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Vorwort

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Darüber hinaus bietet die Reihe ein Forum für die Berichterstattung über die zahlreichen Kooperationsprojekte des Instituts mit Partnern aus Industrie und Wirtschaft.

Berichterstattung heißt hier Dokumentation des Transfers aktueller Ergebnisse aus mathematischer Forschungs- und Entwicklungsarbeit in industrielle Anwendungen und Softwareprodukte – und umgekehrt, denn Probleme der Praxis generieren neue interessante mathematische Fragestellungen.

Prof. Dr. Dieter Prätzel-Wolters
Institutsleiter

Kaiserslautern, im Juni 2001

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ON NUMERICAL UPSCALING FOR FLOWS IN HETEROGENEOUS POROUS MEDIA

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Abstract — A numerical upscaling approach, NU, for solving multiscale elliptic problems is discussed. The main components of this NU are: i) local solve of auxiliary problems in grid blocks and formal upscaling of the obtained results to build a coarse scale equation; ii) global solve of the upscaled coarse scale equation; and iii) reconstruction of a fine scale solution by solving local block problems on a dual coarse grid. By its structure NU is similar to other methods for solving multiscale elliptic problems, such as the multiscale finite element method, the multiscale mixed finite element method, the numerical subgrid upscaling method, heterogeneous multiscale method, and the multiscale finite volume method. The difference with those methods is in the way the coarse scale equation is build and solved, and in the way the fine scale solution is reconstructed. Essential components of the presented here NU approach are the formal homogenization in the coarse blocks and the usage of so called multipoint flux approximation method, MPFA. Unlike the usual usage as MPFA as a discretization method for single scale elliptic problems with tensor discontinuous coefficients, we consider its usage as a part of a numerical upscaling approach. The main aim of this paper is to compare NU with the MsFEM. In particular, it is shown that the resonance effect, which limits the application of the Multiscale FEM, does not appear, or it is significantly relaxed, when the presented here numerical upscaling approach is applied.

2000 Mathematics Subject Classification: 65N22, 65N55, 76S05.

Keywords: numerical upscaling, heterogeneous porous media, single phase flow, Darcy's law, multiscale problem, effective permeability, multipoint flux approximation, anisotropy.

1. Introduction

The problems of subsurface flow simulation are too complicated to be solved analytically. Furthermore, discretizations which resolve all the heterogeneities of the porous media are computationally expensive due to the required memory and computational time. Thus, numerical approaches are applied to solve flow problems in practice. The situation becomes

even more complicated when the porous media have heterogeneities on several length scales. Essential success was achieved during the last decades in studies of problems with clearly separated fine and coarse scales: periodic microstructures and statistically homogeneous porous media [2, 23]. In case the fine and the coarse scales can be decoupled, solving a multiscale problem reduces to one way two-stage procedure: i) solve fine scale cell-problem and use its solution to upscale the effective properties of the multiscale media; ii) solve upscaled coarse scale equation with the calculated effective coefficients. Separation of scales is however not always possible. Although not rigorously justified in general, the procedure of local homogenization (upscaling) is widely used for solving multiscale problems with unseparable scales. In such cases, the computational domain is subdivided into blocks, and effective properties for each block are calculated similar to the periodic or statistically homogenous case. These effective block properties are used then in solving coarse scale problem.

There exists a number of techniques for solving multiscale elliptic problems, which avoid any global solve on the fine scale. The therminology in this area is still not well established, but often the methods are subdivided into two groups, namely, multiscale methods [8, 10, 11, 15, 20] and numerical upscaling approaches [4, 7, 22]. The methods from the latter group consider an analytical form of the coarse scale equations, and provide them with effective (called also homogenized, or averaged) coefficients, while in multiscale methods the coarse scale equations are formed numerically, usually in FE manner, using multiscale basis functions.

Among multiscale methods recently used for solving elliptic problems with oscilatory coefficients, are multiscale finite element method [8, 11, 15], multiscale mixed finite element method [12], multiscale finite volume method [10] and heterogeneous multiscale method [15, 20]. All these methods are based on two-level (fine-coarse scale) algorithms, the main idea of which is to derive coarse scale equations taking into account fine scale heterogeneities. For problems with scale separation multiscale methods save computational resources in comparison to solving all fine scale equations while for problems without scale separation the complexity of the solution by a multiscale method is almost the same as the complexity of solving fine scale discretization. A thorough description of the most of the known multiscale methods, namely, of the multiscale finite element method, the multiscale mixed finite element method, the numerical subgrid upscaling method, and the multiscale finite volume method, can be found in [12], where also a detailed comparison of these methods is presented.

Numerical upscaling algorithms also consist in two-stage procedure. First, fine scale cell-problems are solved and their solutions are used to upscale the effective properties of the multiscale media. Then coarse scale equations with upscaled (tensor) coefficients are solved. Permeability [4, 10, 22] and direct transmissibility [4, 7] upscaling should be distinguished. In the case of direct transmissibility upscaling, local flow problems are solved around the interface between coarse grid cells, the flux through the face is computed and the transmissibility is assigned as the ratio of the flux to the mean of the pressure drop. Permeability upscaling consists in solving local flow problems in each coarse grid block or some extended local subdomain (coarse grid block with overlapping) and using coarse scale Darcy's law for calculating effective tensor which will be constant in each coarse block. The advantage of the permeability upscaling is that it is conservative and it can be easily parallelized.

The discussed here numerical upscaling is based on combination of local solves in coarse blocks in order to calculate effective coefficients for the coarse scale equations, and on MPFA for discretizing obtained coarse scale equations with discontinuous tensor coefficients. It should be mentioned that this approach differs from MsFEM, MsMFEM, subgrid upscaling,

HMM and MsFVM in the way in which the coarse scale equations are formed and solved. The described here upscaling approach is very close to the one implemented in the software used by some of the oil companies, see, e.g., [14]. Note that effective (upscaled) tensor can be a full tensor, even if the fine grid permeability field is described by diagonal (orthotropic) tensor. For accurate solving of elliptic problems with discontinuous tensor coefficient the multipoint flux approximation (MPFA) approach is widely used recently [1, 6, 13, 19, 21]. Theoretical analysis of the method is done only for the case of continuous tensor coefficients [6, 13, 21]. In [9], under certain smoothness assumptions, we show first order approximation for the fluxes in the case of discontinuous tensor coefficients, and show that in the case of constant tensor coefficients MPFA is $O(h^2)$ perturbation of the scheme from [17].

The target of this paper is to compare a numerical upscaling approach with the multiscale FEM, MsFEM. A short discussion on the both approaches can be found in [8], but it concerns only the number of the block solves needed for the both methods, as well as the memory requirements. Only numerical results obtained with MsFEM were shown there. The authors did not found in the literature a direct comparison of the performance of the both methods, and this article aims at filling this gap. A partial motivation for our study was also the article [15], where an example which can not be solved by MsFEM and heterogeneous multiscale method was shown. Furthermore, another target of the paper is to numerically study the role of the boundary conditions for the cell problems. We compare so called linear, oscilatory, and Neumann boundary conditions for the cell problems.

