Polyhedral Analysis of Uncapacitated Single Allocation p-Hub Center Problems

Silke Jütte
Department of Supply Chain Management and Management Science,
University of Cologne, D-50923 Cologne, Germany
silke.juette@uni-koeln.de

Elena O. Gavriliouk*
CSIR0 Mathematical and Information Sciences, Private Bag 33,
South Clayton, VIC 3169, Australia
olena.gavriliouk@cmis.csiro.au

Horst W. Hamacher*
Department of Mathematics, University of Kaiserslautern,
D-67663 Kaiserslautern, Germany
hamacher@mathematik.uni-kl.de

April 25, 2007

Abstract

In contrast to p-hub problems with a summation objective (p-*hub median*), min-max hub problems (p-*hub center*) have not attained much attention in the literature. In this paper, we give a polyhedral analysis of the uncapacitated single allocation p-hub center problem (USApHCP). The analysis will be based on a radius formulation which currently yields the most efficient solution procedures. We show which of the valid inequalities in this formulation are facet-defining and present non-elementary classes of facets, for which we propose separation problems.

A major part in our argumentation will be the close connection between polytopes of the USApHCP and the uncapacitated p-facility location (pUFL). Hence, the new classes of facets can also be used to improve pUFL formulations.

Keywords: hub location, polyhedral analysis, uncapacitated facility location

---

*This research has been partially supported by Deutsche Forschungsgemeinschaft (DFG) grant HA 1737/7 and the Rheinland-Pfalz cluster of excellence "Dependable adaptive systems and mathematical modeling."

1 Introduction

In the last two decades hub location models have enjoyed much attention from researchers in different fields. Research concentrated, however, mostly on hub problems with median objective. It is only in the last few years that hub center problems have been given some attention (see surveys of Alumur and Kara 2007 or Campbell, Ernst, and Krishnamoorthy 2002).

As in all hub problems, pairs of origin-destination (o-d) nodes need to exchange a commodity, e.g., information, passengers, or goods. For this purpose, a subset of the original nodes is selected to serve as *hub nodes*, which are completely interconnected. All the non-hub nodes (so-called *spokes*) are allocated to hubs, where flow is collected, consolidated and distributed. Transportation can profit from economies of scale, thus making the hub network an interesting alternative to a fully interconnected network. The *p-hub center problem* consists of choosing a number of p nodes to become hubs such as to minimize the maximum transportation cost in the resulting hub network. In this treatise, we restrict ourselves to the *single allocation* case, i.e., each spoke node is connected only to one hub node. The optimal allocation of spokes to hubs then becomes part of our problem. We are dealing with an *uncapacitated* optimization problem, meaning that we can allocate an arbitrary number of spokes to a hub. Thus, this special problem type is referred to as the *uncapacitated single allocation p-hub center problem* (USApHCP).

Hub location problems with median objectives have been considered from different perspectives, e.g., polyhedral analysis (Hamacher et al. 2004 and Sonneborn 2002), formulations (Campbell 1996), and heuristics (Ernst and Krishnamoorthy 1998, O'Kelly, Skorin-Kapov, and Skorin-Kapov 1995, etc.).

The study of USApHCP has been initiated by Campbell 1994. Formulations and heuristics for the hub center problem have been considered in Ernst et al. 2002a, Ernst et al. 2002b, and Ernst et al. 2002c. To the best of our knowledge, there are two competing formulations for USApHCP: one due to Kara and Tansel 2000 and the other from Ernst et al. 2002c. In the following, we will concentrate on the latter, the *radius formulation*. This formulation, combined with a hub cover model of Wagner 2004 is also the basis for the currently fastest implementation of a solution algorithm for USApHCP (see Hamacher and Meyer 2006).

Consider an undirected graph $G = (V, E)$ with n supply-demand nodes $V = \{1, \ldots, n\}$. Out of the set V, p nodes have to be chosen as hubs. A cost factor d_{ij} is associated with each o-d pair $i, j \in V$, satisfying (i) $d_{ij} \geq 0 \; \forall \, i, j \in V$ (*non-negativity*), (ii) $d_{ij} = 0 \iff i = j$, (iii) $d_{ij} = d_{ji} \; \forall \, i, j \in V$ (*symmetry*), (iv) $d_{ij} \leq d_{ik} + d_{kj} \; \forall \, i, k, j \in V$ (*triangle inequality*), and (v) $d_{ij} = M \; \forall \, e_{ij} \notin E$, where M is a large positive number. A discount factor $\alpha \in [0, 1]$ represents cost savings resulting from economies of scale on inter-hub arcs. For each hub node k, the *radius* r_k denotes the maximum distance from k to all the spoke nodes allocated to it. For $i, k \in V$, let $X_{i\,k} = 1$ if node i is allocated to hub k, $X_{i\,k} = 0$ else. (Especially, $X_{k\,k} = 1$ if and only if node k is a hub.) Assuming that $M \geq \max_{k,m} (\max_j d_{j\,k} + \max_j d_{j\,m} + \alpha\, d_{k\,m})$ for $j, k, m \in V$, the radius formulation of Ernst et al. 2002c for USApHCP is

$$
\begin{aligned}
&\min z && && \\
\text{s.t.}\quad & z && \geq && r_k + r_m + \alpha\, d_{k\,m} \quad \forall\, k, m \in \{1, \ldots, n\} && (1)\\
& r_k && \geq && d_{i\,k} X_{i\,k} \quad \forall\, i, k \in \{1, \ldots, n\} && (2)\\
& r_k && \leq && M X_{k\,k} \quad \forall\, k \in \{1, \ldots, n\} && (3)\\
& \sum_{k=1}^{n} X_{i\,k} && = && 1 \quad \forall\, i \in \{1, \ldots, n\} && (4)\\
& X_{i\,k} && \leq && X_{k\,k} \quad \forall\, i, k \in \{1, \ldots, n\} && (5)\\
& \sum_{k=1}^{n} X_{k\,k} && = && p && (6)\\
& r_k && \geq && 0 \quad \forall\, k \in \{1, \ldots n\} && (7)\\
& X_{i\,k} && \in && \{0, 1\} \quad \forall\, i, k \in \{1, \ldots n\}. && (8)
\end{aligned}
$$

Constraints (4) and (8) assure that each node is uniquely allocated to a hub, whereas constraints (5) allow allocation to a node only if it is a hub. Constraint (6) requires that exactly p are established. Constraints (2) set up hub radii according to their definition. Constraints (1) together with the objective function work to minimize the maximum distance within an o-d pair. Constraints (3) provide an upper bound on the radii and enforce $r_k = 0$ for spoke nodes. Note that (3) are not included in the original formulation of Ernst et al. 2002c, but have been added since they will be helpful when examining facets.

In the following, we will present a polyhedral analysis for USApHCP using the radius formulation. The rest of this paper is organized as follows: First we examine the polyhedron describing the convex hull of the points satisfying (1)–(8); this examination entails calculating its dimension and determining which of the given constraints are facet-defining. i.e. are *elementary facets*. Then we identify three classes of *non-elementary facets*, i.e., facets of the convex hull other than the ones derived from the constraints. Finally, we propose algorithms for the resulting separation problems.

2 Dimension of the polyhedron

Definition 2.1 *i) The (radius) formulation polyhedron $\mathcal{K}$ of USApHCP is the set of all points $P = (X_{1\,1}, X_{1\,2}, \ldots, X_{n\,n}, r_1, \ldots, r_n, z) \in \mathbb{R}_+^{n^2+n+1}$ such that P satisfies constraints (1)–(7) and $0 \leq X_{i\,k} \leq 1$ for all $i, k \in \{1, \ldots, n\}$.*

ii) The set of all feasible solutions satisfying constraints (1)–(8) is denoted by $\mathcal{X}$, i.e., $\mathcal{X} = \mathcal{K} \cap (\{0,1\}^{n^2} \times \mathbb{R}^{n+1})$.

Since USApHCP has been shown to be NP-hard (see Ernst et al. 2002c), the computation of $\mathrm{conv}(\mathcal{X})$ is out of reach. Hence, in what follows, we present a polyhedral analysis in order to find facet-defining inequalities of $\mathrm{conv}(\mathcal{X})$.

