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On the Completeness and Closure of Vector and Tensor Spherical Harmonics

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RESEARCH ARTICLE

On the Completeness and Closure of Vector and Tensor Spherical Harmonics

Dedicated to the memory of our teacher Claus Müller who died on February 6th, 2008.

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An intrinsically on the 2-sphere formulated proof of the closure and completeness of spherical harmonics is given in vectorial and tensorial framework. The considerations are essentially based on vector and tensor approximation in terms of zonal vector and tensor Bernstein kernels, respectively.

Keywords: Scalar/vectorial/tensorial spherical harmonics, Bernstein kernel, closure, completeness.

AMS Subject Classification: 30F15, 33C55, 35P10, 42C10, 42C30, 43A90

1. Introduction

Spherical harmonics are the analogues of trigonometric functions for Fourier expansion theory on the sphere. They have been introduced in the 1780s to study gravitational theory (cf. [13, 14]). Early publications on the theory of spherical harmonics in their meaning as multipoles can be found, e.g., in [12]. Today the use of spherical harmonics in mathematics, geo- and astrophysics, geodesy and -engineering, etc is a well-established technique, particularly for representing scalar potentials. For example, reference models for the Earth's gravitational or magnetic potential are widely known by tables of coefficients of the spherical harmonic Fourier expansion of their potentials. Moreover, in the second half of the last century a physically motivated approach to the decomposition of vector and tensor fields has been discovered which is based on a spherical variant of the Helmholtz theorem (see, e.g., [1, 2, 17]). Following this concept the tangential part of a spherical vector field is split up into a curl-free and a divergence-free field by use of two differential operators, viz. the surface gradient and the surface curl gradient. Of course, an analogous splitting is valid in tensor theory (cf. [2, 8]). In subsequent publications the vector spherical harmonics are usually written in local coordinate expressions that make mathematical formulations lengthy. In addition, when using local coordinates within a global spherical concept, differential geometry informs us that there is no representation of vector and

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tensor spherical harmonics free of singularities (at the poles). In consequence, the mathematical arrangement involving vector and tensor spherical harmonics has led to an inadequately complex literature. Even more, the concept of zonal kernel functions in a unifying scalar, vector, and tensor framework has not been worked out suitably: addition theorems connecting spherical harmonics and Legendre polynomials have not been generalized in canonical coordinate-free sense to the vector and tensor context. Based on the work due to C. Müller (cf. [18, 19]) and the theses [6, 9, 20], the monograph [8] develops vector and tensor counterparts of the Legendre polynomials and explains their specific role in constructive approximation by use of zonal vector and/or tensor kernels. All material is intrinsically formulated on the sphere. Only the closure and completeness of vector and tensor spherical harmonics are taken from restrictions of properties for homogeneous harmonic vector and tensor polynomials in Euclidean spaces.

In this paper we prove the closure and completeness of vector and tensor spherical harmonics intrinsically on the sphere. For that purpose we use vectorial and tensorial variants of the scalar zonal Bernstein kernels (as presented, e.g., in [5]). Although the approximation of functions by using Bernstein polynomials is one of the classical research topics and their theory is a rich one (see, e.g., [3, 4, 10, 11, 15, 21] and many others), their application within vector and tensor theory of spherical harmonics seems to be unknown. Indeed, the vector and tensor zonal Bernstein kernel approximations can be shown to guarantee the closure property in the space of continuous spherical normal as well as tangential vector and tensor fields, respectively. In consequence, they also assure closure and completeness in the Hilbert space of (Lebesgue-)square-integrable vector and tensor fields, respectively. Essential tools are the theory of Green's function with respect to (iterated) Beltrami operators (see [7]) and the Helmholtz decomposition theorems as presented in [8]. The layout of this paper is as follows: First the spherical Bernstein kernel is introduced. A short approach to spherical Bernstein approximation is presented in scalar theory (in Section 2). Then we consider the vectorial case for which Green's function with respect to the Beltrami operator is introduced and used in the spherical variant of the Helmholtz decomposition theorem. Its approximation by means of the Bernstein kernel yields the closure and completeness of the vector spherical harmonics (in Section 3). In the final part the tensorial version of the Helmholtz decomposition theorem requires Green's function with respect to iterated operators. We present an elementary representation of the occuring Green's function which allows us to establish the closure and completeness properties of the tensor spherical harmonics by approximations in terms of the Bernstein kernel (in Section 4). All three parts – scalar, vectorial as well as tensorial – utilize the spherical Bernstein kernel and its properties as essential tools. The three proofs for the closure and completeness are incorporated in a unified concept. Finally it should be noted that, throughout this paper, the nomenclature is based on that used in the monograph [8] on constructive approximation on the sphere.

2. Closure of Scalar Spherical Harmonics

The point of departure is the so-called *spherical Bernstein kernel of degree n*

$$(\xi,\eta) \mapsto B_n(\xi\cdot\eta) = \frac{n+1}{4\pi}\left(\frac{1+\xi\cdot\eta}{2}\right)^n; \quad \xi,\eta\in\Omega, \tag{1}$$

where Ω denotes the unit sphere in $\mathbb{R}^3$.

Remark 1: The name *Bernstein* is motivated by the fact that the kernel (see Fig. 1) is proportional to the Bernstein polynomial $B_n^\nu(t) = \binom{n}{\nu} t^\nu (1-t)^{n-\nu}$ scaled to the interval $[-1,1]$ with $\nu = n$ (t is the polar distance between ξ and η, i.e., $t = \xi \cdot \eta$).

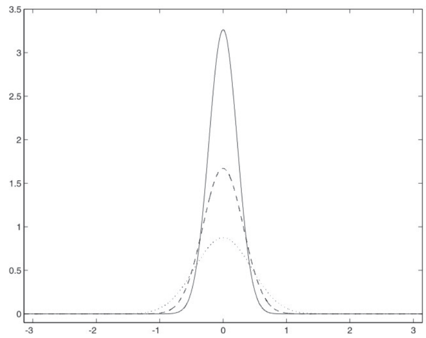

Figure 1. Illustration of the kernel $B_n(\cos(\vartheta)), \vartheta \in [-\pi, \pi]$ for the degrees $n = 10$ (dotted line), $n = 20$ (dashed line), $n = 40$ (solid line)

First we mention some important properties of the Bernstein kernel whose proof results directly from considerations given in [5].

Lemma 2.1: *The spherical Bernstein kernel* $t \mapsto B_n(t) = \frac{n+1}{4\pi}\left(\frac{1+t}{2}\right)^n, t \in [-1,1]$, *satisfies the following properties:*

(i) For all $t \in [-1,1)$

$$\lim_{n\to\infty} B_n(t) = 0. \tag{2}$$

For all $t \in [-1,1]$ *and for all* $n \in \mathbb{N}_0$

$$B_n(t) \geq 0. \tag{3}$$

(ii) For $k = 0, \ldots, n$ *the Legendre coefficient of degree* k *of the Bernstein kernel of degree* n *is given by*

$$B_n^\wedge(k) = 2\pi \int_{-1}^{+1} B_n(t) P_k(t)\ dt = \frac{n!}{(n-k!)} \frac{(n+1)!}{(n+k+1)!} = \frac{\binom{n}{k}}{\binom{n+k+1}{k}}, \tag{4}$$

where P_k *denotes the Legendre polynomial of degree* k*. In particular, the Legendre coefficient of degree* 0 *corresponds to*

$$B_n^\wedge(0) = 2\pi \int_{-1}^{1} B_n(t) dt = 1. \tag{5}$$

The Legendre series representation of the Bernstein kernel of degree n *with the Legendre coefficients from (4) reads as follows*

$$B_n(t) = \sum_{k=0}^{n} \frac{2k+1}{4\pi} \frac{n!}{(n-k!)} \frac{(n+1)!}{(n+k+1)!} P_k(t). \tag{6}$$

(iii) *For $k \in \mathbb{N}$ fixed,*

$$B_n^\wedge(k) < B_{n+1}^\wedge(k), \quad \textit{for all } n \in \mathbb{N}_0. \tag{7}$$

(iv) *For $k \in \mathbb{N}_0$ fixed,*

$$\lim_{n\to\infty} B_n^\wedge(k) = 1. \tag{8}$$

Suppose that F is continuous on Ω. For $\xi \in \Omega$, we observe (5) to assure

$$\int_\Omega B_n(\xi\cdot\eta)F(\eta)d\omega(\eta) = F(\xi) + \int_\Omega B_n(\xi\cdot\eta)\left(F(\eta) - F(\xi)\right)d\omega(\eta).$$

Now, we split the unit sphere Ω into two parts depending on a parameter $\gamma \in (0,1)$ and an arbitrary, but fixed point $\xi \in \Omega$. More explicitly, Ω is divided into $\Omega_\gamma = \{\xi \in \Omega : -1 \le \xi\cdot\eta \le 1-\gamma\}$ and the complementary part $\Omega \setminus \Omega_\gamma$. In consequence, we split the integral into

$$\int_\Omega \ldots = \int_{\Omega_\gamma} \ldots + \int_{\Omega\setminus\Omega_\gamma} \ldots \,.$$

On the one hand side we find with (2) and (3)

$$\begin{aligned}\left|\int_{\Omega_\gamma} B_n(\xi\cdot\eta)F(\eta)\, d\omega(\eta)\right| &\le 4\pi \left\|F\right\|_{\mathrm{C}^{(0)}(\Omega)} \int_{-1}^{1-\gamma} B_n(t)\, dt \\ &\le 2\left\|F\right\|_{\mathrm{C}^{(0)}(\Omega)} \left(1-\frac{\gamma}{2}\right)^{n+1}.\end{aligned}$$

On the other hand side F is uniformly continuous on Ω. Thus there exists a positive function $\mu : \gamma \mapsto \mu(\gamma)$ with $\lim_{\substack{\gamma\to 0\\ \gamma>0}} \mu(\gamma) = 0$ such that $|F(\xi) - F(\eta)| \le \mu(\gamma)$ for all $\eta \in \Omega$ with $1-\gamma \le \xi\cdot\eta \le 1$. Thus it follows by use of (3) that

$$\left|\int_{\Omega\setminus\Omega_\gamma} B_n(\xi\cdot\eta)(F(\eta)-F(\xi))d\omega(\eta)\right| \le \mu(\gamma)\int_\Omega B_n(\xi\cdot\eta)d\omega(\eta) = \mu(\gamma).$$