The paper is organized as follows. Next section is devoted to describing an accurate finite volume discretization for elliptic equations with discontinuous tensor coefficients. Assuming that the flux is piecewise constant in each control volume, we derive interpolating polynomials which satisfy the interface conditions and use them to approximate the solution of the problem. The discretization is derived in the same way as the schemes from [1, 5]. The derivation is based on the finite volume approach (method of balance, [17]) and multipoint flux approximation approach [1]. Writhing the derivation of MPFA in the specific form, we have shown in [9] that the components of the continuous flux are approximated with first order in the midpoints of the edges by the components of the discrete flux.

Third section of the paper treats multiscale problems. Two-level permeability upscaling is discussed here. Different formulae and local flow formulations for calculating effective properties of highly heterogeneous porous media are considered. An algorithm for reconstructing the coarse grid solution into the fine scale is proposed. It should be noted that this reconstruction is done on a dual coarse grid, while the multiscale FEM uses for recosntruction the multiscale basis functions from the primaty grid.

In fourth section, we present the results from the performed numerical experiments. First, we consider a single scale problem and study numerically the convergence of the MPFA discretization for the pressure and for the fluxes in the case of anisotropic discontinuous tensor coefficient. Next, we validate the permeability upscaling procedure comparing it with known analytical results and with other numerical approaches for calculating effective coefficients. Finally, we present results from upscaling two typical multiscale problems with continuous highly oscillating coefficients. Note, that although in this case the porous medium is isotropic at the fine scale and the coefficient is continuous there, we have to deal with anisotropic problem with discontinuous tensor coefficients at the coarse scale. The first example is from the recent article [15], where authors show that multiscale finite element method (MsFEM) and heterogeneous multiscale method (HMM) do not converge for this example. The second example is from [8], where stagnation for MsFEM is reported. In

fact, avoiding this stagnation is the reason for the authors in [8] to consider the so called oversamoling approach. We show that the numerical upscaling approach is more successful in solving these problems even without oversampling. Concluding remarks are presented in the end of the paper.

2. Continuous problem and finite volume discretization

In this section, we describe single scale mathematical model and the discretization approach. The approximation properties of the derived numerical scheme are studied as well.

2.1. Statement of the problem

In this paper, we consider steady state incompressible single phase flow in highly heterogeneous anisotropic porous media. Such flow is described by the equation for the unknown pressure p:

$$-\nabla \cdot (K\nabla p) = f, \ \text{ in } \Omega, \tag{1}$$

subject to the following boundary conditions

$$p = g^D, \ \text{ on } \Gamma_D, \quad K\nabla p \cdot \mathbf{n} = g^N, \ \text{ on } \Gamma_N, \quad \partial\Omega = \Gamma_D \cup \Gamma_N. \tag{2}$$

This problem could be reformulated in a mixed form as a system of the equation $\nabla \cdot \mathbf{v} = f$ expressing mass conservation and the Darcy's law $\mathbf{v} = -K\nabla p$. Here the domain Ω is a parallelepiped with boundaries parallel to the coordinate planes, the set Γ_D is non-empty and has positive surface measure, the permeability tensor is full, symmetric, and uniformly positive definite in Ω:

$$K = \begin{pmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{pmatrix} > 0, \quad k_{12} = k_{21}.$$

The entries of the permeability tensor K may have jump discontinuities along certain interfaces that are parallel to the coordinate planes and along these interfaces the following conditions are satisfied

$$[p] = 0, \quad [K\nabla p \cdot \mathbf{n}] = 0.$$

Here $[u] = u(x_\xi + 0) - u(x_\xi - 0)$ on the interface x_ξ, and $\mathbf{n}$ stands for the normal unit vector.

2.2. Finite volume discretization

The domain Ω is partitioned into blocks Ω_{ij} so that the discontinuities of the permeability tensor K are aligned with cell boundaries. The centers of the cells Ω_{ij} are denoted by (x_i, y_j) and the cell vertexes are the points $(x_i \pm \frac{1}{2}h_x, y_j \pm \frac{1}{2}h_y)$. The mesh that will be used to approximate the pressure will include all cell centers (x_i, y_j). This mesh will be called *primary mesh* $\omega_h = \{(x_i, y_j) : \Omega_{ij}\}$. Similarly we shall use also the mesh of all cell vertexes, called often *dual mesh*. The velocities will be calculated at the points $(x_i \pm \frac{1}{2}h_x, y_j)$ and $(x_i, y_j \pm \frac{1}{2}h_y)$.

The continuity equation ($\nabla \cdot \mathbf{v} = f$) is integrated over control volume Ω_{ij} and making use of the divergence theorem, we obtain

$$\int\limits_{\Omega_{ij}} \nabla \cdot \mathbf{v} d\mathbf{x} = \int\limits_{\Omega_{ij}} f d\mathbf{x} \quad \Rightarrow \quad \int\limits_{\partial\Omega_{ij}} \mathbf{v} \cdot \mathbf{n} ds = \int\limits_{\Omega_{ij}} f d\mathbf{x}. \tag{3}$$

Replacing the velocity $\mathbf{v}$ in (3) by certain approximation involving p, and by using the Darcy's relation $\mathbf{v} = -K\nabla p$ we get a conservative method [17]. In this approximation we assume that the unknowns are the values of the pressure at the cell centers and then use these values to recover the velocity $\mathbf{v}$. According to the multipoint flux approximation (see, e.g., [5, 19]) this is done in the following manner. First we split each control volume

$$\Omega_{ij} = \left(x_i - \frac{1}{2}h_x, x_i + \frac{1}{2}h_x\right) \times \left(y_j - \frac{1}{2}h_y, y_j + \frac{1}{2}h_y\right)$$

into 4 subvolumes $\Omega_{ij}^I = (x_i - \frac{1}{2}h_x, x_i) \times (y_j, y_j + \frac{1}{2}h_y)$, $\Omega_{ij}^{II} = (x_i - \frac{1}{2}h_x, x_i) \times (y_j, y_j - \frac{1}{2}h_y)$, $\Omega_{ij}^{III} = (x_i, x_i + \frac{1}{2}h_x) \times (y_j, y_j + \frac{1}{2}h_y)$, $\Omega_{ij}^{IV} = (x_i + \frac{1}{2}h_x, x_i) \times (y_j, y_j - \frac{1}{2}h_y)$. See Figure 1 for details.