In our analysis, we take advantage of the relation between USApHCP and another optimization problem—the *uncapacitated p-facility location problem*, pUFL (see Cornuejols, Nemhauser, and Wolsey 1990).

Given an undirected graph $G = (V, E)$ with demand nodes $V = \{1, \ldots, n\}$, pUFL consists of choosing p nodes out of V as facilities to serve the other demand nodes. Let $x_{i\,k} = 1$ if node i is served by facility k, otherwise $x_{i\,k} = 0$. Then, equipped with a center objective function, pUFL can be stated as follows:

$$
\begin{aligned}
\min \max_{i,k} \, & d_{i\,k}\, x_{i\,k} \\
\text{s.t.} \sum_{k=1}^{n} x_{i\,k} &= 1 \quad \forall\, i \in \{1, \ldots, n\} && (9)\\
x_{i\,k} &\leq x_{k\,k} \quad \forall\, i, k \in \{1, \ldots, n\} && (10)\\
\sum_{k=1}^{n} x_{k\,k} &= p && (11)\\
x_{i\,k} &\in \{0,1\} \quad \forall\, i, k \in \{1, \ldots n\} && (12)
\end{aligned}
$$

Note that we distinguish between the denotations $x_{i\,k}$ and $X_{i\,k}$ when referring to variables related to pUFL and USApHCP, respectively. Also, when notation becomes cumbersome, we separate the indices with commas, i.e., $X_{i,\,k}$. In this work, both USApHCP and pUFL are considered only for $p \in \{2, \ldots, n-2\}$. Let $\mathcal{X}_{\text{pUFL}}$ denote the set of feasible solutions to pUFL and $\mathrm{conv}(\mathcal{X}_{\text{pUFL}})$ the convex hull of these feasible solutions, and let $\mathcal{K}_{\text{pUFL}}$ be the formulation polytope derived from constraints (9)–(11) and the relaxed binary restrictions. The following relations between USApHCP and pUFL are easy to see.

Proposition 2.2 *Let* $P = (x_{1\,1}, x_{1\,2}, \ldots, x_{n\,n})$ *be a feasible solution to pUFL. Then,* $P^* := (X_{1\,1}, X_{1\,2}, \ldots, X_{n\,n}, r_1, \ldots, r_n, z)$*, with* $X_{i\,k} := x_{i\,k}\ \forall\ i, k$*,* $r_k := \max_i d_{i\,k} X_{i\,k}\ \forall\ k$*, and* $z \geq \max_{k,m} r_k + r_m + \alpha\, d_{k\,m}$ *is a feasible solution to USApHCP, and thus,* $P^* \in \mathrm{conv}(\mathcal{X})$*.*

Proposition 2.3 *Let* $P_1, \ldots, P_m \in \mathrm{conv}(\mathcal{X}_{pUFL})$ *be affinely independent points. Then,* $P_1^*, \ldots, P_m^* \in \mathrm{conv}(\mathcal{X})$ *are affinely independent, where* P_i^* *is constructed from* P_i *as shown in proposition 2.2.*

We have thus shown:

Corollary 2.4 $\dim(\mathrm{conv}(\mathcal{X})) \geq \dim(\mathrm{conv}(\mathcal{X}_{pUFL}))$.

In the derivation of the dimension of $\mathrm{conv}(\mathcal{X})$, we will make use of the following fact due to Cornuejols, Nemhauser, and Wolsey 1990.

Lemma 2.5 $\dim(\mathrm{conv}(\mathcal{X}_{pUFL})) = n^2 - n - 1$.

Now we can determine the dimension of $\mathrm{conv}(\mathcal{X})$. To facilitate the development of the proofs in the following sections, the dimension of $\mathrm{conv}(\mathcal{X})$ will be derived in a direct way, that is by constructing a maximum set of affinely independent points in $\mathrm{conv}(\mathcal{X})$.

In the following, let $X_{i\,k}^P$ denote the value of a variable $X_{i\,k}$ for the point P. The denotations r_k^P and z^P are used analogously.

Theorem 2.6 $\dim(\mathrm{conv}(\mathcal{X})) = n^2$.

Proof: The radius formulation of USApHCP contains n^2+n+1 variables and $n+1$ equations which are linearly independent. Consequently, $\dim(\mathrm{conv}(\mathcal{X})) \leq n^2$. It remains to show that there exist $n^2 + 1$ affinely independent points in $\mathrm{conv}(\mathcal{X})$:

By lemma 2.5, there exist $n^2 - n$ affinely independent points $P_1, \ldots, P_{n^2-n} \in \mathrm{conv}(\mathcal{X}_{\mathrm{pUFL}})$. Thus, by proposition 2.3, we can construct $n^2 - n$ affinely independent points $P_1^*, \ldots, P_{n^2-n}^*$ in $\mathrm{conv}(\mathcal{X})$ as proposed in proposition 2.2, but set $z := 2 \max_{k,m} (r_k + r_m + \alpha\, d_{k\,m})$ for each P_i^*.

Now, use $P_1^*, \ldots, P_{n^2-n}^*$ to construct additional $n+1$ feasible, affinely independent solutions. First, note that for every $k \in \{1, \ldots, n\}$ there is $i \in \{1, \ldots, n^2 - n\}$ such that $r_k^{P_i^*} > 0$ (since $P_1, \ldots, P_{n^2-n}$ are affinely independent).

For every k, choose the minimal index $i(k) \in \{1, \ldots, n^2 - n\}$ with $r_k^{P_{i(k)}^*} > 0$. Let $\mathcal{L} = \{L_1, \ldots, L_s\}$ be the index set of chosen points $P_{L_j}^*$ ordered such that $L_1 <$

$L_2 < \ldots < L_s$, and let $\mathcal{L}_j$ be the set of k-values for which $P^*_{L_j}$ has been chosen, i. e., $\mathcal{L}_j = \{k : i(k) = L_j\}$. Construct new points $\hat{P}_1, \ldots, \hat{P}_n$ with $\hat{P}_k = P^*_{i(k)}$, except that $r_k^{\hat{P}_k} = 2\, r_k^{P^*_{i(k)}}$. Due to the choice of M, the new points $\hat{P}_1, \ldots, \hat{P}_n$ stay feasible.

Next, we show that the points $P^*_1, \ldots, P^*_{n^2-n}, \hat{P}_1, \ldots, \hat{P}_n$ are affinely independent. Assume that

$$\sum_{i=1}^{n^2-n} \alpha^*_i P^*_i + \sum_{k=1}^{n} \hat{\alpha}_k \hat{P}_k = 0 \tag{13}$$

$$\text{with} \quad \sum_{i=1}^{n^2-n} \alpha^*_i + \sum_{k=1}^{n} \hat{\alpha}_k = 0. \tag{14}$$

We want to show that $\alpha^*_i = \hat{\alpha}_k = 0$ for all i, k. Equation (13) induces for all $m, r \in \{1, \ldots, n\}$ the following equations in variables $X_{m\,r}$ for the constructed points:

$$\begin{aligned} \sum_{i=1}^{n^2-n} \alpha^*_i X_{m\,r}^{P^*_i} + \sum_{k=1}^{n} \hat{\alpha}_k X_{m\,r}^{P^*_{i(k)}} &= 0 \\ \Leftrightarrow \quad \sum_{\substack{i=1\\ i\notin\mathcal{L}}}^{n^2-n} \alpha^*_i X_{m\,r}^{P^*_i} + \sum_{j=1}^{s} \Big(\alpha^*_{L_j} + \sum_{l\in\mathcal{L}_j} \hat{\alpha}_l\Big) X_{m\,r}^{P^*_{L_j}} &= 0. \end{aligned} \tag{15}$$

The equivalence follows from the definition of $L_j, \mathcal{L}_j$ and $\mathcal{L}$. Due to the construction of P^*_i $(i = 1, \ldots, n^2-n,\ i \notin \mathcal{L})$ and $P^*_{L_j}$ $(j = 1, \ldots, s)$ and using $P_1, \ldots, P_{n^2-n} \in \operatorname{conv}(\mathcal{X}_{\text{pUFL}})$, we can conclude from (15) that

$$\sum_{\substack{i=1\\ i\notin\mathcal{L}}}^{n^2-n} \alpha^*_i P_i + \sum_{j=1}^{s} \Big(\alpha^*_{L_j} + \sum_{l\in\mathcal{L}_j} \hat{\alpha}_l\Big) P_{L_j} = 0.$$