Summarizing our results we obtain the following estimate

$$\begin{aligned}&\sup_{\xi\in\Omega}\left|\int_\Omega B_n(\xi\cdot\eta)F(\eta)d\omega(\eta) - F(\xi)\right| = \sup_{\xi\in\Omega}\left|\int_\Omega B_n(\xi\cdot\eta)(F(\eta)-F(\xi))d\omega(\eta)\right| \\ &\le \sup_{\xi\in\Omega}\left|\int_{\Omega_\gamma} B_n(\xi\cdot\eta)F(\eta)d\omega(\eta) - F(\xi)\int_{\Omega_\gamma} B_n(\xi\cdot\eta)d\omega(\eta)\right| \\ &\quad + \sup_{\xi\in\Omega}\left|\int_{\Omega\setminus\Omega_\gamma} B_n(\xi\cdot\eta)(F(\eta)-F(\xi))d\omega(\eta)\right| \\ &\le \sup_{\xi\in\Omega}\left|\int_{\Omega_\gamma} B_n(\xi\cdot\eta)F(\eta)d\omega(\eta)\right| + \left\|F\right\|_{\mathrm{C}^{(0)}(\Omega)}\int_{\Omega_\gamma} B_n(\xi\cdot\eta)d\omega(\eta) + \mu(\gamma) \\ &\le 2\left\|F\right\|_{\mathrm{C}^{(0)}(\Omega)}\left(1-\frac{\gamma}{2}\right)^{n+1} + 2\left\|F\right\|_{\mathrm{C}^{(0)}(\Omega)}\left(1-\frac{\gamma}{2}\right)^{n+1} + \mu(\gamma).\end{aligned}$$

Taking the limit with respect to n on both sides we are left with the relation

$$\lim_{n\to\infty} \sup_{\xi\in\Omega} \left| \int_\Omega B_n(\xi\cdot\eta)F(\eta)d\omega(\eta) - F(\xi) \right| \le \mu(\gamma)$$

for every $\gamma \in (0,1)$. Observing $\mu(\gamma) \to 0$ as $\gamma \to 0$ we therefore get the following result.

Theorem 2.2: *For $F \in \mathrm{C}^{(0)}(\Omega)$,*

$$\lim_{n\to\infty} \sup_{\xi\in\Omega} \left| \int_\Omega B_n(\xi\cdot\eta)F(\eta)\ d\omega(\eta) - F(\xi) \right| = 0. \tag{9}$$

In connection with the Legendre series (6) of the Bernstein kernel we find

$$\int_\Omega B_n(\xi\cdot\eta)F(\eta)\ d\omega(\eta) = \sum_{k=0}^{n} B_n^\wedge(k)\frac{2k+1}{4\pi}\int_\Omega P_k(\xi\cdot\eta)F(\eta)\ d\omega(\eta).$$

Hence, the addition theorem for scalar spherical harmonics (see, e.g., [8, 18, 19])

$$\frac{2k+1}{4\pi}P_k(\eta\cdot\xi) = \sum_{j=1}^{2k+1} Y_{k,j}(\eta)Y_{k,j}(\xi),$$

enables us to establish the *'Bernstein summability' of a Fourier series expansion in terms of scalar spherical harmonics.*

Corollary 2.3: *For $F \in \mathrm{C}^{(0)}(\Omega)$,*

$$\lim_{n\to\infty} \sup_{\xi\in\Omega} \left| \sum_{k=0}^{n} B_n^\wedge(k) \sum_{j=1}^{2k+1} F^\wedge(k,j)Y_{k,j}(\xi) - F(\xi) \right| = 0,$$

where $F^\wedge(k,j) = \int_\Omega F(\eta)Y_{k,j}(\eta)d\omega(\eta),\ k = 0,1,...,j = 1,...,2k+1.$

Corollary 2.3 enables us to prove the *closure of the system of spherical harmonics* $\{Y_{k,j}\}_{k=0,1,...,j=1,...,2k+1}$ *in the space* $\mathrm{C}^{(0)}(\Omega)$.

Theorem 2.4: *For any given $\varepsilon > 0$ and each $F \in \mathrm{C}^{(0)}(\Omega)$ there exists a linear combination $\sum_{k=0}^{N}\sum_{j=1}^{2k+1} d_{k,j}Y_{k,j}$ such that*

$$\left\| F - \sum_{k=0}^{N}\sum_{j=1}^{2k+1} d_{k,j}Y_{k,j} \right\|_{\mathrm{C}^{(0)}(\Omega)} \le \varepsilon.$$

In fact, given $F \in \mathrm{C}^{(0)}(\Omega)$, then for any $\varepsilon > 0$ there exists an integer $N = N(\varepsilon)$ because of (2.3) such that

$$\sup_{\xi\in\Omega} \left| \sum_{k=0}^{N}\sum_{j=1}^{2k+1} \underbrace{B_N^\wedge(k)F^\wedge(k,j)}_{=d_{k,j}} Y_{k,j}(\xi) - F(\xi) \right| \le \varepsilon.$$

In addition, by standard arguments, we are able to derive the closure in $\mathrm{L}^2(\Omega)$ as well as the completeness of the scalar spherical harmonics in $\mathrm{L}^2(\Omega)$ (for more details the reader is referred, e.g., to [8]).

3. Closure of Vector Spherical Harmonics

First some information about vector spherical harmonics should be given: We begin with the introduction of the operators $o^{(i)} : \mathrm{C}^{(1)}(\Omega) \to \mathrm{c}^{(0)}(\Omega)$ with the help of the surface gradient ∇^* and the surface curl gradient L^* as follows

$$
\begin{aligned}
o_\xi^{(1)} F(\xi) &= \xi F(\xi), \qquad \xi \in \Omega, \\
o_\xi^{(2)} F(\xi) &= \nabla_\xi^* F(\xi), \qquad \xi \in \Omega, \\
o_\xi^{(3)} F(\xi) &= L_\xi^* F(\xi), \qquad \xi \in \Omega.
\end{aligned}
$$

Obviously, $o^{(1)}F$ is a normal vector field, whereas $o^{(2)}F$ and $o^{(3)}F$ are tangential. Their application to the scalar spherical harmonics defines the orthonormal vector spherical harmonics up to a normalization factor, i.e.,

$$
\begin{aligned}
y_{k,j}^{(1)}(\xi) &= o_\xi^{(1)} Y_{k,j}, \qquad k = 0, 1, \ldots;\ j = 1, \ldots, 2k+1, \\
y_{k,j}^{(i)}(\xi) &= \frac{1}{\sqrt{k(k+1)}}\, o_\xi^{(i)} Y_{k,j}, \qquad i = 2,3;\ k = 1, 2, \ldots;\ j = 1, \ldots, 2k+1.
\end{aligned}
$$

The Helmholtz decomposition theorem for spherical vector fields tells us that a vector field $f \in \mathrm{c}^{(1)}(\Omega)$ can be represented as

$$
f(\xi) = \sum_{i=1}^{3} o^{(i)} F^{(i)}(\xi), \quad \xi \in \Omega,
$$

where the scalar functions $F^{(i)} : \Omega \to \mathbb{R}$ satisfying $\int_\Omega F^{(i)}(\xi) d\omega(\xi) = 0$ for $i = 2, 3$ are uniquely determined by

$$
\begin{aligned}
F^{(1)}(\xi) &= O_\xi^{(1)} f(\xi), \\
F^{(i)}(\xi) &= -\int_\Omega G(\Delta^*; \xi, \eta) O_\eta^{(i)} f(\eta) d\omega(\eta), \qquad i = 2, 3.
\end{aligned}
$$

$O^{(i)}$ denote the adjoint operators associated to the operators $o^{(i)}$, $i \in \{1, 2, 3\}$. The function $(\xi, \eta) \mapsto G(\Delta^*; \xi, \eta)$, $-1 \le \xi \cdot \eta < 1$ is the Green function with respect to the (Laplace-)Beltrami operator Δ^* (cf. [6, 8] for a detailed introduction)

$$
G(\Delta^*; \xi, \eta) = \frac{1}{4\pi} \left(\ln(1 - \xi \cdot \eta) + 1 - \ln(2) \right).
$$

Its bilinear (Legendre) expansion is given by

$$
G(\Delta^*; \xi, \eta) = \frac{1}{4\pi} \sum_{k=1}^{\infty} \frac{2k+1}{-k(k+1)} P_k(\xi \cdot \eta).
$$

For purposes of approximation we convolve the Green function with the Bernstein kernel

$$BG_n(\xi\cdot\eta)=\int\limits_{\Omega} G(\Delta^*;\xi,\alpha)B_n(\alpha\cdot\eta)d\omega(\alpha).$$

In terms of a Legendre series we have

$$BG_n(\xi\cdot\eta)=\sum_{k=1}^{n}\frac{2k+1}{4\pi}\frac{B_n^\wedge(k)}{-k(k+1)}P_k(\xi\cdot\eta).$$

Note that the Bernstein kernel is a polynomial and Green's function is of class $\mathrm{L}^1[-1,1]$, hence, the existence of the convolution integral as bandlimited Legendre expansion is obvious.

Next we are interested in the Bernstein summability of Fourier expansions in terms of vector spherical harmonics. To this end we need some preparatory material (more precisely, Lemma 3.1 and Lemma 3.2). Essential tool of our considerations is the Green function with respect to the Beltrami operator.

Lemma 3.1: *For* $i\in\{1,2,3\}$ *we have*

$$\lim_{n\to\infty}\|F^{(i)}-F_n^{(i)}\|_{\mathrm{C}^{(0)}(\Omega)}=0.$$

Proof: In order to verify these limit relations we introduce the Bernstein approximants of the (uniquely determined) scalar Helmholtz functions $F^{(i)}$ for $i=1,2,3$ given by

$$F_n^{(i)}(\xi)=-\left(BG_n*O^{(i)}f\right)(\xi)=-\int\limits_{\Omega}BG_n(\xi,\eta)O_\eta^{(i)}f(\eta)d\omega(\eta).$$

Clearly, the case $i=1$ of Lemma 3.1 follows immediately from the scalar theory of the previous section. Thus it remains to study $i=2,3$. It is not hard to see that

$$\begin{aligned}\|F^{(i)}-F_n^{(i)}\|_{\mathrm{C}^{(0)}(\Omega)}&=\|G*O^{(i)}f-BG_n*O^{(i)}f\|_{\mathrm{C}^{(0)}(\Omega)}\\&\le\|O^{(i)}f\|_{\mathrm{C}^{(0)}(\Omega)}\|G-BG_n\|_{\mathrm{L}^1[-1,1]}.\end{aligned}$$

Since both kernels G and BG_n are of class $\mathrm{L}^2[-1,1]$ and, for all $k\in\mathbb{N}_0$, the Legendre coefficients of the Bernstein kernel $B_n^\wedge(k)$ converge to 1 for n tending to infinity (see also (8)), we are able to deduce that

$$\lim_{n\to\infty}\|G(\Delta^*;\cdot)-BG_n\|_{\mathrm{L}^2[-1,1]}=0.$$

This implies L^1-convergence as well as $\|F^{(i)}-F_n^{(i)}\|_{\mathrm{C}^{(0)}(\Omega)}\to 0$ for $i=1,2,3$ and $n\to\infty$, as required. □

