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On the Local Multiscale Determination of the Earth's Disturbing Potential From Discrete Deflections of the Vertical

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On the Local Multiscale Determination of the Earth's Disturbing Potential From Discrete Deflections of the Vertical

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Abstract: As a first approximation the Earth is a sphere; as a second approximation it may be considered an ellipsoid of revolution. The deviations of the actual Earth's gravity field from the ellipsoidal 'normal' field are so small that they can be understood to be linear. The splitting of the Earth's gravity field into a 'normal' and a remaining small 'disturbing' field considerably simplifies the problem of its determination.

Under the assumption of an ellipsoidal Earth model high observational accuracy is achievable only if the deviation (deflection of the vertical) of the physical plumb line, to which measurements refer, from the ellipsoidal normal is not ignored. Hence, the determination of the disturbing potential from known deflections of the vertical is a central problem of physical geodesy.

In this paper we propose a new, well-promising method for modelling the disturbing potential locally from the deflections of the vertical. Essential tools are integral formulae on the sphere based on Green's function of the Beltrami operator. The determination of the disturbing potential from deflections of the vertical is formulated as a multiscale procedure involving scale-dependent regularized versions of the surface gradient of the Green function. The modelling process is based on a multiscale framework by use of locally supported surface-curl-free vector wavelets.

1 Physical Background

The purpose in this introductory chapter is to present the fundamentals of gravity field determination, including the relations between the deflections of the vertical and the Earth's disturbing potential. Our intend is to explain the astrogeodetic method of determining both the disturbing potential and the geoidal undulation from deflections of the vertical, avoiding long derivations. Recently, an ellipsoidal reflected approach to gravity field modelling is given by E.W. Grafarend et al. (2006) in physical geodesy (see also references therein). In our approach however we restrict ourselves to the classical spherical approach. For more details the reader is referred to any textbook on classical physical geodesy (e.g., Grafarent et al.(2001,2006), Groten (1979), Heiskanen and Moritz (1967), Torge (1991)).

The *gravity potential* W of the Earth is the sum of the *gravitational potential* V and the *centrifugal potential* Φ, i.e., $W = V + \Phi$. In an Earth's fixed coordinate system the centrifugal potential Φ is explicitly known. Hence, the determination of equipotential surfaces of the potential W is strongly related to the knowledge of the potential V. The *gravity vector* g given by $g(x) = \nabla_x W(x)$, where the point $x \in \mathbb{R}^3$ is located outside and on a sphere around the origin with Earth's mean radius R, is normal to the equipotential surface passing through the same point (for the definition of the mean Earth's radius

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R see, e.g., Groten (1979), Hofmann-Wellenhof and Moritz (2005), Torge (1991)). Thus, equipotential surfaces intuitively express the notion of tangential surfaces, as they are normal to the plumb lines given by the direction of the gravity vector.

Equipotential surfaces of the gravity potential allow in general no simple representation. This is the reason why a reference surface, usually an ellipsoid of revolution, is chosen for the (approximate) construction of the geoid. As a matter of fact, the deviations of the gravity field of the Earth from the normal field of such an ellipsoid are small by typically five orders of magnitude. The remaining parts of the gravity field are gathered in a so-called disturbing gravity field, ∇T, corresponding to the disturbing potential T.

The aim of physical geodesy can, therefore, be seen as the determination of equipotential surfaces of the Earth's gravity field or, equivalently, the determination of the gravity potential W normally (via a linearisation process) involving the disturbing potential T. Knowing the gravity potential, all equipotential surfaces – including the geoid – are given by an equation of the form $W(x) = const$. By introducing U as the normal gravity potential corresponding to the ellipsoidal field and T as the disturbing potential (in the usual Pizetti-Somigliana concept (cf. Pizzetti (1910))) we are led to a decomposition of the gravity potential in the form $W = U + T$ where the zero- and first–order moments of T in terms of spherical harmonics vanish (for details see, e.g., Heiskanen and Moritz (1967)).

A point x on the geoid can be projected onto a point y on the reference ellipsoid by means of the ellipsoidal normal. The *gravity anomaly vector* is defined as the difference between the gravity vector $g(x)$ and the normal gravity vector $\gamma(y)$, $\gamma = \nabla U$, i.e., $g(x) - \gamma(y)$. It is also possible to difference the vectors g and γ at the same point x to get the *gravity disturbance vector* $g(x) - \gamma(x)$ (see Fig. 1).

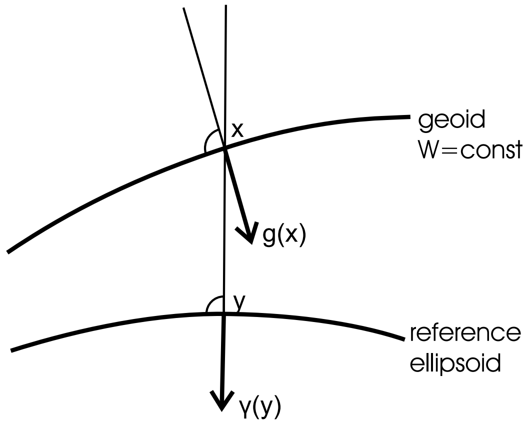

Fig. 1 Illustration of the definition of the gravity anomaly vector $g(x) - \gamma(y)$

There are known several basic mathematical relations between the quantities just mentioned. We only illustrate heuristically the relation of the deflections of the vertical to the surface gradient of the disturbing potential (in spherical approximation). We start by observing that the gravity disturbance vector at the point x can be written as

$$g(x) - \gamma(x) = \nabla_x(W(x) - U(x)) = \nabla_x T(x).$$

Expanding the potential U at x according to Taylor's theorem and truncating the series at the linear term we get (see Fig. 1)

$$U(x) \doteq U(y) + \frac{\partial U}{\partial \nu'}(y)N(x).$$

Here, $\nu'(y)$ is the ellipsoidal normal at y, i.e., $\nu'(y) = -\gamma(y)/|\gamma(y)|$. The geoid undulation $N(x)$, as indicated in Fig. 1, is the distance between x and y, i.e., between the geoid and the reference ellipsoid. Using

$$|\gamma(y)| = -\nu'(y) \cdot \gamma(y) = -\nu'(y) \cdot \nabla_y U(y) = -\frac{\partial U}{\partial \nu'}(y)$$

we arrive at the identity

$$N(x) = \frac{T(x) - (W(x) - U(y))}{|\gamma(y)|}.$$

Letting $U(y) = W(x) = \mathrm{const} = W_0$ (see, e.g. Torge (1991), Eq. (5.38)) we obtain Brun's formula (cf. Bruns (1878))

$$N(x) = \frac{T(x)}{|\gamma(y)|}. \tag{1.1}$$

The last formula (1.1) relates the physical quantity T to the geometric quantity N.

Letting $\nu(x) = -g(x)/|g(x)|$ we find

$$g(x) = \nabla_x W(x) = -|g(x)|\nu(x). \tag{1.2}$$

Furthermore, we have

$$\gamma(x) = \nabla_x U(x) = -|\gamma(x)|\nu'(x).$$

Now, the *deflection of the vertical* $\Theta(x)$ at the point x is defined to be the angular (i.e., tangential) difference between the directions $\nu(x)$ and $\nu'(x)$, i.e., the plumb line and the ellipsoidal normal through the same point:

$$\Theta(x) = \nu(x) - \nu'(x) - ((\nu(x) - \nu'(x)) \cdot \nu'(x))\, \nu'(x). \tag{1.3}$$

Clearly, because of its definition (1.3), $\Theta(x)$ is orthogonal to $\nu'(x)$:

$$\Theta(x) \cdot \nu'(x) = 0.$$

Since the plumb lines are orthogonal to the level surfaces of the geoid and the ellipsoid, respectively, the deflections of the vertical give briefly spoken a measure of the gradient of the level surfaces. This aspect will be described in more detail below: From (1.2) we obtain, in conncetion with (1.3),

$$\begin{aligned} g(x) &= \nabla_x W(x) \\ &= -|g(x)|\left(\Theta(x) + \nu'(x) + ((\nu(x) - \nu'(x)) \cdot \nu'(x))\nu'(x)\right). \end{aligned}$$

Altogether we get for the gravity disturbance vector

$$\begin{aligned} g(x) - \gamma(x) &= \nabla_x T(x) \\ &= -|g(x)|\left(\Theta(x) + ((\nu(x) - \nu'(x)) \cdot \nu'(x))\, \nu'(x)\right) \\ &- (|g(x)| - |\gamma(x)|)\, \nu'(x). \end{aligned} \tag{1.4}$$

The magnitude $|g(x)| - |\gamma(x)|$ is called the *gravity disturbance*. Since the vector $\nu(x) - \nu'(x)$ is (almost) orthogonal to $\nu'(x)$, it can be neglected in (1.4). Hence, it follows that

$$\begin{aligned} g(x) - \gamma(x) &= \nabla_x T(x) \\ &\doteq -|g(x)|\Theta(x) - (|g(x)| - |\gamma(x)|)\, \nu'(x). \end{aligned} \tag{1.5}$$

In spherical approximation

$$x = R\xi,\ R = |x|,\ |\xi| = 1 \tag{1.6}$$

the gradient $\nabla_x T(x)$ can be split into a normal part (pointing into the direction of $\xi = \nu'(x)$) and an angular (tangential) part (characterized by the surface gradient ∇^*) (see, e.g., Freeden et al. (1998) for more details on ∇^*). It follows that

$$\begin{aligned} &\nabla_x T(x) \\ &= \left(\frac{\partial T}{\partial r}(r\xi)\Big|_{r=R}\right)\xi + \frac{1}{R}\nabla^*_\xi T(R\xi) \\ &= \frac{\partial T}{\partial \nu'}(x)\nu'(x) + \frac{1}{R}\nabla^*_\xi T(R\xi). \end{aligned}$$

By comparison of (1.5) and (1.7) we therefore obtain

$$|g(x)| - |\gamma(x)| = -\frac{\partial T}{\partial \nu'}(x),$$

i.e., the gravity disturbance, beside being the difference in magnitude of the actual and the normal gravity vector, is also the normal component of the gravity disturbance vector.

In addition, we are led to the angular, i.e., (tangential) differential equation

$$\frac{1}{R}\nabla^*_\xi T(R\xi) = -|g(x)|\Theta(R\xi).$$

Without loss of precision $-|g(x)|$ can be replaced by $-|\gamma(x)|$. In spherical approximation (with $|\gamma(x)| = kM/R^2$, see, e.g., Heiskanen and Moritz (1966)) this gives us

$$\nabla^*_\xi T(R\xi) = -\frac{kM}{R}\Theta(R\xi), \quad \xi \in \Omega,$$

where k is the gravitational constant and M is the constant of the mass. By virtue of the Brun's formula we finally find

$$\frac{kM}{R^2}\nabla^*_\xi N(R\xi) = -\frac{kM}{R}\Theta(R\xi), \quad \xi \in \Omega,$$

i.e.,

$$\nabla^*_\xi N(R\xi) = -R\Theta(R\xi), \quad \xi \in \Omega. \tag{1.7}$$

In other words, the knowledge of the geoid undulations allows the determination of the deflections of the vertical by taking the surface gradient on the unit sphere.