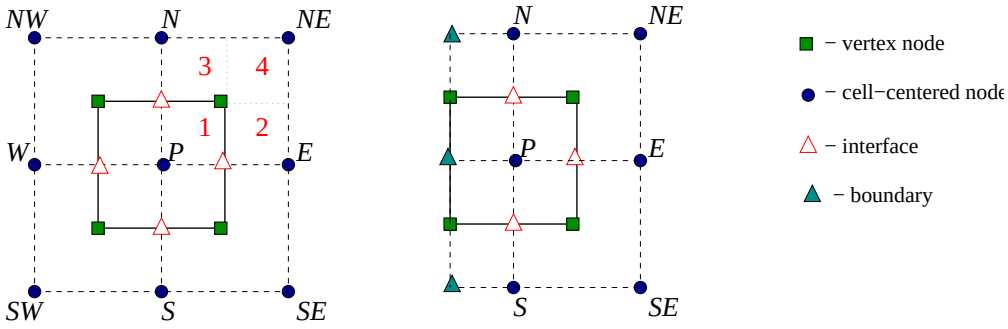

Figure 1. Control volumes: inner and boundary cells

To approximate the pressure, we consider linear function on each subvolume Ω_{ij}^k so that the flux is constant on these subvolumes:

$$q^k(x, y) = a^k x + b^k y + c^k, \ (x, y) \in \Omega^k. \tag{4}$$

On the boundary Γ_D we require these functions to satisfy the prescribed boundary conditions in the boundary grid nodes. The coefficients a^k, b^k and c^k in (4) are determined by the following conditions:
(1c) the polynomials q^k interpolate the pressure values at the volume centers;
(2c) the continuity of these piecewise linear functions at the centers of the faces of the volume Ω_{ij};
(3c) the continuity of the normal component of the approximate velocity $\mathbf{v}$ along the faces of the volume Ω_{ij}.

Conditions (1c)–(3c) correspond to the so called O-method with surface midpoints as continuity points, see, e.g., [1]. These conditions are applied on a cell from the dual grid, i.e. a cell centered at a vertex point from the original grid. These cells are of three categories: corresponding to internal vertices, to boundary vertices; and to the 4 corner points of the domain Ω (Fig. 1).

Consider an internal vertex that is surrounded by four subcells with unknown pressure values p^N, p^{NE}, p^P, p^E at the corners. To find the polynomial coefficients from (4), we use conditions (1c)–(3c). Let the coordinate origin be in the vertex node, so for the considered shifted control volume we have $-h_x/2 \leqslant x \leqslant h_x/2$, $-h_y/2 \leqslant y \leqslant h_y/2$. Using conditions (1c)–(3c) and substituting x, y and the pressure at the cell-centers (P, E, N, NE) into equation (4), we obtain the system

$$Av = w, \quad v = (a^1, b^1, a^3, b^3)^T.$$

The matrix of the system is

$$A = \begin{pmatrix} k_{11}^P + k_{11}^E & k_{12}^P & 0 & k_{12}^E \\ k_{12}^P & k_{22}^P + k_{22}^N & k_{12}^N & 0 \\ 0 & k_{12}^N & k_{11}^N + k_{11}^{NE} & k_{12}^{NE} \\ k_{12}^E & 0 & k_{12}^{NE} & k_{22}^E + k_{22}^{NE} \end{pmatrix}, \tag{5}$$

and the right-hand side is given by

$$w_1 = 2\left(k_{11}^E \frac{p^E - p^P}{h_x} + k_{12}^E \frac{p^{NE} - p^E}{h_y}\right), \quad w_2 = 2\left(k_{21}^N \frac{p^{NE} - p^N}{h_x} + k_{22}^N \frac{p^N - p^P}{h_y}\right),$$
$$w_3 = 2\left(k_{11}^N \frac{p^{NE} - p^N}{h_x} + k_{12}^N \frac{p^N - p^P}{h_y}\right), \quad w_4 = 2\left(k_{12}^E \frac{p^E - p^P}{h_x} + k_{22}^E \frac{p^{NE} - p^E}{h_y}\right).$$

The coefficients a^i, b^i, $i = 2, 4$, can be expressed through a^i, b^i, $i = 1, 3$ and the pressure values at the cell centers

$$a^2 = \frac{p^E - p^P}{0.5h_x} - a^1, \quad b^2 = \frac{p^{NE} - p^E}{0.5h_y} - b^3, \quad a^4 = \frac{p^{NE} - p^N}{0.5h_x} - a^3, \quad b^4 = \frac{p^N - p^P}{0.5h_y} - b^1. \tag{6}$$

Note, that it gives us the expressions for the velocity that is constant over each of the 4 subcells of the vertex-centered volume (Fig. 1). Consider in the same way three other vertex-centered volumes to find the fluxes incoming and outcoming the cell-centered (control) volume Ω_{ij}. These formulas are used to find $\mathbf{v} \cdot \mathbf{n}$ on $\partial\Omega_{ij}$, as needed by relation (3).

For vertex that is on the boundary the situation is simpler. In the case of Neumann boundary conditions the flux is given on the boundary, while in the case of Dirichlet boundary conditions we just do the same procedure as for the inner control volume, but in this case we have only one interface.

Combining this relationship for each neighboring vertex gives us a discrete pressure equation with a 9-point stencil.

Remark 2.1. In the case of diagonal permeability tensor ($k_{12} = 0$), the developed finite volume scheme reduces to the harmonic average scheme

$$\begin{aligned} &\frac{2k_{11}^P k_{11}^E}{k_{11}^P + k_{11}^E} \frac{h_y}{h_x}\left(p^E - p^P\right) - \frac{2k_{11}^P k_{11}^W}{k_{11}^P + k_{11}^W} \frac{h_y}{h_x}\left(p^P - p^W\right) + \\ &\frac{2k_{22}^P k_{22}^N}{k_{22}^P + k_{22}^N} \frac{h_x}{h_y}\left(p^N - p^P\right) - \frac{2k_{22}^P k_{22}^S}{k_{22}^P + k_{22}^S} \frac{h_x}{h_y}\left(p^P - p^S\right) = -f^P h_x h_y. \end{aligned} \tag{7}$$

3. Multiscale problem

In this section we discuss a numerical upscaling algorithm for solving problem (1), (2) in the case of highly oscilating coefficients. The algorithm includes:
i.) decomposition of the computational domain into non-overlapping subdomains,
ii.) solving local problems in each subdomain to calculate (homogenize) effective coefficients for this subdomain (these are so called cell problems);
iii.) solving coarse scale problem with upscaled (tensor) coefficients;
iv.) reconstruction of a fine solution from the coarse one, if needed.

Different boundary conditions for the cell problems are considered, as well as different formulae for calculating the effective coefficients from the solution of the cell problems.

