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O. Iliev, I. Rybak

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Fraunhofer-Institut für Techno- und
Wirtschaftsmathematik ITWM
Fraunhofer-Platz 1

67663 Kaiserslautern
Germany

Telefon: +49(0)631/31600-0
Telefax: +49(0)631/31600-1099
E-Mail: info@itwm.fraunhofer.de
Internet: www.itwm.fraunhofer.de

Vorwort

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In der Reihe »Berichte des Fraunhofer ITWM« soll die Arbeit des Instituts kontinuierlich einer interessierten Öffentlichkeit in Industrie, Wirtschaft und Wissenschaft vorgestellt werden. Durch die enge Verzahnung mit dem Fachbereich Mathematik der Universität Kaiserslautern sowie durch zahlreiche Kooperationen mit internationalen Institutionen und Hochschulen in den Bereichen Ausbildung und Forschung ist ein großes Potenzial für Forschungsberichte vorhanden. In die Berichtreihe sollen sowohl hervorragende Diplom- und Projektarbeiten und Dissertationen als auch Forschungsberichte der Institutsmitarbeiter und Institutsgäste zu aktuellen Fragen der Techno- und Wirtschaftsmathematik aufgenommen werden.

Darüber hinaus bietet die Reihe ein Forum für die Berichterstattung über die zahlreichen Kooperationsprojekte des Instituts mit Partnern aus Industrie und Wirtschaft.

Berichterstattung heißt hier Dokumentation des Transfers aktueller Ergebnisse aus mathematischer Forschungs- und Entwicklungsarbeit in industrielle Anwendungen und Softwareprodukte – und umgekehrt, denn Probleme der Praxis generieren neue interessante mathematische Fragestellungen.

Dieter Prätzel-Wolters

Prof. Dr. Dieter Prätzel-Wolters
Institutsleiter

Kaiserslautern, im Juni 2001

ON APPROXIMATION PROPERTY OF MULTIPOINT FLUX APPROXIMATION METHOD

O.P. ILIEV AND I.V. RYBAK

Abstract. Approximation property of multipoint flux approximation (MPFA) approach for elliptic equations with discontinuous full tensor coefficients is discussed here. Finite volume discretization of the above problem is presented in the case of jump discontinuities for the permeability tensor. First order approximation for the fluxes is proved. Results from numerical experiments are presented and discussed.

Keywords: Multipoint flux approximation, finite volume method, elliptic equation, discontinuous coefficients, anisotropy

1. Introduction

The paper concerns the approximation property of multipoint flux approximation (MPFA) approach applied to second order elliptic equation with discontinuous tensor coefficients. MPFA is a finite volume discretization in which the flux over a face is discretized using several grid points, in contrast to the standard two point approximation for the flux in the case of orthotropic problems. Recall that orthotropic problems are characterized by a diagonal coefficients tensor, while full tensor describes the coefficients of anisotropic problems. One of the first articles introducing MPFA were [8, 10], where MPFA was used in conjunction with simulation of porous media flow. No theoretical analysis was presented there. During the years, the method became popular due to its robustness and good accuracy. It was extensively used for solving applied problems, especially in geoscience. This risen the interest in a theoretical analysis of this approach, and some approximation and convergence results were published recently by Aavatsmark [1, 2], Klausen, Winther [4, 5], Wheeler, Yotov [3, 9]. Although the method was originally derived especially for problems with discontinuous tensor coefficients, the presented theoretical results concern MPFA applied to problems with continuous tensor coefficients. There are only numerical studies of the convergence order for discontinuous coefficients. The theoretical results presented in the above articles were obtained by showing equivalence of MPFA approximation to certain mixed finite element method, what allowed to use the well developed theory of mixed finite element method. First order convergence in W_2^1 was proven, recall that the analysis was done for the case of smooth coefficients.

