega)$. Furthermore, we have the addition theorem for spherical harmonics

$$\sum_{j=-n}^{n} Y_{n,j}(\xi) Y_{n,j}(\eta) = \frac{2n+1}{4\pi} P_n(\xi \cdot \eta); \quad \xi, \eta \in \Omega;$$

where P_n is the Legendre polynomial of degree n. Note that $\int_{-1}^{1} P_n(t)P_m(t)\,\mathrm{d}t = 0$ for $n \neq m$ and $P_n(1) = 1$.
For further details on the theory of spherical harmonics we refer to, for example, [7], [17], and [26].

4. A Complete Orthonormal System in $\mathrm{L}^2(B)$

Let $\{g_k\}_{k\in\mathbb{N}_0}$ be an orthonormal system in $\mathrm{L}^2[0,1]$ with the weight function $w(r) = r^2$ in $[0,1]$, i.e.

$$\int_0^1 r^2 g_k(r) g_l(r) dr = \delta_{k,l}, \quad k,l \in \mathbb{N}_0. \tag{4}$$

We define the sequence $\left\{W^B_{k,n,j}(x)\right\}_{k,n\in\mathbb{N}_0; j=-n,\ldots,n}$ by

$$W^B_{k,n,j}(x) = W^B_{k,n,j}(r_x\xi_x) := \begin{cases} g_k(r_x)Y_{n,j}(\xi_x), & \text{if} \quad x \in B\setminus\{0\}, \\ 1, & \text{if} \quad x = 0, \end{cases}$$

where $r_x = |x|$, $\xi_x = x/|x|$ and $Y_{n,j}$ is the spherical harmonic of degree n and order j. Note that here any other real can be taken as $W^B_{k,n,j}(0)$, too. Throughout this work by r_x and ξ_x we will always denote the norm and the unit vector of $x \in \mathbb{R}^3 \setminus \{0\}$ respectively. Next, we see that

$$\begin{aligned}
\left(W^B_{k_1,n_1,j_1}, W^B_{k_2,n_2,j_2}\right)_{\mathrm{L}^2(B)} &= \int_B W^B_{k_1,n_1,j_1}(x) W^B_{k_2,n_2,j_2}(x) dx \\
&= \int_B (g_{k_1}(r_x)Y_{n_1,j_1}(\xi_x))(g_{k_2}(r_x)Y_{n_2,j_2}(\xi_x)) d(r_x\xi_x) \\
&= \int_0^1 r_x^2 g_{k_1}(r_x) g_{k_2}(r_x) \left(\int_\Omega Y_{n_1,j_1}(\xi_x)Y_{n_2,j_2}(\xi_x) d\omega(\xi_x)\right) dr_x \\
&= \left(\int_0^1 r_x^2 g_{k_1}(r_x) g_{k_2}(r_x) dr\right) \delta_{n_1,n_2}\delta_{j_1,j_2} \\
&= \delta_{k_1,k_2}\delta_{n_1,n_2}\delta_{j_1,j_2},
\end{aligned}$$

where (4) and the orthonormality of $\{Y_{n,j}\}_{n\in\mathbb{N}_0; j=-n,\ldots,n}$ in $\mathrm{L}^2(\Omega)$ have been used. Hence, $W^B := \left\{W^B_{k,n,j}\right\}_{k,n\in\mathbb{N}_0; j=-n,\ldots,n}$ is orthonormal in $\mathrm{L}^2(B)$. Moreover, it can be shown that if $\{g_k\}_{k\in\mathbb{N}_0}$ is complete in $\mathrm{L}^2[0,1]$ then W^B will be complete in $\mathrm{L}^2(B)$. Thus, in order to W^B be a complete orthonormal system in $\mathrm{L}^2(B)$, we need to choose the system $\{g_k\}_{k\in\mathbb{N}_0}$ such that it is complete in $\mathrm{L}^2[0,1]$ and fulfils (4). Therefore, by taking $g_k(r) := \tilde{G}_k(3,3,r)$, W^B will be a complete orthonormal system in $\mathrm{L}^2(B)$.
Using Equations (1) and (2), we obtain $\tilde{G}_k(3,3,r_x) = \sqrt{2k+3}P_k^{(0,2)}(2r_x - 1)$. Hence,

$$W^B_{k,n,j}(x) = W^B_{k,n,j}(r_x\xi_x) := \begin{cases} \sqrt{2k+3}P_k^{(0,2)}(2r_x - 1)Y_{n,j}(\xi_x), & \text{if} \quad x \in B\setminus\{0\}, \\ 1, & \text{if} \quad x = 0, \end{cases} \tag{5}$$

with $k,n \in \mathbb{N}_0; j = -n,\ldots,n$.
Let $\mathrm{C}_0(B)$ be the space of all functions which are continuous on $B\setminus\{0\}$ and bounded

on B. It can be shown that $\mathrm{C}_0(B)$ equipped with the supremum norm

$$\|F\|_\infty := \sup_{x\in B} |F(x)|, \quad F \in \mathrm{C}_0(B),$$

is a Banach space. Moreover, $W^B \subset \mathrm{C}_0(B)$.

5. Splines on the 3-dimensional Ball

In [14], [15], [16], and [17] harmonic spherical splines in a reproducing Sobolev space are introduced. This concept will be used here to develop splines for an interpolation/approximation of the prescribed data on a 3-dimensional ball.

5.1. Sobolev Spaces

Let $\{A_{k,n}\}_{k,n\in\mathbb{N}_0}$ be an arbitrary real sequence. By $\mathcal{E} := \mathcal{E}(\{A_{k,n}\}; B)$ we denote the space of all functions $F \in \mathrm{L}^2(B)$ satisfying

$$\sum_{k=0}^{\infty} \sum_{\substack{n=0\\A_{k,n}\neq 0}}^{\infty} \sum_{j=-n}^{n} A_{k,n}^{-2} \left(F, W_{k,n,j}^B\right)_{\mathrm{L}^2(B)}^2 < +\infty$$

and

$$\left(F, W_{k,n,j}^B\right)_{\mathrm{L}^2(B)} = 0, \quad \text{if} \quad A_{k,n} = 0.$$

Due to the Cauchy-Schwarz inequality $\mathcal{E}$ is a pre-Hilbert space if it is equipped with the inner product

$$(F,G)_{\mathcal{W}(\{A_{k,n}\};B)} := \sum_{k=0}^{\infty} \sum_{\substack{n=0\\A_{k,n}\neq 0}}^{\infty} \sum_{j=-n}^{n} A_{k,n}^{-2} \left(F, W_{k,n,j}^B\right)_{\mathrm{L}^2(B)} \left(G, W_{k,n,j}^B\right)_{\mathrm{L}^2(B)} \qquad F,G \in \mathcal{E}.$$

The Hilbert space $\mathcal{W} := \mathcal{W}(\{A_{k,n}\}; B)$ is defined as the completion of $\mathcal{E}(\{A_{k,n}\}; B)$ with respect to $(.,.)_{\mathcal{W}}$. The induced norm is denoted by $\|F\|_{\mathcal{W}} := \sqrt{(F,F)_{\mathcal{W}}}$.
We will see below (Lemma 5.2) that under certain conditions on the symbol $\{A_{k,n}\}_{k,n\in\mathbb{N}_0}$ the elements of the Sobolev space $\mathcal{W}$ can be identified with piecewise continuous functions with uniformly convergent L^2-Fourier series.
A real sequence $\{A_k\}_{k\in\mathbb{N}_0}$ is called *summable* if the sum

$$\sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \sum_{j=-n}^{n} A_{k,n}^2 \left\|W_{k,n,j}^B\right\|_\infty^2$$

is convergent. In particular, for any $x \in B \setminus \{0\}$

$$\begin{aligned}\sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \sum_{j=-n}^{n} A_{k,n}^2 \left(W_{k,n,j}^B(x)\right)^2 &= \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \sum_{j=-n}^{n} A_{k,n}^2 \left(g_k(r_x)\right)^2 \left(Y_{n,j}(\xi_x)\right)^2 \\ &= \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} A_{k,n}^2 \left(\sqrt{2k+3}P_k^{(0,2)}(2r_x-1)\right)^2 \frac{2n+1}{4\pi}.\end{aligned}$$

Moreover, from (3) follows that

$$\max_{-1\le 2r_x-1\le 1}\left|P_k^{(0,2)}(2r_x-1)\right| = \mathcal{O}(k^2) \quad \text{as} \quad k\to\infty.$$

Therefore, the sequence $\{A_{k,n}\}_{k,n\in\mathbb{N}_0}$ is summable if

$$\sum_{k=0}^{\infty}\sum_{n=0}^{\infty} A_{k,n}^2 k^5 n < \infty.$$

Note that this also includes the case $x=0$.

