A condition that a continuously deformed, simply connected body does not penetrate itself

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Abstract

In this article we give a sufficient condition that a simply connected flexible body does not penetrate itself, if it is subjected to a continuous deformation. It is shown that the deformation map is automatically injective, if it is just locally injective and injective on the boundary of the body. Thereby, it is very remarkable that no higher regularity assumption than continuity for the deformation map is required. The proof exclusively relies on homotopy methods and the Jordan-Brouwer separation theorem.

Keywords. Staticity, Continuum mechanics, Elasticity, Elastoplasticity, Nonlinear/large deformations, Injectivity of mappings, Homotopy lifting, Local and global homeomorphisms, Jordan-Brouwer separation theorem.

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1 Introduction

The basic problem in staticity consists of finding solutions for the deformation $\varphi : \bar{\Omega} \to \varphi(\bar{\Omega}) \subseteq \mathbb{R}^3$ of a flexible body, which is represented by the closure $\bar{\Omega} = \mathrm{cl}\,\Omega$ of a bounded domain $\Omega \subset \mathbb{R}^3$. Cauchy's theorem implies that it has to satisfy the nonlinear boundary value problem

$$\left\{ \begin{array}{rcll} -\operatorname{div}^{\varphi} \sigma^{\varphi}(\varphi(x)) & = & f^{\varphi}(\varphi(x)) & \text{in } \varphi(\Omega) \\ \sigma^{\varphi}(\varphi(x)) n^{\varphi}(\varphi(x)) & = & g^{\varphi}(\varphi(x)) & \text{on } \varphi(\partial_1\Omega) \\ \varphi(x) & = & \varphi_0(x) & \text{on } \partial_0\Omega \end{array} \right. . \qquad (1)$$

These equations for the unknown mapping φ are called the *equations of equilibrium* for the deformed configuration $\varphi(\bar{\Omega})$. Equivalently, the displacement $u = \varphi - \mathrm{id}$ of the body is frequently considered as the unknown function.

- $\sigma^{\varphi} : \bar{\Omega} \to \mathbb{R}^{3\times 3}_{sym}$ is the *Cauchy stress tensor* (living in the deformed configuration $\varphi(\bar{\Omega})$).
- $f^{\varphi} : \varphi(\Omega) \to \mathbb{R}^3$ are prescribed *volume forces* (acting in the deformed configuration $\varphi(\bar{\Omega})$).
- $g^{\varphi} : \varphi(\partial_1\Omega) \to \mathbb{R}^3$ are prescribed *boundary forces* (acting on the deformed boundary part $\varphi(\partial_1\Omega)$).
- $\varphi_0 : \partial_0\Omega \to \mathbb{R}^3$ is a *prescribed deformation* of the (undeformed) part $\partial_0\Omega$ of the boundary of Ω.
- $n^{\varphi} : \varphi(\partial\Omega) \to \mathbb{R}^3$ is the *outer unit normal field* (attached to the deformed boundary $\varphi(\partial\Omega)$)

Here it is assumed that the boundary of Ω is sufficiently smooth and that it decomposes into $\partial\Omega = \mathrm{cl}(\partial_0\Omega) \cup \mathrm{cl}(\partial_1\Omega)$, where the subsets $\partial_0\Omega$ and $\partial_1\Omega$ are relative open and disjoint. The tensor divergence $\mathrm{div}^\varphi\,\sigma^\varphi = \sum_{j=1}^3 \partial\sigma_{ij}^\varphi/\partial x_j^\varphi$ is taken with respect to the deformed coordinates $x^\varphi = \varphi(x)$.
For a derivation of system (1) and a proof of Cauchy's theorem, the reader is referred to Ciarlet [4].

In this context, the following question naturally arises:

> Do there exist some sufficient conditions that the body does *not* penetrate itself, i. e. that the mapping $\varphi : \bar{\Omega} \to \varphi(\bar{\Omega})$ is *injective*?

For *smooth* mappings φ, such sufficient conditions are available, cf. e. g. [1, 4, 12, 13, 16]. We exemplarily present two of them in the following two propositions, which are taken out of Ciarlet [4].