Now, by (14)

$$\sum_{\substack{i=1\\ i\notin\mathcal{L}}}^{n^2-n} \alpha^*_i + \sum_{j=1}^{s} \Big(\alpha^*_{L_j} + \sum_{l\in\mathcal{L}_j} \hat{\alpha}_l\Big) = \sum_{i=1}^{n^2-n} \alpha^*_i + \sum_{k=1}^{n} \hat{\alpha}_k = 0.$$

Since $P_1, \ldots, P_{n^2-n}$ have been chosen affinely independent, we can conclude that

$$\begin{aligned} \alpha^*_i &= 0 \quad \forall\, i \in \{1, \ldots, n^2-n\},\ i \notin \mathcal{L} \\ \text{and} \quad \alpha^*_{L_j} + \sum_{l\in\mathcal{L}_j} \hat{\alpha}_l &= 0 \quad \forall\, j \in \{1, \ldots, s\}. \end{aligned} \tag{16}$$

Thus equation (13) reduces to

$$\sum_{j=1}^{s} \left(\alpha^*_{L_j} P^*_{L_j} + \sum_{l \in \mathcal{L}_j} \hat{\alpha}_l \hat{P}_l \right) = 0.$$

Considering the resulting equations

$$\sum_{j=1}^{s} \left(\alpha^*_{L_j} r_k^{P^*_{L_j}} + \sum_{l \in \mathcal{L}_j} \hat{\alpha}_l r_k^{\hat{P}_l} \right) = 0$$

for the variables r_k $(k \in \{1, \dots, n\})$, we obtain for $k \in \mathcal{L}_s$

$$\begin{aligned} \alpha^*_{L_s} r_k^{P^*_{L_s}} + \sum_{l \in \mathcal{L}_s} \hat{\alpha}_l r_k^{\hat{P}_l} &= 0 \\ \Rightarrow \alpha^*_{L_s} r_k^{P^*_{L_s}} + 2\hat{\alpha}_k r_k^{P^*_{L_s}} + \sum_{\substack{l \in \mathcal{L}_s \\ l \neq k}} \hat{\alpha}_l r_k^{P^*_{L_s}} &= 0 \\ \Leftrightarrow \alpha^*_{L_s} + 2\hat{\alpha}_k + \sum_{\substack{l \in \mathcal{L}_s \\ l \neq k}} \hat{\alpha}_l &= 0 \end{aligned}$$

On the other hand, we know from (16) that

$$\alpha^*_{L_s} + \hat{\alpha}_k + \sum_{\substack{l \in \mathcal{L}_s \\ l \neq k}} \hat{\alpha}_l = 0,$$

and thus, $\hat{\alpha}_k = 0$. In fact, $\hat{\alpha}_k = 0 \; \forall \, k \in \mathcal{L}_s$ which allows us to conclude that $\alpha^*_{L_s} = 0$. In the same manner one can show that $\hat{\alpha}_k = 0 \; \forall \, k \in \mathcal{L}_j \; \forall \, j \in \{1, \dots, s\}$ and $\alpha^*_{L_j} = 0 \; \forall \, j \in \{1, \dots, s\}$.

Hence we have shown that the points $P^*_1, \dots, P^*_{n^2-n}, \hat{P}_1, \dots, \hat{P}_n$ are affinely independent in $\mathrm{conv}(\mathcal{X})$. To complete this proof, consider point $\hat{P}_{n+1}$ with $\hat{P}_{n+1} = P^*_1$ except for $z^{\hat{P}_{n+1}} = \frac{1}{2} z^{P^*_1}$. By choice of $z^{P^*_1}$, $\hat{P}_{n+1}$ stays feasible. Using a similar argumentation as above, it is easy to see that the points $P^*_1, \dots, P^*_{n^2-n}, \hat{P}_1, \dots, \hat{P}_{n+1}$ are affinely independent in $\mathrm{conv}(\mathcal{X})$. $\square$

In the following sections we will present classes of facet-defining inequalities of $\mathrm{conv}(\mathcal{X})$. Since $\dim(\mathrm{conv}(\mathcal{X})) = n^2$, we will look for hyperplanes with dimension $n^2 - 1$. Considering facets in variables $X_{i\,k}$, we can once again make use of the relationship between USApHCP and pUFL, and apply similar arguments as in the proofs of proposition 2.2 and theorem 2.6.

Proposition 2.7 *(i) Given $n^2 - n - 1$ affinely independent points $P_1, P_2, \ldots$ $\ldots, P_{n^2-n-1} \in \operatorname{conv}(\mathcal{X}_{pUFL})$, one can construct n^2 affinely independent points $P^*_1, \ldots, P^*_{n^2-n-1}, \hat{P}_1, \ldots, \hat{P}_{n+1} \in \operatorname{conv}(\mathcal{X})$. Consequently, every facet of $\operatorname{conv}(\mathcal{X}_{pUFL})$ corresponds to a facet of $\operatorname{conv}(\mathcal{X})$.*

(ii) Conversely, if an inequality in variables $x_{i\,k}$ does not define a facet of $\operatorname{conv}(\mathcal{X}_{pUFL})$, then the corresponding inequality in variables $X_{i\,k}$ is not a facet of $\operatorname{conv}(\mathcal{X})$.

Thus, when searching for facets of $\operatorname{conv}(\mathcal{X})$ that include only $X_{i\,k}$ variables, we can restrict ourselves to looking only among the facets of $\operatorname{conv}(\mathcal{X}_{\mathrm{pUFL}})$.

3 Elementary facets

In this section, we examine constraints (1) to (6) to decide which of these are facet-defining for $\operatorname{conv}(\mathcal{X})$. To start off, we present some elementary inequalities of USApHCP that do not represent facets.

Proposition 3.1 *The following valid inequalities do not represent facets of* $\operatorname{conv}(\mathcal{X})$*:*

(i) $X_{k\,k} \geq 0 \quad \forall\, k$,

(ii) $X_{k\,k} \leq 1 \quad \forall\, k$,

(iii) $X_{i\,k} \leq 1 \quad \forall\, i \neq k$.

(iv) $r_k \geq 0 \quad \forall\, k$,

(v) $z \geq r_k + r_m + \alpha\, d_{k\,m}$ *(for* $\alpha \in (0, 1)$*)*

Proof:

(i)-(iii) Obvious.

(iv) If $r_k = 0$ then $X_{i\,k} = 0\; \forall\, i \neq k$ since $d_{i\,k} > 0$.

(v) Assume that $\mathcal{F} := \{P \in \operatorname{conv}(\mathcal{X}) : z = r_k + r_m + \alpha\, d_{k\,m}\}$ defines a facet. Thus, there exist n^2 affinely independent points $P_1, \ldots, P_{n^2} \in \mathcal{F} \cap \mathcal{X}$. Since the points are affinely independent, there exists P_i with $X_{k\,k} = 0$ (and

$X_{k\,i_k} = 1$ for an $i_k \neq k$). But then

$$\begin{aligned}
r_k^{P_i} + r_m^{P_i} + \alpha\, d_{k\,m} &= r_m^{P_i} + \alpha\, d_{k\,m} \\
&< r_m^{P_i} + \alpha\, d_{k\,m} + (1-\alpha) d_{i_k\,k} \\
&= r_m^{P_i} + \alpha (d_{k\,m} - d_{i_k\,k}) + d_{i_k\,k} \\
&\leq r_m^{P_i} + \alpha\, d_{m\,i_k} + d_{i_k\,k} \\
&\leq r_m^{P_i} + \alpha\, d_{m\,i_k} + r_{i_k}^{P_i} \\
&\leq z.
\end{aligned}$$

That is, $z > r_k + r_m + \alpha\, d_{k\,m}$ holds for P_i, and thus, $P_i \notin \mathcal{F}$.

□

Next, we will make use of proposition 2.7 to present elementary facets of $\mathrm{conv}(\mathcal{X})$ in variables $X_{i\,k}$. The following propositions 3.2 and 3.3 are presented without proofs in Cornuejols, Nemhauser, and Wolsey 1990. Here we give the details of the proof of the first result. The reason for this is that most of the proofs presented in the treatise follow a similar pattern. However, the details of those proofs are sometimes very cumbersome and lengthy; therefore, they will be partially omitted, but are available from the authors upon request.