In the current article, we consider MPFA in the context of the theory of finite difference schemes [7, 11, 12]. We prove first order convergence for the MPFA method in discrete W_2^1 norm. More precisely, we write down a specific derivation of MPFA, and show that the components of the continuous flux, $\mathbf{W} = (W_1, W_2)^T$, are approximated with $O(h)$ in the midpoints of the edges by the components of the discrete flux, $\mathbf{W_h} = (W_{1,h}, W_{2,h})^T$. Further on, we use a priori estimates like (14) on

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p.14 in [12]. This estimate allows to obtain convergence result in discrete W_2^1 norm, if the error of a finite difference scheme can be written in divergence form, and if the approximation error for the fluxes is known. as it is often done in the analysis of finite difference schemes, we assume that the solution is piecewise smooth even when the coefficients have jump discontinuity. Obviously, our analysis is valid also for the case applying MPFA method for problems with continuous tensor coefficients. When the coefficients are discontinuous, often the solution has not enough regularity. However, for particular combination of the discontinuous coefficients, the solution may be smooth enough for our needs. We refer to [6] for a detailed discussion on the regularity of the solution in the case of scalar discontinuous coefficient.

So, we show that $W_1(x_{i+1/2}, y_j) - W_{1,h}(x_{i+1/2}, y_j) = O(h)$, where $(x_{i+1/2}, y_j)$ is the middle point of an edge. We proceed with MPFA written in a specific way.

We consider a cell-centered basic grid, as well as a dual grid. The finite volume discretization is obtained on the original grid by applying divergence theorem. The flux over an edge is split into two parts, so that we have two subfluxes contributions over each edge: one for $(x_{i+1/2}, y_j)$, $(x_{i+1/2}, y_{j+1/2})$, and another for $(x_{i+1/2}, y_{j-1/2})$, $(x_{i+1/2}, y_j)$. We will show the approximation only for the first subflux. Now we continue the discretization over the dual grid. For each of the four subcells of the dual cell, we will build **interpolating** linear polynomials. They interpolate the solution in the nodes of the original grid, and in the middle of the edges of the cells from the original grid, e.g., in (x_i, y_j), $(x_{i+1/2}, y_j)$ and in $(x_i, y_{j+1/2})$. In fact, this is similar to one of the ways in which MPFA is derived by solving system of 4 equations with 4 unknowns. Our aim will be to show that the normal component of the flux, W^1 in this case, is approximated with $O(h)$ by the flux of this interpolating polynomial.

2. Continuous problem and finite volume discretization

In this section, we describe the mathematical model and the discretization approach. The derivation is based on the finite volume method (method of balance) and on multipoint flux approximation approach (MPFA).

2.1. **Statement of the problem.** In rectangular domain $\bar{\Omega} = \Omega \cup \partial\Omega$, we consider two-dimensional pressure equation obtained by combining the continuity equation ($\nabla \cdot \mathbf{W} = f$) and Darcy's law ($\mathbf{W} = -K\nabla u$) for steady state incompressible single phase flows in porous media

$$-\nabla \cdot (K\nabla u) = f, \quad \text{in } \Omega. \tag{1}$$

Boundary conditions complete the formulation of the problem

$$u = g^D, \text{ on } \Gamma_D, \quad K\nabla u \cdot \mathbf{n} = g^N, \text{ on } \Gamma_N, \quad \partial\Omega = \Gamma_D \cup \Gamma_N. \tag{2}$$

Here u is unknown pressure, f is the source, the set Γ_D is non-empty and had positive surface measure, the permeability tensor is full, symmetric, and uniformly positive definite in Ω:

$$K = \begin{pmatrix} k_{11} & k_{12} \\ k_{12} & k_{22} \end{pmatrix} > 0, \quad k_{12} \neq 0.$$

The entries of the permeability tensor K may have jump discontinuities along certain interfaces that are parallel to the coordinate planes and along these interfaces the perfect contact interface conditions are satisfied

$$[u] = 0, \quad [K\nabla u \cdot \mathbf{n}] = 0,$$

where $[v] = v(\eta + 0) - v(\eta - 0)$ for the interface η.

We suppose that the following ellipticity conditions are satisfied as well

$$0 < c_1 \sum_{\alpha=1}^{2} \xi_\alpha^2 \le \sum_{\alpha,\,\beta=1}^{2} k_{\alpha\beta}\xi_\alpha\xi_\beta \le c_2 \sum_{\alpha=1}^{2} \xi_\alpha^2, \quad c_1, c_2 > 0, \tag{3}$$

where $c_1,\ c_2 > 0$ are positive constants, $\xi = (\xi_1, \xi_2)$ is an arbitrary nonzero vector $|\xi| = \xi_1^2 + \xi_2^2 \neq 0$.