Let us shortly discuss similarity and differences with MsFEM. The steps i.) and ii.) are almost the same as in multiscale FEM. The difference in ii.) is that here we use the solution of the cell problem in order to calculate effective coefficients for the coarse scale equation, while in MsFEM these local solutions are used to establish a multiscale basis for the FEM. Furthermore, the coarse equation in iii.) is build in different way by the both methods. The used here numerical upscaling use the upscaled coeficients to write a (coarse scale) PDE with discontinuos tensor coefficients, which is after done discretized with MPFA and solved. In MsFEM, the coarse scale equation is not formed analytically, it is obtained numerically within a Galerkin procedure. Finally, in step iv.), MsFEM automatically gets a fine scale solution because multiscale basis functions are used, while presented here numerical approach needs additional block solves on a dual coarse grid in order to recover a fine scale solution. Note, that the usage of a dual grid allows to use more coarse information in the local reconstruction of a fine solution.

3.1. Fine-coarse scale algorithm

Consider decomposition of the domain $\bar{\Omega}$ into the fine grid ω_h defined in the previous section and the coarse grid defined in the same way $\omega_H = \{(x_I, y_J) : \Omega_{IJ} = n_x n_y \Omega_{ij}\}$, where n_x, n_y is the number of fine grid blocks in a coarse one. Both grids are uniform and cell-centered, moreover, the interfaces of each coarse block match the interfaces of the fine blocks (Fig. 2).

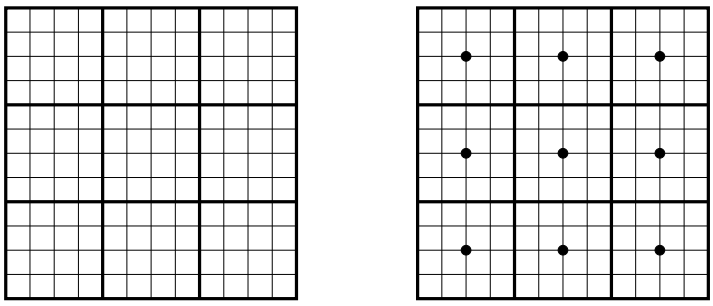

Figure 2. Fine and coarse grids

It was mentioned above that in a number of industrial applications it is reasonable to find effective properties of the medium, i.e., instead of K given in every point on the fine grid, find constant tensor $\tilde{K}$ in each coarse grid block. In the paper, we consider homogenization procedure which allow the coarse scale equation to be of the same form as equation (1) but with permeability K replaced by the coarse scale or effective permeability tensor $\tilde{K}$. In this case instead of equation (1) we will have the coarse scale pressure equation

$$-\nabla \cdot (\tilde{K} \nabla \tilde{p}) = f. \tag{8}$$

In paper[22] it was shown that the homogenized solution $\tilde{p}$ approximates the fine scale solution p in L_2 norm

$$\|p - \tilde{p}\|_\Omega \leqslant C_1 \frac{h}{H} + C_2 H + C_3 h.$$

Note that $\tilde{K}$ depends on heterogeneities only and calculated once can be used in different computational scenarios. Different definitions of $\tilde{K}$ have been proposed [4, 22]. Solutions of

local flow problems in each coarse grid block are postprocessed in order to upscale the permeability tensor. The main differences among various formulations are the boundary conditions imposed on the local flow equation and the averaging processes for computing $\tilde{K}$.

So, the algorithm contains the following two stages: i) solving fine scale cell-problems and using their solutions for upscaling the effective properties of the multiscale media; ii) solving coarse scale equation with the calculated effective coefficients. In case we are interested in fine grid solution in certain subdomains, the algorithm of reconstruction coarse scale solution should be applied.

3.2. Effective permeability

Consider a cubic grid block V. It can be coarse grid block Ω_{ij} or some extended local subdomain (coarse grid block with overlapping). To define $\tilde{K}$ in V we write the coarse scale Darcy's law

$$\langle \mathbf{v} \rangle_V = -\tilde{K} \langle \nabla p \rangle_V , \tag{9}$$

where p and $\mathbf{v}$ are fine scale solutions of the problem $\mathbf{v} = -K \cdot \nabla p,\ \nabla \cdot \mathbf{v} = 0$ in the grid block V with appropriate boundary conditions. Note that $f = 0$, since effective properties should be independent on the source term and global boundary conditions posed on $\partial\Omega$. Here $< . >_V$ is the volume average over V defined by

$$\langle . \rangle_V = \frac{1}{V} \int_V (.) dx.$$

In two-dimensional case, two fine scale flow solutions which provide linearly independent volume averages of the pressure gradients are necessary to determine $\tilde{K}$ from (9) in each grid block

$$\mathbf{v}_i = -K \cdot \nabla p_i,\ \nabla \cdot \mathbf{v}_i = 0, \quad \text{in } V, \quad i = 1, 2. \tag{10}$$

The subscript of $\mathbf{v}$ and ∇p designates the flow problem (1 corresponds to the flow in x-direction, 2 corresponds to y-flow). The solutions of the local flow problems (10) with appropriate boundary conditions are then postprocessed in order to upscale the permeability tensor

$$\langle \mathbf{v}_i \rangle_V = -\tilde{K} \langle \nabla p_i \rangle_V , \quad i = 1, 2. \tag{11}$$

Different local flow formulations are used in practice. The following boundary conditions are usually posed. *Periodic conditions* can be formulated as

$$p_1 = x + \varepsilon, \quad p_2 = y + \varepsilon, \quad \text{periodic on } V. \tag{12}$$

Linear pressure drop conditions are the following

$$p_1 = x, \quad p_2 = y, \quad \text{on } \partial V. \tag{13}$$

Pressure drop no-flow conditions (linear in the direction of the gradient and zero Neumann in normal direction) are given by

$$\begin{aligned} &p_1 = x, \ \text{ on } \Gamma_1, \quad \mathbf{n} \cdot v_1 = 0, \ \text{ on } \Gamma_2, \\ &p_2 = y, \ \text{ on } \Gamma_2, \quad \mathbf{n} \cdot v_2 = 0, \ \text{ on } \Gamma_1, \end{aligned} \tag{14}$$

where Γ_i are the faces of ∂V normal to the unit vector in the ith direction.

Oscillatory boundary conditions are defined in the following way

$$\begin{aligned} p_1 &= x, \ \text{ on } \Gamma_1, \quad p_1 = P(x), \ \text{on } \Gamma_2, \\ p_2 &= y, \ \text{ on } \Gamma_2, \quad p_2 = P(y), \ \text{on } \Gamma_1. \end{aligned} \tag{15}$$

To build the interpolation operator P the following algorithm is used. We solve one-dimensional problems on the edges perpendicular to the boundary Γ_i in order to develop problem dependent interpolation of the values $x,\ y \in \Gamma_i \cap \Gamma_j,\ i \neq j$, in the corners of the considered grid block

$$\frac{\partial}{\partial x}\left(k_{11}\frac{\partial p_1}{\partial x}\right) = 0, \ \text{ in } \Gamma_2, \quad p_1(0) = 0,\ p_1(1) = 1, \tag{16}$$

$$\frac{\partial}{\partial y}\left(k_{22}\frac{\partial p_2}{\partial y}\right) = 0, \ \text{ in } \Gamma_1, \quad p_2(0) = 0,\ p_2(1) = 1. \tag{17}$$

We discretize problems (16), (17) by the harmonic average scheme (7) with 3-point stencil and solve the discrete problem directly using Thomas algorithm. The solutions of the one-dimensional problems (16), (17) provide fine grid pressure values p_i on the boundary of the local subdomain V.