Proposition 3.2 *For any pair* $i, k \in \{1, \ldots, n\}$, $i \neq k$, $\mathcal{F}_I = \{P \in \mathrm{conv}(\mathcal{X}) : X_{i\,k}^P = 0\}$ *is a facet of* $\mathrm{conv}(\mathcal{X})$.

Proof:

Using proposition 2.7, it suffices to show that $\mathcal{F}_I$ is a facet of $\mathrm{conv}(\mathcal{X}_{\text{pUFL}})$. W.l.o.g., $i = 1$ and $k = 2$. Assume that the equation

$$\sum_{j=1}^{n} \sum_{l=1}^{n} a_{j\,l} x_{j\,l} = d \tag{17}$$

holds for all points $P \in \mathcal{F}_I$ feasible for pUFL.

By constructing appropriate points in $\mathcal{F}_I$, we will show that

1. $a_{j\,l} = a_{j\,m} =: a_j \;\forall\, j, l, m$ with $j \notin \{l, m\}$ and $(1, 2) \notin \{(j, l), (j, m)\}$
2. $a_{m\,m} - a_m = a_{l\,l} - a_l := a \;\forall\, l, m$

With the two statements above, (17) can be reformulated as a linear combination of equality constraints (9), (11) and the facet-defining constraint $x_{i\,k} = 0$.

ad 1:

- If $l, m \neq 2$, choose $s_1, \ldots, s_{p-2} \in \{1, \ldots, n\} \backslash \{l, m, j\}$ pairwise different (note that $p \leq n-2$) and set

  P_1: $x_{s_1\, s_1} = \ldots = x_{s_{p-2}\, s_{p-2}} = 1$, $x_{l\, l} = x_{m\, m} = 1$,
  $x_{i\, l} = 1 \quad \forall \quad i \in \{1, \ldots, n\} \backslash \{s_1, \ldots, s_{p-2}, l, m\}$,
  P_2: all values as in P_1, except $x_{j\, l} = 0$, $x_{j\, m} = 1$.

  Since $x_{1\, 2} = 0$ for both points, they lie in $\mathcal{F}_I$, and inserting them into (17) gives $a_{j\, l} = a_{j\, m}$.

- If $l = 2$ (or $m = 2$), construct points as above, but choose $s_1 = 1$ to ensure that $x_{1\, 1} = 1$ (and thus $x_{1\, 2} = 0$).

ad 2: Using statement 1, we have shown that (17) transforms into

$$d - \sum_{j=1}^{n} a_j = \sum_{j=1}^{n} (a_{j\, j} - a_j) x_{j\, j}. \tag{18}$$

- If $l,\ m \neq 1$, choose $s_1 = 1$, $s_2, \ldots, s_{p-1} \in \{2, \ldots, n\} \backslash \{l, m\}$ pairwise different and use the points

  P_1: $x_{s_1\, s_1} = \ldots = x_{s_{p-1}\, s_{p-1}} = 1$, $x_{l\, l} = 1$,
  $x_{r\, s_1} = 1 \quad \forall\, r \in \{1, \ldots, n\} \backslash \{s_1, \ldots, s_{p-1}, l\}$,
  P_2: all values as in P_1, except $x_{l\, l} = 0$, $x_{m\, m} = 1$, $x_{l\, s_1} = 1$, $x_{m\, s_1} = 0$.

  $P_1, P_2 \in \mathcal{F}_I$ since for both points, $x_{1\, 2} = 0$.

- If $1 \in \{l, m\}$, w.l.o.g., $l = 1$. Construct points as above, but choose $s_1, \ldots, s_{p-1} \in \{2, \ldots, n\} \backslash \{m\}$ with $s_1 \neq 2$.

$\square$

Proposition 3.3 *For any pair $i, k \in \{1, \ldots, n\}$, $i \neq k$, $\mathcal{F}_{II} := \{P \in \operatorname{conv}(\mathcal{X}) : X^P_{i\, k} = X^P_{k\, k}\}$ is a facet of $\operatorname{conv}(\mathcal{X})$.*

Next, we examine constraints (2).

Proposition 3.4 *For a fixed $k \in \{1, \ldots, n\}$, let $j \in \{1, \ldots, n\}$ such that $d_{j\, k} := \max_i d_{i\, k}$. Then, $\mathcal{F}_{III} := \{P \in \operatorname{conv}(\mathcal{X}) : r_k = d_{j\, k} X_{j\, k}\}$ is a facet of $\operatorname{conv}(\mathcal{X})$.*

Proof: W.l.o.g., $k = 1$ and $j = 2$. Assume that the equation

$$\sum_{i=1}^{n} \sum_{m=1}^{n} a_{i\, m} X_{i\, m} + \sum_{m=1}^{n} b_m r_m + c\, z = d \tag{19}$$

holds for all USApHCP-solutions $P \in \mathcal{F}_{III}$. By constructing appropriate points in $\mathcal{F}_{III}$, one can show that the following four statements hold:

1. $c = 0$ and $b_m = 0 \quad \forall\, m \neq 1$
2. $a_{i\,l} = a_{i\,m} =: a_i\; \forall\, l, m, i$ with $i \notin \{l, m\}$ and $(2,1) \notin \{(i,l),(i,m)\}$
3. $a_{l\,l} - a_l = a_{m\,m} - a_m =: a\; \forall\, l, m$
4. $a_{2\,1} = a_2 - b_1 d_{2\,1}$

Then equation (19) can be written as a linear combination of the given equations for $\mathcal{F}_{III}$:

$$\begin{aligned}
d \quad &= \sum_{i=1}^{n}\sum_{m=1}^{n} a_{i\,m} X_{i\,m} + \sum_{m=1}^{n} b_m r_m + c\,z \\
&= \sum_{i=1}^{n}\sum_{m=1}^{n} a_{i\,m} X_{i\,m} + b_1 r_1 \\
\Leftrightarrow \quad d - \sum_{i=1}^{n} a_i \quad &= \sum_{m=1}^{n} (a_{m\,m} - a_m) X_{m\,m} + (a_{2\,1} - a_2)\, X_{2\,1} + b_1 r_1 \\
\Leftrightarrow \; d - \sum_{i=1}^{n} a_i - a\,p &= (a_{2\,1} - a_2)\, X_{2\,1} + b_1 r_1 \\
&= b_1 (r_1 - d_{2\,1}\, X_{2\,1}).
\end{aligned}$$

□

However, if $d_{j\,k} < \max_i d_{i\,k}$, then $r_k = d_{j\,k} X_{j\,k}$ implies that $X_{i\,k} = 0$ for all i with $d_{i\,k} > d_{j\,k}$. Consequently, we cannot find n^2 affinely independent points satisfying $r_k \geq d_{j\,k} X_{j\,k}$ with equality, and thus, this inequality does not represent a facet.

Finally, using arguments similar to the ones in the above proofs, we can show that constraints (3) are facet-defining.

Proposition 3.5 *For all k in V, $\mathcal{F}_{IV} := \{P \in \mathrm{conv}(\mathcal{X}) : r_k = M X_{k\,k}\}$ is a facet of $\mathrm{conv}(\mathcal{X})$.*

4 Non-elementary facets

In this section, we present several facet classes of $\mathrm{conv}(\mathcal{X})$ which do not result from constraints (1)–(7).

Theorem 4.1 (Spoke-concentration facets) *Let $k \in \{1, \ldots, n\}$. The inequality*

$$(n-p)X_{k\,k} \geq \sum_{\substack{i=1\\ i\neq k}}^{n} X_{i\,k} \tag{20}$$

is facet-defining for $\mathrm{conv}(\mathcal{X})$*.*

Remark 4.2 There are two types of points that satisfy facet-inequality (20) with equality:

i. points with $X_{k\,k} = 0$ (and thus $X_{i\,k} = 0 \;\forall\; i$), and

ii. points with $X_{k\,k} = 1$. In this case, to fulfill the facet-inequality with equality, we are forced to assign *every* spoke to the hub in k.

