Fast Wavelet Transform by Biorthogonal Locally Supported Radial Basis Functions on Fixed Spherical Grids

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Contents

Introduction

In recent years a considerable amount of research has been devoted to the approximation of functions on the surface of the Earth, from discrete data. These functions can be a representation or a model of environmental phenomena such as magnetic fields, gravitational fields, ocean circulations, melting polar ice caps, storm or hurricane formation and dynamics, etc. The data are acquired by terrain stations spread all over the world or by artificial satellites such as CHAMP, GRACE, and GOCE etc.

Traditionally, the approximation of functions on the sphere (as a model of the Earth) has been done by Fourier theory in form of orthogonal expansions. To be more concrete, the approximation of functions on the sphere was based on the spherical harmonics, which perform a closed orthonormal system of functions in the space of all square integrable functions on the sphere. Because of the orthogonality of the spherical harmonics, they are ideally localized in the frequency domain. Moreover, for those applications with polynomial structure, the spherical harmonics provide a good tool for global approximation. In spite of these attractive properties, the spherical harmonics have some disadvantages. For example, they don't show space localization at all, and a local change of measurements affects all Fourier coefficients. They also show huge oscillation for larger degrees. In addition, the spherical harmonics are not the appropriate tool for approximation of problems with local dense data on the sphere. The opposite extreme to the spherical harmonics, in the sense of ideal frequency localization, is the Dirac functional. Because the Dirac functional contains all frequencies in equal share, it does not show any frequency localization, but its space localization is ideal.

Radial basis functions (RBF) provide a compromise between space and frequency localization. They are not new even on the sphere. Indeed, it should be pointed out that combined polynomial (spherical harmonic) and radial basis function approximations have often been studied especially in the context of conditionally (strictly) positive definite functions.

One advantage of using radial basis functions methods for the approximation of functions is that although the radial basis functions are defined as multivariate functions, they are actually one-dimensional functions depending on the norm of the argument. Because the norm of the argument is a geometric quantity, it is independent of the choice of the coordinates. Therefore, the radial basis functions methods are independent of the choice of the coordinates and consequently, these methods have no artificial boundaries or singularities intrinsic in other methods of the approximation of functions. Another advantage of the radial basis functions is that the localization in frequency/space domain can be adapted to the data situation. Unfortunately, because of the uncertainty principle (cf., [31], [37], [72]), the space and the frequency domain cannot be made arbitrarily small at the same time, i.e., the reduction of the frequency localization leads to an enhancement of the space localization, and vice versa. Thus, the space and the frequency domain localization should be compromised. This can be achieved by the so-called multiscale approximation based on the radial basis functions (see, e.g., [27], [34], [40], [41], [68]). These methods use the radial basis functions at different scales to construct different stages of the space/frequency localization, thus, a trade-off between the space and the frequency localization can be found. This idea led to the wavelet theory during the last decades.

Various concepts of spherical wavelets have been developed by the Geomathematics Group, Technical University of Kaiserslautern ([40], [47], [48], [34], [35]). As in classical wavelet theory, the mother wavelets are based on the spherical radial basis functions, where moving the "center" of the spherical radial basis functions around the sphere, i.e., rotation can be interpreted as counterpart to translation. For the dilation, different approaches have been established: In a first one (cf., [47], [48]), starting with a family of scaling functions corresponding to a family of singular integrals, the dilation is understood as the scaling parameter of the scaling functions. In a second one (as proposed, e.g., [40]), starting with a continuous version of the Legendre transform which is monotonically decreasing on $[0, \infty)$ and continuous at 0 with value 1, say, $\gamma_0 : [0, \infty) \to \mathbb{R}$, the dilation is defined as the usual dilation of this function, i.e., $\gamma_j(x) = \gamma_0(2^{-j}x)$.

It should be mentioned that, there exist other approaches for designing spherical wavelets. For example, by Dahlke, Dahmen, Schmitt and Weinreich ([19], [103]) a $\mathcal{C}^{(1)}$-wavelet basis is constructed in form of a tensor product of two types of refinable functions: the periodized exponential splines and the boundary corrected polynomial B-splines. Lyche and Schumaker ([60], [61]) also have done similar work by using L-splines. Other wavelets on the sphere based on tensor products of Euclidian wavelets involving trigonometric wavelets were proposed by Potts and Tasche [75]. There are several publications based on uniform approximation of the sphere by regular polyhedra. For example, starting with a triangulation of the sphere, the spherical Haar-type wavelets were constructed on triangles (see, e.g., [13], [74], [81], [93], and [99]). A theoretical continuous wavelet transform on the sphere is presented by Dahlke and Maass [20] and Holschneider [52] and Antoine and Vandergheynst [4] and Antoine, Demanet, Jaques and Vandergheynst [3]. A discretization of [3] and [4] is realized by Bogdavova, Vandergheynst, Antoine, Jaques and Morvidone [12]. Recently, Roşca [82] has proposed a wavelet basis on the sphere by means of radial projection.

In this work we have developed new biorthogonal systems of zonal functions (spherical radial functions) which are locally supported. In more detail, we start with an isolatitude spherical gird, e.g., $X_N = \{\xi_{ij} | i \in \mathcal{I},\ j \in \mathcal{J}\}$, where j is corresponding to the latitudes. Then by using an arbitrary family of locally supported kernels, we construct a dual family of locally supported kernels such that the primal and the dual kernels are biorthogonal, i.e., if $\{K_j\}_{j\in\mathcal{J}}$ and $\{\tilde{K}_j\}_{j\in\mathcal{J}}$ are the primal and dual kernels, respectively, then the following conditions should be valid:

$$\Big(K_j(\xi_{ij}\,,\,\cdot)\,,\,\tilde{K}_{j'}(\xi_{i'j'}\,,\,\cdot)\Big)_{\mathcal{L}^2(\Omega)} = \delta_{ii'}\delta_{jj'}, \qquad i,i' \in \mathcal{I},\ j,j' \in \mathcal{J}.$$

This system of biorthogonal scaling functions serves us as scaling functions at the finest level of a multiresolution analysis for a finite dimensional space spanned by the primal or the dual scaling functions at the finest scale.

In addition, the biorthogonal system of zonal functions enables us to construct a new kind of spherical wavelets (see [43]) which are inherently locally supported. Once more, one advantage of these wavelets is that their construction is based on a biorthogonal system of zonal functions, which gives us almost all advantages of an orthogonal approach. Another advantage is that the wavelets and the scaling function are based on zonal kernel functions so that this approach is well–suited for the solution and the regularization of the rotation–invariant pseudodifferential equations. Finally, because the scaling equations are established by only a few coefficients, we end up with a fast and economical wavelet transform which is completely similar to the algorithms known from tensor product approaches of Euclidean wavelet theory.

Outline

The background material which is needed during this thesis is summarized in Chapter 1. The basic notation and definitions and some well-known results useful for an easy understanding of the whole work are briefly recapitulated. Moreover, we introduce some differential operators and special functions like Legendre polynomials, Gegenbauer polynomials, spherical harmonics. Furthermore, we turn to Sobolev spaces and pseudodifferential operators. Finally, spherical singular integrals and their properties are presented.

In Chapter 2 multiscale approximation based on locally supported zonal functions is presented. Spherical radial basis functions on the sphere are introduced. These functions are a powerful tool for the approximation of functions on the sphere. Moreover, necessary and sufficient conditions for the (strictly) positive definiteness of zonal functions on Euclidian spaces and on the sphere are listed. The smoothed Haar functions and their properties are recapitulated. In addition, by using the Fourier transform of the Haar function, an explicit formula for their Legendre transform is developed. We conclude the chapter with the definition of Wendland functions on the sphere. The Wendland functions on the sphere as a new strictly positive definite class of locally supported zonal kernels are developed. Wendland functions are understood as scaling functions in a multiscale procedure. At the end of this chapter, two kinds of wavelets are presented, namely wavelets based on the spherical up functions and spherical difference wavelets.

Chapter 3 deals with the arranging of large (structured) point-sets on the sphere. Some spherical grids like the regular grid, the quadratic grid, the Kurihara grid and the block grid are investigated. All these spherical grids are employed to construct a system of biorthogonal locally supported kernels.

The construction of a system of biorthogonal locally supported zonal kernels on the sphere is the aim of Chapter 4. For a given family of primal locally supported kernels on an isolatitude grid, we construct a family of dual locally supported kernels such that the primal and the dual kernels form a system of biorthogonal locally supported kernels. The method is built in such a way that any dual locally supported kernel is a linear combination of the intermediate kernels with unknowns coefficients. Numerically, the coefficients within the linear combination can be found by solving a moderate linear system of equations (about 15-25 equations) for each dual kernel.

Chapter 5 is devoted to a new type of spherical wavelets based on the biorthogonal locally supported zonal kernels. These wavelets can be constructed on a hierarchical grid like the Kurihara grid, the block grid (cf. [43]) or the so-called HEALPix (see, e.g., [50]). In this thesis, however, we only focus on the wavelets constructed on the block grid. Three kinds of wavelets associated with three directions (east-west, north-south and diagonal) are discussed. A multiresolution analysis for the finite dimensional space spanned by the scaling functions at the scale zero is developed. The chapter ends with examples of the fast wavelet transforms for two trial functions.

Finally, in Chapter 6 we summarize the results obtained throughout this thesis and sketch an outlook for further work and challenges.

Chapter 1

Preliminaries

In this chapter, we briefly introduce the notation required in this work. We review the basic facts which are necessary to motivate and state the main parts of this thesis. For notation and more details, the reader is referred to [34] and the literature therein.

1.1 Notation

We denote the sets of positive integers, integers, and real numbers by $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{R}$, respectively. The set of all non-negative integer numbers is denoted by $\mathbb{N}_0$. Let $\mathbb{R}^3$ denote the three-dimensional Euclidean space. We use $x, y, z, \ldots$ to represent the elements of $\mathbb{R}^3$.

Let $\varepsilon^1, \varepsilon^2, \varepsilon^3$ be the canonical orthonormal basis in $\mathbb{R}^3$:

$$\varepsilon^1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \; \varepsilon^2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \; \varepsilon^3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

If $x, y \in \mathbb{R}^3$ with $x = (x_1,\, x_2,\, x_3)^T$ and $y = (y_1,\, y_2,\, y_3)^T$, then $x \cdot y$ represents the Euclidean inner product, and $x \wedge y$ denotes the vector product. Moreover, the Euclidean norm of x is denoted by $|x|$. In detail,

$$x \cdot y = x^T \cdot y = x_1 y_1 + x_2 y_2 + x_3 y_3,$$

$$x \wedge y = (x_2y_3 - x_3y_2, x_3y_1 - x_1y_3, x_1y_2 - x_2y_1)^T,$$

$$|x| = \sqrt{x \cdot x} = \sqrt{x_1^2 + x_2^2 + x_3^2}.$$

The unit sphere around the origin in $\mathbb{R}^3$ is denoted by Ω. We use the Greek alphabet $\xi, \eta, \dots$ to specify the points of the unit sphere Ω in $\mathbb{R}^3$. Any point $\xi \in \Omega$ can be parameterized by spherical coordinates as follows:

$$\xi = \begin{pmatrix} \sin\vartheta\cos\varphi \\ \sin\vartheta\sin\varphi \\ \cos\vartheta \end{pmatrix}, \quad \varphi \in [0, 2\pi), \ \ \vartheta \in [0, \pi]. \tag{1.1}$$

Using the standard notation $t = \cos\vartheta$, $\vartheta \in [0, \pi]$, we find

$$\xi = \sqrt{1-t^2}(\cos\varphi\varepsilon^1 + \sin\varphi\varepsilon^2) + t\varepsilon^3, \quad t \in [-1, 1], \ \ \varphi \in [0, 2\pi). \tag{1.2}$$

For later use, we introduce local polar coordinates by using the unit vectors $\varepsilon^r, \varepsilon^\varphi$ and $\varepsilon^t = -\varepsilon^\vartheta$. Usually, this system also refers to a local moving triad on the unit sphere Ω. As is well-known, the relation between the local polar coordinates and the standard spherical coordinates on the unit sphere Ω is explicitly given by

$$\varepsilon^r(\varphi, t) = \begin{pmatrix} \sqrt{1-t^2}\cos\varphi \\ \sqrt{1-t^2}\sin\varphi \\ t \end{pmatrix}, \tag{1.3}$$

$$\varepsilon^\varphi(\varphi, t) = \begin{pmatrix} -\sin\varphi \\ \cos\varphi \\ 0 \end{pmatrix}, \tag{1.4}$$

$$\varepsilon^t(\varphi, t) = \begin{pmatrix} -t\cos\varphi \\ -t\sin\varphi \\ \sqrt{1-t^2} \end{pmatrix}, \tag{1.5}$$

where $\varphi \in [0, 2\pi)$ and $t \in [-1, 1]$ with $t = \cos\vartheta$ according to (1.1).

In the following, we briefly introduce some differential operators. A more detailed description of these operators can be found in [34] and [38]. The gradient operator is given by

$$\nabla_x = \left(\frac{\partial}{\partial x_1}, \quad \frac{\partial}{\partial x_2}, \quad \frac{\partial}{\partial x_3} \right)^T, \tag{1.6}$$

and its representation in local polar coordinates $x = r\xi$, $\xi \in \Omega$, is known to be

$$\nabla_x = \frac{\partial}{\partial r}\xi + \frac{1}{r}\nabla^*, \tag{1.7}$$

where ∇^* denotes the surface gradient of the Ω. Its representation in local polar coordinates $x = r\xi$, $\xi \in \Omega$, is given by

$$\nabla^*_\xi = \varepsilon^\varphi \frac{1}{\sqrt{1-t^2}} \frac{\partial}{\partial \varphi} + \varepsilon^t \sqrt{1-t^2} \frac{\partial}{\partial t}. \tag{1.8}$$

Another important differential operator is the Laplace operator Δ in $\mathbb{R}^3$ defined in Cartesian coordinates by

$$\Delta_x = \left(\frac{\partial}{\partial x_1} \right)^2 + \left(\frac{\partial}{\partial x_2} \right)^2 + \left(\frac{\partial}{\partial x_3} \right)^2, \tag{1.9}$$

It can be written in terms of polar coordinates as follows:

$$\Delta_x = \left(\frac{\partial}{\partial r} \right)^2 + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\Delta^*_\xi, \tag{1.10}$$

where Δ^*_ξ is the Beltrami operator on the unit sphere Ω at the point ξ:

$$\Delta^*_\xi = \frac{\partial}{\partial t}(1-t^2)\frac{\partial}{\partial t} + \frac{1}{1-t^2}\left(\frac{\partial}{\partial \varphi} \right)^2. \tag{1.11}$$

Next, we introduce the operator L^*_ξ, i.e., the surface curl gradient on the unit sphere Ω at ξ, as follows:

$$L^*_\xi = -\varepsilon^\varphi \sqrt{1-t^2} \frac{\partial}{\partial t} + \varepsilon^t \frac{1}{\sqrt{1-t^2}} \frac{\partial}{\partial \varphi}. \tag{1.12}$$

Symbol	**Differential operator**
∇_x	Gradient operator at x
$\Delta_x = \nabla_x \cdot \nabla_x$	Laplace operator at x
∇^*_ξ	Surface gradient on Ω at ξ
$L^*_\xi = \xi \wedge \nabla^*_\xi$	Surface curl gradient on Ω at ξ
$\Delta^*_\xi = \nabla^*_\xi \cdot \nabla^*_\xi = L^*_\xi \cdot L^*_\xi$	Beltrami operator on Ω at ξ
$\nabla^* \cdot$	Surface divergence on Ω at ξ
$L^* \cdot$	Surface curl on Ω at ξ

Table 1.1: Differential operators

In Table 1.1, we summarized those differential operators (in coordinate free representation) which are needed in this thesis.

We use capital letters F, G,... for scalar functions and $\mathcal{C}^{(k)}(\Omega), 0 \leq k \leq \infty$, for the space of scaler functions $F : \Omega \to \mathbb{R}$ possessing k times continuous derivatives on the unit sphere Ω. In particular, $\mathcal{C}(\Omega)(= \mathcal{C}^{(0)}(\Omega))$ is the set of all scalar-valued continuous functions on Ω. As is well-known, $\mathcal{C}(\Omega)$ is a normed space equipped with the norm

$$\|F\|_{\mathcal{C}(\Omega)} = \sup_{\xi \in \Omega} | F(\xi) | . \tag{1.13}$$

The space $\mathcal{L}^p(\Omega)$ is the set of all measurable functions $F : \Omega \to \mathbb{R}$ such that the quantity

$$\|F\|_{\mathcal{L}^p(\Omega)} = \left(\int_\Omega | F(\xi) |^p \; d\omega(\xi) \right)^{1/p}, \qquad 1 \leq p < \infty, \tag{1.14}$$

is finite. The space $\mathcal{L}^2(\Omega)$ is a Hilbert space with respect to the inner product given by

$$(F, G)_{\mathcal{L}^2(\Omega)} = \int_\Omega F(\xi) G(\xi) \, d\omega(\xi). \tag{1.15}$$

$\mathcal{L}^2(\Omega)$ is the completion of $\mathcal{C}(\Omega)$ with respect to $\| \cdot \|_{\mathcal{L}^2(\Omega)}$, that is

$$\mathcal{L}^2(\Omega) = \overline{\mathcal{C}(\Omega)}^{\| \cdot \|_{\mathcal{L}^2(\Omega)}}. \tag{1.16}$$

For convenience, $\mathcal{X}(\Omega)$ denotes either the space $\mathcal{C}(\Omega)$ or $\mathcal{L}^2(\Omega)$ with the corresponding inner product.

1.2 Polynomials

In this section we state some important properties of the Legendre polynomials and the Gegenbauer (ultraspherical) polynomials. Both of them can be considered as special cases of *the Jacobi polynomials* (see, e.g., [57] or [100]). In addition, we introduce the scalar spherical harmonics. Some of the most important results of the scalar spherical harmonics are mentioned. More details about spherical harmonics can be found, e.g., in [21], [62], and [100].

We begin our considerations with the Legendre polynomials.

1.2.1 Legendre Polynomials

As mentioned before, Legendre polynomials are special cases of the Jacobi polynomials $P_n^{(\alpha,\beta)}$ by letting $\alpha = \beta = 0$ (For more details see, e.g., [62]). The Legendre polynomial can be uniquely determined by the following properties:

- $P_n,\ n \in \mathbb{N}_0$, is a polynomial of degree n on the interval $[-1, 1]$,
- $\int_{-1}^{1} P_n(t)P_m(t)\ \mathrm{d}t = 0$ for $n \neq m,\ n, m \in \mathbb{N}_0$,
- $P_n(1) = 1$ for all $n \in \mathbb{N}_0$.

The second property guarantees that the Legendre polynomials are orthogonal. Note that they are not orthonormal, since we have

$$\int_{-1}^{1} P_n(t)P_m(t)\ \mathrm{d}x = \frac{2}{2n+1}\,\delta_{nm}, \qquad n, m \in \mathbb{N}_0,$$

where δ_{nm} is *the Kronecker symbol* defined by

$$\delta_{nm} = \begin{cases} 1, & for \quad n = m \\ 0, & for \quad n \neq m \end{cases}. \tag{1.17}$$

We denote the orthonormal Legendre polynomials by P_n^*, i.e.,

$$P_n^*(t) = \sqrt{\frac{2n+1}{2}} P_n(t), \quad n \in \mathbb{N}_0.$$

In other words, the system $\{P_n^*\}_{n=0,1,\ldots}$ forms an orthonormal system with respect to the inner product in $\mathcal{L}^2[-1,1]$,

$$(P_n^*, P_m^*)_{\mathcal{L}^2[-1,1]} = \int_{-1}^{1} P_n^*(x) P_m^*(x) \, \mathrm{d}x = \delta_{nm}.$$

There is another way to define the Legendre polynomials by using the longitude independent part of the Beltrami operator (1.11). The longitude independent part of the Beltrami operator which is referred to the Legendre operator L_t, is defined by:

$$L_t = \frac{\mathrm{d}}{\mathrm{d}t}(1-t^2)\frac{\mathrm{d}}{\mathrm{d}t}. \tag{1.18}$$

The Legendre polynomials $P_n : [-1,1] \to \mathbb{R}$ of degree n, $n \in N_0$ are uniquely defined as the infinitely often differentiable eigenfunctions of the Legendre operator L_t corresponding to the eigenvalues $-n(n+1)$, that is

$$(L_t + n(n+1))P_n(t) = 0, \qquad t \in [-1,1],$$

which satisfy $P_n(1) = 1$.

It can be shown that the Legendre polynomials satisfy the *Rodriguez formula*:

$$P_n(t) = \frac{1}{2^n n!} \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^n (t^2-1)^n. \tag{1.19}$$

Based on this formula, it can be seen that the following relations are valid for $n \geq 1$, $t \in [-1,1]$ (cf., e.g., [34]):

$$(2n+1) \int_{1}^{t} P_n(x) \, \mathrm{d}x = P_{n+1}(t) - P_{n-1}(t) \tag{1.20}$$

$$P'_{n+1}(t) - tP'_n(t) = (n+1)P_n(t), \tag{1.21}$$

$$(t^2-1)P'_n(t) = ntP_n(t) - nP_{n-1}(t), \tag{1.22}$$

$$(n+1)P_{n+1}(t) + nP_{n-1}(t) - (2n+1)tP_n(t) = 0. \tag{1.23}$$

Recall that by using the recursive formula (1.23) and the two first Legendre polynomials, $P_0(t) = 1, P_1(t) = t$, it is also possible to introduce the Legendre polynomials.

From the Rodriguez formula (1.19) we obtain

$$P_n(t) = \frac{1}{2^n} \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-1)^k (2n-2k)!}{k!(n-k)!(n-2k)!} t^{n-2k} \tag{1.24}$$

$$= \frac{1}{2^n} \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n}{k} \binom{2n-2k}{n} t^{n-2k}, \tag{1.25}$$

where $\lfloor r \rfloor$ is *the floor function* that gives the largest integer less than or equal to r. In (1.24) by setting $t = \cos\vartheta$ we obtain

$$|P_n(\cos\vartheta)| \leq \frac{1}{2^n} \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n}{k} \binom{2n-2k}{n} = P_n(\cos 0) = 1, \tag{1.26}$$

therefore, for $n \in \mathbb{N}_0$, it follows that

$$|P_n(t)| \leq 1, \quad -1 \leq t \leq 1. \tag{1.27}$$

From (1.24), it is clear that the Legendre polynomial $P_n(t)$ is an even function if n is even, and is an odd function if n is odd, i.e.,

$$P_n(-t) = (-1)^n P_n(t), \qquad n \in \mathbb{N}_0. \tag{1.28}$$

The set $\{P_n\}_{n\in\mathbb{N}_0}$ is complete in $\mathcal{L}^2[-1,1]$ with respect to $\|\cdot\|_{\mathcal{L}^2[-1,1]}$ and closed in the space of all continuous functions on the interval $[-1,1]$, $\mathcal{C}[-1,1]$, with respect to $\|\cdot\|_{\mathcal{C}[-1,1]}$. By virtue of the completeness and the orthogonality properties of the Legendre polynomials, the Legendre polynomials can be interpreted as a basis in $\mathcal{L}^2[-1,1]$ as follows:

If F is any function of class $\mathcal{L}^2[-1,1]$, then the Legendre series of the function F is given by

$$F = \sum_{n=0}^{\infty} \frac{2n+1}{2} F^\wedge(n) P_n, \tag{1.29}$$

where $F^\wedge(n)$ is the Legendre coefficient of function F given by

$$F^\wedge(n) = (F, P_n)_{\mathcal{L}^2[-1,1]} = 2\pi \int_{-1}^{1} F(t) P_n(t)\, \mathrm{d}t, \quad n \in \mathbb{N}_0. \tag{1.30}$$

Note that, the equality in the relation (1.29) is in the $\mathcal{L}^2[-1,1]$-sense, i.e.,

$$\lim_{N\to\infty} \left\| F - \sum_{n=0}^{N} \frac{2n+1}{2} F^\wedge(n) P_n \right\|_{\mathcal{L}^2[-1,1]} = 0. \tag{1.31}$$

It should be mentioned that the equality in the relation (1.29) is not guaranteed for all $F \in \mathcal{L}^p[-1,1]$ with $p \in [1,\infty]\backslash(\frac{4}{3},4)$. For more details, see, e.g., [34].

An another method to characterize the Legendre polynomials is the following generating series expansion:

$$\frac{1}{\sqrt{1-2rt+r^2}} = \sum_{n=0}^{\infty} r^n P_n(t), \tag{1.32}$$

where $r \in (-1,1)$ and $t \in [-1,1]$.

The following relation, is known as *Abel-Poisson kernel*, is obtained by the differentiation of (1.32) with respect to r, $|r| < 1,\ |t| \le 1$

$$\frac{1}{4\pi} \frac{1-r^2}{(1-2rt+r^2)^{\frac{3}{2}}} = \sum_{n=0}^{\infty} \frac{2n+1}{4\pi} r^n P_n(t). \tag{1.33}$$

For later use, we introduce a class of functions in $\mathcal{L}^2[-1,1]$ which can be derived from the Legendre polynomials.

Definition 1.2.1 (Associated Legendre Functions)
Let $n, m \in \mathbb{N}_0$ and $m \le n$. The function

$$P_{n,m}(t) = (1-t^2)^{m/2} \frac{1}{2^n n!} \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{n+m} (t^2-1)^n = (1-t^2)^{m/2} \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{m} P_n(t), \tag{1.34}$$

is called *the associated Legendre function of degree n and order m.*

The associated Legendre functions with negative orders are defined as follows:

$$P_{n,-m}(t) = (-1)^m \frac{(n+m)!}{(n-m)!} P_{n,m}(t), \quad n, m \in \mathbb{N}_0. \tag{1.35}$$

As we mentioned before, for a fixed m the associated Legendre functions are orthogonal but not orthonormal in $\mathcal{L}^2[-1,1]$, i.e.,

$$(P_{n,m}, P_{l,m})_{\mathcal{L}^2[-1,1]} = \frac{2}{2n+1} \frac{(n+m)!}{(n-m)!} \delta_{n,l}, \quad m \le n, l. \tag{1.36}$$

We denote the orthonormal associated Legendre functions by $P^*_{n,m}$, i.e.,

$$P^*_{n,m}(t) = \sqrt{\frac{2n+1}{2} \frac{(n-m)!}{(n+m)!}} P_{n,m}(t), \quad m \le n, \tag{1.37}$$

thus, we have

$$(P^*_{n,m}, P^*_{l,m})_{\mathcal{L}^2[-1,1]} = \delta_{nl}, \quad m \le n, l.$$

Finally, we mention the relation between the associated Legendre functions and the Gegenbauer polynomials (see Section 1.2.2)

$$P_{n,m}(t) = (-1)^m \frac{(2m)!}{2^m m!} \left(1 - t^2\right)^{\frac{m}{2}} C^{m+\frac{1}{2}}_{n-m}(t), \quad m \le n, \tag{1.38}$$

for $n, m \in \mathbb{N}_0$. More details about the associated Legendre functions and especially the definition of $P_{\nu,\mu}$ with unrestricted μ and ν can be found in, e.g., [62].

1.2.2 Gegenbauer Polynomials

Gegenbauer polynomials $C^\lambda_n(t)$ are special cases of the Jacobi polynomials $P^{(\alpha,\beta)}_n$ (see, e.g., [2], [100]) by letting $\alpha = \beta = \lambda - \frac{1}{2}$ under the normalization

$$C^\lambda_n(1) = \binom{n+2\lambda-1}{n}, \tag{1.39}$$

where $\lambda > -\frac{1}{2}$. The Gegenbauer polynomials are also called ultraspherical polynomials.

The Gegenbauer polynomials can be directly introduced by the expansion

$$\frac{1}{(1-2rt+r^2)^\lambda} = \sum_{n=0}^{\infty} C_n^\lambda(t) r^n. \tag{1.40}$$

If $\lambda = 0$ then $C_n^\lambda(t) \equiv 0$. The Gegenbauer polynomials are orthogonal with respect to the weight function $(1-t^2)^{\lambda-\frac{1}{2}}$ when $\lambda > -\frac{1}{2}$. The orthogonality relation for $\lambda > -\frac{1}{2}$ reads as follows:

$$\int_{-1}^{1} C_n^\lambda(t) C_m^\lambda(t)(1-t^2)^{\lambda-\frac{1}{2}}\ \mathrm{d}t = \frac{\sqrt{\pi}\,\Gamma(\lambda+\frac{1}{2})}{(\lambda+n)\Gamma(\lambda)} C_n^\lambda(1)\ \delta_{nm}, \quad \lambda \neq 0. \tag{1.41}$$

The tree-term recurrence relation for the Gegenbauer polynomial is given by

$$nC_n^\lambda(t) - 2(n+\lambda-1)\, t\, C_{n-1}^\lambda(t) + (n+2\lambda-2)C_{n-2}^\lambda(t) = 0, \tag{1.42}$$

for $n \geq 2$ and $C_0^\lambda(t) = 1$, $C_1^\lambda(t) = 2\lambda t$.

It is well known that the Gegenbauer polynomials are a solution of the following differential equation

$$(1-t^2)\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} - (2\lambda+1)t\frac{\mathrm{d}y}{\mathrm{d}t} + n(n+2\lambda)y = 0. \tag{1.43}$$

If $t \in [-1,1]$ and $\lambda > 0$, then

$$|C_n^\lambda(t)| \leq C_n^\lambda(1). \tag{1.44}$$

If $T_n(t)$ denotes the Chebyshev polynomial of the first kind (see, e.g., [62],[1]) then we have

$$\lim_{\lambda \to 0} \frac{C_n^\lambda(t)}{C_n^\lambda(1)} = T_n(t). \tag{1.45}$$

Formulas for the integral and the derivative of the Gegenbauer polynomials are

$$2(n+\lambda)\int C_n^\lambda(t)\ \mathrm{d}t = C_{n+1}^\lambda(t) - C_{n-1}^\lambda(t), \tag{1.46}$$

$$\int C_n^\lambda(t)\ \mathrm{d}t = \frac{1}{2(\lambda-1)} C_{n+1}^{\lambda-1}(t), \tag{1.47}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} C_n^\lambda(t) = 2\lambda C_{n-1}^{\lambda+1}(t). \tag{1.48}$$

Finally, the relation between the Gegenbauer polynomials and the associated Legendre functions (see Section 1.2.1) is known to be

$$C_n^{\lambda}(t) = \binom{n+2\lambda-1}{n}\Gamma(\lambda+\frac{1}{2})\left(\frac{t^2-1}{4}\right)^{\frac{1}{4}-\frac{\lambda}{2}} P^{\frac{1}{2}-\lambda}_{n+\lambda-\frac{1}{2}}(t). \tag{1.49}$$

Especially, if $\lambda = \frac{1}{2}$, then we have

$$C_n^{\frac{1}{2}}(t) = P_n(t), \qquad t \in \mathbb{R}. \tag{1.50}$$

1.2.3 Spherical Harmonics

This section is devoted to the definition of spherical harmonics and the recapitulation of their main properties. There are various ways to introduce spherical harmonics. The standard way to introduce spherical harmonics is the restriction of homogeneous harmonic polynomial to the sphere Ω (cf. [34]). For more details and further references to the literature, see [34], [70], [71] and the references therein.

Let H_n be a homogeneous polynomial of degree n in $\mathbb{R}^3$, i.e.,

$$H_n(\lambda x) = \lambda^n H_n(x), \quad x \in \mathbb{R}^3, \quad \lambda \in \mathbb{R},$$

Furthermore, let H_n be harmonic, that is H_n satisfies the Laplace differential equation

$$\Delta_x H_n(x) = 0, \quad x \in \mathbb{R}^3.$$

Then the set of all homogeneous harmonic polynomial of degree n is denoted by $Harm_n(\mathbb{R}^3)$. The dimension of $Harm_n(\mathbb{R}^3)$ is known to be $2n+1$.

Definition 1.2.2 (Spherical Harmonics)
Let H_n be in $Harm_n(\mathbb{R}^3)$. The restriction $Y_n = H_n|_\Omega$ is called a spherical harmonic of degree n. The space of all spherical harmonics of degree n is denoted by $Harm_n(\Omega)$.

Suppose that $H_n \in Harm_n(\mathbb{R}^3)$ and $H_m \in Harm_m(\mathbb{R}^3)$. By Green theorem we have

$$\begin{aligned} 0 &= \int_{\|x\|\le 1} \left(H_n(x)\Delta_x H_m(x) - H_m(x)\Delta_x H_n(x)\right)\, \mathrm{d}x \\ &= \int_\Omega \left(H_n(\xi)\frac{\partial}{\partial r}H_m(r\xi) - H_m(\xi)\frac{\partial}{\partial r}H_n(r\xi)\right)\Big|_{r=1} \mathrm{d}\omega(\xi) \\ &= (n-m)\int_\Omega Y_n(\xi)Y_m(\xi)\, \mathrm{d}\omega(\xi). \end{aligned}$$

Thus, it is clear that spherical harmonics of different degrees are orthogonal in the sense of the $\mathcal{L}^2(\Omega)-$inner product. The dimension of $Harm_n(\Omega)$ is equal to the dimension of $Harm_n(\mathbb{R}^3)$, i.e., $dim(Harm_n(\Omega)) = 2n+1$. Note that any polynomial in $\mathbb{R}^3$ with degree $\le n,\ n \in \mathbb{N}_0$ restricted to the sphere Ω can be decomposed into a direct sum of the spherical harmonics of degrees $i,\ i = 0,\dots,n$. If we denote the space of all spherical harmonics of degree $\le n,\ n \in \mathbb{N}_0$, by $Harm_{0,\dots,n}(\Omega)$, then because of the $\mathcal{L}^2(\Omega)-$orthogonality, we have

$$Harm_{0,\dots,n}(\Omega) = \bigoplus_{j=0}^{n} Harm_j(\Omega), \tag{1.51}$$

and $dim(Harm_{0,\dots,n}(\Omega)) = (n+1)^2$. By observing this result, the Hilbert space $\mathcal{L}^2(\Omega)$ can be decomposed into a direct sum of the spaces of spherical harmonics. This fact will be discussed in detail later in the *fundamental theorem of spherical harmonic expansions.*

An explicit formula for an orthonormal basis of the space $Harm_n(\Omega)$ with respect to $(\cdot,\cdot)_{\mathcal{L}^2(\Omega)}$ is presented in the following definition.

Definition 1.2.3 (Orthonormal Spherical Harmonics)
Let $n \in \mathbb{N}_0$ and $k \in \mathbb{Z}$ with $-n \le k \le n$. The function

$$Y_{n,k}(\xi) = \sqrt{\frac{1}{\pi(1+\delta_{0k})}} \begin{cases} P^*_{n,k}(\cos\vartheta)\cos(k\varphi), & k \ge 0 \\ \\ P^*_{n,|k|}(\cos\vartheta)\sin(|k|\varphi), & k < 0 \end{cases} \tag{1.52}$$

is called the normalized spherical harmonic of degree n and order k, where $\vartheta \in [0,\pi]$ and $\varphi \in [0,2\pi)$ are the spherical coordinates of $\xi \in \Omega$ and P_n^{k*} are the normalized associated Legendre functions defined in (1.37).

From now on, we denote an orthonormal basis of the space $Harm_n(\Omega)$ with respect to $(\cdot,\cdot)_{\mathcal{L}^2(\Omega)}$ by $\{Y_{n,k}\}_{k=-n,\ldots,n}$.

Obviously, $Harm_n(\Omega)$ is the eigenspace of the Beltrami operator Δ^*_ξ as defined in (1.11) corresponding to the eigenvalues $(\Delta^*)^\wedge(n) = -n(n+1)$, i.e.,

$$(\Delta^*_\xi - (\Delta^*)^\wedge(n))Y_n(\xi) = 0, \qquad \xi \in \Omega, \qquad Y_n \in Harm_n(\Omega).$$

The sequence $\{(\Delta^*)^\wedge(n)\}_{n=0,1,\ldots}$ is called the spherical symbol of the Beltrami operator.

Next, we state *the addition theorem for spherical harmonics*. This theorem is a bridge between the Legendre polynomials and the spherical harmonics.

Theorem 1.2.4 (Addition Theorem)
Let $\{Y_{n,k}\}_{k=-n,\ldots,n}$, $n \in \mathbb{N}_0$, be a system of orthonormal spherical harmonics of degree n with respect to $(\cdot,\cdot)_{\mathcal{L}^2(\Omega)}$, and let P_n be the Legendre polynomial of degree n. Then, for all $\xi,\eta \in \Omega$,

$$\sum_{k=-n}^{n} Y_{n,k}(\xi)Y_{n,k}(\eta) = \frac{2n+1}{4\pi}P_n(\xi\cdot\eta). \tag{1.53}$$

An immediate consequence is

$$\sum_{k=-n}^{n} (Y_{n,k}(\xi))^2 = \frac{2n+1}{4\pi}. \tag{1.54}$$

In the following *the Funk-Hecke Formula* is presented. This formula yields a connection between the integral over the surface of the sphere Ω and the integral over the interval $[-1,1]$.

Theorem 1.2.5 (Funk-Hecke formula)
Let $G \in \mathcal{L}^1[-1,1]$ and let P_n be the Legendre polynomial. Then, for all $\xi,\eta \in \Omega$ and $n \in \mathbb{N}_0$,

$$\int_\Omega G(\xi\cdot\zeta)P_n(\eta\cdot\zeta)\,\mathrm{d}\omega(\zeta) = G^\wedge(n)P_n(\xi\cdot\eta), \tag{1.55}$$

where $G^\wedge(n)$ is the Legendre coefficient of G, i.e.,

$$G^\wedge(n) = (G,P_n)_{\mathcal{L}^2[-1,1]}.$$

As a useful result of Funk-Hecke formula, we mention

$$\int_{\Omega} G(\xi\cdot\eta)Y_n(\eta)\,\mathrm{d}\omega(\eta) = G^{\wedge}(n)Y_n(\xi), \qquad Y_n \in Harm_n(\Omega), \tag{1.56}$$

for all $\xi \in \Omega$. The relation (1.56) leads us the concept of spherical convolutions, see, e.g., [11] or [34].

Definition 1.2.6 (Spherical Convolution)
Assume that $F \in \mathcal{L}^2(\Omega)$ and $G \in \mathcal{L}^2[-1,1]$. Then the function

$$(F * G)(\xi) = \int_{\Omega} G(\xi\cdot\eta)F(\eta)\,\mathrm{d}\omega(\eta), \qquad \xi \in \Omega, \tag{1.57}$$

is called the spherical convolution of F and G.

We can rewrite (1.56) by using the spherical convolution as follows:

$$(G * Y_n)(\xi) = G^{\wedge}(n)Y_n(\xi), \quad G \in \mathcal{L}^2[-1,1],\ Y_n \in Harm_n(\Omega),\ \xi \in \Omega. \tag{1.58}$$

For later use, we define the spherical convolution of a function with itself as follows:

Definition 1.2.7 (Iterated Spherical Convolution)
Assume that $G \in \mathcal{L}^2[-1,1]$. The spherical convolution of function G with itself is denoted by $G^{(2)}$ and defined by

$$G^{(2)}(\xi\cdot\zeta) = (G * G)(\xi\cdot\zeta) = \int_{\Omega} G(\xi\cdot\eta)G(\eta\cdot\zeta)\,\mathrm{d}\omega(\eta), \qquad \xi,\ \zeta \in \Omega, \tag{1.59}$$

and the k^{th} iterated spherical convolution of G is denoted and defined by

$$G^{(k)}(\xi\cdot\zeta) = (G * G^{(k-1)})(\xi\cdot\zeta) = \int_{\Omega} G(\xi\cdot\eta)G^{(k-1)}(\eta\cdot\zeta)\,\mathrm{d}\omega(\eta), \qquad \xi,\ \zeta \in \Omega, \tag{1.60}$$

for $k = 3, 4, \ldots$.

Clearly, it follows that

$$(G^{(k)})^{\wedge}(n) = (G^{\wedge}(n))^k, \quad n = 0, 1, \ldots,\ k = 2, 3, \ldots. \tag{1.61}$$

Remark 1.2.8
It should be noted that the concept of a spherical convolution (1.57) is fundamental for the theory of *singular integrals* that will be introduced in Section 1.4, and the singular integrals form the essential concept of *spherical wavelets* (cf., e.g., [40], [46], [47], [48], [108], [35], and [92]).

The system $\{Y_{n,k}\}_{k=-n,\dots,n}$, $n \in \mathbb{N}_0$, is closed in $(\mathcal{C}(\Omega), \|\cdot\|_{\mathcal{C}(\Omega)})$. That means for each $F \in \mathcal{C}(\Omega)$ and any $\varepsilon > 0$ there exist numbers N_ε and $d_{n,k}$ such that

$$\left\| F - \sum_{n=0}^{N_\varepsilon} \sum_{k=-n}^{n} d_{n,k} Y_{n,k} \right\|_{\mathcal{C}(\Omega)} \leq \varepsilon. \tag{1.62}$$

The system $\{Y_{n,k}\}_{k=-n,\dots,n}$, $n \in \mathbb{N}_0$ is closed in $(\mathcal{L}^2(\Omega), \|\cdot\|_{\mathcal{L}^2(\Omega)})$. Especially

$$F = \sum_{n=0}^{\infty} \sum_{k=-n}^{n} F_{n,k} Y_{n,k}, \tag{1.63}$$

for all $F \in \mathcal{L}^2(\Omega)$ with respect to $\|\cdot\|_{\mathcal{L}^2(\Omega)}$, where $F_{n,k}$ is called *(spherical) Fourier coefficients* of F

$$F_{n,k} = \int_\Omega F(\eta) Y_{n,k}(\eta)\, d\omega(\eta). \tag{1.64}$$

The relation (1.63) is called the orthogonal expansion (or the Fourier expansion in terms of spherical harmonics) of F.

Most of the aforementioned results are summarized in the *fundamental theorem of spherical harmonic expansions* (cf. [34]):

Theorem 1.2.9 (Fundamental Theorem of Spherical Harmonic Expansions)
The following seven statements are equivalent:

1. $\{Y_{n,k}\}_{k=-n,\dots,n}$, $n \in \mathbb{N}_0$ *is closed in* $\mathcal{L}^2(\Omega)$ *(closure property).*

2. *The orthogonal expansion of any* $F \in \mathcal{L}^2(\Omega)$ *converges in the* $\mathcal{L}^2(\Omega)$*−norm to* F*, i.e.,*

$$\lim_{N\to\infty} \left\| F - \sum_{n=0}^{N} \sum_{k=-n}^{n} F_{n,k} Y_{n,k} \right\|_{\mathcal{L}^2(\Omega)} = 0.$$

3. *Parseval's identity holds, that is*

$$\|F\|^2_{\mathcal{L}^2(\Omega)} = (F,F)_{\mathcal{L}^2(\Omega)} = \sum_{n=0}^{\infty}\sum_{k=-n}^{n} |(F,Y_{n,k})_{\mathcal{L}^2(\Omega)}|^2,$$

for all $F \in \mathcal{L}^2(\Omega)$.