In physical geodesy (see e.g., Groten (1981), Hofmann-Wellenhof and Moritz (2005), Rummel (1992), Torge (1991)), the deflection of the vertical, which is a (tangential) vector field, is usually decomposed into mutually perpendicular scalar components. Conventionally, their representation is given in terms of Stokes function (cf. Stokes (1849))

In fact, there are various distinctions in the introduction of the deflections of the vertical (see, e.g., Featherstone and Rüger (2000), Grafarend (2001, 2006), Jekeli (1999), Torge (1991)).

2 The Problem

In what follows we understand $T(R\cdot)$ and $\Theta(R\cdot)$ as functions defined on the unit sphere Ω. In other words $T(R\cdot)$ and $\nabla^* T(R\cdot) = \Theta(R\cdot)$ are assumed to be of class $C^{(1)}(\Omega)$ and $c^{(0)}(\Omega)$, respectively.
The disturbing potential $T(R\cdot)$ is conventionally represented on the whole Earth's surface, i.e. on the (unit) sphere Ω, by a Fourier (orthogonal) expansion in terms of spherical harmonics $Y_{n,j}$ (see, e.g., Hofmann-Wellendorf and Moritz (2005), Torge (1991)) thereby assuming (in accordance with the Pizetti-Somigliana concept)

(i) that the center of the reference ellipsoid coincides with the center of gravity of the Earth,

(ii) that the difference of the mass of the Earth and the mass of the ellipsoid is zero.

In other words we are confronted with the following (astrogeodetic) problem: Given the deflections Θ of the vertical on the unit sphere $\Omega \subset \mathbb{R}^3$. Find the disturbing potential $T(R\cdot) : \Omega \to \mathbb{R}$ satisfying

$$\nabla^* T(R\xi) = -\frac{kM}{R}\Theta(R\xi), \quad \xi \in \Omega$$

and

(i) $\displaystyle\int_\Omega T(R\eta)Y_{0,1}(\eta)d\omega(\eta) = \frac{1}{\sqrt{4\pi}}\int_\Omega T(R\eta)d\omega(\eta) = 0$

(ii) $\displaystyle\int_\Omega T(R\eta)Y_{1,j}(\eta)d\omega(\eta) = 0, \quad j = 1,2,3.$

($d\omega$ denotes the surface element). The theory of this problem was known even before the satellite era. As a matter of fact, the series expansion of $T(R\cdot)$

$$T(R\cdot) = \sum_{n=2}^{\infty}\int_\Omega T(R\eta)Y_{n,j}(\eta)d\omega(\eta)Y_{n,j}$$

leads to an orthogonal series expansion of the deflections of the vertical Θ in terms of surface curl free vector spherical harmonics $\nabla^* Y_{n,j}$ on the whole spherical Earth Ω:

$$-\frac{kM}{R}\Theta(R\xi) = \sum_{n=2}^{\infty}\int_\Omega T(R\eta)Y_{n,j}(\eta)d\omega(\eta)\nabla^* Y_{n,j}.$$

But what is lacking are data globally distributed over Ω. Hence, the classical approaches and many others attempt to give global models of $T(R\cdot)$ and ignore the local availability of the data. Even more, the vector types $\nabla^* Y_{n,j}$ of polynomial functions are far from being suitable for local purposes, since the constituting ingredients of spherical harmonics show certain phenomena of (global) periodicity at least when the classical basis system involving associated Legendre functions is used. Moreover, singularities at the poles occur when polar coordinates come into play. Furthermore, boundary effects like the Gibbs phenomenon along the local area, where the data are given, are not avoidable by use of spherical harmonics, i.e., by use of non-spacelocalizing polynomials. In consequence it is really required in geoscientific practice to develop a new approach to the classical (astrogeodetic) problem of determining the potential $T(R\cdot)$ from deflections of the vertical thereby using specific spacelocalizing (trial) kernel vector functions for local applications.

In this paper we are concerned with the following local (discrete) variants of the astrogeodetic method on physical geodesy of determining the disturbing potential $T(R\cdot)$ from deflections Θ of the vertical on a so-called normal subregion Γ of the Earth Ω.

(1) **The potential problem:** Let the vector deflections of the vertical be known for a finite subset of points $\{\eta_1, \cdots, \eta_N\}$ on Γ. Assume, further, that the (scalar) disturbing potential $T(R\cdot)$ is known for a set $\{\widetilde{\eta_1}, \cdots, \widetilde{\eta_N}\}$ on the boundary $\partial\Gamma$ of Γ. Find an approximation of $T(R\cdot)$ from the discrete data $\{\eta_i, \nabla^* T(R\eta_i)\}_{i=1,\cdots,N}$ and $\{\widetilde{\eta_i}, T(R\widetilde{\eta_i})\}_{i=1,\cdots,\widetilde{N}}$ on the domain $\overline{\Gamma} = \Gamma \cup \partial\Gamma$.

(2) **The epoch problem:** Given two datasets for different time epochs, namely $\{\eta_i, \nabla^* T_{t'}(R\eta_i)\}_{i=1,\cdots,N}$ and $\{\eta_i, \nabla^* T_{t''}(R\eta_i)\}_{i=1,\cdots,N}$ at the same data knots $\{\eta_1, \cdots, \eta_N\} \subset \Gamma$. Find the 'epoch difference potential' $D : \overline{\Gamma} \to \Omega$ with $D = T_{t'}(R\cdot) - T_{t''}(R\cdot)$.

3 Green's Theorems on (Normal) Regions of the Sphere

Throughout this paper we need a number of differential operators on the unit sphere $\Omega \subset \mathbb{R}^3$ which are listed in Table 1 (see, e.g., [7] for more details).

Table 1 Differential operators

Symbol	**Differential Operator**
∇_x	Gradient at x
$\Delta_x = \nabla_x \cdot \nabla_x$	Laplace operator at x
∇^*_ξ	Surface gradient on the unit sphere Ω at ξ
$L^*_\xi = \xi \wedge \nabla^*_\xi$	Surface curl gradient on the unit sphere Ω at ξ
$\Delta^*_\xi = \nabla^*_\xi \cdot \nabla^*_\xi = L^*_\xi \cdot L^*_\xi$	Beltrami operator on the unit sphere Ω at ξ
$\nabla^*_\xi \cdot$	Surface divergence on the unit sphere Ω at ξ
$L^*_\xi \cdot$	Surface curl on the unit sphere Ω at ξ

It should be noted that the operators ∇^*, L^*, and Δ^* will be always used in a coordinate-free representation, thereby avoiding any kind of singularities at the poles. Moreover, following the nomenclature of [7] we denote by $\nabla^* \cdot$ the surface divergence on Ω and by $L^* \cdot$ the surface curl on Ω. Clearly, $\Delta^* = \nabla^* \cdot \nabla^* = L^* \cdot L^*$. Note that the operators ∇^*, L^*, Δ^* show special features in certain situations (for more details the reader is referred to [7]). For example, let $\eta \in \Omega$ be fixed. If G is of class $C^{(1)}[-1,1]$ and G' denotes its derivative, then we find

$$\begin{aligned} \nabla^*_\xi G(\xi \cdot \eta) &= G'(\xi \cdot \eta)(\eta - (\xi \cdot \eta)\xi), \quad \xi \in \Omega, \\ L^*_\xi G(\xi \cdot \eta) &= G'(\xi \cdot \eta)(\xi \wedge \eta), \quad \xi \in \Omega, \end{aligned} \tag{3.1}$$

whereas for $G \in C^{(2)}[-1,1]$

$$\Delta^*_\xi G(\xi \cdot \eta) = (\nabla^*_\xi \cdot \nabla^*_\xi) G(\xi \cdot \eta) = -2(\xi \cdot \eta) G'(\xi \cdot \eta) + (1 - (\xi \cdot \eta)^2) G''(\xi \cdot \eta), \quad \xi \in \Omega.$$

As an essential tool, for our considerations, we first introduce the definition and discuss some properties of Green's function with respect to the Beltrami operator Δ^* (see [4]).

Definition 3.1 A function $G(\Delta^*; \cdot, \cdot) : (\xi, \eta) \mapsto G(\Delta^*; \xi, \eta)$, $\xi, \eta \in \Omega$ with $-1 \leq \xi \cdot \eta < 1$, is called *Green's function* on Ω with respect to the operator Δ^*, if it satisfies the following properties:

1. (differential equation) For every point $\xi \in \Omega$, $\eta \mapsto G(\Delta^*; \xi, \eta)$ is twice continuously differentiable on $\{\eta \in \Omega : -1 \leq \xi \cdot \eta < 1\}$, and we have

$$\Delta^*_\eta G(\Delta^*; \xi, \eta) = -\frac{1}{4\pi}, \quad -1 \leq \xi \cdot \eta < 1.$$

2. (characteristic singularity) For every $\xi \in \Omega$, the function

$$\eta \mapsto G(\Delta^*;\xi,\eta) - \frac{1}{4\pi}\ln(1-\xi\cdot\eta)$$

is continuously differentiable on Ω.

3. (rotational symmetry) For all orthogonal transformations $\mathbf{t}$

$$G(\Delta^*;\mathbf{t}\xi,\mathbf{t}\eta) = G(\Delta^*;\xi,\eta).$$

4. (normalization) For every $\xi \in \Omega$,

$$\int_\Omega G(\Delta^*;\xi,\eta)\, d\omega(\eta) = 0.$$

Following [4, 5] the uniqueness of Green's function with respect to Δ^* is guaranteed. In fact, the function

$$G(\Delta^*;\xi,\eta) = \frac{1}{4\pi}\ln(1-\xi\cdot\eta) + \frac{1}{4\pi} - \frac{1}{4\pi}\ln 2, \quad -1 \le \xi\cdot\eta < 1,$$

is an explicit representation of Green's function with respect to the Beltrami operator Δ^*. In connection with (3.1) we obtain

$$\nabla^*_\eta G(\Delta^*;\xi,\eta) = -\frac{1}{4\pi}\frac{(\xi-(\xi\cdot\eta)\eta)}{1-\xi\cdot\eta}, \quad -1 \le \xi\cdot\eta < 1. \tag{3.2}$$

Next, we explain some geometrical assumptions imposed on subsets of the unit sphere $\Omega \subset \mathbb{R}^3$ under consideration in our work.

Definition 3.2 A region, i.e., an open and connected set $\Gamma \subset \Omega$, is called *normal* if the surface theorem of Gauss

$$\int_\Gamma \nabla^*_\xi \cdot f(\xi) d\omega(\xi) = \int_{\partial\Gamma} \nu_\xi \cdot f(\xi) d\sigma(\xi)$$

is valid for all continuously differentiable vector fields $f \in c^{(1)}(\Omega)$, where $\partial\Gamma$ is the boundary curve of Γ, ν is the unit surface field pointing normal to $\partial\Gamma$ and σ is the arc length along $\partial\Gamma$. A normal region $\Gamma \subset \Omega$ is called *regular*, if its boundary $\partial\Gamma$ has a continuously differentiable unit normal field $\nu : \partial\Gamma \to \mathbb{R}^3$ pointing outward of Γ, i.e., into $\Omega\backslash\overline{\Gamma}$.