Thus the facets of theorem 4.1 represent all points with "trivial" spoke allocation in the sense that all spokes are allocated to a single hub; we call them *spoke-concentration facets*.

Proof: Due to proposition 2.7, it suffices to show that (20) is valid and facet-defining for $\mathrm{conv}(\mathcal{X}_{\text{pUFL}})$.

Validity:

If $x_{k\,k} = 0$, inequality (20) trivially holds with equality. Now, assume $x_{k\,k} = 1$. Since the number of hubs is fixed to p, only $n-p$ of the remaining nodes are spokes and could thus be allocated to k. Thus,

$\sum_{i=1, i\neq k}^{n} x_{i\,k} \leq n - p = (n-p)\,x_{k\,k}$.

Facet-defining:

For ease of notation, assume w.l.o.g. that $k = 1$. Set

$$\mathcal{F} = \{P \in \mathrm{conv}(\mathcal{X}_{\text{pUFL}}) : (n-p)x_{1\,1} = \sum_{i=2}^{n} x_{i\,1}\}.$$

Assume that there is a further equation

$$\sum_{i=1}^{n}\sum_{k=1}^{n} a_{i\,k}x_{i\,k} = d \tag{21}$$

that is satisfied by all points in $\mathcal{F}$. It is possible to construct appropriate points in $\mathcal{F}$ to prove that the following statements hold:

1. $a_{i\,k} = a_{i\,l} =: a_i \; \forall\, i, k, l$ with $k, l \notin \{1, i\}$
2. $a_{k\,k} + a_k = a_{l\,l} + a_l =: a \; \forall\, k, l \geq 2$
3. $a_{i\,1} - a_i = a_{j\,1} - a_j =: b \; \forall\, i, j \geq 2$
4. $a_{1\,1} = a + a_1 - b\,(n - p)$

Then, we can reformulate (21) as follows:

$$\begin{aligned} d &= \sum_{i=1}^{n} \sum_{k=1}^{n} a_{i\,k} x_{i\,k} \\ &= \sum_{i=1}^{n} a_i \sum_{\substack{k=2 \\ k \neq i}}^{n} x_{i\,k} + \sum_{k=1}^{n} a_{k\,k} x_{k\,k} + \sum_{i=2}^{n} a_{i\,1} x_{i\,1} \quad \text{(by 1)}. \end{aligned} \tag{22}$$

From (22) and the fact that $\sum_k x_{i\,k} = 1$, we have

$$\begin{aligned} d - \sum_{i=1}^{n} a_i &= \sum_{k=1}^{n} (a_{k\,k} - a_k) x_{k\,k} + \sum_{i=2}^{n} (a_{i\,1} - a_i) x_{i\,1} \\ &= a \sum_{k=2}^{n} x_{k\,k} + (a_{1\,1} - a_1) x_{1\,1} + \sum_{i=2}^{n} (a_{i\,1} - a_i) x_{i\,1} \quad \text{(by 2)}, \end{aligned} \tag{23}$$

which, using the fact that $\sum_k x_{k\,k} = p$, can be rewritten as

$$\begin{aligned} d - \sum_{i=1}^{n} a_i - a\,p &= (a_{1\,1} - a_1 - a) x_{1\,1} + \sum_{i=2}^{n} (a_{i\,1} - a_i) x_{i\,1} \\ &= -b\,(n - p)\, x_{1\,1} + b \sum_{i=2}^{n} x_{i\,1} \quad \text{(by 3 and 4)} \\ &= b\big(rhs(20) - lhs(20)\big), \end{aligned}$$

where rhs (lhs) denotes the right-hand side (left-hand side) of the given equation. $\square$

Next we present another class of facet-defining inequalities in variables $X_{i\,k}$. Let $A \subset \{1, \ldots, n\}$ and a be some fixed node in A. The set A will be subdivided into two sets A^* and $\bar{A}^* = A \backslash A^* \ni a$. For the elements $k \in A^*$, summands $X_{b_k\,k}$ will be introduced, where $b_k \in \bar{A}$.

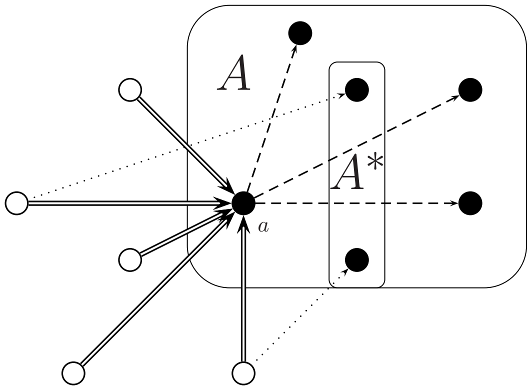

Figure 1: Graphical interpretation of focus-element facets

Theorem 4.3 (Focus-element facets) *Let* $A \subset \{1, \dots, n\}$ *with* $|A| = n - p$, $a \in A$, *and* $A^* \subset A \backslash \{a\}$ *with* $|A^*| \in \{0, \dots, n-p-2\}$, *and let* $\{b_k : k \in A^*\}$ *be pairwise different elements of* $\bar{A} := \{1, \dots, n\} \backslash A$. *Then,*

$$\sum_{i \in A} X_{i\,i} \geq \sum_{j \in \bar{A}} X_{j\,a} + \sum_{i \in A \backslash (\{a\} \cup A^*)} X_{a\,i} + \sum_{k \in A^*} X_{b_k\,k} \tag{24}$$

is a facet of $\operatorname{conv}(\mathcal{X})$.

Remark 4.4 Since the facets described in theorem 4.3 concentrate on a single element $a \in A$, we refer to them as *focus-element facets*. Figure 1 gives an interpretation of the facet class presented in theorem 4.3. Black nodes denote elements of A, white nodes are elements of $\bar{A}$. The arcs that contribute to $\sum_{i \in \bar{A}} X_{i\,a}$ are marked by doubled arrows. Dashed arrows denote arcs contributing to $\sum_{j \in A \backslash (A^* \cup \{a\})} X_{a\,j}$. The arcs that contribute to $\sum_{k \in A^*} X_{b_k\,k}$ are marked by dotted arrows.

Proof: It suffices to show validity and the facet-defining property of (24) for $\operatorname{conv}(\mathcal{X}_{\text{pUFL}})$. W.l.o.g., assume that $A = \{1, \dots, n-p\}$, $a = 1$ and $A^* = \{2, \dots, t\}$ with $t \leq n - p - 1$ ($A^* = \emptyset$ is possible). Then, inequality (24) can be written as

$$\sum_{k=1}^{n-p} x_{k\,k} \geq \sum_{i=n-p+1}^{n} x_{i\,1} + \sum_{k=t+1}^{n-p} x_{1\,k} + \sum_{k=2}^{t} x_{b_k\,k}. \tag{25}$$

Validity: If $x_{1\,1} = 0$, then, (25) is valid since $x_{b_k\,k} \leq x_{k\,k}$ and $x_{1\,k} \leq x_{k\,k}$ for all k. Now, assume $x_{1\,1} = 1$. Set $\sum_{k=2}^{n-p} x_{k\,k} = s$. Then, due to $\sum_{k=1}^{n} x_{k\,k} = p$ and

$x_{1\,1} = 1$, we have that $\sum_{k=n-p+1}^{n} x_{k\,k} = p - s - 1$, and consequently,

$$
\begin{aligned}
rhs(25) &= \sum_{k=2}^{t} x_{b_k\,k} + \sum_{i=n-p+1}^{n} x_{i\,1} \\
&= \sum_{k=2}^{t} (x_{b_k\,k} + x_{b_k\,1}) + \sum_{\substack{i=n-p+1 \\ i \notin \{b_2,\dots,b_t\}}}^{n} x_{i\,1} \\
&\leq \underbrace{\sum_{k=2}^{t} (1 - x_{b_k\,b_k})}_{\text{\# spokes in } \{b_2,\dots,b_t\}} + \underbrace{\sum_{\substack{i=n-p+1 \\ i \notin \{b_2,\dots,b_t\}}}^{n} (1 - x_{i\,i})}_{\substack{\text{\# spokes in} \\ \{n-p+1,\dots,n\} \setminus \{b_2,\dots,b_t\}}} \\
&= \underbrace{p}_{\substack{\text{\# nodes in} \\ \{n-p+1,\dots,n\}}} - \underbrace{(p - s - 1)}_{\substack{\text{\# hubs in} \\ \{n-p+1,\dots,n\}}} = s + 1 = lhs(25),
\end{aligned}
$$

where the inequality is valid since $x_{b_k\,k} + x_{b_k\,1} \leq 1$ and $x_{b_k\,k} = x_{b_k\,1} = 0$ if b_k is a hub.