Note that the upscaled permeability tensor $\tilde{K}$ computed via equations (11) can be non-symmetric. Various procedures are applied to enforce symmetry. The simplest approach is to use the mean value of the cross terms $(\tilde{k}_{12}+\tilde{k}_{21})/2$. The second disadvantage of the above mentioned approach is that it doesn't guarantee the positive definiteness of the upscaled tensor $\tilde{K}$.

In the paper [22] it was shown that for periodic and linear boundary conditions (12), (13) formula (11) can be simplified as

$$\tilde{K}\mathbf{e}_i = \langle \mathbf{v}_i \rangle_V , \tag{18}$$

where $\mathbf{e}_i$ is the unit vector in the ith direction. So, one local flow problem gives us components $\tilde{k}_{11},\ \tilde{k}_{12}$ of the upscaled permeability tensor and in the case of isotropic media ($k_{11} = k_{22}$) there is no need to solve two local flow problems. Another nice property of the boundary conditions (12) and (13) is that they lead to symmetric and positive definite $\tilde{K}$. In this case, we have another way to compute homogenized permeability tensor

$$\mathbf{e}_i \cdot \tilde{K}\mathbf{e}_j = \langle \nabla p_i \cdot K \nabla p_j \rangle_V , \tag{19}$$

From formula (19) it is easy to see that in this case $\tilde{K}$ is symmetric and positive definite.

3.3. Reconstruction of the solution

Since we deal with discontinuous and strongly varying by several orders of magnitude coefficients in the domain of interest, the problem dependent prolongation operator has to be used [18] to reconstruct the solution from the coarse grid.

Suppose that the coarse scale pressure values $\tilde{p} = \tilde{p}_{I,J}$ are known in the nodes (x_I, y_J) of the coarse grid ω_H, and these values have to be interpolated to the fine grid ω_h. Consider four neighbouring coarse grid nodes (x_I, y_J), (x_{I+1}, y_J), (x_I, y_{J+1}), (x_{I+1}, y_{J+1}) forming a rectangle V. To reconstruct the solution in the considered subdomain the following two-stage algorithm is used. First, we solve four one-dimensional problems

$$\frac{\partial}{\partial x}\left(k_{11}\frac{\partial p^{edge}}{\partial x}\right) = 0, \ \text{ in } \Gamma_2, \quad p^{edge}(x,y) = \tilde{p}, \ \text{ in } \Gamma_1 \cap \Gamma_2, \tag{20}$$

$$\frac{\partial}{\partial y}\left(k_{22}\frac{\partial p^{edge}}{\partial y}\right)=0,\ \ \text{in } \Gamma_1,\quad p^{edge}(x,y)=\tilde{p},\ \ \text{in } \Gamma_1\cap\Gamma_2. \tag{21}$$

Solutions of problems (20), (21) give problem dependent interpolation of the values $\tilde{p}_{I,J}$ along the boundary ∂V of the considered vertex-centered grid block V. Then solutions p^{edge} are used as Dirichlet boundary conditions for two-dimensional problem in block V:

$$\frac{\partial}{\partial x}\left(k_{11}\frac{\partial p^c_{rec}}{\partial x}\right)+\frac{\partial}{\partial y}\left(k_{22}\frac{\partial p^c_{rec}}{\partial y}\right)=0,\ \ \text{in } V, \tag{22}$$

$$p^c_{rec}=p^{edge},\ \ \text{on } \partial V. \tag{23}$$

Solution of problem (22), (23) provides the reconstructed solution p^c_{rec} in subdomain V under consideration. For subdomains near the boundary $\partial\Omega$ we use global boundary conditions on $\partial V\cap\partial\Omega$.

4. Numerical results

In this section, we validate the developed single grid finite volume discretization for problems with discontinuous tensor coefficients, calculate effective properties of the media and solve multiscale problem using combination of permeability upscaling and multipoint flux approximation approach.

4.1. Validation of the discretization for single scale problems

To validate the developed finite volume scheme we perform numerical experiments for Dirichlet boundary-value problem ($p=g^D$ on $\partial\Omega$) for the pressure equation (1) with the coefficients discontinuous along the interfaces $x=x_\xi$ and $y=y_\eta$. The permeability tensor is given by

$$K(x,y)=\begin{cases}\alpha\cdot I, & (x,y)\in K_1\cup K_4,\\ \beta\cdot I, & (x,y)\in K_2\cup K_3,\end{cases},\quad \text{where } I=\begin{pmatrix}1 & 0.5\\ 0.5 & 1\end{pmatrix},$$

where $\lambda=\beta/\alpha$ is the contrast of discontinuity (Fig. 3).

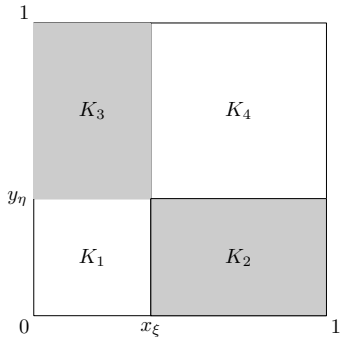

Figure 3. Discontinuity surface

We choose the exact solution which satisfies the continuity conditions of the pressure and the fluxes along these interfaces

$$p=(x-x_\xi)^2(y-y_\xi)^2\sin(\pi(x+y)).$$

Source f and boundary conditions g^D are defined by (1) and (2) accordingly.

Convergence rate for the pressure and the velocity is established by running cases for several levels of grid refinement, refining the grid steps by factor of two for each successive level. The following notations are used

$$\|\delta p_h\|_C = \frac{\|p - p_h\|_C}{\|p\|_C}, \quad \|\delta p_h\|_{L_2} = \frac{\|p - p_h\|_{L_2}}{\|p\|_{L_2}},$$

where p is the (exact) solution of the differential problem (1), (2), and p_h is the (approximate) solution of the MPFA finite volume scheme. In this particular problem $v_x = v_y = v$, so we define in the same way

$$\|\delta v_h\|_C = \frac{\|v - v_h\|_C}{\|v\|_C}, \quad \|\delta v_h\|_{L_2} = \frac{\|v - v_h\|_{L_2}}{\|v\|_{L_2}}.$$

For these experiments the following parameters are used $x_\xi = 0.4$, $y_\eta = 0.4$, $\alpha = 1$, $\beta = 1000$. Table 1 shows the difference between the analytic and the numerical solutions in the corresponding energy and maximum norms.