Considering the $o^{(i)}$-derivatives we have to prove

Lemma 3.2: *For $i \in \{1,2,3\}$ we have*

$$\lim_{n\to\infty} \sup_{\xi\in\Omega} \left| o_\xi^{(i)} F^{(i)}(\xi) - o_\xi^{(i)} F_n^{(i)}(\xi) \right| = 0.$$

Proof: Clearly, we have

$$\begin{aligned}
&\| o_\xi^{(i)} F^{(i)}(\xi) - o_\xi^{(i)} F_n^{(i)}(\xi) \|_{\mathrm{c}^{(0)}(\Omega)} \\
&\quad = \sup_{\xi\in\Omega} \left| -o_\xi^{(i)} \int_\Omega G(\Delta^*;\xi,\eta) O_\eta^{(i)} f(\eta) d\omega(\eta) + o_\xi^{(i)} \int_\Omega BG_n(\xi,\eta) O_\eta^{(i)} f(\eta) d\omega(\eta) \right| \\
&\quad = \sup_{\xi\in\Omega} \left| \int_\Omega o_\xi^{(i)} G(\Delta^*;\xi,\eta) O_\eta^{(i)} f(\eta) d\omega(\eta) - \int_\Omega o_\xi^{(i)} BG_n(\xi,\eta) O_\eta^{(i)} f(\eta) d\omega(\eta) \right|,
\end{aligned}$$

where it can be easily seen that the operator $o^{(i)}$ can be drawn inside the two integrals. This leads us to following estimate

$$\begin{aligned}
&\sup_{\xi\in\Omega} \left| \int_\Omega o_\xi^{(i)} G(\Delta^*;\xi,\eta) O_\eta^{(i)} f(\eta) d\omega(\eta) - \int_\Omega o_\xi^{(i)} BG_n(\xi,\eta) O_\eta^{(i)} f(\eta) d\omega(\eta) \right| \\
&\quad \leq \sup_{\xi\in\Omega} \int_\Omega \left| o_\xi^{(i)} G(\Delta^*;\xi,\eta) - o_\xi^{(i)} BG_n(\xi,\eta) \right| \left| O_\eta^{(i)} f(\eta) \right| d\omega(\eta) \\
&\quad \leq \| O^{(i)} f \|_{\mathrm{C}^{(0)}(\Omega)} \int_\Omega \left| o_\xi^{(i)} G(\Delta^*;\xi,\eta) - o_\xi^{(i)} BG_n(\xi,\eta) \right| d\omega(\eta).
\end{aligned}$$

We need to study the convergence of the last integral. In more detail, we have to show

$$\lim_{n\to\infty} \int_\Omega \left| o_\xi^{(i)} G(\Delta^*;\xi,\eta) - o_\xi^{(i)} BG_n(\xi,\eta) \right| d\omega(\eta) = 0.$$

To this end, we notice that the vectorial Bernstein kernels $o_\xi^{(i)} BG_n(\xi\cdot\eta),\ i = 2,3,$ admit the following (Legendre) series expansions

$$\begin{aligned}
o_\xi^{(2)} BG_n(\xi\cdot\eta) &= \sum_{k=1}^{n} \frac{2k+1}{4\pi} \frac{B_n^\wedge(k)}{-k(k+1)} P_k'(\xi\cdot\eta) \left(\eta - (\xi\cdot\eta)\xi\right), \\
o_\xi^{(3)} BG_n(\xi\cdot\eta) &= \sum_{k=1}^{n} \frac{2k+1}{4\pi} \frac{B_n^\wedge(k)}{-k(k+1)} P_k'(\xi\cdot\eta) \left(\xi\wedge\eta\right).
\end{aligned}$$

Moreover, an easy calculation shows that the application of the $o^{(i)}$-operators, $i = 2,3$, to the Green function with respect to the Beltrami operator leads to

$$o_\xi^{(2)} G(\Delta^*;\xi,\eta) = \frac{-1}{4\pi} \frac{\eta - (\eta\cdot\xi)\xi}{1-\eta\cdot\xi}, \qquad o_\xi^{(3)} G(\Delta^*;\xi,\eta) = \frac{-1}{4\pi} \frac{\xi\wedge\eta}{1-\eta\cdot\xi}.$$

Consequently, our integral can be expressed in the form

$$\begin{aligned}
&\int_\Omega \left| o_\xi^{(i)} G(\Delta^*;\xi,\eta) - o_\xi^{(i)} BG_n(\xi,\eta) \right| d\omega(\eta) \\
&= \int_\Omega \left| \frac{-1}{4\pi} \frac{o_\xi^{(i)}(\xi\cdot\eta)}{1-\xi\cdot\eta} - \frac{-1}{4\pi} \sum_{k=1}^{n} \frac{2k+1}{k(k+1)} B_n^\wedge(k) P_k'(\xi\cdot\eta) o_\xi^{(i)}(\xi\cdot\eta) \right| d\omega(\eta) \\
&= \frac{1}{4\pi} \int_\Omega \left| o_\xi^{(i)}(\xi\cdot\eta) \right| \left| \frac{1}{1-\xi\cdot\eta} - \sum_{k=1}^{n} \frac{2k+1}{k(k+1)} B_n^\wedge(k) P_k'(\xi\cdot\eta) \right| d\omega(\eta) \\
&= \frac{1}{2} \int_{-1}^{1} \sqrt{1-t^2} \left| \frac{1}{1-t} - \sum_{k=1}^{n} \frac{2k+1}{k(k+1)} B_n^\wedge(k) P_k'(t) \right| dt.
\end{aligned}$$

At this point we use a well-known recurrence relation for the Legendre polynomials (and their derivatives) (see, e.g., [16]), namely

$$(t^2-1)P_k'(t) = \frac{k(k+1)}{2k+1}(P_{k+1}(t) - P_{k-1}(t)). \tag{10}$$

This gives us the identity

$$\begin{aligned}
&\int_\Omega \left| o_\xi^{(i)} G(\Delta^*;\xi,\eta) - o_\xi^{(i)} BG_n(\xi,\eta) \right| d\omega(\eta) \\
&= \frac{1}{2} \int_{-1}^{1} \sqrt{\frac{1+t}{1-t}} \left| 1 - (1-t) \sum_{k=1}^{n} B_n^\wedge(k) \frac{P_{k+1}(t) - P_{k-1}(t)}{t^2-1} \right| dt \\
&= \frac{1}{2} \int_{-1}^{1} \sqrt{\frac{1+t}{1-t}} \left| 1 + \frac{1}{1+t} \sum_{k=1}^{n} B_n^\wedge(k) \left(P_{k+1}(t) - P_{k-1}(t)\right) \right| dt\,. \qquad (11)
\end{aligned}$$

For the occurring sum it follows that

$$\begin{aligned}
&\sum_{k=1}^{n} B_n^\wedge(k) \left(P_{k+1}(t) - P_{k-1}(t)\right) = \\
&\qquad B_n^\wedge(n) P_{n+1}(t) + B_n^\wedge(n-1) P_n(t) - B_n^\wedge(2) P_1(t) - B_n^\wedge(1) P_0(t) \\
&\qquad + \sum_{k=2}^{n-1} \left(B_n^\wedge(k-1) - B_n^\wedge(k+1)\right) P_k(t)\,, \qquad (12)
\end{aligned}$$

where a simple calculation using (4) shows that

$$B_n^\wedge(k-1) - B_n^\wedge(k+1) = B_{n+1}^\wedge(k)(2k+1)\frac{2}{(n+2)}. \tag{13}$$

Together with (4) and the Legendre series (6) we plug (13) into (12) getting the following result

$$\begin{aligned}
&\sum_{k=1}^{n} B_n^{\wedge}(k)\,(P_{k+1}(t) - P_{k-1}(t)) \\
&\qquad = B_n^{\wedge}(n)P_{n+1}(t) + B_n^{\wedge}(n-1)P_n(t) - B_n^{\wedge}(2)\,P_1(t) - B_n^{\wedge}(1)P_0(t) \\
&\qquad\quad + \frac{2}{n+2}\sum_{k=2}^{n-1} B_{n+1}^{\wedge}(k)(2k+1)P_k(t) \\
&\qquad = \frac{2}{n+2}\sum_{k=0}^{n+1} B_{n+1}^{\wedge}(k)(2k+1)P_k(t) - (1+t) \\
&\qquad = \frac{2}{n+2}(n+1)\left(\frac{1+t}{2}\right)^{n+1} - (1+t)\,.
\end{aligned}$$

With this result in mind we return to the integral (11). As a matter of fact, (11) can be rewritten in the form

$$\begin{aligned}
&\frac{1}{2}\int_{-1}^{1}\sqrt{\frac{1+t}{1-t}}\left|1 + \frac{1}{1+t}\sum_{k=1}^{n} B_n^{\wedge}(k)\,(P_{k+1}(t) - P_{k-1}(t))\right| dt \\
&= \frac{1}{2}\int_{-1}^{1}\sqrt{\frac{1+t}{1-t}}\left|\frac{n+1}{n+2}\left(\frac{1+t}{2}\right)^{n}\right| dt.
\end{aligned}$$

Clearly, the Bernstein kernel is non-negative (see (3)) such that we are left with the integral expression

$$\begin{aligned}
\int_{\Omega}\left|o_{\xi}^{(i)}G(\Delta^*;\xi,\eta) - o_{\xi}^{(i)}BG_n(\xi,\eta)\right| d\omega(\eta) &= \frac{1}{2}\frac{n+1}{n+2}\int_{-1}^{1}\sqrt{\frac{1+t}{1-t}}\left(\frac{1+t}{2}\right)^{n} dt \\
&= \frac{\Gamma(n+\frac{3}{2})}{\Gamma(\frac{1}{2})\,\Gamma(n+2)},
\end{aligned}$$

(which can be proven by induction). It is well-known that the value of our integral can be estimated as follows

$$\frac{1}{\sqrt{2n+2}} < \frac{\Gamma(n+\frac{3}{2})}{\Gamma(\frac{1}{2})\,\Gamma(n+2)} < \frac{2}{\sqrt{2n+2}}\,.$$

Therefore, we immediately obtain convergence of our integral for $n \to \infty$. We even know the speed of convergence, i.e.,

$$\int_{\Omega}\left|o_{\xi}^{(i)}G(\Delta^*;\xi,\eta) - o_{\xi}^{(i)}BG_n(\xi,\eta)\right| d\omega(\eta) = \mathcal{O}(n^{-1/2}).$$

□

We are now in position to establish the *'Bernstein summability' of Fourier series* *in terms of vector spherical harmonics.*

Theorem 3.3: *For any vector field* $f \in \mathrm{c}^{(1)}(\Omega)$,

$$\lim_{n\to\infty} \sup_{\xi\in\Omega} \left| f(\xi) - \sum_{i=1}^{3} \sum_{k=0_i}^{n} \sum_{j=1}^{2k+1} B_n^\wedge(k) \left(f^{(i)}\right)^\wedge (k,j) y_{k,j}^{(i)}(\xi) \right| = 0,$$

where we have used the abbreviation $0_1 = 0$ *and* $0_i = 1,\ i = 2,3$.