1.1 Proposition (Meisters/Olech/Weinstein) *Let $\hat{\Omega}$ be an open subset of $\mathbb{R}^n$ and $\Omega \subset \mathbb{R}^n$ be a bounded domain such that its (compact) closure $K = \bar{\Omega}$ satisfies $K \subset \hat{\Omega}$. Let further a mapping $\varphi : \hat{\Omega} \to \mathbb{R}^n$ be given that satisfies*

- $\varphi \in C^1(\hat{\Omega}, \mathbb{R}^n)$,
- $\det \nabla\varphi > 0$ *in* $\mathrm{int}\, K$, *except possibly on a finite subset,*
- $\det \nabla\varphi(x) > 0$ *for at least one point* x *of* ∂K,
- $\varphi|_{\partial K}$ *is injective.*

Then the mapping $\varphi : K = \bar{\Omega} \to \mathbb{R}^n$ is injective.

Proof: See Meisters/Olech [12] for the case that the boundary ∂K is connected. Weinstein [16] showed that this additional assumption is not required and may therefore be dropped. ■

1.2 Proposition (Ciarlet) *Let $\Omega \subset \mathbb{R}^n$ be a bounded open connected subset of $\mathbb{R}^n$ such that $\mathrm{int}(\mathrm{cl}\,\Omega) = \Omega$. Let further mappings ϕ, $\varphi : \bar{\Omega} \to \mathbb{R}^n$ be given that satisfy*

- $\phi \in C^0(\bar{\Omega}, \mathbb{R}^n)$ *is injective,*
- $\varphi \in C^0(\bar{\Omega}, \mathbb{R}^n) \cap C^1(\Omega, \mathbb{R}^n)$,
- $\det \nabla\varphi > 0$ *in* Ω,
- $\varphi = \phi$ *on* $\partial\Omega$.

Then the mapping $\varphi : \bar{\Omega} \to \varphi(\bar{\Omega})$ is a homeomorphism, the mapping $\varphi|_\Omega : \Omega \to \varphi(\Omega)$ is a C^1-diffeomorphism, and finally,

$$\varphi(\Omega) = \varphi_0(\Omega), \qquad \varphi(\bar{\Omega}) = \varphi_0(\bar{\Omega}).$$

In particular, the mapping $\varphi : \bar{\Omega} \to \mathbb{R}^n$ is injective.

Proof: See theorem 5.5-2 in Ciarlet [4]. ■

In the proofs of propositions 1.1 and 1.2, the presumption of continuously differentiability is needed in order to express the Brouwer mapping degree in terms of the derivative $\nabla\varphi$ and to use this representation to establish conditions on the derivative that assure that φ is one-to-one. Unfortunately, when dealing with weak solutions of system (1), sufficient smoothness of φ, i. e. regularity degree C^1 above, is usually not granted. We give two examples.

Example 1. If the system (1) is considered for St. Venant-Kirchhoff material, solutions φ are typically regular up to class $W^{2,p}(\Omega, \mathbb{R}^3)$, $p > 3$, see e. g. [4, 5, 6]. In this case, the continuous Sobolev embedding

$$W^{k,p}(\Omega, \cdot) \hookrightarrow C^{0,1}(\bar{\Omega}, \cdot) \qquad \text{if } \frac{n}{p} + 1 < k$$

for $k = 2$, $p > 3$ and $n = 3$ implies the Hölder continuity with exponent equal to one, i. e. Lipschitz continuity. Rademacher's theorem guarantees the differentiability of φ a. e. in Ω, but *not* the continuously differentiability. Here we are lucky, since extensions of proposition 1.2 to Lipschitz continuous functions $\varphi \in C^{0,1}(\bar{\Omega}, \mathbb{R}^n)$ can be found in Pourciau [13]. □

Example 2. When considering the problem of linearised elastoplasticity for linear kinematic (plus isotropic) hardening material, the solutions φ are typically of class $W^{1,2}(\Omega, \mathbb{R}^3)$, see e. g. [8, 9, 10, 11]. Extensions of proposition 1.2 to Sobolev functions $\varphi \in W^{1,p}(\Omega, \mathbb{R}^n)$, where $p > n$, are given in Ball [1], but the latter results would not be applicable in this case, since $k = 1$, $p = 2$ and $n = 3$. Even the plane case $n = 2$ could not be covered. □

The main theorem of this article is the following theorem 1.3, which generalises the result of Meisters and Olech. It considers deformation maps φ of a simply connected domain Ω, but does require *neither* higher regularity than C^0 for φ *nor* higher regularity for $\partial\Omega$.