Remark. Equation (1) describes also heat conductivity in composite materials, diffusion in heterogeneous media, stationary distribution of electric and magnetic field, etc.

2.2. **Finite volume discretization.** The domain Ω is partitioned into blocks Ω_{ij} so that the discontinuities of the permeability tensor K are aligned with cell boundaries. The centers of the cells Ω_{ij} are denoted by (x_i, y_j) and the cell vertexes are the points $(x_i \pm \frac{1}{2}h_1, y_j \pm \frac{1}{2}h_2)$. The mesh that will be used to approximate the pressure will include all cell centers (x_i, y_j). This mesh will be called *primary mesh* $\omega_h = \{(x_i, y_j) : \Omega_{ij}\}$. Similarly we shall use also the mesh of all cell vertexes, called often *dual mesh*. The velocities will be calculated at the points $(x_i \pm \frac{1}{2}h_1, y_j)$ and $(x_i, y_j \pm \frac{1}{2}h_2)$.

The continuity equation ($\nabla \cdot \mathbf{W} = f$) is integrated over control volume Ω_{ij} and making use of the divergence theorem, we obtain

$$\int\limits_{\Omega_{ij}} \nabla \cdot \mathbf{W} d\mathbf{x} = \int\limits_{\Omega_{ij}} f d\mathbf{x} \quad \Rightarrow \quad \int\limits_{\partial\Omega_{ij}} \mathbf{W} \cdot \mathbf{n} ds = \int\limits_{\Omega_{ij}} f d\mathbf{x}. \tag{4}$$

Replacing the velocity $\mathbf{W}$ in (4) by certain approximation involving u by using the Darcy's relation $\mathbf{W} = -K\nabla u$ we get a conservative method [7]. In this approximation we assume that the unknowns (or degrees of freedom) are the values of the pressure at the cell centers and then use these values to recover the velocity $\mathbf{W}$. According to the multipoint flux approximation this is done in the following manner. First we split each control volume

$$\Omega_{ij} = \left(x_i - \frac{1}{2}h_1, x_i + \frac{1}{2}h_1\right) \times \left(y_j - \frac{1}{2}h_2, y_j + \frac{1}{2}h_2\right)$$

into 4 subvolumes $\Omega_{ij}^{I} = (x_i - \frac{1}{2}h_1, x_i) \times (y_j, y_j + \frac{1}{2}h_2)$, $\Omega_{ij}^{II} = (x_i - \frac{1}{2}h_1, x_i) \times (y_j, y_j - \frac{1}{2}h_2)$, $\Omega_{ij}^{III} = (x_i, x_i + \frac{1}{2}h_1) \times (y_j, y_j + \frac{1}{2}h_2)$, $\Omega_{ij}^{IV} = (x_i + \frac{1}{2}h_1, x_i) \times (y_j, y_j - \frac{1}{2}h_2)$.

We take the pressure to be a linear function on each subvolume Ω_{ij}^{k} so that

$$u = \alpha^k x + \beta^k y + \gamma^k, \quad k = \overline{1, 4}. \tag{5}$$

The coefficients α^k, β^k and γ^k in (5) determined by the following conditions:
(1c) the polynomials interpolate pressure values at the volume centers;
(2c) the continuity of the pressure at the centers of the faces of the volume Ω_{ij} and the pressure data on faces that are part of Γ_D;
(3c) the continuity of the normal component of the velocity $\mathbf{v}$ at the centers of the faces of the volume Ω_{ij} and the boundary data for the normal velocity on faces on Γ_N.