Proof: From Lemma 3.2 we know that for $f \in \mathrm{c}^{(1)}(\Omega)$

$$\begin{aligned}\lim_{n\to\infty} \sup_{\xi\in\Omega} \left| f(\xi) - \sum_{i=1}^{3} o_\xi^{(i)} F_n^{(i)}(\xi) \right| &= \lim_{n\to\infty} \sup_{\xi\in\Omega} \left| \sum_{i=1}^{3} o_\xi^{(i)} F^{(i)}(\xi) - \sum_{i=1}^{3} o_\xi^{(i)} F_n^{(i)}(\xi) \right| \\ &\le \sum_{i=1}^{3} \lim_{n\to\infty} \sup_{\xi\in\Omega} \left| o_\xi^{(i)} F^{(i)}(\xi) - o_\xi^{(i)} F_n^{(i)}(\xi) \right| = 0.\end{aligned} \tag{14}$$

The expression $o_\xi^{(1)} F_n^{(1)}(\xi)$ can be written in the form

$$\begin{aligned} o_\xi^{(1)} F_n^{(1)}(\xi) &= o_\xi^{(1)} \int_\Omega B_n(\xi\cdot\eta) O_\eta^{(1)} f(\eta) d\omega(\eta) \\ &= \sum_{k=0}^{n} B_n^\wedge(k) o_\xi^{(1)} \sum_{j=1}^{2k+1} \left(O^{(1)} f\right)^\wedge (k,j) Y_{k,j}(\xi) \\ &= \sum_{k=0}^{n} \sum_{j=1}^{2k+1} B_n^\wedge(k) \left(O^{(1)} f\right)^\wedge (k,j) y_{k,j}^{(1)}(\xi), \end{aligned} \tag{15}$$

where

$$\begin{aligned} \left(O^{(1)} f\right)^\wedge (k,j) &= \int_\Omega O_\eta^{(1)} f(\eta) Y_{k,j}(\eta) d\omega(\eta) \\ &= \int_\Omega f(\eta) \cdot \underbrace{o_\eta^{(1)} Y_{k,j}(\eta)}_{=y_{k,j}^{(1)}(\eta)} d\omega(\eta) = \left(f^{(1)}\right)^\wedge (k,j). \end{aligned} \tag{16}$$

Furthermore, for $i = 2,3$, we have

$$\begin{aligned} o_\xi^{(i)} F_n^{(i)}(\xi) &= -o_\xi^{(i)} \int_\Omega BG_n(\xi\cdot\eta) O_\eta^{(i)} f(\eta) d\omega(\eta) \\ &= \sum_{k=1}^{n} \frac{B_n^\wedge(k)}{k(k+1)} o_\xi^{(i)} \sum_{j=1}^{2k+1} \left(O^{(i)} f\right)^\wedge (k,j) Y_{k,j}(\xi) \\ &= \sum_{k=1}^{n} \sum_{j=1}^{2k+1} \frac{B_n^\wedge(k)}{\sqrt{k(k+1)}} \left(O^{(i)} f\right)^\wedge (k,j) y_{k,j}^{(i)}(\xi). \end{aligned} \tag{17}$$

Taking a look at the coefficients $\left(O^{(i)}f\right)^{\wedge}(k,j)$ we find

$$\left(O^{(i)}f\right)^{\wedge}(k,j) = \int\limits_{\Omega} O^{(i)}_{\eta} f(\eta) Y_{k,j}(\eta) d\omega(\eta) = \int\limits_{\Omega} f(\eta) \cdot o^{(i)}_{\eta} Y_{k,j}(\eta) d\omega(\eta)$$

$$= \sqrt{k(k+1)} \int\limits_{\Omega} f(\eta) \cdot y^{(i)}_{k,j}(\eta) d\omega(\eta) = \sqrt{k(k+1)} \left(f^{(i)}\right)^{\wedge}(k,j). \quad (18)$$

The identities (15) and (16) as well as (17) and (18) allow us to conclude

$$o^{(i)}_{\xi} F^{(i)}_{n}(\xi) = \sum_{k=1}^{n} \sum_{j=1}^{2k+1} B^{\wedge}_{n}(k) \left(f^{(i)}\right)^{\wedge}(k,j) y^{(i)}_{k,j}(\xi), \qquad i = 1,2,3.$$

In connection with (14) we therefore obtain

$$\lim_{n\to\infty} \sup_{\xi\in\Omega} \left| f(\xi) - \sum_{i=1}^{3} o^{(i)}_{\xi} F^{(i)}_{n}(\xi) \right|$$

$$= \lim_{n\to\infty} \sup_{\xi\in\Omega} \left| f(\xi) - \sum_{i=1}^{3} \sum_{k=0_i}^{n} \sum_{j=1}^{2k+1} B^{\wedge}_{n}(k) \left(f^{(i)}\right)^{\wedge}(k,j) y^{(i)}_{k,j}(\xi) \right| = 0. \quad (19)$$

provided that $f \in \mathrm{c}^{(1)}(\Omega)$. This is the desired result. □

A well-known density argument enables us to verify the *closure of the vector spherical harmonics* $\left\{y^{(i)}_{k,j}\right\}_{i,k,j}$ *in the space* $\mathrm{c}^{(0)}(\Omega)$.

Theorem 3.4: *For any given* $\varepsilon > 0$ *and each* $f \in \mathrm{c}^{(0)}(\Omega)$ *there exists a linear combination* $\sum_{i=1}^{3}\sum_{k=0_i}^{N}\sum_{j=1}^{2k+1} d^{(i)}_{k,j} y^{(i)}_{k,j}$, *such that*

$$\left\| f - \sum_{i=1}^{3} \sum_{k=0_i}^{N} \sum_{j=1}^{2k+1} d^{(i)}_{k,j} y^{(i)}_{k,j} \right\|_{\mathrm{c}^{(0)}(\Omega)} \le \varepsilon.$$

Indeed, if we take any $g \in \mathrm{c}^{(0)}(\Omega)$ and any $\varepsilon > 0$, we find a field $f \in \mathrm{c}^{(1)}(\Omega)$ such that $\sup_{\xi\in\Omega} |g(\xi) - f(\xi)| < \frac{\varepsilon}{2}$. Due to Theorem 3.3 there also exists an $N \in \mathbb{N}$ with

$$\sup_{\xi\in\Omega} \left| f(\xi) - \sum_{i=1}^{3} \sum_{k=0_i}^{N} \sum_{j=1}^{2k+1} B^{\wedge}_{n}(k) \left(f^{(i)}\right)^{\wedge}(k,j) y^{(i)}_{k,j}(\xi) \right| < \frac{\varepsilon}{2}.$$

Combining both inequalities we obtain

$$\sup_{\xi\in\Omega} \left| g(\xi) - \sum_{i=1}^{3} \sum_{k=0_i}^{N} \sum_{j=1}^{2k+1} \underbrace{B^{\wedge}_{n}(k) \left(f^{(i)}\right)^{\wedge}(k,j)}_{d^{(i)}_{k,j}} y^{(i)}_{k,j}(\xi) \right| < \varepsilon.$$

By standard arguments this immediately gives us the closure in $\mathrm{c}^{(0)}(\Omega)$ with respect to $\|\cdot\|_{\mathrm{l}^2(\Omega)}$ as well as in $\mathrm{l}^2(\Omega)$ which in turn leads to completeness of the system $\left\{y^{(i)}_{k,j}\right\}_{i,k,j}$ in $\mathrm{l}^2(\Omega)$ (see, e.g., [8]).

4. Closure of Tensor Spherical Harmonics

First we list some preliminaries on tensor spherical harmonics: The operators $\mathbf{o}^{(i,k)}$: $\mathrm{C}^{(2)}(\Omega) \to \mathbf{c}^{(0)}(\Omega)$ are defined as follows (again using the surface gradient ∇^* and the surface curl gradient L^*)

$$\begin{aligned}
\mathbf{o}^{(1,1)}_\xi F(\xi) &= \xi \otimes \xi F(\xi),\\
\mathbf{o}^{(1,2)}_\xi F(\xi) &= \xi \otimes \nabla^*_\xi F(\xi),\\
\mathbf{o}^{(1,3)}_\xi F(\xi) &= \xi \otimes L^*_\xi F(\xi),\\
\mathbf{o}^{(2,1)}_\xi F(\xi) &= \nabla^*_\xi F(\xi) \otimes \xi,\\
\mathbf{o}^{(3,1)}_\xi F(\xi) &= L^*_\xi F(\xi) \otimes \xi,\\
\mathbf{o}^{(2,2)}_\xi F(\xi) &= \mathbf{i}_{\mathrm{tan}}(\xi) F(\xi),\\
\mathbf{o}^{(2,3)}_\xi F(\xi) &= \left(\nabla^*_\xi \otimes \nabla^*_\xi - L^*_\xi \otimes L^*_\xi\right) F(\xi) + 2\nabla^*_\xi F(\xi) \otimes \xi,\\
\mathbf{o}^{(3,2)}_\xi F(\xi) &= \left(\nabla^*_\xi \otimes L^*_\xi + L^*_\xi \otimes \nabla^*_\xi\right) F(\xi) + 2L^*_\xi F(\xi) \otimes \xi,\\
\mathbf{o}^{(3,3)}_\xi F(\xi) &= \mathbf{j}_{\mathrm{tan}}(\xi) F(\xi).
\end{aligned}$$

(for more details see [8]). Note that the surface identity tensor field is defined by $\mathbf{i}_{\mathrm{tan}}(\xi) = \mathbf{i} - \xi \otimes \xi$ (where $\mathbf{i}$ is the identity tensor) and the surface rotation tensor field by $\mathbf{j}_{\mathrm{tan}}(\xi) = \sum_{i=1}^{3} (\xi \wedge \varepsilon^i) \otimes \varepsilon^i$ (ε^1, ε^2, ε^3 denote the Cartesian unit vectors in $\mathbb{R}^3$). For notational convenience we introduce the abbreviation

$$0_{i,k} = \begin{cases} 0 & \text{for} \quad (i,k) = (1,1), (2,2), (3,3);\\ 1 & \text{for} \quad (i,k) = (1,2), (1,3), (2,1), (3,1);\\ 2 & \text{for} \quad (i,k) = (2,3), (3,2). \end{cases}$$