By choosing $f = F\nabla^* Q, F \in C^{(1)}(\overline{\Gamma}), Q \in C^{(2)}(\overline{\Gamma})$ in the surface theorem of Gauss we get Green's surface identity for the operator ∇^*, viz.

$$\int_\Gamma \left(\nabla^*_\eta F(\eta)\cdot\nabla^*_\eta Q(\eta) + F(\eta)\Delta^*_\eta Q(\eta)\right)\, d\omega(\eta) = \int_{\partial\Gamma} F(\eta)\nu_\eta\cdot\nabla^*_\eta Q(\eta)\, d\sigma(\eta). \tag{3.3}$$

Let the function $F : \overline{\Gamma} \to \mathbb{R}$ be continuously differentiable and $\xi \in \overline{\Gamma}$ be fixed. Applying Green's surface identity to F and Green's function $G(\Delta^*;\xi,\cdot)$ on the region $\{\eta \in \partial\Gamma : |\xi-\eta| \ge \varepsilon\}$ we obtain for sufficiently small $\varepsilon > 0$

$$\begin{aligned} &\int_{|\xi-\eta|\ge\varepsilon,\,\eta\in\Gamma} \left(\nabla^*_\eta F(\eta)\cdot\nabla^*_\eta G(\Delta^*;\xi,\eta) + F(\eta)\Delta^*_\eta G(\Delta^*;\xi,\eta)\right)\, d\omega(\eta) \\ &\quad = \int_{|\xi-\eta|=\varepsilon,\,\eta\in\Gamma} F(\eta)\nu_\eta\cdot\nabla^*_\eta G(\Delta^*;\xi,\eta)\, d\sigma(\eta) + \int_{|\xi-\eta|\ge\varepsilon,\,\eta\in\partial\Gamma} F(\eta)\nu_\eta\cdot\nabla^*_\eta G(\Delta^*;\xi,\eta)\, d\sigma(\eta), \end{aligned} \tag{3.4}$$

where σ denotes the arc length along $\partial\Gamma$ and $\{\eta \in \Gamma : |\xi-\eta| = \varepsilon\}$, while ν is the unit surface vector normal to $\{\eta \in \overline{\Gamma} : |\xi-\eta| = \varepsilon\}$ or $\{\eta \in \partial\Gamma : |\xi-\eta| \ge \varepsilon\}$, respectively. Using Property 1 of Definition 3.1 equation (3.4) can be rewritten as follows

$$\begin{aligned} &\int_{|\xi-\eta|\ge\varepsilon,\,\eta\in\Gamma} \nabla^*_\eta F(\eta)\cdot\nabla^*_\eta G(\Delta^*;\xi,\eta)\, d\omega(\eta) - \frac{1}{4\pi}\int_{|\xi-\eta|\ge\varepsilon,\,\eta\in\Gamma} F(\eta)\, d\omega(\eta) \\ &\quad = \int_{|\xi-\eta|=\varepsilon,\,\eta\in\Gamma} F(\eta)\nu_\eta\cdot\nabla^*_\eta G(\Delta^*;\xi,\eta)\, d\sigma(\eta) + \int_{|\xi-\eta|\ge\varepsilon,\,\eta\in\partial\Gamma} F(\eta)\nu_\eta\cdot\nabla^*_\eta G(\Delta^*;\xi,\eta)\, d\sigma(\eta). \end{aligned} \tag{3.5}$$

Next, we concentrate on the integral

$$I_\varepsilon(\xi) = \int_{|\xi-\eta|=\varepsilon,\, \eta\in\Gamma} F(\eta)\nu_\eta \cdot \nabla^*_\eta G(\Delta^*;\xi,\eta)\; d\sigma(\eta).$$

For each point $\eta \in \Gamma$ with $|\xi - \eta| = \varepsilon$, we have

$$\nu_\eta = -\frac{\xi - (\xi\cdot\eta)\eta}{\sqrt{1-(\xi\cdot\eta)^2}}. \tag{3.6}$$

Hence, we find with (3.2)

$$I_\varepsilon(\xi) = -\frac{1}{4\pi}\int_{|\xi-\eta|=\varepsilon,\, \eta\in\Gamma} F(\eta)\frac{\sqrt{1-(\xi\cdot\eta)^2}}{1-\xi\cdot\eta}\; d\sigma(\eta).$$

Letting $\varepsilon \to 0$ we obtain, in analogy to well-known results of potential theory (see e.g. [5, 17]),

$$\lim_{\varepsilon\to 0} I_\varepsilon(\xi) = -\frac{\alpha(\xi)}{2\pi}F(\xi),$$

where $\alpha(\xi)$ is the solid angle subtended at $\xi \in \overline{\Gamma}$. This finally leads to the following integral formula.

Theorem 3.3 (Fundamental Theorem for ∇^* on Normal Regions) *Let Γ be a normal region with boundary $\partial\Gamma$. Suppose that F is a continuously differentiable function on $\overline{\Gamma}$, i.e., $F \in C^{(1)}(\overline{\Gamma})$. Then, for every point $\xi \in \Omega$, we have*

$$\frac{\alpha(\xi)}{2\pi}F(\xi) = \frac{1}{4\pi}\int_\Gamma F(\eta)\; d\omega(\eta) - \int_\Gamma \nabla^*_\eta F(\eta)\cdot\nabla^*_\eta G(\Delta^*;\xi,\eta)\; d\omega(\eta) + \int_{\partial\Gamma} F(\eta)\nu_\eta\cdot\nabla^*_\eta G(\Delta^*;\xi,\eta)\; d\sigma(\eta).$$

Setting, particularly, $F = 1$ on $\overline{\Gamma}$ we immediately get from Theorem 3.3

$$\alpha(\xi) = \frac{\|\Gamma\|}{2} + 2\pi\int_{\partial\Gamma}\nu_\eta\cdot\nabla^*_\eta G(\Delta^*;\xi,\eta) d\sigma(\eta), \quad \|\Gamma\| = \int_\Gamma d\omega. \tag{3.7}$$

Clearly, in case of a regular region, $\alpha(\xi) = 2\pi$ for all $\xi \in \Gamma$ and $\alpha(\xi) = \pi$ for all $\xi \in \partial\Gamma$. Furthermore, for the whole sphere Ω we have the following result (cf. [5, 7]).

Corollary 3.4 (Fundamental Theorem for ∇^* on Ω) *Suppose that F is of class $C^{(1)}(\Omega)$. Then, for every $\xi \in \Omega$,*

$$F(\xi) = \frac{1}{4\pi}\int_\Omega F(\eta)\; d\omega(\eta) - \int_\Omega \nabla^*_\eta F(\eta)\cdot\nabla^*_\eta G(\Delta^*;\xi,\eta)\; d\omega(\eta).$$

4 Potential and Stream Functions

Let us consider a continuous spherical vector field f of class $c^{(0)}(\Omega)$. For all $\xi \in \Omega$ we call $\xi \to f_{nor}(\xi) = (f(\xi)\cdot\xi)\xi$ the *normal field* of f, while $\xi \to f_{tan}(\xi) = f - f_{nor}(\xi)$, is called the *tangential field* of f. Obviously, $f(\xi) = f_{nor}(\xi) + f_{tan}(\xi)$ and $f_{nor}(\xi)\cdot f_{tan}(\xi) = 0$. Furthermore, for $f, g \in \mathrm{c}^{(0)}(\Omega)$ and $\xi \in \Omega$, $f(\xi)\cdot g(\xi) = f_{\mathrm{nor}}(\xi)\cdot g_{\mathrm{nor}}(\xi) + f_{\mathrm{tan}}(\xi)\cdot g_{\mathrm{tan}}(\xi)$.

Lemma 4.1 *The tangential field of f vanishes, i.e., $f_{\mathrm{tan}}(\xi) = 0$, $\xi \in \Omega$, if and only if $f(\xi)\cdot\hat{\tau}(\xi) = 0$ for every unit vector $\hat{\tau}(\xi)$ that is perpendicular to ξ, i.e., for which $\xi\cdot\hat{\tau}(\xi) = 0$, $\xi \in \Omega$.*

P r o o f. First, assume $f_{\mathrm{tan}} = 0$. Then for all $\xi \in \Omega$, we have $f(\xi)\cdot\hat{\tau}(\xi) = 0$. Conversely, assume that the tangential field is non-vanishing, i.e., $f_{\mathrm{tan}}(\xi) = f(\xi) - (f(\xi)\cdot\xi)\xi \neq 0$. Then it follows that $f_{\mathrm{tan}}(\xi)|f_{\mathrm{tan}}(\xi)|^{-1}$ is a unit vector field perpendicular to ξ. Hence, by our hypothesis, $f_{\mathrm{tan}}(\xi)\cdot f_{\mathrm{tan}}(\xi)|f_{\mathrm{tan}}(\xi)|^{-1} = 0$. This implies $|f_{\mathrm{tan}}(\xi)| = 0$ which is a contradiction. Thus it follows that $f_{\mathrm{tan}}(\xi) = 0$, as required. □

Lemma 4.2 *Suppose that f is continuous on a simply connected normal region $\Gamma \subset \Omega$. Moreover, let*

$$\int_{\mathcal{C}} \nu_\xi \cdot f(\xi)\; d\sigma(\xi) = 0$$

for every curve $\mathcal{C}$ on Γ. Then $f_{\mathrm{tan}}(\xi) = 0$ for all $\xi \in \Gamma$, i.e., the tangential field of f vanishes for all $\xi \in \Gamma$.

P r o o f. Choose any point $\xi_0 \in \Gamma$. Let ν_{ξ_0} be any unit vector satisfying $\nu_{\xi_0} \cdot \xi_0 = 0$. Then there is a curve $\mathcal{C}$ on Γ passing through ξ_0 whose unit normal vector at ξ_0 is just ν_{ξ_0}. Let $\mathcal{C}_{\rm sub}^{\xi_0}$ be any subset of $\mathcal{C}$ containing ξ_0. Then, in accordance with our assumption,

$$\int_{\mathcal{C}_{\rm sub}^{\xi_0}} \nu_\xi \cdot f(\xi) \, d\sigma(\xi) = 0.$$

Hence, letting the length of $\mathcal{C}_{\rm sub}^{\xi_0}$ tend to zero we find $\nu_{\xi_0} \cdot f(\xi_0) = 0$. Lemma 4.1 then yields $f_{\tan}(\xi_0) = f(\xi_0) - (f(\xi_0) \cdot \xi_0)\xi_0 = 0$. Since ξ_0 can be any point on Γ, we have $f_{\tan}(\xi) = f(\xi) - (f(\xi) \cdot \xi)\xi = 0$ for all $\xi \in \Gamma$. This is the desired result. □

The surface curl gradient acts like an ordinary gradient in $\mathbb{R}^3$ when we integrate it along lines on Γ. In more detail, suppose F is continuously differentiable in an open set in $\mathbb{R}^3$ containing Γ, and $\mathcal{C}$ is any curve lying on Γ, starting at ξ_0 and ending at ξ_1. Suppose that ν_ξ is the unit normal vector at ξ on $\mathcal{C}$ pointing from ξ_0 to ξ_1. Then

$$F(\xi_1) - F(\xi_0) = \int_{\mathcal{C}} \nu_\xi \cdot L_\xi^* F(\xi) \, d\sigma(\xi) \tag{4.1}$$

(observe that $\nu_\xi \cdot L_\xi F(\xi) = \nu_\xi \cdot L_\xi^* F(\xi)$, $\xi \in \Gamma$, see, e.g., [1, 7]). This result enables us to show the following lemma.