4. *The extended Parseval's identity holds, i.e.,*

$$(F,G)_{\mathcal{L}^2(\Omega)} = \sum_{n=0}^{\infty}\sum_{k=-n}^{n} F_{n,k} G_{n,k}$$

holds for all $F, G \in \mathcal{L}^2(\Omega)$.

5. *There is no strictly larger orthonormal system containing the orthonormal system* $\{Y_{n,k}\}_{k=-n,\ldots,n}$, $n \in \mathbb{N}_0$.

6. *The system* $\{Y_{n,k}\}_{k=-n,\ldots,n}$, $n \in \mathbb{N}_0$ *has the completeness property, i.e., if* $F \in \mathcal{L}^2(\Omega)$ *and* $F_{n,k} = 0$ *for all* $n \in \mathbb{N}_0$ *and* $k = -n,\ldots,n$, *then* $F = 0$.

7. *Any element* $F \in \mathcal{L}^2(\Omega)$ *is uniquely determined by its (spherical) Fourier coefficients. That means if* $F_{n,k} = G_{n,k}$ *for all* $n \in \mathbb{N}_0$ *and* $k = -n,\ldots,n$, *then* $F = G$.

Proof:
See any monograph on functional analysis, for example, [21]. □

1.3 Sobolev Spaces and Pseudodifferential Operators

In this section, we discuss pseudodifferential operators and their so-called native spaces. Pseudodifferential operators are generalizations of differential and integral operators. To specify the reference spaces of the pseudodifferential operators, i.e., *the Sobolev spaces*, there are at least two approaches. The first approach is based on the fact that the sphere Ω is a two-dimensional differentiable manifold. By using this approach one can define the Sobolev spaces on an open subset of the sphere Ω, too. For details on the Euclidian case, see, e.g., [53] and for the spherical case see, e.g., [98]. The second one is based on the Fourier theory. This approach, in our nomenclature, is much easier than the first one. Therefore, we use the second approach to introduce the Sobolev spaces on the sphere Ω. Our presentation owes much to [27], [29], [34] and [32] for the extension to the harmonic case. These papers and textbooks develop, in a considerably accurate way, the Sobolev spaces and the pseudodifferential operators on the sphere Ω and provide their application preferably in geosciences.

To introduce the Sobolev spaces, let $\{A_n\}$ be a sequence of real numbers with $A_n \neq 0$ for all $n \in \mathbb{N}_0$. Consider the set $\mathcal{E}(\{A_n\};\Omega)$ of all functions $F \in \mathcal{C}^{(\infty)}(\Omega)$ of the form

$$F = \sum_{n=0}^{\infty} \sum_{k=-n}^{n} F_{n,k} Y_{n,k},$$

satisfying

$$\sum_{n=0}^{\infty} \sum_{k=-n}^{n} A_n^2 F_{n,k}^2 < \infty. \tag{1.65}$$

We impose an inner product $(\cdot,\cdot)_{\mathcal{H}(\{A_n\};\Omega)}$ on the space $\mathcal{E}(\{A_n\};\Omega)$ defined by

$$(F,G)_{\mathcal{H}(\{A_n\};\Omega)} = \sum_{n=0}^{\infty} \sum_{k=-n}^{n} A_n^2 F_{n,k} G_{n,k}\,. \tag{1.66}$$

The associated norm is given by

$$\|F\|_{\mathcal{H}(\{A_n\};\Omega)} = \left(\sum_{n=0}^{\infty} \sum_{k=-n}^{n} A_n^2 F_{n,k}^2 \right)^{1/2}. \tag{1.67}$$

The Sobolev space is now introduced as follows:

Definition 1.3.1 (Sobolev Spaces)
The Sobolev space $\mathcal{H}(\{A_n\};\Omega)$ is the completion of $\mathcal{E}(\{A_n\};\Omega)$ under the norm defined in (1.67), i.e.,

$$\mathcal{H}(\{A_n\};\Omega) = \overline{\mathcal{E}(\{A_n\};\Omega)}^{\|\cdot\|_{\mathcal{H}(\{A_n\};\Omega)}}.$$

Of course, $\mathcal{H}(\{A_n\};\Omega)$ with the inner product given by (1.66) is a Hilbert space. For convenience, we will simply write

$$\mathcal{H}_s(\Omega) = H\left(\left\{\left(n+\frac{1}{2}\right)^s;\ \Omega\right\}\right), \quad s \in \mathbb{R}. \tag{1.68}$$

The relation between the norm in $\mathcal{H}_s(\Omega)$ and $\mathcal{L}^2(\Omega)$-norm is given by

$$\|F\|^2_{\mathcal{H}_s(\Omega)} = \|(-\Delta^* + \frac{1}{4})^{\frac{s}{2}} F\|^2_{\mathcal{L}^2(\Omega)}. \tag{1.69}$$

In particular, we have $\mathcal{H}_0(\Omega) = \mathcal{H}(\{1\};\Omega) = \mathcal{L}^2(\Omega)$. Furthermore, if $t < s$ then $\mathcal{H}_s(\Omega) \subset \mathcal{H}_t(\Omega)$ and $\|F\|_{\mathcal{H}_t(\Omega)} \leq \|F\|_{\mathcal{H}_s(\Omega)}$.

The next lemma states that under certain circumstances we are still dealing with continuous functions. In order to explain this result we need, as proposed in [34], the concept of summable sequences.

Definition 1.3.2 (Summable Sequences)
A sequence $\{A_n\}_{n\in\mathbb{N}_0}$ is called summable if

$$\sum_{n\in\mathcal{N}(A_n)}^{\infty} \frac{2n+1}{A_n^2} < \infty, \tag{1.70}$$

where $\mathcal{N}(A_n)$ is the set of all $n \in \mathbb{N}_0$ such that $A_n \neq 0$.

Lemma 1.3.3 (Sobolev Lemma)
Let $\{A_n\}$ be summable. Then any $F \in \mathcal{H}(\{A_n\};\Omega)$ corresponds to a continuous function on Ω. If, further, $F \in \mathcal{H}_s(\Omega)$ for $s > k+1$, then F corresponds to a function of class $\mathcal{C}^{(k)}(\Omega)$.

For more details on Sobolev spaces and the proof of the Sobolev Lemma, see [34] for the spherical case and [32] for the case of harmonic functions inside/outside a sphere.

In connection to Sobolev spaces, we introduce (invariant) pseudodifferential operators.

Definition 1.3.4 (Pseudodifferential Operators)
Let $\{\Lambda_n\}_{n\in\mathbb{N}_0}$ be a sequence of real numbers. The operator $\Lambda : \mathcal{H}_s(\Omega) \longrightarrow \mathcal{H}_{s-t}(\Omega)$ defined by

$$\Lambda F = \sum_{n=0}^{\infty} \sum_{k=-n}^{n} \Lambda_n F_{n,k} Y_{n,k}, \tag{1.71}$$

is called a *pseudodifferential operator of order t*, if

$$\lim_{n\to\infty} \frac{|\Lambda_n|}{(n+\frac{1}{2})^t} = const \neq 0, \tag{1.72}$$

for some $t \in \mathbb{R}$. The sequence $\{\Lambda_n\}_{n\in\mathbb{N}_0}$ is called the symbol of Λ. Moreover, if the limit relation

$$\lim_{n\to\infty} \frac{|\Lambda_n|}{(n+\frac{1}{2})^t} = 0 \tag{1.73}$$

holds for all $t \in \mathbb{R}$, then the operator $\Lambda : \mathcal{H}_s(\Omega) \longrightarrow \mathcal{C}^{(\infty)}(\Omega)$ is called a *pseudodifferential operator of order* $-\infty$.

It should be mentioned that the equality in (1.71) is understood in the $\mathcal{H}_{s-t}(\Omega)$-topology.

Some interesting properties of the pseudodifferential operators are valid:

$$(\Lambda' + \Lambda'')_n = \Lambda'_n + \Lambda''_n, \quad n \in \mathbb{N}_0, \tag{1.74}$$

$$(\Lambda'\Lambda'')_n = \Lambda'_n\Lambda''_n, \quad n \in \mathbb{N}_0. \tag{1.75}$$

In addition, we have

$$\Lambda Y_{n,k} = \Lambda_n Y_{n,k}, \quad n = 0, 1, \ldots, \; j = 1, \ldots, 2n+1. \tag{1.76}$$

The property (1.76) states that the symbol of an pseudodifferential operator as defined by Definition 1.3.4 is independent of the order of the spherical harmonic $Y_{n,k}$, i.e., for an arbitrary but fixed $n \in \mathbb{N}_0$ we have $\Lambda_{n,k} = \Lambda_n$, for $k = -n, \ldots, n$.

Remark 1.3.5
Because of the property (1.76), we sometimes call an operator Λ in Definition 1.3.4 the *invariant* pseudodifferential operator.

Finally, we mention that for all invertible operators Λ on $\mathcal{H}_s(\Omega)$, i.e., $\Lambda_n \neq 0$ for all $n \in \mathbb{N}_0$, we have

$$\|F\|_{\mathcal{H}(\{\Lambda_n A_n\};\Omega)} = \|\Lambda F\|_{\mathcal{H}(\{A_n\};\Omega)}, \quad F \in \mathcal{H}(\{\Lambda_n A_n\};\Omega). \tag{1.77}$$

In this case, we have $\mathcal{H}(\{\Lambda_n A_n\};\Omega) = \Lambda^{-1}\mathcal{H}(\{A_n\};\Omega)$. A more detailed discussion on the pseudodifferential operators on the sphere Ω can be found in [34], [16], and [17].

1.4 Spherical Singular Integrals

As we already stated in Subsection 1.2.3 the concept of the spherical convolution (1.2.6) enables us to introduce a powerful tool in approximation theory, the so-called *spherical singular integrals* (cf., e.g., [8] and [34]).

Definition 1.4.1 (Spherical Singular Integrals)
Let $\{K_h\}_{h\in(-1,1)}$ be a family of functions in $\mathcal{X}(\Omega)$ satisfying the conditions $K_h^\wedge(0) = 1$ for all $h \in (-1,1)$. The bounded linear operator $I_h : \mathcal{X}(\Omega) \to \mathcal{X}(\Omega)$ given by

$$I_h(F) = K_h * F, \qquad F \in \mathcal{X}(\Omega), \tag{1.78}$$

is called a spherical singular integral and $\{I_h\}_{h\in(-1,1)}$ is called a family of spherical singular integrals in $\mathcal{X}(\Omega)$ and $\{K_h\}_{h\in(-1,1)}$ is called a family of kernels of the spherical singular integrals.

Definition 1.4.2 (Spherical Approximate Identity)
Assume that $\{I_h\}_{h\in(-1,1)}$ is a family of spherical singular integrals in $\mathcal{X}(\Omega)$. Then $\{I_h\}_{h\in(-1,1)}$ is called an approximate Identity in $\mathcal{X}(\Omega)$ if

$$\lim_{h\to 1^-} \|I_h(F) - F\|_{\mathcal{X}(\Omega)} = 0, \tag{1.79}$$

for all $F \in \mathcal{X}(\Omega)$.

Recall that in Definition 1.4.1 and Definition 1.4.2, if $\{K_h\}_{h\in(-1,1)} \subset \mathcal{L}^1[-1,1]$ then $\mathcal{X}(\Omega) = \mathcal{C}(\Omega)$, and if $\{K_h\}_{h\in(-1,1)} \subset \mathcal{L}^2[-1,1]$ then $\mathcal{X}(\Omega) = \mathcal{L}^2(\Omega)$.

The following theorem presents a necessary and sufficient condition for a spherical singular integral to be an approximate identity.

Theorem 1.4.3
Let $\{K_h\}_{h\in(-1,1)}$ be a family of kernels of singular integrals $\{I_h\}_{h\in(-1,1)}$ in $\mathcal{X}(\Omega)$. Assume that $\{K_h\}_{h\in(-1,1)}$ is uniformly bounded, i.e., there is a constant M, independent of h, such that

$$2\pi \int_{-1}^{1} |K_h(t)| \, \mathrm{d}t \le M, \quad h \in (-1,1). \tag{1.80}$$

Then $\{I_h\}_{h\in(-1,1)}$ *is an approximate identity in* $\mathcal{X}(\Omega)$ *if and only if*

$$\lim_{h\to 1^-} K_h^\wedge(n) = 1, \quad n \in \mathbb{N}_0. \tag{1.81}$$

Proof:
If $\{I_h\}_{h\in(-1,1)}$ is an approximate identity in $\mathcal{X}(\Omega)$, then (1.79) holds for every $F \in \mathcal{X}(\Omega)$, especially for all spherical harmonics Y_n of degree n:

$$\lim_{h\to 1^-} \|I_h(Y_n) - Y_n\|_{\mathcal{X}(\Omega)} = 0, \quad n \in \mathbb{N}_0. \tag{1.82}$$

By the Funk-Hecke formula we have

$$I_h(Y_n)(\xi) = K_n^\wedge(n) Y_n(\xi), \quad \xi \in \Omega, \tag{1.83}$$

thus

$$0 = \lim_{h\to 1^-} \|I_h(Y_n) - Y_n\|_{\mathcal{X}(\Omega)} = \lim_{h\to 1^-} |K_h^\wedge(n) - 1| \, \|Y_n\|_{\mathcal{X}(\Omega)}, \quad n \in \mathbb{N}_0. \tag{1.84}$$

Because $\|Y_n\|_{\mathcal{X}(\Omega)} \neq 0$ for all $Y_n \neq 0,\ n \in \mathbb{N}_0$, it follows that $\lim_{h\to 1^-} \tilde{K}_h^\wedge(n) = 1,\ n \in \mathbb{N}_0$.

Conversely, suppose that (1.81) holds true. To prove that $\{I_h\}_{h\in(-1,1)}$ is an approximate identity in $\mathcal{X}(\Omega)$ we have to consider two cases as follows:

Case 1: $\{K_h\}_{h\in(-1,1)} \subset \mathcal{L}^1[-1,1]$.

Let Y_n be an arbitrary spherical harmonic of degree $n \in \mathbb{N}_0$. Then similar to (1.84) we have

$$\lim_{h\to 1^-} \|I_h(Y_n) - Y_n\|_{\mathcal{C}(\Omega)} = \lim_{h\to 1^-} |K_h^\wedge(n) - 1| \, \|Y_n\|_{\mathcal{C}(\Omega)} = 0, \quad n \in \mathbb{N}_0. \tag{1.85}$$

Suppose $F \in \mathcal{C}(\Omega)$ is arbitrary. Let $\varepsilon > 0$ be given. By the triangle inequality, we have

$$\|F - I_F\|_{\mathcal{C}(\Omega)} \le \|F - L_F\|_{\mathcal{C}(\Omega)} + \|L_F - I_h(L_F)\|_{\mathcal{C}(\Omega)} + \|I_h(L_F) - I_h(F)\|_{\mathcal{C}(\Omega)} . \tag{1.86}$$

Because the system $\{Y_{n,k}\}_{k=-n,\ldots,n}$, $n \in \mathbb{N}_0$, is closed in $(\mathcal{C}(\Omega), \|\cdot\|_{\mathcal{C}(\Omega)})$ (see (1.62)), then there exists a linear combination

$$L_F = \sum_{n=0}^{N_\varepsilon} \sum_{k=-n}^{n} d_{n,k} Y_{n,k} \tag{1.87}$$

such that

$$\|F - L_F\|_{\mathcal{C}(\Omega)} \leq min\{\frac{\varepsilon}{3}, \frac{\varepsilon}{3M}\}, \tag{1.88}$$

where M is the constant given in (1.80). Therefore, the first summand in (1.86) can be estimated by $\varepsilon/3$.

Now, let

$$C = \max_{\substack{n=0,\ldots,N_\varepsilon \\ k=-n,\ldots,n}} |d_{n,k}|,$$

then because of (1.85) there exists some h_0 such that for all $h \in [h_0, 1)$, we have

$$\|I_h(Y_{n,k}) - Y_{n,k}\|_{\mathcal{C}(\Omega)} \leq \frac{\varepsilon}{3C(N_\varepsilon + 1)^2}, \quad n = 0, \ldots, N_\varepsilon, \;\; k = -n, \ldots, n. \tag{1.89}$$

Hence, for the second summand in (1.86), we get

$$\|L_F - I_h(L_F)\|_{\mathcal{C}(\Omega)} \leq \sum_{n=0}^{N_\varepsilon} \sum_{k=-n}^{n} |d_{n,k}| \|I_h(Y_{n,k}) - Y_{n,k}\|_{\mathcal{C}(\Omega)} \leq \frac{\varepsilon}{3}. \tag{1.90}$$

Finally, to estimate the last summand in (1.86), we observe the uniform boundedness of $\{K_h\}_{h\in(-1,1)}$ as follows:

$$\begin{aligned}
\|I_h(L_F)(\xi) - I_h(F)(\xi)\|_{\mathcal{C}(\Omega)} &= \|I_h(L_F - F)\|_{\mathcal{C}(\Omega)} \\
&= \|K_h * (L_F - F)\|_{\mathcal{C}(\Omega)} \\
&\leq \|L_F - F\|_{\mathcal{C}(\Omega)} \; \|K_h\|_{\mathcal{L}^1[-1,1]} \\
&= \|L_F - F\|_{\mathcal{C}(\Omega)} \; 2\pi \int_{-1}^{1} |K_h(t)| \; \mathrm{d}t \\
&\leq \frac{\varepsilon}{3M} M = \frac{\varepsilon}{3}
\end{aligned}$$

Case 2: $\{K_h\}_{h\in(-1,1)} \subset \mathcal{L}^2[-1,1]$.

From the uniform boundedness of $\{K_h\}_{h\in(-1,1)}$ and (1.27) it follows that

$$\begin{aligned}
|K_h^\wedge(n)| &= \left| 2\pi \int_{-1}^{1} K_h(t) \; P_n(t) \; \mathrm{d}t \right| \\
&\leq 2\pi \int_{-1}^{1} |K_h(t)| \; |P_n(t)| \; \mathrm{d}t \\
&\leq 2\pi \int_{-1}^{1} |K_h(t)| \; \mathrm{d}t \leq M,
\end{aligned}$$

for all $h \in (-1, 1)$ and for all $n \in \mathbb{N}_0$. Therefore,

$$\|F - I_h(F)\|^2_{\mathcal{L}^2(\Omega)} = \sum_{n=0}^{\infty} \sum_{k=-n}^{n} (1 - K_h^\wedge(n))^2 F_{n,k}^2 \leq (M+1)^2 \|F\|^2_{\mathcal{L}^2(\Omega)},$$

for all $h \in (-1, 1)$ and for all $F \in \mathcal{L}^2(\Omega)$. Since the upper bound $(M + 1)$ of $|1 - K_h^\wedge(n)|$ is independent of h, in other words, the $\lim_{h \to 1^-}$ and the series may be interchanged,

$$\lim_{h \to 1^-} \|F - I_h(F)\|_{\mathcal{L}^2(\Omega)} = \left(\sum_{n=0}^{\infty} \sum_{k=-n}^{n} \lim_{h \to 1^-} (1 - K_h^\wedge(n))^2 F_{n,k}^2 \right)^{\frac{1}{2}} = 0$$

for all $F \in \mathcal{L}^2(\Omega)$. □

We point out that for the non-negative kernels $\{K_h\}_{h \in (-1,1)}$ the condition $K_h^\wedge(0) = 1$ implies that $M = 1$ in (1.80).

The following theorem lists the equivalent conditions for an approximate identity with non-negative kernels.

Theorem 1.4.4
Let $\{K_h\}_{h \in (-1,1)}$ be a family of non-negative kernels in $\mathcal{X}(\Omega)$ with $K_h^\wedge(0) = 1$. Suppose that $\{I_h\}_{h \in (-1,1)}$ is the spherical singular integral corresponding to the kernels $\{K_h\}_{h \in (-1,1)}$. Then the following statements are equivalent:

(i) $\{I_h\}_{h \in (-1,1)}$ *is an approximate identity.*

(ii) $\lim_{h \to 1^-} K_h^\wedge(n) = 1 \quad n \in \mathbb{N}_0$.

(iii) $\lim_{h \to 1^-} K_h^\wedge(1) = 1$.

(iv) $\{K_h\}_{h \in (-1,1)}$ *satisfies the "localization property":*

$$\lim_{h \to 1^-} \int_{-1}^{\delta} K_h(t)\, \mathrm{d}t = 0, \qquad \textit{for all } \delta \in (-1, 1).$$

Proof:
See [35] or [40]. □

In this work, we call the non-negative family of functions $\{K_h\}_{h\in(-1,1)}$ which satisfy one of the condition (i)-(iv), as stated in Theorem 1.4.4, a family of scaling functions in $\mathcal{X}(\Omega)$. In other words, a family of scaling functions generates a family of approximate identity operators.

Chapter 2

Multiscale Approximation by Locally Supported Zonal Kernels

During the last decades, many geoscientists have been using satellites to gather data from the Earth. These scientific satellites collect a huge amount of data and send them stations on the Earth's surface. This information must be analyzed. There are various methods to analyze these data, and clearly the specification of an adequate method to analyze these data is very important (see, e.g., [32] for the determination of the gravity field). For example, consider the problem of constructing a smooth function over the sphere which interpolates a set of scattered points with associated real values. In other words, given a set $X_N = \{\xi_1, \ldots, \xi_N\}$ of distinct points on the sphere Ω and a target function $F : \Omega \to \mathbb{R}$, the problem is to find an *interpolant* $S : \Omega \to \mathbb{R}$ such that

$$S(\xi_i) = F(\xi_i), \quad i = 1, \ldots, N. \tag{2.1}$$

There are different approaches to find the solution of this interpolation problem. One of the most powerful and popular tools used to find an interpolant S that satisfies the interpolation conditions (2.1) is the radial basis functions (RBF) approach for the sphere, and this is the main topic of this chapter (note that in first approximation the Earth's surface may be understood to be spherical).

2.1 Spherical Radial Basis Functions

A radial function, say $\Psi : \mathbb{R}^d \to \mathbb{R}$, depends only on the distance between two elements of $\mathbb{R}^d$. In other words, $\Psi : \mathbb{R}^d \to \mathbb{R}$ is a radial function, when $\Psi(x) = \psi(d(x,x))$, where $\psi : \mathbb{R}^+ \to \mathbb{R}$ and d is a metric on $\mathbb{R}^d$, e.g., the Euclidean metric on $\mathbb{R}^d$ (for more details about metrics and metric spaces, see, e.g., [83]). From the geometric point of view, a radial function Ψ in $\mathbb{R}^3$ can be generated by rotating the graph of a one-dimensional function $\psi : \mathbb{R}^+ \to \mathbb{R}$ around the axis $x = 0$.

In this work, we are interested in the concept of spherical radial basis functions (SRBF). We call the function $\Phi : \Omega \to \mathbb{R}$ a spherical radial function, if Φ depends on the geodetic distance of two points on the sphere Ω, where the geodetic distance is defined as follows:

Definition 2.1.1 (Geodetic Distance (Metric))
The function $d : \Omega^2 \to [0, \pi]$, given by

$$d(\xi, \eta) = \cos^{-1}(\xi \cdot \eta), \quad \xi, \eta \in \Omega, \tag{2.2}$$

is called the geodetic distance (metric).

Remark 2.1.2
It should be mentioned that because of the equation

$$\|\xi - \eta\| = \sqrt{2(1 - \xi \cdot \eta)}, \quad \xi, \eta \in \Omega, \tag{2.3}$$

the restriction of a radial function to the sphere Ω is a spherical radial function, and vice versa.

Definition 2.1.3 (Zonal Function)
For given $\phi : [-1, 1] \to \mathbb{R}$, the function of the form $K_\xi : \Omega \to \mathbb{R}$ defined by

$$K_\xi(\eta) = \phi(\xi \cdot \eta), \qquad \eta \in \Omega,$$

is called a ξ-zonal function on Ω.

It should be mentioned that because of Remark 2.1.2, if $\xi \in \Omega$ is fixed, then each spherical radial function is a zonal function on the sphere Ω, and vice versa.

Usually, the interpolant S in the SRBF approach is chosen to be a linear combination of translates (rotations) of a zonal function, i.e.,

$$S(\xi) = \sum_{j=1}^{N} \mu_j \phi(\xi \cdot \xi_j), \quad \xi \in \Omega, \tag{2.4}$$

where ϕ is a *zonal kernel* on Ω and μ_j are unknowns. If we apply the interpolation conditions (2.1) to (2.4) then we get the following linear system of equations:

$$\mathbf{A}\mu = \mathbf{b}, \tag{2.5}$$

where

$$\mathbf{A} \in \mathbb{R}^{N\times N}, \mu = \begin{pmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_N \end{pmatrix} \in \mathbb{R}^{1\times N}, \ \mathbf{b} = \begin{pmatrix} F(\xi_1) \\ F(\xi_2) \\ \vdots \\ F(\xi_N) \end{pmatrix} \in \mathbb{R}^{1\times N}. \tag{2.6}$$

The matrix $\mathbf{A}$ is sometimes called the interpolation matrix and its elements are given by

$$a_{ij} = \phi(\xi_i \cdot \xi_j), \quad \xi_i, \ \xi_j \in X_N \subset \Omega. \tag{2.7}$$

To find uniquely determined unknowns $\mu_j, \ j = 1, \ldots, N$, in (2.4), the interpolation matrix $\mathbf{A}$ should be non-singular. The non-singularity of $\mathbf{A}$ is dependent on

- the position of points X_N on the sphere Ω
- the special choice of the zonal kernel ϕ.

If for a given zonal kernel, there is a system of points X_N such that the interpolation matrix $\mathbf{A}$ is non-singular then this system of points is called *the fundamental system of points relative to the space*

$$V = \text{span } \{\phi(\xi_i \cdot), \ i = 1, \ldots, N\}. \tag{2.8}$$

In addition, if a system of points contains a fundamental system of points relative to the space V then the interpolation problem is clearly solvable. Such a system of points that contains a fundamental system of points relative to the space V is called an *admissible system of points relative to the space* V.

Fundamental systems of points relative to $Harm_{0,\ldots,n}$ have been investigated by several authors. For example, Xu [110], [111] has provided some fundamental systems of points relative to $Harm_{0,\ldots,n}$. Further work about specifying fundamental systems of points relative to $Harm_{0,\ldots,n}$ can be found in [114] and [25].

As we mentioned before, the solvability (2.5) is dependent on the choice of the zonal kernel function. In the next section, we will characterize suitable functions to be used in interpolation in more detail.

2.2 Positive Definiteness of Locally Supported Kernel Functions

Schonenberg [89] in 1942 investigated the property of kernel functions on the sphere such that the interpolation problem (2.5) is solvable. His work is based on the orthogonal expansion of the zonal function in terms of the Gegenbauer (ultraspherical) polynomials. Because the Legendre polynomials are easier to handle and much more known tools than the Gegenbauer polynomials, we shall use the orthogonal expansion of the zonal function in terms of the Legendre polynomials.

If $K \in \mathcal{L}^2[-1,1]$, then the zonal function K_ξ given by $\eta \longmapsto K_\xi(\eta) = K(\xi \cdot \eta)$ is in $\mathcal{L}^2(\Omega)$. Therefore, by Theorem 1.2.9, we have

$$K_\xi(\eta) = \sum_{n=0}^{\infty} \sum_{k=-n}^{n} K_\xi^\wedge(n,k) Y_{n,k}(\eta), \quad \eta \in \Omega. \tag{2.9}$$

By the Funk-Hecke formula, we have

$$K_\xi^\wedge(n,k) = K_\xi^\wedge(n) Y_{n,k}(\eta), \quad \eta \in \Omega, \tag{2.10}$$

where

$$K_\xi^\wedge(n) = \int_\Omega K_\xi(\eta) P_n(\xi \cdot \eta)\, \mathrm{d}\omega(\eta) = 2\pi \int_{-1}^{1} \phi(t) P_n(t)\, \mathrm{d}t, \quad n \in \mathbb{N}_0. \tag{2.11}$$

After substituting (2.10) in (2.9) and applying the addition theorem (Theorem 1.2.4), we obtain

$$K_\xi(\eta) = \sum_{n=0}^{\infty} \frac{2n+1}{4\pi} K_\xi^\wedge(n) P_n(\xi \cdot \eta), \quad \eta \in \Omega. \tag{2.12}$$

Obviously, the function in terms of Legendre polynomials reads as follows:

$$\phi(t) = \sum_{n=0}^{\infty} \frac{2n+1}{4\pi} \phi^{\wedge}(n) P_n(t), \quad t \in [-1, 1]. \tag{2.13}$$

Now, we are able to define *(strictly) positive definite functions.*

Definition 2.2.1 (Positive Definite Function)
A continuous function $\phi : [-1, 1] \to \mathbb{R}$ is said to be positive definite (PD) on the sphere Ω if, for any set $X_N = \{\xi_1, \ldots, \xi_N\}$ of distinct points on the sphere Ω and an arbitrary vector $\mu = (\mu_1, \ldots, \mu_N)^T$, the quadratic form

$$\mu^T A \mu = \sum_{i=1}^{N} \sum_{j=1}^{N} \mu_i \mu_j \phi(\xi_i \cdot \xi_j) \tag{2.14}$$

is non-negative.

Definition 2.2.2 (Strictly Positive Definite Function)
A continuous function $\phi : [-1, 1] \to \mathbb{R}$ is said to be strictly positive definite (SPD) on the sphere Ω if, for any set $X_N = \{\xi_1, \ldots, \xi_N\}$ of distinct points on the sphere Ω and an arbitrary non-zero vector $\mu = (\mu_1, \ldots, \mu_N)^T$, the quadratic form

$$\mu^T A \mu = \sum_{i=1}^{N} \sum_{j=1}^{N} \mu_i \mu_j \phi(\xi_i \cdot \xi_j) \tag{2.15}$$

is positive.

Remark 2.2.3 (Native Space of (S)PD Function)
A (strictly) positive definite function can be considered as the reproducing kernel of a uniquely determined Hilbert space (this is a theorem by Aronszajn [5, Sec. 2], although he ascribed it to Moore [69]). We call the Hilbert space associated with the (strictly) positive definite function φ as the native space of φ, and denote it by $\mathcal{N}_\varphi$.

Sometimes we would like to point out that if the interpolation data of F in (2.1) come from a spherical harmonic of degree $\leq\ m$ then the interpolant S be exactly equivalent to F in every point on the sphere Ω (see, e.g., [27]). In other words, we would like that S has the polynomial precision of order m. In such a case, we usually add to S in (2.4) a spherical harmonic from $Harm_{0,\ldots,m}(\Omega)$, i.e.,

$$S(\xi) = \sum_{j=1}^{N} \mu_j \phi(\xi \cdot \xi_j) + \sum_{n=0}^{m} \sum_{k=-n}^{n} \nu_{n,k} Y_{n,k}(\xi), \quad \xi \in \Omega, \tag{2.16}$$

From the interpolation conditions (2.1), we get N linear equations in $N+M$ unknowns, where M is $dim(Harm_{0,\ldots,m}(\Omega)) = (m+1)^2$, and therefore, there are M degrees of freedom. These extra degrees of freedom can be absorbed by adding the following constraints

$$\sum_{j=1}^{N} \mu_j Y_{n,k}(\xi_j) = 0, \qquad n = 0, \ldots, m, \; k = 1, \ldots, 2n+1. \tag{2.17}$$

Thus we have the following system of equations

$$\sum_{j=1}^{N} \mu_j \phi(\xi_i \cdot \xi_j) + \sum_{n=0}^{m} \sum_{k=-n}^{n} \nu_{n,k} Y_{n,k}(\xi_i) = F(\xi_i), \quad i = 1, \ldots, N,$$

$$\sum_{j=1}^{N} \mu_j Y_{n,k}(\xi_j) = 0, \quad n = 0, \ldots, m, \; k = 1, \ldots, 2n+1. \tag{2.18}$$

The system of equations (2.18) can be written in matrix form as follows:

$$\begin{pmatrix} A & Y \\ Y^T & 0 \end{pmatrix} \begin{pmatrix} \mu \\ \nu \end{pmatrix} = \begin{pmatrix} b \\ 0 \end{pmatrix} \tag{2.19}$$

where A, b and μ are the same as in (2.6) and $Y \in \mathbb{R}^{N \times M}$ is the coefficient matrix of (2.17). Thus the interpolant S in (2.16) can be uniquely found if and only if the matrix

$$\begin{pmatrix} A & Y \\ Y^T & 0 \end{pmatrix} \tag{2.20}$$

is regular.

Definition 2.2.4 (Conditionally Positive Definite Function)
A continuous function $\phi : [-1,1] \to \mathbb{R}$ is said to be conditionally positive definite of order m on the sphere Ω if, for any set $X_N = \{\xi_1, \ldots, \xi_N\}$ of

distinct points on the sphere Ω and all vectors $\mu = (\mu_1, \ldots, \mu_N)^T$ satisfying

$$\sum_{j=1}^{N} \mu_j Y_{n,k}(\xi_j) = 0, \quad n = 0, \ldots, m, \; k = 1, \ldots, 2n+1, \tag{2.21}$$

the quadratic form

$$\mu^T A \mu = \sum_{i=1}^{N} \sum_{j=1}^{N} \mu_i \mu_j \phi(\xi_i \cdot \xi_j) \tag{2.22}$$

is non-negative.

Definition 2.2.5 (Conditionally Strictly Positive Definite Function)
A continuous function $\phi : [-1, 1] \to \mathbb{R}$ is said to be conditionally strictly positive definite of order m on the sphere Ω if, for any set $X_N = \{\xi_1, \ldots, \xi_N\}$ of distinct points on the sphere Ω and all non-zero vectors $\mu = (\mu_1, \ldots, \mu_N)^T$ satisfying

$$\sum_{j=1}^{N} \mu_j Y_{n,k}(\xi_j) = 0, \quad n = 0, \ldots, m, \; k = 1, \ldots, 2n+1, \tag{2.23}$$

the quadratic form

$$\mu^T A \mu = \sum_{i=1}^{N} \sum_{j=1}^{N} \mu_i \mu_j \phi(\xi_i \cdot \xi_j) \tag{2.24}$$

is positive.

In particular, a conditionally (strictly) positive definite function of order $m = -1$ is understood to be a (strictly) positive definite function.

Now the question is: under which conditions is a zonal function (strictly) positive definite? As we mentioned before, the first work in this context is due to Schoenberg [89]. The following theorem is Schoenberg's result formulated in terms of the Legendre polynomials expansion.

Theorem 2.2.6 (Necessary and Sufficient Conditions for PD)
Let $\phi : [-1, 1] \to \mathbb{R}$ be continuous. Suppose that the Legendre coefficients of ϕ satisfy

$$\sum_{n=0}^{\infty} \frac{2n+1}{4\pi} \phi^\wedge(n) < \infty. \tag{2.25}$$

Then, ϕ is positive definite if and only if $\phi^\wedge(n) \geq 0$.

As pointed out in the last section, for the solvability of (2.5) we need the strictly positive definiteness of the zonal kernel ϕ. The following theorem states an equivalent condition for the strictly positive definiteness.

Theorem 2.2.7
Let $\phi : [-1, 1] \to \mathbb{R}$ be continuous. Suppose that the Legendre coefficients of ϕ satisfy

$$\sum_{n=0}^{\infty} \frac{2n+1}{4\pi} \phi^\wedge(n) < \infty. \tag{2.26}$$

Then ϕ is strictly positive definite if and only if the set $\{\phi(\xi_1 \cdot), \ldots, \phi(\xi_N \cdot)\}$ is linearly independent for any choice of pairwise distinct points $\xi_1, \ldots, \xi_N \in \Omega$.

Proof:
See [34] or [91]. □

In [112], Xu and Cheney have shown that if all the Legendre coefficients $\phi^\wedge(n)$ in (2.13) are positive, then the function ϕ is strictly positive definite on the sphere Ω. In [91], Schreiner has improved the result of Xu and Cheney: if the function ϕ is positive definite and finitely many of the Legendre coefficients are zero then ϕ is strictly positive definite on the sphere Ω. Another important result for strictly positive definiteness is obtained by Chen, Menegatto and Sun [15]. In the next theorem, we state their result for the sphere Ω.

Theorem 2.2.8 (Necessary and Sufficient Conditions for SPD)
Let $\phi : [-1, 1] \to \mathbb{R}$ be continuous. Suppose that ϕ admits the uniformly convergent series expansion

$$\phi(t) = \sum_{n=0}^{\infty} \frac{2n+1}{4\pi} \phi^\wedge(n) P_n(t), \quad t \in [-1, 1]. \tag{2.27}$$

Then, ϕ is strictly positive definite if and only if the set of indices $\{n \in \mathbb{N}_0 | \phi^\wedge(n) > 0\}$ contains infinitely many odd integers as well as infinitely many even integers.

For more discussion on the (strictly) positive definite function on the sphere, we refer to [80], [79], [64], [65], [15] and the references therein.

Some remarks should be made:

Remark 2.2.9
Theorem 2.2.8 is also valid for the m-dimensional sphere, $m \geq 2$, but it is not valid for the one-dimensional sphere. In the paper [97] a sufficient condition for strictly positive definiteness on the circle is given. Moreover, a necessary and sufficient condition for strictly positive definite kernels on a subset of the complex plane can be found in [66].

Remark 2.2.10
The restriction of a strictly positive definite function on $\mathbb{R}^3$ to the sphere Ω is a strictly positive definite function on the sphere Ω. In other word if ψ is a strictly positive definite function on $\mathbb{R}^3$ then $\phi(t) = \psi(\sqrt{2-2t}),\ t \in [-1,1]$ is a strictly positive definite function on the sphere Ω. This is also valid for the restriction of conditionally strictly positive definite functions to the sphere.

According to Remark 2.2.10, it is possible to extend all results valid in $\mathbb{R}^3$ to the sphere. Before we extend some of these results to the sphere, we mention the following definition (see, e.g., [24]).

Definition 2.2.11 (Completely Monotone on the Sphere)
A continuous function $\phi : [-1,1] \to \mathbb{R}$ is said to be completely monotone on the sphere Ω if $\phi \in \mathcal{C}^\infty(0,\infty)$ and $(-1)^k \frac{\mathrm{d}^k}{\mathrm{d}t^k}\phi(\sqrt{t}) \geq 0,\ t \in (0,\infty)$, for every $k \in \mathbb{N}_0$.

Schoenberg [88] has characterized positive definite functions on $\mathbb{R}^d$. As an extension of Schoenberg's work, Micchelli [67] stated a sufficient condition for conditionally positive definite functions on $\mathbb{R}^d$. According to Remark 2.2.10 and Micchelli's work, we conclude the following theorem.

Theorem 2.2.12
Let ϕ be continuous on $[0,\infty)$ and $(-1)^m\phi^{(m)}$ be completely monotone on the sphere Ω and $\frac{\mathrm{d}^{m+1}}{\mathrm{d}t^{m+1}}\phi(\sqrt{t}) \neq const$. Then $\phi(\sqrt{2-2t}),\ t \in [-1,1]$ is conditionally strictly positive definite of order m on the sphere Ω.

Gue et al. [51] have proved that the Micchelli's conditions are also necessary for the conditionally positive definiteness of a function on $\mathbb{R}^d$.

Another method to characterize positive definite functions is based on the Fourier transforms of functions. One of the most celebrated works in this context was established by Bochner [9], [10], and [11]. Here we present a modified version of Bochner's characterization for the radial functions on $\mathbb{R}^d$.

Theorem 2.2.13 (Modified Bochner's Conditions for PD on $\mathbb{R}^d$)
Let $\Phi \in \mathcal{L}^1(\mathbb{R}^d)$ be a continuous radial function on $\mathbb{R}^d$. Then Φ is positive definite on $\mathbb{R}^d$ if and only if Φ is bounded and the Fourier transform of Φ, denoted by $\hat{\Phi}$ and defined by

$$\hat{\Phi}(r) = r^{-\frac{d-2}{2}} \int_0^\infty \Phi(t) t^{\frac{d}{2}} J_{\frac{d-2}{2}}(rt)\, \mathrm{d}t, \tag{2.28}$$

is non-negative and non-vanishing, where $J_\alpha(t)$ is the Bessel function of the first kind.

Proof:
See [107]. □

Remark 2.2.14
We point out that if $\Phi \in \mathcal{L}^1(\mathbb{R}^d)$ is a continuous radial function on $\mathbb{R}^d$ then from (2.28) it is clear that the Fourier transform of Φ is also a radial function.

A list of conditionally (strictly) positive definite functions on the sphere with their applications in geosciences can be found in Freeden et al. [34] and [45]. For a similar list of conditionally (strictly) positive definite functions on $\mathbb{R}^d$, one can refer to, e.g., [76] or [14].

Our next aim is to present a linkage between the Fourier transform of a radial function $\Phi(\|\cdot\|)$ on $\mathbb{R}^d$ and the Legendre transform of the restriction of Φ to the sphere. This relation is provided by [73] and [115]. In the next theorem, we follow the work by [115]. In our proof, we consider $d = 3$, where the general case, $\mathbb{R}^d$, is similar.

Theorem 2.2.15 (Relation Between Fourier and Legendre Transform)
Let $\Phi(\|\cdot\|)$ and $\hat{\Phi}(\|\cdot\|)$ be in $\mathcal{L}^1(\mathbb{R}^3)$, also suppose that

$$I_n(\hat{\Phi}) = \int_0^\infty J^2_{\frac{1}{2}+n}(t) \hat{\Phi}(\|t\|) t\, \mathrm{d}t, \tag{2.29}$$

exists for all $n \in \mathbb{N}_0$. Then

$$\phi^\wedge(n) = (2\pi)^{\frac{3}{2}} I_{d,n}(\hat{\Phi}), \quad n \in \mathbb{N}_0, \tag{2.30}$$

where $\phi^\wedge(n)$ is the Legendre transform of $\phi(x \cdot y) = \Phi(\|x - y\|)|_{x,y \in \Omega}$.