Error	$N = 20$	$N = 40$	$N = 80$	$N = 160$	$N = 320$
$\|\delta p_h\|_{L_2}$	1.20E-2	3.06E-3	7.77E-4	1.96E-4	4.92E-5
	1.97	1.98	1.98	1.99	
$\|\delta v_h\|_{L_2}$	1.08E-2	4.39E-3	1.56E-3	4.95E-4	1.49E-4
	1.29	1.49	1.66	1.73	
$\|\delta p_h\|_C$	1.73E-2	4.91E-3	1.30E-3	3.37E-4	8.66E-5
	1.82	1.92	1.95	1.96	
$\|\delta v_h\|_C$	1.70E-2	1.04E-2	5.59E-3	2.87E-3	1.45E-3
	0.70	0.89	0.96	0.99	

Table 1. Errors and orders of convergence for problem (1), (2) in L_2 and C norms

From Tab. 1 it is easy to see that the approximate solution p_h converges to the exact solution p with the second order both in the maximum and energy norms, while for the fluxes we have first order convergence in C norm and second order convergence in L_2 norm.

4.2. Efficient calculations of effective properties

As was mentioned above, isotropic inhomogeneity may lead to a dense tensor. To demonstrate this, J. Bourgat considered the L-shaped region [3] with the following permeability tensor

$$K(x,y) = \begin{cases} 1 \cdot I, & (x,y) \in \Omega_0, \\ 10 \cdot I, & (x,y) \in \Omega_1, \end{cases}, \quad \text{where } I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Different techniques were used to find the homogenized tensor $\tilde{K}$ which is constant in the whole domain $\Omega_0 \cup \Omega_1$. The asymptotic computation of Bourgat gives

$$\tilde{K}_{as} = \begin{pmatrix} 1.915 & -0.101 \\ -0.101 & 1.915 \end{pmatrix}.$$

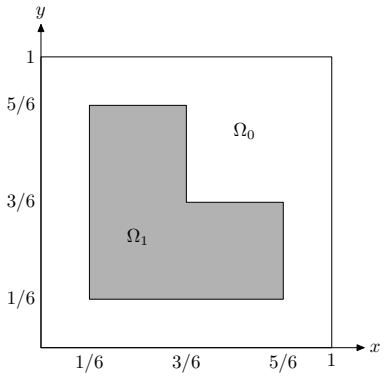

Figure 4. *L*-shaped inhomogeneity

Black box homogenization [16] also provides a full tensor, specifically for grid 768×768 effective tensor is equal

$$\tilde{K}_{bb} = \begin{pmatrix} 1.959 & -0.153 \\ -0.153 & 1.959 \end{pmatrix}.$$

We calculated effective permeability numerically on the same grid 768×768 using different formulas and local flow formulations

$$\tilde{K}_{dl} = \begin{pmatrix} 1.937 & -0.136 \\ -0.136 & 1.937 \end{pmatrix}, \quad \tilde{K}_{el} = \begin{pmatrix} 1.935 & -0.135 \\ -0.135 & 1.935 \end{pmatrix}, \quad \tilde{K}_{dn} = \begin{pmatrix} 1.887 & -0.127 \\ -0.127 & 1.887 \end{pmatrix}.$$

Effective permeability tensors $\tilde{K}_{dl}$, $\tilde{K}_{dn}$ are calculated by means of formula (11) while $\tilde{K}_{el}$ is computed via formula (19). For calculating $\tilde{K}_{dl}$ and $\tilde{K}_{el}$ linear boundary conditions (13) are posed, for $\tilde{K}_{dl}$ – pressure drop no-flow conditions (14). For particularly this case oscillatory boundary conditions (15) give the same result as linear boundary conditions (13) since there are no inhomogeneities on the domain boundary. Note that for this case all the approaches give quite similar results.

4.3. Computation of multiscale problems

In order to compare permeability upscaling with multiscale finite element method (MsFEM) and heterogeneous multiscale method (HMM), we consider the multiscale problems with unseparable scales posed in the paper by P. Ming and X. Yue [15]. The authors reported that MsFEM and HMM fail in solving this problem.

In domain $\bar{\Omega} = [0,1] \times [0,1]$ consider pressure equation (1) with homogeneous Dirichlet boundary conditions ($p = 0,$ on $\partial\Omega$) and constant right-hand side ($\mathrm{f} = 10$). The problem is characterized by highly heterogeneous isotropic continuous coefficients

$$K = \begin{pmatrix} a^{\varepsilon}(x,y) & 0 \\ 0 & a^{\varepsilon}(x,y) \end{pmatrix},$$

with the components

$$\begin{aligned} a^{\varepsilon}(x,y) = \frac{1}{6}\Bigg(& \frac{1.1+\sin(2\pi x/\varepsilon_1)}{1.1+\sin(2\pi y/\varepsilon_1)} + \frac{1.1+\sin(2\pi y/\varepsilon_2)}{1.1+\cos(2\pi x/\varepsilon_2)} + \frac{1.1+\cos(2\pi x/\varepsilon_3)}{1.1+\sin(2\pi y/\varepsilon_3)} \\ & + \frac{1.1+\sin(2\pi y/\varepsilon_4)}{1.1+\cos(2\pi x/\varepsilon_4)} + \frac{1.1+\cos(2\pi x/\varepsilon_5)}{1.1+\sin(2\pi y/\varepsilon_5)} + \sin\left(4x^2y^2\right) + 1\Bigg), \end{aligned} \tag{24}$$

where $\varepsilon_1 = 1/5$, $\varepsilon_2 = 1/13$, $\varepsilon_3 = 1/17$, $\varepsilon_4 = 1/31$, $\varepsilon_5 = 1/65$ (Fig. 5).

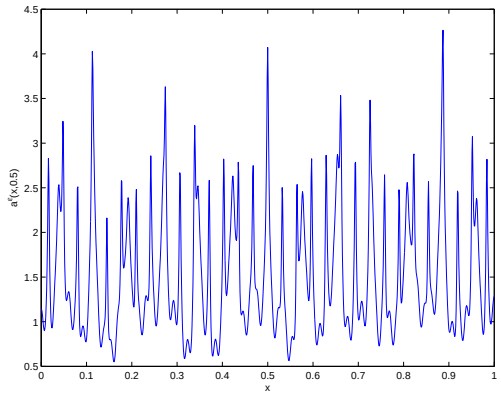

Figure 5. Coefficient (27) at y = 0.5

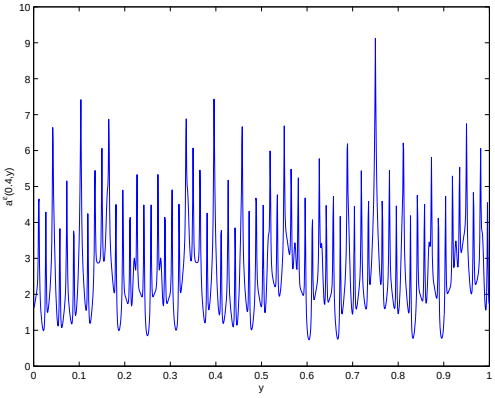

Figure 6. Coefficient (27) at x = 0.4

We solve local flow problems with pressure drop no-flow boundary conditions (14) and use formulas (11) for calculating effective permeability tensor.