Lemma 4.3 *Suppose that Γ is a simply connected normal region. Let F be of class $\mathrm{C}^{(1)}(\Gamma)$, then $L_\xi^* F(\xi) = 0$ if and only if F is constant.*

P r o o f. If $L_\xi^* F(\xi) = 0$, then we obtain, in connection with (4.1), $F(\xi_1) = F(\xi_0)$ for any ξ_0, ξ_1 on Γ. Conversely, if F is constant, the identity (4.1) shows that $f = L^* F$ fulfills

$$\int_{\mathcal{C}} \nu_\xi \cdot f(\xi) \, d\sigma(\xi) = 0$$

for every curve $\mathcal{C}$ lying on Γ. Consequently, following Lemma 4.2, $f_{\tan}(\xi) = 0$ for all $\xi \in \Gamma$. This shows that $f_{\tan}(\xi) = f(\xi) - (f(\xi) \cdot \xi)\xi = f(\xi) = L_\xi^* F(\xi) = 0$ for all $\xi \in \Gamma$. □

Next we prove the following result of spherical vector analysis (see, e.g., [1]).

Lemma 4.4 *Let $f \in \mathrm{c}^{(0)}(\Gamma)$ be a tangent vector field on a simply connected region Γ, i.e., $f(\xi) = f_{\tan}(\xi), \xi \in \Gamma$. Furthermore, suppose that*

$$\int_{\mathcal{C}} \nu_\xi \cdot f(\xi) \, d\sigma(\xi) = 0$$

for every closed curve on Γ. Then there is a scalar field P on Γ such that

$$f(\xi) = L_\xi^* P(\xi), \quad \xi \in \Gamma.$$

The field P is continuously differentiable and is unique up to a constant.

P r o o f. Take an arbitrary, but fixed $\xi_0 \in \Gamma$. We let

$$P(\xi) = \int_{\xi_0}^{\xi} \nu_\zeta \cdot f(\zeta) \, d\sigma(\zeta),$$

be the integral along any curve $\mathcal{C}$ that starts at $\xi_0 \in \Gamma$ and ends at $\xi \in \Gamma$. Then, for any two points ξ_0, ξ on Γ and any curve $\mathcal{C}$ lying on Γ and starting at ξ_0 and ending at ξ_1,

$$P(\xi_1) - P(\xi_0) = \int_{\xi_0}^{\xi_1} \nu_\zeta \cdot f(\zeta) \, d\sigma(\zeta). \tag{4.2}$$

Observing (4.1) we find

$$P(\xi_1) - P(\xi_0) = \int_{\xi_0}^{\xi_1} \nu_\zeta \cdot L_\zeta^* P(\zeta) \, d\sigma(\zeta). \tag{4.3}$$

Combining (4.2) and (4.3) we obtain

$$\int_{\xi_0}^{\xi_1} \tau_\zeta \cdot \left(f(\zeta) - L^*_\zeta P(\zeta)\right)\, d\sigma(\zeta) = 0$$

for any curve $\mathcal{C}$ on Γ. Lemma 4.2, therefore, shows us that $f(\xi) - L^*_\xi P(\xi) = 0$, $\xi \in \Gamma$. The proof that P is continuously differentiable on Γ is omitted. The easiest way to construct such a proof is to take P constant on each straight line passing through Γ in the normal direction (see, e.g., [1]). In order to verify that P is unique up to a constant, we observe that $L^*_\xi P_1(\xi) = L^*_\xi P_2(\xi), \xi \in \Gamma$, implies $L^*_\xi (P_1 - P_2)(\xi) = 0$, $\xi \in \Gamma$, i.e., by virtue of Lemma 4.3, $P_1 - P_2 =$ const. □

Now we are able to verify the following important theorem.

Theorem 4.5 *Let $f \in \mathrm{c}^{(1)}(\Gamma)$ be a tangential field on a simply connected normal region Γ, i.e., $f(\xi) = f_{\tan}(\xi)$ for all $\xi \in \Gamma$. Then $L^*_\xi \cdot f(\xi) = 0$, $\xi \in \Gamma$, if and only if there is a scalar field P such that*

$$f(\xi) = \nabla^*_\xi P(\xi), \quad \xi \in \Gamma,$$

and P is unique up to an additive constant (P is called potential function *for f).*

Similarly, $\nabla^*_\xi \cdot f(\xi) = 0$, $\xi \in \Gamma$, if and only if there is a scalar field S such that

$$f(\xi) = L^*_\xi S(\xi), \quad \xi \in \Gamma,$$

and S is unique up to an additive constant (S is called stream function *for f).*

Proof. The condition $f = L^* P$ implies $\nabla^* \cdot f = 0$, and $f = L^* S$ implies $\nabla^* \cdot f = 0$.

Conversely, assume that $\nabla^*_\xi \cdot f(\xi) = 0$, $\xi \in \Gamma$. Then the surface theorem of Gauss implies

$$\int_{\mathcal{C}} \nu_\xi \cdot f(\xi)\, d\sigma(\xi) = 0$$

for every closed curve $\mathcal{C}$ on Γ. From Lemma 4.4 it follows that there exists a scalar field P such that $f = L^* P$. Furthermore, P is unique up to an additive constant.

Finally, suppose $L^* \cdot f = 0$. Then $\nabla^*_\xi \cdot (-\xi \wedge f(\xi)) = 0$, $\xi \in \Gamma$, hence, there is a scalar field S, unique up to a constant, such that $\xi \wedge f(\xi) = L^*_\xi S(\xi)$, $\xi \in \Gamma$. This is equivalent to $-\xi \wedge (\xi \wedge f(\xi)) = (-\xi \wedge L^*_\xi) S(\xi), \quad \xi \in \Gamma$, or $f = \nabla^* S$ on Γ. This proves Theorem 4.5. □

From Lemma 4.3 we are immediately able to deduce the following statement.

Lemma 4.6 *Let F be of class $C^{(1)}(\Gamma)$, then $\nabla^*_\xi F(\xi) = 0$ if and only if F is constant.*

Proof. If $\nabla^*_\xi F(\xi) = 0$, we find $F =$const.

Conversely, if F is constant, then $\nabla^*_\xi F(\xi) = 0$ for all $\xi \in \Gamma$. This proves our assertion. □

5 The Differential Equations of the Surface Gradient

In what follows we give two conditions for the uniqueness of a solution for the differential equation (1.7) of the surface gradient ∇^*. First, based on the results of Chapter 3 we formulate a certain integrability condition to assure uniqueness.

Theorem 5.1 *Given $f = \nabla^* F \in c^{(0)}(\Omega)$. Then the scalar function F is uniquely determined by the condition:*

$$\frac{1}{4\pi} \int_\Omega F(\eta) d\omega(\eta) = C_0, \quad C_0 \in \mathbb{R}.$$

Proof. We suppose that $F_1, F_2 \in C^{(1)}(\Omega)$ are functions satisfying the condition above. Then the difference $D = F_1 - F_2$ satisfies $\nabla^* D = 0$ on Ω and $\frac{1}{4\pi}\int_\Omega D(\eta) d\omega(\eta) = 0$. Consequently, from Corollary 3.4, we obtain $D(\xi) = 0$ for all $\xi \in \Omega$. Therefore, $F_1 = F_2$, as required. □

Remark 5.2 Following the Pizetti-Somigliana approach the disturbing potential $T(R\cdot)$ is uniquely determined on Ω from the deflections of the vertical by the condition $\int_\Omega T(R\eta) d\omega(\eta) = 0$.

Theorem 5.3 *Let $\Gamma \subset \Omega$ be a simply connected normal region. Given $f = \nabla^* F \in c^{(0)}(\Omega)$. Then the scalar function F is uniquely determined by the condition taken at one point $\xi_0 \in \overline{\Gamma}$:*

$$\frac{1}{4\pi}\int_{\Gamma} F(\eta)d\omega(\eta) + \int_{\partial\Gamma} F(\eta)\nu_\eta \cdot G(\Delta^*;\xi_0,\eta)d\sigma(\eta) = C_0, \quad C_0 \in \mathbb{R}.$$

Proof. We look at the difference D of two solutions which satisfies $\nabla^* D = 0$ in Γ. By Lemma 4.6 we find $D(\xi) = \mathrm{const} = \mathrm{C}$ for all $\xi \in \overline{\Gamma}$. In connection with Theorem 3.3 we have

$$\frac{C}{2\pi}\left(\frac{\|\Gamma\|}{2} + 2\pi\int_{\partial\Gamma} \nu_\eta \cdot \nabla^*_\eta G(\Delta^*;\xi_0,\eta)d\sigma(\eta)\right) = 0.$$

Using (3.7) we, therefore, find $C = 0$, i.e., $D = 0$ on $\overline{\Gamma}$, as required. □

Second, based on the results of Chapter 4, we are able to formulate a uniqueness condition by fixing a certain functional value.

Theorem 5.4 *Given $f = \nabla^* F \in c^{(0)}(\Omega)$. Then the scalar function F is uniquely determined by the condition taken at one point $\xi_0 \in \Omega$:*

$$F(\xi_0) = C_0, \quad C_0 \in \mathbb{R}.$$

Proof. The constant difference D of two functions satisfying the conditions is equal to $D(\xi_0) = C_0 - C_0 = 0 = D(\xi)$ for all $\xi \in \Omega$. □

Remark 5.5 The disturbing potential $T(R\cdot)$ on Ω is uniquely determined from the deflections of the vertical, if its value is known at one point of $\overline{\Gamma}$.

Theorem 5.6 *Suppose that Γ is a simply connected normal region. Given $f = \nabla^* F \in c^{(0)}(\Omega)$. Then the scalar function F is uniquely determined by the condition taken at one point $\xi_0 \in \overline{\Gamma}$:*

$$F(\xi_0) = C_0, \quad C_0 \in \mathbb{R}.$$

Proof. D is constant on $\overline{\Gamma}$ with $D(\xi_0) = 0$. Hence $D(\xi_0) = 0$ in $\overline{\Gamma}$. □

6 Regularized Green's Theorems on (Normal) Regions on the Sphere

In the following we first introduce the regularized Green function with respect to Δ^*. We state its definition together with some properties which are needed for the discussion of spherical wavelets on regular regions.