Proof:
Let $x, y \in \mathbb{R}^3$ and $a = \|x\|$, $b = \|y\|$ and $c = \|x - y\|$ form a triangle. For convenience, let $\Phi(\|z\|) = \varphi(r)$, where $r = \|z\|$. Then from (2.28) we have

$$\hat{\hat{\varphi}}(c) = \varphi(c) = \frac{1}{\sqrt{c}} \int_0^\infty \hat{\varphi}(t) t^{\frac{3}{2}} J_{\frac{1}{2}}(ct)\, \mathrm{d}t$$

By using Gegenbauer's addition theorem for Bessel functions (see [102, Sec. 11.4, Eq. (3)])

$$\frac{\sin c}{c} = \sqrt{\frac{\pi}{2}} \frac{J_{\frac{1}{2}}(c)}{\sqrt{c}} = \pi \sum_{n=0}^{\infty} (n + \frac{1}{2}) \frac{J_{n+\frac{1}{2}}(a)}{\sqrt{a}} \frac{J_{n+\frac{1}{2}}(b)}{\sqrt{b}} P_n(\cos\vartheta), \tag{2.31}$$

where ϑ is the angle between a and b, we have

$$\varphi(c) = \sqrt{2\pi} \sum_{n=0}^{\infty} (n + \frac{1}{2})\, P_n(\frac{x \cdot y}{ab}) \int_0^\infty \hat{\varphi}(t) t^2 \frac{J_{n+\frac{1}{2}}(at)}{\sqrt{at}} \frac{J_{n+\frac{1}{2}}(bt)}{\sqrt{bt}}\, \mathrm{d}t. \tag{2.32}$$

Now, let x and y be on the sphere Ω, from (2.32) we obtain

$$\varphi(\sqrt{2 - 2x \cdot y}) = (2\pi)^{\frac{3}{2}} \sum_{n=0}^{\infty} \frac{2n+1}{4\pi} P_n(x \cdot y) \int_0^\infty \hat{\varphi}(t) t\, J^2_{n+\frac{1}{2}}(t)\, \mathrm{d}t. \tag{2.33}$$

Because

$$\phi(\sqrt{2 - 2x \cdot y}) = \varphi(\sqrt{2 - 2x \cdot y})|_{x,y\in\Omega} = \Phi(\|x - y\|)|_{x,y\in\Omega}\ ,$$

by comparing the coefficients in (2.33) and (2.12) we arrive at

$$\phi^\wedge(n) = (2\pi)^{\frac{3}{2}} \int_0^\infty \hat{\varphi}(t) t\, J^2_{n+\frac{1}{2}}(t)\, \mathrm{d}t. \tag{2.34}$$

This is the desired result. □

An immediate consequence of the last theorem is as follows:

Corollary 2.2.16 (Legendre Transforms of Two Radial Functions)
Let Φ and Ψ be two radial functions such that Φ, Ψ, $\hat{\Phi}$ and $\hat{\Psi}$ are in $\mathcal{L}^1(\mathbb{R}^d)$, also suppose that $\hat{\Phi}$ and $\hat{\Psi}$ are strictly positive and $\hat{\Phi} \le c\hat{\Psi}$. Then

$$0 < \phi^\wedge(n) \le \psi^\wedge(n), \quad n \in \mathbb{N}_0, \tag{2.35}$$

where $\phi^\wedge(n)$ and $\psi^\wedge(n)$ are the Legendre transforms of the restriction of Φ and Ψ to the sphere, respectively.

Proof:
Because Φ and Ψ are strictly positive definite on $\mathbb{R}^d$, the integral (2.2.15) for Φ and Ψ always exists. By applying (2.30) to Φ and Ψ, we are led to the result. □

2.3 Zonal Finite Elements

In this section, we focus on a family of locally supported kernels, the so-called *isotropic finite elements* on the sphere. In Chapter 4, we shall need them for the construction of biorthogonal kernels. These kernels have been discussed in more detail by [98], [87], [36], [18], [45] and [90] but, for our presentation, we follow the work of Freeden et al. [34] and [44].

We start from the definition of the so-called smoothed Haar functions.

Definition 2.3.1 (Smoothed Haar Functions)
For $h \in (-1, 1)$ and $\lambda > -1$, the piecewise polynomial function $B_{h,\lambda} : [-1, 1] \to \mathbb{R}$ given by

$$B_{h,\lambda}(t) = \begin{cases} 0 & \text{for} \quad t \in [-1, h] \\ \left(\frac{t-h}{1-h}\right)^{\lambda} & \text{for} \quad t \in (h, 1] \end{cases} \tag{2.36}$$

is called the smoothed Haar function.

Remark 2.3.2
It should be noted that for $-1 < \lambda < 0$ the function $B_{h,\lambda}$ is unbounded. Nevertheless it is of the class $L^1[-1, 1]$.

Let $\xi \in \Omega$ be fixed. Then, similar to (2.12), the ξ-zonal function $B_{h,\lambda}(\xi \cdot) : \Omega \to \mathbb{R}$ admits the following Legendre series expansion

$$B_{h,\lambda}(\xi \cdot) = \sum_{n=0}^{\infty} \frac{2n+1}{4\pi} B^{\wedge}_{h,\lambda}(n) P_n(\xi \cdot), \tag{2.37}$$

where

$$B^{\wedge}_{h,\lambda}(n) = 2\pi \int_{-1}^{1} B_{h,\lambda}(t) P_n(t) \, \mathrm{d}t, \tag{2.38}$$

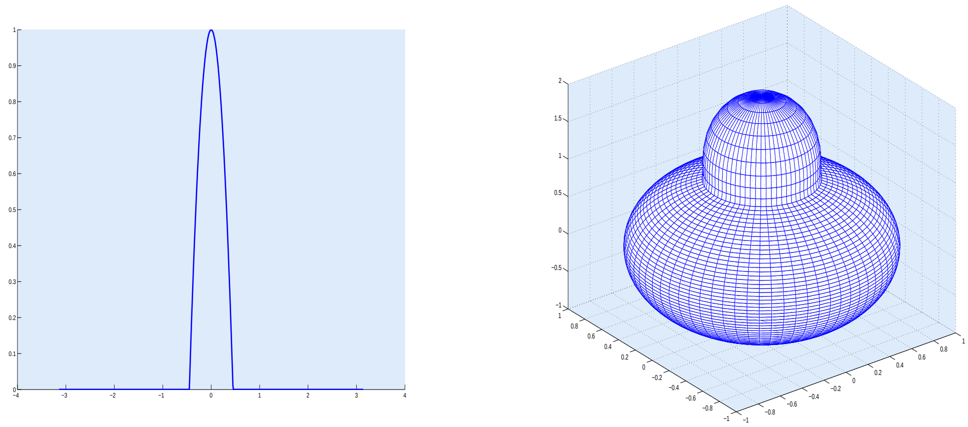


Figure 2.1: The smoothed Haar function $B_{h,\lambda}$ for $h = 0.9$ and $\lambda = 1$
Left: $\vartheta \mapsto B_{h,\lambda}(\cos\vartheta)$, $\vartheta \in [-\pi, \pi]$.
Right: $\eta \mapsto B_{h,\lambda}(\xi \cdot \eta)$ on the sphere Ω, where ξ is the North pole.

and the local support of $B_{h,\lambda}$ is

$$\text{supp } B_{h,\lambda}(\xi\cdot) = \{\eta \in \Omega | h \le \xi \cdot \eta \le 1\}. \tag{2.39}$$

Figure 2.1 illustrates $B_{0.9,1}$ in the plane and on the sphere Ω.

The following lemma, which is of great importance for practical purposes, yields a recursion formula for the Legendre transform of the smoothed Haar functions (see [17], [36]).

Lemma 2.3.3 (Recursion Formula for the Legendre Transforms)
For $h \in (-1, 1)$ and $\lambda > -1$, the Legendre transforms of $B_{h,\lambda}$ satisfy the following recursion formula

$$B_{h,\lambda}^{\wedge}(0) = 2\pi \frac{1-h}{1+\lambda} \tag{2.40}$$

$$B_{h,\lambda}^{\wedge}(1) = \frac{\lambda + h + 1}{\lambda + 2} B_{h,\lambda}^{\wedge}(0) \tag{2.41}$$

$$B_{h,\lambda}^{\wedge}(n+1) = \frac{2n+1}{n+\lambda+2} h\, B_{h,\lambda}^{\wedge}(n) + \frac{\lambda+1-n}{n+\lambda+2} B_{h,\lambda}^{\wedge}(n-1), \quad n \ge 1. \tag{2.42}$$

Proof:
$B_{h,\lambda}^{\wedge}(0)$ and $B_{h,\lambda}^{\wedge}(1)$ can be calculated by straightforward integration. From (1.23), it follows that

$$\int_h^1 B_{h,\lambda}(t)\left((n+1)P_{n+1}(t) + nP_{n-1}(t) - (2n+1)tP_n(t)\right)\, \mathrm{d}t = 0, \quad n \geq 1. \tag{2.43}$$

Therefore,

$$(n+1)B_{h,\lambda}^{\wedge}(n+1) + nB_{h,\lambda}^{\wedge}(n-1) - (2n+1)(B_{h,\lambda+1}^{\wedge}(n) - hB_{h,\lambda}^{\wedge}(n)) = 0, \tag{2.44}$$

for $n \geq 1$. By using (1.20) and integration by parts, we obtain

$$(2n+1)(B_{h,\lambda+1}^{\wedge}(n) = -(\lambda+1)(B_{h,\lambda}^{\wedge}(n+1) - B_{h,\lambda}^{\wedge}(n-1)), \quad n \geq 1. \tag{2.45}$$

Inserting (2.45) in (2.44) we obtain the following recurrence formula

$$(n+\lambda+2)B_{h,\lambda}^{\wedge}(n+1) - (2n+1)h\, B_{h,\lambda}^{\wedge}(n) - (\lambda+1-n)\, B_{h,\lambda}^{\wedge}(n-1) = 0, \tag{2.46}$$

for $n \geq 1$. □

Later, we shall construct an approximate identity from the smoothed Haar kernels, therefore, we normalize the kernel $B_{h,\lambda}$ in the sense that its Legendre transform of order zero be one.

Definition 2.3.4 (Normalized Smoothed Haar Functions)
For $h \in (-1,1)$ and $\lambda > -1$, the function $L_{h,\lambda} : [-1,1] \to \mathbb{R}$ given by

$$L_{h,\lambda}(t) = \frac{1}{B_{h,\lambda}^{\wedge}(0)} B_{h,\lambda}(t) = \begin{cases} 0 & \text{for} \quad t \in [-1,h] \\ \frac{\lambda+1}{2\pi(1-h)^{\lambda+1}}(t-h)^{\lambda} & \text{for} \quad t \in (h,1] \end{cases} \tag{2.47}$$

is called normalized smoothed Haar function.

Similar to Lemma 2.3.3, there is also a recursion formula for the kernels $L_{h,\lambda}$ as follows:

$$L_{h,\lambda}^{\wedge}(0) = 1, \quad L_{h,\lambda}^{\wedge}(1) = \frac{\lambda+h+1}{\lambda+2}, \tag{2.48}$$

$$L_{h,\lambda}^{\wedge}(n+1) = \frac{2n+1}{n+\lambda+2} h\, L_{h,\lambda}^{\wedge}(n) + \frac{\lambda+1-n}{n+\lambda+2} L_{h,\lambda}^{\wedge}(n-1), \quad n \geq 1, \tag{2.49}$$

for $h \in (-1,1)$ and $\lambda > -1$.

The following lemma presents the lower and upper bounds of the Legendre transforms of $L_{h,\lambda}$ In addition, it is shown that the upper bound is the least upper bound, too.

Lemma 2.3.5 (Bounds for the Legendre Transforms of $L_{h,\lambda}$)
Let $h \in (-1,1)$ and $\lambda > -1$. Then

(i) $|L^{\wedge}_{h,\lambda}(n)| < 1, \qquad n \in \mathbb{N},$

(ii) $\lim_{h\to 1^-} L^{\wedge}_{h,\lambda}(n) = 1, \qquad n \in \mathbb{N}_0.$

Proof:
Part(i) follows from the definition of the Legendre transform and (1.27).

To prove part(ii), from $B_{h,\lambda}(t) \geq 0,\ t \in [-1,1]$, and by using the Second Mean Value Theorem for integration we find

$$\begin{aligned}\lim_{h\to 1^-} L^{\wedge}_{h,\lambda}(n) &= \lim_{h\to 1^-} \frac{2\pi}{B^{\wedge}_{h,\lambda}(0)} \int_{-1}^{1} B_{h,\lambda}(t) P_n(t)\, \mathrm{d}t \\ &= \lim_{h\to 1^-} \frac{2\pi P_n(t_0)}{B^{\wedge}_{h,\lambda}(0)} \int_{h}^{1} B_{h,\lambda}(t)\, \mathrm{d}t,\end{aligned}$$

where $t_0 \in [h,1]$. Therefor, the desired result follows by $P_n(1) = 1$. □

Now, Lemma 2.3.5 and the concept of spherical convolution enable us to introduce a singular integral on the unit sphere such that this singular integral is an approximate identity in $\mathcal{X}(\Omega)$.

Theorem 2.3.6
Let $\lambda > -1$. Suppose that $\{L_{h,\lambda}\}_{h\in(-1,1)}$ is a family of kernels defined by Definition 2.3.4. Then the singular integral I_h, $h \in (-1,1)$, defined by

$$I_h(F) = L_{h,\lambda} * F, \qquad F \in \mathcal{X}(\Omega), \tag{2.50}$$

is an approximate identity in $\mathcal{X}(\Omega)$, i.e.,

$$\lim_{h\to 1^-} \|F - I_h(F)\|_{\mathcal{X}(\Omega)} = 0, \qquad F \in \mathcal{X}(\Omega). \tag{2.51}$$

Next, we would like to find the Legendre transform of $B_{h,\lambda}$ for $\lambda > -1$.

2.3.1 Legendre Transform of Smoothed Haar Functions

Schreiner [92] has developed an explicit expression for the Legendre transform of $B_{h,\lambda}$ for $\lambda \in \mathbb{N}$. In this section, we extend his work thereby assuming $\lambda > -1$. The outline of our approach is as follows: first we extend the smoothed Haar functions to the Euclidean space $\mathbb{R}^3$. Then we compute the Fourier transform of the generalized smoothed Haar functions. By using Theorem 2.2.15 we are able to find the Legendre transform of the generalized smoothed Haar functions. Finally, we come back to the sphere Ω, i.e., we determine the Legendre transform of $B_{h,\lambda}$.

Definition 2.3.7 (Generalized Smoothed Haar Functions on $\mathbb{R}^3$)
For $\lambda > -1$, the generalized smoothed Haar function in $\mathbb{R}^3$ is denoted by Φ_λ and defined by

$$\Phi_\lambda(x) = \phi_\lambda(\|x\|) = (1 - \|x\|^2)_+^\lambda, \quad x \in \mathbb{R}^3, \tag{2.52}$$

where the (truncated power) function $(t)_+$ is defined by

$$(t)_+ = \begin{cases} t & \text{for} \quad t \geq 0 \\ 0 & \text{for} \quad t < 0 \end{cases}. \tag{2.53}$$

It is clear that the support of Φ_λ is

$$\mathit{supp}\ \Phi_\lambda = \{x \in \mathbb{R}^3;\ \|x\|^2 \leq 1\}.$$

Later, we will show how to control the support of Φ_λ.

The following lemma gives us the Fourier transform of the radial function ϕ_λ.

Lemma 2.3.8
Let ϕ_λ be the generalized smoothed Haar functions as defined in (2.52), where $\lambda > -1$. Then the Fourier transform of ϕ_λ is

$$\hat{\phi_\lambda}(r) = 2^\lambda \Gamma(\lambda + 1) \frac{J_{\lambda+\frac{3}{2}}(r)}{r^{\lambda+\frac{3}{2}}}. \tag{2.54}$$

Proof:
We use (2.28) to compute the Fourier transform of ϕ_λ as follows:

$$\begin{aligned}
\hat{\phi}_\lambda(r) &= \frac{1}{\sqrt{r}} \int_0^\infty \phi_\lambda(t) t^{\frac{3}{2}} J_{\frac{1}{2}}(rt)\,\mathrm{d}t \\
&= \frac{1}{\sqrt{r}} \int_0^\infty \phi_\lambda(\frac{u}{r}) \left(\frac{u}{r}\right)^{\frac{3}{2}} J_{\frac{1}{2}}(u)\, \frac{1}{r}\,\mathrm{d}u \\
&= \frac{1}{r^3} \int_0^\infty \left(1 - \frac{u^2}{r^2}\right)_+^\lambda u^{\frac{3}{2}} J_{\frac{1}{2}}(u)\,\mathrm{d}u \\
&= r^{-2\lambda-3}\, I_\lambda(r),
\end{aligned}$$

where

$$I_\lambda(r) = \int_0^r \left(r^2 - u^2\right)^\lambda\; u^{\frac{3}{2}} J_{\frac{1}{2}}(u)\,\mathrm{d}u.$$

This integral may be found in [49, eq. 2.20]:

$$I_\lambda(r) = \frac{\Gamma(\lambda+1)}{2^{3/2}\Gamma(\lambda+\frac{5}{2})} r^{2\lambda+3} {}_1F_2\left(\frac{3}{2};\ \frac{3}{2},\ \lambda+\frac{5}{2};\ \frac{-r^2}{4}\right),$$

where ${}_pF_q$ are the hypergeometric functions (see, e.g., [1] or [102]) defined by

$${}_pF_q(a_1,\ \ldots, a_p;\ b_1,\ \ldots, b_q;\ x) = \sum_{r=0}^\infty \frac{(a_1)_r \ldots (a_p)_r}{(b_1)_r \ldots (b_q)_r} x^r, \tag{2.55}$$

where the Pochhammer's symbol $(a)_r$ is defined by

$$(a)_r = a(a+1)\ldots(a+r-1) \quad r \geq 1, \tag{2.56}$$

and $(a)_0 = 1$. Therefore, we have

$$\hat{\phi}_\lambda(r) = \frac{\Gamma(\lambda+1)}{2^{3/2}\Gamma(\lambda+\frac{5}{2})} {}_1F_2\left(\frac{3}{2};\ \frac{3}{2},\ \lambda+\frac{5}{2};\ \frac{-r^2}{4}\right). \tag{2.57}$$

By using the cancellation rule of the hypergeometric functions

$${}_{p+1}F_{q+1}(a_1,\ \ldots, a_p, c;\ b_1,\ \ldots, b_q, c;\ x) = {}_pF_q(a_1,\ \ldots, a_p;\ b_1,\ \ldots, b_q;\ x),$$

and the formula ([1, eq. 9.1.69])

$${}_0F_1(\nu+1; \frac{-r^2}{4}) = \frac{\Gamma(\nu+1)}{(r/2)^\nu} J_\nu(r),$$

we have

$$\hat{\phi}_\lambda(r) = 2^\lambda \Gamma(\lambda+1) r^{-\frac{3}{2}-\lambda} J_{\lambda+\frac{3}{2}}(r).$$

This is the assertion of this lemma. □

Theorem 2.2.15 together with Lemma 2.3.8 enables us to determine the Legendre transforms of ϕ_λ as follows:

Theorem 2.3.9
Let ϕ_λ be the generalized smoothed Haar functions as defined in (2.52), where $\lambda > -1$. Furthermore, suppose that the integral

$$\int_0^\infty J^2_{n+\frac{1}{2}}(t) \hat{\phi}_\lambda(t) t \, \mathrm{d}t, \tag{2.58}$$

exists, then the Legendre transform of ϕ_λ is given by

$$\phi_\lambda^\wedge(n) = 3^{\frac{\lambda+1}{2}} \pi P_{n,-\lambda-1}(1/2), \quad n \in \mathbb{N}_0. \tag{2.59}$$

Proof:
From Theorem 2.2.15 we have

$$\phi_\lambda^\wedge(n) = (2\pi)^{3/2} \int_0^\infty J^2_{n+\frac{1}{2}}(t) \hat{\phi}_\lambda(t) t \, \mathrm{d}t,$$

for each $n \in \mathbb{N}_0$. By substituting (2.54) in the above relation we get

$$\begin{aligned} \phi_\lambda^\wedge(n) &= (2\pi)^{3/2} \int_0^\infty J^2_{n+\frac{1}{2}}(t) 2^\lambda \Gamma(\lambda+1) \frac{J_{\lambda+\frac{3}{2}}(t)}{t^{\lambda+\frac{3}{2}}} t \, \mathrm{d}t \\ &= (2\pi)^{3/2} 2^\lambda \Gamma(\lambda+1) \int_0^\infty J^2_{n+\frac{1}{2}}(t) J_{\lambda+\frac{3}{2}}(t) t^{1-(\lambda+\frac{3}{2})} \, \mathrm{d}t, \end{aligned}$$

for each $n \in \mathbb{N}_0$. The integral in the last relation is known from [77, Sec. 2.12.42 Eq. 22] and with our notation we obtain

$$\int_0^\infty J^2_{n+\frac{1}{2}}(t) J_{\lambda+\frac{3}{2}}(t) t^{1-(\lambda+\frac{3}{2})} \, \mathrm{d}t = \frac{3^{\frac{\lambda+1}{2}}}{2^{\frac{2\lambda+3}{2}} \sqrt{\pi}} P_{n,-\lambda-1}(1/2), \tag{2.60}$$

for $\lambda > -1$, where $P_{\nu,\mu}$ is the generalized associated Legendre function as defined in (1.34) (the definition of $P_{\nu,\mu}$ with unrestricted μ and ν can be found in, e.g., [62]).

Thus

$$\phi_\lambda^\wedge(n) = 3^{\frac{\lambda+1}{2}} \pi P_{n,-\lambda-1}(1/2), \quad n \in \mathbb{N}_0.$$

□

As we stated before, we would like the support of Φ_λ to be controllable with respect to its size. Therefore, we define the generalized smoothed Haar functions with controllable support as follows:

$$\Phi_{\lambda,\,\rho}(x) = \phi_{\lambda,\,\rho}(\|x\|) = \left(1 - \frac{\|x\|^2}{\rho^2}\right)_+^\lambda, \quad x \in \mathbb{R}^3. \tag{2.61}$$

where $\lambda > -1$ and $\rho \in (0, 2)$.

Let $\rho = \sqrt{2-2h}$ where $h \in (-1, 1)$. Then the restriction of Φ_λ to the sphere Ω is $B_{h,\lambda}$, i.e.,

$$\phi_{\lambda,\,\sqrt{2-2h}}\left(\sqrt{2-2t}\right) = B_{h,\lambda}(t), \quad t \in [-1, 1]. \tag{2.62}$$

In particular, $\phi_\lambda = B_{1/2,\lambda}$. Therefore, from Theorem 2.3.9 we have

Corollary 2.3.10
Let $\lambda > -1$. Then

$$B_{1/2,\lambda}^\wedge(n) = 3^{\frac{\lambda+1}{2}} \pi P_{n,-\lambda-1}(1/2), \quad n \in \mathbb{N}_0. \tag{2.63}$$

It should be noted that in analogy to Theorem 2.3.9, the Legendre transform of $\Phi_{\lambda,\,\sqrt{2-2h}}$ for arbitrary $\lambda > -1$ and $h \in (-1, 1)$ can be detected, but we postpone it for later.

2.4 Zonal Wendland Kernel Functions

In this chapter, we would like to discuss a special class of locally supported positive definite functions on Ω. This class of functions was firstly constructed in the Euclidian space $\mathbb{R}^d$ by Wendland [104], [105],[106]. In this work, we restrict ourselves to functions on the sphere Ω. Our manipulation enables us to control the support of these functions. To define the Wendland functions, we need to introduce the following operators.

Definition 2.4.1 (Operators $\mathcal{I}$ and $\mathcal{D}$)

(i) Let φ be a function such that $t \mapsto t\varphi(t)$ is of the class $\mathcal{L}^1[0,\infty)$. Then we define

$$\mathcal{I}(\varphi)(t) = \int_t^\infty r\varphi(r)\, \mathrm{d}r, \tag{2.64}$$

for every $t \geq 0$.

(ii) Let φ be a function of class $\varphi \in \mathcal{C}^2(\mathbb{R})$. Then we let

$$\mathcal{D}(\varphi)(t) = -\frac{1}{t}\frac{\mathrm{d}}{\mathrm{d}t}\varphi(t) \tag{2.65}$$

for every $t \geq 0$.

In the following, the relation between the operators $\mathcal{I}$ and $\mathcal{D}$ and their Fourier transforms is presented.

Lemma 2.4.2 (Properties of Operators $\mathcal{I}$ and $\mathcal{D}$)
Let Φ be a radial function on $\mathbb{R}^d$, i.e., $\Phi(x) = \varphi(\|x\|)$, $x \in \mathbb{R}^d$. Suppose that Φ is a continuous function on $\mathbb{R}^d$. If $\hat{\varphi}^d$ denotes the Fourier transform of φ on $\mathbb{R}^d$ as defined in (2.28), then the following statements are valid:

(i) If $t \mapsto t\varphi(t)$ is of the class $\mathcal{L}^1[0,\infty)$ then $\mathcal{D}\mathcal{I}\varphi = \varphi$.

(ii) If $\varphi \in \mathcal{C}^2(\mathbb{R})$ and also $\varphi'(t) \in \mathcal{L}^1[0,\infty)$ then $\mathcal{I}\mathcal{D}\varphi = \varphi$.

(iii) If $\Phi \in \mathcal{L}^1(\mathbb{R}^d)$ then $\hat{\varphi}^d = \mathcal{I}\hat{\varphi}^{d-2}$, $d \geq 3$.

(iv) If $\varphi \in \mathcal{C}^2(\mathbb{R})$ and also $t^d\varphi' \in \mathcal{L}^1[0,\infty)$ then $\hat{\varphi}^d = \mathcal{D}\hat{\varphi}^{d-2}$.

(v) If $\Phi \in \mathcal{L}^1(\mathbb{R}^d)$, $d \geq 3$, then Φ is positive definite on $\mathbb{R}^d$ if and only if $\mathcal{I}\varphi$ is positive definite on $\mathbb{R}^{d-2}$.

(vi) If $\varphi \in \mathcal{C}^2(\mathbb{R})$ and also $t^d\varphi' \in \mathcal{L}^1[0,\infty)$ then Φ is positive definite on $\mathbb{R}^d$ if and only if $\mathcal{D}\varphi$ is positive definite on $\mathbb{R}^{d+2}$.

Proof:
See [86] and [107]. □

Indeed, the operators $\mathcal{I}$ and $\mathcal{D}$ are inverse in the sense of Lemma 2.4.2. Moreover, they walk through the space dimension in steps of width 2. An extension

of these operators with steps of arbitrary width can be found in [86] and [109]. In this work, we restrict ourselves to $\mathbb{R}^3$. The general case can be found in [104].
Next, by using the operator that has been defined in (2.4.1), we define the Wendland functions as follows:

Definition 2.4.3 (Wendland functions on $\mathbb{R}^3$)
For $k \in \mathbb{N}_0$, the radial function $\Phi_k : \mathbb{R}^3 \to \mathbb{R}$ is defined by

$$\Phi_k(x) = \varphi_k(\|x\|) = \mathcal{I}^k(1 - \|x\|)_+^{k+2} \tag{2.66}$$

where the function $(t)_+$ is given by (2.53).

From Definition 2.4.3, it is clear that supp $\varphi_k = [0, 1]$.

We next list some important properties of the function φ_k defined in (2.66). Further results can be found in [105] or [104].

Theorem 2.4.4 (Properties of Wendland functions)

(i) *The function Φ_k is positive definite on $\mathbb{R}^3$.*

(ii) *Φ_k is of class $C^{2k}(\mathbb{R}^3)$*

(iii) *φ_k is of the form*

$$\varphi_k(r) = \begin{cases} p_k(r) & \text{for} \quad 0 \le r \le 1 \\ 0 & \text{otherwise} \end{cases} \tag{2.67}$$

with a univariate polynomial p_k of degree 3k+2. The function φ_k is of minimal degree for given smoothness $2k$ on $\mathbb{R}^3$ and is up to a constant factor uniquely determined by this setting.

(iv) *The polynomial p_k in (2.67) has the representation*

$$p_k(r) = \sum_{j=0}^{3k+2} d_{j,k} r^j,$$

where the coefficients are recursively calculable for $0 \leq s \leq k-1$*:*

$$d_{j,0} = (-1)^j \binom{k+2}{j}$$

$$d_{0,s+1} = \sum_{j=0}^{k+2s+2} \frac{d_{j,s}}{j+2}\ , \qquad d_{1,s+1} = 0 \qquad s \geq 0$$

$$d_{j,s+1} = -\frac{d_{j-2,s}}{j}\ , \qquad s \geq 0, \quad 2 \leq j \leq k+2s+4$$

Furthermore, precisely the first k *odd coefficients* $d_{j,k}$ *vanish.*

(v) *There exist constants* $c_1,\ c_2 > 0$*, depending only on* k*, such that the Fourier transforms of* $\Phi_k,\ k \in \mathbb{N}_0$*, satisfy in the following bounds:*

$$\frac{c_1}{(1+r^2)^{k+2}} \leq \hat{\varphi}_k(r) \leq \frac{c_2}{(1+r^2)^{k+2}}, \quad k \in \mathbb{N}_0, \tag{2.68}$$

for each $r \geq 0$*, where* $\Phi_k(x) = \varphi_k(\|x\|)$*,* $x \in \mathbb{R}^3$ *and* $k \in \mathbb{N}_0$*.*

Proof:
See [104]. □

The next section is devoted to the restriction of Wendland functions on the sphere Ω. Moreover, we also discuss the behavior of the Legendre transform of Wendland functions.

2.4.1 Wendland Functions on the Sphere

Because in this work we are interested in the functions on the unit sphere, thus we define

Definition 2.4.5 (Restriction of φ_k to Ω)
For $k \in \mathbb{N}_0$, the restriction of φ_k to the sphere Ω is defined by

$$\phi_k(\xi \cdot \eta) = \varphi_k(\sqrt{2 - 2\xi \cdot \eta}) = \mathcal{I}^k(1 - \sqrt{2 - 2\xi \cdot \eta})_+^{k+2}, \tag{2.69}$$

where ξ and η are elements of the sphere Ω.

k	$\phi_k(t)$
$k=0$	$\phi_0(t) = (1-\sqrt{2-2t})_+^2$
$k=1$	$\phi_1(t) \doteq \left(1-\sqrt{2-2t}\right)_+^4 (4\sqrt{2-2t}+1)$
$k=2$	$\phi_2(t) \doteq \left(1-\sqrt{2-2t}\right)_+^6 \left(18\sqrt{2-2t}-70t+73\right)$
$k=3$	$\phi_3(t) \doteq \left(1-\sqrt{2-2t}\right)_+^8 \left(32\sqrt{(2-2t)^3}+8\sqrt{2-2t}-50t+51\right)$

Table 2.1: The functions $t \mapsto \phi_k(t), \quad k=0,1,2,3$
($\doteq$ denotes equality up to a constant).

Moreover, let $\xi \cdot \eta = t$ then we obtain from (2.69)

$$\phi_k(t) = \varphi_k(\sqrt{2-2t}) = \mathcal{I}^k(1-\sqrt{2-2t})_+^{k+2}, \quad k \in \mathbb{N}_0. \tag{2.70}$$

Clearly, the support of ϕ_k is $[\frac{1}{2}, 1]$. Later we shall see how can we control the support of ϕ_k.

In Table 2.1 the functions $\phi_k, \quad k = 0,1,2,3$, are listed.

Remark 2.4.6
According to Theorem 2.4.4, the functions $\Phi_k, \quad k \in \mathbb{N}_0$ and consequently $\varphi_k, \ k \in \mathbb{N}_0$ are positive definite on $\mathbb{R}^3$. Therefore, by using Remark 2.2.10, the functions $\phi_k, \ k \in \mathbb{N}_0$ are positive definite on the sphere Ω, too. In addition, $\phi_k, \ k \in \mathbb{N}_0$, possesses $2k$ continuous derivatives on the sphere Ω.

Figure 2.2 illustrates the functions $\vartheta \mapsto \phi_k(\cos\vartheta), \quad \vartheta \in [\frac{-\pi}{2}, \frac{\pi}{2}]$, for $k = 0,1,2,3$.

Example 2.4.7
In this example, we would like to discuss φ_0 in more detail. We have

$$\Phi_0(x) = \varphi_0(\|x\|) = (1-\|x\|)_+^2, \quad x \in \mathbb{R}^3. \tag{2.71}$$

By restricting φ_0 to the sphere Ω in the form

$$\phi_0(\xi \cdot \eta) = \varphi_0(\|\xi - \eta\|)|_{\xi,\eta \in \Omega},$$

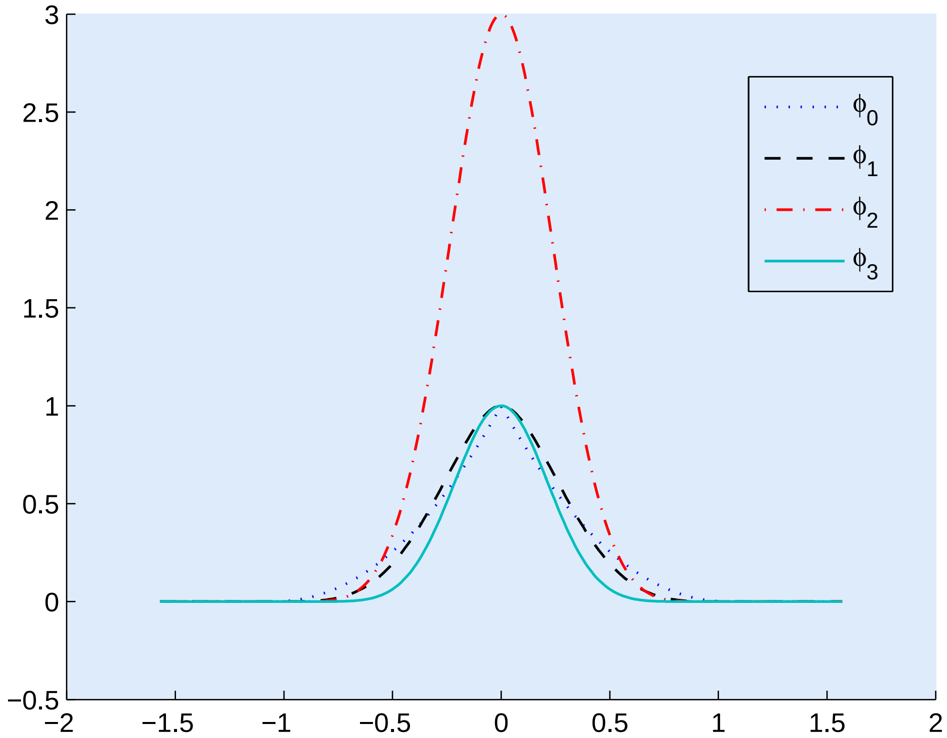


Figure 2.2: The function $\vartheta \mapsto \phi_k(\cos\vartheta)$, $\vartheta \in [\frac{-\pi}{2}, \frac{\pi}{2}]$, for $k = 0, 1, 2, 3$.

we are led to

$$\phi_0(\xi \cdot \eta) = (1 - \|\xi - \eta\|)_+^2 = (1 - \sqrt{2(1 - \xi \cdot \eta)})_+^2, \quad \xi, \eta \in \Omega. \tag{2.72}$$

Setting $t = \xi \cdot \eta$, we, therefore, obtain

$$\phi_0(t) = (1 - \sqrt{2(1-t)})_+^2 = \begin{cases} 3 - 2t - 2\sqrt{2-2t} & \text{for} \quad \frac{1}{2} \le t \le 1 \\ 0 & \text{for} \quad -1 \le t < \frac{1}{2} \end{cases} . \tag{2.73}$$

Figure 2.3 illustrates the function defined in (2.72) for fixed ξ (the North pole) on the sphere Ω.

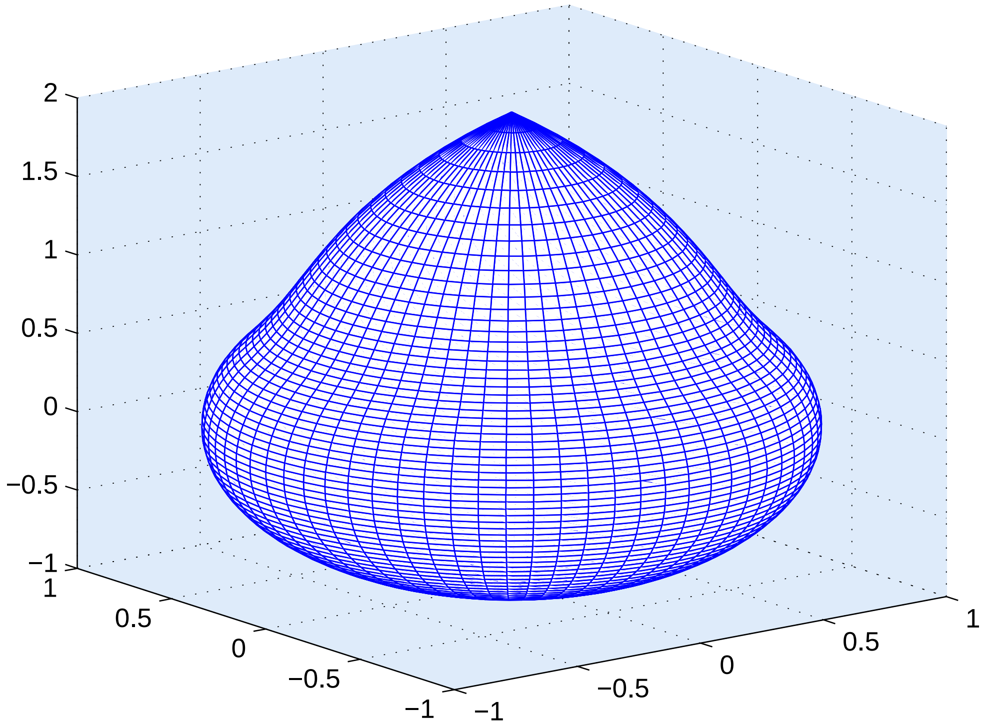


Figure 2.3: The function $\eta \mapsto \phi_0(\xi \cdot \eta)$ on the sphere Ω,
Where ξ is the North pole.

We now turn to the problem of finding the Fourier transforms of φ_0. In the next lemma, for convenience, we simply replace φ_0 by φ.

Lemma 2.4.8
Let φ be the function defined in (2.71). Then the Fourier transform of φ is

$$\hat{\varphi}(r) = \frac{1}{15\sqrt{2\pi}}\, {}_1F_2\left(2;\ 3,\ \frac{7}{2};\ \frac{-r^2}{4}\right), \tag{2.74}$$

where ${}_pF_q$ is the hypergeometric function (see, e.g., [102]) defined in (2.55).

Proof:
We use (2.28) to compute the Fourier transform of φ as follows:

$$\begin{aligned}
\hat{\varphi}(r) &= \frac{1}{\sqrt{r}} \int_0^\infty \varphi(t) t^{\frac{3}{2}} J_{\frac{1}{2}}(rt)\, \mathrm{d}t \\
&= \frac{1}{\sqrt{r}} \int_0^\infty \varphi(\frac{u}{r}) \left(\frac{u}{r}\right)^{\frac{3}{2}} J_{\frac{1}{2}}(u)\, \frac{1}{r}\, \mathrm{d}u \\
&= \frac{1}{r^3} \int_0^\infty \left(1 - \frac{u}{r}\right)_+^2 u^{\frac{3}{2}} J_{\frac{1}{2}}(u)\, \mathrm{d}u \\
&= r^{-5}\, I(r),
\end{aligned}$$

where

$$I(r) = \int_0^r (r-u)^2\ u^{\frac{3}{2}} J_{\frac{1}{2}}(u)\, \mathrm{d}u. \tag{2.75}$$

This integral is done in [23, Sec. 13.1, Eq. 56]:

$$I(r) = \sqrt{\frac{2}{\pi}}\, \frac{r^5}{30}\, {}_1F_2\left(\frac{3}{2},\ 2;\ \frac{3}{2},\ 3,\ \frac{7}{2};\ \frac{-r^2}{4}\right).$$

Therefore, we have

$$\hat{\varphi}(r) = \frac{1}{15\sqrt{2\pi}}\, {}_1F_2\left(2;\ 3,\ \frac{7}{2};\ \frac{-r^2}{4}\right),$$

□

It should be noted here that the integral (2.75) has another representation (see [49, Eq. 2.15]):

$$I(r) = \int_0^r (1 - \cos u)(1 - \cos(r-u))\, \mathrm{d}u.$$

Thus another equivalent form for the Fourier transform of φ can be found:

$$\hat{\varphi}(r) = \frac{2\sqrt{2}}{r^5\sqrt{\pi}}(2r + r\cos r - 3\sin r). \tag{2.76}$$

Remark 2.4.9 (Strictly Positive Definiteness of ϕ_k)
We know that from Theorem 2.4.4, $\Phi_k,\ k \in \mathbb{N}_0$ are positive definite on $\mathbb{R}^3$ and by using Remark 2.2.14 the Fourier transforms of $\Phi_k,\ k \in \mathbb{N}_0$ are also radial functions. In addition, from (2.68), it follows that the function $r \mapsto r^2 \hat{\varphi}_k(t)$ is of the class $\mathcal{L}^1[0,\infty),\ k \in \mathbb{N}_0$ or equivalently $\hat{\Phi}_k \in \mathcal{L}^1(\mathbb{R}^3),\ k \in \mathbb{N}_0$. Therefore, from Theorem 2.2.15, it implies that ϕ_k, the restriction of Φ_k to the sphere Ω, is strictly positive definite on the sphere Ω.

Asymptotic Behavior of Legendre Transform

To find the asymptotic behavior of the Legendre transform of $\phi_k,\ k \in \mathbb{N}_0$, let

$$\Psi_s(x) = \int_{\mathbb{R}^3} \frac{e^{ix\cdot y}}{(1+\|y\|^2)^s}\, \mathrm{d}y, \quad x \in \mathbb{R}^3. \tag{2.77}$$

Clearly, Ψ is strictly positive definite on $\mathbb{R}^3$ and

$$\hat{\Psi}_s(r) = \frac{1}{(1+r^2)^s}, \quad r \geq 0.$$

In [73], it is proved that the Legendre transforms of Ψ have the following asymptotic behavior:

$$\psi_s^\wedge(n) = \mathcal{O}(n^{-2s-\frac{3}{2}}), \tag{2.78}$$

for large $n \in \mathbb{N}_0$, where ψ_s is the restriction of Ψ_s to the sphere Ω. Therefore, from (2.68) and Corollary 2.2.16, it follows that

$$\phi_k^\wedge(n) = \mathcal{O}(n^{-2k-\frac{3}{2}}), \tag{2.79}$$

for large $n \in \mathbb{N}_0$.

Figure 2.4 illustrates the asymptotic behavior of the Legendre transform of ϕ_0.

Remark 2.4.10 (Native Spaces of ϕ_k)
By using (2.68) and from the classical theory of Sobolev spaces (cf., e.g., [95], [84] or [113]), it can be deduced

$$\mathcal{N}_{\varphi_k} = \mathcal{H}_{k+2}(\mathbb{R}^3), \quad k \in \mathbb{N}_0. \tag{2.80}$$

In addition, the asymptotic behavior of the Legendre transform of $\phi_k,\ k \in \mathbb{N}_0$, implies that

$$\mathcal{N}_{\phi_k} = \mathcal{H}_{k+\frac{3}{2}}(\Omega), \quad k \in \mathbb{N}_0, \tag{2.81}$$

where ϕ_k, defined in (2.69), is the restriction of the functions $\Phi_k,\ k \in \mathbb{N}_0$ to the sphere Ω. Therefore, we can conclude that the restriction of the functions $\Phi_k,\ k \in \mathbb{N}_0$ to the sphere Ω changes the space $\mathcal{H}_s(\mathbb{R}^3)$ to $\mathcal{H}_{s-\frac{1}{2}}(\Omega)$.