Facet-defining:

Let $\mathcal{F} := \left\{ P \in \operatorname{conv}(\mathcal{X}_{\text{pUFL}}) : \sum_{k=1}^{n-p} x_{k\,k} = \sum_{i=n-p+1}^{n} x_{i\,1} + \sum_{k=t+1}^{n-p} x_{1\,k} + \sum_{k=2}^{t} x_{b_k\,k} \right\}$.

Assume that

$$\sum_{i=1}^{n} \sum_{k=1}^{n} a_{i\,k} x_{i\,k} = d \tag{26}$$

is a further equation that is satisfied by all points in $\mathcal{F}$. By constructing appropriate points in $\mathcal{F}$, we can show that

1. (a) $a_{1\,k} = a_{1\,l} =: a_1 \;\forall\, k, l \geq n - p + 1$
   (b) $a_{1\,k} = a_1 \;\forall\, k \in \{2, \dots, t\}$
   (c) $a_{i\,k} = a_{i\,l} =: a_i \;\forall\, i \in \{2, \dots, n - p\}$ and $k, l \in \{1, \dots, n\} \setminus \{i\}$
   (d) $a_{i\,k} = a_{i\,l} =: a_i \;\forall\, i \geq n - p + 1$ and $k, l \in \{t + 1, \dots, n\} \setminus \{i\}$
   (e) $a_{i\,k} = a_i \;\forall\, i \in \{n - p + 1, \dots, n\} \setminus \{b_2, \dots, b_t\}$ and $k \in \{2, \dots, t\}$

2. $a_{k\,k} - a_k = a_{l\,l} - a_l =: a \;\forall\, k, l \geq n - p + 1$

3. (a) $a_{i\,1} - a_i = a_{j\,1} - a_j =: -b \;\forall\, i, j \geq n - p + 1$
   (b) $a_{b_k\,k} - a_{b_k} = -b \;\forall\, k \in \{2, \dots, t\}$

(c) $a_{1\,k} - a_1 = -b \; \forall\, k \in \{t+1, \ldots, n-p\}$

4. $a_{k\,k} - a_k - a = b \; \forall\, k \leq n-p$

Then equation (26) can be reformulated as follows:

$$
\begin{aligned}
&\sum_{k=2}^{t} a_{1\,k} x_{1\,k} + \sum_{k=t+1}^{n-p} a_{1\,k} x_{1\,k} + \sum_{k=n-p+1}^{n} a_{1\,k} x_{1\,k} \\
&+ \sum_{i=2}^{n-p} \sum_{\substack{k=1 \\ k \neq i}}^{n} a_{i\,k} x_{i\,k} + \sum_{i=n-p+1}^{n} a_{i\,1} x_{i\,1} + \sum_{\substack{i=n-p+1 \\ i \notin \{b_2, \ldots, b_t\}}}^{n} \sum_{k=2}^{t} a_{i\,k} x_{i\,k} \\
&+ \sum_{k=2}^{t} a_{b_k\,k} x_{b_k\,k} + \sum_{i=n-p+1}^{n} \sum_{\substack{k=t+1 \\ k \neq i}}^{n} a_{i\,k} x_{i\,k} + \sum_{k=1}^{n} a_{k\,k} x_{k\,k} \quad = d.
\end{aligned}
\tag{27}
$$

Using statements 1a–1e, (27) can be rewritten as

$$
\begin{aligned}
&\sum_{k=t+1}^{n-p} (a_{1\,k} - a_1) x_{1\,k} + \sum_{i=n-p+1}^{n} (a_{i\,1} - a_i) x_{i\,1} \\
&+ \sum_{k=2}^{t} (a_{b_k\,k} - a_{b_k}) x_{b_k\,k} + \sum_{k=1}^{n} (a_{k\,k} - a_k) x_{k\,k} \quad = \quad d - \sum_{i=1}^{n} a_{i\,k}.
\end{aligned}
\tag{28}
$$

Applying statement 2, equation (28) becomes

$$
\begin{aligned}
&\sum_{k=t+1}^{n-p} (a_{1\,k} - a_1) x_{1\,k} + \sum_{i=n-p+1}^{n} (a_{i\,1} - a_i) x_{i\,1} \quad + \quad \sum_{k=2}^{t} (a_{b_k\,k} - a_{b_k}) x_{b_k\,k} \\
&\qquad + \sum_{k=1}^{n-p} (a_{k\,k} - a_k - a) x_{k\,k} \quad = \quad d - \sum_{i=1}^{n} a_{i\,k} - a\,p.
\end{aligned}
\tag{29}
$$

Statements 3a–3c transform (29) into

$$
\begin{aligned}
&- b \sum_{k=t+1}^{n-p} x_{1\,k} - b \sum_{i=n-p+1}^{n} x_{i\,1} - b \sum_{k=2}^{t} x_{b_k\,k} \\
&\qquad + \sum_{k=1}^{n-p} (a_{k\,k} - a_k - a) x_{k\,k} \quad = \quad d - \sum_{i=1}^{n} a_{i\,k} - a\,p
\end{aligned}
\tag{30}
$$

which, using statement 4, can be rewritten as

$$b\Big(lhs(25) - rhs(25)\Big) = d - \sum_{i=1}^{n} a_{i\,k} - a\,p.$$

$\square$

As stated in section 3, constraint $r_k \geq d_{i\,k} X_{i\,k}$ represents a facet if and only if $d_{i\,k} = \max_j d_{j\,k}$ (see proposition 3.4). Now we present new facet-defining inequalities that generalize this constraint.

Theorem 4.5 (Increasing-distances facets) *Let* $p \leq \lfloor \frac{n}{2} \rfloor$, $k \in \{1, \ldots, n\}$ *and* $i_k := \mathrm{argmax}_i\{d_{i\,k} : i = 1, \ldots, n\}$. *Let* $A = \{a_1, \ldots, a_t\} \subset \{1, \ldots, n\} \backslash \{k, i_k\}$ *with* $|A| = t \in \{1, \ldots, p-1\}$, *where* $d_{a_i\,k} \leq d_{a_{i+1},\,k} \quad \forall\ i = 1, \ldots, t-1$. *Consider inequality*

$$r_k \geq d_{a_1\,k} X_{a_1\,k} + \sum_{i=2}^{t} (d_{a_i\,k} - d_{a_{i-1},\,k}) X_{a_i\,k} + (d_{i_k\,k} - d_{a_t\,k}) X_{i_k\,k}. \tag{31}$$

(i) Inequality (31) is valid for $\mathrm{conv}(\mathcal{X})$.

(ii) If $t \leq p-2$*, then (31) represents a facet of* $\mathrm{conv}(\mathcal{X})$.

(iii) If $t = p-1$ *and* $p \leq \lfloor \frac{n-1}{2} \rfloor$ *or* $d_{i\,k} \leq d_{a_t\,k}\ \forall\, i \notin A$, *then (31) represents a facet of* $\mathrm{conv}(\mathcal{X})$.

Remark 4.6 Due to the ordering of the nodes in increasing distances to node k, we refer to the facets presented in theorem 4.5 as *increasing-distances facets*. The right hand side term of inequality (31) considers, starting from the node a_1 that is nearest to k, the increase of the radius r_k when stepwise allocating new spokes to hub k, each one being further away from k than the ones considered so far. Figure 2 marks the distances that are added up in the increasing-distances facets (doubled lines) for an example with $t = 3$.

Proof: W.l.o.g., assume that $k = 1$, $i_k = n$ and $A = \{a_1 = n-t,\, a_2 = n-t+1, \ldots, a_t = n-1\}$. Then, inequality (31) can be written as

$$r_1 \geq d_{n-t,\,1} X_{n-t,\,1} + \sum_{i=n-t+1}^{n} (d_{i\,1} - d_{i-1,\,1}) X_{i\,1}. \tag{32}$$

Validity: If $X_{i\,1} = 0$ for all $i \geq n-t+1$, then (32) reduces to constraint (2) and thus is valid.