Definition 6.1 Given $\rho \in (0,2)$, the regularized Green function with respect to Δ^* is defined for all $\xi,\eta \in \Omega$ by

$$G_\rho(\Delta^*;\xi,\eta) = \begin{cases} \frac{1}{4\pi}\ln(1-\xi\cdot\eta) + \frac{1}{4\pi} - \frac{1}{4\pi}\ln 2, & 1-\xi\cdot\eta > \rho, \\ \frac{1}{4\pi\rho}(1-\xi\cdot\eta) + \frac{1}{4\pi}\ln(\rho) - \frac{1}{4\pi}\ln 2, & 1-\xi\cdot\eta \le \rho. \end{cases}$$

The regularized Green function with respect to the Beltrami operator $(\xi,\eta) \to G_\rho(\Delta^*;\xi,\eta)$ only depends on the inner product of ξ and η, hence, it is a radial basis function, i.e., $G_\rho(\Delta^*;\mathbf{t}\xi,\mathbf{t}\eta) = G_\rho(\Delta^*;\xi,\eta)$ holds true for all orthogonal transformations $\mathbf{t}$. Figure 2 gives an illustration of the regularized Green function with respect to Δ^*. Note that, by construction, this kernel function represents an approximation of the original Green's function, i.e., it converges pointwise to Green's function as ρ tends to 0.

We immediately realize that the regularized Green function with respect to Δ^* is continuously differentiable. Applying the surface curl gradient ∇^* to the second variable yields to the so-called *regularized Green function with respect to* ∇^*. Obviously, for $\rho \in (0,2)$, we obtain for all $\xi,\eta \in \Omega$

$$g_\rho^{\nabla^*}(\xi,\eta) = \nabla^*_\eta G_\rho(\Delta^*;\xi,\eta) = \begin{cases} \frac{1}{4\pi}\frac{1}{1-\xi\cdot\eta}(\xi-(\xi\cdot\eta)\eta), & 1-\xi\cdot\eta > \rho, \\ \frac{1}{4\pi}\frac{1}{\rho}(\xi-(\xi\cdot\eta)\eta), & 1-\xi\cdot\eta \le \rho. \end{cases} \tag{6.1}$$

Observing the equation $|\xi-(\xi\cdot\eta)\eta| = \sqrt{1-(\xi\cdot\eta)^2}$ we derive for all $\xi,\eta \in \Omega$ and $\rho \in (0,2)$

$$|g_\rho^{\nabla^*}(\xi,\eta)| = \begin{cases} \frac{1}{4\pi}\sqrt{\frac{1+\xi\cdot\eta}{1-\xi\cdot\eta}}, & 1-\xi\cdot\eta > \rho, \\ \frac{1}{4\pi}\sqrt{1-(\xi\cdot\eta)^2}, & 1-\xi\cdot\eta \le \rho. \end{cases}$$

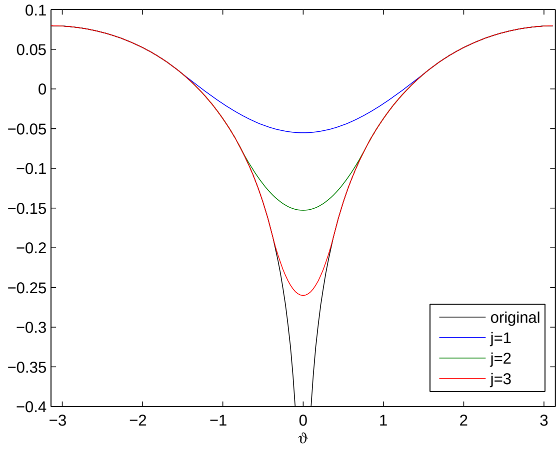

Fig. 2 The regularized Green function $\vartheta \mapsto G_\rho(\Delta^*; \cos\vartheta)$ for $\rho = 1 - \cos(\pi 2^{-j})$ with $j = 1, 2, 3$ and the original Green function $G_\rho(\Delta^*; \xi \cdot \eta)$ with respect to the Beltrami operator Δ^*. Note that $\xi \cdot \eta = \cos\vartheta, \vartheta = \measuredangle(\xi, \eta)$.

A graphical impression of the norm of the regularized Green function with respect to ∇^* and the norm of the surface curl gradient of Green's function with respect to Δ^* is illustrated in Figure 3. By similar arguments as known from potential theory (see e.g. [17]) we obtain the following counterpart of the integral formula developed in Chapter 3.

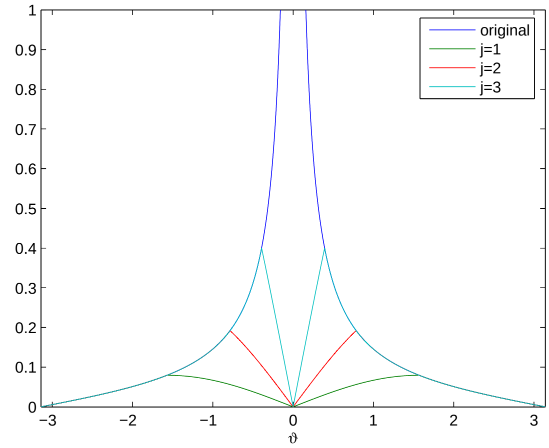

Fig. 3 The norm of the regularized Green function $\vartheta \to |g_\rho^{\nabla^*}(\cos(\vartheta))|$ for $\rho = 1 - \cos(\pi 2^{-j})$ with $j = 1, 2, 3$ and the norm of the surface curl gradient of Green's function with respect to Δ^*.

Theorem 6.2 *For $F \in C^{(1)}(\Omega)$ we have*

$$\lim_{\rho \to 0} \sup_{\xi \in \Omega} \left| F(\xi) - \frac{1}{4\pi} \int_\Omega F(\eta) d\omega(\eta) + \int_\Omega g_\rho^{\nabla^*}(\xi, \eta) \cdot \nabla^* F(\eta) d\omega(\eta) \right| = 0.$$

After deriving the regularized version of the integral theorem for ∇^* on Ω we now turn to the regularized integral theorem for ∇^* on normal regions Γ. For that purpose we introduce the following settings.

Definition 6.3 Let $\Gamma \subset \Omega$ be a normal region of the unit sphere Ω. For $F \in C^{(1)}(\overline{\Gamma})$ we let

$$S_\rho(F)(\xi) = \int_\Gamma g_\rho^{\nabla^*}(\xi, \eta) \cdot \nabla_\eta^* F(\eta) d\omega(\eta) - \int_{\partial\Gamma} g_\rho^{\nabla^*}(\xi, \eta) \cdot F(\eta) \nu_\eta d\sigma(\eta), \quad \rho \in (0, 2),$$

as a counterpart of

$$S(F)(\xi) = \int_\Gamma \nabla_\eta^* G(\Delta^*; \xi, \eta) \cdot \nabla_\eta^* F(\eta) d\omega(\eta) - \int_{\partial\Gamma} \nabla_\eta^* G(\Delta^*; \xi, \eta) \cdot F(\eta) \nu_\eta d\sigma(\eta).$$

Clearly, it is not hard to show that

$$\lim_{\rho\to 0}\sup_{\xi\in\Omega}|S(F)(\xi)-S_\rho(F)(\xi)|=0. \tag{6.2}$$

Theorem 6.4 (Regularized Integral Theorem for ∇^* on Γ) *Let $\Gamma\subset\Omega$ be a normal region with boundary $\partial\Gamma$. Suppose that F is a continuously differentiable function on $\overline{\Gamma}$, i.e., $F\in C^{(1)}(\overline{\Gamma})$. Then*

$$\lim_{\rho\to 0}\sup_{\xi\in\overline{\Gamma}}\left|\frac{\alpha(\xi)}{2\pi}F(\xi)-\frac{1}{4\pi}\int_\Gamma F(\eta)d\omega(\eta)+S_\rho(F)(\xi)\right|=0,$$

where $\alpha(\xi)$ denotes, as usually, the solid angle subtended at $\xi\in\overline{\Gamma}$.

7 Vector Spherical Wavelets on Normal Regions

We turn our attention to the introduction of vector spherical wavelets, where $\Gamma\subset\Omega$ is supposed to be a normal region. We choose a sequence which divides the continuous scale interval $(0,2)$ into discrete pieces. More explicitly, $(\rho_j)_{j\in\mathbb{N}_0}$ denotes a sequence of real numbers satisfying $\lim_{j\to\infty}\rho_j=0$ and $\lim_{j\to 0}\rho_j=2$. For example, we can choose $\rho_j=2^{1-j}$ or $\rho_j=1-\cos(\pi 2^{-j})$, $j\in\mathbb{N}_0$. The point of departure for our considerations on normal regions Γ is Theorem 6.4 in the form

$$\frac{\alpha(\xi)}{2\pi}F(\xi)-F^\Gamma_{mean}=-\lim_{j\to\infty}S_{\rho_j}(F)(\xi),\quad F^\Gamma_{mean}=\frac{1}{4\pi}\int_\Gamma F(\eta)d\omega(\eta),\quad \xi\in\overline{\Gamma}. \tag{7.1}$$

Note that the discrete steps in this approximation process are called *scales*, i.e., the value j takes the role of the scale parameter, i.e., the parameter to model out more and more local features. By using discrete regularization parameters we are naturally led to the following type of scale discretized Green wavelets.

Definition 7.1 Let $\{g^{\nabla^*}_{\rho_j}\}_{j\in\mathbb{N}_0}$ be the regularized Green function with respect to ∇^* (see (6.1)). Then the scale discretized regularized Green wavelet function with respect to ∇^* is defined by

$$\psi_{\rho_j}\ =\ g^{\nabla^*}_{\rho_{j+1}}-g^{\nabla^*}_{\rho_j}=\nabla^*_\eta G_{\rho_{j+1}}(\Delta^*;\xi,\eta)-\nabla^*_\eta G_{\rho_j}(\Delta^*;\xi,\eta),\quad j\in\mathbb{N}_0. \tag{7.2}$$

In fact, the difference of two consecutive scales of regularized Green functions with respect to Δ^* reads

$$\begin{aligned}&G_{\rho_{j+1}}(\Delta^*;\xi,\eta)-G_{\rho_j}(\Delta^*;\xi,\eta)\\ &=\begin{cases}0, & 1-\xi\cdot\eta\ >\rho_j,\\ \frac{1}{4\pi}\ln(1-\xi\cdot\eta)-\frac{1}{4\pi\rho_j}(1-\xi\cdot\eta)+\frac{1}{4\pi}(1-\ln(\rho_j)), & \rho_j>\ 1-\xi\cdot\eta\ >\rho_{j+1},\\ \left(\frac{1}{4\pi\rho_{j+1}}-\frac{1}{4\pi\rho_j}\right)(1-\xi\cdot\eta)+\frac{1}{4\pi}(\ln(\rho_{j+1})-\ln(\rho_j)), & 1-\xi\cdot\eta\ \le\rho_{j+1},\end{cases}\end{aligned}$$

such that

$$\psi_{\rho_j}(\xi,\eta)=g^{\nabla^*}_{\rho_{j+1}}(\xi,\eta)-g^{\nabla^*}_{\rho_j}(\xi,\eta)=\begin{cases}0, & 1-\xi\cdot\eta\ >\rho_j,\\ \frac{1}{4\pi}\left(\frac{1}{\rho_j}-\frac{1}{1-\xi\cdot\eta}\right)(\xi-(\xi\cdot\eta)\eta), & \rho_j>\ 1-\xi\cdot\eta\ >\rho_{j+1},\\ \frac{1}{4\pi}\left(\frac{1}{\rho_j}-\frac{1}{\rho_{j+1}}\right)(\xi-(\xi\cdot\eta)\eta), & 1-\xi\cdot\eta\ \le\rho_{j+1}.\end{cases}$$

A graph of the norm of the scale discretized regularized Green wavelet function with respect to ∇^* for the discretization parameters $\rho=1-\cos(\pi 2^{-j})$ with $j=0,1,2,3$ is shown in Figure 4. Note, that the functions ψ_{ρ_j} have a local support. $S_{\rho_j}(F)(\xi)$, as given by Definition 6.3, is called the *scale discrete regularized Green scaling function transform* with respect to ∇^*. Let $\{\psi_{\rho_j}\}_{j\in\mathbb{N}_0}$ be the scale discretized regularized Green function with respect to ∇^*. The scale discretized regularized Green wavelet transform with respect to ∇^* is defined by

$$W_{\rho_j}(F)(\xi)=\int_\Gamma\nabla^*_\eta F(\eta)\cdot\psi_{\rho_j}(\xi,\eta)d\omega(\eta)-\int_{\partial\Gamma}F(\eta)\nu_\eta\cdot\psi_{\rho_j}(\xi,\eta)d\sigma(\eta).$$

We arrive at the following theorem, that is of basic interest for our computation.