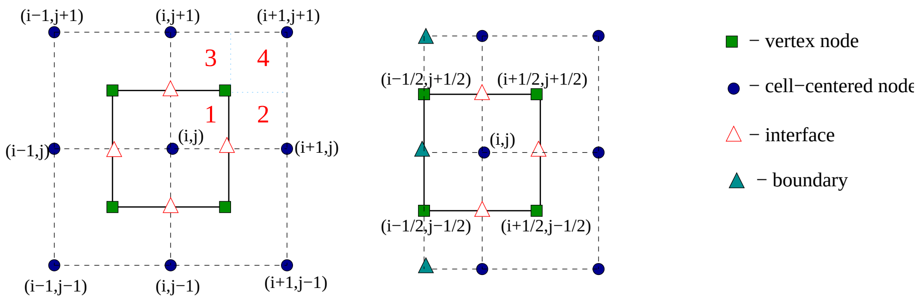

FIGURE 1. Control volume: inner and boundary cells

Conditions (1c)–(3c) correspond to O-method with surface midpoints as continuity points [1]. These conditions are applied on a cell from the dual grid, i.e. a cell centered at a vertex point from the dual grid (see interaction volume on Fig. 1). These cells are of three categories: cells corresponding to internal vertices, cells corresponding to boundary vertices, and 4 corner points of the domain Ω.

Consider an internal vertex that is surrounded by four subcells with $u_{i,j+1}$, $u_{i+1,j+1}$, $u_{i,j}$, $u_{i+1,j}$ at the corners. To find the polynomial coefficients from (5), we use conditions (1c)–(3c). Note that four interaction volumes should be considered in order to find α^k, β^k, γ^k, $k=\overline{1,4}$.

The interpolating polynomials are defined by

$$
\begin{aligned}
P^1(x,y) &= \frac{u_{i+1/2,j}-u_{i,j}}{0.5h_1}(x+0.5h_1)+\frac{u_{i,j+1/2}-u_{i,j}}{0.5h_2}(y+0.5h_2)+u_{i,j}\\
&= a^1x+b^1y+c^1,\\
P^2(x,y) &= \frac{u_{i+1,j}-u_{i+1/2,j}}{0.5h_1}(x-0.5h_1)+\frac{u_{i+1,j+1/2}-u_{i+1,j}}{0.5h_2}(y+0.5h_2)\\
&\quad +u_{i+1,j}=a^2x+b^2y+c^2,\\
P^3(x,y) &= \frac{u_{i+1,j+1}-u_{i+1/2,j+1}}{0.5h_1}(x-0.5h_1)+\frac{u_{i+1,j+1}-u_{i+1,j+1/2}}{0.5h_2}(y-0.5h_2)\\
&\quad +u_{i+1,j+1}=a^3x+b^3y+c^3,\\
P^4(x,y) &= \frac{u_{i+1/2,j+1}-u_{i,j+1}}{0.5h_1}(x+0.5h_1)+\frac{u_{i,j+1}-u_{i,j+1/2}}{0.5h_2}(y-0.5h_2)\\
&\quad +u_{i,j+1}=a^4x+b^4y+c^4,
\end{aligned}
\tag{6}
$$

where $u_{i+1/2,j}=u(x_{i+1/2},y_j)$, $u_{i,j}=u(x_i,y_j)$, etc.

Similar to the condition for continuity of normal component of fluxes of the solution, we require these *interpolating* polynomials also to satisfy condition for continuity of normal component of their fluxes:

$$
\begin{aligned}
&k_{11}\frac{u_{i+1/2,j}-u_{i,j}}{0.5h_1}+k_{12}\frac{u_{i,j+1/2}-u_{i,j}}{0.5h_2}=\\
=&k_{11}^{(+11)}\frac{u_{i+1,j}-u_{i+1/2,j}}{0.5h_1}+k_{12}^{(+11)}\frac{u_{i+1,j+1/2}-u_{i+1,j}}{0.5h_2},\\
&k_{12}\frac{u_{i+1/2,j}-u_{i,j}}{0.5h_1}+k_{22}\frac{u_{i,j+1/2}-u_{i,j}}{0.5h_2}=\\
=&k_{12}^{(+12)}\frac{u_{i+1/2,j+1}-u_{i,j+1}}{0.5h_1}+k_{22}^{(+12)}\frac{u_{i,j+1}-u_{i,j+1/2}}{0.5h_2},
\end{aligned}
\tag{7}
$$