The application of the operators $\mathbf{o}^{(i,k)}$ to the scalar spherical harmonics defines the tensor spherical harmonics

$$\mathbf{y}^{(i,k)}_{m,j}(\xi) = \frac{1}{\sqrt{\mu^{(i,k)}_m}} \mathbf{o}^{(i,k)}_\xi Y_{m,j}(\xi),$$

where the normalization factor $\mu^{(i,k)}_m$ (to establish orthonormality) is given by

$$\mu^{(i,k)}_m = \begin{cases} 1 & \text{if} \quad (i,k) = (1,1);\\ 2 & \text{if} \quad (i,k) = (2,2), (3,3);\\ m(m+1) & \text{if} \quad (i,k) = (1,2), (1,3), (2,1), (3,1);\\ 2(m-1)m(m+1)(m+2) & \text{if} \quad (i,k) = (2,3), (3,2). \end{cases}$$

The tensorial Helmholtz decomposition theorem (see [2] for the (geo)physical background and [8] for the mathematical details) allows the decomposition of any given tensorial field $\mathbf{f}$ of class $\mathbf{c}^{(2)}(\Omega)$. In more detail, there exist the unique scalar functions $F^{(i,k)} \in \mathrm{C}^{(2)}(\Omega)$, $(i,k) \in \{(1,1),(1,2),\ldots,(3,3)\}$ with $\int_\Omega F^{(i,k)}(\xi)d\omega(\xi) = 0$ for $(i,k) = (1,2),(1,3),(2,1),(3,1),(2,3),(3,2)$ and $\int_\Omega F^{(i,k)}(\xi)Y_{1,l}d\omega(\xi) = 0$ for all $l = 1,2,3$ and for $(i,k) = (2,3),(3,2)$ such that

$$\mathbf{f} = \sum_{i,k=1}^{3} \mathbf{o}^{(i,k)} F^{(i,k)}.$$

These functions are determined by

$$\begin{aligned}
F^{(1,1)}(\xi) &= O_\xi^{(1,1)}\mathbf{f}(\xi)\\
F^{(2,2)}(\xi) &= \frac{1}{2}O_\xi^{(2,2)}\mathbf{f}(\xi)\\
F^{(3,3)}(\xi) &= \frac{1}{2}O_\xi^{(3,3)}\mathbf{f}(\xi)\\
F^{(1,2)}(\xi) &= -\left(G(\Delta^*;\cdot,\cdot) * O^{(1,2)}\mathbf{f}\right)(\xi)\\
F^{(1,3)}(\xi) &= -\left(G(\Delta^*;\cdot,\cdot) * O^{(1,3)}\mathbf{f}\right)(\xi)\\
F^{(2,1)}(\xi) &= -\left(G(\Delta^*;\cdot,\cdot) * O^{(2,1)}\mathbf{f}\right)(\xi)\\
F^{(3,1)}(\xi) &= -\left(G(\Delta^*;\cdot,\cdot) * O^{(3,1)}\mathbf{f}\right)(\xi)\\
F^{(2,3)}(\xi) &= \frac{1}{2}\left(G(\Delta^*(\Delta^*+2);\cdot,\cdot) * O^{(2,3)}\mathbf{f}\right)(\xi)\\
F^{(3,2)}(\xi) &= \frac{1}{2}\left(G(\Delta^*(\Delta^*+2);\cdot,\cdot) * O^{(3,2)}\mathbf{f}\right)(\xi).
\end{aligned}$$

$O^{(i,k)}$ denote the adjoint operators associated to the operators $\mathbf{o}^{(i,k)}$ (see [8] for a detailed description), and $(\xi,\eta) \mapsto G(\Delta^*(\Delta^*+2);\xi,\eta)$, $-1 \le \xi\cdot\eta \le 1$, is the Green function with respect to the operator $\Delta^*(\Delta^*+2)$. For more details about the general concept of Green's functions on the sphere with respect to iterated Beltrami operators the reader is referred to [7, 8].

Since the elementary representation of the Green function with respect to the operator $\Delta^*(\Delta^*+2)$ seems to be unknown, we first deal with the following lemma.

Lemma 4.1: *For $t \in [-1,1]$,*

$$G(\Delta^*(\Delta^*+2);t) = \frac{1}{8\pi}(1-t)\ln(1-t) + \frac{1}{4\pi}\left(\frac{1}{12}+\frac{\ln(2)}{2}\right)t + \frac{1}{4\pi}\left(\frac{1}{4}-\frac{\ln(2)}{2}\right).$$

Proof: We use the bilinear expansion of $t \mapsto G(\Delta^*(\Delta^*+2);t)$, $t \in [-1,1]$ (with $t = \xi\cdot\eta$) given by

$$G(\Delta^*(\Delta^*+2);t) = \frac{1}{4\pi}\sum_{k=2}^{\infty}\frac{2k+1}{k(k+1)(k-1)(k+2)}P_k(t).$$

For simplicity we introduce the abbreviation $G(\Delta^*(\Delta^*+2);t) = \frac{1}{4\pi}G(t)$. With this simplified notation we have

$$G(t) = \sum_{k=2}^{\infty} \frac{2k+1}{(k-1)k(k+1)(k+2)} P_k(t).$$

Considering the derivatives of G we obtain in a first step with (10)

$$\begin{aligned} G'(t) &= \sum_{k=2}^{\infty} \frac{2k+1}{(k-1)k(k+1)(k+2)} P_k'(t) \\ &= \frac{1}{t^2-1} \sum_{k=2}^{\infty} \frac{1}{(k-1)(k+2)} \left(P_{k+1}(t) - P_{k-1}(t)\right). \end{aligned}$$

By index shifts we are able to conclude that

$$G'(t) = \frac{1}{t^2-1} \left(\sum_{k=3}^{\infty} \frac{2(2k+1)}{(k-2)k(k+1)(k+3)} P_k(t) - \frac{1}{4} P_1(t) - \frac{1}{10} P_2(t) \right).$$

We derive this expression once again:

$$\begin{aligned} G''(t) = \frac{-2t}{(t^2-1)^2} &\left(\sum_{k=3}^{\infty} \frac{2(2k+1)}{(k-2)k(k+1)(k+3)} P_k(t) - \frac{1}{4} t - \frac{1}{10} P_2(t) \right) \\ + \frac{1}{t^2-1} &\left(\sum_{k=3}^{\infty} \frac{2(2k+1)}{(k-2)k(k+1)(k+3)} P_k'(t) - \frac{1}{4} - \frac{3}{10} t \right). \end{aligned}$$

The second sum can be transformed by use of the recurrence relation (10)

$$\begin{aligned} \sum_{k=3}^{\infty} \frac{2(2k+1)}{(k-2)k(k+1)(k+3)} P_k'(t) &= \frac{1}{t^2-1} \sum_{k=3}^{\infty} \frac{2}{(k-2)(k+3)} \left(P_{k+1}(t) - P_{k-1}(t)\right) \\ = \frac{1}{t^2-1} &\left(\sum_{k=4}^{\infty} \frac{4(2k+1)}{(k-3)(k-1)(k+2)(k+4)} P_k(t) - \frac{1}{3} P_2(t) - \frac{1}{7} P_3(t) \right). \end{aligned}$$

This provides us with the following representation of the second derivative of G

$$\begin{aligned} G''(t) = \frac{1}{(t^2-1)^2} &\left(\sum_{k=3}^{\infty} \frac{-4(2k+1)}{(k-2)k(k+1)(k+3)} t P_k(t) + \frac{1}{2} t^2 + \frac{1}{5} t P_2(t) \right) \\ + \frac{1}{(t^2-1)^2} &\left(\sum_{k=4}^{\infty} \frac{4(2k+1)}{(k-3)(k-1)(k+2)(k+4)} P_k(t) - \frac{1}{3} P_2(t) - \frac{1}{7} P_3(t) \right. \\ &\left. \qquad - \frac{1}{4}(t^2-1) - \frac{3}{10} t (t^2-1) \right). \end{aligned}$$

In the first sum the three-term-recurrence of the Legendre polynomials (see, e.g.,

[16]) is applied, i.e., $(2k+1)tP_k(t) = (k+1)P_{k+1}(t) + kP_{k-1}(t)$, which yields

$$\begin{aligned}
&\sum_{k=3}^{\infty} \frac{2k+1}{(k-2)k(k+1)(k+3)} tP_k(t) \\
&\quad = \sum_{k=3}^{\infty} \frac{1}{(k-2)k(k+3)} P_{k+1}(t) + \sum_{k=3}^{\infty} \frac{1}{(k-2)(k+1)(k+3)} P_{k-1}(t) \\
&\quad = \sum_{k=4}^{\infty} \frac{2k+1}{(k-3)(k-1)(k+2)(k+4)} P_k(t) + \frac{1}{24} P_2(t) + \frac{1}{70} P_3(t).
\end{aligned}$$

Summarizing we have the second derivative given by

$$\begin{aligned}
G''(t) = \frac{1}{(t^2-1)^2} \Bigg(&-4 \sum_{k=4}^{\infty} \frac{2k+1}{(k-3)(k-1)(k+2)(k+4)} P_k(t) - \frac{1}{6} P_2(t) - \frac{2}{35} P_3(t) \\
&+ \frac{1}{2} t^2 + \frac{1}{5} t P_2(t) + 4 \sum_{k=4}^{\infty} \frac{(2k+1)}{(k-3)(k-1)(k+2)(k+4)} P_k(t) \\
&- \frac{1}{3} P_2(t) - \frac{1}{7} P_3(t) - \frac{1}{4}(t^2-1) - \frac{3}{10} t(t^2-1) \Bigg) \\
= \frac{1}{(t^2-1)^2} &\left(-\frac{1}{2}(t-1)(t+1)^2 \right) = \frac{1}{2} \frac{1}{1-t} .
\end{aligned}$$

Now an elementary representation of G can be recovered by integration. Of course, this requires certain values to determine the constants of integration. We use initial conditions at $t = -1$ (see, e.g., [16])

$$\begin{aligned}
G(-1) &= \sum_{k=2}^{\infty} \frac{2k+1}{(k-1)k(k+1)(k+2)} P_k(-1) \\
&= \sum_{k=2}^{\infty} \frac{2k+1}{(k-1)k(k+1)(k+2)} (-1)^k = \frac{1}{6}, \\
G'(-1) &= \sum_{k=2}^{\infty} \frac{2k+1}{(k-1)k(k+1)(k+2)} P_k'(-1) \\
&= \sum_{k=2}^{\infty} \frac{2k+1}{(k-1)k(k+1)(k+2)} \frac{k(k+1)}{2} (-1)^{k+1} \\
&= \sum_{k=2}^{\infty} \frac{2k+1}{2(k-1)(k+2)} (-1)^{k+1} = -\frac{5}{12}.
\end{aligned}$$