1.3 Theorem. *Let $\hat{\Omega} \subseteq \mathbb{R}^n$ be open, Ω a bounded simply connected open subset of $\hat{\Omega}$ such that $\bar{\Omega} \subseteq \hat{\Omega}$. Let further a mapping $\varphi \in C^0(\hat{\Omega}, \mathbb{R}^n)$ be given with the following properties.*

- *The mapping $\varphi : \hat{\Omega} \to \varphi(\hat{\Omega})$ is a local homeomorphism.*
- *The restriction $\varphi|_{\partial\Omega}$ on the boundary $\partial\Omega$ is injective.*

Then the following assertions hold.

(a) The restriction $\varphi|_{\bar{\Omega}}$ is injective on the whole closure $\bar{\Omega}$.

(b) There exists an open neighbourhood $\tilde{\Omega}$ of $\bar{\Omega}$ such that $\bar{\Omega} \subseteq \tilde{\Omega} \subseteq \hat{\Omega}$ and that the restriction

$$_{\varphi(\tilde{\Omega})}|\varphi|_{\tilde{\Omega}} : \tilde{\Omega} \to \varphi(\tilde{\Omega}) \tag{2}$$

is a global homeomorphism.

Loosely speaking, a local homeomorphism of a simply connected domain Ω is automatically a global one, if it is injective just on the boundary $\partial\Omega$.

Its proof, which is delayed to section 3, exclusively relies on topological techniques and arguments, since its assertion is of intrinsic topological nature and has nothing to do with differentiability. (In particular, fractals such as Koch's snowflake in dimension $n = 2$ or Alexander's horned sphere in dimension $n = 3$ would be admissible examples for $\partial\Omega$, even if they don't occur in applications, where system (1) makes sense.) In order to apply homotopy methods, we cannot help assuming the simply connectedness of Ω.
The topological tools needed are more or less elementary, except for the famous Jordan-Brouwer separation theorem for $\mathbb{R}^n$, which is known as hard to prove. They are provided in section 2. We hope that our theorem will provide a useful tool in analytical continuum mechanics.

2 Some devices from topology

In this section, we summarise some facts from topology. The reader will find these theorems in every basic textbook about this subject. In what follows,

$$S^{n-1} = \partial B_1^n(0) = \{x \in \mathbb{R}^n : \|x\|^2 = \langle x, x\rangle = 1\}$$

denotes the unit sphere of $\mathbb{R}^n$. The following theorem is clearly a highlight of topology.

2.1 Theorem (Jordan-Brouwer separation theorem) *Every Jordan manifold in the Euklidian space $\mathbb{R}^n$, i. e. the image $\psi(S^{n-1})$ of a*

$$\textit{continuous injective mapping } \psi : S^{n-1} \to \mathbb{R}^n,$$

separates its complement $\psi(S^{n-1})^c = \mathbb{R}^n \setminus \psi(S^{n-1})$ into

- *exactly one bounded domain U_ψ, named the* interior of ψ,
- *exactly one unbounded domain U_ψ^∞, named the* exterior of ψ.

It is the common boundary of both domains, i. e.

$$\mathbb{R}^n = U_\psi \,\dot\cup\, \psi(S^{n-1}) \,\dot\cup\, U_\psi^\infty, \qquad \partial U_\psi = \psi(S^{n-1}) = \partial U_\psi^\infty.$$

Proof: See Brouwer [2]. The reader finds one of the shortest proofs for the special case $n = 2$ in Schmidt [14] or Carathéodory [3], where as well the simply connectedness of the interior is proven. ■

In order to formulate the following two propositions, we first need some definitions. Let therefore X, Y, Z denote some topological spaces and $p : Y \to Z$, $f : X \to Z$ some continuous mappings.