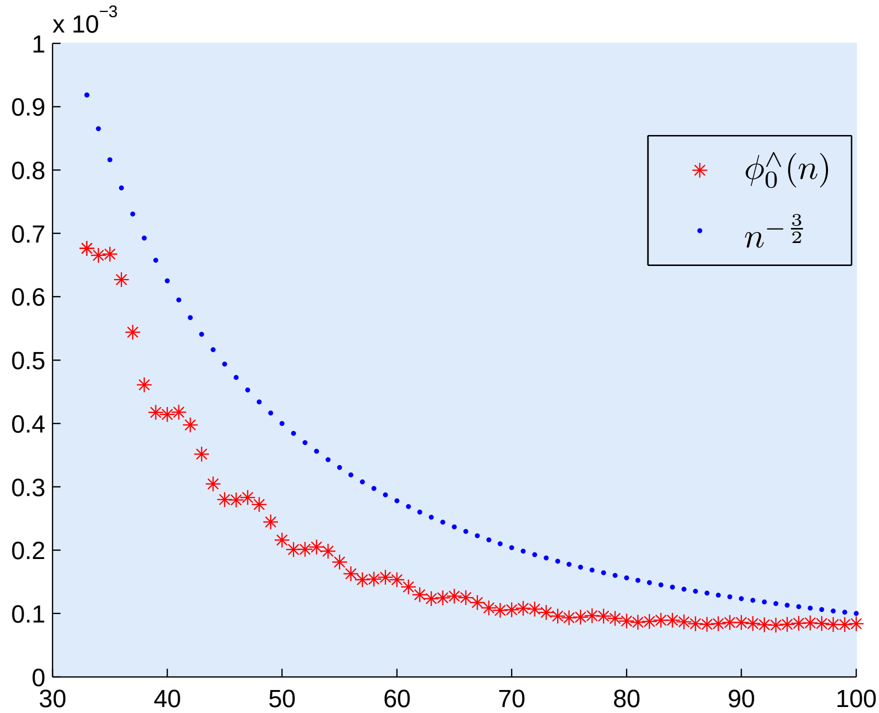


Figure 2.4: Comparison of $\phi_0^\wedge(n)$ and $n^{-\frac{3}{2}}$ for $n = 30, \ldots, 100$.

We here point out that it is possible to perform an error analysis for the spherical spline with the kernel $\phi_k,\ k \in \mathbb{N}_0$ but we postpone it for future work (for detailed accounts on the well-developed theory of spherical spline and its application in geosciences, we refer to [27], [26], [101], [28], [39], [29], and [30]).

Wendland Functions as Scaling Functions

In this section, we would like to interpret the Wendland functions as scaling functions. We start by the following definition.

Definition 2.4.11
For $k \in \mathbb{N}_0$ and $h \in [\frac{1}{2}, \infty)$, the function $\phi_{k,h} : [-1, 1] \to \mathbb{R}$ is defined by

$$\phi_{k,h}(t) = \mathcal{I}^k (1 - h\sqrt{2(1-t)})_+^{k+2}, \tag{2.82}$$

where the operator $\mathcal{I}$ is defined by (2.4.1).

Note that $\phi_{k,h}$ inherits the essential property of ϕ_k. In particular, $\phi_{k,h}$ possesses $2k$ continuous derivatives and also $\phi_{k,h}$ is strictly positive definite on the sphere Ω.

One can see by definition (2.4.11) that

$$\text{supp } \phi_{k,h} = \left[1 - \frac{1}{2h^2}, 1\right],$$

for $k \in \mathbb{N}_0$ and $h \in [\frac{1}{2}, \infty)$. Clearly, if $h_1 < h_2$ then

$$\text{supp } \phi_{k,h_2} \subset \text{supp } \phi_{k,h_1}.$$

Figure 2.5 illustrates the functions $\vartheta \mapsto \phi_{2,h}(\cos\vartheta), \quad \vartheta \in [-\pi, \pi]$, for $h = \frac{1}{2}, 1, 2$.

From now on, because of numerical purposes, we only focus on $\phi_{0,h}$. For simplicity, we denote $\phi_{0,h}$ by ϕ_h. It should be noted that many of the following results are also valid for $\phi_{k,h}, \ k \in \mathbb{N}_0$. In particular, Theorem 2.4.16 is also true for the general case.

From Definition 2.4.11 for $k = 0$, it follows that

$$\phi_h(t) = \begin{cases} 1 + 2h^2 - 2h^2 t - 2h\sqrt{2-2t} & \text{for} \quad 1 - \frac{1}{2h^2} \le t \le 1 \\ 0 & \text{for} \quad -1 \le t < 1 - \frac{1}{2h^2} \end{cases}. \tag{2.83}$$

It is clear that if $t \in [-1, 1]$, then

$$0 \le \phi_h(t) \le 1, \quad h \in [\frac{1}{2}, \infty). \tag{2.84}$$

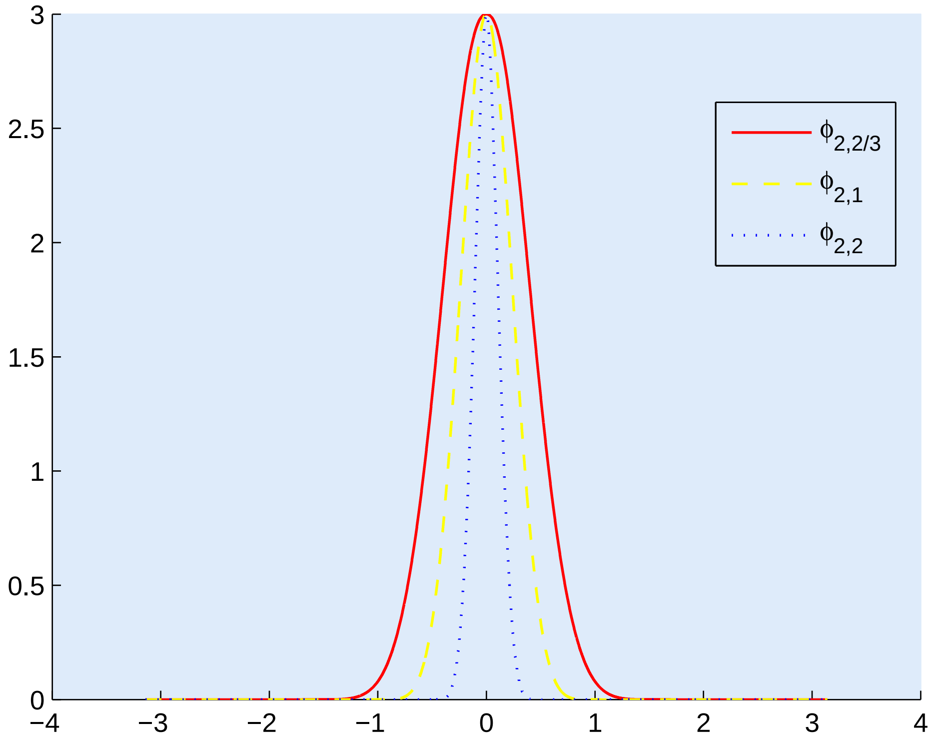


Figure 2.5: The function $\vartheta \mapsto \phi_{2,h}(\cos\vartheta)$, $\vartheta \in [-\pi,\pi]$, for $h = \frac{2}{3}, 1, 2$. Note that $\phi_{2,h}$ possesses four continuous derivatives.

If $t \in \left(1 - \frac{1}{2h^2}, 1\right)$, then we have

$$0 < \phi_h(t) < 1,$$

for $h \in [\frac{1}{2}, \infty)$.

The following lemma shows us that, for a fixed $n \in \mathbb{N}_0$, the Legendre transform ϕ_h is a monotonically decreasing function with respect to variable $h \in [\frac{1}{2}, \infty)$.

Lemma 2.4.12 (Monotonicity of $\phi_h^\wedge(n)$ w.r.t. h)
For any $n \in \mathbb{N}_0$, there exists a value $h_0 \in [1,\infty)$ such that, for every h_1, h_2 with $h_1 \geq h_0$ and $h_2 \geq h_0$

$$h_1 < h_2 \quad \textit{implies} \quad \phi_{h_1}^\wedge(n) > \phi_{h_2}^\wedge(n).$$

Proof:
Let $n \in \mathbb{N}_0$ be fixed. Because the Legendre polynomial, is continuous at $t = 1$ and $P_n(1) = 1$, there is an $\varepsilon > 0$ such that

$$P_n(t) > 0, \quad t \in [1-\varepsilon, 1].$$

Now, we choose h_0 such that supp $\phi_{h_0} \subset [1-\varepsilon, 1]$. Therefore, if $h_0 \leq h_1 < h_2$ we have

$$\begin{aligned}
\phi_{h_1}^{\wedge}(n) - \phi_{h_2}^{\wedge}(n) &= 2\pi \int_{-1}^{1} (\phi_{h_1}(t) - \phi_{h_2}(t)) P_n(t) \, \mathrm{d}t \\
&= 2\pi \int_{1-\frac{1}{2h_1^2}}^{1} (\phi_{h_1}(t) - \phi_{h_2}(t)) P_n(t) \, \mathrm{d}t \quad > 0.
\end{aligned}$$

□

An elementary integration yields

$$\phi_h^{\wedge}(0) = \frac{\pi}{6h^2}, \quad \phi_h^{\wedge}(1) = \pi \frac{10h^2 - 1}{60h^4},$$

for $h \in [\frac{1}{2}, \infty)$.

Our aim is to build a scaling function from ϕ_h. According our definition of singular integrals in Section 1.4, a necessary condition for a family of functions to be a family of scaling functions is that the Legendre transform of order zero of these functions is one. For this reason, we normalize the functions ϕ_h in the sense that its Legendre transform of order zero be one.

Definition 2.4.13 (Normalization of $\phi_h(t)$)
For $h \in [\frac{1}{2}, \infty)$, the function $K_h : [-1, 1] \to \mathbb{R}$ is defined by

$$K_h(t) = \frac{6h^2}{\pi} \phi_h(t).$$

Figure 2.6 illustrates the functions $\vartheta \mapsto K_h(\cos\vartheta)$, $\vartheta \in [-\pi, \pi]$, for different values h.

An immediate result from (2.84) is that $K_h(t)$ is uniformly bounded:

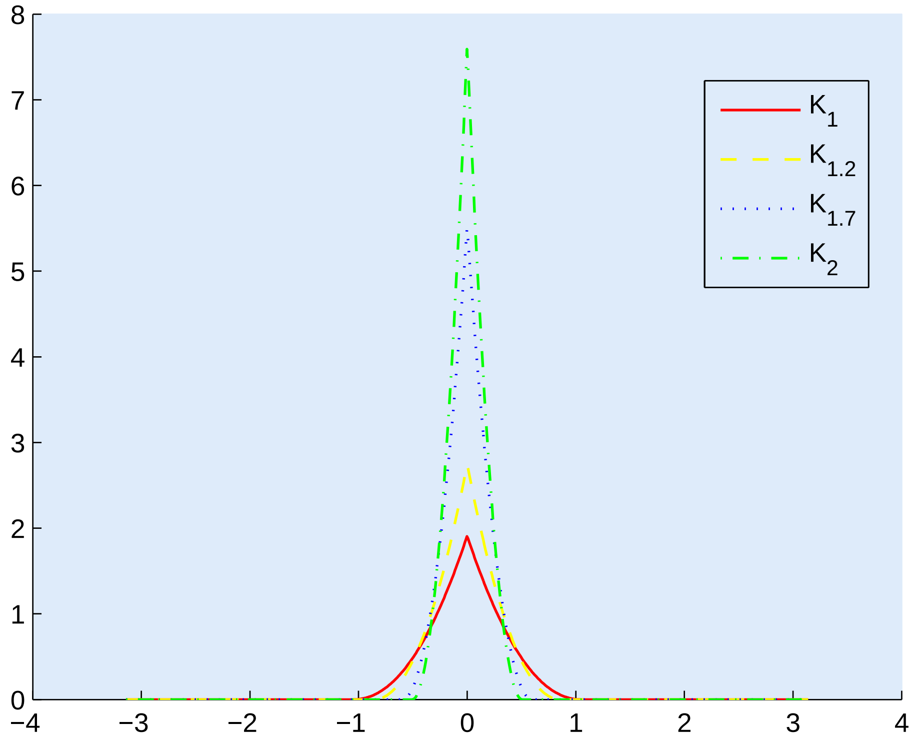


Figure 2.6: The function $\vartheta \mapsto K_h(\cos\vartheta)$, $\vartheta \in [-\pi, \pi]$, for $h = 1, 1.2, 1.7, 2$.

Lemma 2.4.14 (Uniformly Boundedness of K_h)
The function defined in Definition 2.4.13 is uniformly bounded, i.e., there exists a positive constant M, independent of h, such that

$$2\pi \int_{-1}^{1} |K_h(t)| \, \mathrm{d}t \leq M,$$

for $h \in [\frac{1}{2}, \infty)$.

Proof:
Let $h \in [\frac{1}{2}, \infty)$. From (2.84), we can conclude that

$$0 \leq K_h(t) \leq \frac{6h^2}{\pi}, \qquad h \in [\frac{1}{2}, \infty),$$

for $t \in [-1, 1]$. Thus

$$\begin{aligned} 2\pi \int_{-1}^{1} |K_h(t)| \, \mathrm{d}t &= 2\pi \int_{1-\frac{1}{2h^2}}^{1} K_h(t) \, \mathrm{d}t \\ &= K_h^\wedge(0) = 1. \end{aligned}$$

□

Next, we drive a bound for the Legendre transform of K_h. In addition, we show that this bound is the least upper bound. In other words, we prove that the functions $\{K_h\}_{h \in [\frac{1}{2}, \infty)}$ are scaling functions.

Lemma 2.4.15
Let $h \in \left[\frac{1}{2}, \infty\right)$. Then

(i) $\quad 0 < K_h^\wedge(n) < 1, \quad n \in \mathbb{N}$

(ii) $\quad \lim_{h\to\infty} K_h^\wedge(n) = 1, \quad n \in \mathbb{N}_0.$

Proof:
Part(i) is easy to verify, since $\mid P_n(t) \mid < 1, \qquad t \in (-1, 1)$.

To prove part(ii), we have by Definition 2.4.11 that $K_h(t) \geq 0$, then

$$\begin{aligned} \lim_{h\to\infty} K_h^\wedge(n) &= \lim_{h\to\infty} 12h^2 \int_{-1}^{1} \phi_h(t) P_n(t) \, \mathrm{d}t \\ &= \lim_{h\to\infty} 12h^2 \int_{1-\frac{1}{2h^2}}^{1} \phi_h(t) P_n(t) \, \mathrm{d}t \\ &= \lim_{h\to\infty} P_n(t_0) 12h^2 \int_{1-\frac{1}{2h^2}}^{1} \phi_h(t) \, \mathrm{d}t, \end{aligned}$$

where $t_0 \in [1 - \frac{1}{2h^2}, 1]$. The desired result follows from $P_n(1) = 1$ and $\phi_h^\wedge(0) = \frac{\pi}{6h^2}$. □

Figure 2.7 shows the Legendre transform of K_h for different values h.

We arrive at the point that we can realize our aim of this section: By Lemma 2.4.15 and the concept of spherical convolution we can introduce a singular integral on the sphere Ω such that this singular integral is an approximate identity in $\mathcal{X}(\Omega)$.

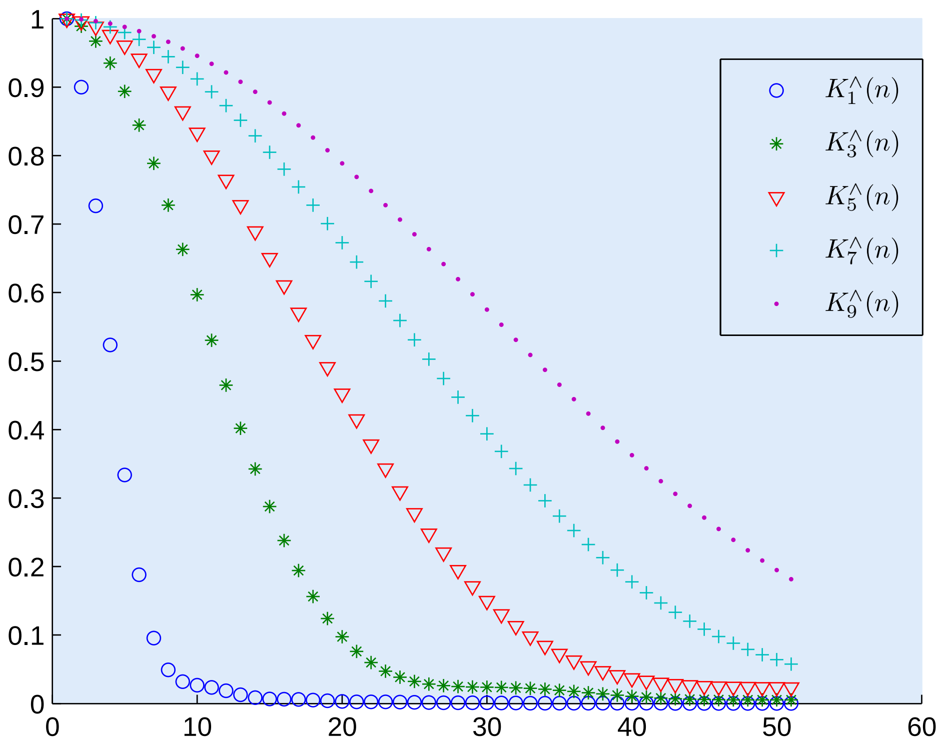


Figure 2.7: The Legendre transform of K_h for $n = 0, \ldots, 50$ and $h = 1,\ 3,\ 5,\ 7,\ 9$.

Theorem 2.4.16
Let K_h be a subfamily defined by Definition (2.4.13). Then the singular integral $I_h,\ h \in [\frac{1}{2}, \infty)$, defined by

$$I_h(F) = K_h * F, \qquad F \in \mathcal{X}(\Omega)$$

is an approximate identity in $\mathcal{X}(\Omega)$, i.e.,

$$\lim_{h \to \infty} \|F - I_h(F)\|_{\mathcal{X}(\Omega)} = 0, \qquad F \in \mathcal{X}(\Omega).$$

Later, we will see that by constructing the so-called *up-function*, it is possible to establish a multiresolution analysis based on the Wendland functions.

2.5 Infinite Spherical Convolution of the Locally Supported Zonal Kernels

Now, we study the so-called *up-function*. The classical variant of the up-function is defined for the one dimensional Euclidean space (see, e.g., [85]) and it was developed for the spherical case by [34], [44], and [92]. The main idea of the up-function is to construct an infinite spherical convolution of locally supported kernels.

In the following, we first define the up-function then we use them to construct the so-called multiresolution analysis based on the smoothed Haar functions (cf. [44]) and the Wendland functions.

Let $\chi_k(x) : \mathbb{R} \to \mathbb{R}$, be given by

$$\chi_k(x) = \begin{cases} 2^k & \text{for} \quad 0 \le x \le 2^{-k} \\ 0 & \quad\text{otherwise} \end{cases},$$

where $k \in \mathbb{N}_0$. Then the up-function is defined by the infinite spherical convolution product

$$Up = \chi_0 * \chi_1 * \dots .$$

It follows from this definition that the up-function is supported in the interval $[0, 2]$ and is infinitely smooth.

Dyn and Ron [22] have used the up-function in a multiresolution analysis on the Euclidean space $\mathbb{R}$ and Freeden et al. [34], Freeden and Schreiner [44] and Schreiner [92] have developed a multiresolution analysis on the unit sphere. Similar to [44], we use the locally supported kernels to build the up-function.

Definition 2.5.1 (Spherical Up-function)
Suppose that $h \in (-1, 1)$ and $0 < q \le \frac{1}{2}$. Let $\varphi_0 = \arccos h$ and $h_i = \cos \varphi_i$, $i \in \mathbb{N}$, where $\varphi_i = q^i \varphi_0$, $i \in \mathbb{N}$. Moreover, assume that $\{\mathcal{K}_{h_i}\}_{i\in\mathbb{N}}$ is a family of locally supported scaling functions such that $0 < \mathcal{K}^{\wedge}_{h_i}(n) \le 1$ and $\operatorname{supp} \mathcal{K}_{h_i} = [h_i, 1]$. We introduce

$$\mathcal{K}^j_h = \mathcal{K}_{h_1} * \mathcal{K}_{h_2} * \dots * \mathcal{K}_{h_j} = \mathop{\ast}_{i=1}^{j} \mathcal{K}_{h_i}.$$

Then the function $\mathcal{K}p_h$ defined by

$$\mathcal{K}p_h = \lim_{j\to\infty} \mathcal{K}_h^j = \mathop{\ast}_{i=1}^{\infty} \mathcal{K}_{h_i}$$

is called spherical up-function.

It should be mentioned that by changing q, it is possible to control "speed decay" of the radius of the support $\mathcal{K}_{h_i}$. Furthermore, because in our study q in Definition 2.5.1 is fixed, for convenience, we don't use q in our notation for the spherical up-function.

Clearly, the function $\vartheta \mapsto \mathcal{K}p_h(\cos\vartheta)$ has the support $[0, \sum_{i=1}^{\infty}\varphi_i] = [0, \frac{\varphi_1}{1-q}]$. Thus supp $\mathcal{K}p_h(t) = [\arccos(\frac{\varphi_1}{1-q}), 1]$.

From (1.61), it follows that

$$(\mathcal{K}_h^j)^\wedge(n) = \prod_{i=1}^{j} \mathcal{K}_{h_i}^\wedge(n), \tag{2.85}$$

for all $n \in N_0$. Therefore, by Theorem 1.4.4, we have

$$\lim_{h\to 1^-} (\mathcal{K}_h^j)^\wedge(n) = 1, \qquad n \in \mathbb{N}_0. \tag{2.86}$$

From Definition 2.5.1 and (2.86), it is clear that

$$\mathcal{K}p_h^\wedge(n) = \prod_{i=1}^{\infty} \mathcal{K}_{h_i}^\wedge(n), \qquad n \in \mathbb{N}_0, \tag{2.87}$$

and

$$\lim_{h\to 1^-} \mathcal{K}p_h^\wedge(n) = 1, \qquad n \in \mathbb{N}_0. \tag{2.88}$$

Moreover, from (2.79) and (2.85), it follows that the Legendre transform of $\mathcal{K}p_h$ decays for $n \to \infty$, faster than any rational function. In other words, $\mathcal{K}p_h(\eta.) \in \mathcal{C}^{(\infty)}(\Omega)$ for every $\eta \in \Omega$.

In addition, $\mathcal{K}p_h$, as a zonal function, can be expressed by the following uniformly convergent series.

$$\mathcal{K}p_h(t) = \sum_{n=0}^{\infty} \frac{2n+1}{4\pi} \mathcal{K}p_h^\wedge(n) P_n(t), \tag{2.89}$$

for all $t \in [-1, 1]$.

Finally, because of (2.88), it is possible to introduce a family of singular integrals with the kernels $\mathcal{K}p_h$ such that this family of singular integrals is an approximate identity in $\mathcal{X}(\Omega)$. The following theorem state this fact precisely.

Theorem 2.5.2
Suppose that $\{\mathcal{K}p_h\}_{h\in(-1,1)}$ is a family of kernels defined by Definition 2.5.1. Then the singular integral I_h, $h \in (-1, 1)$, defined by:

$$I_h(F) = \mathcal{K}p_h * F, \qquad F \in \mathcal{X}(\Omega), \tag{2.90}$$

is an approximate identity in $\mathcal{X}(\Omega)$, i.e.,

$$\lim_{h\to 1^-} \|F - I_h(F)\|_{\mathcal{X}(\Omega)} = 0, \qquad F \in \mathcal{X}(\Omega). \tag{2.91}$$

From the numerical point of view, it is impossible to realize an infinite spherical convolution in the definition of $\mathcal{K}p_h$. To overcome this problem, we have to replace the infinite spherical convolution with the finite one with arbitrary accuracy. In other words, let $\varepsilon > 0$ be arbitrary and for $N \in \mathbb{N}_0$ we split $\mathcal{K}p_h$ as follows:

$$\mathcal{K}p_h(t) = \mathcal{K}_h^N * \mathop{\Huge *}_{i=N+1}^{\infty} \mathcal{K}_{h_i}(t), \quad t \in [-1, 1],$$

then we are interested in finding $N \in \mathbb{N}_0$, if it is possible, such that

$$|\mathcal{K}_h^N(t) - \mathcal{K}p_h(t)| < \varepsilon,$$

for all $t \in [-1, 1]$. Freeden and Schreiner [44] have done it for the smoothed Haar functions as choice for the scaling function. Here, we use their approach for the general cases. We start with a series of lemmata.

Lemma 2.5.3
Suppose that $\mathcal{K}$ is a non-negative locally supported zonal function in $\mathcal{X}(\Omega)$ with $\mathcal{K}^\wedge(0) = 1$ and supp $\mathcal{K} = [h, 1]$. Then, for every $F \in \mathcal{C}(\Omega)$,

$$\|\mathcal{K} * F - F\|_{\mathcal{C}(\Omega)} \le \max_{\xi\cdot\eta\ge h} |F(\eta) - F(\xi)|. \tag{2.92}$$

Proof:
For $\xi \in \Omega$ we have

$$\begin{aligned}
|\mathcal{K} * F(\xi) - F(\xi)| &= \left| \int_\Omega \mathcal{K}(\xi \cdot \eta) F(\eta) \, d\omega(\eta) - F(\xi) \right| \\
&= \left| \int_\Omega \mathcal{K}(\xi \cdot \eta) [F(\eta) - F(\xi)] \, d\omega(\eta) \right| \\
&\leq \int_\Omega \mathcal{K}(\xi \cdot \eta) \, d\omega(\eta) \max_{\xi \cdot \eta \geq h} |F(\eta) - F(\xi)| \\
&= \max_{\xi \cdot \eta \geq h} |F(\eta) - F(\xi)|.
\end{aligned}$$

□

Lemma 2.5.4
Suppose that $\mathcal{K}$ is a non-negative locally supported zonal function in $\mathcal{X}(\Omega)$ with $\mathcal{K}^\wedge(0) = 1$ and supp $\mathcal{K} = [h, 1]$. Assume that $H \in \mathcal{C}(\Omega)$. Then, for every $t \in [-1, 1]$,

$$|\mathcal{K} * H(t) - H(t)| \leq \sqrt{2}\sqrt{1 - h^2} \max_{\tau \in [-1,1]} |H'(\tau)|. \tag{2.93}$$

Proof:
We deduce from the last lemma, that for every $\xi, \eta \in \Omega$

$$|\mathcal{K} * H(\xi \cdot \eta) - H(\xi \cdot \eta)| \leq \max_{\eta \cdot \zeta \geq h} |H(\xi \cdot \eta) - H(\xi \cdot \zeta)|.$$

For $\eta \cdot \zeta \geq h$ we have

$$\begin{aligned}
|\xi \cdot \eta - \xi \cdot \zeta| = |\xi \cdot (\eta - \zeta)| &\leq \sqrt{(\eta - \zeta)^2} \\
&= \sqrt{2 - 2\eta \cdot \zeta} \\
&\leq \sqrt{2}\sqrt{1 - h^2}.
\end{aligned}$$

Hence, the result stated in Lemma 2.5.4 easily follows from the Mean Value Theorem. □

Lemma 2.5.5
Let $\mathcal{K} \in \mathcal{H}_2(\Omega)$, i.e.,

$$\sum_{n=0}^{\infty} \frac{2n + 1}{4\pi} (\mathcal{K}^\wedge(n))^2 \left(n + \frac{1}{2} \right)^4 < \infty.$$

Assume further, that $\mathcal{K}^\wedge(n) \geq 0$ for all $n \in \mathbb{N}_0$. Then $\mathcal{K}$ is continuously differentiable and

$$|\mathcal{K}'(t)| \leq \mathcal{K}'(1) = \sum_{n=0}^{\infty} \frac{2n+1}{4\pi} \frac{n(n+1)}{2} \mathcal{K}^\wedge(n).$$

Proof:
It follows from the Sobolev Lemma 1.3.3 that $\mathcal{K}$ is continuously differentiable. Furthermore, we obtain the uniformly convergent series

$$\begin{aligned} |\mathcal{K}'(t)| &= \left| \sum_{n=0}^{\infty} \frac{2n+1}{4\pi} \mathcal{K}^\wedge(n) P_n'(t) \right| \\ &\leq \sum_{n=0}^{\infty} \frac{2n+1}{4\pi} \mathcal{K}^\wedge(n) P_n'(1) \\ &= \sum_{n=0}^{\infty} \frac{2n+1}{4\pi} \frac{n(n+1)}{2} \mathcal{K}^\wedge(n) \\ &= \mathcal{K}'(1), \end{aligned}$$

□

By using the above results, we are able to prove our promised result as follows:

Theorem 2.5.6
Let $\mathcal{K}p_h$ and $\mathcal{K}_h^j$ be the same functions as defined by Definition 2.5.1. For a given $\varepsilon > 0$ choose $N \in \mathbb{N}_0$ such that $\mathcal{K}_h^N \in \mathcal{H}_2(\Omega)$ and

$$\sqrt{1 - h_N^2} \leq \frac{\varepsilon}{\sqrt{2} \frac{\mathrm{d}}{\mathrm{d}t} \mathcal{K}_h^N(1)}.$$

Then

$$|\mathcal{K}_h^N(t) - \mathcal{K}p_h(t)| < \varepsilon,$$

for all $t \in [-1, 1]$.

Proof:
The proof follows easily by applying Lemma 2.5.4. □

By using the spherical up-function, we are able to make a multiresolution analysis. In the next two sections, we do it for the smoothed Haar functions and Wendland functions. A multiresolution analysis based on the smoothed Haar functions is known from [34], [44], and [92], but a multiresolution analysis based on Wendland functions seems to be new.

2.5.1 Multiresolution Analysis by Means of Up-function

As we mentioned before in Section 2.4.1, if the Legendre transforms of the locally supported scaling functions are not monotonically decreasing with respect to h, then it is impossible to have a multiresolution analysis based on the locally supported scaling functions. To prevail this trouble, we use the concept of the up-function.

Definition 2.5.7 (Complementary of $\mathcal{K}_h^j$)
Suppose that $\{\mathcal{K}_{h_i}\}_{i\in\mathbb{N}}$ is a family of locally supported scaling functions as defined in Definition 2.5.1. We define

$$\bar{\mathcal{K}}_h^j = \mathcal{K}_{h_{j+1}} * \mathcal{K}_{h_{j+2}} * \ldots = \mathop{\ast}_{i=j+1}^{\infty} \mathcal{K}_{h_i}. \tag{2.94}$$

By Definition 2.5.1, it follows that

$$\bar{\mathcal{K}}_h^0 = \mathcal{K}p_h.$$

Moreover, we have

$$\mathcal{K}_h^j * \bar{\mathcal{K}}_h^j = \mathcal{K}p_h.$$

It is also clear that supp $\bar{\mathcal{K}}_h^j = [\arccos(\frac{\varphi_{j+1}}{1-q}), 1]$ and we have the refinement equation

$$\bar{\mathcal{K}}_h^{j+1} * \mathcal{K}_{h_{j+1}} = \bar{\mathcal{K}}_h^j \tag{2.95}$$

Similarly to Theorem 2.5.2, we are able to define a singular integral based on $\bar{\mathcal{K}}_h^j$ as follows:

Theorem 2.5.8
Let $\bar{\mathcal{K}}_h^j$ defined by (2.94). Then the singular integral $\mathcal{I}_j$ defined by

$$\mathcal{I}_j(F) = \bar{\mathcal{K}}_h^j * F, \qquad F \in \mathcal{L}^2(\Omega)$$

is an approximate identity in $\mathcal{L}^2(\Omega)$*, i.e.,*

$$\lim_{j\to\infty} \|F - \mathcal{I}_j(F)\|_{\mathcal{L}^2(\Omega)} = 0, \qquad F \in \mathcal{L}^2(\Omega).$$

Proof:
We show that

$$\lim_{j\to\infty} (\bar{\mathcal{K}}_h^j)^\wedge(n) = 1,$$

for all $n \in \mathbb{N}_0$. By using Definition 2.5.7, we have

$$\lim_{j\to\infty} (\bar{\mathcal{K}}_h^j)^\wedge(n) = \lim_{j\to\infty} \frac{\mathcal{K}p_{h_1}^\wedge(n)}{(\mathcal{K}_{h_1}^{j-1})^\wedge(n)} = \frac{\mathcal{K}p_{h_1}^\wedge(n)}{\mathcal{K}p_{h_1}^\wedge(n)} = 1. \tag{2.96}$$

Recall that, according to Definition 2.5.7, $(\mathcal{K}_{h_1}^{j-1})^\wedge(n) \neq 0,\ j \in \mathbb{N}$. □

Since $(\bar{\mathcal{K}}_h^j)^\wedge(n) \leq (\bar{\mathcal{K}}_h^{j+1})^\wedge(n)$, then it follows that

$$\|\bar{\mathcal{K}}_h^j * F\|_{\mathcal{L}^2(\Omega)} \leq \|\bar{\mathcal{K}}_h^{j+1} * F\|_{\mathcal{L}^2(\Omega)} \tag{2.97}$$

for every $F \in \mathcal{L}^2(\Omega)$. Moreover, by *Young's inequality* (cf., e.g., [34] or [55]) and the fact $(\bar{\mathcal{K}}_h^j)^\wedge(0) = 1$ we have

$$\|\bar{\mathcal{K}}_h^j * F\|_{\mathcal{L}^2(\Omega)} \leq \|F\|_{\mathcal{L}^2(\Omega)}, \tag{2.98}$$

for every $F \in \mathcal{L}^2(\Omega)$.

Now, we introduce the linear bounded operator $T_j : \mathcal{L}^2(\Omega) \to \mathcal{L}^2(\Omega)$ by

$$T_j(F) = \mathcal{I}_{h_j}(F) = \bar{\mathcal{K}}_h^j * F, \quad j \in \mathbb{N}_0. \tag{2.99}$$

From Theorem 2.5.8, it follows that $T_j(f)$ is an approximation of F at the scale j. Thus $\bar{\mathcal{K}}_h^j$ can be interpreted as a low-pass filter and consequently we can introduce the scale spaces V_j as follows:

$$V_j = \{T_j(F) | F \in \mathcal{L}^2(\Omega)\}, \quad j \in \mathbb{N}_0. \tag{2.100}$$

In other words, the scale space V_j is the image of $\mathcal{L}^2(\Omega)$ under the operator T_j. Finally, the scale space V_j defines a *multiresolution analysis(MRA)* in the following sense:

Theorem 2.5.9 (MRA by Mean of the Up-function)
Let $\{h_j\},\ j \in \mathbb{N}_0$ *be a monotonically increasing sequence in* $(-1, 1)$*. The family of scale spaces*

$$V_j = \{T_j(F) = \bar{\mathcal{K}}_h^j * F | F \in \mathcal{L}^2(\Omega)\}, \qquad j \in \mathbb{N}_0$$

defines a multiresolution of $\mathcal{L}^2(\Omega)$ *in the following sense:*

(i) $V_j \subset V_{j'} \subset \mathcal{L}^2(\Omega), \qquad j < j',\ j, j' \in \mathbb{N}_0$

(ii) $\overline{\bigcup_{j=0}^{\infty} V_j} = \mathcal{L}^2(\Omega)$

(iii) $\bigcap_{j=0}^{\infty} V_j = V_0$

Proof:
The statement (i) follows from (2.97) and (2.98). The statement (ii) follows from Theorem 2.5.8 and (iii) is trivial. □

Now, we have a multiresolution analysis based on scaling functions $\bar{\mathcal{K}}_h^j$, where these scaling functions, as said before, are a sequence of low-pass filters. The difference between two low-pass filters gives us a band-pass filter. More precisely, based on the refinement equation (2.95), we can construct locally supported spherical wavelets. More details about this kind of wavelets can be found in [44].

2.5.2 Examples

We present two examples for the MRI by means of the up-function. In the first example, we use the smoothed Haar functions as the scaling functions and in the second example, we use the Wendland's functions as the scaling functions. The first example is known from [44], but the second one seems to be new.

Multiresolution Analysis Based on Infinite Spherical Convolution of Smoothed Haar Functions

We use the iterated spherical convolution of the normalized smoothed Haar functions defined by Definition 2.3.4 to construct a multiresolution analysis. We follow the work by Freeden and Schreiner [44].

Definition 2.5.10 (Up-function Based on Smoothed Haar Functions) Let $L^{(2)}_{h,\lambda}$ be the iterated spherical convolution of the normalized smoothed Haar functions defined by Definition 2.3.4. Suppose that $h \in (-1,1)$ and $\lambda > -1$. Let $\varphi_0 = \arccos h$. We introduce

$$\varphi_i = 2^{-i}\varphi_0,\ h_i = \cos\frac{\varphi_i}{2},\ i = 1, 2, \ldots,$$

and

$$U^j_{h,\lambda} = L^{(2)}_{h_1,\lambda} * L^{(2)}_{h_2,\lambda} * \ldots L^{(2)}_{h_j,\lambda}, \quad j \in \mathbb{N}.$$

Then $\mathit{Up}_{h,\lambda}$ defined by

$$\mathit{Up}_{h,\lambda} = \lim_{j\to\infty} U^j_{h,\lambda} = L^{(2)}_{h_1,\lambda} * L^{(2)}_{h_2,\lambda} * \ldots = \mathop{\ast}_{i=1}^{\infty} L^{(2)}_{h_i,\lambda}$$

is called up-function based on the smoothed Haar functions. We also define the complementary of $U^j_{h,\lambda}$ by

$$\bar{U}^j_{h,\lambda} = L^{(2)}_{h_{j+1},\lambda} * L^{(2)}_{h_{j+2},\lambda} * \ldots = \mathop{\ast}_{i=j+1}^{\infty} L^{(2)}_{h_i,\lambda}, \quad j \in \mathbb{N}.$$

Figure 2.8 shows the functions $\bar{U}^j_{h,\lambda}$ for different values of j.

Because the support of $L^{(2)}_{h_i,\lambda}(t)$ is $[h_i, 1]$, it follows that supp $\mathit{Up}_{h,\lambda} = [h, 1]$. Moreover, in [44] is shown that for every $F \in \mathcal{L}^2(\Omega)$

$$\lim_{j\to\infty} \|F - \bar{U}^j_{h,\lambda} * F\| = 0.$$

Therefore, the operators $T_j : \mathcal{L}^2(\Omega) \to \mathcal{L}^2(\Omega)$ defined by

$$T_j(F) = \mathcal{I}_j(F) = \bar{U}^j_{h,\lambda} * F, \quad j \in \mathbb{N}_0,\ \lambda > -1$$

construct a family of approximate identity on $\mathcal{L}^2(\Omega)$. In addition, if we define the scaling spaces V_j as follows:

$$V_j = \left\{T_j(F) | F \in \mathcal{L}^2(\Omega)\right\},$$

for all $j \in \mathbb{N}_0$, then we get a multiresolution analysis of $\mathcal{L}^2(\Omega)$ in the following sense:

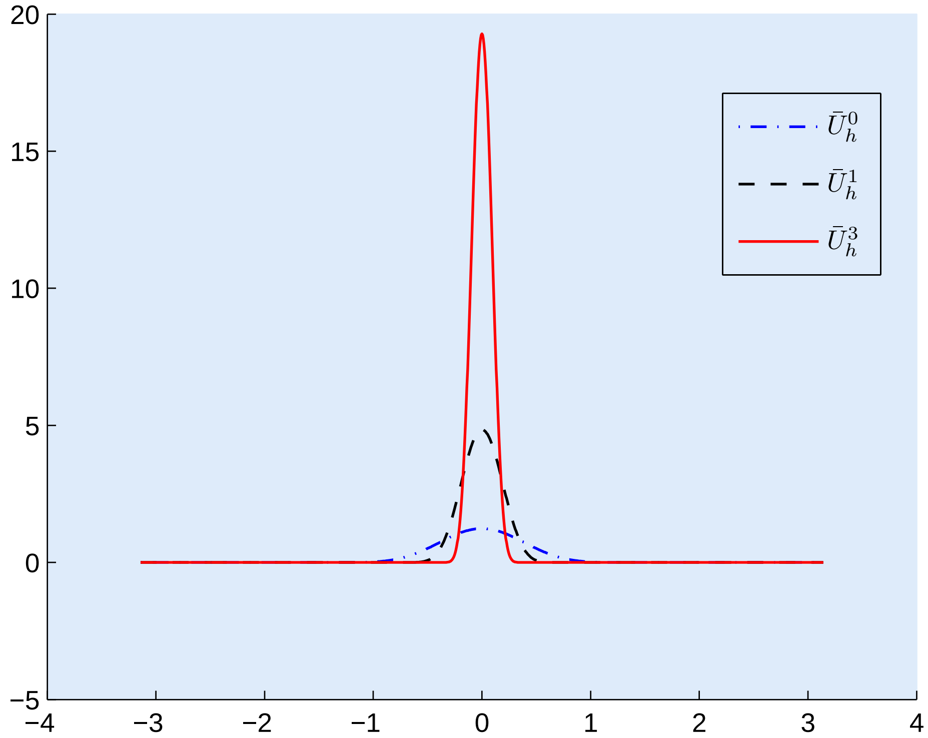


Figure 2.8: The functions $\vartheta \mapsto \bar{U}^j_{h,\lambda}(\cos\vartheta)$, $\vartheta \in [-\pi/2, \pi/2]$, for $j = 0, 1, 3$, and $\lambda = 1$, where $h_j = \cos(\frac{\pi}{2^j 3})$ for $j \in \mathbb{N}_0$.

(i) $V_j \subset V_{j'} \subset \mathcal{L}^2(\Omega), \qquad j < j', \qquad j, j' \in \mathbb{N}_0$

(ii) $\overline{\bigcup_{j=0}^{\infty} V_j} = \mathcal{L}^2(\Omega)$

(iii) $\bigcap_{j=0}^{\infty} V_j = V_0$

Multiresolution Analysis Based on Infinite Spherical Convolution of Wendland Functions

As we stated before in Subsection 2.4.1, the Legendre transforms of the Wendland's function are not monotonically decreasing with respect to h. Therefore, it is impossible to construct a multiresolution analysis of $\mathcal{L}^2(\Omega)$ by using the Wendland functions. To prevail this trouble, we use the concept of the up-function introduced in Subsection 2.5.1.