Ordinary integration gives us

$$G'(t) = \int_{-1}^{t} \frac{1}{2} \frac{1}{1-s} ds = -\frac{1}{2} \ln(1-t) + \frac{1}{2} \ln(2),$$

which does not yet fit the demanded value at $t = -1$. This is the reason why we choose

$$G'(t) = -\frac{1}{2}\ln(1-t) + \frac{1}{2}\ln(2) - \frac{5}{12}$$

for the second integration which results in

$$\begin{aligned} G(t) &= \int_{-1}^{t} -\frac{1}{2}\ln(1-s) + \frac{1}{2}\ln(2) - \frac{5}{12} ds \\ &= \frac{1}{2}(1-t)\ln(1-t) - \frac{1}{2} + \frac{1}{2}t + \frac{\ln(2)}{2}t - \frac{5}{12}t - \frac{1}{2}\ln(2) + 1 - \frac{5}{12}. \end{aligned}$$

Once again a constant needs to be added, i.e.,

$$G(t) = \frac{1}{2}(1-t)\ln(1-t) - \frac{1}{2} + \frac{1}{2}t + \frac{\ln(2)}{2}t - \frac{5}{12}t - \frac{1}{2}\ln(2) + 1 - \frac{5}{12} + \frac{1}{6}.$$

This yields the required result of Lemma 4.1. □

Next we introduce the 'Bernstein convolutions' to the nine Helmholtz functions. More explicitly,

$$\begin{aligned} F_n^{(1,1)}(\xi) &= \left(B_n * O^{(1,1)}\mathbf{f}\right)(\xi) \\ F_n^{(2,2)}(\xi) &= \frac{1}{2}\left(B_n * O^{(2,2)}\mathbf{f}\right)(\xi) \\ F_n^{(3,3)}(\xi) &= \frac{1}{2}\left(B_n * O^{(3,3)}\mathbf{f}\right)(\xi) \\ F_n^{(1,2)}(\xi) &= -\left(BG_n * O^{(1,2)}\mathbf{f}\right)(\xi) \\ F_n^{(1,3)}(\xi) &= -\left(BG_n * O^{(1,3)}\mathbf{f}\right)(\xi) \\ F_n^{(2,1)}(\xi) &= -\left(BG_n * O^{(2,1)}\mathbf{f}\right)(\xi) \\ F_n^{(3,1)}(\xi) &= -\left(BG_n * O^{(3,1)}\mathbf{f}\right)(\xi) \\ F_n^{(2,3)}(\xi) &= \left(BG_n^{(2)} * O^{(2,3)}\mathbf{f}\right)(\xi) \\ F_n^{(3,2)}(\xi) &= \left(BG_n^{(2)} * O^{(3,2)}\mathbf{f}\right)(\xi), \end{aligned}$$

where

$$\begin{aligned} BG_n^{(2)}(\xi,\eta) &= \frac{1}{2}\int_{\Omega} G(\Delta^*(\Delta^*+2);\xi,\alpha)B_n(\alpha\cdot\eta)d\omega(\alpha) \\ &= \sum_{k=2}^{n} \frac{2k+1}{4\pi}\frac{B_n^\wedge(k)}{2(k-1)k(k+1)(k+2)}P_k(\xi\cdot\eta). \end{aligned}$$

Our interest now is the 'Bernstein summability' of Fourier expansions in terms of tensor spherical harmonics. Again we need some preparatory results (viz., Lemma 4.2 and Lemma 4.3).

Lemma 4.2: *For $i,k \in \{1,2,3\}$ we have*

$$\lim_{n\to\infty} \sup_{\xi\in\Omega} \left| F^{(i,k)}(\xi) - F_n^{(i,k)}(\xi) \right| = 0.$$

Proof: Since both kernels $G(\Delta^*(\Delta^*+2);\cdot)$ and $BG_n^{(2)}(\cdot)$ are in $\mathrm{L}^2[-1,1]$ and the Legendre coefficients of the Bernstein kernel $B_n^\wedge(k)$ converge to 1 as $n \to \infty$ (see (8)) for all $k \in \mathbb{N}_0$, we obtain

$$\lim_{n\to\infty} \|\frac{1}{2}G(\Delta^*(\Delta^*+2);\cdot) - BG_n^{(2)}\|_{\mathrm{L}^2[-1,1]} = 0$$

The last limit also holds true in L^1-metric. Consequently we are able to deduce that $\|F^{(i,k)} - F_n^{(i,k)}\|_{C^{(0)}(\Omega)} \longrightarrow 0$ for $i,k \in \{1,2,3\}$ as $n \longrightarrow \infty$. □

Including the tensorial operators and considering their corresponding differences we are led to the following result.

Lemma 4.3: *For $i,k \in \{1,2,3\}$ we have*

$$\lim_{n\to\infty} \sup_{\xi\in\Omega} \|\mathbf{o}^{(i,k)} F^{(i,k)} - \mathbf{o}^{(i,k)} F_n^{(i,k)}\|_{\mathbf{c}^{(0)}(\Omega)} = 0.$$

Proof: For the types $(i,k) = (1,1),(2,2),(3,3)$ we obtain the required convergence of $\|\mathbf{o}^{(i,k)} F^{(i,k)} - \mathbf{o}^{(i,k)} F_n^{(i,k)}\|_{\mathbf{c}^{(0)}(\Omega)}$ as in the scalar case, and for the types $(i,k) = (1,2),(1,3),(2,1),(3,1)$ as in the vectorial case because of the structure of the corresponding operators $\mathbf{o}^{(i,k)}$. This leaves us with the two types $(i,k) = (2,3),(3,2)$.

$$\begin{aligned}
&\|\mathbf{o}_\xi^{(i,k)} F^{(i,k)}(\xi) - \mathbf{o}_\xi^{(i,k)} F_n^{(i,k)}(\xi)\|_{\mathbf{c}^{(0)}(\Omega)} \\
&\quad = \sup_{\xi\in\Omega} \left| \mathbf{o}_\xi^{(i,k)} \int_\Omega \frac{1}{2} G(\Delta^*(\Delta^*+2);\xi,\eta) O_\eta^{(i,k)} \mathbf{f}(\eta)\, d\omega(\eta) \right. \\
&\qquad\qquad \left. - \mathbf{o}_\xi^{(i,k)} \int_\Omega BG_n^{(2)}(\xi,\eta) O_\eta^{(i,k)} \mathbf{f}(\eta)\, d\omega(\eta) \right| \\
&\quad = \sup_{\xi\in\Omega} \left| \int_\Omega \mathbf{o}_\xi^{(i,k)} \frac{1}{2} G(\Delta^*(\Delta^*+2);\xi,\eta) O_\eta^{(i,k)} \mathbf{f}(\eta)\, d\omega(\eta) \right. \\
&\qquad\qquad \left. - \int_\Omega \mathbf{o}_\xi^{(i,k)} BG_n^{(2)}(\xi,\eta) O_\eta^{(i,k)} \mathbf{f}(\eta)\, d\omega(\eta) \right|,
\end{aligned}$$

where the operator $\mathbf{o}^{(i,k)}$ can be put inside both integrals. By obvious manipulations we find

$$\begin{aligned}
&\sup_{\xi\in\Omega}\left|\int_{\Omega}\left(\mathbf{o}_{\xi}^{(i,k)}\frac{1}{2}G(\Delta^*(\Delta^*+2);\xi,\eta)-\mathbf{o}_{\xi}^{(i,k)}BG_n^{(2)}(\xi,\eta)\right)O_{\eta}^{(i,k)}\mathbf{f}(\eta)d\omega(\eta)\right|\\
&\leq\sup_{\xi\in\Omega}\int_{\Omega}\left|\mathbf{o}_{\xi}^{(i,k)}\frac{1}{2}G(\Delta^*(\Delta^*+2);\xi,\eta)-\mathbf{o}_{\xi}^{(i,k)}BG_n^{(2)}(\xi,\eta)\right|\left|O_{\eta}^{(i,k)}\mathbf{f}(\eta)\right|d\omega(\eta)\\
&\leq\|O^{(i,k)}\mathbf{f}\|_{\mathrm{C}^{(0)}(\Omega)}\int_{\Omega}\left|\mathbf{o}_{\xi}^{(i,k)}\frac{1}{2}G(\Delta^*(\Delta^*+2);\xi,\eta)-\mathbf{o}_{\xi}^{(i,k)}BG_n^{(2)}(\xi,\eta)\right|d\omega(\eta). \quad (20)
\end{aligned}$$

Therefore, we just need to prove the convergence of the last integral, i.e., the $\mathbf{l}^1$-norm. Application of the tensorial operators $\mathbf{o}^{(2,3)}$ and $\mathbf{o}^{(3,2)}$ to the corresponding Green's function results in the identities

$$\begin{aligned}
\mathbf{o}_{\xi}^{(2,3)}G(\Delta^*(\Delta^*+2);&\xi,\eta)=\\
&\frac{1}{4\pi}G''(\xi\cdot\eta)\left[(\eta-(\xi\cdot\eta)\,\xi)\otimes(\eta-(\xi\cdot\eta)\,\xi)-(\xi\wedge\eta)\otimes(\xi\wedge\eta)\right],\\
\mathbf{o}_{\xi}^{(3,2)}G(\Delta^*(\Delta^*+2);&\xi,\eta)=\\
&\frac{1}{4\pi}G''(\xi\cdot\eta)\left[(\eta-(\xi\cdot\eta)\,\xi)\otimes(\xi\wedge\eta)+(\xi\wedge\eta)\otimes(\eta-(\xi\cdot\eta)\,\xi)\right].
\end{aligned}$$