$$
\begin{aligned}
&k_{11}^{(+12)}\frac{u_{i+1/2,j+1}-u_{i,j+1}}{0.5h_1}+k_{12}^{(+12)}\frac{u_{i,j+1}-u_{i,j+1/2}}{0.5h_2}=\\
=&k_{11}^{(+11,+12)}\frac{u_{i+1,j+1}-u_{i+1/2,j+1}}{0.5h_1}+k_{12}^{(+11,+12)}\frac{u_{i+1,j+1}-u_{i+1,j+1/2}}{0.5h_2},\\
&k_{12}^{(+11)}\frac{u_{i+1,j}-u_{i+1/2,j}}{0.5h_1}+k_{22}^{(+11)}\frac{u_{i+1,j+1/2}-u_{i+1,j}}{0.5h_2}=\\
=&k_{12}^{(+11,+12)}\frac{u_{i+1,j+1}-u_{i+1/2,j+1}}{0.5h_1}+k_{22}^{(+11,+12)}\frac{u_{i+1,j+1}-u_{i+1,j+1/2}}{0.5h_2}.
\end{aligned}
$$

As an auxiliary step, we write the above equalities as a system (7) with respect to unknown values at the midpoint of edges:

$$
Av=w, \tag{8}
$$

where $v=(u_{i+1/2,j},u_{i,j+1/2},u_{i+1/2,j+1},u_{i+1,j+1/2})^T$, $w=(w_1,w_2,w_3,w_4)^T$,

$$
A=\begin{pmatrix}
\frac{k_{11}+k_{11}^{(+11)}}{h_1} & \frac{k_{12}}{h_2} & 0 & -\frac{k_{12}^{(+11)}}{h_2}\\
\frac{k_{12}}{h_1} & \frac{k_{22}+k_{22}^{(+12)}}{h_2} & -\frac{k_{12}^{(+12)}}{h_1} & 0\\
0 & -\frac{k_{12}^{(+12)}}{h_2} & \frac{k_{11}^{(+12)}+k_{11}^{(+11,+12)}}{h_1} & \frac{k_{12}^{(+11,+12)}}{h_2}\\
-\frac{k_{12}^{(+11)}}{h_1} & 0 & \frac{k_{12}^{(+11,+12)}}{h_1} & \frac{k_{22}^{(+11)}+k_{22}^{(+11,+12)}}{h_2}
\end{pmatrix}.
$$

Here

$$
\begin{aligned}
w_1&=\left(\frac{k_{11}^{(+11)}}{h_1}-\frac{k_{12}^{(+11)}}{h_2}\right)u_{i+1,j}+\left(\frac{k_{11}}{h_1}+\frac{k_{12}}{h_2}\right)u_{i,j},\\
w_2&=\left(\frac{k_{12}}{h_1}+\frac{k_{22}}{h_2}\right)u_{i,j}-\left(\frac{k_{12}^{(+12)}}{h_1}-\frac{k_{22}^{(+12)}}{h_2}\right)u_{i,j+1},\\
w_3&=\left(\frac{k_{11}^{(+12)}}{h_1}-\frac{k_{12}^{(+12)}}{h_2}\right)u_{i,j+1}+\left(\frac{k_{11}^{(+11,+12)}}{h_1}+\frac{k_{12}^{(+11,+12)}}{h_2}\right)u_{i+1,j+1},\\
w_4&=\left(-\frac{k_{12}^{(+11)}}{h_1}+\frac{k_{22}^{(+11)}}{h_2}\right)u_{i+1,j}+\left(\frac{k_{12}^{(+11,+12)}}{h_1}+\frac{k_{22}^{(+11,+12)}}{h_2}\right)u_{i+1,j+1}.
\end{aligned}
$$

Now, from the auxiliary step we come to reformulating the system in the form we need. The quantities which are of interest for us at this stage are the approximations to the derivatives of the solution. These new variables coincide with the coefficients of the polynomials $P^i(x,y)$):

$$
a^1=\frac{u_{i+1/2,j}-u_{i,j}}{0.5h_1},\quad b^1=\frac{u_{i,j+1/2}-u_{i,j}}{0.5h_2},
$$
$$
a^3=\frac{u_{i+1,j+1}-u_{i+1/2,j+1}}{0.5h_1},\quad b^3=\frac{u_{i+1,j+1}-u_{i+1,j+1/2}}{0.5h_2}.
$$