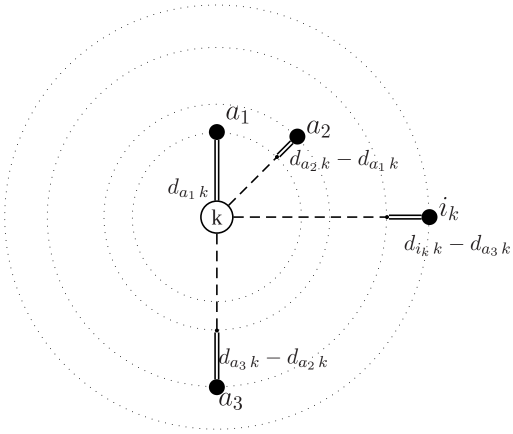

Figure 2: Graphical interpretation of increasing-distances facets

Now, consider the case that at least one of the nodes in $\{n-t+1,\ldots,n\}$ is allocated to 1, i.e.,
$\exists\, i_1,\ldots,i_r \in \{n-t+1,\ldots,n\}$ (with $r \geq 1$ and $i_1 < i_2 < \ldots < i_r$)
such that $X_{i_1\,1} = X_{i_2\,1} = \ldots = X_{i_r\,1} = 1$ and
$X_{i\,1} = 0 \quad \forall\, i \in \{n-t+1,\ldots,n\} \backslash \{i_1,\ldots,i_r\}$.

The right hand side of inequality (32) can be transformed as follows:

$$
\begin{aligned}
& d_{n-t,\,1} X_{n-t,\,1} + \sum_{i=n-t+1}^{n} (d_{i,\,1} - d_{i-1,\,1}) X_{i,\,1} \\
= \;& d_{n-t,\,1} X_{n-t,\,1} \\
& + (d_{i_1,\,1} - d_{i_1-1,\,1}) + \ldots + (d_{i_{r-1},\,1} - d_{i_{r-1}-1,\,1}) + (d_{i_r,\,1} - d_{i_r-1,\,1}) \\
= \;& d_{i_r,\,1} + (d_{i_{r-1},\,1} - d_{i_r-1,\,1}) + (d_{i_{r-2},\,1} - d_{i_{r-1}-1,\,1}) + \ldots \\
& + (d_{i_1,\,1} - d_{i_2-1,\,1}) - d_{i_1-1,\,1} + d_{n-t,\,1} X_{n-t,\,1}.
\end{aligned}
$$

Due to the assumption that $i_1 < i_2 < \ldots < i_r$, it holds that $i_1 \leq i_2 - 1, \ldots,$ $i_{r-1} \leq i_r - 1$, and thus by the general assumption that $d_{i,\,k} \leq d_{i+1,\,k} \quad \forall\, i = n-t,\ldots,n-1$ we obtain $d_{i_{r-1},\,1} - d_{i_r-1,\,1} \leq 0, \ldots, d_{i_1,\,1} - d_{i_2-1,\,1} \leq 0$.

Similarly, since $n - t \leq i_1 - 1$, we conclude $-d_{i_1-1,\,1} + d_{n-t,\,1} X_{n-t,\,1} \leq 0$, and thus arrive at $rhs(32) \leq d_{i_r\,1} \leq r_1$.

Facet-defining (case $t \leq p-2$): We will show that if all points lying on the face

$$
\mathcal{F} := \{P \in \operatorname{conv}(\mathcal{X}) : r_1 = d_{n-t,\,1} X_{n-t,\,1} + \sum_{i=n-t+1}^{n} (d_{i\,1} - d_{i-1\,1}) X_{i\,1}\}
$$

satisfy

$$\sum_{i=1}^{n}\sum_{k=1}^{n} a_{i\,k} X_{i\,k} + \sum_{k=1}^{n} b_k r_k + c\,z = d, \tag{33}$$

then (33) is a linear combination of (31).

First, one can construct points in $\mathcal{F}$ to show that the following statements hold:

1. (a) $b_k = 0\ \forall\, k \geq 2$
   (b) $c = 0$
2. (a) $a_{i\,k} = a_{i\,l} =: a_i\ \forall\, i \leq n-t-1$ and $k, l \in \{1, \ldots, n\} \backslash \{i\}$
   (b) $a_{i\,k} = a_{i\,l} =: a_i\ \forall\, i \geq n-t$ and $k, l \in \{2, \ldots, n\} \backslash \{i\}$
3. $a_{k\,k} - a_k = a_{l\,l} - a_l =: a\ \forall\, k, l \in \{1, \ldots, n\}$
4. (a) $a_{n-t,\,1} - a_{n-t} = -b_1\, d_{n-t,\,1}$
   (b) $a_{i\,1} - a_i = -b_1\,(d_{i\,1} - d_{i-1,\,1})\ \forall\, i \geq n-t+1$

Using claims 1a and 1b, equation (33) can be reformulated to obtain

$$\begin{aligned}
\sum_{i=1}^{n}\sum_{k=1}^{n} a_{i\,k} X_{i\,k} + b_1\, r_1 \quad &= \quad d \\
\Rightarrow \sum_{i=1}^{n-t-1} a_i \sum_{k=1}^{n} X_{i\,k} + \sum_{k=1}^{n-t-1} (a_{k\,k} - a_k) X_{k\,k} & \\
+ \sum_{i=n-t}^{n}\sum_{k=1}^{n} a_{i\,k} X_{i\,k} + b_1\, r_1 \quad &= \quad d \quad \text{(by 2a)} \\
\Rightarrow \sum_{i=1}^{n} a_i \sum_{k=1}^{n} X_{i\,k} + \sum_{k=1}^{n} (a_{k\,k} - a_k) X_{k\,k} & \\
+ \sum_{i=n-t}^{n} (a_{i\,1} - a_i) X_{i\,1} + b_1\, r_1 \quad &= \quad d \quad \text{(due to 2b)} \\
\Leftrightarrow \sum_{i=n-t}^{n} (a_{i\,1} - a_i) X_{i\,1} + \sum_{k=1}^{n} (a_{k\,k} - a_k) X_{k\,k} + b_1\, r_1 \quad &= \quad d - \sum_{i=1}^{n} a_i. \\
\Rightarrow \sum_{i=n-t}^{n} (a_{i\,1} - a_i) X_{i\,1} + b_1\, r_1 \quad &= \quad d - \sum_{i=1}^{n} a_i - a\,p \\
\Rightarrow b_1 (r_1 - rhs(32)) \quad &= \quad d - \sum_{i=1}^{n} a_i - a\,p \\
& \qquad \text{(by 4a and 4b).}
\end{aligned}$$

Facet-defining (case $t = p - 1$):

The proof of this case is analogous to that with $t \leq p - 2$ (except that additional assumptions $p \leq \lfloor \frac{n-1}{2} \rfloor$ or $d_{i\,k} \leq d_{a_t\,k}\ \forall\, i \notin A$ must be used). The same four claims can be shown to hold true, and then with a derivation similar to that above, one can prove that (31) is a facet in this case as well. $\square$

5 Separation

In this section we consider the respective separation problems for the different classes of facets we have obtained.

Separating spoke-concentration facets

Since there are only n inequalities in the class of spoke-concentration facets, a simple enumeration procedure can solve the corresponding separation problem efficiently.

Separating focus-element facets

In the case of the focus-element facets, the number of inequalities is exponentially large. To find violated focus-element facets, we will first restrict ourselves to searching among those facets with $A^* = \emptyset$. In the following, we give a polynomial time exact solution algorithm for this case.

Given a point $P^* = (\mathbf{X}^*, \mathbf{r}^*, z^*)$, we have to identify an inequality, if any, in the focus-element facet class that is violated by P^*, i.e., find a set A of $n - p$ nodes and a node $a \in A$ such that the term
$\sum\limits_{j \notin A} X^*_{j\,a} + \sum\limits_{i \in A \setminus \{a\}} X^*_{a\,i} - \sum\limits_{i \in A} X^*_{i\,i}$
is maximized; if the value of the term is strictly larger than zero, a violated inequality has been identified. It turns out that this separation problem can be solved in polynomial time using a greedy strategy.