Definition 2.5.11 (Up-function Based on Wendland Functions)
Let $h \in [\frac{1}{2}, \infty)$ and $0 < q \leq \frac{1}{2}$ and $\varphi_0 = \arccos(1 - \frac{1}{2h^2})$ and $h_i = (\sqrt{2 - 2\cos\varphi_i})^{-1}$, where $\varphi_i = q^i \varphi_0$, $i \in \mathbb{N}$. Suppose that $\{K_{h_i}\}_{i \in \mathbb{N}_0}$ is a family of the Wendland scaling functions defined in (2.4.13). We introduce

$$K_h^j = K_{h_1} * K_{h_2} * \ldots K_{h_j}, \quad j \in \mathbb{N}.$$

Then Kp_h defined by

$$Kp_h = \lim_{j \to \infty} K_h^j = K_{h_1} * K_{h_2} * \ldots = \mathop{\ast}_{i=1}^{\infty} K_{h_i}$$

is called up-function based on the Wendland functions. We also define the complementary of K_h^j by

$$\bar{K}_h^j = K_{h_{j+1}} * K_{h_{j+2}} * \ldots = \mathop{\ast}_{i=j+1}^{\infty} K_{h_i},$$

for all $j \in \mathbb{N}_0$.

Clearly, the function $\vartheta \mapsto Kp_h(\cos\vartheta)$ has the support $[0, \sum_{i=1}^{\infty} \varphi_i] = [0, \frac{q\varphi_0}{1-q}]$. Thus supp $Kp_h(t) = [\arccos(\frac{q\varphi_0}{1-q}), 1]$. Similarly, supp $\bar{K}_h^j(t) = [\arccos(\frac{q\varphi_j}{1-q}), 1]$.

Figure 2.9 shows the functions $\bar{K}_h^j$ for different values of j.

It is clear that we have the refinement equation

$$\bar{K}_h^{j+1} * K_{h_{j+1}} = \bar{K}_h^j \tag{2.101}$$

Because $\{\bar{K}_h^j\}_{j \in \mathbb{N}_0}$ is a family of scaling functions, we can define an approximate identity based on $\bar{K}_h^j$ in $\mathcal{L}^2(\Omega)$ as follows:

$$\mathcal{I}_j(F) = \bar{K}_h^j * F, \qquad F \in \mathcal{L}^2(\Omega)$$

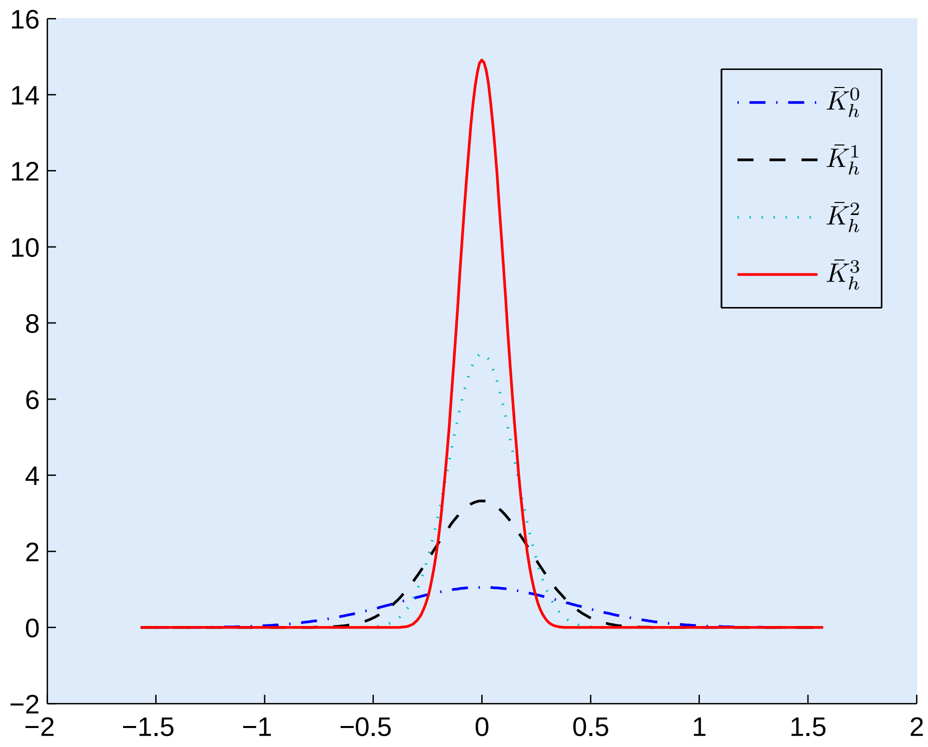


Figure 2.9: The functions $\vartheta \mapsto \bar{K}_h^j(\cos\vartheta)$, $\vartheta \in [-\pi/2, \pi/2]$, for $j = 0, 1, 2, 3$, where $h_j = j + 1$ for $j \in \mathbb{N}_0$. It shows also when $h \to \infty$ then $\bar{K}_h^j$ converges to the *Dirac delta* function.

Similar to last example, if we define the scaling spaces V_j as follows:

$$V_j = \left\{ \bar{K}_h^j * F | F \in \mathcal{L}^2(\Omega) \right\},$$

for all $j \in \mathbb{N}_0$, then we get a multiresolution analysis of $\mathcal{L}^2(\Omega)$ in the following sense:

(i) $V_j \subset V_{j'} \subset \mathcal{L}^2(\Omega), \qquad j < j', \qquad j, j' \in \mathbb{N}_0$

(ii) $\overline{\bigcup_{j=0}^{\infty} V_j} = \mathcal{L}^2(\Omega)$

(iii) $\bigcap_{j=0}^{\infty} V_j = V_0$

Finally, we mentioned that based on the refinement equation (2.101), we are able to construct locally supported spherical wavelets based on Wendland functions. We postpone this topic for later work.

2.6 Spherical Difference Wavelets

In this section, our interest is to introduce the so-called spherical difference wavelets. These kinds of wavelets are developed, for the first time, by Freeden and Schreiner [40]. The idea of construction of these wavelets is as follows: As we stated before in Theorem 1.4.4 a family of non-negative scaling functions $\{K_h\}_{h\in(-1,1)}$ can generate an approximate identity. This approximate identity provides nothing else than a sequence of low-pass filters. The difference between these low pass filters provide band pass filters. We consider these band pass filters as the spherical difference wavelets. The presentation in this section follows the paper by Freeden and Hesse [35].

In Subsection 2.6.1 the basic definitions will be given and the decomposition and the reconstruction of the approximation of F with spherical difference wavelets will be developed. In Subsection 2.6.2 the spherical difference wavelets will be computed for the generalized smoothed Haar kernels introduced in the last section.

2.6.1 Decomposition and Reconstruction Formula

Definition 2.6.1 (Spherical Difference Wavelets)
Let $\{I_h\}_{h\in(-1,1)}$ be an approximate identity in $\mathcal{C}(\Omega)$ or $\mathcal{L}^2(\Omega)$, generated by the scaling function $\{K_h\}_{h\in(-1,1)} \subset \mathcal{L}^1[-1,1]$, and $\{K_h\}_{h\in(-1,1)} \subset \mathcal{L}^2[-1,1]$, respectively. Suppose that $\{h_j\}_{j\in\mathbb{N}_0} \subset (-1,1]$ is a strict monotonically increasing sequence with $\lim_{j\to\infty} h_j = 1$. Define the sequence $\{T_j\}_{j\in\mathbb{N}_0}$ of bounded linear operators

$$T_j : \mathcal{X}(\Omega) \to \mathcal{X}(\Omega),\ F \mapsto T_j(F) = I_{h_j}(F) = K_{h_j} * F,$$

where $\mathcal{X}(\Omega) = \mathcal{C}(\Omega)$ for $\{K_h\}_{h\in(-1,1)} \subset \mathcal{L}^1[-1,1]$, and $\mathcal{X}(\Omega) = \mathcal{L}^2(\Omega)$ for $\{K_h\}_{h\in(-1,1)} \subset \mathcal{L}^2[-1,1]$, respectively. The family $\{\Psi_j\}_{j\in\mathbb{N}_0} \subset \mathcal{L}^1[-1,1]$, and $\{\Psi_j\}_{j\in\mathbb{N}_0} \subset \mathcal{L}^2[-1,1]$, respectively, given by

$$\Psi_j = K_{h_{j+1}} - K_{h_j} \tag{2.102}$$

is called *spherical difference wavelet* corresponding to the scaling function $\{K_{h_j}\}_{j\in\mathbb{N}_0}$. Furthermore, define a family $\{R_j\}_{j\in\mathbb{N}_0}$ of bounded linear operators

$$R_j : \mathcal{X}(\Omega) \to \mathcal{X}(\Omega),\ F \mapsto R_j(F) = \Psi_j * F.$$

Remark 2.6.2 (Locally Supported Spherical Difference Wavelets)
Note that it is also possible to define the locally supported spherical difference wavelet based on the locally supported scaling functions. To be more precise, let $\{K_h\}_{h\in(-1,1)} \subset \mathcal{L}^1[-1,1]$ or $\{K_h\}_{h\in(-1,1)} \subset \mathcal{L}^2[-1,1]$ be a locally supported scaling function, and let $\{h_j\}_{j\in\mathbb{N}_0} \subset (-1,1]$ be a strict monotonically increasing sequence with $\lim_{j\to\infty} h_j = 1$. Then $\{\Psi_j\}_{j\in\mathbb{N}_0}$, defined by

$$\Psi_j = K_{h_{j+1}} - K_{h_j}, \quad j \in \mathbb{N}_0$$

is called a *locally supported spherical difference wavelet* corresponding to the locally supported scaling function $\{K_{h_j}\}_{j\in\mathbb{N}_0}$. The operators T_j and R_j for the locally supported difference wavelets are the same as Definition 2.6.1.

From Section 1.4, it is clear that for each family of non-negative scaling functions $\{K_h\}_{h\in(-1,1)}$ we have $(K_h)^\wedge(0) = 1$. Therefore, the spherical difference wavelets satisfy in the zero mean property of wavelets as follows:

Lemma 2.6.3 (Zero Mean Property of Difference Wavelets)
Let $\{\Psi_j\}_{j\in\mathbb{N}_0}$ be the spherical difference wavelets defined in Definition 2.6.1 and Remark 2.6.2. Then these wavelets satisfy the zero mean property of wavelets:

$$\int_{-1}^{1} \Psi_j(t)\ \mathrm{d}t = 0$$

for all $j \in \mathbb{N}_0$.

The next theorem shows, that the low-pass filter T_J, $J \in \mathbb{N}_0$, can be decomposed into a sum of the low-pass filter T_{J_0} and the band-pass filters R_j, $j \in \{J_0, J_0+1, \ldots, J-1\}$ and, thus, be reconstructed as a sum of the latter.

Theorem 2.6.4 (Decomposition and Reconstruction Formula)
Let the assumptions and the notation be as in Definition 2.6.1. Then,

$$\lim_{J\to\infty} \|F - T_J(F)\|_{\mathcal{X}(\Omega)} = 0,$$

for all $F \in \mathcal{X}(\Omega)$*, where*

$$T_J(F) = T_{J-1}(F) + R_{J-1}(F) = T_{J_0}(F) + \sum_{j=J_0}^{J-1} R_j(F) \tag{2.103}$$

for all $J, J_0 \in \mathbb{N}_0$*,* $0 \leq J_0 < J$*.*

Equation (2.103) is called a reconstruction of the approximation $T_J(F)$*. Particularly,*

$$F = T_{J_0}(F) + \sum_{j=J_0}^{\infty} R_j(F) \tag{2.104}$$

in $\mathcal{X}(\Omega)$*-sense.*

Proof:
Equation (2.103) is a consequence of the definitions of the operators T_j and R_j:

$$T_J(F) = K_{h_J} * F = (K_{h_{J-1}} + \Psi_{J-1}) * F = T_{J-1}(F) + R_{J-1}(F).$$

This proves the first equality. The second equality follows analogously by repeating this process for $T_{J-1}(F), \ldots, T_{J_0+1}(F)$. Equation (2.104) is a consequence of Equation (2.103) and the fact that $T_J(F)$ converges to F (in $\mathcal{X}(\Omega)$) for $J \to \infty$. □

In the following, the spherical difference wavelets are computed for the locally supported scaling functions defined by Definition 2.3.4.

2.6.2 Locally Supported Difference Wavelets Based on Normalized Smoothed Haar Kernels

Let $\lambda \in \mathbb{N}_0$. To emphasize the assumptions that λ is an integer, we denote the normalized smoothed Haar scaling functions defined in Definition 2.3.4 by $\{L_{h_j,k}\}_{j\in\mathbb{N}_0}$. Moreover, suppose that $h_j = 1 - \frac{1}{2^j}$ then the normalized smoothed Haar scaling functions $\{L_{h_j,k}\}_{j\in\mathbb{N}_0} \subset \mathcal{C}^{k-1}[-1,1]$, $k \in \mathbb{N}_0$, are

$$L_{h_j,k}(t) = \begin{cases} 0 & \text{for} \quad t \in [-1, 1 - \frac{1}{2^j}) \\ \frac{2^{j(k+1)-1}(k+1)}{\pi}(t - 1 + \frac{1}{2^j})^k & \text{for} \quad t \in [1 - \frac{1}{2^j}, 1] \end{cases} .$$

The corresponding smoothed Haar wavelets are the families $\{\Psi_{j,k}\}_{j\in\mathbb{N}_0} \subset \mathcal{C}^{k-1}[-1,1]$, $k \in \mathbb{N}_0$, of functions $\Psi_{j,k} : [-1,1] \to \mathbb{R}$, $t \mapsto \Psi_{j,k}(t)$, given by

$$\Psi_{j,k}(t) = \begin{cases} 0 & \text{for } t \in [-1, 1-\frac{1}{2^j}) \\ -\frac{2^{j(k+1)-1}(k+1)}{\pi}(t-1+\frac{1}{2^j})^k & \text{for } t \in [1-\frac{1}{2^j}, 1-\frac{1}{2^{j+1}}) \\ \frac{2^{j(k+1)-1}(k+1)}{\pi}(2^{k+1}(t-1+\frac{1}{2^{j+1}})^k - & \\ (t-1+\frac{1}{2^j})^k) & \text{for } t \in [1-\frac{1}{2^{j+1}}, 1] \end{cases} .$$

In Figure 2.10 we show Haar scaling functions $L_{h_j,0}$ for $j = 1,2,3$ and the corresponding Haar wavelets $\Psi_{j,0}$ for $j = 1,2,3$. Figure 2.11 illustrates smoothing Haar scaling functions $L_{h_j,2}$ for $j = 1,2,3$ and the corresponding smoothing Haar wavelets $\Psi_{j,2}$ for $j = 1,2,3$.

Remark 2.6.5
We point out that, the formulation of difference wavelets for the Wendland functions is straightforward and is omitted.

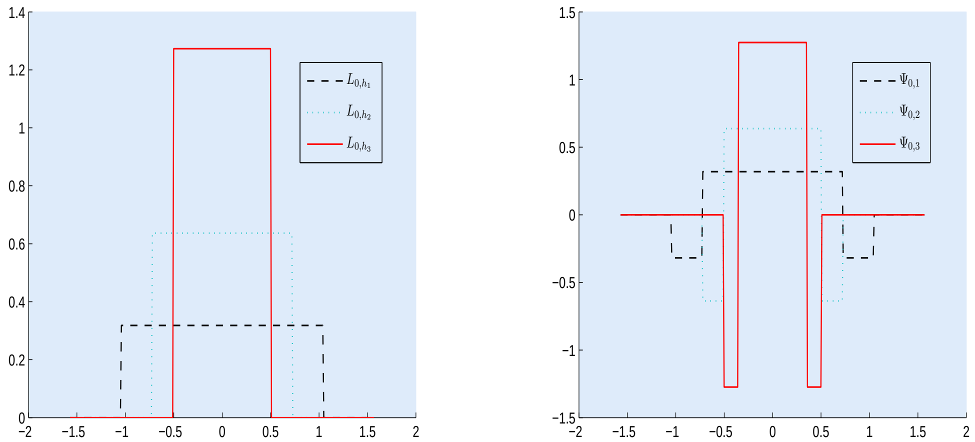


Figure 2.10: The normalized Haar functions $L_{h_j,0}$ and the Haar wavelets $\Psi_{j,0}$.
Left: $\vartheta \mapsto L_{h_j,0}(\cos\vartheta)$, $\vartheta \in [\frac{-\pi}{2}, \frac{\pi}{2}]$ for $j = 1,2,3$.
Right: $\vartheta \mapsto \Psi_{j,0}(\cos\vartheta)$, $\vartheta \in [\frac{-\pi}{2}, \frac{\pi}{2}]$ for $j = 1,2,3$.

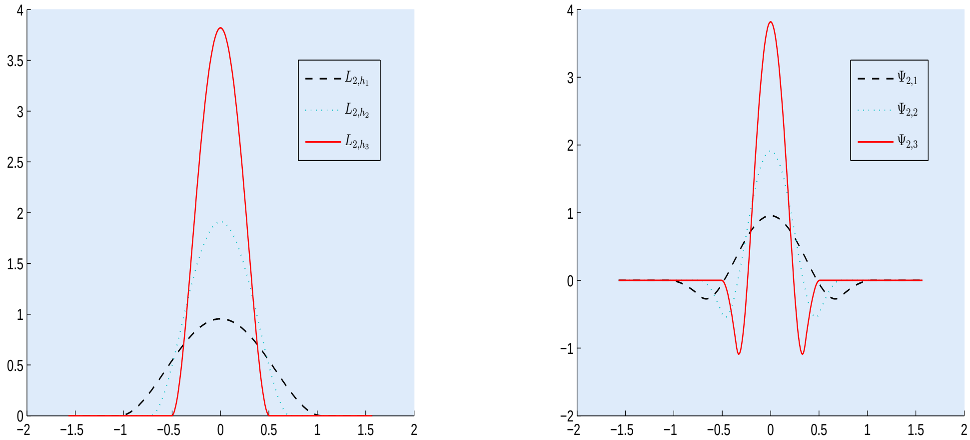


Figure 2.11: The normalized smoothed Haar functions $L_{h_j,2}$ and the Haar wavelets $\Psi_{j,2}$

Left: $\vartheta \mapsto L_{h_j,2}(\cos\vartheta), \quad \vartheta \in [\frac{-\pi}{2}, \frac{\pi}{2}]$ for $j = 1, 2, 3$.

Right: $\vartheta \mapsto \Psi_{j,2}(\cos\vartheta), \quad \vartheta \in [\frac{-\pi}{2}, \frac{\pi}{2}]$ for $j = 1, 2, 3$.

Chapter 3

Spherical Grids

The problem of arranging a dense structured lattice of points over the surface of a sphere is an interesting and widely studied problem. This problem has numerous applications in various areas of science such as crystallography, tomography, molecular structure and especially, in our interest, geosciences. Clearly, every field has different conditions for distributing a certain number of points over the surface of the sphere and, consequently, it yields different grids on the sphere Ω. In geosciences, the adequate condition for distributing points on the sphere Ω is a better approximation of a function on the Earth. For example, the choice of a spherical grid is very important for approximate integration by using mean values of a function on the grid. For more discussion of grids and adequate conditions see [34], [56], [78], and [94]. The purpose of this study is to compare some grids and to look for a grid such that the variation in the number of points within every spherical cap centered by a point of the grid with a prescribed radius is minimized. In other words, if $X_N = \{\xi_1, \ldots, \xi_N\}$ is a set of pairwise distinct points on the sphere Ω and $D_i = \{\xi_j \in X_N | \; d(\xi_i, \xi_j) \leq r_i\}$, where r_i, $1 \leq i \leq N$, is a prescribed radius, then the aim is to find a grid on the sphere Ω such that

$$\delta = max_{1 \leq i \leq N} \# D_i - min_{1 \leq i \leq N} \# D_i, \tag{3.1}$$

is minimized in this grid. On the other hand, if $\delta^{'}$, the total number of distances between every two points in the grid, is defined as follows:

$$\delta^{'} = \# \left\{ d(\xi_i, \xi_j) | \quad \xi_i, \xi_j \in X_N \right\}, \tag{3.2}$$

then we are interested in amounts of $\delta^{'}$ as small as possible. As we will notice in Chapter 4, the amount of $\delta^{'}$ is related to the number of equations in a system of equations that should be solved. Because we have to solve a lot of such systems, the numerical efforts will be decreased if we can find a grid with smaller amount of δ'.

In the following, we first explain the *regular grid*. This grid is obtained from the projection of a regular grid from the plane onto the sphere Ω. Then we develop a latitude-longitude grid on the sphere Ω. Because each cell in this grid is as quadratic as possible, we call it *quadratic grid*. In Section 3.3 we describe *Kurihara grid* on the sphere Ω. This grid shows more homogeneous distribution of points for the sphere Ω than other ones. Finally we discuss the *block grid* on the sphere Ω. The block grid was introduced for the first time by Freeden and Schreiner [43].

3.1 Regular Grid

Let $\varphi \in [0, 2\pi)$, $\theta \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$ and M_φ, N_θ be the number of longitudes and latitudes, respectively. To construct the longitudes, we divide $[0, 2\pi)$ into M_φ equal share as follows:

$$\varphi_i = \frac{2\pi}{M_\varphi} i, \quad i = 0, \ldots, M_\varphi - 1. \tag{3.3}$$

For the latitudes, also, we divide $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$ into N_θ equal share as follows:

$$\theta_j = \frac{\pi}{N_\theta} j, \quad j = 0, \ldots, N_\theta. \tag{3.4}$$

The regular grid is the simplest grid on the sphere Ω but unfortunately, for large M_φ and N_θ, there are too many points near the poles than near the Equator.

Figure 3.1 shows one octant of the regular grid.

3.2 Quadratic Grid

Let $\varphi \in [0, 2\pi)$, $\theta \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$ and M_φ and N_θ be the number of longitudes and latitudes, respectively. Moreover, suppose that N_θ is an odd number. To construct the longitudes, we divide $[0, 2\pi)$ into M_φ equal share as follows:

$$\varphi_i = \frac{2\pi}{M_\varphi} i, \quad i = 0, \ldots, M_\varphi - 1. \tag{3.5}$$

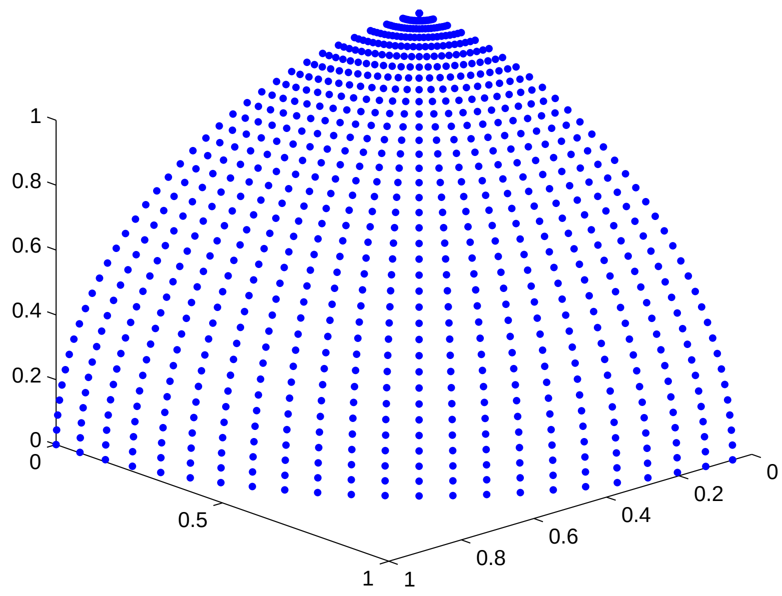


Figure 3.1: One octant of the regular grid for $M_\varphi = 91$ and $N_\theta = 71$.

For the latitudes, first we arrange all the latitudes in the northern hemisphere and after that, in a flip way, we build the grid points in the southern hemisphere. We start from the Equator, $\theta'_0 = 0$, and move toward the North pole. We set

$$\theta'_{j+1} = \theta'_j + \Delta\theta'_j, \qquad j = 0, \dots, \frac{N_\theta - 3}{2}, \tag{3.6}$$

$$\theta'_{\frac{N_\theta - 1}{2}} = \frac{\pi}{2}, \tag{3.7}$$

where $\Delta\theta'_j$ in (3.6) is defined by

$$\Delta\theta'_j = 2 \arcsin\left(cos(\theta'_j) sin(\frac{\pi}{M_\varphi}) \right), \qquad j = 0, \dots, \frac{N_\theta - 3}{2}. \tag{3.8}$$

Note that it is possible that for some j in (3.6), say $j = j_0$, we get $\theta'_{j_0+1} \geq \frac{\pi}{2}$. In such a case we should change the amount of N_θ with a new N_θ^{new} provided that

$$N_\theta^{new} \leq 2j_0 + 3.$$

Now, we extend the latitudes over the whole of the sphere Ω as follows:

$$\theta_0 = -\frac{\pi}{2}, \theta_1 = -\theta'_{\frac{N_\theta - 3}{2}}, \ldots, \theta_{\frac{N_\theta - 1}{2}} = 0, \theta_{\frac{N_\theta + 1}{2}} = \theta'_1, \ldots, \theta_{N_\theta - 1} = \frac{\pi}{2}. \tag{3.9}$$

Finally, the points ξ_{ij} defined by

$$\xi_{ij} = \begin{pmatrix} \cos\varphi_i \cos\theta_j \\ \sin\varphi_i \cos\theta_j \\ \sin\theta_j \end{pmatrix}, \quad i = 0, \ldots, M_\theta - 1, \; j = 0, \ldots, N_\theta - 1. \tag{3.10}$$

generate the quadratic grid on the sphere Ω. For a given M_φ and N_θ, the total number of points in the quadratic grid is $M_\varphi(N_\theta - 2) + 2$.

Algorithm 1 generates the quadratic grid on the surface of the sphere. Figure 3.2 shows one octant of quadratic grid.

Algorithm 1 (Quadratic Grid)

Given M_φ and N_θ, the number of longitudes and latitudes, respectively. The purpose of the algorithm is to produce the quadratic grid $X_N = \{\xi_1, \ldots, \xi_N\}$ in spherical coordinates $(\theta_i, \varphi_i)_{i=1,\ldots,N}$.

Start:

- **for** $i = 0$ to $M_\varphi - 1$ **do**
  - $\varphi_i = i\frac{2\pi}{M_\varphi}$
- **end for**
- $\theta'_0 = 0$
- **for** $j = 0$ to $\frac{N_\theta - 3}{2}$ **do**
  - $\Delta D = 2Arcsin\left(cos(\theta'_j)sin(\frac{\pi}{M_\varphi})\right)$
  - $\theta'_{j+1} = \theta'_j + \Delta D$
- **end for**
- $\theta'_{\frac{N_\theta - 1}{2}} = \frac{\pi}{2}$
- **for** $j = 0$ to $\frac{N_\theta - 1}{2}$ **do**
  - $\theta_j = -\theta'_{\frac{N_\theta - 1}{2} - j}$
- **end for**
- **for** $j = \frac{N_\theta + 1}{2}$ to $N_\theta - 1$ **do**
  - $\theta_j = \theta'_{j - \frac{N_\theta - 1}{2}}$
- **end for**
- $X_N = \{\theta_i \mid \; 0 \le i \le N_\theta - 1\} \times \{\varphi_i \mid \; 0 \le i \le M_\varphi - 1\}$

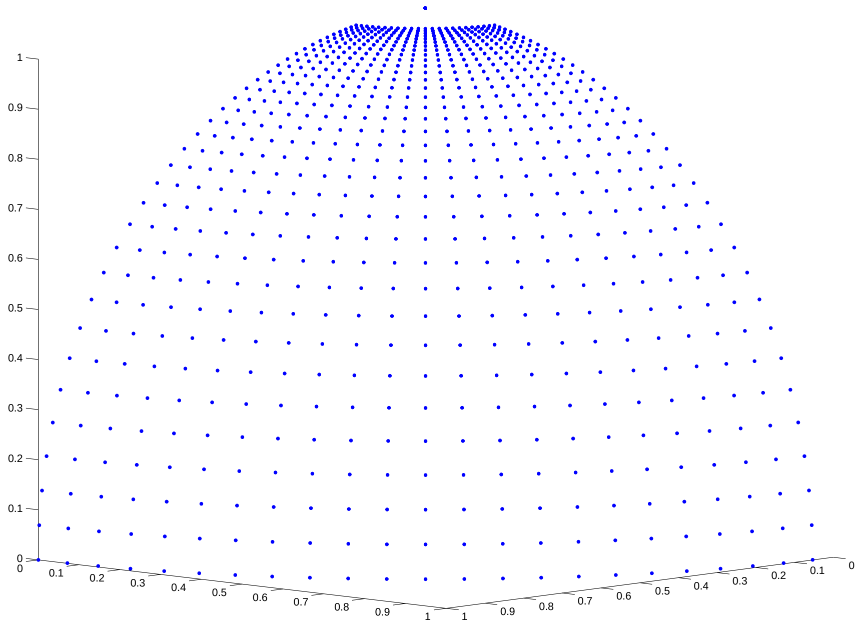


Figure 3.2: One octant of quadratic grid for $M_\varphi = 91$ and $N_\theta = 71$.

The advantage of the quadratic grid is that all cells except "pizza slices" ringing each pole are as possible as quadratic. This property is the reason that every cap centered by grid points has the same number of points as another one, except for those caps containing one or more of the "pizza slices" around the poles. In Figures 3.3 and 3.4, one can see the difference between the quadratic grid and the regular grid.

We mention here that the quadratic grid has "the pole problem", too. That means, there are too many points of the grid near the poles than near the Equator. Also this grid is not a hierarchical grid, that means the grid with higher number of points does not include another one with smaller number of points.

In the next section, we will introduce Kurihara grid on the surface of sphere. This grid is more uniform than the quadratic grid and it is a hierarchical grid.

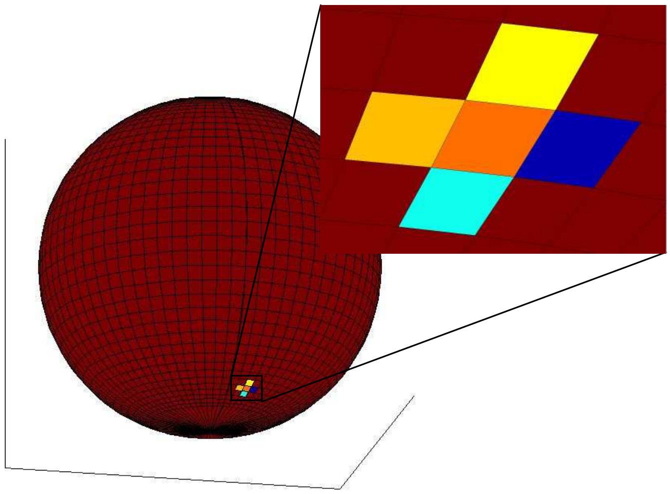


Figure 3.3: Quadratic grid
In this example, the number of longitudes and latitudes for $M_\varphi = 91$ and $N_\theta = 71$, respectively. In the quadratic grid, every cap contains the same number of points as another one.

3.3 Kurihara grid

In 1965 Kurihara [56] proposed a grid such that its density on the surface of the sphere is nearly homogeneous. In this grid, the pole problem does not occur. It is overcome by placing smaller number of points at those latitudes that are close to the poles. Let $\varphi \in [0, 2\pi)$ and $\theta \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$. Suppose that the resolution of the grid, denoted by N, is given. To construct the latitudes of the grid, we divide $\left[0, \frac{\pi}{2}\right]$ to N equal share as follows:

$$\theta_j = \frac{\pi}{2}\left(1 - \frac{j}{N}\right), \qquad j = 0, \ldots, N, \tag{3.11}$$

where θ_0, as a latitude, is the North pole and θ_N is the Equator. Then for $j = 0$ we set $\varphi_{0,0} = 0$ and for $j = 1, \ldots, N$ we set

$$\varphi_{i,j} = \frac{\pi}{2}\frac{i}{j}, \quad i = 0, \ldots, 4j - 1. \tag{3.12}$$

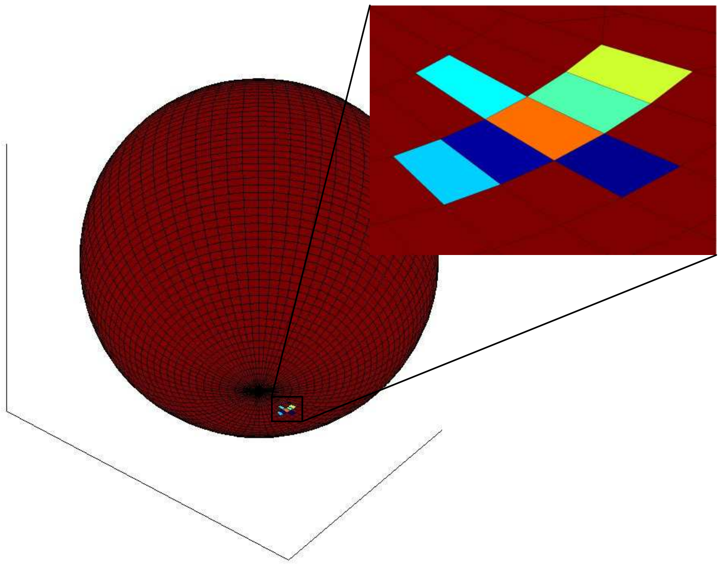


Figure 3.4: Regular grid
In this example, the number of longitudes and latitudes are $M_\varphi = 91$ and $N_\theta = 71$, respectively. In the regular grid, the caps near the poles have more number of points than the caps near the Equator.

The points ξ_{ij} defined by

$$\xi_{ij} = \begin{pmatrix} \cos\varphi_{i,j}\cos\theta_j \\ \sin\varphi_{i,j}\cos\theta_j \\ \sin\theta_j \end{pmatrix}, \quad j = 0, \ldots, N, \; i = 0, \ldots, 4j-1. \tag{3.13}$$

generate the Kurihara grid on the northern hemisphere. The grid points for the southern hemisphere can be easily determined in a flip way of the grid points of the northern hemisphere. For a given N, the resolution of the grid, the total number of grid points on the surface of the sphere is $4N^2 + 2$.

Algorithm 2 generates the Kurihara grid on the surface of the sphere.

Algorithm 2 (Kurihara Grid)

Given N, the resolution of the grid. The purpose of the algorithm is to produce the Kurihara grid $X_M = \{\xi_1, \ldots, \xi_M\}$ in spherical coordinates, where M is $4N^2 + 2$.

Start:

$\theta_0 = \frac{\pi}{2}, \quad \varphi_{0,0} = 0$ *(North Pole)*

for $j = 1$ to N **do**

$\theta_j = \frac{\pi}{2}\left(1 - \frac{j}{N}\right)$

for $i = 0$ to $4j - 1$ **do**

$\varphi_{i,j} = \frac{\pi}{2}\left(\frac{i}{j}\right)$

end for

end for

for $j = 1$ to $N - 1$ **do**

$\theta_{N+j} = -\theta_{N-j}$

for $i = 0$ to $4(N-j) - 1$ **do**

$\varphi_{i,N+j} = \varphi_{i,N-j}$

end for

end for

$\theta_{2N} = -\frac{\pi}{2}, \quad \varphi_{1,2N} = 0$ *(South Pole)*

$$\begin{aligned} X_M &= \{(\theta_i, \varphi_{i,j}) \mid \quad 0 \le j \le N, \quad 0 \le i \le 4j - 1\} \\ &\quad \cup \{(\theta_i, \varphi_{i,j}) \mid \quad N + 1 \le j \le 2N, \quad 0 \le i \le 4N - 4j - 1\} \end{aligned}$$

As we mentioned before, the Kurihara grid is a hierarchical grid. That means if we double the resolution of the grid, N, we get a hierarchical grid. In other words, if K_N denotes the Kurihara grid for the resolution N, then we have

$$\ldots \subset K_{\frac{N}{2}} \subset K_N \subset K_{2N} \subset K_{4N} \ldots . \tag{3.14}$$

Note that by using the hierarchical grids, one can construct a multiresolution analysis. Later, we will describe this method in Chapter 4. Figure 3.5 shows one octant of Kurihara grid.

3.4 Block Grid

The block grid was proposed for the first time by Freeden and Schreiner [43]. One of the nice properties of this grid is that the amount of δ in (3.2) is

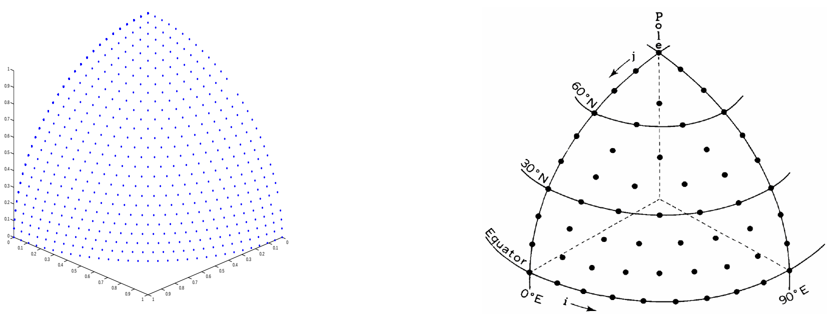


Figure 3.5: One octant of the Kurihara grid
The Kurihara grid for resolution $N = 30$ (left) and $N = 10$ (right).
The figure on the right hand side is from [56].

smaller than the quadratic grid or the regular grid. On the other hand, the block grid has solved the "the pole problem". In other words, the block grid has all advantages of the three aforementioned grids: the Kurihara grid , the quadratic grid and the regular grid. The block grid is constructed based on this idea: first, we start with a grid similar to the regular grid and divide its latitudes in some blocks. Then we delete some longitudes in each block in such a way that we eliminate more longitudes in those blocks which are near the poles rather than those blocks which are near the Equator. In the following we describe this idea for the northern hemisphere in detail (cf. [43]). The southern hemisphere grid is a mirror of the northern hemisphere grid.

Suppose that the resolution of grid, $N,\ N \geq 2$ is fixed. We divide $[0, 2\pi)$ to 2^{N+2} equal share, then we have

$$\varphi_i = i\pi 2^{-(N+1)},\ i = 0, \ldots, 2^{N+2} - 1. \tag{3.15}$$

For the latitudes, we arrange $2^N - 1$ points between $\left(0, \frac{\pi}{2}\right)$ as latitudes in the following way

$$0 < \theta_0 < \ldots < \theta_{2^N - 2} < \frac{\pi}{2}. \tag{3.16}$$

The points ${}^m\eta_{ij}$ on the the sphere Ω are defined by

$${}^m\eta_{ij} = \begin{pmatrix} \cos\varphi_i \cos\theta_j \\ \sin\varphi_i \cos\theta_j \\ (-1)^m \sin\theta_j \end{pmatrix}, \quad i = 0, \ldots, 2^{N+1} - 1,\ j = 0, \ldots, 2^N - 2, \tag{3.17}$$

where $m = 0$ is for the northern hemisphere and $m = 1$ is for the southern hemisphere.

Note that to obtain the block grid, too many of the points ${}^m\eta_{ij}$ should be deleted. Now in a south–north direction, for the finest level, $l=0$, we separate the latitudes in N blocks such that the k^{th} block has 2^{N-k-1} latitudes for $k=0,\dots,N-1$. For the coarser level, $l=1,\dots,N$, we reduce the number of latitudes and longitudes by factor 2. In more detail, for level l and for blocks $k,\ k=0,\dots,N-l-1,$, we define the index sets as follows:

$$\begin{aligned}\mathcal{I}_k^{(l)} &= \left\{i|\ i=0,2^{k+l},\dots,2^{N+1}-2^{k+l}\right\}\times\left\{j|\ j=2^N-2^{N-k},\right.\\ &\quad \left.2^N-2^{N-k}+2^l,\dots,2^N-2^{N-k-1}-2^l\right\},\end{aligned} \tag{3.18}$$

and for the special situation $k=N-l$ when $l>0$, the index set $\mathcal{I}_{N-1}^{(l)}$ would be empty using the relation (3.18). For this reason, we set

$$\mathcal{I}_{N-l}^{(l)}=\left\{(0,2^N-2^l),(2^N,2^N-2^l)\right\},\quad l=1,\dots,N. \tag{3.19}$$

By definition, all other $\mathcal{I}_k^{(l)}$ are equal to the empty set. In addition, we let

$$\mathcal{I}^{(l)}=\bigcup_{k=0}^{N-l}\mathcal{I}_k^{(l)}. \tag{3.20}$$

For a given N, we have, therefore, defined the following non-empty index sets:

$$\mathcal{I}_0^{(0)},\dots,\mathcal{I}_{N-1}^{(0)},\ \mathcal{I}_1^{(1)},\dots,\ \mathcal{I}_{N-1}^{(1)},\ \mathcal{I}_1^{(2)},\dots,\mathcal{I}_{N-2}^{(2)},\ \dots,\mathcal{I}_0^{(N)}.$$

As an example, we have listed the index sets for the case $N=3$ in Table 3.1.

Based on the points ${}^m\eta_{ij}$ defined in (3.17) and the index sets defined in (3.18) and (3.19), the grid points in the block k and the level l for the northern hemisphere are defined as follows:

$${}^NB_k^{(l)}=\left\{{}^0\eta_{2i+(2^k-1)\,j}\mid(i,j)\in\mathcal{I}_k^{(l)}\right\},\qquad k=0,\dots,N-l-1,\ l=0,\dots,N. \tag{3.21}$$

Clearly, the collection of grid points of all blocks in the set

$${}^NB^{(l)}=\bigcup_{k=0}^{N-l-1}{}^NB_k^{(l)},\qquad l=0,\dots,N, \tag{3.22}$$

gives us a block grid for the northern hemisphere in the level l, $l=0,\dots,N$.

Level	Block	$\mathcal{I}_k^{(l)}$
$l=0$	$k=0$	$\{0,1,2,\ldots,15\}\times\{0,1,2,3\}$
	$k=1$	$\{0,2,4,\ldots,14\}\times\{4,5\}$
	$k=2$	$\{0,4,8,12\}\times\{6\}$
$l=1$	$k=0$	$\{0,2,4,\ldots,14\}\times\{0,2\}$
	$k=1$	$\{0,4,8,12\}\times\{4\}$
	$k=2$	$\{0,8\}\times\{6\}$
$l=2$	$k=0$	$\{0,4,8,12\}\times\{0\}$
	$k=1$	$\{0,8\}\times\{4\}$
$l=3$	$k=0$	$\{0,8\}\times\{0\}$

Table 3.1: Index sets for the case $N=3$

As aforementioned, to extend the block grid to the whole sphere Ω, it suffices to mirror the northern hemisphere grid. To be more precise, if ${}^S B_k^{(l)}$ denotes the grid points in the block k and the level l in the southern hemisphere then

$$ {}^S B_k^{(l)} = \left\{ {}^1\eta_{2i+(2^k-1)\,j} \mid (i,j) \in \mathcal{I}_k^{(l)} \right\}, \qquad k=0,\ldots,N-l-1,\ l=0,\ldots,N. \tag{3.23} $$

Similar to the northern hemisphere, if ${}^S B^{(l)}$ denotes the grid points in the southern hemisphere for the level l then we have

$$ {}^S B^{(l)} = \bigcup_{k=0}^{N-l-1} {}^S B_k^{(l)}, \qquad l=0,\ldots,N. \tag{3.24} $$

Therefore, if the block grid for the whole of the sphere Ω for the level l is denoted by $B^{(l)}$ then we have

$$ B^{(l)} = {}^N B^{(l)} \cup {}^S B^{(l)}, \qquad l=0,\ldots,N. \tag{3.25} $$

Figure 3.6 illustrates one octant of the block grid $N=4$.