A topological subspace $U \subseteq Z$ is called *trivially covered by* p, iff there exists a discrete topological space F_U and a homeomorphism

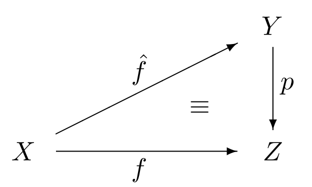
$$\Phi_U : p^{-1}(U) \to U \times F_U$$

that is compatible with the canonical projection $\pi_U : U \times F_U \to U$, i. e. with the property

$$p = \pi_U \circ \Phi_U \qquad \text{on } p^{-1}(U).$$

A mapping $p : Y \to Z$ is called *covering*, iff for each $z \in Z$ there is an open neighbourhood $U \subseteq Z$ of z that is trivially covered by p.

A *lifting of f in the covering* $p : Y \to Z$ is a continuous mapping $\hat f : X \to Y$ such that the diagram



commutes, i. e. such that

$$p \circ \hat f = f$$

holds. Now the following two propositions are standard results of topology. For proofs, see [7, 15] for example.

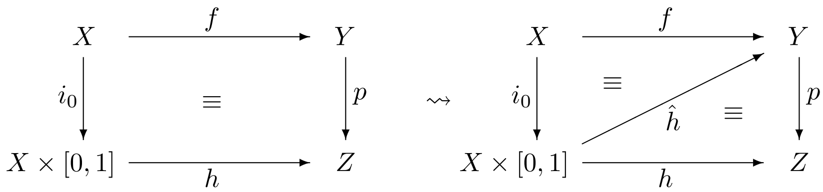
2.2 Proposition (Uniqueness of liftings) *Let X, Y, Z be topological spaces and $\hat{f}_1$, $\hat{f}_2 : X \to Y$ two liftings of $f : X \to Z$ in the covering $p : Y \to Z$. If X is connected and if $\hat{f}_1(x) = \hat{f}_2(x)$ for at least one $x \in X$, then there follows $\hat{f}_1 = \hat{f}_2$.*

2.3 Proposition (Homotopy-lifting-property) *Let X, Y, Z be topological spaces. Then every covering $p : Y \to Z$ has the* homotopy-lifting-property for X. *This means:*

If the mappings $f : X \to Y$, $h : X \times [0,1] \to Z$ are continuous and $i_0 : X \to X \times [0,1]$, $x \mapsto (x, 0)$ is a mapping satisfying $h \circ i_0 = p \circ f$, then there exists exactly one continuous mapping $\hat{h} : X \times [0,1] \to Y$ such that

$$p \circ \hat{h} = h, \qquad \hat{h} \circ i_0 = f.$$

Demonstratively, the situations are reflected in the following two commutative diagrams.



Especially, $\hat{h}$ is a lifting of the homotopy h.

3 Proof of the main theorem

Before starting, some remarks about notations. For any set X and a subset $\Xi \subseteq X$, Ξ^c is the complement $\Xi^c = X \setminus \Xi$ of Ξ in X. For any space X with metric d,

$$B_\epsilon(x) = \left\{\xi \in X : d(\xi, x) < \epsilon\right\}$$

denotes the open ball with centre $x \in X$ and radius $\epsilon > 0$. The distance of two subsets X_1, $X_2 \subseteq X$ is defined as

$$d(X_1, X_2) = \inf\left\{d(x_1, x_2) : x_1 \in X_1,\, x_2 \in X_2\right\}.$$

Especially for $X_1 = \{x_1\} \subseteq X$ and $X_2 \subseteq X$, the distance of x_1 and X_2 is

$$d(x_1, X_2) = \inf\left\{d(x_1, x_2) : x_2 \in X_2\right\}.$$

The space $\mathbb{R}^n$ is equipped with the topology, induced by the standard Euklidian metric d, which is given by

$$d^2(x_1, x_2) = \|x_1 - x_2\|^2 = \left(x_1^{(1)} - x_2^{(1)}\right)^2 + \ldots + \left(x_1^{(n)} - x_2^{(n)}\right)^2$$

for points $x_1 = (x_1^{(1)}, \ldots, x_1^{(n)})$, $x_2 = (x_2^{(1)}, \ldots, x_2^{(n)}) \in \mathbb{R}^n$. We start with a useful lemma.