Assumption 5.1 *We always assume that the used sequences $\{A_{k,n}\}_{k,n\in\mathbb{N}_0}$ are summable.*

For numerical implementations it is convenient to write $\{A_{k,n}\}_{k,n\in\mathbb{N}_0}$ in the form of a product of two sequences, i.e. $A_{k,n}=B_kC_n$, $k,n\in\mathbb{N}_0$. Clearly, in this case the sequence $\{A_{k,n}\}_{k,n\in\mathbb{N}_0}$ will be summable if the series $\sum_{k=0}^{\infty} B_k^2k^5$ and $\sum_{n=0}^{\infty} C_n^2 n$ are summable. For example $\{A_{k,n}\}_{k,n\in\mathbb{N}_0}$ is summable if $B_k = h_1^{k(k+1)/2}$, with $h_1\in(0,1)$ (Gauß–Weierstraß sequence) and $C_n = h_2^{n/2}$ with $h_2\in(0,1)$ (Abel–Poisson sequence).

The following lemma is an analog of the Sobolev lemma.

Lemma 5.2 $\mathcal{W}(\{A_{k,n}\};B)\subset \mathrm{C}_0(B)$ *and for every $F\in\mathcal{W}(\{A_{k,n}\};B)$ the Fourier series*

$$F(x)=\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\sum_{j=-n}^{n}\left(F,W_{k,n,j}^B\right)_{\mathrm{L}^2(B)}W_{k,n,j}^B(x) \tag{6}$$

is uniformly convergent on B.

Proof: Application of the Cauchy-Schwarz inequality yields for $F\in\mathcal{W}$ the estimate

$$\begin{aligned}
\left|\sum_{k=K}^{\infty}\sum_{n=N}^{\infty}\sum_{j=-n}^{n}\left(F,W_{k,n,j}^B\right)_{\mathrm{L}^2(B)}W_{k,n,j}^B(x)\right| &= \left|\sum_{k=K}^{\infty}\sum_{\substack{n=N\\ A_{k,n}\neq 0}}^{\infty}\sum_{j=-n}^{n}\left(F,W_{k,n,j}^B\right)_{\mathrm{L}^2(B)}A_{k,n}^{-1}A_{k,n}W_{k,n,j}^B(x)\right| \\
&\le \left(\sum_{k=K}^{\infty}\sum_{\substack{n=N\\ A_{k,n}\neq 0}}^{\infty}\sum_{j=-n}^{n}\left(F,W_{k,n,j}^B\right)_{\mathrm{L}^2(B)}^2 A_{k,n}^{-2}\right)^{1/2}\left(\sum_{k=K}^{\infty}\sum_{\substack{n=N\\ A_{k,n}\neq 0}}^{\infty}\sum_{j=-n}^{n}A_{k,n}^2(W_{k,n,j}^B(x))^2\right)^{1/2} \\
&\le \|F\|_{\mathcal{W}}\left(\sum_{k=K}^{\infty}\sum_{\substack{n=N\\ A_{k,n}\neq 0}}^{\infty}\sum_{j=-n}^{n}A_{k,n}^2\left\|W_{k,n,j}^B\right\|_{\infty}^2\right)^{1/2}\underset{K,N\to\infty}{\longrightarrow} 0,
\end{aligned}$$

where the right hand side converges as $K\to\infty$ and $N\to\infty$ uniformly and absolutely with respect to $x\in B$ due to the summability condition. Finally, from $W_{k,n,j}^B\in\mathrm{C}_0(B)$, $k,n\in\mathbb{N}_0$, $j=-n,...,n$ and from the uniform convergence of the series in (6) follows that $F \in \mathrm{C}_0(B)$. ■

From this proof it also follows that

$$\sup_{x\in B\setminus\{0\}} |F(x)| \leq \|F\|_{\mathcal{W}} \left(\sum_{k=0}^{\infty} \sum_{n=0}^{\infty} A_{k,n}^2 \left(\sqrt{2k+3} \left\| P_k^{(0,2)} \right\|_{\mathrm{C}[-1,1]} \right)^2 \frac{2n+1}{4\pi} \right)^{1/2}$$

for all $F \in \mathcal{W}$. However, using property *(iii)* in the definition of $P_k^{(\alpha,\beta)}$ we obtain that $\left\| P_k^{(0,2)} \right\|_{\mathrm{C}[-1,1]} \geq 1$. Therefore, taking into account the fact that

$$|F(0)| \leq \|F\|_{\mathcal{W}} \left(\sum_{k=0}^{\infty} \sum_{n=0}^{\infty} A_{k,n}^2 (2n+1) \right)^{1/2},$$

we obtain that

$$\|F\|_{\infty} \leq \|F\|_{\mathcal{W}} \left(\sum_{k=0}^{\infty} \sum_{n=0}^{\infty} A_{k,n}^2 \left(\sqrt{2k+3} \left\| P_k^{(0,2)} \right\|_{\mathrm{C}[-1,1]} \right)^2 \frac{2n+1}{2} \right)^{1/2}. \tag{7}$$

Under the given assumptions we can prove the existence of a reproducing kernel, see also [8] for further details on such kernels.

Definition 5.3 *The function $K_{\mathcal{W}} : B \times B \to \mathbb{R}$ is called a reproducing kernel of $\mathcal{W}$ if*
(i) $K_{\mathcal{W}}(x,\cdot) \in \mathcal{W}$ for all $x \in B$.
(ii) $(F(\cdot), K_{\mathcal{W}}(x,\cdot))_{\mathcal{W}} = F(x)$ for all $F \in \mathcal{W}$ and for all $x \in B$ (reproducing property).

Theorem 5.4 *If $\{A_{k,n}\}_{k,n\in\mathbb{N}_0}$ is summable, then $\mathcal{W}$ has a unique reproducing kernel $K_{\mathcal{W}} : B \times B \to \mathbb{R}$ given by*

$$K_{\mathcal{W}}(x,y) = \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \sum_{j=-n}^{n} A_{k,n}^2 W_{k,n,j}^B(x) W_{k,n,j}^B(y)$$

This theorem can be proven in analogy to the corresponding theorem in [8] and [16].

Theorem 5.5 *Let $\mathcal{F}$ be a bounded linear functional on $\mathcal{W}$. Then the function $y \mapsto \mathcal{F}_x K_{\mathcal{W}}(x,y)$ is in $\mathcal{W}$ and*

$$\mathcal{F}(F) = (F, \mathcal{F}_x K_{\mathcal{W}}(x,\cdot))_{\mathcal{W}}$$

for all $F \in \mathcal{W}$.

(Here, $\mathcal{F}_x K_{\mathcal{W}}(x,\cdot)$ means that $\mathcal{F}$ is applied to the function $x \mapsto K_{\mathcal{W}}(x,y)$ where y is arbitrary but fixed.)

This is a general property of reproducing kernel Hilbert spaces, see [8]. This theorem implies that we can define an inner product in the dual space $\mathcal{W}^*$ of $\mathcal{W}$ as

$$(\mathcal{F}, \mathcal{G})_{\mathcal{W}^*} := (\mathcal{F}_x K_{\mathcal{W}}(x,\cdot), \mathcal{G}_x K_{\mathcal{W}}(x,\cdot))_{\mathcal{W}} = \mathcal{F}\mathcal{G} K_{\mathcal{W}}(\cdot,\cdot).$$

$\mathcal{W}^*$ is a Hilbert space with respect to $(\cdot,\cdot)_{\mathcal{W}^*}$. The spaces $\mathcal{W}$ and $\mathcal{W}^*$ are known to be isomorphic and isometric (see e.g. [8]).

The following theorem shows that in $\mathcal{W}(\{A_{k,n}\}; B)$ complete sets of functions can be generated from complete sets of functionals.

Theorem 5.6 *([8]) The sequence $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$ of bounded linear functionals is complete in $\mathcal{W}^*$, i.e. $f \in \mathcal{W}, \mathcal{F}_n(f) = 0, n = 1, 2, ...,$ implies $f \equiv 0$, if and only if the functions*

$$g_n(y) := (\mathcal{F}_n)_x\, K_{\mathcal{W}}(x, y), \quad y \in X, n = 1, 2, ...$$

form a complete set for $\mathcal{W}$.

Since in Hilbert spaces closure and completeness are equivalent concepts, we get the following result.

Corollary 5.7 *The system of bounded linear functionals $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$ is complete in $\mathcal{W}^*$ if and only if*

$$\overline{\mathrm{span}_{n\in\mathbb{N}}\{(\mathcal{F}_n)_x\, K_{\mathcal{W}}(x,\cdot)\}}^{\|\cdot\|_{\mathcal{W}}} = \mathcal{W}. \tag{8}$$

In Figure 1 the localization character of $K_{\mathcal{W}}(x, y)$, with $A_{k,n} = B_k C_n$, $k, n \in \mathbb{N}_0$ for some B_k and C_n is demonstrated, where $x = (0, x_2, x_3)$, $y = (0, y_2, y_3)$, and the reproducing kernel $K_{\mathcal{W}}(x, y)$ is plotted in dependence of y_2 and y_3, with $y_2^2 + y_3^2 \leq 1$ and the value of $K_{\mathcal{W}}(0, 0)$ is ignored.