The block grid is a hierarchical grid. That means

$$ B^{(N)} \subset B^{(N-1)} \subset \ldots \subset B^{(0)}. \tag{3.26} $$

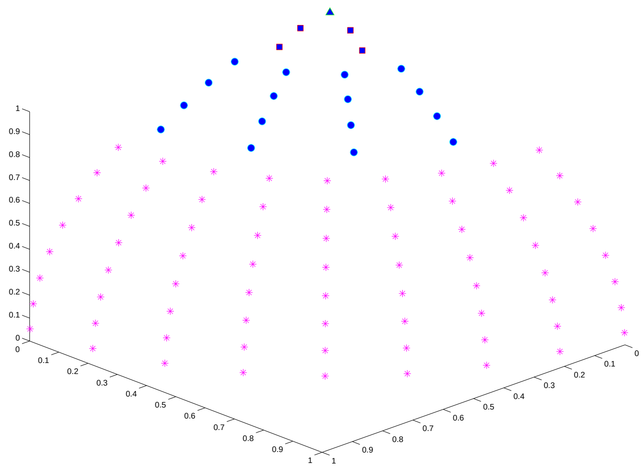


Figure 3.6: One octant of the block grid for $N = 4$.
Block 0: $*$, block 1: $\circ$, block 2: $\square$, block 3: $\triangle$.

By using the relation (3.26) it is possible to construct a multiresolution analysis and therefore, according to the work of Freeden and Schreiner [43], it is possible to construct locally supported wavelets.

Now, to obtain the number of points in block grid, we have

$$\begin{aligned}
\#B^{(l)} &= \#\,^{N}B^{(l)} + \#\,^{S}B^{(l)} \\
&= 2\#\,^{N}B^{(l)} \\
&= 2\#\mathcal{I}^{(l)} \\
&= 2\sum_{k=0}^{N-l} \#\mathcal{I}_k^{(l)} \\
&= 2\sum_{k=0}^{N-l-1} 4^{N-k-l} + (1-\delta_{l0}) \\
&= \frac{8}{3}(4^{N-l}-1) + (1-\delta_{l0}),
\end{aligned}$$

where δ_{ij} is the Kronecker symbol defined in (1.17). This yields the following lemma.

Lemma 3.4.1
For $N \geq 2$, the total number of points in the block grid in the level l is

$$\#B^{(l)} = \frac{8}{3}(4^{N-l} - 1) + (1 - \delta_{l0}), \qquad l = 0, \dots, N. \tag{3.27}$$

Finally, we summarize the method of generating the block grid in Algorithm 3. Note that in this algorithm, we start with a regular grid but one can start with a grid in which the interval $(0, \frac{\pi}{2})$ is not divided in equal share.

Algorithm 3 (Block Grid)

Given N, the resolution of the grid and l, the level of the grid. The purpose of the algorithm is to produce the block grid $B^{(l)} = \{\xi_1, \dots, \xi_{N_l}\}$ in spherical coordinates where N_l is $\frac{8}{3}(4^{N-l} - 1) + (1 - \delta_{l0})$.

Start:

- **for** $i = 0$ to $2^{N+2} - 1$ **do**
  - $\varphi_i = i \frac{\pi}{2^{N+1}}$
- **end for**
- $Const = \frac{\pi}{2^{(N+1)} - 2}$
- **for** $j = 0$ to $2^N - 2$ **do**
  - $\theta_j = Const * (j + \frac{1}{2})$
- **end for**
- **for** $k = 0$ to $N - l - 1$ **do**
  - $I_k^{(l)} = \left\{i |\ i = 0, 2^{k+l}, \dots, 2^{N+1} - 2^{k+l}\right\}$
  - $J_k^{(l)} = \left\{j |\ j = 2^N - 2^{N-k}, 2^N - 2^{N-k} + 2^l, \dots, 2^N - 2^{N-k-1} - 2^l\ \right\}$
  - ${}^N B_k^{(l)} = \{(\varphi_{2i+(2^k-1)}, \theta_j) | i \in I_k^{(l)}, j \in J_k^{(l)}\}$
- **end for**
- **if** $l \# 0$ **then**
  - ${}^N B_{N-l}^{(l)} = (\varphi_{2^{N-l}-1}, \theta_{2^N - 2^l})$
- **end if**
- ${}^N B^{(l)} = \bigcup_{k=0}^{N-l-1} {}^N B_k^{(l)}$ %The block grid for the northern hemisphere.
- ${}^S B^{(l)} = mirror({}^N B^{(l)})$ %The block grid for the southern hemisphere.
- $B^{(l)} = {}^N B^{(l)} \cup {}^S B^{(l)}$

Chapter 4

Biorthogonal Locally Supported Radial Basis Functions on the Sphere

In many problems of approximation theory on the sphere, there are matrices with elements similar to

$$K_i * K_j(\xi_i^m \cdot \xi_j^n), \tag{4.1}$$

where K_i and K_j are zonal functions and ξ_i^m and ξ_j^n are points from a spherical grid. For example, these elements appear in the interpolation matrices, the least square matrices or in the stiffness matrices (overlap matrices). If there is a system of biorthogonal kernels then these aforementioned matrices are diagonal or block diagonal matrices and, therefore, working with them is very easy, fast and efficient. How can one construct such biorthogonal kernels on the sphere? This chapter is devoted to answering this question.

4.1 Biorthogonal Locally Supported Zonal Kernels

Let $X_N = \{\xi_{ij} | \ (i,j) \in \Gamma\} \subset \Omega$ be an isolatitude grid (cf. Chapter 3) on the sphere Ω, where Γ is a set of ordered pairs indices corresponding to longitudes and latitudes, respectively, with $\#\Gamma = N$. Let $\mathcal{J}$ be the set of all latitudes indices, i.e.,

$$\mathcal{J} = \{j \mid (i,j) \in \Gamma\}.$$

Now, we introduce

Definition 4.1.1 (Biorthogonal Kernels)
Let X_N be an isolatitude grid on the sphere Ω. Two families of kernels $\{K_j\}_{j\in\mathcal{J}}$ and $\{\tilde{K}_j\}_{j\in\mathcal{J}}$ in $\mathcal{L}^2[-1,1]$ are called a system of biorthogonal kernels on the sphere Ω if the following conditions are satisfied:

$$\left(K_j * \tilde{K}_{j'}\right)(\xi_{ij}\,,\,\xi_{i'j'}) = \left(K_j(\xi_{ij}\,,\,\cdot)\,,\,\tilde{K}_{j'}(\xi_{i'j'}\,,\,\cdot)\right)_{\mathcal{L}^2(\Omega)} = \delta_{ii'}\delta_{jj'}, \tag{4.2}$$

for all $i, i' \in \mathcal{I}$ and all $j, j' \in \mathcal{J}$.

From now on, we call $\{K_j\}_{j\in\mathcal{J}}$ and $\{\tilde{K}_j\}_{j\in\mathcal{J}}$ *primal* and *dual* kernels. Our purpose is to construct a system of biorthogonal locally supported zonal kernels by using families of locally supported zonal kernels in $\mathcal{L}^2[-1,1]$. To begin our approach, let $\{K_j\}_{j\in\mathcal{J}}$ be an arbitrary family of locally supported zonal kernels in $\mathcal{L}^2[-1,1]$, where the support of K_j is $[h_j, 1]$. As we will describe later, the value of h_j is important to have a good approximation and on the other hand to minimize computational efforts of finding the biorthogonal kernels.

Suppose that $j \in \mathcal{J}$ is arbitrary and fixed. Let $\{\tilde{K}_{jl}\}_{l\in\mathcal{S}_j}$ be an arbitrary family of locally supported zonal kernels in $\mathcal{L}^2[-1,1]$, where $\mathcal{S}_j$ is an index set. Moreover, suppose that the support of $\tilde{K}_{jl}$ is a subset of the support of K_j for all $l \in \mathcal{S}_j$. For convenience, we call the locally supported zonal kernels $\{\tilde{K}_{jl}\}_{l\in\mathcal{S}_j}$ the *intermediate* locally supported zonal kernels.

To construct dual kernels, we replace $\tilde{K}_j$ in (4.2) with a linear combination of intermediate locally supported zonal kernels with unknown coefficients. In other words, we substitute $\tilde{K}_j$ in (4.2) by the following term:

$$\sum_{l=1}^{s_j} x_{jl}\tilde{K}_{jl}, \tag{4.3}$$

where $x_{jl},\ l = 1,\dots,s_j$, are unknowns and $s_j,\ j \in \mathcal{J}$, is the number of unknowns and $\tilde{K}_{jl},\ l = 1,\dots,s_j,\ j \in \mathcal{J}$, are the intermediate locally supported zonal kernels in $\mathcal{L}^2[-1,1]$ such that their support fulfills the following conditions:

$$\text{supp}\ \tilde{K}_{j1} = \text{supp}\ K_j, \qquad j \in \mathcal{J}, \tag{4.4}$$

$$\text{supp}\ \tilde{K}_{jl} \subset \text{supp}\ K_j, \qquad l = 2,\dots,s_j, \quad j \in \mathcal{J}. \tag{4.5}$$

After substituting (4.3) instead of $\tilde{K}_j$ in (4.2), we get the following linear systems of equations:

$$\sum_{l=1}^{s_{j'}} x_{j'l} \left(K_j * \tilde{K}_{j'l} \right) (\xi_{ij}\,,\, \xi_{i'j'}) = \delta_{ii'}\delta_{jj'}, \quad i, i' \in \mathcal{I},\; j' \in \mathcal{J}, \tag{4.6}$$

for all $j \in \mathcal{J}$.

Remark 4.1.2
It should be noted that if $i \neq i'$ and $j \neq j'$ and also

$$d(\xi_{ij}\,,\, \xi_{i'j'}) \geq \cos^{-1}(h_j) + \cos^{-1}(h'_j), \tag{4.7}$$

where $d(\xi_{ij}\,,\, \xi_{i'j'})$ is the geodetic distance between ξ_{ij} and $\xi_{i'j'}$, then the supports of the corresponding kernels don't have any intersection with each other. Therefore, the corresponding equation in (4.6) is already satisfied.

Remark 4.1.3
Because $K_j * \tilde{K}_{j'l},\ l = 1, \ldots, s_j$ and $j, j' \in \mathcal{J}$ are zonal functions some equations in (4.6) are repeated. These equations are corresponding to those pair of points $(\xi_{ij}\,,\, \xi_{i'j'})$ that the geodetic distance between them are equal for a fixed $j \in \mathcal{J}$.

After eliminating the satisfied and repeated equations from (4.6), we obtain $\#\mathcal{J}$ number of linear systems of equations as follows:

$$\mathbf{A_j X_j} = \mathbf{b_j}, \qquad j \in \mathcal{J}, \tag{4.8}$$

where

$$\mathbf{A_j} \in \mathbb{R}^{q_j \times s_j}, \mathbf{X_j} = \begin{pmatrix} x_{j1} \\ x_{j2} \\ \vdots \\ x_{js_j} \end{pmatrix}, \mathbf{b_j} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in \mathbb{R}^{1 \times q_j}, \tag{4.9}$$

and q_j is the number of equations corresponding to the j^{th} latitude. As we shall see in our examples, the amount of q_j is not big (about 15-25 for our examples). Therefore, in the case of solvability the linear systems of equations (4.8), numerical effort for solving them is relatively small.

Note that for an adequate choice of primal and intermediate kernels, the linear systems of equations (4.8) are solvable. However, for a certain choice of primal and intermediate kernels, it is possible that some of linear systems of equations in (4.8) are not solvable. In such a case, increasing the number of unknowns, s_j, may be help us to solve these linear system of equations. Note that, the number of equations in (4.8) does not depend on the number of unknowns, i.e., increasing the number of unknowns do not change the number of equations.

By using the solution of (4.8), we can define the dual locally supported zonal kernels as follows:

Definition 4.1.4 (Dual Locally Supported Kernel)
Let $x_{jl},\ l = 1, \dots, s_j,\ j \in \mathcal{J}$, be the solution of (4.8). Then the dual locally supported zonal kernel of $K_j,\ j \in \mathcal{J}$, is denoted by $\tilde{K}_j,\ j \in \mathcal{J}$, and defined by

$$\tilde{K}_j = \sum_{l=1}^{s_j} x_{jl} \tilde{K}_{jl}, \tag{4.10}$$

for $j \in \mathcal{J}$.

Now, we are able to state the main theorem of this section.

Theorem 4.1.5 (Biorthogonal Locally Supported Zonal Kernels)
Suppose that $\{K_j\}_{j\in\mathcal{J}}$ *and* $\{\tilde{K}_{jl}\}_{l=1,\dots,s_j,\, j\in\mathcal{J}}$ *are families of locally supported kernels in* $\mathcal{X}(\Omega)$. *Let*

$$\tilde{K}_j = \sum_{l=1}^{s_j} x_{jl} \tilde{K}_{jl}, \quad j \in \mathcal{J}, \tag{4.11}$$

where the coefficient $x_{jl},\ l = 1, \dots, s_j$, *are solutions of systems of linear equation (4.8). Then the system* $\left\{K_j, \tilde{K}_j\right\}_{j\in\mathcal{J}}$ *is a biorthogonal system of locally supported zonal kernels on the sphere* Ω *in the sense of the* $\mathcal{L}^2(\Omega)-$*inner product.*

If at the beginning of our biorthogonalization process, we start with a family of primal and intermediate zonal kernels with $K_j^{\wedge}(0) = \tilde{K}_{jl}^{\wedge}(0) = 1$, and we would like that $\tilde{K}_j^{\wedge}(0) = 1$, then we should impose the following constraint to the unknowns of (4.6):

$$\sum_{l=1}^{s_j} x_{jl} = 1. \tag{4.12}$$

Remark 4.1.6
It should be noted that in the biorthogonalization approach if we start with the (strictly) positive definite primal and intermediate kernels then it is NOT necessary that the dual kernels are (strictly) positive definite. However, if we would like that the dual kernels $\tilde{K}_j,\ j \in \mathcal{J}$ be (strictly) positive definite then we must start with the (strictly) positive definite primal and intermediate kernels and also we should impose the following constraint to the unknowns of the system of linear equations (4.6):

$$x_{jl} \geq 0, \quad l = 1, \ldots, s_j. \tag{4.13}$$

Solving such a problem can be done by using linear programming. However, the solvability of this linear programming problem must be discussed and it can be a future work.

Next, we present some examples of the construction of biorthogonal kernels on the sphere Ω.

4.1.1 Biorthogonal Kernels on the Quadratic Grid

In this section, we use the quadratic grid on the sphere Ω (see Section 3.2) as the system of points on the sphere Ω. In addition, we choose the piecewise polynomial locally supported zonal kernels introduced in Section 2.3 as the primal zonal kernels, i.e.,

$$K_j(t) = L_{h_j,\lambda}(t), \quad j \in \mathcal{J}, \tag{4.14}$$

where the support of $L_{h_j,\lambda}$ is $[h_j, 1],\ j \in \mathcal{J}$. Suppose that $j \in \mathcal{J}$ is fixed. As we mentioned before, the amount of h_j is of great importance for practical aims. In detail, as in Figure 4.1 is shown, the support of $K_j(\xi_{ij}, \cdot)$ and consequently the support of $L_{h_j,\lambda}(\xi_{ij}, \cdot)$ is a spherical cap with center ξ_{ij} and radius r_j, where the relation between r_j and h_j is:

$$r_j = \cos^{-1}(h_j), \quad j \in \mathcal{J}. \tag{4.15}$$

Note that in (4.15), we use the geodetic metric for the distance between two points on the sphere Ω. If h_j is near 1 then the amount of r_j is very small. Then the support of the kernel only contains the center of the spherical cap and, therefore, an approximation by using this kernel maybe contains an unacceptable error. On the other hand, if h_j is near -1 then the support of

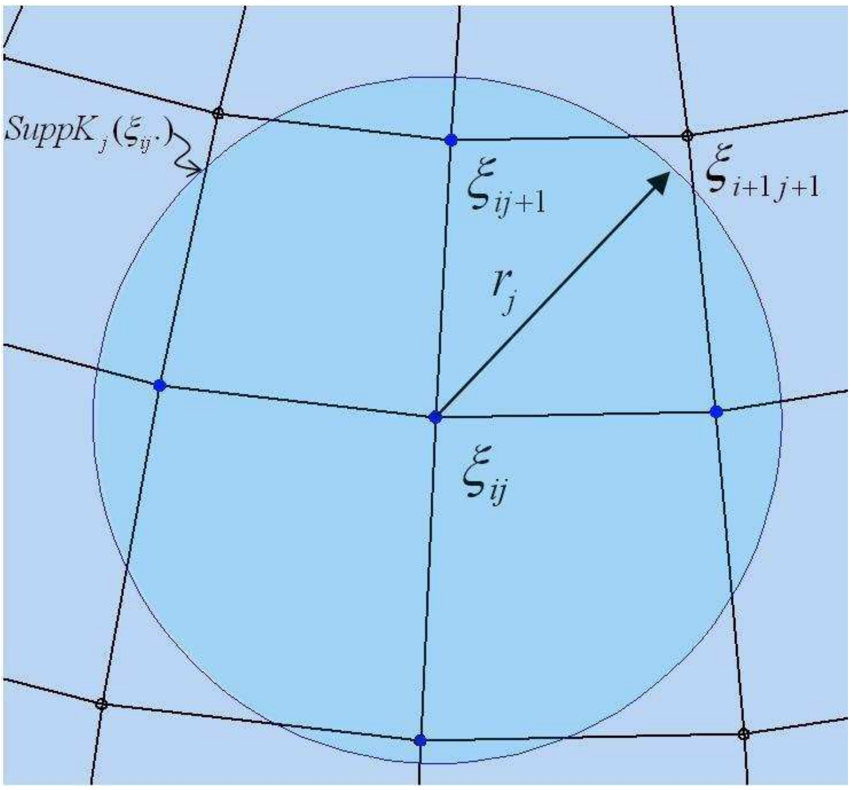


Figure 4.1: The support of $\eta \mapsto K_j(\xi_{ij}, \cdot\eta)$ in the quadratic grid.
The support of $\vartheta \mapsto K_j(\xi_{ij}, \cdot\eta)$ is a spherical cap with the center ξ_{ij} and the radius $r_j = \cos^{-1}(h_j)$.

the kernel contains a lot of grid points and consequently the linear system of equations (4.8) will be very big. Therefore, in our example, we have chosen the amount of h_j such that only those points of grid which are the "neighborhood" of the center of the spherical cap should be in the support of the kernel.

Figure 4.1 illustrates a "spherical support cap" with radius r_j. The neighborhood points of the center and the center of spherical cap are shown by the solid circles.

The succeeding step would be the selection of kernels $\tilde{K}_{jl},\ l = 1, \dots, s_j$. We can do it in two different ways. One way is the picking of kernels $\tilde{K}_{jl},\ l = 1, \dots, s_j$ the same as K_j but with different radius of support cap. In other words, we choose $\tilde{K}_{j1} = K_j$ and for $\tilde{K}_{j2}, \dots, K_{js_j}$ the same as K_j but with smaller radius of the support cap, i.e., if $\tilde{r}_{jl}$ denotes the radius of the support cap of $\tilde{K}_{jl}$ then

$$\tilde{r}_{j1} = r_j > \tilde{r}_{j2} > \ldots > \tilde{r}_{js_j} > 0 \tag{4.16}$$

Remark 4.1.7
Note that with a "bad" choice of $\{\tilde{r}_{jl}\}_{l=1,\ldots,s_j}$, it is possible that in (4.8), for example, we get some equations that only one coefficient is non-zero and all the rest coefficients are zero. Clearly, the solution of such equations is zero but we don't have any interest in these kinds of solutions.

Another way is the picking of the kernels $\tilde{K}_{jl},\ l = 1, \ldots, s_j$, with the same radius of the support cap but with different smoothness, i.e.,

$$\tilde{K}_{jl} = L_{h_j,\lambda+l-1}, \quad l = 1, \ldots, s_j. \tag{4.17}$$

It should be mentioned that with this choice of kernels, we do not have the problem explained by Remark 4.1.7.

After selecting the primal kernels and intermediate kernels $\left\{\tilde{K}_{jl}\right\}_{l=1,\ldots,s_j}$ we can get the dual kernels, $\tilde{K}$, by solving the linear system of equations (4.8). For this example, the system (4.8) has about 10–25 equations and s_j unknowns.

Figure 4.2 shows the dual kernels for the quadratic grid with $M_\varphi = 21$ and $N_\theta = 17$.

4.1.2 Biorthogonal Kernels on the Block Grid

In this section, we use the block grid on the sphere Ω (see Section 3.4) as the system of points on the sphere Ω. Similar to the previous section, we choose the piecewise polynomial locally supported zonal kernels $L_{h_j,\lambda}$ introduced in Section 2.3 as the primal kernels: $K_j(t) = L_{h_j,\lambda}(t),\ j \in \mathcal{J}$ with the support $[h_j, 1]$. Again, we choose the intermediate kernels $\left\{\tilde{K}_{jl}\right\}_{l=1,\ldots,s_j}$ as in the last section:

$$\tilde{K}_{jl} = L_{h_j,\lambda+l-1}, \quad l = 1, \ldots, s_j.$$

After solving the linear system of equations (4.8), we get the dual kernel $\tilde{K}_j,\ j \in \mathcal{J}$, by letting

$$\tilde{K}_j = \sum_{l=1}^{s_j} x_{jl} \tilde{K}_{jl}, \quad j \in \mathcal{J}.$$

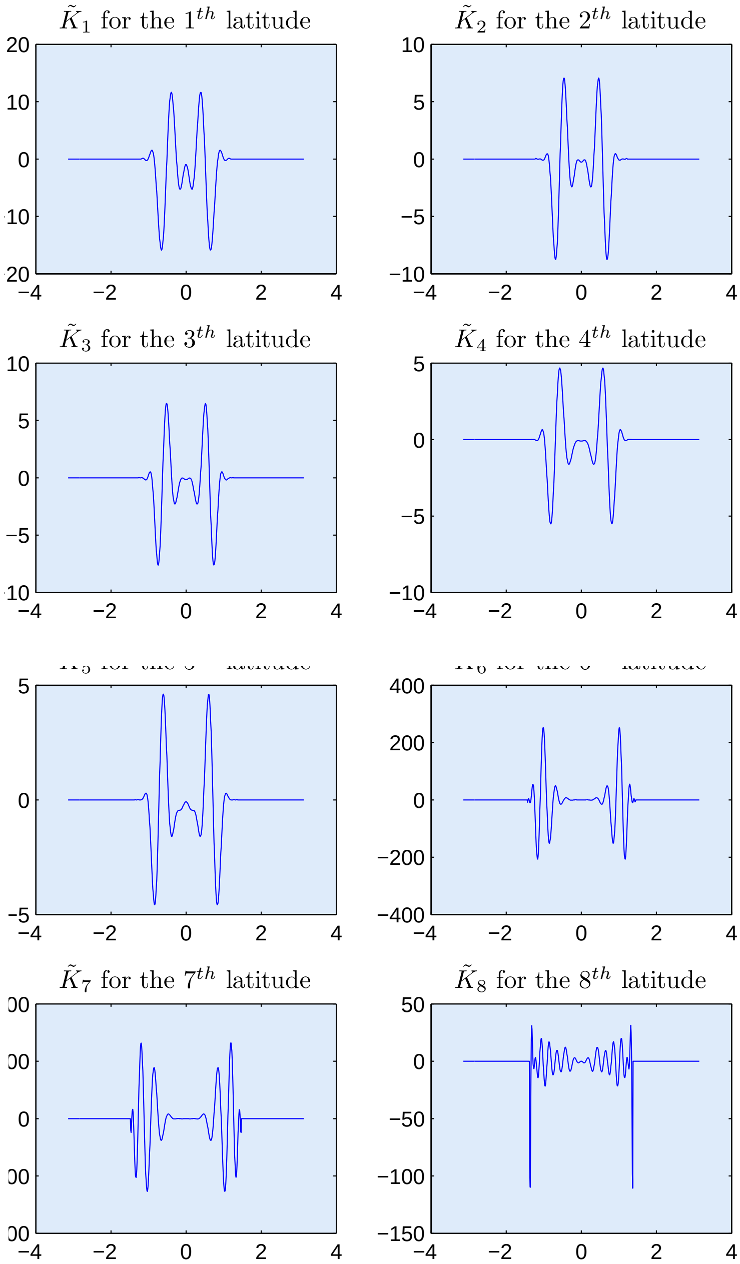


Figure 4.2: The dual kernels $\tilde{K}_j$ on the quadratic grid.
The dual kernels $\tilde{K}_j$ in the northern hemisphere for the quadratic grid with $M_\varphi = 21$ and $N_\theta = 17$. The first dual kernel, $\tilde{K}_1$, is corresponding to the Equator and the $\tilde{K}_8$ is corresponding to the last latitude near the North pole.

Figures 4.3 and 4.4 show the dual kernels for the block grid with the grid resolution $N = 3$. The dual kernels are only computed for the northern hemisphere. To compute the dual kernels in Figure 4.3, we don't impose the constraint (4.12) to the unknowns. But for computing the dual kernels in Figure 4.4, we enforce the restriction (4.12) to the system of linear equations (4.8).

4.2 Approximation Using Biorthogonal Kernels

In this section, we describe how we can apply the biorthogonal locally supported zonal kernels to approximate a function on the sphere Ω. Again let $X_N = \{\xi_{ij} | \ (i,j) \in \Gamma\} \subset \Omega$ be an isolatitude grid on the sphere Ω (cf. Chapter 3) where Γ is a set of ordered pair indices corresponding to longitudes and latitudes, respectively. Suppose that $\{K_j, \tilde{K}_j\}_{j\in\mathcal{J}}$ is a family of biorthogonal locally supported zonal kernels on the sphere Ω. Now, for every point of the grid X_N we define two zonal functions K_{ij} and $\tilde{K}_{ij}$ as follows:

$$K_{ij} = K_j(\xi_{ij}, \cdot), \quad (i,j) \in \Gamma, \tag{4.18}$$
$$\tilde{K}_{ij} = \tilde{K}_j(\xi_{ij}, \cdot), \quad (i,j) \in \Gamma. \tag{4.19}$$

By using the zonal functions $\tilde{K}_{ij}$, we define the space V with respect to the $\mathcal{L}^2(\Omega)-$inner product as follows:

$$V = \operatorname*{span}_{(i,j)\in\Gamma} \left\{ \tilde{K}_{ij} \right\}. \tag{4.20}$$

Also, we define the projection operator $P : \mathcal{L}^2(\Omega) \to V$ as follows:

$$P(F) = \sum_{(i,j)\in\Gamma} (F, K_{ij})_{\mathcal{L}^2(\Omega)} \tilde{K}_{ij}. \tag{4.21}$$

The projection operator P defined in (4.21) is an orthogonal projection operator from $\mathcal{L}^2(\Omega)$ to the space V, because

$$\begin{aligned}
(F - P(F), K_{ij})_{\mathcal{L}^2(\Omega)} &= (F, K_{ij})_{\mathcal{L}^2(\Omega)} - (P(F), K_{ij})_{\mathcal{L}^2(\Omega)} \\
&= (F, K_{ij})_{\mathcal{L}^2(\Omega)} - \sum_{(i',j')\in\Gamma} (F, K_{i'j'})_{\mathcal{L}^2(\Omega)} (\tilde{K}_{i'j'}, K_{ij})_{\mathcal{L}^2(\Omega)} \\
&= (F, K_{ij})_{\mathcal{L}^2(\Omega)} - \sum_{(i',j')\in\Gamma} (F, K_{i'j'})_{\mathcal{L}^2(\Omega)} \delta_{ii'}\delta_{jj'} \\
&= 0.
\end{aligned}$$

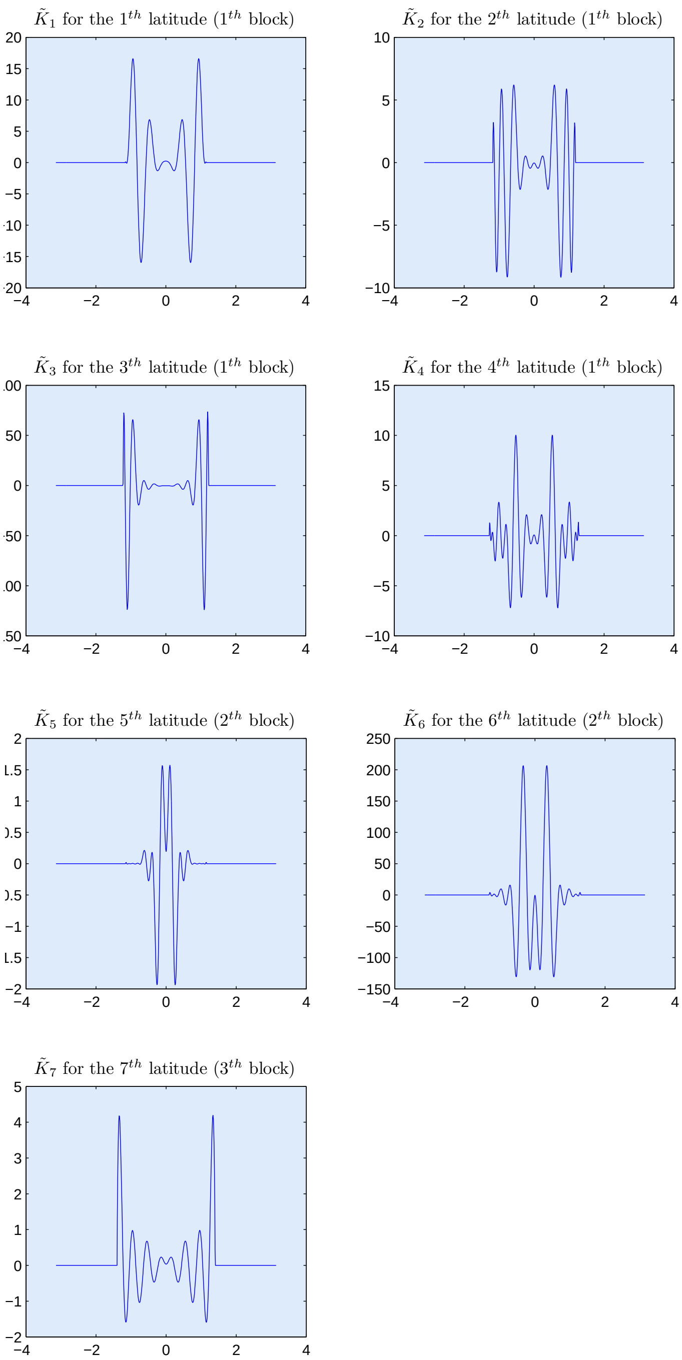


Figure 4.3: The dual kernels $\tilde{K}_j$ for the block grid.
The dual kernels $\tilde{K}_j$ for the northern hemisphere. In this example, the system of points on the sphere Ω is the block grid with the resolution of grid $N = 3$.

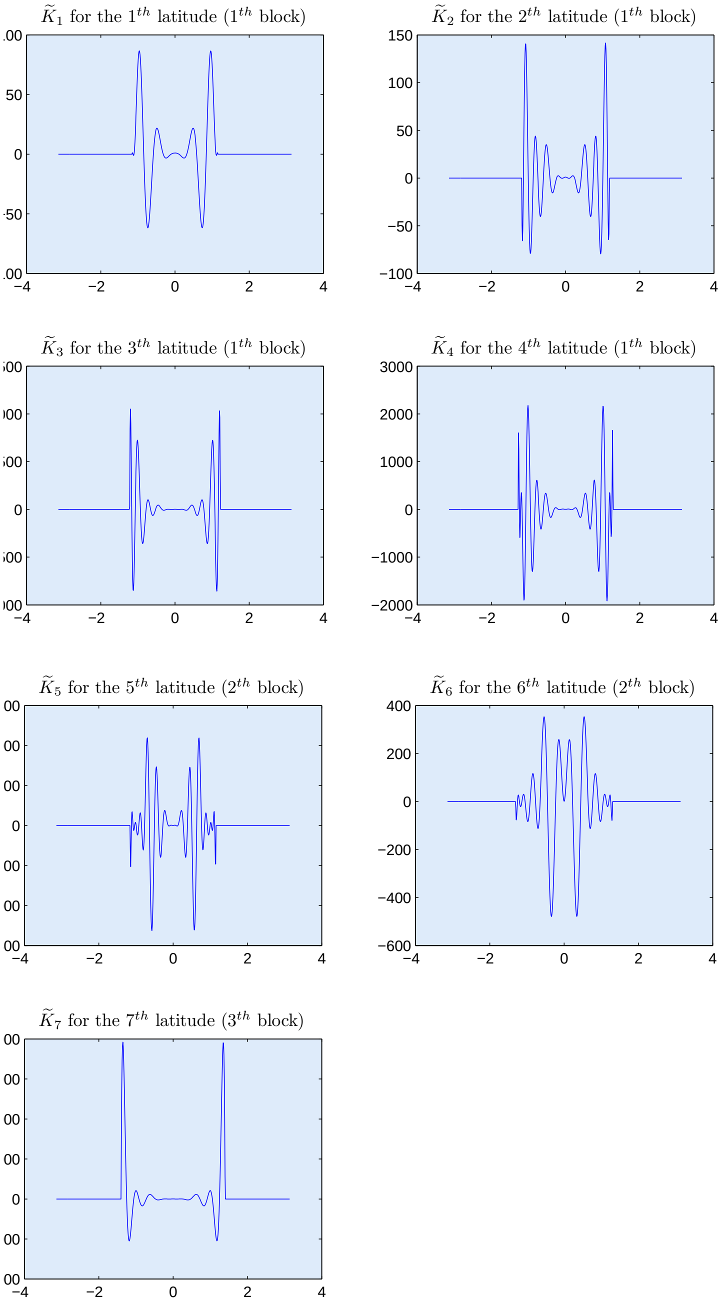


Figure 4.4: The dual kernels $\tilde{K}_j$ for the block grid.
The dual kernels $\tilde{K}_j$ for the northern hemisphere for the block grid. The resolution of grid is $N = 3$. In this example we impose the constraint (4.12) to the system of equations (4.8).

Remark 4.2.1
It should be mentioned that the operator P defined in (4.21) approximate the function $F \in V$ precisely, i.e., if $F \in V$ then $P(F) = F$.

In the next chapter, we will introduce three kinds of wavelets based on the system of biorthogonal locally supported kernels.

Chapter 5

Fast Spherical Wavelet Transform Based on Biorthogonal Zonal Kernels

In this chapter we discuss a new kind of spherical wavelets which were constructed, for the first time, by Freeden and Schreiner [43]. These wavelets are based on the biorthogonal locally supported zonal kernels constructed in Chapter 4. In comparison with Euclidean wavelet theory, these kinds of wavelets are similar to the tensor product wavelets (for detailed literature on the tensor product wavelets in Euclidean space see [59] or [63]). Because of their construction, these kinds of wavelets are locally supported and also are easy to derive from scaling functions. Another property which can be an advantage or a disadvantage is that these wavelets are not isotropic. This property enables us to detect point singularities.

Although we can construct the biorthogonal zonal kernels on all the grids introduced in Chapter 3, it is not possible to construct these wavelets on non-hierarchical grids. In other words, these wavelets are based on a hierarchical grid like Kurihara grid, the block grid (see Chapter 3) or HEALPix (see, e.g., [50]). In this work, we only focus on the wavelets constructed on the block grid.

5.1 Biorthogonal Scaling Functions

Let $\{K_j, \tilde{K}_j\}_{j\in\mathcal{J}}$ be a biorthogonal system of locally supported zonal kernels on the block grid (see Section 4.1.2). By using this biorthogonal system of kernels, we define the scaling function for the scale $l = 0$ as follows:

Definition 5.1.1 (Scaling Function for the Scale $l = 0$)
Let $\{K_j, \tilde{K}_j\}_{j\in\mathcal{J}}$ be a biorthogonal system that comes from Theorem 4.1.5 with $K_j^\wedge(0) = \tilde{K}_j^\wedge(0) = 1,\ j \in \mathcal{J}$. The primal scaling function for the scale $l = 0$ is defined by

$$ {}^m\phi_{ij}^{(0)} = K_j\left({}^m\eta_{2i+(2^k-1)j}\cdot\right), \quad (i,j) \in \mathcal{I}^{(0)},\ m = 0, 1, \tag{5.1} $$

similarly, the dual scaling function for the scale $l = 0$ is defined by

$$ {}^m\tilde{\phi}_{ij}^{(0)} = \tilde{K}_j\left({}^m\eta_{2i+(2^k-1)j}\cdot\right), \quad (i,j) \in \mathcal{I}^{(0)},\ m = 0, 1, \tag{5.2} $$

where the index set $\mathcal{I}^{(0)}$ is defined by (3.20).

According to our construction, it is clear that

$$ \int_\Omega {}^m\phi_{ij}^{(0)}(\xi)\, d\omega(\xi) = \int_\Omega {}^m\tilde{\phi}_{ij}^{(0)}(\xi)\, \mathrm{d}\omega(\xi) = 1, $$

for all $(i,j) \in \mathcal{I}^{(0)}$. Moreover, from the construction of the biorthogonal system $\{K_j, \tilde{K}_j\}_{j\in\mathcal{J}}$, it implies that both ${}^m\phi_{ij}^{(0)}$ and ${}^m\tilde{\phi}_{ij}^{(0)}$ are locally supported zonal kernels with

$$ \mathrm{supp}\ {}^m\phi_{ij}^{(0)} = \mathrm{supp}\ {}^m\tilde{\phi}_{ij}^{(0)} = [h_j, 1], $$

for all $(i,j) \in \mathcal{I}^{(0)}$.

We construct inductively the primal and dual scaling functions for other scales. To find the scaling function for the scale l, we compute the mean of the four scaling functions at the scale $l-1$ corresponding to four neighboring indices. It should be noted that there are some cases that there are no four scaling functions at the scale $l-1$ in a block. In such cases, we compute the mean of the scaling functions in this block and the neighboring block, if it exists, in the direction of the North (South) pole for the northern (southern) hemisphere. Figure 5.1 illustrates the idea of constructing the scaling function for the scale l.

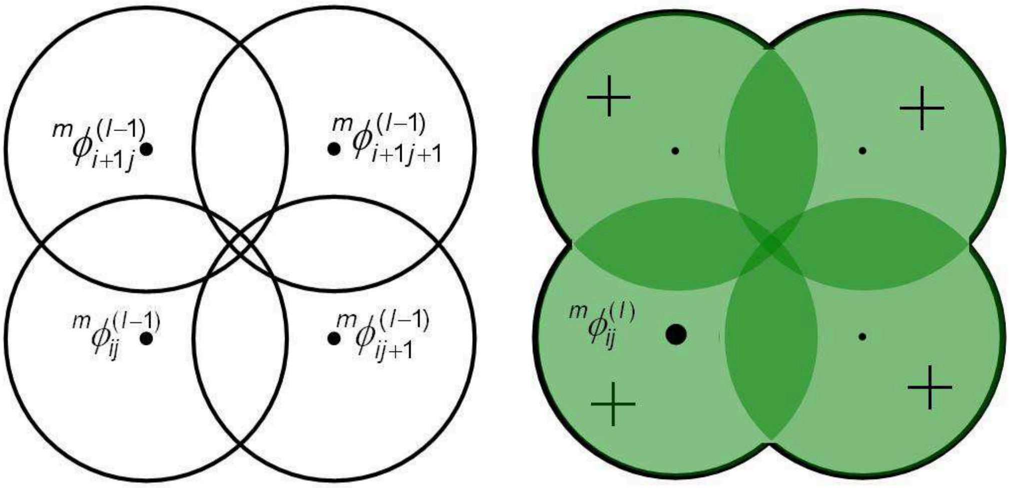


Figure 5.1: Scaling function for the scale l.
The scaling function for the scale l, except for some special cases, is the mean of four scaling functions at the scale $l-1$ corresponding to four neighboring indices.

Definition 5.1.2 (Scaling Function for the Scale l)
Suppose that the primal and dual scaling functions for the scale $l-1$ are given. Then for $k = 0, \ldots, N-l-1$ and $m = 0, 1$ and $(i,j) \in \mathcal{I}_k^{(l)}$ we define the primal scaling functions for the scale l as the following

$$ {}^{m}\phi^{(l)}_{ij} = \frac{1}{2}\left({}^{m}\phi^{(l-1)}_{ij} + {}^{m}\phi^{(l-1)}_{i+2^{k+l-1}\;j} + {}^{m}\phi^{(l-1)}_{i\;j+2^{l-1}} + {}^{m}\phi^{(l-1)}_{i+2^{k+l-1}\;j+2^{l-1}} \right), \tag{5.3} $$

and also for the dual scaling functions for the scale l we define

$$ {}^{m}\tilde{\phi}^{(l)}_{ij} = \frac{1}{2}\left({}^{m}\tilde{\phi}^{(l-1)}_{ij} + {}^{m}\tilde{\phi}^{(l-1)}_{i+2^{k+l-1}\;j} + {}^{m}\tilde{\phi}^{(l-1)}_{i\;j+2^{l-1}} + {}^{m}\tilde{\phi}^{(l-1)}_{i+2^{k+l-1}\;j+2^{l-1}} \right). \tag{5.4} $$

For $k = N-l$ we have to distinguish two cases.

- If $l = 1$, for the primal scaling function we assign

$$ {}^{m}\phi^{(l)}_{ij} = \frac{1}{\sqrt{2}}\left({}^{m}\phi^{(l-1)}_{ij} + {}^{m}\phi^{(l-1)}_{i+2^{k+l-1}\;j} \right), \quad (i,j) \in \mathcal{I}^{(1)}_{N-1}, \tag{5.5} $$

  and for the dual scaling function we assign

$$ {}^{m}\tilde{\phi}^{(l)}_{ij} = \frac{1}{\sqrt{2}}\left({}^{m}\tilde{\phi}^{(l-1)}_{ij} + {}^{m}\tilde{\phi}^{(l-1)}_{i+2^{k+l-1}\;j} \right), \quad (i,j) \in \mathcal{I}^{(1)}_{N-1}. \tag{5.6} $$

- If $l \neq 1$ then we set for the primal scaling function

$$^{m}\phi_{ij}^{(l)} = \frac{1}{2}\left({}^{m}\phi_{ij}^{(l-1)} + {}^{m}\phi_{i+2^{k+l-1}\,j}^{(l-1)} + 2\,{}^{m}\phi_{i\,j+2^{l-1}}^{(l-1)}\right), \quad (i,j) \in \mathcal{I}_{N-l}^{(l)}, \tag{5.7}$$

and similarly for the dual scaling function:

$$^{m}\tilde{\phi}_{ij}^{(l)} = \frac{1}{2}\left({}^{m}\tilde{\phi}_{ij}^{(l-1)} + {}^{m}\tilde{\phi}_{i+2^{k+l-1}\,j}^{(l-1)} + 2\,{}^{m}\tilde{\phi}_{i\,j+2^{l-1}}^{(l-1)}\right), \quad (i,j) \in \mathcal{I}_{N-l}^{(l)}. \tag{5.8}$$

Note that the primal and the dual scaling functions inherit being biorthogonal from K_j and $\tilde{K}_j$. The following theorem states this fact precisely.