3.1 Lemma. *Let $\hat{\Omega} \subseteq \mathbb{R}^n$ be open, Ω a bounded open subset of $\hat{\Omega}$ such that $\bar{\Omega} \subseteq \hat{\Omega}$ and $\partial\Omega$ is a Jordan manifold. Let further a mapping $\varphi \in C^0(\hat{\Omega}, \mathbb{R}^n)$ be given with the following properties.*

- *The mapping $\varphi : \hat{\Omega} \to \varphi(\hat{\Omega})$ is a local homeomorphism.*

- *The restriction $\varphi|_{\partial\Omega}$ on the boundary $\partial\Omega$ is injective.*

Then

$$\varphi(\partial\Omega) \subseteq \mathbb{R}^n \qquad \text{is a Jordan manifold,} \tag{3}$$

which separates $\mathbb{R}^n$ into an interior U and an exterior U^∞, and is common boundary of both domains

$$\mathbb{R}^n = U \,\dot\cup\, \varphi(\partial\Omega) \,\dot\cup\, U^\infty, \qquad \partial U = \partial U^\infty = \varphi(\partial\Omega). \tag{4}$$

In addition, there holds

$$\varphi(\partial\Omega) = \partial\varphi(\Omega), \qquad U = \varphi(\Omega), \qquad U^\infty = \varphi(\Omega)^c. \tag{5}$$

Proof: According to our premise, there exists a continuous injective mapping $\psi : S^{n-1} \to \mathbb{R}^n$ such that $\partial\Omega = \psi(S^{n-1})$. As the restriction $\varphi|_{\partial\Omega}$ is injective and continuous, the composition

$$\varphi \circ \psi : S^{n-1} \xrightarrow{\psi} \partial\Omega \xrightarrow{\varphi} \mathbb{R}^n$$

has the same properties. Thus (3) holds and theorem 2.1 implies the uniquely existence of such domains U and U^∞ with the properties (4). The image $\varphi(\Omega)$ is

- open (since local homeomorphisms are open mappings),
- bounded (as a subset of the compact set $\varphi(\bar\Omega)$),
- connected (as continuous image of a connected set).

We now show assertion (5). Note that the third identity therein easily follows from the preceding two. We proceed in four steps.

STEP 1. There holds

$$\partial\varphi(\Omega) \subseteq \varphi(\partial\Omega). \tag{6}$$

For that purpose, let $w \in \partial\varphi(\Omega)$. Thus, there exits a sequence $(x_n) \subseteq \Omega$ such that $\varphi(x_n) \xrightarrow{n} w$. As Ω is relatively compact, there exists $x \in \bar\Omega = \Omega \,\dot\cup\, \partial\Omega$ such that for a subsequence, which is denoted again by (x_n), there holds $x_n \xrightarrow{n} x$. By virtue of the continuity of φ, we conclude $\varphi(x_n) \xrightarrow{n} \varphi(x)$, especially $w = \varphi(x)$. We assume $x \in \Omega$. Then there exists an open neighbourhood $V \subseteq \Omega$ of x such that $\varphi|_V$ is injective. Then $\varphi(V)$ is an open neighbourhood of $\varphi(x)$. But then $w \in \varphi(V) \subseteq \varphi(\Omega)$, in contradiction to $w \in \mathrm{cl}(\varphi(\Omega)) \setminus \varphi(\Omega)$, where we note that $\varphi(\Omega)$ is open. Thus $x \in \partial\Omega$, and therefore $w = \varphi(x) \in \varphi(\partial\Omega)$.

STEP 2. There holds

$$U^\infty \cap \varphi(\Omega) = \emptyset. \tag{7}$$

Otherwise, there exists an $x \in \Omega$ such that $\varphi(x) \in U^\infty$. As U^∞ is unbounded, there exists a continuous function $\gamma : [0,1) \to U^\infty$ such that $\gamma(0) = \varphi(x)$ and $\lim_{t\to 1} \gamma(t) = \infty$. Then we have

$$\tau := \sup\big\{t \in [0,1) : \gamma(t) \in \varphi(\Omega)\big\} < 1,$$

since $\varphi(\Omega)$ is bounded. Let $(t_n) \subseteq [0,\tau)$ such that $t_n \xrightarrow{n} \tau$ and $x_n \in \Omega$ such that $\varphi(x_n) = \gamma(t_n)$. Since Ω is relatively compact, we may assume that the (x_n) converge in $\bar\Omega$. Because of the continuity of φ there holds $\gamma(t_n) \xrightarrow{n} \gamma(\tau)$. Consequently, there holds