Considering the absolute value of these two we find that

$$\begin{aligned}
&\left|\mathbf{o}_{\xi}^{(2,3)}G(\Delta^*(\Delta^*+2);\xi,\eta)\right|\\
&\qquad=\frac{1}{4\pi}\left|G''(\xi\cdot\eta)\right|\left|(\eta-(\xi\cdot\eta)\,\xi)\otimes(\eta-(\xi\cdot\eta)\,\xi)-(\xi\wedge\eta)\otimes(\xi\wedge\eta)\right|\\
&\qquad=\frac{1}{8\pi}\frac{1}{1-\xi\cdot\eta}\sqrt{2}(1-(\xi\cdot\eta)^2)=\frac{1}{4\pi}\frac{1}{\sqrt{2}}(1+\xi\cdot\eta)
\end{aligned}$$

and

$$\begin{aligned}
&\left|\mathbf{o}_{\xi}^{(3,2)}G(\Delta^*(\Delta^*+2);\xi,\eta)\right|\\
&\qquad=\frac{1}{4\pi}\left|G''(\xi\cdot\eta)\right|\left|(\eta-(\xi\cdot\eta)\,\xi)\otimes(\xi\wedge\eta)+(\xi\wedge\eta)\otimes(\eta-(\xi\cdot\eta)\,\xi)\right|\\
&\qquad=\frac{1}{8\pi}\frac{1}{1-\xi\cdot\eta}\sqrt{2}(1-(\xi\cdot\eta)^2)=\frac{1}{4\pi}\frac{1}{\sqrt{2}}(1+\xi\cdot\eta).
\end{aligned}$$

Note that for the first operator we use the relation $|x\otimes x-y\otimes y|^2=|x|^4+|y|^4-2(x\cdot y)^2$ with $x=\eta-(\xi\cdot\eta)\,\xi$ and $y=\xi\wedge\eta$. It should be remarked that $(\eta-(\xi\cdot\eta)\,\xi)\cdot(\xi\wedge\eta)=0$ and $|\eta-(\xi\cdot\eta)\,\xi|^2=|\xi\wedge\eta|^2=1-(\xi\cdot\eta)^2$. For the second operator a slightly different relation is required, i.e., $|x\otimes y+y\otimes x|^2=2(x\cdot y)^2+2|x|^2|y|^2$ with $x=\eta-(\xi\cdot\eta)\,\xi$ and $y=\xi\wedge\eta$.

Thus, we can conclude both that $\mathbf{o}^{(3,2)}G(\Delta^*(\Delta^*+2);\cdot,\eta)$ is of class $\mathbf{l}^2(\Omega)$ and $\mathbf{o}^{(2,3)}G(\Delta^*(\Delta^*+2);\cdot,\eta)$ is of class $\mathbf{l}^2(\Omega)$ for all $\eta\in\Omega$.
In consequence, the desired $\mathbf{l}^1$-convergence results from the $\mathbf{l}^2$-convergence of the

two kernels (both are in $\mathbf{l}^2(\Omega)$ and $B_n^{\wedge}(k)$ tends to 1 as presented in (8)). Thus, Lemma 4.3 is guaranteed for all types (i,k). □

Next we come to the promised *'Bernstein summability' of Fourier series in terms of tensor spherical harmonics.*

Theorem 4.4: *For any tensor field* $\mathbf{f} \in \mathbf{c}^{(2)}(\Omega)$,

$$\lim_{n\to\infty} \sup_{\xi\in\Omega} \left| \mathbf{f}(\xi) - \sum_{i,k=1}^{3} \sum_{m=0_{i,k}}^{n} \sum_{j=1}^{2m+1} B_n^{\wedge}(m) \left(f^{(i,k)}\right)^{\wedge}(m,j)\mathbf{y}_{m,j}^{(i,k)}(\xi) \right| = 0.$$

Proof: From Lemma 4.3 we have for any tensorial field $\mathbf{f} \in \mathbf{c}^{(2)}(\Omega)$

$$\begin{aligned}
&\lim_{n\to\infty} \sup_{\xi\in\Omega} \left| \mathbf{f}(\xi) - \sum_{i,k=1}^{3} \mathbf{o}_\xi^{(i,k)} F_n^{(i,k)}(\xi) \right| \\
&\quad = \lim_{n\to\infty} \sup_{\xi\in\Omega} \left| \sum_{i,k=1}^{3} \mathbf{o}_\xi^{(i,k)} F^{(i,k)}(\xi) - \sum_{i,k=1}^{3} \mathbf{o}_\xi^{(i,k)} F_n^{(i,k)}(\xi) \right| \\
&\quad \le \sum_{i,k=1}^{3} \lim_{n\to\infty} \sup_{\xi\in\Omega} \left| \mathbf{o}_\xi^{(i,k)} F^{(i,k)}(\xi) - \mathbf{o}_\xi^{(i,k)} F_n^{(i,k)}(\xi) \right| = 0.
\end{aligned} \tag{21}$$

Let us consider the term $\mathbf{o}_\xi^{(1,1)} F_n^{(1,1)}(\xi)$ in more detail. A simple calculation yields

$$\begin{aligned}
\mathbf{o}_\xi^{(1,1)} F_n^{(1,1)}(\xi) &= \mathbf{o}_\xi^{(1,1)} \int_\Omega B_n(\xi\cdot\eta) O_\eta^{(1,1)} \mathbf{f}(\eta) d\omega(\eta) \\
&= \sum_{k=0}^{n} B_n^{\wedge}(k) \frac{2k+1}{4\pi} \mathbf{o}_\xi^{(1,1)} \int_\Omega P_k(\xi\cdot\eta) O_\eta^{(1,1)} \mathbf{f}(\eta) d\omega(\eta) \\
&= \sum_{k=0}^{n} B_n^{\wedge}(k) \mathbf{o}_\xi^{(1,1)} \sum_{j=1}^{2k+1} \left(O^{(1,1)}\mathbf{f}\right)^{\wedge}(k,j) Y_{k,j}(\xi) \\
&= \sum_{k=0}^{n} \sum_{j=1}^{2k+1} B_n^{\wedge}(k) \left(O^{(1,1)}\mathbf{f}\right)^{\wedge}(k,j) \mathbf{y}_{k,j}^{(1,1)}(\xi).
\end{aligned} \tag{22}$$

Note that

$$\begin{aligned}
\left(O^{(1,1)}\mathbf{f}\right)^{\wedge}(k,j) &= \int_\Omega O_\eta^{(1,1)} \mathbf{f}(\eta) Y_{k,j}(\eta) d\omega(\eta) \\
&= \int_\Omega \mathbf{f}(\eta) \cdot \underbrace{\mathbf{o}_\eta^{(1,1)} Y_{k,j}(\eta)}_{=\mathbf{y}_{k,j}^{(1,1)}(\eta)} d\omega(\eta) = \left(\mathbf{f}^{(1,1)}\right)^{\wedge}(k,j)
\end{aligned} \tag{23}$$

such that from (22) and (23) we are able to conclude for type $(1,1)$ that

$$\mathbf{o}_\xi^{(1,1)} F_n^{(1,1)}(\xi) = \sum_{k=0}^{n} \sum_{j=1}^{2k+1} B_n^\wedge(k) \left(\mathbf{f}^{(1,1)}\right)^\wedge (k,j) \mathbf{y}_{k,j}^{(1,1)}(\xi). \tag{24}$$

For the cases $(i,k) = (2,2), (3,3)$ we have

$$\begin{aligned}
\mathbf{o}_\xi^{(i,k)} F_n^{(i,k)}(\xi) &= \mathbf{o}_\xi^{(i,k)} \frac{1}{2} \int_\Omega B_n(\xi\cdot\eta) O_\eta^{(i,k)} \mathbf{f}(\eta) d\omega(\eta) \\
&= \frac{1}{2} \sum_{m=0}^{n} B_n^\wedge(m) \frac{2m+1}{4\pi} \mathbf{o}_\xi^{(i,k)} \int_\Omega P_m(\xi\cdot\eta) O_\eta^{(i,k)} \mathbf{f}(\eta) d\omega(\eta) \\
&= \frac{1}{2} \sum_{m=0}^{n} B_n^\wedge(m) \mathbf{o}_\xi^{(i,k)} \sum_{j=1}^{2m+1} \left(O^{(i,k)}\mathbf{f}\right)^\wedge (m,j) Y_{m,j}(\xi) \\
&= \frac{1}{\sqrt{2}} \sum_{m=0}^{n} \sum_{j=1}^{2m+1} B_n^\wedge(m) \left(O^{(i,k)}\mathbf{f}\right)^\wedge (m,j) \mathbf{y}_{m,j}^{(i,k)}(\xi).
\end{aligned} \tag{25}$$

Observe that

$$\begin{aligned}
\left(O^{(i,k)}\mathbf{f}\right)^\wedge (m,j) &= \int_\Omega O_\eta^{(i,k)} \mathbf{f}(\eta) Y_{m,j}(\eta) d\omega(\eta) \\
&= \int_\Omega \mathbf{f}(\eta) \cdot \underbrace{\mathbf{o}_\eta^{(i,k)} Y_{m,j}(\eta)}_{=\sqrt{2}\mathbf{y}_{m,j}^{(i,k)}(\eta)} d\omega(\eta) = \sqrt{2} \left(\mathbf{f}^{(i,k)}\right)^\wedge (m,j).
\end{aligned} \tag{26}$$

Combining (25) and (26) we find for $(i,k) = (2,2), (3,3)$

$$\mathbf{o}_\xi^{(i,k)} F_n^{(i,k)}(\xi) = \sum_{m=0}^{n} \sum_{j=1}^{2m+1} B_n^\wedge(m) \left(\mathbf{f}^{(i,k)}\right)^\wedge (m,j) \mathbf{y}_{m,j}^{(i,k)}(\xi). \tag{27}$$

For $(i,k) = (1,2), (1,3), (2,1), (3,1)$ we have

$$\begin{aligned}
\mathbf{o}_\xi^{(i,k)} F_n^{(i,k)}(\xi) &= -\mathbf{o}_\xi^{(i,k)} \int_\Omega BG_n(\xi\cdot\eta) O_\eta^{(i,k)} \mathbf{f}(\eta) d\omega(\eta) \\
&= \sum_{m=1}^{n} \frac{B_n^\wedge(m)}{m(m+1)} \frac{2m+1}{4\pi} \mathbf{o}_\xi^{(i,k)} \int_\Omega P_m(\xi\cdot\eta) O_\eta^{(i,k)} \mathbf{f}(\eta) d\omega(\eta) \\
&= \sum_{m=1}^{n} \frac{B_n^\wedge(m)}{m(m+1)} \mathbf{o}_\xi^{(i,k)} \sum_{j=1}^{2m+1} \left(O^{(i,k)}\mathbf{f}\right)^\wedge (m,j) Y_{m,j}(\xi) \\
&= \sum_{m=1}^{n} \sum_{j=1}^{2m+1} \frac{B_n^\wedge(m)}{\sqrt{m(m+1)}} \left(O^{(i,k)}\mathbf{f}\right)^\wedge (m,j) \mathbf{y}_{m,j}^{(i,k)}(\xi).
\end{aligned} \tag{28}$$