Theorem 5.1.3 (Biorthogonalization of Scaling Functions)
For each scale l,

$$\left({}^{m}\phi_{ij}^{(l)},\ {}^{m'}\tilde{\phi}_{i'j'}^{(l)}\right)_{\mathcal{L}^2(\Omega)} = \delta_{ii'}\delta_{jj'}\delta_{mm'}. \tag{5.9}$$

It should be mentioned that the support of the primal (dual) scaling functions for the scale greater than zero is the union of the support of the primal (dual) scaling functions which it is built with (see Figure 5.1).

Because of the construction of the scaling functions, only the scaling functions for the scale $l = 0$ are zonal functions and the scaling functions for the scale greater than zero are not more zonal functions.

5.2 Wavelets Based on the Biorthogonal Scaling Functions

In this section, we construct three kinds of wavelets by using the biorthogonal scaling functions introduced in the last section. The idea of construction is as follows: Suppose that we made the scaling functions at each point of the block grid at the level $l-1$. Then consider a point of block grid at the level l. We compute the mean of four scaling functions at the scale $l-1$ of this point and three of its neighboring points from the block grid at the level $l-1$, where two of them are multiplied by -1. Depending on in which direction we multiply the scaling function of the scale $l-1$ with -1, we get East-West wavelets, North-South wavelets or Diagonal wavelets. In the following, for each of these wavelets, this idea is separately elaborated in detail (see also [43]). Note that, during our construction, we suppose that the scale l, $l = 1, \ldots, N$ is fixed.

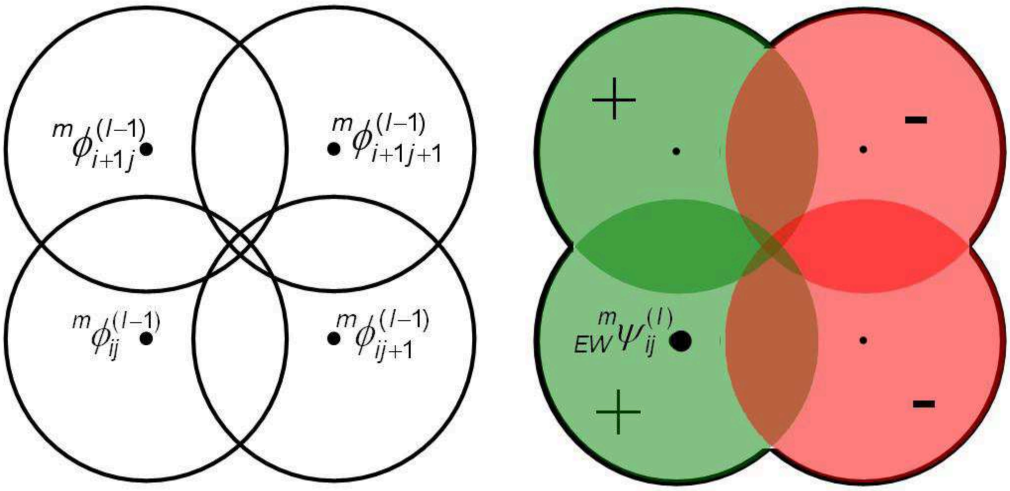


Figure 5.2: The East-West wavelets for the scale l.

5.2.1 East-West Wavelets

We construct the East-West wavelets for the scale l from the scaling functions of the scale $l-1$. The East-West wavelet for the scale l, except for some special cases, is the mean of four scaling functions at the scale $l-1$ corresponding to four neighboring indices where two scaling functions in the East are multiplied by -1. The idea of the construction of East-West wavelets is shown in Figure 5.2.

The primal East-West wavelets for the scale l are the differences between the primal scaling functions for the scale $l-1$ in direction East-West as follows:

$$ {}^{m}_{EW}\psi^{(l)}_{ij} = \frac{1}{2}\left({}^{m}\phi^{(l-1)}_{ij} - {}^{m}\phi^{(l-1)}_{i+2^{k+l-1}\,j} + {}^{m}\phi^{(l-1)}_{i\,j+2^{l-1}} - {}^{m}\phi^{(l-1)}_{i+2^{k+l-1}\,j+2^{l-1}} \right), \quad (5.10) $$

where $(i,j) \in \mathcal{I}^{(l)}_k$ for $k = 0, \ldots, N-l-1$. For the dual wavelets for the scale l, we define

$$ {}^{m}_{EW}\tilde{\psi}^{(l)}_{ij} = \frac{1}{2}\left({}^{m}\tilde{\phi}^{(l-1)}_{ij} - {}^{m}\tilde{\phi}^{(l-1)}_{i+2^{k+l-1}\,j} + {}^{m}\tilde{\phi}^{(l-1)}_{i\,j+2^{l-1}} - {}^{m}\tilde{\phi}^{(l-1)}_{i+2^{k+l-1}\,j+2^{l-1}} \right), \quad (5.11) $$

where $(i,j) \in \mathcal{I}^{(l)}_k$ for $k = 0, \ldots, N-l-1$.

Similar to the scaling function we have to consider the special case $k = N-l$ separately as follows: The primal wavelets are

$$ {}^{m}_{EW}\psi^{(l)}_{ij} = \frac{1}{\sqrt{2}}\left({}^{m}\phi^{(l-1)}_{ij} - {}^{m}\phi^{(l-1)}_{i+2^{k+l-1}\,j} \right), \quad (i,j) \in \mathcal{I}^{(1)}_{N-l}, \quad (5.12) $$

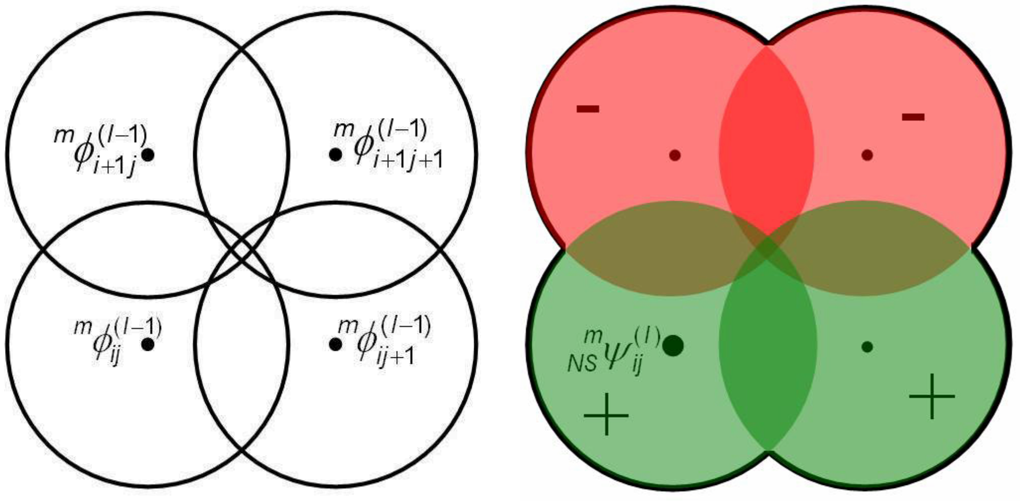


Figure 5.3: The North-South wavelets for the scale l.

and the dual wavelets are

$$ {}^{m}_{EW}\tilde{\psi}^{(l)}_{ij} = \frac{1}{\sqrt{2}} \left({}^{m}\tilde{\phi}^{(l-1)}_{ij} - {}^{m}\tilde{\phi}^{(l-1)}_{i+2^{k+l-1}\ j} \right), \quad (i,j) \in \mathcal{I}^{(1)}_{N-l}. \tag{5.13} $$

5.2.2 North-South Wavelets

Analogously to the East-West wavelets, we construct the North-South wavelets for the scale l from the scaling functions of the scale $l-1$. The North-South wavelet for the scale l, except for some special cases, is the mean of four scaling functions at the scale $l-1$ corresponding to four neighboring indices where two scaling functions in the south are multiplied by -1. The idea of the construction of North-South wavelets is shown in Figure 5.3.

The primal North-South wavelets for the scale l are the differences between the primal scaling functions for the scale $l-1$ in the North-South direction as follows:

$$ {}^{m}_{NS}\psi^{(l)}_{ij} = \frac{(-1)^m}{2} \left({}^{m}\phi^{(l-1)}_{ij} + {}^{m}\phi^{(l-1)}_{i+2^{k+l-1}\ j} - {}^{m}\phi^{(l-1)}_{i\ j+2^{l-1}} - {}^{m}\phi^{(l-1)}_{i+2^{k+l-1}\ j+2^{l-1}} \right), \tag{5.14} $$

where $(i,j) \in \mathcal{I}^{(l)}_{k}$ for $k = 0, \ldots, N-l-1$. For the dual wavelets for the scale l we define

$$ {}^{m}_{NS}\tilde{\psi}^{(l)}_{ij} = \frac{(-1)^m}{2} \left({}^{m}\tilde{\phi}^{(l-1)}_{ij} + {}^{m}\tilde{\phi}^{(l-1)}_{i+2^{k+l-1}\ j} - {}^{m}\tilde{\phi}^{(l-1)}_{i\ j+2^{l-1}} - {}^{m}\tilde{\phi}^{(l-1)}_{i+2^{k+l-1}\ j+2^{l-1}} \right), \tag{5.15} $$

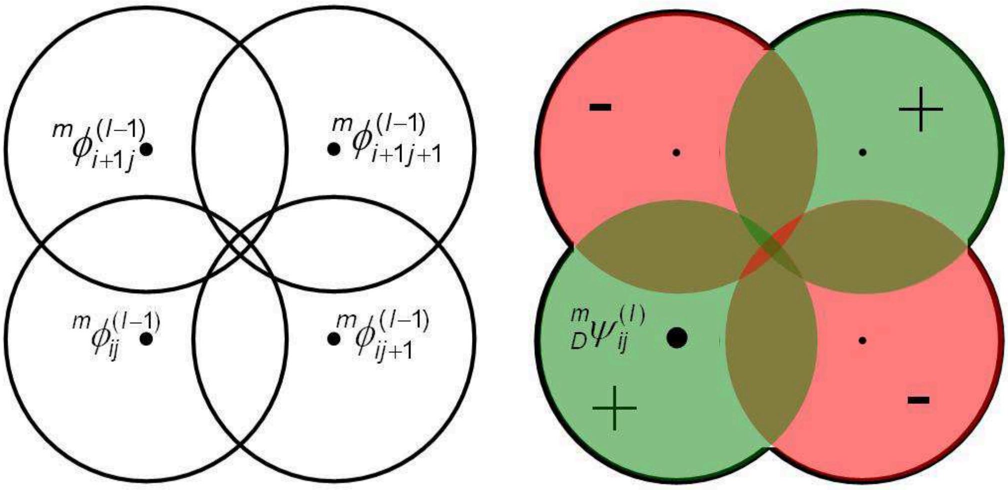


Figure 5.4: The diagonal wavelets for the scale l.

where $(i,j) \in \mathcal{I}^{(l)}_k$ for $k = 0, \ldots, N-l-1$.

Similar to the scaling function we have to consider the special case $k = N - l$ separately as follows:

- If $l = 1$, because the block ${}^N B^{(0)}_N$ (or ${}^S B^{(0)}_N$ for the southern hemisphere) does not exist, we cannot define the North-South wavelet for this case.
- If $l \neq 1$ then we set

$$ {}^m_{NS}\psi^{(l)}_{ij} = \frac{(-1)^m}{2}\left({}^m\phi^{(l-1)}_{ij} + {}^m\phi^{(l-1)}_{i+2^{k+l-1}\,j} - 2\,{}^m\phi^{(l-1)}_{i\,j+2^{l-1}}\right), \quad (i,j) \in \mathcal{I}^{(l)}_{N-l}, \tag{5.16} $$

and similarly for the dual wavelets:

$$ {}^m_{NS}\tilde{\psi}^{(l)}_{ij} = \frac{(-1)^m}{2}\left({}^m\tilde{\phi}^{(l-1)}_{ij} + {}^m\tilde{\phi}^{(l-1)}_{i+2^{k+l-1}\,j} - 2\,{}^m\tilde{\phi}^{(l-1)}_{i\,j+2^{l-1}}\right), \quad (i,j) \in \mathcal{I}^{(l)}_{N-l}. \tag{5.17} $$

5.2.3 Diagonal Wavelets

The diagonal wavelet for the scale l, except for some special cases, is the mean of four scaling functions at the scale $l-1$ corresponding to four neighboring indices where two scaling functions in the diagonal are multiplied by -1. The idea of the construction of Diagonal wavelets is shown in Figure 5.4.

We define the primal diagonal wavelets for the scale l as follows:

$$ {}^{m}_{D}\psi^{(l)}_{ij} = \frac{(-1)^m}{2}\left({}^{m}\phi^{(l-1)}_{ij} - {}^{m}\phi^{(l-1)}_{i+2^{k+l-1}\,j} - {}^{m}\phi^{(l-1)}_{i\,j+2^{l-1}} + {}^{m}\phi^{(l-1)}_{i+2^{k+l-1}\,j+2^{l-1}} \right), \tag{5.18} $$

where $(i,j) \in \mathcal{I}^{(l)}_k$ for $k = 0, \ldots, N-l-1$. For the dual diagonal wavelets for the scale l we define

$$ {}^{m}_{D}\tilde{\psi}^{(l)}_{ij} = \frac{(-1)^m}{2}\left({}^{m}\tilde{\phi}^{(l-1)}_{ij} - {}^{m}\tilde{\phi}^{(l-1)}_{i+2^{k+l-1}\,j} - {}^{m}\tilde{\phi}^{(l-1)}_{i\,j+2^{l-1}} + {}^{m}\tilde{\phi}^{(l-1)}_{i+2^{k+l-1}\,j+2^{l-1}} \right), \tag{5.19} $$

where $(i,j) \in \mathcal{I}^{(l)}_k$ for $k = 0, \ldots, N-l-1$.

Since for the case $k = N - l$ there are only two scaling functions in a block, we cannot construct the diagonal wavelets for this case.

Remark 5.2.1 (Zero Mean Property of the X-Wavelets)
It should be mentioned that all of the three kinds of wavelets constructed above have the zero mean property, i.e., for $l = 1, \ldots, N$ we have

$$ \int_\Omega {}^{m}_{X}\psi^{(l)}_{ij}(\xi)\,\mathrm{d}\omega(\xi) = \int_\Omega {}^{m}_{X}\tilde{\psi}^{(l)}_{ij}(\xi)\,\mathrm{d}\omega(\xi) = 0, \quad (i,j) \in \mathcal{I}^{(l)}_k,\ k = 0, \ldots, N-l-1, \tag{5.20} $$

where $X \in \{EW, NS, D\}$.

5.3 Multiresolution Analysis

In this section a multiresolution analysis for the space V defined by (4.20) will be introduced. Most of the multiresolution analysis are for the space of all square integrable functions, but here, our multiresolution analysis is for a finite dimension subspace of the space of all square integrable functions. Indeed, from a numerical point of view, this is what has to be done for practical applications (cf. [43]).

Similar to (4.20), we introduce the scaling spaces as follows:

Definition 5.3.1 (Scaling Spaces)
Let $N \geq 2$ be fixed, and ${}^{m}\tilde{\phi}^{(l)}_{ij}$ be the scaling functions defined in Definitions 5.1.1 and 5.1.2. Then we define the scaling space V_l by

$$ V_l = \operatorname*{span}_{(i,j)\in\mathcal{I}^{(l)}} \left\{ {}^{0}\tilde{\phi}^{(l)}_{ij},\ {}^{1}\tilde{\phi}^{(l)}_{ij} \right\}, \tag{5.21} $$

for $l = 1, \ldots, N$.

Note that the scaling spaces form a nested sequence of subspaces in the form

$$\{0\} \subset V_N \subset V_{N-1} \subset \ldots \subset V_1 \subset V_0 \subset \mathcal{L}^2(\Omega). \tag{5.22}$$

Again, similar to (4.21), by using the biorthogonality property of the scaling functions we define projection operators $P_l : \mathcal{L}^2(\Omega) \to V_l$ by

$$P_l(F) = \sum_{(i,j)\in\mathcal{I}^{(l)}} \left({}^0\phi_{ij}^{(l)}, F \right) {}^0\tilde{\phi}_{ij}^{(l)} + \sum_{(i,j)\in\mathcal{I}^{(l)}} \left({}^1\phi_{ij}^{(l)}, F \right) {}^1\tilde{\phi}_{ij}^{(l)}, \tag{5.23}$$

for $l = 0, \ldots, N-1$, where, as always, $(\cdot, \cdot)$ is understood in the topology of $\mathcal{L}^2(\Omega)$.

Recall that similar to Remark 4.2.1, if $F \in V_l$ then $P_l(F) = F$. In the sense of signal processing, the projection operators P_l can be associated with low–pass filtering. The difference between two succeeding scale spaces is collected in a detail space. Therefore, we define

Definition 5.3.2 (Wavelet Spaces)
Let ${}^m_X\tilde{\psi}_{ij}^{(l)}$, $X \in \{EW,\ NS,\ D\}$ be the available dual wavelets defined in Section 5.2. Then we define the wavelet space W_l by

$$W_l = \operatorname*{span}_{\substack{(i,j)\in\mathcal{I}^{(l)} \\ X\in\{EW,\ NS,\ D\}}} \left\{ {}^0_X\tilde{\psi}_{ij}^{(l)},\ {}^1_X\tilde{\psi}_{ij}^{(l)}\ |\text{when available} \right\}, \tag{5.24}$$

for $l = 1, \ldots, N$.

It should be mentioned that, for example, in the definition of wavelet spaces by using the diagonal wavelets, there are not any diagonal wavelets in the last block near the pole. In such a case, we use another available kind of wavelets, e.g., East-West wavelets, to define the diagonal wavelet spaces.

Because of the decomposition of V_{l-1} as a direct sum of W_l and V_l, i.e.,

$$W_l \oplus V_l = V_{l-1},\ \ l = 1, \ldots, N, \tag{5.25}$$

we are able to deduce

Lemma 5.3.3
For $N \geq 2$ fixed,

$$V_0 = V_N \oplus \bigoplus_{l=1}^{N} W_l. \tag{5.26}$$

The space W_l contains the detail information of a signal F. In fact, our method enables a dynamical space-varying frequency distribution of a function $F \in \mathcal{L}^2(\Omega)$. Consequently, the wavelet analysis is not only related to a frequency band (according to the scale l), but also scale–dependent spatial information is provided.

The analysis is performed by the wavelets transform that is defined as follows: For the scale l and the index $(i,j) \in \mathcal{I}^{(l)}$,

$${}^m_X\mathrm{WT}(l; i, j; F) = ({}^m_X\psi^{(l)}_{ij}, F), \;\; F \in \mathcal{L}^2(\Omega), \tag{5.27}$$

where $m = 0, 1$ and $X \in \{EW, NS, D\}$ (when defined). Due to the biorthogonality of the wavelets and the dual wavelets, we are able to introduce the operators $R_l : \mathcal{L}^2(\Omega) \longrightarrow W_l$ by

$$R_l(F) = \sum_{m=0}^{1} \sum_{(i,j)\in\mathcal{I}^{(l)}} \sum_{X\in\{EW,NS,D\}} ({}^m_X\psi^{(l)}_{ij}, F)\, {}^m_X\tilde{\psi}^{(l)}_{ij}. \tag{5.28}$$

These operators act as band–pass filters on a signal $F \in \mathcal{L}^2(\Omega)$.

The wavelet analysis and the reconstruction can be summarized in the following theorem:

Theorem 5.3.4 (Reconstruction Formula)
For $F \in \mathcal{L}^2(\Omega)$,

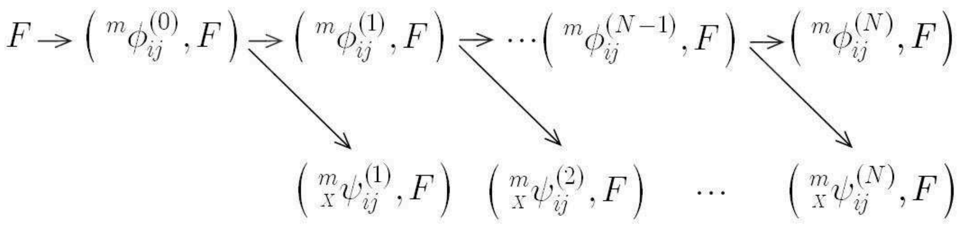
$$\begin{aligned} P_0(F) \;=\; & \sum_{m=0}^{1} \left(\sum_{(i,j)\in\mathcal{I}^{(0)}} ({}^m\phi^{(N)}_{ij}, F)\, {}^m\tilde{\phi}^{(n)}_{ij} \right. \\ & \left. + \sum_{l=1}^{N} \sum_{(i,j)\in\mathcal{I}^{(l)}} \sum_{X\in\{EW,NS,D\}} ({}^m_X\psi^{(l)}_{ij}, F)\, {}^m_X\tilde{\psi}^{(l)}_{ij} \right). \end{aligned}$$

The wavelet analysis and the reconstruction can be organized as a fast wavelet transform. Basis for these algorithms are the filter coefficients in the scale relation, which are implicitly defined in Definition 5.1.1 and 5.1.2. For example, it follows from (5.10) that

$$\begin{aligned} {}^{m}_{EW}\mathrm{WT}(l;i,j;F) &= \frac{1}{2}\Big(({}^{m}\phi_{ij}^{(l-1)}, F) - ({}^{m}\phi_{i+2^{k+l-1}\; j}^{(l-1)}, F) \\ &+ ({}^{m}\phi_{i\; j+2^{l-1}}^{(l-1)}, F) - ({}^{m}\phi_{i+2^{k+l-1}\; j+2^{l-1}}^{(l-1)}, F)\Big) \,. \end{aligned}$$

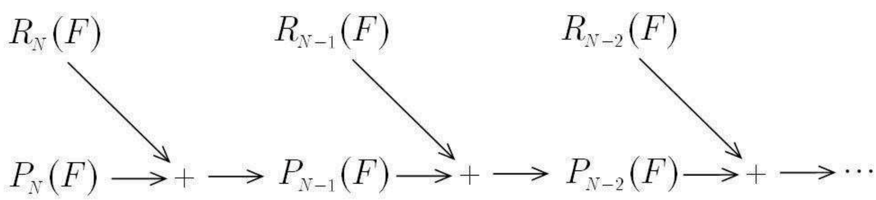
In fact, we end up with a fast tree algorithm of the following structure:

Wavelet Decomposition



The reconstruction can be organized as follows:

Wavelet Reconstruction



5.4 Examples

In this section, we present two examples for the wavelet techniques described in Section 5.2. Both examples illustrate the gravity potential of buried point masses with different depths. The first example is harmonic and globally supported while the second example is non-harmonic and locally supported.

Example 5.4.1
The function $\xi \mapsto F(\xi)$ defined by

$$F(\xi) = \sum_{i=1}^{4} \frac{1}{\|\xi - h_i \eta_i\|}, \quad \xi \in \Omega \tag{5.29}$$

is chosen as the trial function, where $\eta_i,\ i = 1, \ldots, 4$ are four fixed points on the sphere Ω. The fixed parameters $h_i,\ i = 1, \ldots, 4$ are equal to $0.9,\ 0.8,\ 0.7,\ 0.6$, respectively. Figure 5.5 illustrates this function on the sphere Ω. As we mentioned before, this function is harmonic and globally supported on the sphere Ω. The function is sampled at the block grid with $N = 11$ with about 11,200,000 points on the sphere. Because the grid is rather dense for this function, we decided to approximate $({}^m\phi_{ij}^{(0)}, F)$ just by the value $F({}^m\eta_{2i+2^k-1,j})$. The scaling functions at the different scales are represented in Figure 5.6. The wavelet transforms at different scales of types EW, NS and D are plotted in the figures 5.7, 5.8 and 5.9, respectively. Note that the wavelet transforms at the scale j are multiplied with a factor 2^{-j} to get comparable results.

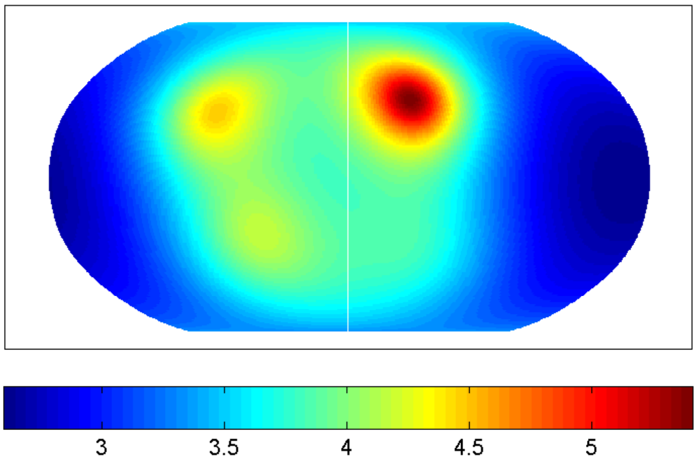


Figure 5.5: The function defind by (5.29),
where $h_1 = 0.9$, $h_2 = 0.8$, $h_3 = 0.7$ and $h_4 = 0.6$.

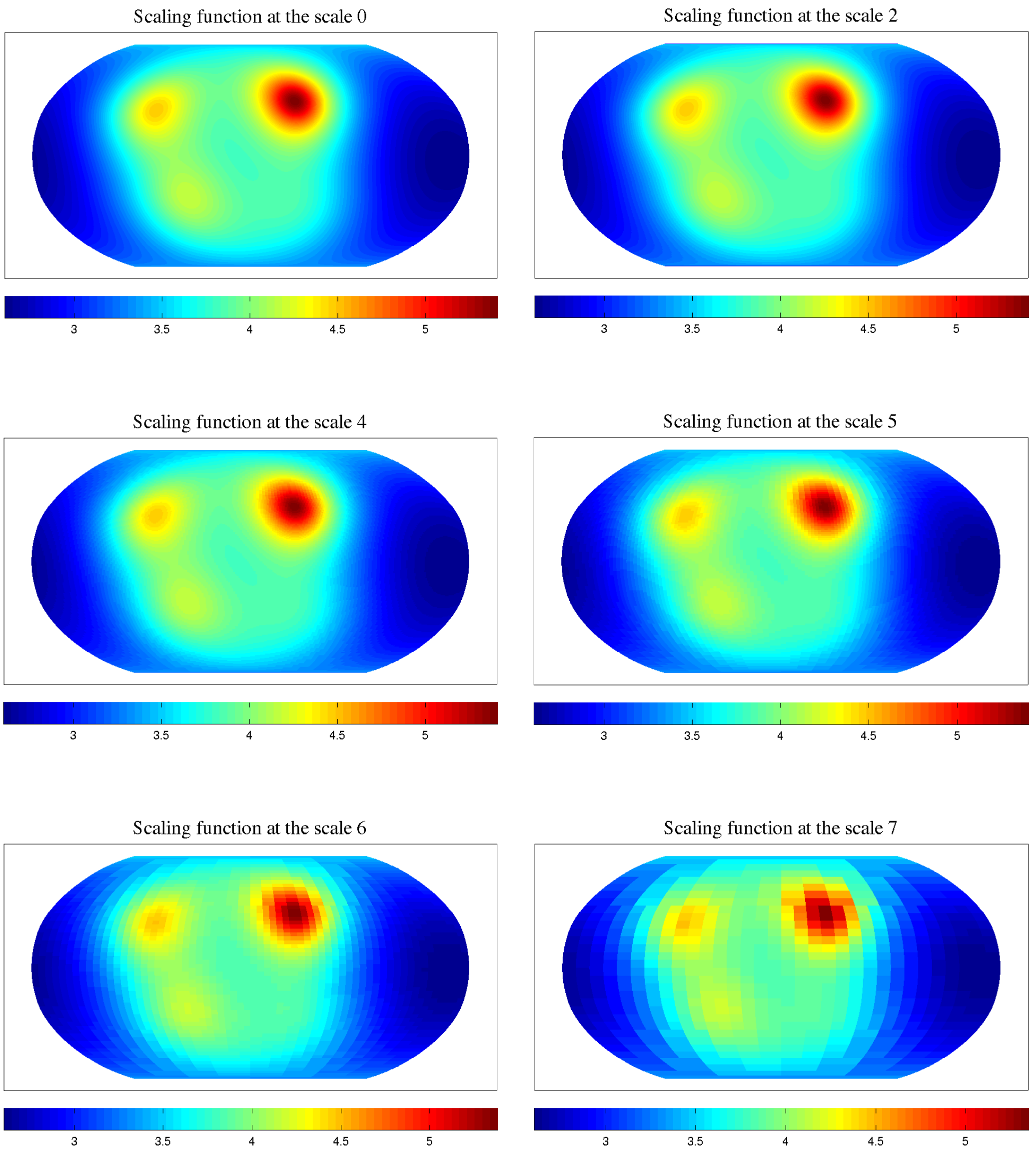


Figure 5.6: The scaling functions of Example 5.4.1
at the scales 0, 2, 4, 5, 6, 7.

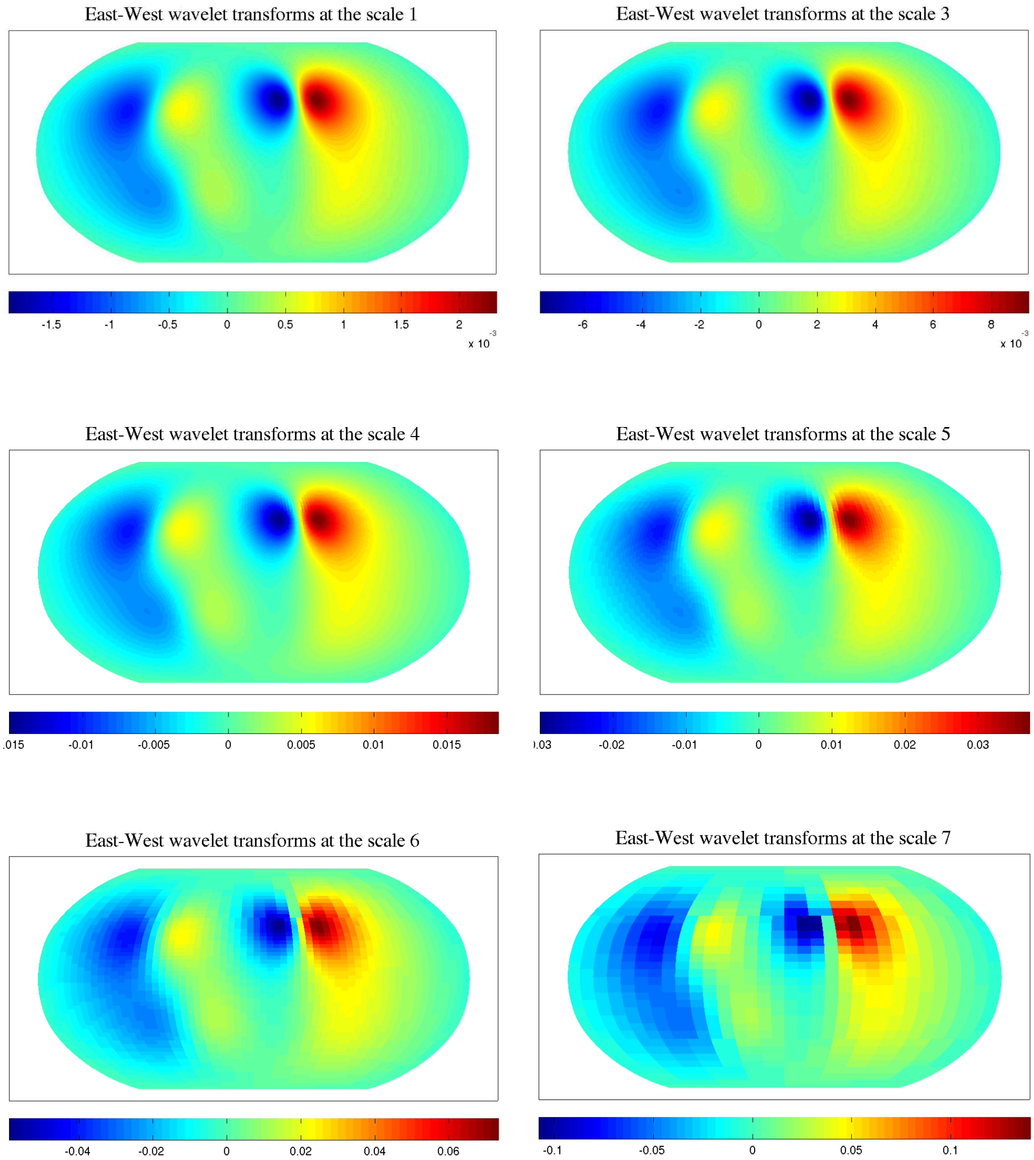


Figure 5.7: The value of East-West wavelet transforms for the trial function of Example 5.4.1 at the scales 1, 3, 4, 5, 6, 7.

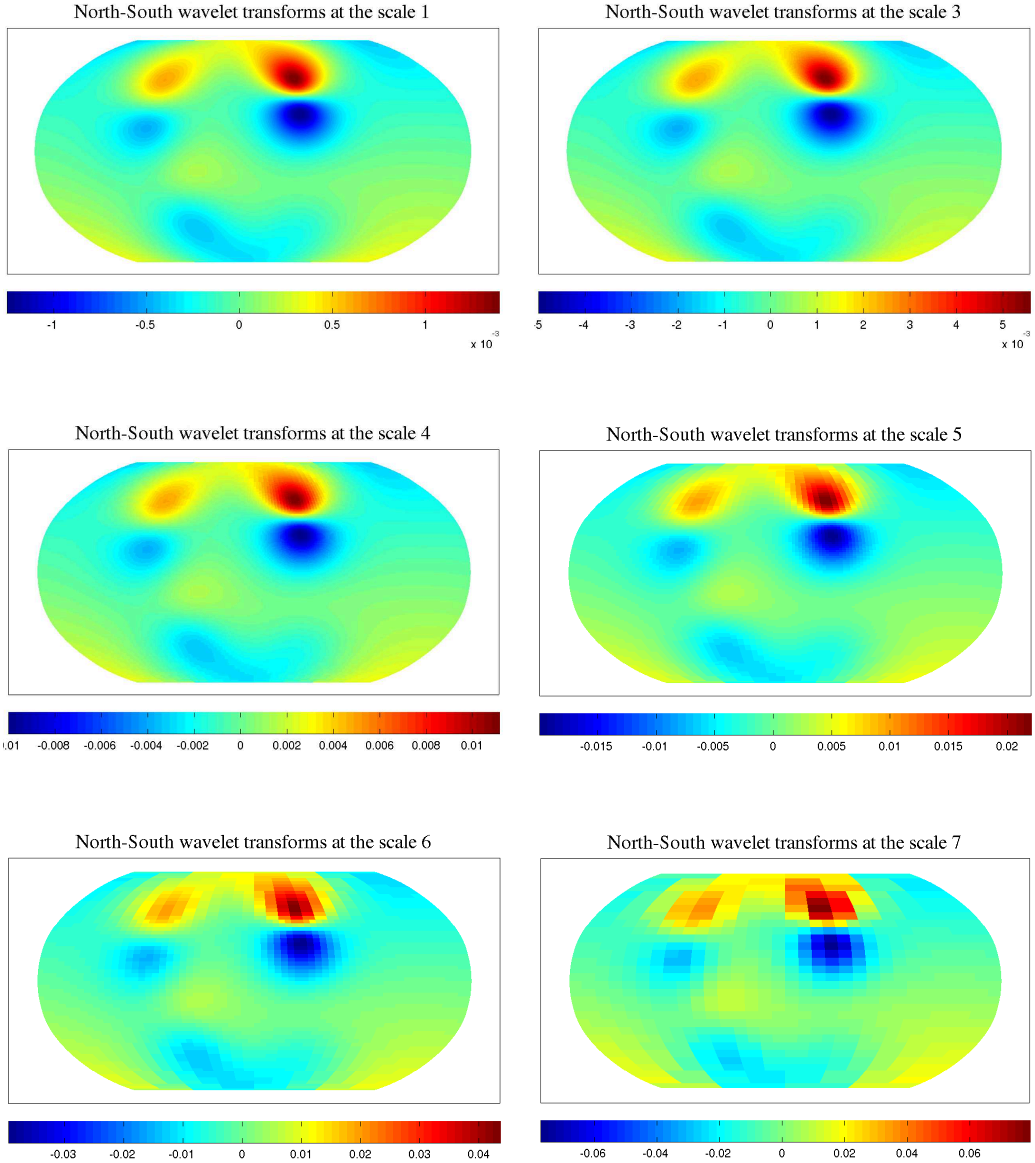


Figure 5.8: The value of North-South wavelet transforms for the trial function of Example 5.4.1 at the scales 1, 3, 4, 5, 6, 7.

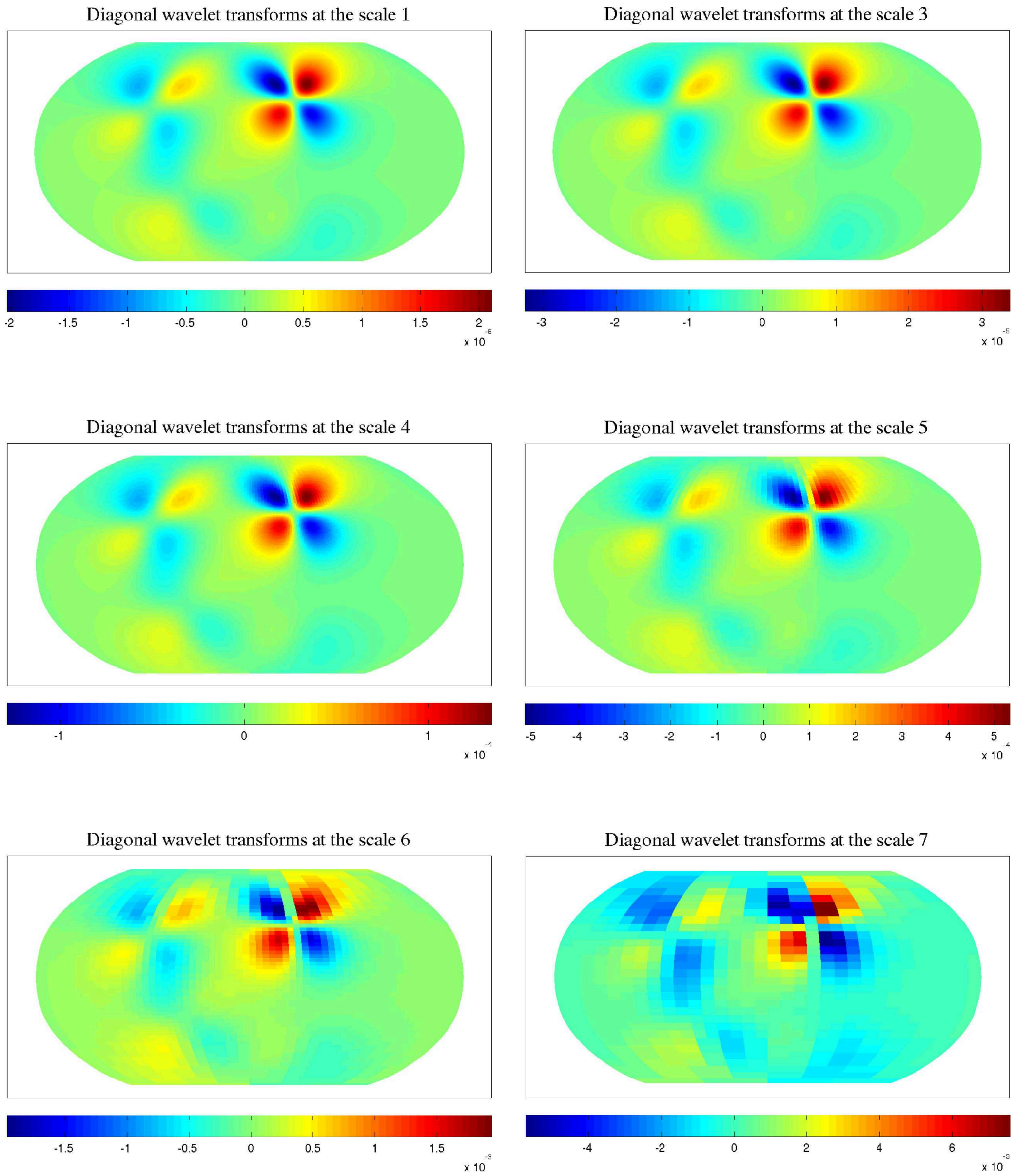


Figure 5.9: The value of Diagonal wavelet transforms for the trial function of Example 5.4.1 at the scales 1, 3, 4, 5, 6, 7.