Theorem 7.2 *Let $\{g^{\nabla^*}_{\rho_j}\}_{j\in\mathbb{N}_0}$ be the regularized Green function with respect to ∇^*. Then the multiscale reconstruction of a function $F\in C^{(1)}(\Gamma)$ is given by*

$$\frac{\alpha(\xi)}{2\pi}F(\xi)-F^\Gamma_{mean}=-\sum_{j=0}^{\infty}W_{\rho_j}(F)(\xi),\quad \xi\in\Gamma,$$

where the equality holds in the $\|\cdot\|_{C(\overline{\Gamma})}$-sense.

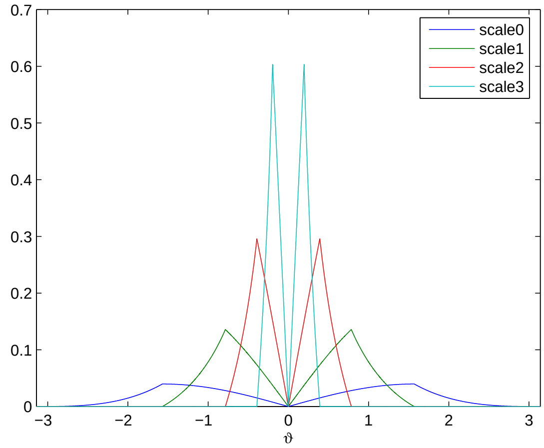

Fig. 4 The norm of the regularized Green wavelet function $\vartheta \rightarrow |\Psi_{\rho_j}(\cos(\vartheta))|$ with respect to ∇^* for $\rho = 1 - \cos(\pi 2^{-j})$ with scale $j = 0, 1, 2, 3$.

By observing the definition of the scaling transform $S_{\rho_j}(F)(\xi)$, Theorem 7.2 admits the following reformulation.

Corollary 7.3 *Under the assumptions of Theorem 7.2*

$$\frac{\alpha(\xi)}{2\pi} F(\xi) - F^{\Gamma}_{mean} + S_{\rho_J}(F)(\xi) = -\sum_{j=J}^{\infty} W_{\rho_j}(F)(\xi), \quad \xi \in \overline{\Gamma},$$

for every $J \in \mathbb{N}_0$ *in the* $\|\cdot\|_{C(\overline{\Gamma})}$*-sense.*

These reconstruction formula will now be applied to the modelling of oceanic circulation.

8 Multiscale Modelling of the Disturbing Potential From Deflections of the Vertical

Our considerations have shown that the disturbing potential $T(R\cdot) \in C^{(1)}(\Omega)$ can be uniquely determined on Ω from the deflections of the vertical $\Theta \in c^{(0)}(\Omega)$ in terms of the integral formula.

$$T(R\xi) = \frac{kM}{R} \int_{\Omega} \nabla^*_{\eta} G(\Delta^*; \xi, \eta) \cdot \Theta(R\eta)\, d\omega(\eta), \quad \xi \in \Omega.$$

In addition, $T(R\cdot)$ can be approximated as follows

$$T(R\xi) = \lim_{j\to\infty} \frac{kM}{R} \int_{\Omega} \nabla^*_{\eta} G_{\rho_j}(\Delta^*; \xi, \eta) \cdot \Theta(R\eta)\, d\omega(\eta), \quad \xi \in \Omega$$

For numerical purposes it suffices to have an (in the sense of Weyl) equidistributed nodal set $(\eta_i, \Theta(\eta_i))$, $\eta_i \in \Omega$, $i = 1, \cdots, N$, to discretize the integral on the right hand side and to establish a multiscale approximation of the geopotential $T(R\cdot)$ on Ω. But - as already mentioned - what is lacking are data on the whole sphere Ω.

Consequently, we are often confronted with the problem of determining the disturbing potential $T(R\cdot)$ on a certain subdomain Γ of Ω (e.g., caps, squares or rectangles), where suitable discrete data information about the deflections of the vertical is available. When we are interested in solving that problem numerically from discrete data our approach shows that we have to know, in addition, the disturbing potential $T(R\cdot)$ of the boundary $\partial\Gamma$.

Even more, our numerical calculation based on discrete data is only unique up to a constant $T^{\Gamma}_{mean} = -T^{\Omega\setminus\Gamma}_{mean}$ (confer the considerations given in Chapter 5):

$$\frac{\alpha(\xi)}{2\pi} T(R\xi) - T^{\Gamma}_{mean} = -\lim_{j\to\infty} \left(\int_{\Gamma} \nabla^*_{\eta} T(R\eta) \cdot g^{\nabla^*}_{\rho_j}(\xi, \eta) d\omega(\eta) - \int_{\partial\Gamma} T(R\eta)\nu_{\eta} \cdot g^{\nabla^*}_{\rho_j}(\xi, \eta) d\sigma(\eta) \right).$$

8.1 Multiscale Solution of the Potential Problem

In what follows, particular attention is paid to the numerical stability caused by the specific observation of the boundary terms in our numerical calculation. We consider a spherical cap Γ, i.e., a (special) regular region as a reference area.

The spherical cap under consideration is defined by its center $\zeta \in \Omega$ and its radius $r > 0$, more precisely we let

$$\Gamma_r(\zeta) = \{\eta \in \Omega : |\zeta - \eta| < r\}.$$

In this case the normal unit vector ν_η is explicitly given for all $\eta \in \partial\Gamma_r(\zeta)$ by (3.6). Substituting the tangential unit vector in the equations above we obtain

$$\frac{\alpha(\xi)}{2\pi} T(R\xi) - T^{\Gamma_r}_{mean} = -\lim_{j\to\infty} \left(\int_{\Gamma_r(\zeta)} \nabla^*_\eta T(R\eta) \cdot g^{\nabla^*}_{\rho_j}(\xi,\eta)\, d\omega(\eta) + \int_{\partial\Gamma_r(\zeta)} T(R\eta) \frac{g^{\nabla^*}_{\rho_j}(\xi,\eta) \cdot (\zeta - (\zeta\cdot\eta)\eta)}{\sqrt{1-(\zeta\cdot\eta)^2}}\, d\sigma(\eta) \right).$$

In more detail, the region of interest in our first example is a spherical cap Γ_{30} where $30°$ denotes the apex angle of the cap. Furthermore, we assume that the vertical deflection measurements are not continuously given, but on an equiangular longitude-latitude grid with a step size of $0.12°$. The potential $T(R\cdot)$ is prescribed at a finite set of boundary points that are sampled with an angular distance of $0.0051°$. Both data sets have been generated from the EGM96 up to degree 200 (see Lemoine et al(1998)).

Figure 5 illustrates the disturbing potential which is used to calculate the input dataset for our numerical tests below, i.e., the deflections of the vertical are obtained from EGM96, they are shown in Figure 5. Since we are especially interested in boundary effects, we always plot the spherical cap together with its surrounding environment.

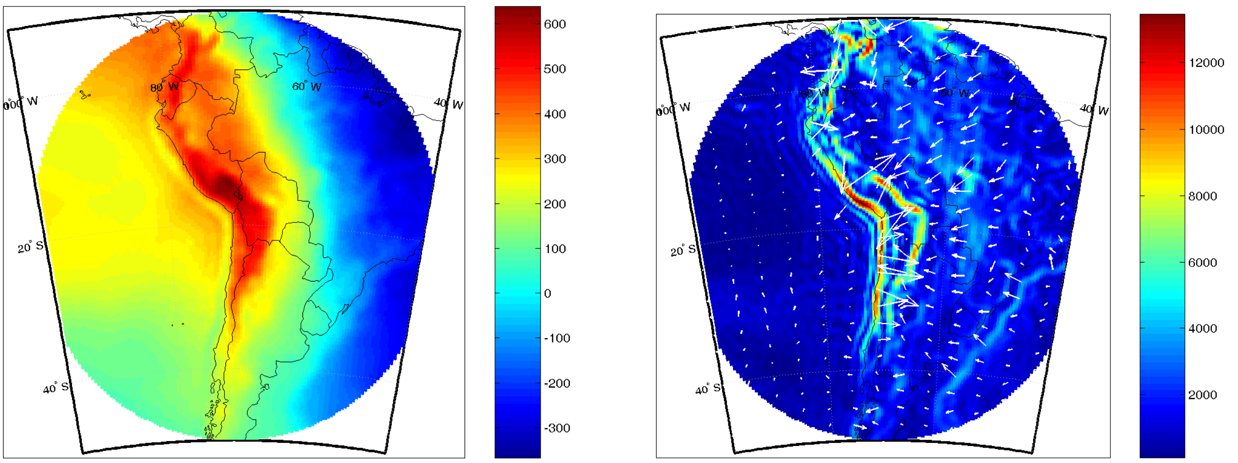

Fig. 5 Plot of the geopotential in $[\frac{m^2}{s^2}]$(left), which is used to calculate the input dataset, i.e., the deflections of the vertical (right) in$[\frac{m}{s^2}]$.

For the modelling of the disturbing potential EGM96 from its deflections of the vertical in the particularly interesting area of South America we have to discretize the scale interval $(0, 2)$. In our computations we let $\rho_j = 2^{1-j}$. In Figure 6 the wavelet decomposition for the scale 8 to 12 is illustrated graphically. At first sight the approximated disturbing potential is close to the original potential inside the spherical cap even for a moderately small scale parameter j, whereas the potential at a certain strip around the boundary shows essentially larger error effects. However, it should be noted, that (i) the boundary errors and the diameter of the strip become smaller for increasing scale parameters and (ii) any kind of phenomena of oscillation for the approximated potential outside the boundary strip can be avoided. In other words, by taking into account additional potential values on the boundary of the domain under consideration (in our case the circle $\partial\Gamma_{30}$ of the cap Γ_{30}) a stabilizing process can be detected within the multiscale reconstruction, where the stabilization correlates to the scale level to be realized in the numerical computation.