So, we rewrite system (8) in new variables $\tilde{A}\tilde{v}=\tilde{w}$ with $\tilde{v}=(a^1,b^1,a^3,b^3)^T$. The matrix of the system is

$$
\tilde{A}=\begin{pmatrix}
k_{11}+k_{11}^{(+11)} & k_{12} & 0 & k_{12}^{(+11)}\\
k_{12} & k_{22}+k_{22}^{(+12)} & k_{12}^{(+12)} & 0\\
0 & k_{12}^{(+12)} & k_{11}^{(+12)}+k_{11}^{(+11,+12)} & k_{12}^{(+11,+12)}\\
k_{12}^{(+11)} & 0 & k_{12}^{(+11,+12)} & k_{22}^{(+11)}+k_{22}^{(+11,+12)}
\end{pmatrix}, \tag{9}
$$

and the right-hand side is given by

$$\tilde{w}_1 = 2\left(k_{11}^{(+11)}\frac{u_{i+1,j}-u_{i,j}}{h_1} + k_{12}^{(+11)}\frac{u_{i+1,j+1}-u_{i+1,j}}{h_2}\right),$$
$$\tilde{w}_2 = 2\left(k_{yx}^{(+12)}\frac{u_{i+1,j+1}-u_{i,j+1}}{h_1} + k_{22}^{(+12)}\frac{u_{i,j+1}-u_{i,j}}{h_2}\right),$$
$$\tilde{w}_3 = 2\left(k_{11}^{(+12)}\frac{u_{i+1,j+1}-u_{i,j+1}}{h_1} + k_{12}^{(+12)}\frac{u_{i,j+1}-u_{i,j}}{h_2}\right),$$
$$\tilde{w}_4 = 2\left(k_{12}^{(+11)}\frac{u_{i+1,j}-u_{i,j}}{h_1} + k_{22}^{(+11)}\frac{u_{i+1,j+1}-u_{i+1,j}}{h_2}\right).$$

Coefficients a^i, b^i, $i = 2, 4$, can be expressed through a^i, b^i, $i = 1, 3$ and the pressure values at the cell centers

$$\begin{aligned} a^2 &= \frac{u_{i+1,j}-u_{i,j}}{0.5h_1} - a^1, \quad b^2 = \frac{u_{i+1,j+1}-u_{i+1,j}}{0.5h_2} - b^3 \\ a^4 &= \frac{u_{i+1,j+1}-u_{i,j+1}}{0.5h_1} - a^3, \quad b^4 = \frac{u_{i,j+1}-u_{i,j}}{0.5h_2} - b^1. \end{aligned} \tag{10}$$

Note, that it gives us the expressions for the velocity that is constant over each of the 4 subcells of the vertex-centered volume. Consider in the same way three other vertex-centered volumes to find the fluxes incoming and outcoming the cell-centered (control volume) Ω_{ij}. These formulas are used to find $\mathbf{W}\cdot\mathbf{n}$ on $\partial\Omega_{ij}$, as needed by the relation (4).

We use the midpoint rule for calculating the integrals in the balance method (4), that provides

$$h_2 W_{1,h}^{out} - h_2 W_{1,h}^{in} + h_1 W_{2,h}^{out} - h_1 W_{2,h}^{in} = h_1 h_2 f.$$

Here $W_{i,h}^{in}$, $W_{i,h}^{out}$, $i = 1, 2$ are obtained by summing the two fluxes incoming and leaving the considered control volume (Fig. 1). Thus, the difference scheme for the pressure equation (1) can be written in the following form

$$\begin{aligned} &\frac{h_2}{2}\left(k_{11}a^2 + k_{12}b^2 + k_{11}^{(+11)}\breve{a}^3 + k_{12}^{(+11)}\breve{b}^3\right) - \\ &\frac{h_2}{2}\left(k_{11}^{(-11)}\bar{a}^2 + k_{12}^{(-11)}\bar{b}^2 + k_{11}\breve{\bar{a}}^3 + k_{12}\breve{\bar{b}}^3\right) + \\ &\frac{h_1}{2}\left(k_{12}a^2 + k_{22}b^2 + k_{12}^{(+12)}\bar{a}^3 + k_{22}^{(+12)}\bar{b}^3\right) - \\ &\frac{h_2}{2}\left(k_{12}^{(-12)}\breve{a}^2 - k_{22}^{(-12)}\breve{b}^2 - k_{12}\breve{\bar{a}}^3 - k_{22}\breve{\bar{b}}^3\right) = -f h_1 h_2, \end{aligned} \tag{11}$$

where coefficients $\bar{a}^i$, $\bar{b}^i$ are calculated by the same formulas as the coefficients a^i, b^i, but in the grid block $u_{i-1,j+1}u_{i,j+1}u_{i,j}u_{i-1,j}$. The coefficients $\breve{a}^i$, $\breve{b}^i$ and $\breve{\bar{a}}^i$, $\breve{\bar{b}}^i$ are calculated in the grid blocks $u_{i,j}u_{i+1,j}u_{i+1,j-1}u_{i,j-1}$ and $u_{i-1,j}u_{i,j}u_{i,j-1}u_{i-1,j-1}$ respectively.