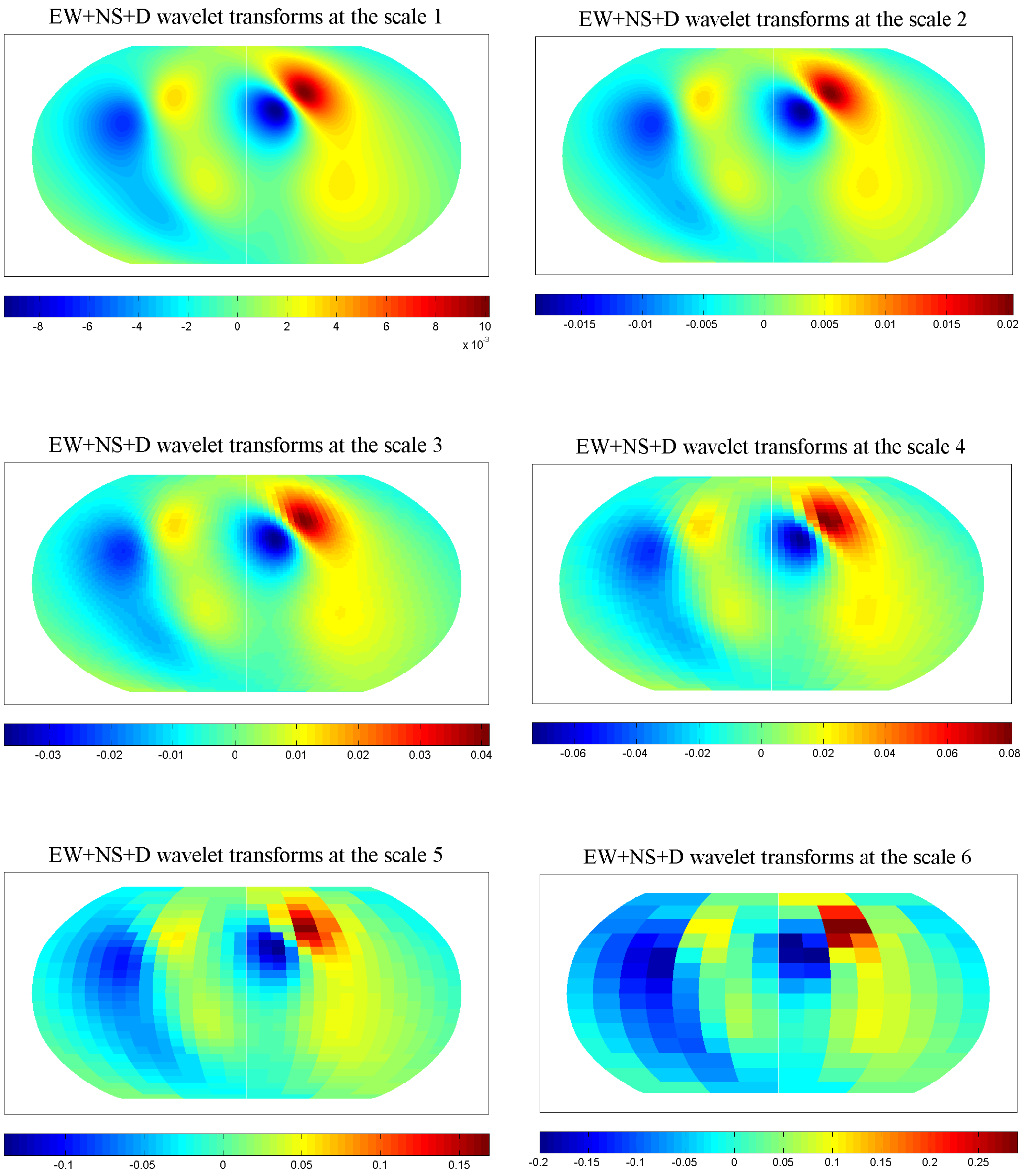


Figure 5.10: The value of EW+NS+D wavelet transforms of Example 5.4.1 at the scales 1, 3, 4, 5, 6, 7.

Example 5.4.2
In this example, the trial function is composed by three normalized Wendland functions. The function is illustrated in Figure 5.11 on the sphere Ω and in Figure 5.12 in two-dimensional plane. The function is sampled at the block grid with $N = 10$. Thus we have about 2,800,000 points on the sphere. Similar to Example 5.4.1, we approximated $({}^m\phi^{(0)}_{ij}, F)$ just by the value $F({}^m\eta_{2i+2^k-1,j})$. Figure 5.13 illustrates the scaling functions at the different scales. The wavelet transforms at different scales of types EW, NS and D are plotted in the figures 5.14, 5.15 and 5.16, respectively.

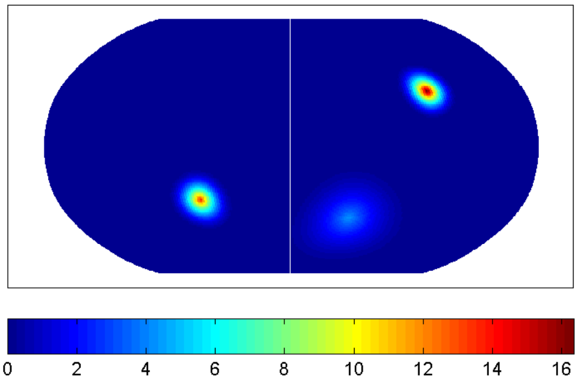


Figure 5.11: The trial function of Example 5.4.2 composed of three Wendland's functions: $K_{1.5}$, $K_{2.7}$ and K_3.

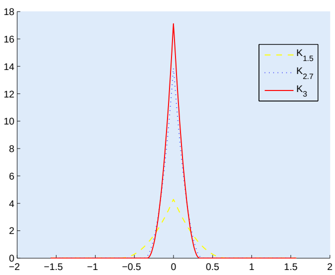


Figure 5.12: The Wendland Function $\vartheta \mapsto K_h(\cos\vartheta)$, $\vartheta \in [-\pi/2, \pi/2]$, for $h = 1.5, 2.7, 3$.

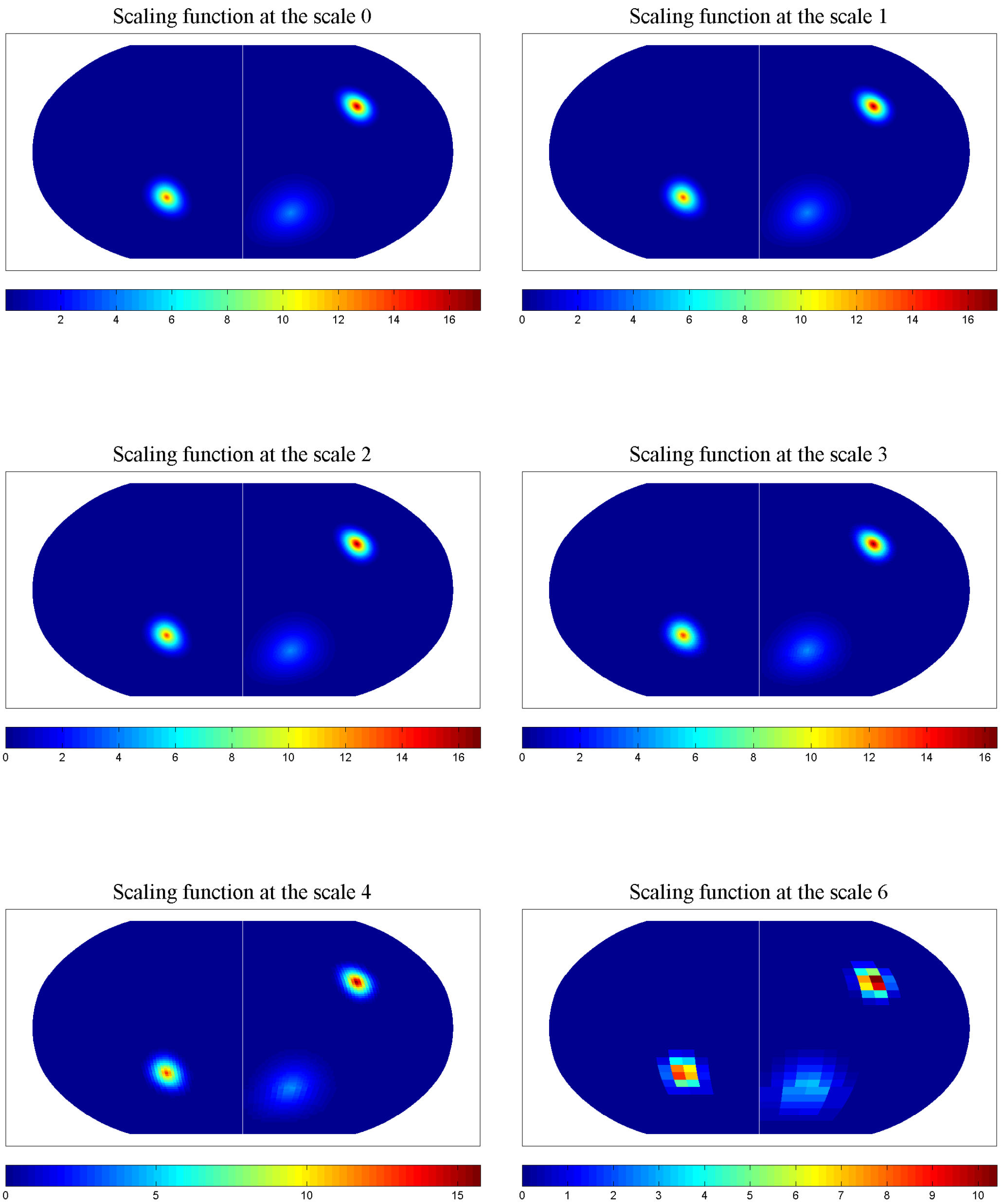


Figure 5.13: The scaling functions of Example 5.4.2
at the scales 0, 1, 2, 3, 4, 6.

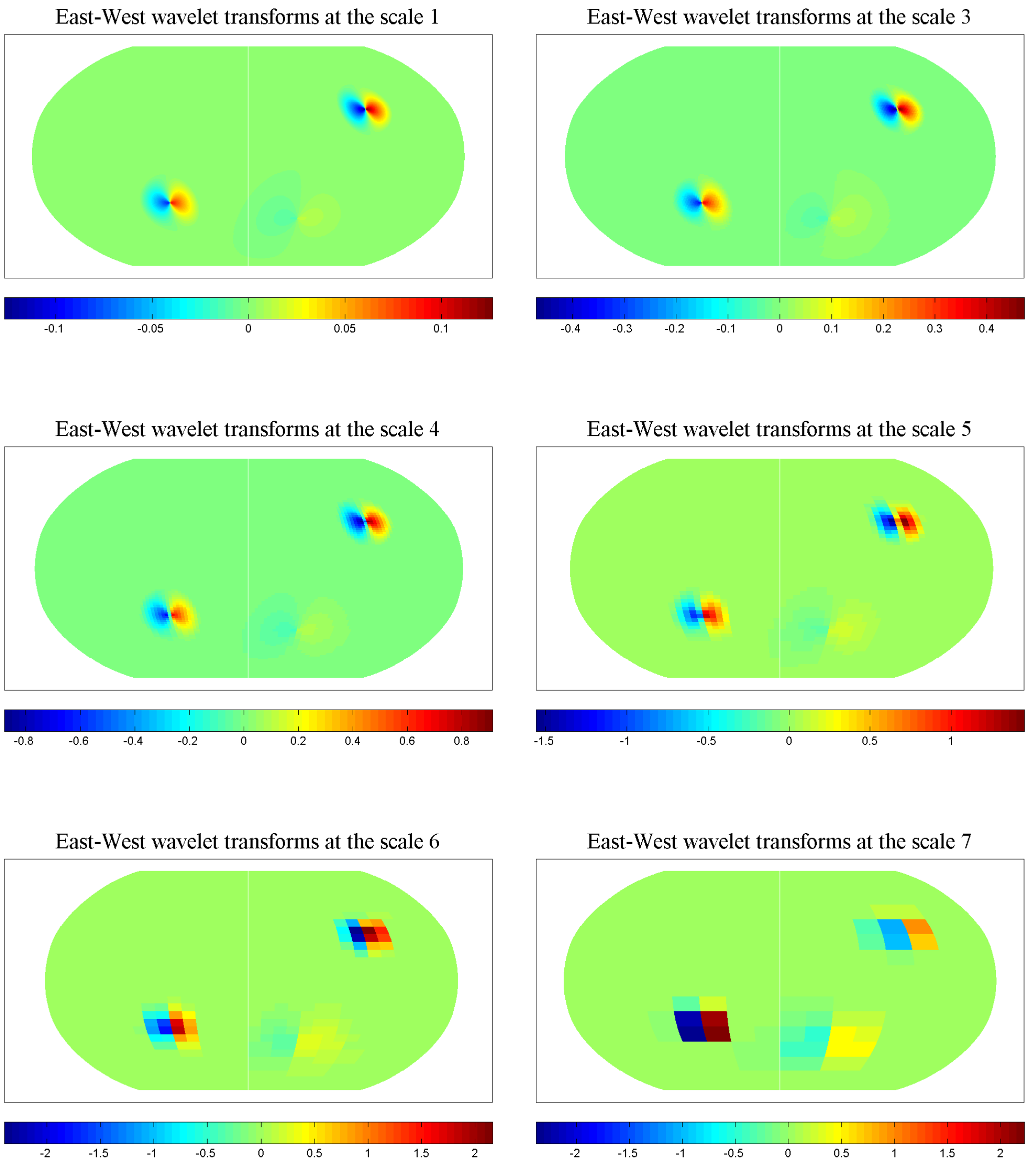


Figure 5.14: The value of East-West wavelet transforms of Example 5.4.2 at the scales 1, 3, 4, 5, 6, 7.

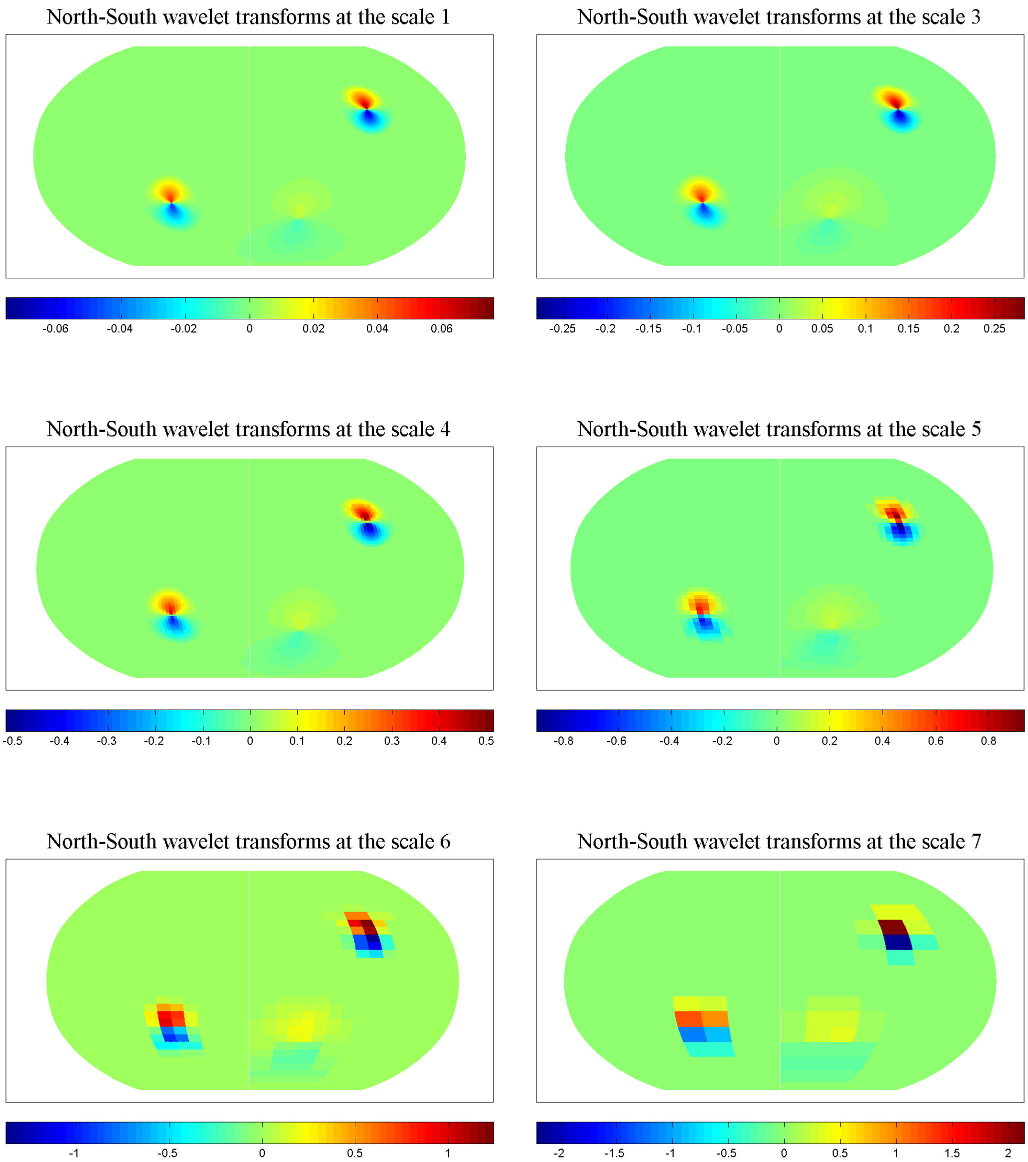


Figure 5.15: The value of North-South wavelet transforms of Example 5.4.2 at the scales 1, 3, 4, 5, 6, 7.

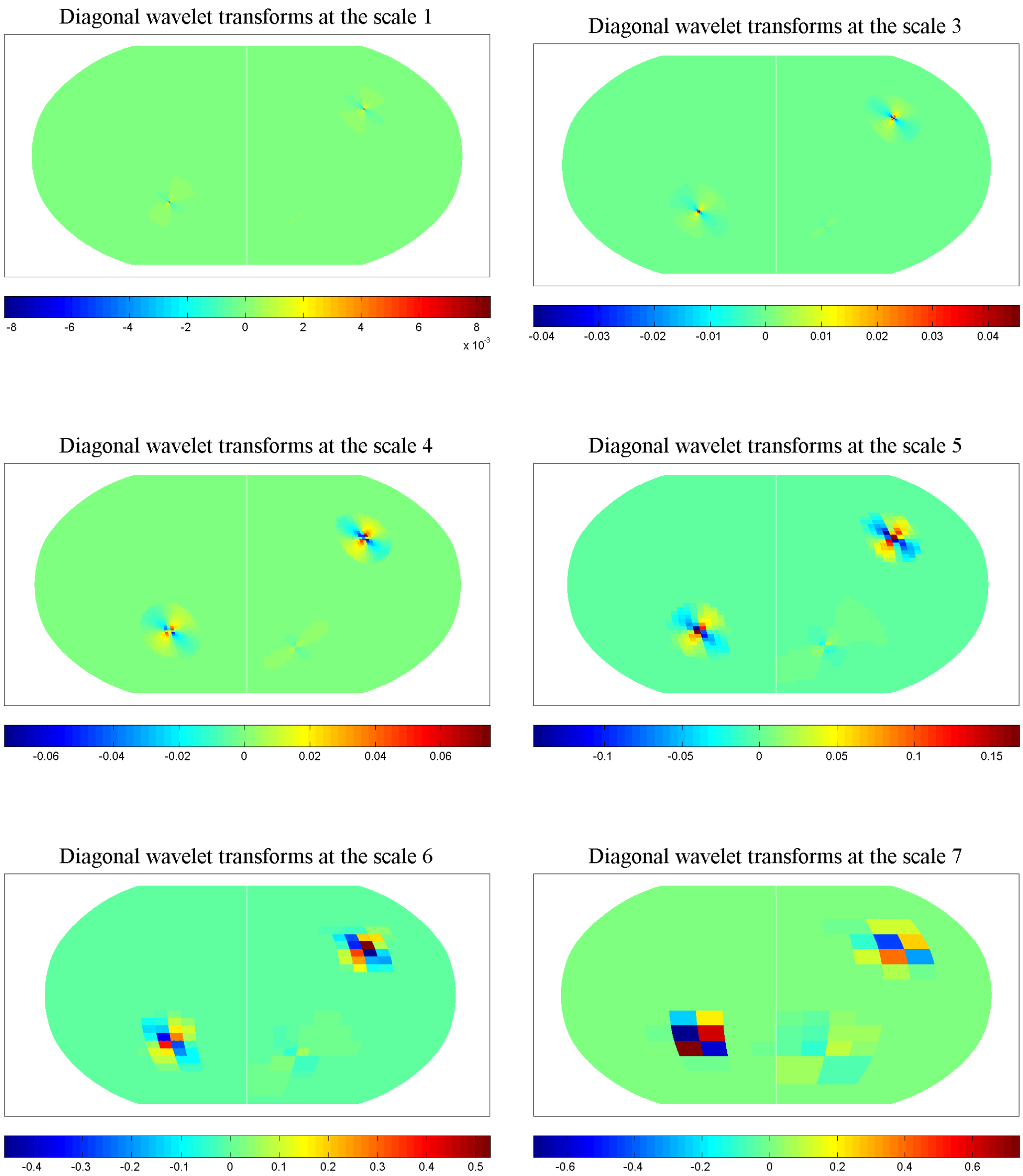


Figure 5.16: The value of Diagonal wavelet transforms of Example 5.4.2 at the scales 1, 3, 4, 5, 6, 7.

Chapter 6

Summary and Outlook

In this thesis we have discussed locally supported zonal kernels as a powerful tool for the multiscale approximation of functions on the sphere. We have investigated the smoothed Haar functions and, by a new technique, we have extended the explicit expressions for the Legendre transforms of the smoothed Haar functions proposed by [92]. Moreover, we have extended the Wendland functions to the sphere. These functions have appealing properties: They are strictly positive definite on the sphere and are locally supported on the sphere and their native space is known, thus an error analysis is possible.

Based on the locally supported zonal kernels, we have developed a system of biorthogonal locally supported zonal kernels on the sphere. This system of biorthogonal zonal kernels has almost all advantages of an orthogonal system of functions. In fact, the numerical effort for the construction of this system is low, and also, all the tasks are easy to perform.

We have used the system of biorthogonal zonal kernels as the scaling functions at the scale 0 for a biorthogonal multiscale analysis on the sphere. Upon the biorthogonal scaling functions, we have investigated a system of biorthogonal locally supported wavelets. Because the wavelet analysis benefits the local support and biorthogonal properties of the scaling functions and the wavelets, we have therefore obtained fast algorithms that are easy to implement, especially, we have established a fast wavelet transform. Furthermore, this method enables fast approximations of local phenomena, too. Even more, it should be mentioned that it is possible to apply the described concept in a fully local framework.

Further investigations need to be done for the implementation of the fast wavelet analysis for the problem involving the rotational invariant pseudodifferential operators, especially with real satellite data. For example, a future task is to formulate this method for the problem of downward continuation by means of inverse Abel–Poisson–type operators (see [32], [34], [42], [92]).

Bibliography

[1] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, Dover Publication Inc., New York, 1964.

[2] G. Andrews, R. Askey, and R. Roy, *Special Functions*, Cambridge University Press, Cambridge, England, 1999.

[3] J.-P. Antoine, L. Demanet, L. Jacques, and P. Vandergheynst, *Wavelets on the Sphere: Implementation and Approximations*, Applied and Computational Harmonic Analysis **13** (2002), no. 3, 177–200.

[4] J.-P. Antoine and P. Vandergheynst, *Wavelets on the* 2*-sphere: A Group-theoretical Approach*, Applied and Computational Harmonic Analysis **7** (1999), no. 3, 262–291.

[5] N. Aronszajn, *Theory of Reproducing Kernels*, Transactions of the American Mathematical Society **68** (1950), 337–404.

[6] S. Axler, P. Bourdon, and W. Ramey, *Harmonic Function Theory*, second ed., Graduate Texts in Mathematics, Vol. 137, Springer, New York, 2001.

[7] F. Bauer, W. Freeden, and M. Schreiner, *A Tree Algorithm for Isotropic Finite Elements on the Sphere*, Numerical Functional Analysis and Optimization **27** (2006), no. 1, 1–24.

[8] H. Berens, P. L. Butzer, and S. Pawelke, *Limitierungsverfahren von Reihen mehrdimensionaler Kugelfunktionen und deren Saturationsverhalten*, Publications of the Research Institute for Mathematical Sciences **4** (1968), 201–268.

[9] S. Bochner, *Vorlesungen über Fouriersche Integrale*, Monatshefte für Mathematik **40** (1932), no. 1.

[10] S. Bochner, *Monotone Funktionen, Stieltjessche Integrale und harmonische Analyse*, Mathematische Annalen **108** (1933), no. 1, 378–410.

[11] S. Bochner, *Positive Zonal Functions on Spheres*, Proceedings of the National Academy of Sciences of the United States of America, **40** (1954), no. 12, 1141–1147.

[12] I. Bogdanova, P. Vandergheynst, J-P. Antoine, L. Jacques, and M. Morvidone, *Stereographic Wavelet Frames on the Sphere*, Applied and Computational Harmonic Analysis **19** (2005), no. 2, 223–252.

[13] G. P. Bonneau, *Optimal Triangular Haar Bases for Spherical Data*, VIS '99: Proceedings of the conference on Visualization '99 (Los Alamitos, CA, USA), IEEE Computer Society Press, 1999, pp. 279–284.

[14] M. D. Buhmann, *Radial Basis Functions: Theory and Implementations*, Cambridge Monographs on Applied and Computational Mathematics, Vol. 12, Cambridge University Press, Cambridge, 2003.

[15] D. Chen, V. A. Menegatto, and X. Sun, *A Necessary and Sufficient Condition for Strictly Positive Definite Functions on Spheres*, Proceedings of the American Mathematical Society **131** (2003), no. 9, 2733–2740.

[16] J. Cui, *Finite Pointset Methods on the Sphere and Their Application in Physical Geodesy*, Ph.D. thesis, Geomathematics Group, Department of Mathematics, University of Kaiserslautern, Germany, 1995.

[17] J. Cui and W. Freeden, *Equidistribution on the Sphere*, SIAM Journal on Scientific Computing **18** (1997), no. 2, 595–609.

[18] J. Cui, W. Freeden, and B. Witte, *Gleichmässige Approximation mittels sphärischer Finite-Elemente und ihre Anwendung auf die Geodäsie*, Zeitschrift für Vermessungswesen **117** (1992), 266–278.

[19] S. Dahlke, W. Dahmen, I. Weinreich, and E. Schmitt, *Multiresolution Analysis and Wavelets on S^2 and S^3*, Numerical Functional Analysis and Optimization **16** (1995), no. 1-2, 19–41.

[20] S. Dahlke and P. Maass, *Continuous Wavelet Transforms with Applications to Analyzing Functions on Spheres*, The Journal of Fourier Analysis and Applications **2** (1996), no. 4, 379–396.

[21] P.J. Davis, *Interpolation and Approximation*, Dover Publications Inc., New York, 1975.

[22] N. Dyn and A. Ron, *Multiresolution Analysis by Infinitely Differentiable Compactly Supported Functions*, Applied and Computational Harmonic Analysis **2** (1995), 15–20.

[23] A. Erdélyi, W. Magnus, F. Oberhettinger, and F. G. Tricomi, *Tables of Integral Transforms. Vol. II*, McGraw-Hill Book Company, Inc., New York-Toronto-London, 1954.

[24] Cheney E.W. and Will L., *A Course in Approximation Theory*, Brooks/Cole Publishing Company, Pacific Grove, 1999.

[25] N. L. Fernandez, *Polynomial Bases on the Sphere*, Ph.D. thesis, Institute of Mathematics, Universität zu Lübeck, Logos Verlag, Berlin, Germany, 2003.

[26] W. Freeden, *On Approximation by Harmonic Splines*, Manuscripta Geodaetica **6** (1981), 193–244.

[27] W. Freeden, *On Spherical Spline Interpolation and Approximation*, Mathematical Methods in the Applied Sciences **3** (1981), no. 4, 551–575.

[28] W. Freeden, *Spline Methods in Geodetic Approximation Problems*, Mathematical Methods in the Applied Sciences **4** (1982), no. 3, 382–396.

[29] W. Freeden, *Spherical Spline Interpolation—Basic Theory and Computational Aspects*, Journal of Computational and Applied Mathematics **11** (1984), no. 3, 367–375.

[30] W. Freeden, *A Spline Interpolation Method for Solving Boundary Value Problems of Potential Theory from Discretely Given Data*, Numerical Methods for Partial Differential Equations **3** (1987), no. 4, 375–398.

[31] W. Freeden, *The Uncertainty Principle and Its Role in Physical Geodesy*, In: Progress in Geodetic Science, Shaker, 225-236, 1998.

[32] W. Freeden, *Multiscale Modelling of Spaceborne Geodata*, B.G. Teubner, Stuttgart, Leipzig, 1999.

[33] W. Freeden, T. Gervens, and M. Schreiner, *Tensor Spherical Harmonics and Tensor Spherical Splines*, Manuscripta Geodaetica **19** (1994), 70–100.

[34] W. Freeden, T. Gervens, and M. Schreiner, *Constructive Approximation on the Sphere (With Applications to Geomathematics)*, Oxford Sciences Publication, Clarendon, Oxford, 1998.

[35] W. Freeden and K. Hesse, *On the Multiscale Solution of Satellite Problems by Use of Locally Supported Kernel Functions Corresponding to Equidistributed Data on Spherical Orbits*, Studia Scientiarum Mathematicarum Hungarica **39** (2002), 37–74.

[36] W. Freeden and J. C. Mason, *Uniform Piecewise Approximation on the Sphere*, Algorithms for approximation, II (Shrivenham, 1988), Chapman and Hall, London, 1990, pp. 320–333.

[37] W. Freeden and V. Michel, *Constructive Approximation and Numerical Methods in Geodetic Research Today — an Attempt at a Categorization Based on an Uncertainty Principle*, Journal of Geodesy **73** (1999), 452–465.

[38] W. Freeden and V. Michel, *Multiscale Potential Theory (With Applications to Geoscience)*, Birkhäuser Boston Inc., Boston, MA, 2004.

[39] W. Freeden and R. Reuter, *Spherical Harmonic Splines: Theoretical and Computational Aspects*, Approximation and Optimization in Mathematical Physics (Oberwolfach, 1982), Vol. 27, Methoden und Verfahren der Mathematischen Physik, Lang, Frankfurt am Main, 1983, pp. 79–103.

[40] W. Freeden and M. Schreiner, *Non-Orthogonal Expansions on the Sphere*, Mathematical Methods in the Applied Sciences **18** (1995), no. 2, 83–120.

[41] W. Freeden and M. Schreiner, *Orthogonal and Nonorthogonal Multiresolution Analysis, Scale Discrete and Exact Fully Discrete Wavelet Transform on the Sphere*, Constructive Approximation **14** (1998), no. 4, 493–515.

[42] W. Freeden and M. Schreiner, *Spaceborne Gravitational Field Determination by Means of Locally Supported Wavelets*, Journal of Geodesy **79** (2005), 431–446.

[43] W. Freeden and M. Schreiner, *Biorthogonal Locally Supported Wavelets on the Sphere Based on Zonal Kernel Functions*, Journal of Fourier Analysis and Applications (2006), (accepted for publication).

[44] W. Freeden and M. Schreiner, *Multiresolution Analysis by Spherical Up Functions*, Constructive Approximation **23** (2006), 241–259.

[45] W. Freeden, M. Schreiner, and R. Franke, *A Survey on Spherical Spline Approximation*, Surveys on Mathematics for Industry **7** (1997), no. 1, 29–85.

[46] W. Freeden and U. Windheuser, *Earth's Gravitational Potential and its MRA Approximation by Harmonic Singular Integrals*, Zeitschrift für Angewandte Mathematik und Mechanik (ZAMM), SII **75** (1995), 633–634.

[47] W. Freeden and U. Windheuser, *Spherical Wavelet Transform and its Discretization*, Advances in Computational Mathematics **5** (1996), no. 1, 51–94.

[48] W. Freeden and U. Windheuser, *Combined Spherical Harmonic and Wavelet Expansion—A Future Concept in Earth's Gravitational Determination*, Applied and Computational Harmonic Analysis **4** (1997), no. 1, 1–37.

[49] G. Gasper, *Positive Integrals of Bessel Functions*, SIAM Journal on Mathematical Analysis **6** (1975), no. 5, 868–881.

[50] K. M. Górski, E. Hivon, A. J. Banday, B. D. Wandelt, F. K. Hansen, M. Reinecke, and M. Bartelmann, *HEALPix: A Framework for High-Resolution Discretization and Fast Analysis of Data Distributed on the Sphere*, The Astrophysical Journal **622** (2005), 759–771.

[51] K. Guo, S. Hu, and X. Sun, *Conditionally Positive Definite Functions and Laplace-Stieltjes Integrals*, Journal of Approximation Theory **74** (1993), no. 3, 249–265.

[52] M. Holschneider, *Continuous Wavelet Transforms on the Sphere*, Journal of Mathematical Physics **37** (1996), no. 8, 4156–4165.

[53] L. Hörmander, *Linear Partial Differential Operators*, Die Grundlehren der mathematischen Wissenschaften, Bd. 116, Springer-Verlag, Berlin-Heidelberg, 1963.

[54] S. Hubbert, *Radial Basis Function Interpolation on the Sphere*, Ph.D. thesis, Imperial College London, 2002.

[55] S. Igari, *Real Analysis—with An Introduction to Wavelet Theory*, Translations of Mathematical Monographs, Vol. 177, American Mathematical Society, Providence, RI, 1998.

[56] Y. Kurihara, *Numerical Integration of the Primitive Equations on a Spherical Grid*, Monthly Weather Review **93** (1965), no. 7, 399–415.

[57] N. N. Lebedew, *Spezielle Funktionen und ihre Anwendung*, Bibliographisches Institut, Mannheim, Germany, 1973.

[58] J. L. Lions and E. Magenes, *Non-Homogeneous Boundary Value Problems and Applications* , vol. 1, Springer-Verlag, Berlin and Heidelberg, New York, 1972.

[59] A. K. Louis, P. Maaß, and A. Rieder, *Wavelets: Theory and Applications*, second ed., John Wiley & Sons, Chichester, UK, 1997.

[60] T. Lyche and L. Schumaker, *L-spline Wavelets*, Wavelets: Theory, Algorithms, and Applications (Taormina, 1993), Wavelet Analysis and its Applications, Vol. 5, Academic Press, San Diego, CA, 1994, pp. 197–212.

[61] T. Lyche and L. Schumaker, *A Multiresolution Tensor Spline Method for Fitting Functions on the Sphere*, SIAM Journal on Scientific Computing **22** (2000), no. 2, 724–746.

[62] W. Magnus, F. Oberbettinger, and R.P. Soni, *Formulas and Theorems for the Special Functions of Mathematical Physics*, Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen, vol. 52, Springer-Verlag, Berlin and Heidelberg, New York, 1966.

[63] S. Mallat, *A Wavelet Tour of Signal Processing*, Academic Press Inc., San Diego, CA, 1998.

[64] V. A. Menegatto, *Fundamental Sets of Functions on Spheres*, Methods and Applications of Analysis **5** (1998), no. 4, 387–398.

[65] V. A. Menegatto, *Strict Positive Definiteness on Spheres*, Analysis. International Mathematical Journal of Analysis and its Applications **19** (1999), no. 3, 217–233.

[66] V. A. Menegatto, C. P. Oliveira, and A. P. Peron, *Strictly Positive Definite Kernels on Subsets of the Complex Plane*, Computers & Mathematics with Applications. An International Journal **51** (2006), no. 8, 1233–1250.

[67] C. A. Micchelli, *Interpolation of Scattered Data: Distance Matrices and Conditionally Positive Definite Functions*, Constructive Approximation **2** (1986), no. 1, 11–22.

[68] V. Michel, *A Multiscale Approximation for Operator Equations in Separable Hilbert Spaces – Case Study: Reconstruction and Description of the Earth's Interior*, Habilitation Thesis, Department of Mathematics, University of Kaiserslautern, Geomathematics Group, Shaker Verlag, Aachen, Germany, 2002.

[69] E. H. Moore, *General Analysis*, Memoirs of the American Philosophical Society, Philadelphia, 1935, Part I and Part II.

[70] C. Müller, *Spherical Harmonics*, Lecture Notes in Mathematics, Vol. 17, Springer-Verlag, Berlin, 1966.

[71] C. Müller, *Foundations of the Mathematical Theory of Electromagnetic Waves*, Revised and enlarged translation from the German. Die Grundlehren der mathematischen Wissenschaften, Band 155, Springer-Verlag, New York, 1969.

[72] F. J. Narcowich and J. D. Ward, *Nonstationary Wavelets on the m-sphere for Scattered Data*, Applied and Computational Harmonic Analysis **3** (1996), no. 4, 324–336.

[73] F. J. Narcowich and J. D. Ward, *Scattered-Data Interpolation on the Sphere: Error Estimates and Locally Supported Basis Functions*, SIAM Journal on Mathematical Analysis **33** (2002), no. 6, 1393–1410.

[74] G. M. Nielson, I. Jung, and J. Sung, *Haar Wavelets Over Triangular Domains With Applications to Multiresolution Models for Flow Over a Sphere*, VIS '97: Proceedings of the 8th conference on Visualization '97 (Los Alamitos, CA, USA), IEEE Computer Society Press, 1997, pp. 143–150.

[75] D. Potts and M. Tasche, *Interpolatory Wavelets on the Sphere*, Approximation theory VIII, Vol. 2 (College Station, TX, 1995), World Scientific Publisher, River Edge, NJ, 1995, pp. 335–342.

[76] M. J. D. Powell, *The Theory of Radial Basis Function Approximation in 1990*, Advances in numerical analysis, Vol. II (Lancaster, 1990), Oxford Sci. Publ., Oxford Univ. Press, New York, 1992, pp. 105–210.

[77] A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev, *Integrals and Series. Vol. 2*, Gordon & Breach Science Publishers, New York, Second Edition, 1988, Special functions, Translated from the Russian by N. M. Queen.

[78] E. A. Rakhmanov, E. B. Saff, and Y. M. Zhou, *Minimal Discrete Energy on the Sphere*, Mathematical Research Letters **1** (1994), no. 6, 647–662.

[79] A. Ron and X. Sun, *Strictly Positive Definite Functions on Spheres*, Tech. Report CMS TR 96–4, University of Wisconsin-Madison, 1994.

[80] A. Ron and X. Sun, *Strictly Positive Definite Functions on Spheres in Euclidean Spaces*, Mathematics of Computation **65** (1996), no. 216, 1513–1530.

[81] D. Roşca, *Haar Wavelets on Spherical Triangulations*, Advances in multiresolution for geometric modelling, Springer, Berlin, 2005, pp. 405–417.

[82] D. Roşca, *Wavelet Bases on the Sphere Obtained by Radial Projections*, Journal of Fourier Analysis and Applications **13** (2007), no. 4, 421–434.

[83] W. Rudin, *Principles of Mathematical Analysis (International Series in Pure & Applied Mathematics)*, McGraw-Hill Publishing Co., 1976.

[84] W. Rudin, *Functional Analysis*, International Series in Pure and Applied Mathematics, McGraw-Hill Inc., New York, Second Edition, 1991.

[85] V. A. Rvachev, *Compactly Supported Solutions of Functional-Differential Equations and Their Applications*, Russian Math. Surveys **45** (1990), no. 1, 87–120.

[86] R. Schaback and Z. Wu, *Operators on Radial Functions*, Journal of Computational and Applied Mathematics **73** (1996), 257–270.

[87] H.J. Schaffeld, *Eine Finite–Elemente–Methode und ihre Anwendung zur Erstellung von Digitalen Geländemodellen.*, Ph.D. thesis, Geodätisches Institut der RWTH, University of Aachen, Germany, 1988.

[88] I. J. Schoenberg, *Metric Spaces and Completely Monotone Functions*, Annals of Mathematics. Second Series **39** (1938), no. 4, 811–841.

[89] I.J. Schoenberg, *Positive Definite Functions on Spheres*, Duke Mathematical Journal **9** (1942), no. 1, 96–108.

[90] M. Schreiner, *Locally Supported Kernels for Spherical Spline Interpolation*, Journal of Approximation Theory **89** (1997), no. 2, 172–194.

[91] M. Schreiner, *On a New Condition for Strictly Positive Definite Functions on Spheres*, Proceedings of the American Mathematical Society **125** (1997), no. 2, 531–539.

[92] M. Schreiner, *Wavelet Approximation by Spherical Up Function*, Habilitation Thesis, Department of Mathematics, University of Kaiserslautern, Geomathematics Group, Shaker Verlag, Aachen, Germany, 2004.

[93] P. Schröder and W. Sweldens, *Spherical Wavelets: Efficiently Representing Functions on the Sphere*, in: *Computer Graphics Proceedings*, Annual Conference Series, ACM SIGGRAPH, 161–175, 1995.

[94] I.H. Sloan and R.S. Womersley, *Extremal Systems of Points and Numerical Integration on the Sphere*, Advances in Computational Mathematics **21** (2004), no. 1-2, 102–125.

[95] E. M. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton Mathematical Series, No. 30, Princeton University Press, Princeton, N.J., 1970.

[96] J. Stoer and R. Bulirsch, *Introduction to Numerical Analysis*, Third ed., Texts in Applied Mathematics, Vol. 12, Springer-Verlag, New York, 2002.

[97] X. Sun, *Strictly Positive Definite Functions on the Unit Circle*, Mathematics of Computation **74** (2005), no. 250, 709–721.

[98] S. L. Svensson, *Finite Elements on the Sphere*, Journal of Approximation Theory **40** (1984), no. 3, 246–260.

[99] W. Sweldens, *The Lifting Scheme: A Construction of Second Generation Wavelets*, SIAM Journal on Mathematical Analysis **29** (1998), no. 2, 511–546.

[100] G. Szegö, *Orthogonal Polynomials*, American Mathematical Society, Vol. 23, American Mathematical Society Providence, Vol. 23, 1959.

[101] G. Wahba, *Spline Interpolation and Smoothing on the Sphere*, SIAM Journal on Scientific and Statistical Computing **2** (1981), no. 1, 5–16, Also errata: SIAM Journal on Scientific and Statistical Computing, 3, 385–386.

[102] G. N. Watson, *A Treatise on the Theory of Bessel Functions*, Cambridge University Press, Cambridge, England, 1944.

[103] I. Weinreich, *A Construction of C^1-wavelets on the Two-dimensional Sphere*, Applied and Computational Harmonic Analysis **10** (2001), no. 1, 1–26.

[104] H. Wendland, *Piecewise, Positive Definite and Compactly Supported Radial Functions of Minimal Degree*, Advances in Computational Mathematics **4** (1995), 389–396.

[105] H. Wendland, *Konstruktion und Untersuchung radialer Basisfunktionen mit kompaktem Träger*, Ph.D. thesis, NAM Göttingen, Göttingen, Germany, 1996.

[106] H. Wendland, *Error Estimates for Interpolation by Compactly Supported Radial Basis Functions of Minimal Degree*, Journal of Approximation Theory **93** (1998), 258–272.

[107] H. Wendland, *Scattered Data Approximation*, Cambridge Monographs on Applied and Computational Mathematics, Vol. 17, Cambridge University Press, Cambridge, 2005.

[108] U. Windheuser, *Sphärische Wavelets: Theorie und Anwendung in der Physikalischen Geodäsie*, Ph.D. thesis, Geomathematics Group, Department of Mathematics, University of Kaiserslautern, Germany, 1995.

[109] Zong Min Wu, *Compactly Supported Positive Definite Radial Functions*, Advances in Computational Mathematics **4** (1995), no. 3, 283–292.

[110] Y. Xu, *Polynomial Interpolation on the Unit Sphere*, SIAM Journal on Numerical Analysis **41** (2003), no. 2, 751–766.

[111] Y. Xu, *Polynomial Interpolation on the Unit Sphere and on the Unit Ball*, Advances in Computational Mathematics **20** (2004), no. 2, 247–260.

[112] Y. Xu and E. W. Cheney, *Strictly Positive Definite Functions on Spheres*, Proceedings of the American Mathematical Society **116** (1992), no. 4, 977–981.

[113] K. Yosida, *Functional Analysis*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], Vol. 123, Springer-Verlag, Berlin, Sixth Edition, 1980.

[114] W. zu Castell, N. L. Fernandez, and Y. Xu, *Polynomial Interpolation on the Unit Sphere II*, Advances in Computational Mathematics (2006), to appear.

[115] W. zu Castell and F. Filbir, *Radial Basis Functions and Corresponding Zonal Series Expansions on the Sphere*, Journal of Approximation Theory **134** (2005), no. 1, 65–79.

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