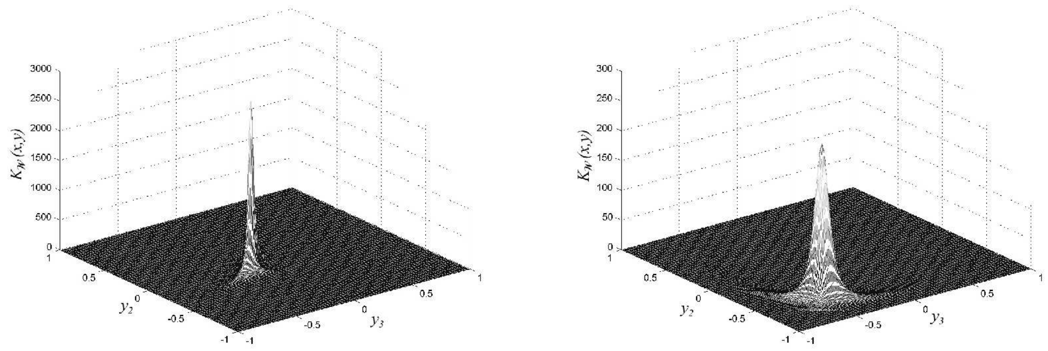


Figure 1. The reproducing kernel $K_{\mathcal{W}}(x, y)$ with $B_k = e^{-0.1k}$, $C_n = e^{-0.1n}$, $x_2 = -0.1$, $x_3 = -0.2$ (left), $B_k = e^{-0.05k(k+1)}$, $C_n = e^{-0.1n}$, $x_2 = -0.6$, $x_3 = -0.5$ (right)

5.2. Splines

Let $\mathcal{F}^N := \{\mathcal{F}_n\}_{n=1,...,N}$ be a linearly independent system of linear continuous functionals on $\mathcal{W}$.

Definition 5.8 *A function $S \in \mathcal{W}$ of the form*

$$S(x) = \sum_{k=1}^{N} a_k \mathcal{F}_k K_{\mathcal{W}}(\cdot, x), \quad x \in B,$$

$a = (a_1, ..., a_N)^{\mathrm{T}} \in \mathbb{R}^N$ *is called spline in $\mathcal{W}$ relative to $\mathcal{F}^N$. The scalars $a_1, ..., a_N$ are called the coefficients of the spline S. Such splines are collected in the space* $\mathrm{Spline}(\{A_{k,n}\}; \mathcal{F}^N)$ *or simply* $\mathrm{Spl}_{\mathcal{F}^N}$.

A spline interpolation problem can be formulated as follows.

Problem 5.9 *For a given linearly independent system $\mathcal{F}^N = \{\mathcal{F}_1, ..., \mathcal{F}_N\}$ of linear continuous functionals and a vector $y = (y_1, ..., y_N)^{\mathrm{T}} \in \mathbb{R}^N$ determine $S \in$* $\mathrm{Spline}(\{A_{k,n}\}; \mathcal{F}^N)$ *such that*

$$\mathcal{F}_i S = y_i \quad \text{for all} \quad i = 1, ..., N$$

Or, equivalently, determine $a \in \mathbb{R}^N$ such that

$$\sum_{j=1}^{N} a_j \mathcal{F}_i \mathcal{F}_j K_{\mathcal{W}}(\cdot, \cdot) = y_i \quad \text{for all} \quad i = 1, ..., N \tag{9}$$

This yields a linear equation system with the matrix

$$\mathbf{k}_N = (\mathcal{F}_i \mathcal{F}_j K_{\mathcal{W}}(\cdot, \cdot))_{i,j=1,...,N} \tag{10}$$

which is positive definite according to the following theorem, which can be proved in analogy to the spherical spline theory.

Theorem 5.10 *Let $\mathcal{F}^N := \{\mathcal{F}_1, ..., \mathcal{F}_N\}$ be a system of bounded linear functionals on $\mathcal{W}$. This system is linearly independent if and only if the matrix $\boldsymbol{k}_N$ is positive definite.*

As a consequence we obtain the following theorem.

Theorem 5.11 *The formulated (spline interpolation) Problem 5.9 is always uniquely solvable.*

Remark 5.12 *Theorem 5.10 implies that the system $\{\mathcal{F}_1 K_{\mathcal{W}}(x, \cdot), ..., \mathcal{F}_N K_{\mathcal{W}}(x, \cdot)\}$ is linearly independent, and therefore,* $\mathrm{Spline}(\{A_{k,n}\}; \mathcal{F}^N)$ *is an N-dimensional subspace of $\mathcal{W}$.*

As an immediate consequence of Theorem 5.5 we get the following lemma.

Lemma 5.13 ***($\mathcal{W}$-spline formula)*** *Let $S \in \mathrm{Spl}_{\mathcal{F}^N}$ with $S(x) = \sum_{l=1}^{N} a_l \mathcal{F}_l K_{\mathcal{W}}(., x), \quad x \in B$. Then, for arbitrary $F \in \mathcal{W}$, $(F, S)_{\mathcal{W}} = \sum_{l=1}^{N} a_l \mathcal{F}_l F$.*

The following two theorems can be proven in analogy to the corresponding theorems in [14] as a result of Lemma 5.13.

Theorem 5.14 ***(1st Minimum Property)*** *Let $y \in \mathbb{R}^N$ be given and $\mathcal{F}^N := \{\mathcal{F}_1, ..., \mathcal{F}_N\} \subset \mathcal{W}^*$ be linearly independent. If $S^* = \sum_{i=1}^N a_i(\mathcal{F}_i)_x K_{\mathcal{W}}(\cdot, x)$ is the unique spline satisfying $\mathcal{F}_i S^* = y_i$ for all $i = 1, ..., N$ then S^* is the unique minimizer of*

$$\|S^*\|_{\mathcal{W}} = \min\{\|F\|_{\mathcal{W}} | F \in \mathcal{W}, \mathcal{F}_i F = y_i \;\; \forall i = 1, ..., N\}.$$

The obtained result shows that the formulated (spline interpolation) Problem 5.9 is equivalent to the *minimum norm interpolation* problem:

Problem 5.15 *Let $\mathcal{F}^N = \{\mathcal{F}_1, ..., \mathcal{F}_N\}$ be a linearly independent system of linear bounded functionals on $\mathcal{W}$ and $y = (y_1, ..., y_N)^{\mathrm{T}} \in \mathbb{R}^N$. Let also $F \in \mathcal{W}$ with $\mathcal{F}_i F = y_i$ for $i = 1, ..., N$.*
Determine $S^F_{\mathcal{F}^N} \in \mathcal{W}$ such that

$$\left\|S^F_{\mathcal{F}^N}\right\|_{\mathcal{W}} = \inf_{G \in \mathcal{I}_N(y)} \|G\|_{\mathcal{W}}, \tag{11}$$

where

$$\mathcal{I}_N(y) = \{G \in \mathcal{W} | \mathcal{F}_i G = \mathcal{F}_i F = y_i, i = 1, ..., N\} \tag{12}$$

Theorem 5.16 ***(2nd Minimum Property)*** *Let $F \in \mathcal{W}$ be given and $\mathcal{F}^N := \{\mathcal{F}_1, ..., \mathcal{F}_N\} \subset \mathcal{W}^*$ be linearly independent. If $S^* \in \mathrm{Spline}(\{A_k\}; \mathcal{F}^N)$ is the unique spline satisfying $\mathcal{F}_i S^* = \mathcal{F}_i F$ for all $i = 1, ..., N$, then S^* is the unique minimizer of*

$$\|F - S^*\|_{\mathcal{W}} = \min\{\|F - S\|_{\mathcal{W}} | S \in \mathrm{Spline}(\{A_k\}; \mathcal{F}^N)\}.$$

Thus, if F represents an unknown function in $\mathcal{W}$, the interpolating spline S^* represents the best possible approximation to F among all splines, measured with respect to the metric induced by the Sobolev norm $\|\cdot\|_{\mathcal{W}}$. Moreover, among all functions in $\mathcal{W}$ that fit to the known data y_i the spline S^* is the 'smoothest' (in $\|\cdot\|_{\mathcal{W}}$-sense).

Summarizing our results we obtain the following theorem.

Theorem 5.17 *Problem 5.15 is well-posed, in the sense that its solution exists, is unique, and depends continuously on the data $y_1, ..., y_N$. The uniquely determined solution is given by*

$$S^F_{\mathcal{F}^N}(x) = \sum_{i=1}^N a_i \mathcal{F}_i K_{\mathcal{W}}(\cdot, x) \quad x \in B,$$

where the coefficients $a_1, ..., a_N$ satisfy the linear equation system (9).