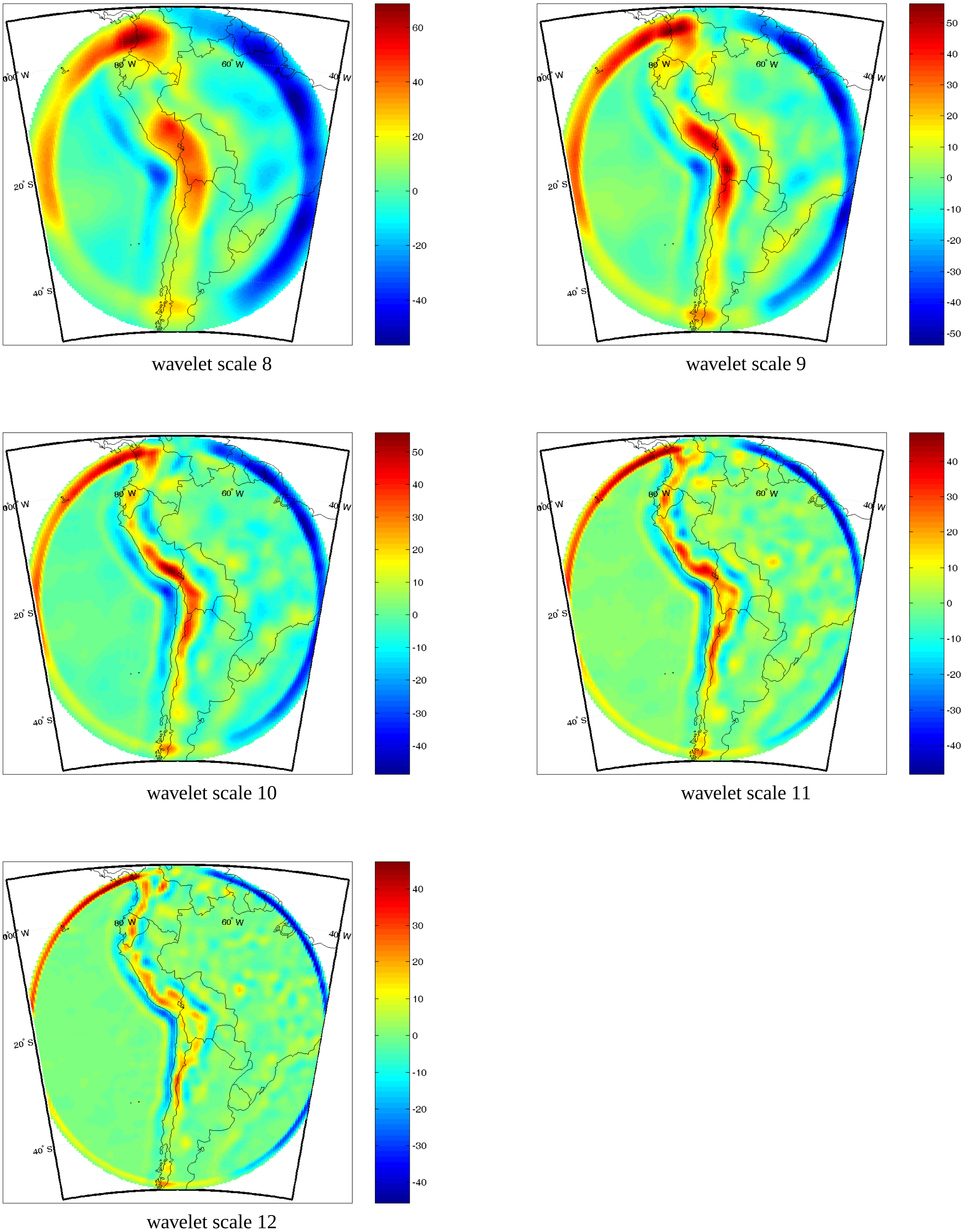

Fig. 6 Wavelet reconstruction of the geopotential in $[\frac{m^2}{s^2}]$ on the spherical cap Γ_{30} at certain scales using the scale discretized regularized Green function.

8.2 Multiscale Solution of the Epoch Problem

A significant application of our multiscale technique the detection of (artificial) disturbances of the potential within a local area of geophysical interest. For the difference potential D before and after the perturbation we have

$$\frac{\alpha(\xi)}{2\pi} D(\xi) - D^{\Gamma_r}_{mean} = - \lim_{j \to \infty} \int_{\Gamma} \nabla^*_{\eta} D(\eta) \cdot g^{\nabla^*}_{\rho_j}(\xi, \eta) \, d\omega(\eta).$$

The problem is to characterize certain geophysical features (like mass pole perturbations) by their (scale-dependent) space evolution detected in the wavelet coefficients. To be more concrete, we take the dataset of deflections of the vertical from EGM96 (up to degree 200) and disturbed the data artificially by several unit mass points in a depth of 50km up to 120km. Figure 7 give a graphical illustration of the disturbed dataset. In detail, we positioned 6 mass points along a (linear) curve in the Pacific Ocean at depth 50km (for the point in the North) up to a depth 100km (for the point in the South). Furthermore, we put two irregularities by buried mass points over land (at depth 70km and 120km, respectively).

Again, the discrete scales $\rho_j = 2^{1-j}$ have been chosen to get a detailed information by the wavelet spaces to prepare out the disturbances. In Figure 8 the wavelet decomposition of the disturbed EGM96-potential is shown. The positions of the disturbances can be easily detected in the difference plot between Figure 8 and Figure 6. The difference is illustrated in Figure 9.

When looking at the differences caused by the eight buried mass points we are confronted with the following situation: The height of the "bumb" in the error plot corresponds to the depth of the mass points: the smaller the depth the larger the maximum of the gravitational perturbation for the different scales. Figure 10 illustrates this phenomenon in dependence of the scale. Even more, the diameter of the "bumb", i.e., the horizontal distance between maximal and minimal value of the gravitational perturbation is of larger value, the deeper the point is situated. With increasing scale the diameter is decreasing in (horizontal) size (see Figure 11). Furthermore, we notice a clear difference in the diameter for comparable mass point perturbations on continent and ocean. In other words, there is a correlation between the gravitational effect of the perturbations and the density distribution inside the Earth.

Altogether, the multiscale solution realized for the epoch problem involving disturbances by buried mass points offers a palette of mathematical indicators to specify the location and to classify the depth of the gravitational perturbation. A more careful and detailed multiscale investigation of even more complicated gravitational perturbations (e.g., inner line and surface disturbances) based on locally supported outer wavelets is certainly a great challenge for future work.

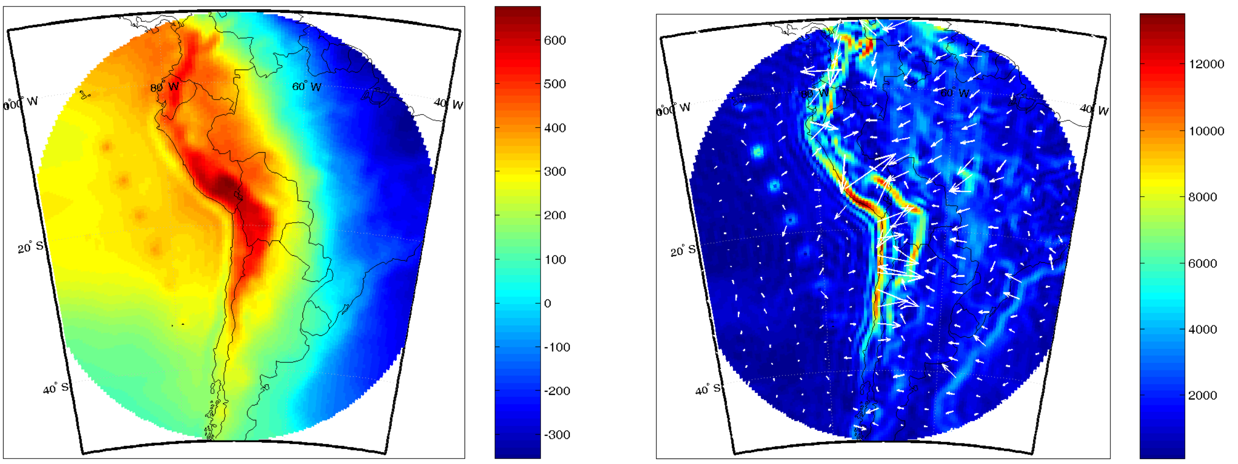

Fig. 7 Plot of the geopotential in $[\frac{m^2}{s^2}]$ (left) and the vertical deflection in $[\frac{m}{s^2}]$ (right) of EGM96 disturbed by buried mass points.

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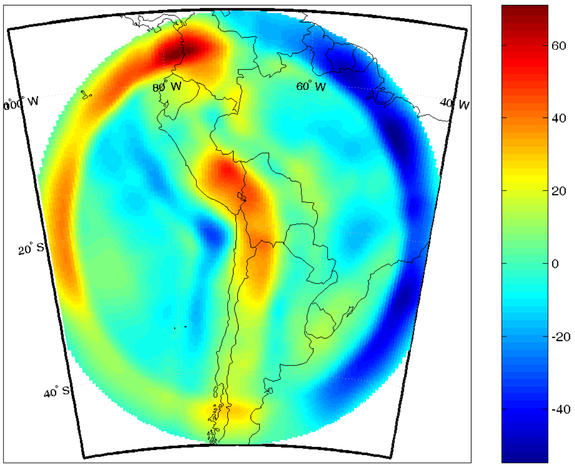
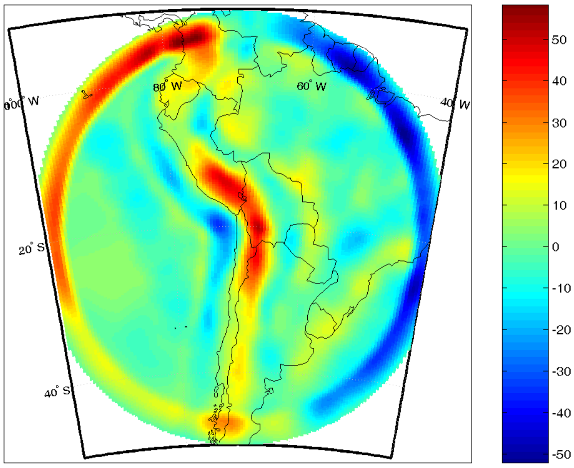
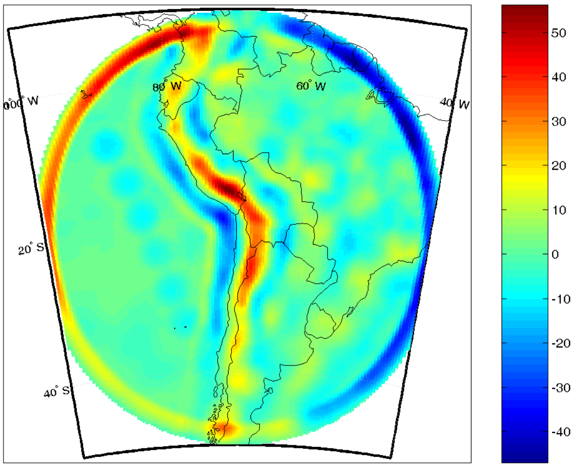