(i) on the one hand $\gamma(\tau) \in \operatorname{cl} \varphi(\Omega)$,

(ii) on the other hand $\gamma\big((\tau, 1)\big) \subseteq \varphi(\Omega)^c$, thus $\gamma(\tau) \in \operatorname{cl}\left(\varphi(\Omega)^c\right)$.

We have derived the contradiction

$$\gamma(\tau) \in \operatorname{cl}(\varphi(\Omega)) \cap \operatorname{cl}(\varphi(\Omega)^c) \cap U^\infty = \partial\varphi(\Omega) \cap U^\infty \overset{(6)}{\subseteq} \varphi(\partial\Omega) \cap U^\infty \overset{(4)}{=} \emptyset.$$

STEP 3. There holds

$$\varphi(\Omega) \cap U \neq \emptyset, \tag{8}$$

because (4) and (7) imply $\varphi(\Omega) \subseteq \varphi(\partial\Omega) \,\dot\cup\, U$. But $\varphi(\Omega) \subseteq \varphi(\partial\Omega)$ is impossible, since $\varphi(\Omega)$ is open and $\varphi(\partial\Omega)$ is a one-dimensional manifold.

STEP 4. There holds

$$\varphi(\partial\Omega) \subseteq \partial\varphi(\Omega). \tag{9}$$

Otherwise, there exists $w \in \varphi(\partial\Omega)$ such that $w \notin \partial\varphi(\Omega)$. Then there exists an open ball $B = B_\epsilon(w)$, $0 < \epsilon \ll 1$, such that either $B \subseteq \varphi(\Omega)$ or $B \subseteq \varphi(\Omega)^c$.

(i) The case $B \subseteq \varphi(\Omega)$ implies the contradiction $\emptyset \neq B \cap U^\infty \subseteq \varphi(\Omega) \cap U^\infty = \emptyset$, since B is a neighbourhood of $w \in \varphi(\partial\Omega) = \partial U^\infty$.

(ii) In the case $B \subseteq \varphi(\Omega)^c$, we connect a point of $U \cap \varphi(\Omega)$ – which exists according to (8) – by a continuous curve $\gamma : [0, 1] \to U$ to w. Through a completely analogous supremum argument as in step 2, we find a point in $\partial\varphi(\Omega) \cap U$. And we have derived the contradiction

$$\emptyset \neq \partial\varphi(\Omega) \cap U \overset{(6)}{\subseteq} \varphi(\partial\Omega) \cap U \overset{(4)}{=} \emptyset.$$

From (6) and (9) we have $\partial\varphi(\Omega) = \varphi(\partial\Omega)$. There now follows easily $U = \varphi(\Omega)$ and $U^\infty = \varphi(\Omega)^c$ in (5). ■

Proof of theorem 1.3. Due to the presumptions on Ω, the boundary $\partial\Omega \subset \mathbb{R}^n$ is a Jordan manifold. As the restriction $\varphi|_{\partial\Omega}$ is injective, $\varphi(\partial\Omega) \subset \mathbb{R}^n$ is a Jordan manifold as well.

PART A. By virtue of lemma 3.1, the space $\mathbb{R}^n$ is disjointly separated into $\varphi(\partial\Omega)$, an interior U and an exterior U^∞ in a unique fashion such that the identities (4) and (5) hold. By virtue of (5), it is seen that the interior $U = \varphi(\Omega)$ is even *simply* connected, since Ω is simply connected.

PART B. We show that

$$\Phi := {}_{\varphi(\Omega)}|\varphi|_\Omega : \Omega \to \varphi(\Omega) \tag{10}$$

is a covering. To this end, let $u \in \varphi(\Omega)$ be arbitrarily given. Then $\Phi^{-1}(\{u\})$ is finite, since otherwise the elements would have a cluster point in the the compact set $\bar\Omega$, which would contradict the local injectivity of φ. Let

$$\Phi^{-1}\big(\{u\}\big) = \big\{x_1, \ldots, x_N\big\}$$

with pairwise distinct x_n. Now choose $0 < \delta \ll 1$ small enough such that the following conditions are satisfied.