5.3. Error Estimates and Convergence Results

Theorem 5.18 *Let F be a function in $\mathcal{W}$, $y = (y_1, ..., y_N)^{\mathrm{T}} \in \mathbb{R}^N$ and let $\mathcal{F}^N = \{\mathcal{F}_1, ..., \mathcal{F}_N\} \subset \mathcal{W}^*$ be a linearly independent system. Denote by $S^F_{\mathcal{F}^N} \in \mathcal{W}$ the uniquely determined solution of Problem 5.15. Then*

$$\sup_{\substack{\mathcal{L} \in \mathcal{W}^* \\ \|\mathcal{L}\|_{\mathcal{W}^*} = 1}} \left|\mathcal{L}F - \mathcal{L}S^F_{\mathcal{F}^N}\right| \le 2\Lambda_{\mathcal{F}^N} \|F\|_{\mathcal{W}}, \tag{13}$$

where the $\mathcal{F}^N-$width $\Lambda_{\mathcal{F}^N}$ is defined by

$$\Lambda_{\mathcal{F}^N} := \sup_{\substack{\mathcal{L}\in\mathcal{W}^* \\ \|\mathcal{L}\|_{\mathcal{W}^*}=1}} \left(\min_{\mathcal{J}\in \mathrm{span}(\mathcal{F}^N)} \|\mathcal{L}-\mathcal{J}\|_{\mathcal{W}^*} \right).$$

Remark 5.19 *Note that in the definition of $\Lambda_{\mathcal{F}^N}$ the "min" exists due to the finite dimension of* $\mathrm{span}(\mathcal{F}^N)$. *Moreover, for any $\mathcal{L}\in\mathcal{W}^*$ with $\|\mathcal{L}\|_{\mathcal{W}^*}=1$*

$$\min_{\mathcal{J}\in \mathrm{span}(\mathcal{F}^N)} \|\mathcal{L}-\mathcal{J}\|_{\mathcal{W}^*} \le \|\mathcal{L}\|_{\mathcal{W}^*} = 1.$$

Thus, for arbitrary $\mathcal{F}^N \subset \mathcal{W}^*$

$$0 \le \Lambda_{\mathcal{F}^N} \le 1.$$

Hence, we see that (13) is a more precise version of the fact that for all $\mathcal{L}\in\mathcal{W}^*$ with $\|\mathcal{L}\|_{\mathcal{W}^*}=1$ and for all $F\in\mathcal{W}$

$$\left|\mathcal{L}F - \mathcal{L}S^F_{\mathcal{F}^N}\right| \le \|\mathcal{L}\|_{\mathcal{W}^*}\left\|F - S^F_{\mathcal{F}^N}\right\|_{\mathcal{W}} \le \|F\|_{\mathcal{W}} + \left\|S^F_{\mathcal{F}^N}\right\|_{\mathcal{W}} \le 2\,\|F\|_{\mathcal{W}}.$$

Proof of Theorem 5.18: For any $\mathcal{L}\in\mathcal{W}^*$ with $\|\mathcal{L}\|_{\mathcal{W}^*}=1$ there exists $\mathcal{J}_{\mathcal{L}}\in\mathrm{span}(\mathcal{F}^N)$ such that $\|\mathcal{L}-\mathcal{J}_{\mathcal{L}}\|_{\mathcal{W}^*} \le \Lambda_{\mathcal{F}^N}$. Since $\mathcal{F}_kF = \mathcal{F}_kS^F_{\mathcal{F}^N}$ for all $k=1,...,N$, hence $\mathcal{J}_{\mathcal{L}}F = \mathcal{J}_{\mathcal{L}}S^F_{\mathcal{F}^N}$, and therefore

$$\mathcal{L}F - \mathcal{L}S^F_{\mathcal{F}^N} = \mathcal{L}F - \mathcal{J}_{\mathcal{L}}F + \mathcal{J}_{\mathcal{L}}S^F_{\mathcal{F}^N} - \mathcal{L}S^F_{\mathcal{F}^N} = (\mathcal{L}-\mathcal{J}_{\mathcal{L}})F - (\mathcal{L}-\mathcal{J}_{\mathcal{L}})S^F_{\mathcal{F}^N}.$$

From Theorem 5.5 follows that

$$\begin{aligned}(\mathcal{L}-\mathcal{J}_{\mathcal{L}})F &= (F, (\mathcal{L}-\mathcal{J}_{\mathcal{L}})_x K_{\mathcal{W}}(x,\cdot))_{\mathcal{W}} \\ (\mathcal{L}-\mathcal{J}_{\mathcal{L}})S^F_{\mathcal{F}^N} &= \left(S^F_{\mathcal{F}^N}, (\mathcal{L}-\mathcal{J}_{\mathcal{L}})_x K_{\mathcal{W}}(x,\cdot)\right)\Big)_{\mathcal{W}}\end{aligned}$$

Next, using the Cauchy-Schwarz inequality we obtain

$$\begin{aligned}|(F, (\mathcal{L}-\mathcal{J}_{\mathcal{L}})_x K_{\mathcal{W}}(x,\cdot))_{\mathcal{W}}| &\le \|F\|_{\mathcal{W}}\,(\kappa_{\mathcal{W}}(\mathcal{L},\mathcal{J}_{\mathcal{L}}))^{1/2} \\ \left|(S^F_{\mathcal{F}^N}, (\mathcal{L}-\mathcal{J}_{\mathcal{L}})_x K_{\mathcal{W}}(x,\cdot))_{\mathcal{W}}\right| &\le \left\|S^F_{\mathcal{F}^N}\right\|_{\mathcal{W}}(\kappa_{\mathcal{W}}(\mathcal{L},\mathcal{J}_{\mathcal{L}}))^{1/2}\end{aligned}$$

where

$$\kappa_{\mathcal{W}}(\mathcal{L},\mathcal{J}_{\mathcal{L}}) = ((\mathcal{L}-\mathcal{J}_{\mathcal{L}})_x K_{\mathcal{W}}(x,\cdot), (\mathcal{L}-\mathcal{J}_{\mathcal{L}})_x K_{\mathcal{W}}(x,\cdot))_{\mathcal{W}}.$$

Therefore, again using Theorem 5.5 we obtain

$$(\kappa_{\mathcal{W}}(\mathcal{L},\mathcal{J}_{\mathcal{L}}))^{1/2} = ((\mathcal{L}-\mathcal{J}_{\mathcal{L}})(\mathcal{L}-\mathcal{J}_{\mathcal{L}})K_{\mathcal{W}}(\cdot,\cdot))^{1/2} = \|\mathcal{L}-\mathcal{J}_{\mathcal{L}}\|_{\mathcal{W}^*} \le \Lambda_{\mathcal{F}^N}.$$

Now, since $S^F_{\mathcal{F}^N}$ is the 'smoothest' interpolant (see Theorem 5.14), thus

$$\left\|S^F_{\mathcal{F}^N}\right\|_{\mathcal{W}} \le \|F\|_{\mathcal{W}}.$$

Therefore, summarizing our results we obtain

$$\left|\mathcal{L}F - \mathcal{L}S^F_{\mathcal{F}^N}\right| \le 2\Lambda_{\mathcal{F}^N}\|F\|_{\mathcal{W}}$$

which proves the theorem, since $\mathcal{L}\in\mathcal{W}^*$ with $\|\mathcal{L}\|_{\mathcal{W}^*}=1$ was arbitrary. ■

Theorem 5.20 *Let F be a function in $\mathcal{W}$, $y = (y_1, ..., y_N)^{\mathrm{T}} \in \mathbb{R}^N$ and let $\mathcal{F}^N = \{\mathcal{F}_1, ..., \mathcal{F}_N\} \subset \mathcal{W}^*$ be a linearly independent system. Then*

$$\left\| F - S^F_{\mathcal{F}^N} \right\|_{\mathcal{W}} \leq 2\Lambda^{1/2}_{\mathcal{F}^N} \|F\|_{\mathcal{W}}, \tag{14}$$

where $S^F_{\mathcal{F}^N}$ and $\Lambda_{\mathcal{F}^N}$ are defined in Theorem 5.18.