For a fixed node a in $\{1, \ldots, n\}$, the separation problem reduces to finding $n - p - 1$ further nodes to obtain a set A so that $\sum\limits_{j \notin A} X^*_{j\,a} + \sum\limits_{i \in A \setminus \{a\}} X^*_{a\,i} - \sum\limits_{i \in A} X^*_{i\,i}$ is maximized. Assume w.l.o.g. that $a = n$. The set A is characterized using variables $A_1, \ldots, A_{n-1}$ by $A_i = 1$ if node i is chosen as further element of A, $A_i = 0$ else.

With the help of the $(n-1)$-dimensional vector $\mathcal{A} = (A_1, \ldots, A_{n-1})$, the separation problem can be formulated as follows:

$$\max_{\mathcal{A}} \quad \Big(\sum_{j=1}^{n-1} X^*_{j\,a}(1-A_j) + \sum_{i=1}^{n-1} X^*_{a\,i}\,A_i - \sum_{i=1}^{n-1} X^*_{i\,i}\,A_i - X^*_{a\,a}\Big)$$

$$s.t. \quad \sum_{i=1}^{n-1} A_i = n-p-1, \quad A_i \in \{0;1\} \quad \forall\, i.$$

The above objective function can be reformulated as

$$\max_{\mathcal{A}} \Big[\sum_{i=1}^{n-1}(X^*_{a\,i} - X^*_{i\,a} - X^*_{i\,i})\,A_i\Big] + \sum_{i=1}^{n-1} X^*_{i\,a} - X^*_{a\,a}.$$

The corresponding integer optimization problem

$$\max_{\mathcal{A}} \Big\{ \sum_{i=1}^{n-1}(X^*_{a\,i} - X^*_{i\,a} - X^*_{i\,i})\,A_i \;\Big|\; \sum_{i=1}^{n-1} A_i = n-p-1, A_i \in \{0;1\}\ \forall\, i \Big\}$$

can be solved by setting $A_i = 1$ for the $n-p-1$ values of i with highest coefficients $X^*_{a\,i} - X^*_{i\,a} - X^*_{i\,i}$. As this greedy strategy has to be applied for all $a \in \{1,\dots,n\}$, the complete separation algorithm has a complexity of $\mathcal{O}(n^2)$.

Suppose that the above procedure yields no violating inequality. Then, we check if P^* violates an inequality of the focus-element facet class with $A^* \neq \emptyset$.

For a given set A with element a, it is quite easy to determine $A^* \subset A$ and nodes $\{b_k : k \in A^*\} \subset \bar{A}$ such that the term

$$\sum_{j \notin A} X^*_{j\,a} + \sum_{i \in A \setminus (\{a\} \cup A^*)} X^*_{a\,i} + \sum_{k \in A^*} X^*_{b_k\,k} - \sum_{i \in A} X^*_{i\,i} \tag{34}$$

is maximized. We proceed as follows:

Start with $A^* = \emptyset$. For every node $k \in A$, determine a node $n_k \in \bar{A}$ with maximum value $X^*_{n_k\,k}$. Starting with a node $k \in A$ with the largest value of $X^*_{n_k\,k} - X^*_{a\,k}$, check if $X^*_{n_k\,k} > X^*_{a\,k}$; if so, substitute those summands in the above term, i.e., set $A^* := A^* \cup \{k\}$ and $b_k := n_k$. Stop if either $n-p-2$ nodes have been substituted or the value of (34) exceeds zero.

However, as soon as the set A is not given, it does not, in general, suffice to choose A which is optimal for the case $A^* = \emptyset$ and then apply the above procedure.

The general problem is that the sets A and A^* and the nodes b_k have to be determined simultaneously, but optimal choices of b_k can only be made once the sets A and A^* are known. A heuristic to deal with the separation problem for this general case is proposed in Baumgartner 2003.

Separating increasing-distances facets

Lastly, we consider the separation problem for the increasing-distances facets. For given k and P^*, we have to determine a number t and a set $A = \{a_1, \ldots, a_t\}$ of nodes such that the term

$$d_{a_1\,k} X^*_{a_1\,k} + \sum_{i=2}^{t} (d_{a_i\,k} - d_{a_{i-1},\,k}) X^*_{a_i\,k} + (d_{i_k\,k} - d_{a_t\,k}) X^*_{i_k\,k}$$

is maximized. If this value is larger than r^*_k, a violated inequality is found. Note that the above separation problem exhibits similarities to the knapsack problem. Since the "benefit" of each element in A depends on this particular element and its neighbors, the problem can be solved using a shortest path algorithm on a graph.

The construction of the desired graph is the following:

- Each node of the shortest path graph corresponds to a node of the hub network that can be chosen as an element of A.
- For each node i, include edges to all nodes j with $d_{j\,k} > d_{i\,k}$.
- Edges from $i \neq k$ to j have costs $-(d_{j\,k} - d_{i\,k}) X^*_{j\,k}$.
  The edge from k to i has the cost $-d_{i\,k} X^*_{i\,k}$.

For every path from k to i_k in the constructed graph, the nodes that have been traversed correspond to the choice of elements in A. The cost of a path is equal to $-rhs(31)$ for the particular choice of A. Hence the shortest path in the constructed graph yields a set A with maximum value $rhs(31)$. If this value is larger than $r_k = lhs(31)$, a violated increasing-distances inequality has been identified.

References

Alumur, S., and B. Y. Kara. 2007. "Network hub location problems: the state of the art." *European Journal of Operational Research*. To appear.

Baumgartner, S. (Jütte). 2003. "Polyhedral Analysis of Hub Center Problems." Diploma thesis, Technische Universität Kaiserslautern.

Campbell, J. F. 1994. "Integer programming formulations of discrete hub location problems." *European Journal of Operational Research* 72:387–405.

———. 1996. "Hub location and the p-hub median problem." *Operations Research* 44 (6): 923–935.

Campbell, J. F., A. T. Ernst, and M. Krishnamoorthy. 2002. “Hub location problems.” In *Facility Location: Applications and Theory*, edited by Z. Drezner and H. W. Hamacher, 373–407. Springer Verlag.

Cornuejols, G., G. L. Nemhauser, and L. A. Wolsey. 1990. “The uncapacitated facility location problem.” In *Discrete Location Theory*, edited by P. B. Mirchandani and R. L. Francis, 119–171. Wiley-Interscience.

Ernst, A., H. W. Hamacher, H. Jiang, M. Krishnamoorthy, and G. Woeginger. 2002a. “Formulations, complexity and heuristic algorithms for hub center problems.” Technical Report, CSIRO, Melbourne, Australia.

———. 2002b. “Heuristic algorithms for the uncapacitated hub center single allocation problem.” Technical Report, CSIRO, Melbourne, Australia.

———. 2002c. “Uncapacitated single and multiple allocation p-hub center problems.” Technical Report, CSIRO, Melbourne, Australia.

Ernst, A. T., and M. Krishnamoorthy. 1998. “Exact and heuristic algorithms for the uncapacitated multiple allocation p-hub median problem.” *European Journal of Operational Research* 104:100–112.

Hamacher, H. W., M. Labbé, S. Nickel, and T. Sonneborn. 2004. “Adapting polyhedral properties from facility to hub location problems.” *Discrete Applied Mathematics* 145 (1): 104–116.

Hamacher, H. W., and T. Meyer. 2006. “Hub cover and hub center problems.” Technical Report, Technische Universität Kaiserslautern. Report in Wirtschaftsmathematik Nr. 98/2006.

Kara, B. Y., and B. C. Tansel. 2000. “On the single assignment p-hub center problem.” *European Journal of Operational Research* 125:648–655.

O’Kelly, M., D. Skorin-Kapov, and J. Skorin-Kapov. 1995. “Lower bounds for the hub location problem.” *Management Science* 41 (4): 713–721.

Sonneborn, T. 2002. “The Uncapacitated Hub Location Problem: Polyhedral Analysis and Applications.” Dissertation, Technische Universität Kaiserslautern.

Wagner, B. 2004. “Model formulations for hub covering problems.” Technical Report. Schriften zur Quantitativen Betriebswirtschaftslehre Nr. 3.