- There holds $B_\delta(x_n) \subseteq \Omega$ for $n = 1, \ldots, N$.
- For each $n = 1, \ldots, N$, the mapping $\varphi\big|_{B_\delta(x_n)}$ is a local homeomorphism.

- The balls $B_\delta(x_1), \ldots, B_\delta(x_N)$ are pairwise disjoint.

Then let $0 < \epsilon \ll 1$ be such that $B_\epsilon(u) \subseteq U$ and

$$\Phi^{-1}\big(B_\epsilon(u)\big) \subseteq B_\delta(x_1) \,\dot\cup\, \ldots \,\dot\cup\, B_\delta(x_N). \tag{11}$$

Such a number ϵ exists, since otherwise, we could find for each sequence $(\epsilon_k)_{k\in\mathbb{N}}$ in $(0,\infty)$, that converges to zero, a sequence

$$(\xi_k)_{k\in\mathbb{N}} \subseteq \bigcap_{n=1}^{N} \big(\Omega \setminus B_\delta(x_n)\big)$$

such that $\Phi(\xi_k) \in B_{\epsilon_k}(u)$. We may assume – maybe by choosing an appropriate subsequence – that $\xi := \lim_{k\to\infty} \xi_k \in \bar\Omega$ exists, since $\bar\Omega$ is compact. There follows

$$\varphi(\xi) = \lim_{k\to\infty} \varphi(\xi_k) = \lim_{k\to\infty} \Phi(\xi_k) = u.$$

Since $\xi \in \partial\Omega$ is impossible due to (5), there follows $\xi \in \Phi^{-1}(\{u\})$, which cannot be true, as the limit must satisfy $\|\xi - x_n\| \geq \delta$.
Now, $B_\epsilon(u)$ is a trivially covered neighbourhood of u, since it is easy to see with the aid of (11) that the mapping

$$\Phi^{-1}\big(B_\epsilon(u)\big) \to B_\epsilon(u) \times \big\{1, \ldots, N\big\}, \qquad x \mapsto \big(\varphi(x),\, n\big) \quad \text{if } x \in B_\delta(x_n)$$

is a homeomorphism that is compatible with the canonical projection onto the first component,

$$B_\epsilon(u) \times \big\{1, \ldots, N\big\} \ni (\zeta,\, n) \mapsto \zeta \in B_\epsilon(u).$$

Part C. Now we show the injectivity of Φ from (10). If this would not be the case, there exist

$$x_0 \neq x_1 \in \Omega \qquad \text{s.t.} \qquad w := \Phi(x_0) = \Phi(x_1).$$

Since Ω is pathwise connected, we may connect point x_0 to point x_1 by a continuous curve

$$\gamma : [0,1] \to \Omega, \qquad \gamma(0) = x_0, \quad \gamma(1) = x_1.$$

Then, $\Phi \circ \gamma$ is a closed curve in $\varphi(\Omega)$ with the property

$$(\Phi \circ \gamma)(0) = w = (\Phi \circ \gamma)(1).$$

As U is simply connected due to part A, there exists a homotopy $h : [0,1] \times [0,1] \to \varphi(\Omega)$ with the property

$$h(t,\, 0) = (\Phi \circ \gamma)(t), \qquad h(t,\, 1) = w, \qquad h(0,\, s) = w = h(1,\, s). \tag{12}$$

Since Φ is a covering by virtue of part B, proposition 2.3 gives us a lifting $\hat h$: $[0,1] \times [0,1] \to \Omega$ such that

$$\Phi\big(\hat h(t,\, s)\big) = h(t,\, s), \qquad \hat h(t,\, 0) = \gamma(t). \tag{13}$$

(i) On the one hand, this implies

$$\Phi\big(\hat h(0,\, s)\big) \overset{(13)}{=} h(0,\, s) \overset{(12)}{=} w \overset{(12)}{=} h(1,\, s) \overset{(13)}{=} \Phi\big(\hat h(1,\, s)\big).$$

As Φ is injective on a neighbourhood of x_0 resp. x_1, there follows

$$\hat h(0,\, s) = x_0, \qquad \hat h(1,\, s) = x_1.$$

(ii) On the other hand, we have

$$\Phi\big(\hat{h}(t,\,1)\big) \overset{(13)}{=} h(t,\,1) \overset{(12)}{=} w.$$

Therefore, $\hat{h}(\cdot,\,1)$ is a path from x_0 to x_1 satisfying $\Phi\big(\hat{h}(\cdot,\,1)\big) \equiv w$.