Proof: Due to Riesz's representation theorem (see e.g. [39]) for every $F \in \mathcal{W}$ and for the corresponding $S^F_{\mathcal{F}^N}$ there exists $\mathcal{L} \in \mathcal{W}^*$ such that $F - S^F_{\mathcal{F}^N}$ is the representer of $\mathcal{L}$, i.e. for any $G \in \mathcal{W}$ we have $\mathcal{L}G = (G, F - S^F_{\mathcal{F}^N})_{\mathcal{W}}$. By taking $G = K_{\mathcal{W}}(x, \cdot)$, we will have

$$\mathcal{L}K_{\mathcal{W}}(x, \cdot) = (K_{\mathcal{W}}(x, \cdot), F - S^F_{\mathcal{F}^N})_{\mathcal{W}} = \left(F - S^F_{\mathcal{F}^N}\right)(x).$$

Note that since $\mathcal{L}$ is the representer of $F - S^F_{\mathcal{F}^N}$ and due to Theorem 5.14

$$\|\mathcal{L}\|_{\mathcal{W}^*} = \left\| F - S^F_{\mathcal{F}^N} \right\|_{\mathcal{W}} \leq \|F\|_{\mathcal{W}} + \left\| S^F_{\mathcal{F}^N} \right\|_{\mathcal{W}} \leq 2\|F\|_{\mathcal{W}}.$$

Let $\|F - S^F_{\mathcal{F}^N}\|_{\mathcal{W}} \neq 0$ (otherwise there is nothing to prove, since the right hand side of (14) is non-negative). We set $\mathcal{L}_0 := \mathcal{L}/\|\mathcal{L}\|_{\mathcal{W}^*}$, so $\mathcal{L}_0 \in \mathcal{W}^*$ and $\|\mathcal{L}_0\|_{\mathcal{W}^*} = 1$. Hence, we obtain

$$\begin{aligned}\left\| F - S^F_{\mathcal{F}^N} \right\|_{\mathcal{W}} &= \left(F - S^F_{\mathcal{F}^N}, F - S^F_{\mathcal{F}^N}\right)^{1/2}_{\mathcal{W}} = \left(\mathcal{L}\left(F - S^F_{\mathcal{F}^N}\right)\right)^{1/2} = \|\mathcal{L}\|^{1/2}_{\mathcal{W}^*} \left(\mathcal{L}_0\left(F - S^F_{\mathcal{F}^N}\right)\right)^{1/2} \\ &= \|\mathcal{L}\|^{1/2}_{\mathcal{W}^*} \left(\mathcal{L}_0 F - \mathcal{L}_0 S^F_{\mathcal{F}^N}\right)^{1/2} \leq \|\mathcal{L}\|^{1/2}_{\mathcal{W}^*} \left(2\Lambda_{\mathcal{F}^N}\|F\|_{\mathcal{W}}\right)^{1/2} \leq 2\Lambda^{1/2}_{\mathcal{F}^N}\|F\|_{\mathcal{W}},\end{aligned}$$

where we used Theorem 5.5 and Theorem 5.18. ■

One of the important questions of every interpolation problem is whether (and under which circumstances) the interpolating function converges to the initial function. Here we obtain a necessary and sufficient condition, under which the sequence of interpolating splines converges to the initial function, in the sense of a strong as well as a weak convergence.

Let $F \in \mathcal{W}$ be arbitrary and $\mathcal{F} := \{\mathcal{F}_1, \mathcal{F}_2, ...\}$ be a sequence of linearly independent bounded linear functionals on $\mathcal{W}$. For any $N \in \mathbb{N}$ define $\mathcal{F}^N := \{\mathcal{F}_1, ..., \mathcal{F}_N\}$ and consider the sequence $\left\{S^F_{\mathcal{F}^N}\right\}_{N\in\mathbb{N}}$ of the (uniquely determined) solutions of the spline interpolation problems

$$\left\| S^F_{\mathcal{F}^N} \right\|_{\mathcal{W}} = \min_{\substack{G\in\mathcal{W} \\ \mathcal{F}_i G = \mathcal{F}_i F, i=1,...,N}} \|G\|_{\mathcal{W}}, \quad N \in \mathbb{N}. \tag{15}$$

Theorem 5.21 *The following statements are equivalent*

$$\begin{aligned}&(i) \lim_{N\to\infty} \left|\mathcal{L}F - \mathcal{L}S^F_{\mathcal{F}^N}\right| = 0 \quad \textit{for any} \quad F \in \mathcal{W}, \quad \textit{and for any} \quad \mathcal{L} \in \mathcal{W}^*, \\ &(ii) \lim_{N\to\infty} \left\| F - S^F_{\mathcal{F}^N} \right\|_{\mathcal{W}} = 0 \quad \textit{for any} \quad F \in \mathcal{W}, \\ &(iii) \textit{the system } \{\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, ...\} \textit{ is complete in } \mathcal{W}^*,\end{aligned}$$

where for any $N \in \mathbb{N}$, $S^F_{\mathcal{F}^N} \in \mathcal{W}$ is the unique solution of the interpolation problem (15).

Proof: First, let us show that (ii) is equivalent to (iii). Using Corollary 5.7 we obtain that (iii) is equivalent to

$$\overline{\mathrm{span}_{N\in\mathbb{N}}\{(\mathcal{F}_N)_y K(\cdot,y)\}}^{\|\cdot\|_{\mathcal{W}}} = \mathcal{W}.$$

Next, it is clear that if (ii) holds, then

$$\overline{\bigcup_{N=1}^{\infty} \mathrm{Spl}_{\mathcal{F}^N}}^{\|\cdot\|_{\mathcal{W}}} = \mathcal{W}, \tag{16}$$

However, (16) means that for any $F \in \mathcal{W}$ and for any $\varepsilon > 0$ there exists $N_0 \in \mathbb{N}$ and $S_{N_0} \in \mathrm{Spl}_{\mathcal{F}^{N_0}}$ such that $\|F - S_{N_0}\|_{\mathcal{W}} \leq \varepsilon$. Therefore, using Theorem 5.16 we obtain that

$$\left\|F - S^F_{\mathcal{F}^N}\right\|_{\mathcal{W}} \leq \left\|F - S^F_{\mathcal{F}^{N_0}}\right\|_{\mathcal{W}} \leq \|F - S_{N_0}\|_{\mathcal{W}} \leq \varepsilon \quad \text{for all} \quad N > N_0.$$

Hence, (ii) is equivalent to (16). Observing the fact that

$$\overline{\bigcup_{N=1}^{\infty} \mathrm{Spl}_{\mathcal{F}^N}}^{\|\cdot\|_{\mathcal{W}}} = \overline{\mathrm{span}_{N\in\mathbb{N}}\{(\mathcal{F}_N)_y K(\cdot,y)\}}^{\|\cdot\|_{\mathcal{W}}},$$

we obtain the equivalency of (ii) and (iii). Now we will show the equivalency of (i) and (iii). Taking into account the fact that from the strong convergence of a sequence follows the weak convergence of one, and using the equivalency of (ii) and (iii) we obtain that (iii) implies (i). So, to finish the proof of the theorem, it is enough to show that (i) implies (iii), or equivalently $\mathsf{Not}\,(iii)$ implies $\mathsf{Not}\,(i)$. Assume now that (iii) is not true, i.e. there exists $G \in \mathcal{W}$ such that $\mathcal{F}_i G = 0$, $i \in \mathbb{N}$, but $G \neq 0$. Denote by $\mathcal{L}_G$ the functional, whose representer is G. In this case using Lemma 5.13 we obtain

$$\mathcal{L}_G S^G_{\mathcal{F}^N} = \left(S^G_{\mathcal{F}^N}, G\right)_{\mathcal{W}} = \sum_{i=1}^{N} a_i^N \mathcal{F}_i G = 0, \quad \text{for any} \quad N \in \mathbb{N},$$

where for any $N \in \mathbb{N}$, $a_1^N, ..., a_N^N$ are the coefficients of the spline $S^G_{\mathcal{F}^N}$. Hence,

$$\lim_{N\to\infty} \left|\mathcal{L}_G G - \mathcal{L}_G S^G_{\mathcal{F}^N}\right| = |\mathcal{L}_G G| = |(G,G)_{\mathcal{W}}| = \|G\|^2_{\mathcal{W}} \neq 0.$$

That is, $\mathsf{Not}\,(iii)$ implies $\mathsf{Not}\,(i)$. ■

One can combine the interpolation conditions with a smoothing condition to obtain an approximation problem. This is realized by adding positive constants to the diagonal of the matrix ([17], [24]). More precisely, by solving the modified linear equation system

$$\sum_{k=1}^{N} a_k \left(\mathcal{F}_n\right)_x \left(\mathcal{F}_k\right)_y K_{\mathcal{W}}(y,x) + \rho a_n = b_n; \quad n = 1, ..., N;$$

for $\rho > 0$ the spline

$$S = \sum_{k=1}^{N} a_k \left(\mathcal{F}_k\right)_y K_{\mathcal{W}}(y, .)$$

is the unique minimizer of the functional (see also [2])

$$\mathcal{W} \ni F \mapsto \sum_{n=1}^{N} |\mathcal{F}_n F - b_n|^2 + \rho(F, F)_{\mathcal{W}}.$$

6. Application to Seismic Body Wave Tomography

The task of seismic body tomography is to determine the slowness function $\tilde{S}$ of body wave propagation out of the travel times T_q of seismic body waves along corresponding seismic rays γ_q; $q = 1, ..., N$; between E_q (epicenter) and R_q (seismometer, receiver). The relation is given by the path integral

$$\left(\mathcal{F}_q \tilde{S} :=\right) \int_{\gamma_q} \tilde{S}(\xi)\, \mathrm{d}\sigma(\xi) = T_q, \quad q = 1, ..., N.$$

We discuss the linear variant of the seismic traveltime tomography problem, where the seismic rays $\gamma_q; q = 1, ..., N$ are constructed according to a given (slowness) reference model S_0 and are independent from $\tilde{S}$ (see e.g. [5], [22], [29]).
We will assume the following properties.