To compare fine grid solution with the coarse grid one the following criterion is used. We will check convergence of the coarse grid solution p^c to the reference solution p^{rf} calculated on the single grid 2048×2048. First we consider how close the volume average of the fine grid solution $\langle p^{rf} \rangle$ is to the solution p^c obtained on the coarse grid

$$\frac{\left\| \langle p^{rf} \rangle - p^c \right\|_{L_2}}{\left\| p^{rf} \right\|_{L_2}}, \quad \frac{\left\| \langle p^{rf} \rangle - p^c \right\|_{C}}{\left\| p^{rf} \right\|_{C}}. \tag{25}$$

Then we reconstruct the solution from the coarse grid into the fine grid by approach described in Section 3.3. The reconstructed solution is denoted by p^c_{rec}.

$$\frac{\left\| p^{rf} - p^c_{rec} \right\|_{L_2}}{\left\| p^{rf} \right\|_{L_2}}, \quad \frac{\left\| p^{rf} - p^c_{rec} \right\|_{C}}{\left\| p^{rf} \right\|_{C}}. \tag{26}$$

Numerical results demonstrating convergence of the pressure with the first order on the coarse grid $O(H)$ are summarized in Tab. 2. The following notations are used: $Nx \times Ny$ is the number of coarse blocks in the domain Ω, $nx \times ny$ is the number of fine blocks in the coarse one.

From Tab. 2 it is easy to see that in this case, the coarse grid solution converges to fine grid one with $O(H)$. Fig. 7, 8 visualize the results from Tab. 2. From these pictures we can see that discussed here two-level permeability upscaling converges for considered problem while MsFEM and HMM diverges (see [15]).

We also consider influence of different local flow formulations on the convergence of the method. From Fig. 9, 10 it is easy to see that pressure drop no-flow boundary conditions (14) and oscillatory boundary conditions (15) give better results, and linear local flow boundary conditions (13) don't provide the convergence.

To be more sure that permeability upscaling gives more accurate results, we consider the example from the paper [8]. By this example, we compare performance of two-level upscaling procedure with multiscale finite element method. In domain $\bar{\Omega} = [0,1] \times [0,1]$ consider pressure equation (1) with homogeneous Dirichlet boundary conditions ($p = 0$, on $\partial\Omega$) and

$N_x \times N_y$	$n_x \times n_y$	$\frac{\left\|\langle p^{rf}\rangle - p^c\right\|_{L_2}}{\left\|p^{rf}\right\|_{L_2}}$	$\frac{\left\|\langle p^{rf}\rangle - p^c\right\|_{C}}{\left\|p^{rf}\right\|_{C}}$	$\frac{\left\|p^{rf} - p^c_{rec}\right\|_{L_2}}{\left\|p^{rf}\right\|_{L_2}}$	$\frac{\left\|p^{rf} - p^c_{rec}\right\|_{C}}{\left\|p^{rf}\right\|_{C}}$
4×4	8×8	0.1889	0.1714	0.0643	0.1362
8×8	8×8	0.1372	0.1210	0.0904	0.1267
16×16	8×8	0.0195	0.0280	0.0227	0.0500
32×32	8×8	0.0132	0.0203	0.0132	0.0380
64×64	8×8	0.0068	0.0114	0.0080	0.0253
128×128	8×8	0.0048	0.0060	0.0051	0.0123
256×256	8×8	0.0031	0.0035	0.0032	0.0059
4×4	16×16	0.2544	0.1898	0.0961	0.1750
8×8	16×16	0.0515	0.0507	0.0251	0.0634
16×16	16×16	0.0214	0.0286	0.0175	0.0475
32×32	16×16	0.0107	0.0171	0.0114	0.0345
64×64	16×16	0.0061	0.0107	0.0075	0.0256
128×128	16×16	0.0046	0.0059	0.0049	0.0122

Table 2. Convergence of the pressure for problem (8)

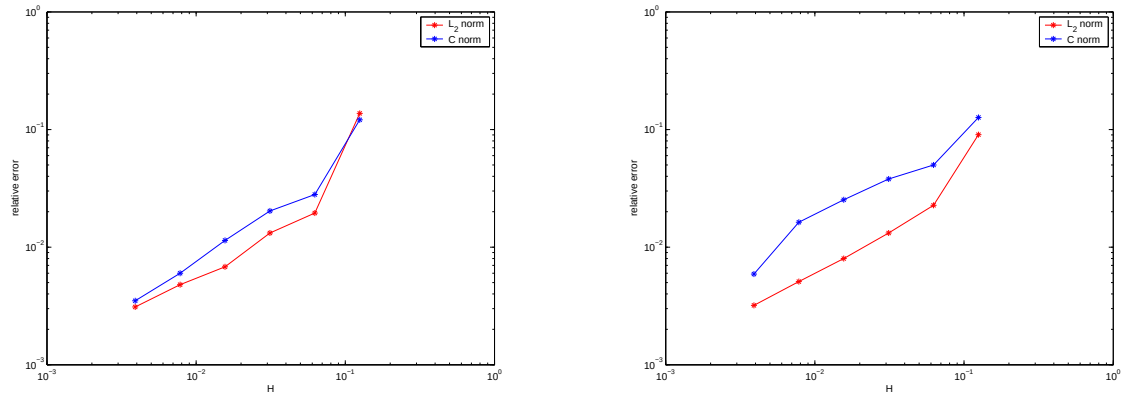

Figure 7. Results for $n = 8$ according to criterium (25) (left) and criterium (26) (right)