But this contradicts the local injectivity of Φ. Therefore, Φ has to be globally injective.

PART D. We now finally prove the assertions (a) and (b) of the theorem.

(a) The injectivity of $\varphi|_{\bar{\Omega}}$ follows directly from the injectivity of the functions $\varphi|_{\partial\Omega}$, which is granted by premise, and $\Phi = \varphi|_{\Omega}$, see part C, together with (5) from lemma 3.1.

(b) Let $\epsilon := d(\bar{\Omega},\,\partial\hat{\Omega})$. Then $\epsilon > 0$, since $\bar{\Omega}$ is compact, $\partial\hat{\Omega}$ is closed and both sets $\bar{\Omega}$, $\partial\hat{\Omega}$ are disjoint. The set $\Omega' := \{x \in \mathbb{R}^2 : d(x,\bar{\Omega}) < \epsilon\}$ is open and relatively compact, satisfying $\bar{\Omega}' \subset \hat{\Omega}$. If we set $K = \bar{\Omega}'$ (compact), $C = \bar{\Omega}$ (closed) in the following lemma 3.2 for injective extension, the latter gives us a set $U = \tilde{\Omega}$ (open) such that $\bar{\Omega} \subseteq \tilde{\Omega} \subseteq \bar{\Omega}' \subset \hat{\Omega}$ and (2) holds.

The proof is finished. ■

3.2 Lemma (Injective extension) *Let $\varphi : K \to T$ be a continuous, locally injective mapping of a compact metric space $(K,\,d)$ into a topological space T and $C \subseteq K$ a closed set. If $\varphi|_C$ is injective, there exists an open set U such that $C \subseteq U \subseteq K$ and $\varphi|_U$ is injective.*

Proof: We assume the converse. Then we consider

$$U_n := \bigcup_{x \in C} B_{1/n}(x) = \big\{x \in K : d(x, C) < 1/n\big\}$$

for $n \in \mathbb{N}$. These are K-open neighbourhoods of C, thus by assumption, there exist for each $n \in \mathbb{N}$ points

$$x_1^{(n)},\, x_2^{(n)} \in U_n \qquad \text{s.t.} \qquad x_1^{(n)} \neq x_2^{(n)} \quad \text{but} \quad \varphi(x_1^{(n)}) = \varphi(x_2^{(n)}). \tag{14}$$

By choosing an appropriate subsequence – by virtue of the compactness of K – and relabeling if necessary, we may assume

$$x_1^{(n)} \xrightarrow{\;n\;} x_1, \qquad x_2^{(n)} \xrightarrow{\;n\;} x_2 \tag{15}$$

with certain elements $x_1,\, x_2 \in K$. Because of

$$d(x_1^{(n)}, C) \xrightarrow{\;n\;} 0, \qquad d(x_2^{(n)}, C) \xrightarrow{\;n\;} 0$$

there must hold $x_1,\, x_2 \in C$, since C is closed. Due to the continuity of φ, we conclude

$$\varphi(x_1) \xleftarrow{\;n\;} \varphi(x_1^{(n)}) = \varphi(x_2^{(n)}) \xrightarrow{\;n\;} \varphi(x_2),$$

thus $\varphi(x_1) = \varphi(x_2)$ and $x := x_1 = x_2$, since the restriction $\varphi|_C$ is injective according to our premise. Due to the local injectivity of φ, we find a K-open neighbourhood V of x, on which φ is injective. But if we choose a $k \in \mathbb{N}$ large enough, we may achieve that $x_1^{(k)},\, x_2^{(k)} \in V$ because of (15). And, by virtue of (14) for $n = k$, we have a contradiction to the injectivity of $\varphi|_V$. ■

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