Assumption 6.1 *$\gamma_i \neq \gamma_j$, if $i \neq j$, $i, j = 1, ..., N$.*

Assumption 6.2 *There exists an integer L such that for any $i, j = 1, ..., N$, with $i \neq j$ the number of intersection points of γ_i and γ_j is smaller than L.*

Assumption 6.3 *The lengths of seismic rays are uniformly bounded, i.e. there exists $M \in \mathbb{R}$ such that*

$$\text{length}\,(\gamma_q) < M, \qquad q = 1, ..., N.$$

The functionals $\mathcal{F}_q$ are obviously linear, due to the linearity of the integral, and continuous on $\mathcal{W} \subset \mathrm{C}_0(B)$ since

$$\begin{aligned} |\mathcal{F}_q F| &\leq \|F\|_\infty \cdot \text{length}\,(\gamma_q) \\ &\leq \|F\|_{\mathcal{W}} \left(\sum_{k=0}^{\infty} \sum_{n=0}^{\infty} A_{k,n}^2 \left(\sqrt{2k+3} \left\| P_k^{(0,2)} \right\|_{\mathrm{C}[-1,1]} \right)^2 \frac{2n+1}{2} \right)^{1/2} M, \end{aligned}$$

for all $F \in \mathcal{W}$, where we have used Equation (7) and Assumption 6.3.

Theorem 6.4 *The system of functionals $\{\mathcal{F}_1, \mathcal{F}_2, ..., \mathcal{F}_N\}$ is linearly independent.*

Proof: Let Assumption 6.1 hold, i.e. $\gamma_i \neq \gamma_j$, if $i \neq j$, $i, j = 1, ..., N$, but $\{\mathcal{F}_1, \mathcal{F}_2, ..., \mathcal{F}_N\}$ is linearly dependent. That is there exist coefficients $a_1, ..., a_N$ where at least one of them is not 0, such that $\sum_{k=1}^N a_k \mathcal{F}_k = 0$. However, this means that for any $F \in \mathcal{W}$

$$\sum_{k=1}^N a_k \mathcal{F}_k F = 0. \tag{17}$$

Let $a_{i_0} \neq 0$. Assume without loss of generality that $a_{i_0} > 0$. We will construct a function in $\mathcal{W}$ for which (17) does not hold. As we have already mentioned by r_x and ξ_x we will always denote the norm and the unit vector of $x \in \mathbb{R}^3 \setminus \{0\}$, respectively. Clearly, from Assumptions 6.1 and 6.2 follows that there exists $x_0 \in \gamma_{i_0}$, with $x_0 \neq 0$ and $\varepsilon > 0$ such that $x_0(\varepsilon) \cap \gamma_i = \emptyset$ if $i \neq i_0$, where $x_0(\varepsilon)$ is the ε-neighborhood of x_0. Now, it is not hard to check that for an arbitrary real $M_0 > 0$ we can construct $u_1 \in \mathrm{C}[0,1]$ and $v_1 \in \mathrm{C}(\Omega)$ such that for $F_1(x) = F_1(r_x \xi_x) := u_1(r_x) v_1(\xi_x)$, $x \in B \setminus \{0\}$ we have that $F_1(x) \geq 0$, $x \in B$ and

$$F_1(x) = \begin{cases} M_0, & \text{if} \quad x \in x_0(\varepsilon/n_0) \\ 0, & \text{if} \quad x \in B \backslash x_0(\varepsilon), \end{cases} \tag{18}$$

where n_0 is some fixed integer.
Hence,

$$\lambda_1 := \sum_{k=1}^N a_k \int_{\gamma_k} F_1(x) \mathrm{d}\sigma(x) = a_{i_0} \int_{\gamma_{i_0}} F_1(x) \mathrm{d}\sigma(x) > \frac{a_{i_0} M_0 \varepsilon}{2 n_0} =: M_1 > 0. \tag{19}$$

Now since length(γ_i), $i = 1, ..., N$ is bounded

$$M_2 := \sum_{k=1}^N |a_k| \, \mathrm{length}(\gamma_k) < \infty.$$

Let $p := \max(\|u_1\|_\infty, \|v_1\|_\infty)$ and $g_k(r) := \tilde{G}_k(3,3,r)$, $k \in \mathbb{N}$, $r \in [0,1]$. Since the system $\{g_k\}_{k \in \mathbb{N}_0}$ is closed in $\mathrm{C}[0,1]$ (see e.g. [34]) and the system $\{Y_{n,j}\}_{n \in \mathbb{N}_0; j=-n,\dots,n}$ is closed in $\mathrm{C}(\Omega)$ (see e.g. [17]), for $\delta := M_1/(2M_2)$ and for $\delta_1 < \min(p, \delta/(3p))$ there exist linear combinations

$$\tilde{g} := \sum_{k=0}^{k_0} b_k g_k \quad \text{and} \quad \tilde{Y} := \sum_{n=0}^{n_0} \sum_{j=-n}^{n} c_{n,j} Y_{n,j}$$

such that

$$\|u_1 - \tilde{g}\|_\infty \leq \delta_1 \quad \text{and} \quad \|v_1 - \tilde{Y}\|_\infty \leq \delta_1.$$

Hence, if we denote $F_2(x) = F_2(r_x \xi_x) = \tilde{g}(r_x) \tilde{Y}(\xi_x)$, $x \in B \setminus \{0\}$ and $F_2(0)$ appropriate, then clearly, $F_2 \in \mathcal{W}$ and

$$\begin{aligned}
&\sup_{x\in B\setminus\{0\}} |F_2(x) - F_1(x)| = \sup_{x\in B\setminus\{0\}} \left|\tilde{g}(r_x)\tilde{Y}(\xi_x) - u_1(r_x)v_1(\xi_x)\right| \\
&= \sup_{x\in B\setminus\{0\}} \Big|(\tilde{g}(r_x) - u_1(r_x))(\tilde{Y}(\xi_x) - v_1(\xi_x)) \\
&\quad + v_1(\xi_x)(\tilde{g}(r_x) - u_1(r_x)) + u_1(r_x)(\tilde{Y}(\xi_x) - v_1(\xi_x))\Big| \\
&\leq \sup_{r\in(0,1]} |\tilde{g}(r) - u_1(r)| \sup_{\xi\in\Omega} \left|\tilde{Y}(\xi) - v_1(\xi)\right| \\
&\quad + \sup_{\xi\in\Omega} |v_1(\xi)| \sup_{r\in(0,1]} |\tilde{g}(r) - u_1(r)| + \sup_{r\in(0,1]} |u_1(r)| \sup_{\xi\in\Omega} \left|\tilde{Y}(\xi) - v_1(\xi)\right| \\
&\leq \delta_1^2 + 2p\delta_1 \leq 3p\delta_1 \\
&\leq \delta.
\end{aligned}$$

Thus, if we denote

$$\lambda_2 := \sum_{k=1}^{N} a_k \mathcal{F}_k F_2 = \sum_{k=1}^{N} a_k \int_{\gamma_k} F_2(x) \mathrm{d}\sigma(x),$$

then using in the case of $0 \in \gamma_k$ the fact that path integrals are invariant w.r.t. changes of the function at one single point

$$\begin{aligned}
|\lambda_1 - \lambda_2| &= \left| \sum_{k=1}^{N} a_k \int_{\gamma_k} (F_1 - F_2)(x) \mathrm{d}\sigma(x) \right| \\
&\leq \sup_{x\in B\setminus\{0\}} |F_1(x) - F_2(x)| \sum_{k=1}^{N} |a_k| \operatorname{length}(\gamma_k) \\
&\leq \delta M_2 = \frac{M_1}{2}.
\end{aligned}$$

That is

$$\lambda_1 - M_1/2 \leq \lambda_2 \leq \lambda_1 + M_1/2,$$

such that using (19) we obtain that

$$\sum_{k=1}^{N} a_k \mathcal{F}_k F_2 = \lambda_2 > M_1 - \frac{M_1}{2} = \frac{M_1}{2} > 0.$$

However, this is a contradiction to (17), hence, $\{\mathcal{F}_1, \mathcal{F}_2, ..., \mathcal{F}_N\}$ is linearly independent. ■

The idea that we follow here is to approximate $\tilde{S}$ by a spline $S \in \mathcal{W}$ based on a system $\{\mathcal{F}_1, \mathcal{F}_2, ..., \mathcal{F}_N\}$, i.e. by a spline of the form

$$S(x) = \sum_{k=1}^{N} a_k \mathcal{F}_k K_{\mathcal{W}}(.,x), \quad x \in B.$$

It is known that if L is a curve parameterized by a $\mathrm{C}^{(1)}([a,b],\mathbb{R}^3)$–function l, and F is a continuous scalar field, then

$$\int_L F(x)\,\mathrm{d}\sigma(x) = \int_a^b F(l(t))\,|l'(t)|\,\mathrm{d}t\,.$$

Hence, knowing parametric equations of the raypaths $\gamma_q; q = 1,...,N$ we can calculate the matrix components corresponding to our spline interpolation problem:

$$(\mathcal{F}_l)_x\,(\mathcal{F}_q)_y\,K_{\mathcal{W}}(y,x) = \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} A^2_{k,n} \sum_{j=-n}^{n} \int_{\gamma_l} W^B_{k,n,j}(x)\,\mathrm{d}\sigma(x)\,\int_{\gamma_q} W^B_{k,n,j}(y)\mathrm{d}\sigma(y).$$

Note that here we can change the order of integration and summation, since the discussed functionals $\mathcal{F}_q$ are linear and continuous.
Thus, by solving the linear equation system

$$\sum_{q=1}^{N} a_q\,(\mathcal{F}_l)_x\,(\mathcal{F}_q)_y\,K_{\mathcal{W}}(y,x) = T_l \text{ for all } l = 1,...,N \tag{20}$$

we obtain the coefficients $(a_q)_{q=1,...,N}$ of the spline

$$S(x) = \sum_{q=1}^{N} a_q\,(\mathcal{F}_q)_y\,K_{\mathcal{W}}(y,x) = \sum_{q=1}^{N} a_q \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} A^2_{k,n} \sum_{j=-n}^{n} \int_{\gamma_q} W^B_{k,n,j}(y)\,\mathrm{d}\sigma(y)\,W^B_{k,n,j}(x)$$

approximating the function $\tilde{S}$.
Methods of determining the parametric equations of the raypaths $\gamma_q; q = 1,...,N$ can be found e.g. in [5].