Again we take a look at the coefficients $\left(O^{(i,k)}\mathbf{f}\right)^{\wedge}(m,j)$ and find

$$\left(O^{(i,k)}\mathbf{f}\right)^{\wedge}(m,j) = \int\limits_{\Omega} O^{(i,k)}_{\eta}\mathbf{f}(\eta)Y_{m,j}(\eta)d\omega(\eta) = \int\limits_{\Omega} \mathbf{f}(\eta)\cdot \mathbf{o}^{(i,k)}_{\eta}Y_{m,j}(\eta)d\omega(\eta)$$
$$= \sqrt{m(m+1)}\int\limits_{\Omega} \mathbf{f}(\eta)\cdot \mathbf{y}^{(i,k)}_{m,j}(\eta)d\omega(\eta) = \sqrt{m(m+1)}\left(\mathbf{f}^{(i,k)}\right)^{\wedge}(m,j). \quad (29)$$

Putting together (28) and (29) we are able to see that, for $(i,k) = (1,2),(1,3),(2,1),(3,1)$,

$$\mathbf{o}^{(i,k)}_{\xi}F^{(i,k)}_{n}(\xi) = \sum_{m=1}^{n}\sum_{j=1}^{2m+1} B^{\wedge}_{n}(m)\left(\mathbf{f}^{(i,k)}\right)^{\wedge}(m,j)\mathbf{y}^{(i,k)}_{m,j}(\xi). \quad (30)$$

Finally, we treat $(i,k) = (2,3),(3,2)$. It is not hard to verify that

$$\mathbf{o}^{(i,k)}_{\xi}F^{(i,k)}_{n}(\xi) = \mathbf{o}^{(i,k)}_{\xi}\int\limits_{\Omega} BG^{(2)}_{n}(\xi\cdot\eta)O^{(i,k)}_{\eta}\mathbf{f}(\eta)d\omega(\eta)$$
$$= \sum_{m=2}^{n}\frac{B^{\wedge}_{n}(m)}{2m(m+1)(m(m+1)-2)}\frac{2m+1}{4\pi}\mathbf{o}^{(i,k)}_{\xi}\int\limits_{\Omega} P_{m}(\xi\cdot\eta)O^{(i,k)}_{\eta}\mathbf{f}(\eta)d\omega(\eta)$$
$$= \sum_{m=2}^{n}\frac{B^{\wedge}_{n}(m)}{2m(m+1)(m(m+1)-2)}\mathbf{o}^{(i,k)}_{\xi}\sum_{j=1}^{2m+1}\left(O^{(i,k)}\mathbf{f}\right)^{\wedge}(m,j)Y_{m,j}(\xi)$$
$$= \sum_{m=2}^{n}\sum_{j=1}^{2m+1}\frac{B^{\wedge}_{n}(m)}{\sqrt{2m(m+1)(m(m+1)-2)}}\left(O^{(i,k)}\mathbf{f}\right)^{\wedge}(m,j)\mathbf{y}^{(i,k)}_{m,j}(\xi). \quad (31)$$

This enables us to rewrite the coefficients $\left(O^{(i,k)}\mathbf{f}\right)^{\wedge}(m,j)$ as follows

$$\left(O^{(i,k)}\mathbf{f}\right)^{\wedge}(m,j) = \int\limits_{\Omega} O^{(i,k)}_{\eta}\mathbf{f}(\eta)Y_{m,j}(\eta)d\omega(\eta) = \int\limits_{\Omega} \mathbf{f}(\eta)\cdot \mathbf{o}^{(i,k)}_{\eta}Y_{m,j}(\eta)d\omega(\eta)$$
$$= \sqrt{2m(m+1)(m(m+1)-2)}\int\limits_{\Omega} \mathbf{f}(\eta)\cdot \mathbf{y}^{(i,k)}_{m,j}(\eta)d\omega(\eta)$$
$$= \sqrt{2m(m+1)(m(m+1)-2)}\left(\mathbf{f}^{(i,k)}\right)^{\wedge}(m,j). \quad (32)$$

Consequently, (31) and (32) lead to the conclusion that for $(i,k) = (2,3),(3,2)$

$$\mathbf{o}^{(i,k)}_{\xi}F^{(i,k)}_{n}(\xi) = \sum_{m=2}^{n}\sum_{j=1}^{2m+1} B^{\wedge}_{n}(m)\left(\mathbf{f}^{(i,k)}\right)^{\wedge}(m,j)\mathbf{y}^{(i,k)}_{m,j}(\xi). \quad (33)$$

Altogether, the identities (24), (27), (30) and (33) in connection with (21) yield the desired summability of tensor spherical harmonics, i.e.,

$$\lim_{n\to\infty}\sup_{\xi\in\Omega}\left|\mathbf{f}(\xi)-\sum_{i,k=1}^{3}\mathbf{o}_{\xi}^{(i,k)}F_n^{(i,k)}(\xi)\right|$$

$$=\lim_{n\to\infty}\sup_{\xi\in\Omega}\left|\mathbf{f}(\xi)-\sum_{i,k=1}^{3}\sum_{m=0_{i,k}}^{n}\sum_{j=1}^{2m+1}B_n^{\wedge}(m)\left(f^{(i,k)}\right)^{\wedge}(m,j)\mathbf{y}_{m,j}^{(i,k)}(\xi)\right|=0,$$

provided that $\mathbf{f}\in\mathbf{c}^{(2)}(\Omega)$. □

As in the vector case, based on a density argument, the *closure of the tensor spherical harmonics* $\left\{\mathbf{y}_{m,j}^{(i,k)}\right\}_{i,k,m,j}$ *in the space* $\mathbf{c}^{(0)}(\Omega)$ becomes obvious.

Theorem 4.5: *For any given $\varepsilon>0$ and each $\mathbf{f}\in\mathbf{c}^{(0)}(\Omega)$ there exists a linear combination $\sum_{i,k=1}^{3}\sum_{m=0_{i,k}}^{N}\sum_{j=1}^{2m+1}d_{m,j}^{(i,k)}\mathbf{y}_{m,j}^{(i,k)}$, such that*

$$\left\|\mathbf{f}-\sum_{i,k=1}^{3}\sum_{m=0_{i,k}}^{N}\sum_{j=1}^{2m+1}d_{m,j}^{(i,k)}\mathbf{y}_{m,j}^{(i,k)}\right\|_{\mathbf{c}^{(0)}(\Omega)}\leq\varepsilon.$$

Again, standard arguments guarantee the closure in $\mathbf{c}^{(0)}(\Omega)$ with respect to $\|\cdot\|_{\mathbf{l}^2(\Omega)}$ as well as in $\mathbf{l}^2(\Omega)$ which in turn shows the completeness of the system $\left\{\mathbf{y}_{m,j}^{(i,k)}\right\}_{i,k,m,j}$ in $\mathbf{l}^2(\Omega)$.

References

[1] G.E. Backus, *Potentials for Tangent Tensor Fields on Spheroids*, Archs. Ration. Mech. Anal. 22 (1966), pp. 210–252.

[2] ———, *Converting Vector and Tensor Equations to Scalar Equations in Spherical Coordinates*, Geophys. J. R. astr. Soc. 13 (1967), pp. 71–101.

[3] S.N. Bernstein, *Sur l'ordre de la meilleure approximation des fonctions continues par des polynômes de degré donné*, Mém. Cl. Sci. Acad. Roy. Belg. (2) 4 (1912), pp. 1–104.

[4] ———, *Sur un procédé de sommation des séries trigonométriques*, C. R. Acad. Sci. Paris 191 (1930), pp. 976–979.

[5] M.J. Fengler, W. Freeden, and M. Gutting, *The Spherical Bernstein Wavelet*, International Journal of Pure and Applied Mathematics (IJPAM) 31 (2006), pp. 209–230.

[6] W. Freeden, *Über eine Klasse von Integralformeln der Mathematischen Geodäsie*, Veröff. Geod. Inst. RWTH Aachen, Heft 27 (1979).

[7] ———, *On Integral Formulas of the (Unit) Sphere and Their Application to Numerical Computation of Integrals*, Computing 25 (1980), pp. 131–146.

[8] W. Freeden, T. Gervens, and M. Schreiner, *Constructive Approximation on the Sphere (With Applications to Geomathematics)*, Oxford Science Publications, Clarendon (1998).

[9] T. Gervens, *Vektorkugelfunktionen mit Anwendungen in der Theorie der elastischen Verformungen für die Kugel*, Ph.D. thesis, Rheinisch-Westfälische Technische Hochschule (RWTH), Aachen, Germany (1989).

[10] H.H. Gonska and J. Meier, *A Bibliography on Approximation of Functions by Bernstein Type Operators (1955-1982)*, in "Approximation Theory IV", C. K. Chui, L. L. Schumaker, J. D. Ward (eds.), Academic Press, New York (1983), pp. 739–785.

[11] H.H. Gonska and J. Meier-Gonska, *A Bibliography on Approximation of Functions by Bernstein Type Operators (Supplement 1986)*, in "Approximation Theory V", C. K. Chui, L. L. Schumaker, J. D. Ward (eds.), Academic Press, New York (1986), pp. 621–654.

[12] E. Heine, *Handbuch der Kugelfunktionen*, G. Reimer (1878).

[13] P.S. Laplace, *Théorie des attractions des sphéroïds et de la figure des planètes*, Tech. rep., Mèm. de l'Acad., Paris (1785).

[14] A.M. Legendre, *Recherches sur l'attraction des sphéroides homogènes*, Mém. math. phys. prés. á l'Acad. Aci. par. divers savants 10 (1785), pp. 411–434.

[15] G.G. Lorentz, *Bernstein Polynomials*, University of Toronto, Toronto, Canada (1953).

[16] W. Magnus, F. Oberhettinger, and R.P. Soni, *Formulas and Theorems for the Special Functions of Mathematical Physics*, *Die Grundlehren der mathematischen Wissenschaften*, vol. 52, 3rd ed., Springer, New York (1966).

[17] P.M. Morse and H. Feshbach, *Methods of Theoretical Physics*, McGraw-Hill, New York (1953).

[18] C. Müller, *Spherical Harmonics*, *Lecture Notes in Mathematics*, vol. 17, Springer, Berlin, Heidelberg, New York (1966).

[19] ———, *Analysis of Spherical Symmetries in Euclidean Spaces* , *Applied mathematical sciences*, vol. 129, Springer, New York (1998).

[20] M. Schreiner, *Tensor Spherical Harmonics and Their Application in Satellite Gradiometry*, Ph.D. thesis, Geomathematics Group, Department of Mathematics, University of Kaiserslautern (1994).

[21] E.I. Stark, *Bernstein-Polynome, 1912-1955*, in "Functionals Analysis and Approximation", P. L. Butzer, B. Sz. Nagy, E. Görlich (eds.), Birkhäuser, Basel (1981), pp. 443–461.

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