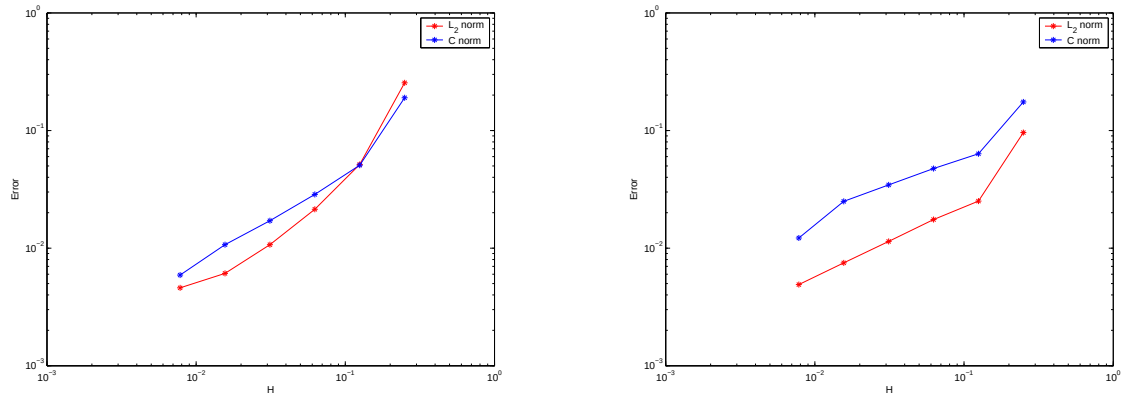

Figure 8. Results for $n = 16$ according to criterium (25) (left) and criterium (26) (right)

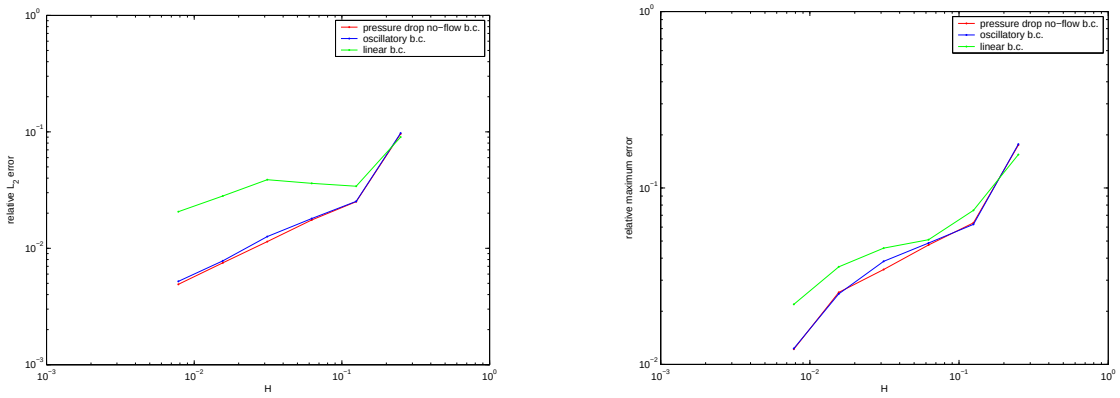

Figure 9. Compare volume average of reference solution with coarse solution for $n = 16$ (criterium (25))

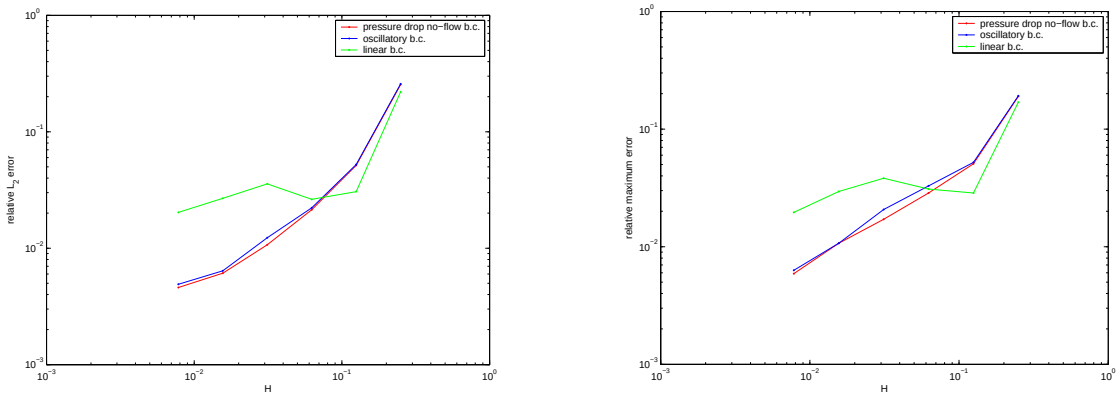

Figure 10. Compare reference solution with reconstructed for $n = 16$ (criterium (26))

constant right-hand side (f = -1). The problem is characterized by highly heterogeneous isotropic continuous coefficients

$$a^{\varepsilon}(x,y) = \frac{2 + P\sin(2\pi x/\varepsilon)}{2 + P\cos(2\pi y/\varepsilon)} + \frac{2 + \sin(2\pi y/\varepsilon)}{2 + P\sin(2\pi x/\varepsilon)}, \quad P = 1.8. \tag{27}$$

In Tab. 3, 4 we compare results from two-level upscaling procedure with the results obtain by MsFEM in the paper [8]. These results are marked by **(HW)**.

N	n	$\|p - p_h\|_{L_2}$ **(HW)**	$\|p - p_h\|_C$ **(HW)**	$\|\langle p^{rf}\rangle - p^c\|_{L_2}$	$\|\langle p^{rf}\rangle - p^c\|_C$
32	64	2.52e-5	4.89e-5	3.79e-5	7.80e-5
64	32	5.79e-5	1.06e-4	1.35e-5	4.87e-5
128	16	9.65e-5	1.74e-4	7.40e-6	2.64e-5
256	8	2.10e-4	3.76e-4	1.28e-5	5.16e-5
512	4	9.88e-5	1.77e-4	7.11e-5	1.27e-4

Table 3. Results for $\varepsilon = 0.005$

From Tab. 3 we see that two-level upscaling procedure gives more accurate results. (Different error estimates for $h < \varepsilon$ and $h > \varepsilon$).

From Tab. 4 it is easy to see that we don't observe here resonance effect like in MsFEM.

N	ε	$\|p - p_h\|_{L_2}$ **(HW)**	$\left\|\langle p^{rf}\rangle - p^c\right\|_{L_2}$	$\left\|\langle p^{rf}\rangle - p^c\right\|_C$
16	64	6.23e-5	1.83e-4	2.99e-4
32	32	8.43e-5	7.08e-5	1.31e-4
64	16	9.32e-5	3.39e-5	6.95e-5
128	8	9.65e-5	7.40e-6	2.65e-5

Table 4. Results for $\varepsilon/h = 0.64$ and $n = 16$

5. Conclusions

The purpose of the paper was to develop and apply fine-coarse scale algorithm for solving elliptic problems describing flows in strongly heterogeneous porous media with unseparable scales.

The method combines multipoint flux approximation approach and permeability upscaling technique. Convergence properties of the discretization on a single grid is studied numerically for pressure and fluxes in energy and maximum norms.

Different local flow formulations are considered at the pre-processing step for efficient calculating the effective properties of the heterogeneous porous media. Numerical results are compared with known analytical solutions as well as with other numerical techniques.

Finally, developed fine-coarse scale algorithm is applied for solving multiscale problem with unseparable scales, for which MsFEM and HMM do not converge. We show that the developed algorithm successfully solves this problem.

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