7. Numerical Tests

Let V_0 be the P-wave velocity function according to PREM ([9]). In numerical tests we take $S_1 := 1/V_1$ as a reference slowness model , where V_1 is an approximation to V_0 with a function which stepwise is of the form (see Figure 2):

$$V(r) = A\,r^{(1-b)}, \quad r \in [0,1], \quad A, b = \text{const.} \tag{21}$$

More precisely, we get V_1 by dividing [0,1] into 48 parts and in each of these parts approximating V_0 with a function of a form (21). Thus, rays should be generated according to the slowness model S_1. In this case the parametric equations of the raypaths $\gamma_q; q = 1,...,N$ can be written in a simple analytic form (see [5], p. 177).
As a sequence $\{A_{k,n}\}_{k,n\in\mathbb{N}_0}$ we took $A^2_{k,n} = B^2_k C^2_n$, $k,n \in \mathbb{N}_0$, where $B^2_k = e^{-\lambda_1\,k(k+1)}$ is the Gauß–Weierstraß symbol, and $C^2_n = e^{-\lambda_2\,n}$ is the Abel-Poisson symbol. In this case our reproducing kernel $K_{\mathcal{H}}(\cdot,\cdot)$ can be written as (see (5) and [17], p. 45),

$$K_{\mathcal{H}}(x,y) = \sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\sum_{j=-n}^{n} A^2_{k,n} W^B_{k,n,j}(x) W^B_{k,n,j}(y) = \frac{1}{4\pi}\frac{1-h^2}{(1+h^2-2h(\frac{x}{|x|}\cdot\frac{y}{|y|}))^{(3/2)}}$$

$$\times \sum_{k=0}^{\infty} B^2_k (2k+3) P^{(0,2)}_k\,(2|x|-1)\,P^{(0,2)}_k\,(2|y|-1) = K_1\,(x/|x|, y/|y|)\;K_2\,(|x|,|y|)\,,$$

where

$$K_1\left(x/|x|, y/|y|\right) = K_1\left(\xi_x, \xi_y\right) := \frac{1}{4\pi}\frac{1-h^2}{(1+h^2-2h(\xi_x\cdot\xi_y))^{(3/2)}},$$

with $h := C_1^2 = e^{-\lambda_2}$ and

$$K_2\left(|x|, |y|\right) = K_2\left(r_x, r_y\right) := \sum_{k=0}^{\infty} B_k^2(2k+3)P_k^{(0,2)}(2r_x-1)P_k^{(0,2)}(2r_y-1) \tag{22}$$

(for similar kernels see [35]). We see that for fixed $x_0 \in B$, K_1 only depends on ξ_y, i.e. on the unit vector of y, and K_2 only depends on r_y, i.e. on the radius of y. This suggests that we can choose the parameters λ_1, λ_2 independently to control the localization character (hat-width) of $K_{\mathcal{H}}$ in the direction of r_y and ξ_y, respectively. The last point is particularly important in body wave tomography, since here the unknown (velocity) function has strong variations in the direction of r_y and relatively small variations in the direction of ξ_y. In this case the matrix components corresponding to our spline interpolation problem can be written as

$$\begin{aligned}\left(\mathcal{F}_l\right)_x\left(\mathcal{F}_q\right)_y K_{\mathcal{H}}(y,x) = \frac{1-h^2}{4\pi}\int_{\gamma_l}\int_{\gamma_q}\Bigg(&\frac{1}{(1+h^2-2h(\frac{x}{|x|}\cdot\frac{y}{|y|}))^{(3/2)}}\times\\ &\sum_{k=0}^{\infty} B_k^2(2k+3)P_k^{(0,2)}\left(2|x|-1\right)P_k^{(0,2)}\left(2|y|-1\right)\Bigg)\mathrm{d}\sigma(x)\mathrm{d}\sigma(y).\end{aligned}$$

Thus, by solving the linear equation system (20) we obtain the coefficients $(a_q)_{q=1,\dots,N}$ of the spline

$$\begin{aligned}S(x) = \sum_{q=1}^{N} a_q\left(\mathcal{F}_q\right)_y K_{\mathcal{H}}(y,x) = \frac{1-h^2}{4\pi}\sum_{q=1}^{N} a_q\int_{\gamma_q}\Bigg(&\frac{1}{(1+h^2-2h(\frac{x}{|x|}\cdot\frac{y}{|y|}))^{(3/2)}}\times\\ \sum_{k=0}^{\infty} B_k^2(2k+3)P_k^{(0,2)}\left(2|x|-1\right)P_k^{(0,2)}\left(2|y|-1\right)\Bigg)\mathrm{d}\sigma(y).\end{aligned}$$

approximating the unkown function. The representation of $x \in B$ in the spherical coordinates will be denoted by $\bar{x}(r,\theta,\phi)$, where $r \in [0,1], \theta \in [0,\pi]$ and $\phi \in [0,2\pi)$.

Here we run several numerical tests. In Figure 4 we present a reconstruction of $V_1(r)$ in the segment $r \in [0.65,1]$ with $\theta = 120°$ and $\phi = 90°$ using the synthetic ray system presented in Figure 3(a). In Figures 6 and 5 we present an approximaton of the function $V_2(\bar{x}(r,\theta,\phi)) := 5 + 0.1\,\sin(5r)\cos(20\theta)$ at $r = 0.98$, $r = 0.99$, $\theta \in [100°, 125°]$ and $\phi = 90°$ using the synthetic ray system presented in Figure 3(b). To see how the measurement errors affect the result, we recalculate the models in Figures 4, 5 and 6, where we add a random noise of one percent to the corresponding traveltimes (see Figures 7, 8 and 9). The integral terms representing the matrix components and the spline basis have been calculated approximately with the trapezoidal rule, where the series in (22) has been truncated at level 50. Moreover, a smoothing (regularization) of the linear equation system, with a smoothing parameter ρ, was done.

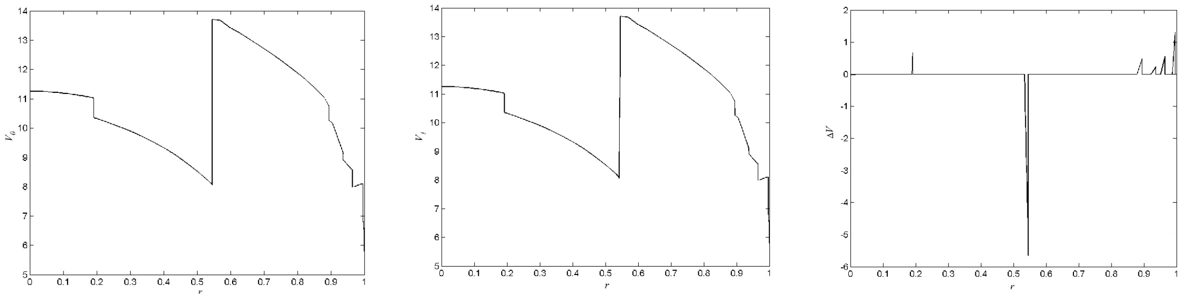
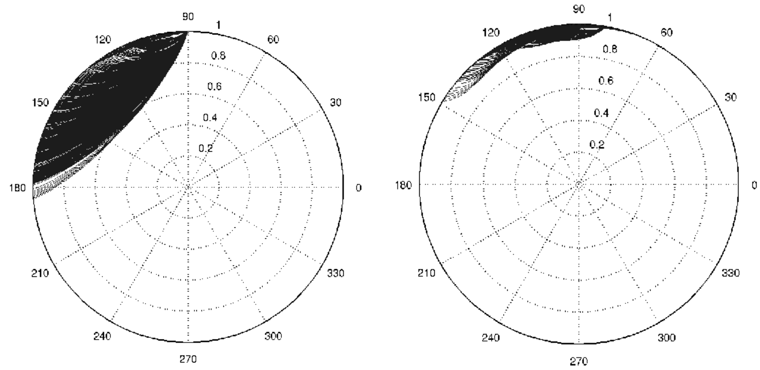
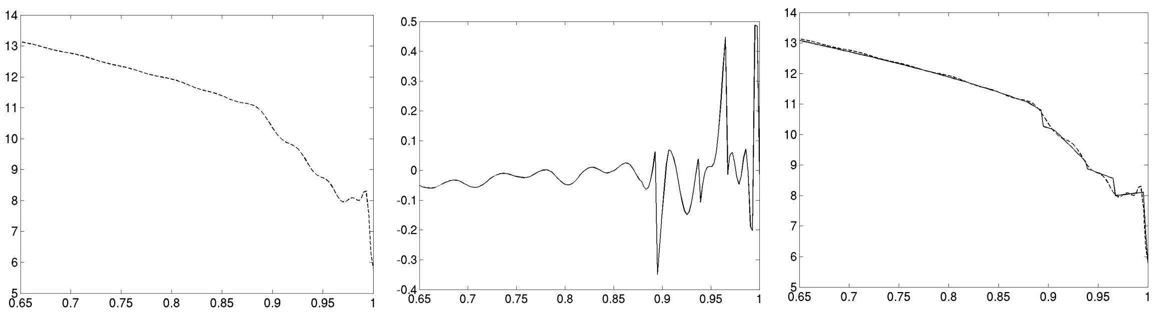