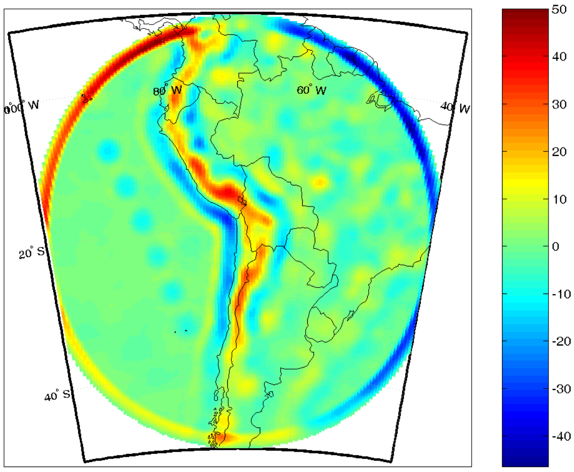
wavelet scale 8



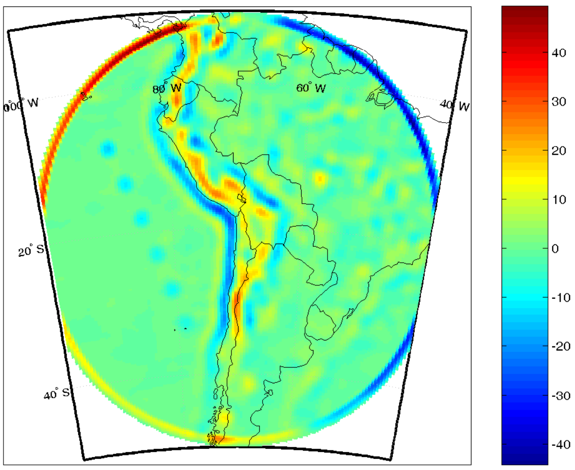
wavelet scale 9



wavelet scale 10

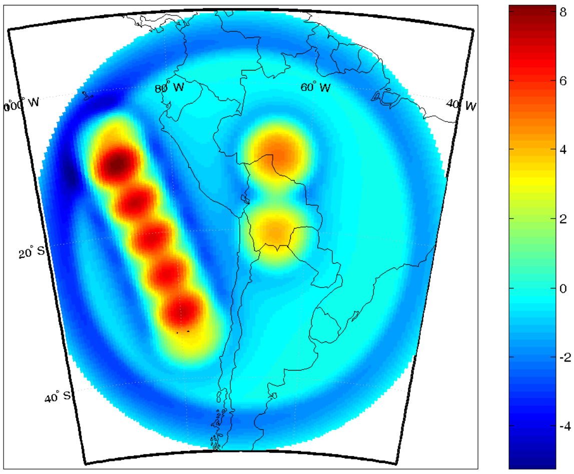


wavelet scale 11

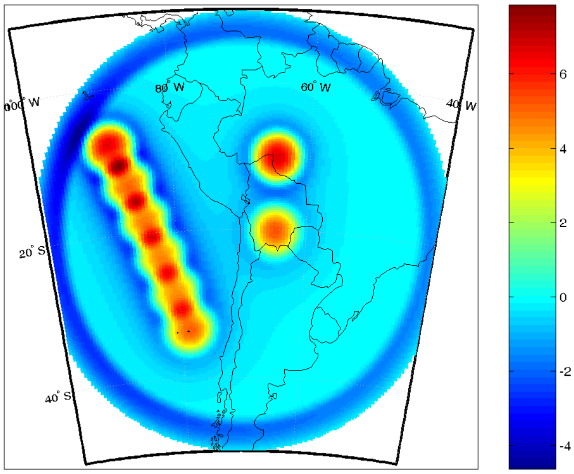


wavelet scale 12

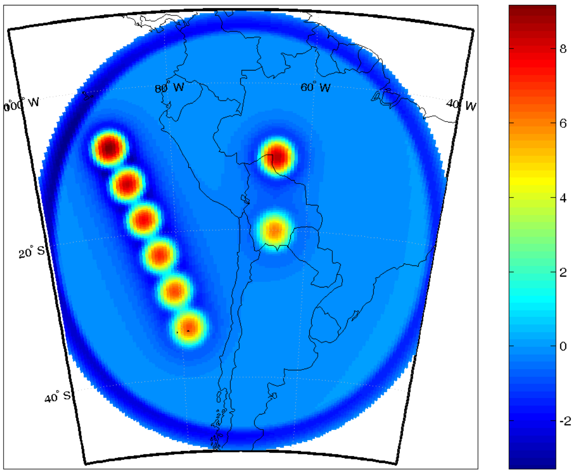
Fig. 8 Wavelet reconstruction of the disturbed geopotential in $[\frac{m^2}{s^2}]$ to detect the mass points on the spherical cap Γ_{30} at certain scales using the scale discretized regularized Green function.



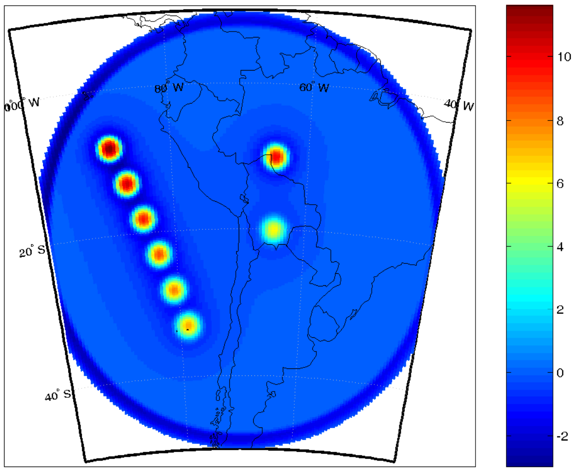
wavelet scale 8



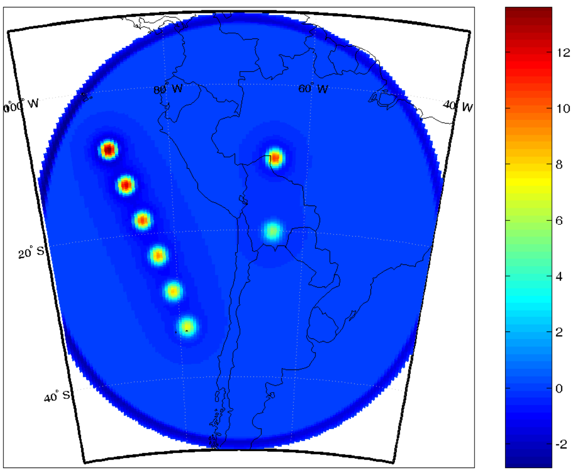
wavelet scale 9



wavelet scale 10



wavelet scale 11



wavelet scale 12

Fig. 9 Illustration of the difference of the original potential and the by mass points disturbed potential in $[\frac{m^2}{s^2}]$.

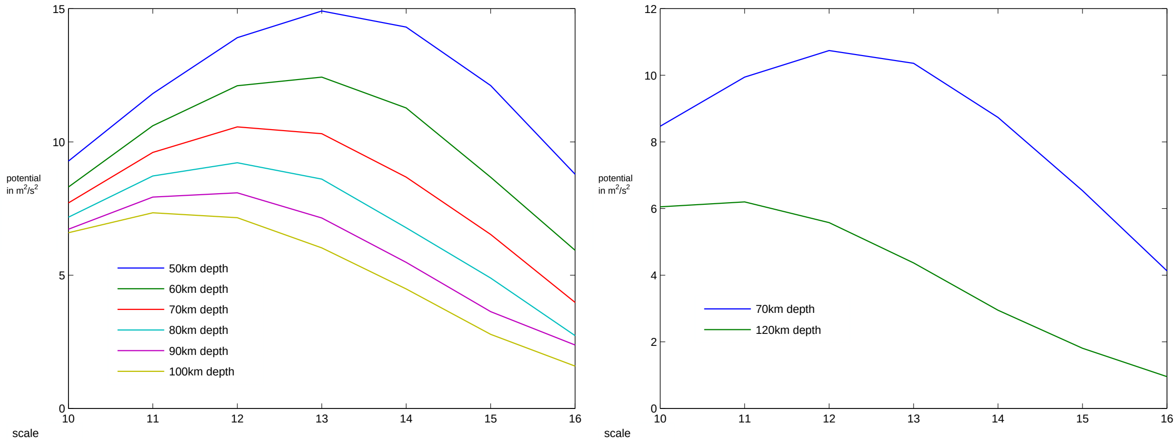


Fig. 10 Plot of the scale dependent maxima of the mass points in the ocean (left) and over land (right).

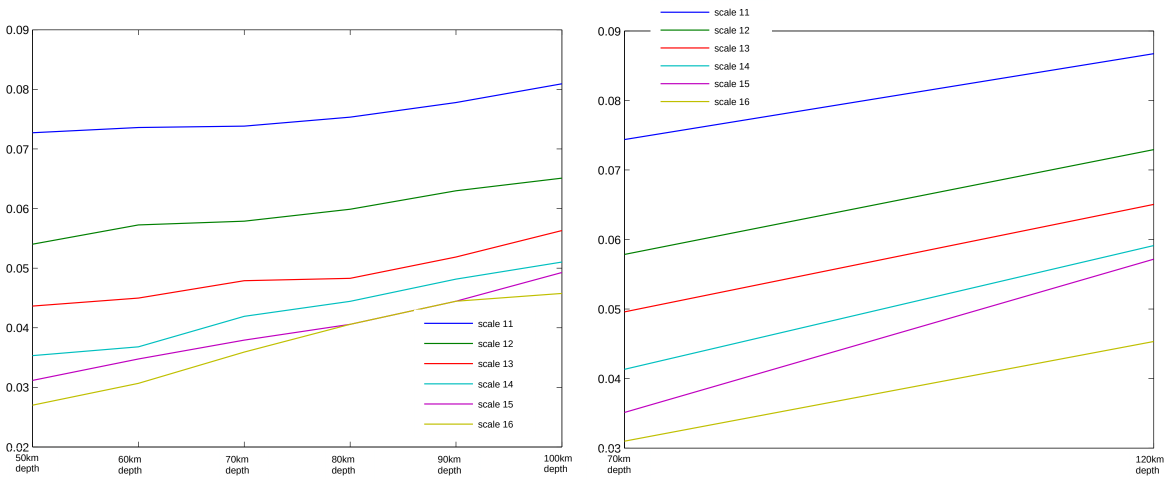


Fig. 11 Plot of the diameter of the mass points at certain scales in the ocean (left) and over land (right).

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