Figure 2. P-Wave velocity V_0 (according to PREM) (left), approximation of V_0, with a function V_1 which stepwise is of the form (21) (center), difference of V_1 and V_0, $\Delta V = V_1 - V_0$ (right).



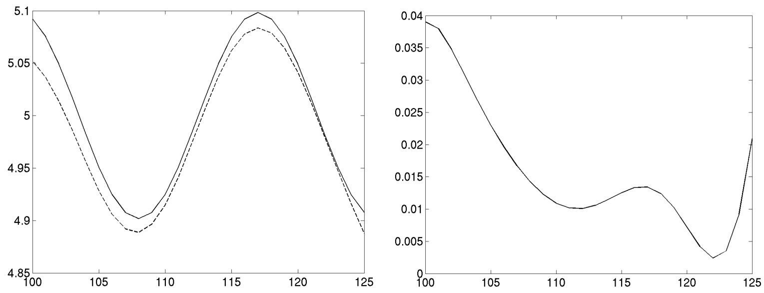
(a) paths of 360 synthetic rays (b) paths of 300 synthetic rays

Figure 3. paths of synthetic rays generated according to V_1 and plotted on the plane $\phi = 90°$



(a) reconstruction of V_1 (b) error of the reconstruction of V_1 (c) comparison of the profiles of V_1 (solid line) and its reconstruction (dashed line)

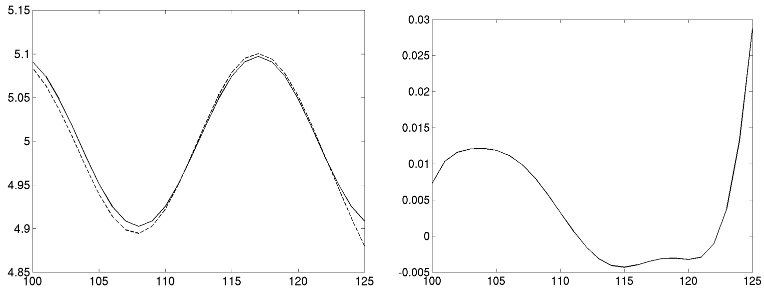
Figure 4. reconstruction and corresponding error of V_1 and comparison of their profiles, using the rays in Figure 3(a), with $\lambda_1 = 0.001$, $\lambda_2 = 10$, $\rho = 10^{-6}$



(a) comparison of the profiles of V_2 (solid line) and its reconstruction (dashed line)

(b) error of the reconstruction of V_2

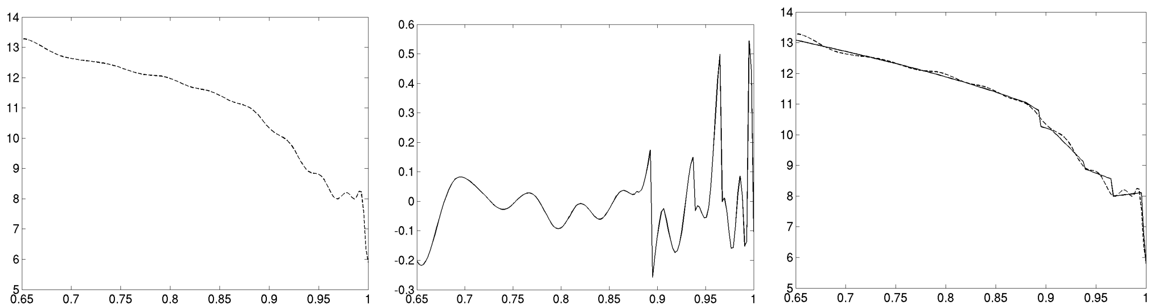
Figure 5. reconstruction of $V_2(r, \theta)$ using the rays in Figure 3(b), with $\lambda_1 = 0.2$, $\lambda_2 = 0.3, \rho = 0.04$ at $r = 0.98$, $\theta \in [100°, 125°]$ and $\phi = 90°$



(a) comparison of the profiles of V_2 (solid line) and its reconstruction (dashed line)

(b) error of the reconstruction of V_2

Figure 6. reconstruction of $V_2(r, \theta)$ using the rays in Figure 3(b), with $\lambda_1 = 0.2$, $\lambda_2 = 0.3, \rho = 0.04$ at $r = 0.99$, $\theta \in [100°, 125°]$ and $\phi = 90°$

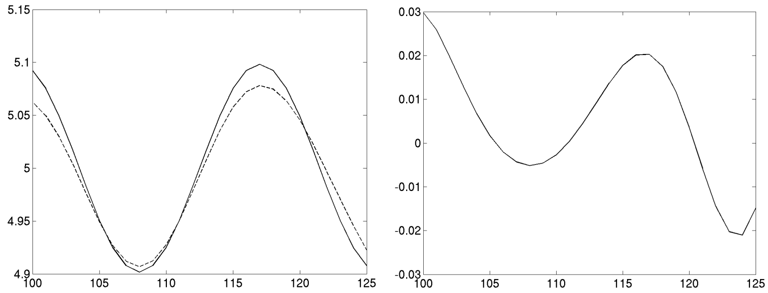


(a) reconstruction of V_1

(b) error of the reconstruction of V_1

(c) comparison of the profiles of V_1 (solid line) and its reconstruction (dashed line)

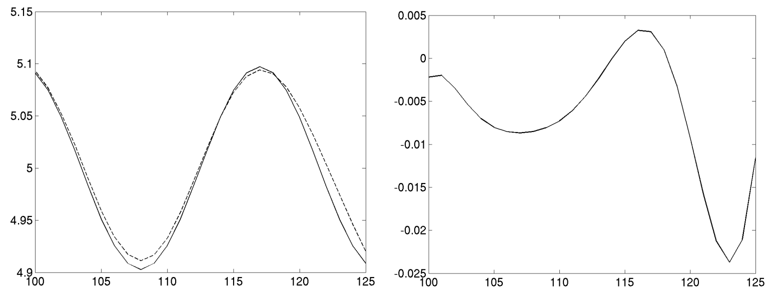
Figure 7. reconstruction and corresponding error of V_1 and comparison of their profiles, using the rays in Figure 3(a), with $\lambda_1 = 0.001$, $\lambda_2 = 10$, $\rho = 10^{-6}$ and with 1% random error in the traveltimes



(a) comparison of the profiles of V_2 (solid line) and its reconstruction (dashed line)

(b) error of the reconstruction of V_2

Figure 8. reconstruction of $V_2(r, \theta)$ using the rays in Figure 3(b), with $\lambda_1 = 0.2$, $\lambda_2 = 0.3, \rho = 0.04$ at $r = 0.98$, $\theta \in [100°, 125°]$, $\phi = 90°$ and with 1% random error in the traveltimes



(a) comparison of the profiles of V_2 (solid line) and its reconstruction (dashed line)

(b) error of the reconstruction of V_2

Figure 9. reconstruction of $V_2(r, \theta)$ using the rays in Figure 3(b), with $\lambda_1 = 0.2$, $\lambda_2 = 0.3, \rho = 0.04$ at $r = 0.99$, $\theta \in [100°, 125°]$, $\phi = 90°$ and with 1% random error in the traveltimes

8. Conclusions

The results demonstrate that with the described spline method we are able to obtain a good approximation for a relatively smooth laterally heterogeneous model (see Figures 6 and 5) as well as for a model with rather big radial variations (see Figure 4). Moreover, Figures 7, 8 and 9 demonstrate the stability of our method with respect to measurement errors. Hence, the described spline approximation method proved to be a worthy tool for interpolating/approximating a function in a ball, in particular for body wave tomography.

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