For vertex that is on the boundary the situation is simpler. In the case of Neumann boundary conditions the flux is given on the boundary, while in the case of Dirichlet boundary conditions we just do the same procedure as for the inner control volume, but in this case we have only one interface.

Combining this relationship for each neighboring vertex gives us a discrete pressure equation with a 9-point stencil.

Remark. In the case of diagonal (orthotropic) permeability tensor ($k_{12} = 0$), scheme (11) reduces to the well-known harmonic averaging finite difference scheme

$$\frac{1}{h_1}\left(\frac{2k_{11}k_{11}^{(+11)}}{k_{11}+k_{11}^{(+11)}}\frac{u_{i+1,j}-u_{i,j}}{h_1} - \frac{2k_{11}k_{11}^{(-11)}}{k_{11}+k_{11}^{(-11)}}\frac{u_{i,j}-u_{i-1,j}}{h_1}\right) +$$
$$\frac{1}{h_2}\left(\frac{2k_{22}k_{22}^{(+12)}}{k_{22}+k_{22}^{(+12)}}\frac{u_{i,j+1}-u_{i,j}}{h_2} - \frac{2k_{22}k_{22}^{(-12)}}{k_{22}+k_{22}^{(-12)}}\frac{u_{i,j}-u_{i,j-1}}{h_2}\right) = -f.$$

Remark. In the case, the coefficients k_{11}, k_{12}, k_{22} are constant, the difference scheme (11) can be written in the following form

$$k_{11}u_{x\bar{x}} + k_{12}\frac{u_{\bar{x}y}+u_{x\bar{y}}}{2} + k_{12}\frac{u_{\bar{x}\bar{y}}+u_{xy}}{2} + k_{xy}u_{y\bar{y}} + R = -f,$$

where

$$R = \frac{k_{12}^2}{4}\left(\frac{h_1^2}{k_{11}} + \frac{h_2^2}{k_{22}}\right)u_{\bar{x}x\bar{y}y} = O\left(h^2\right).$$

So, in this case the finite volume scheme (11) is equivalent to the second-order finite difference scheme from [7]:

$$k_{11}u_{x\bar{x}} + 0.5k_{12}\left(u_{\bar{x}y} + u_{x\bar{y}} + u_{\bar{x}\bar{y}} + u_{xy}\right) + k_{22}u_{y\bar{y}} = -f.$$

3. Approximation and convergence properties

In this section, we prove the first order approximation for the fluxes in the middle points of the edges in the case of discontinuous tensor coefficients.

3.1. **Approximation order of fluxes.** We examine the order of the approximation of the fluxes in the case of discontinuous tensor coefficients and piecewise smooth solution. To do this, consider the condition for the continuity of normal component of fluxes of the PDE solution at midpoints of edges, and its discrete approximation be the normal components of fluxes of the above derived piecewise linear polynomials. By considering the difference of the both, we will get expressions for the approximation error. After certain manipulations, we show that the approximation is $O(h)$.

To start, consider the flux continuity condition at the point $(x_{i+1/2}, y_j)$:

$$\begin{aligned} &k_{11}\frac{\partial u}{\partial x}\Big|_{(x_{i+1/2-0},y_j)} + k_{12}\frac{\partial u}{\partial y}\Big|_{(x_{i+1/2-0},y_j)} = \\ &k_{11}^{(+11)}\frac{\partial u}{\partial x}\Big|_{(x_{i+1/2+0},y_j)} + k_{12}^{(+11)}\frac{\partial u}{\partial y}\Big|_{(x_{i+1/2+0},y_j)}, \end{aligned} \tag{12}$$

and its discrete approximation (see formula (7)):

$$\begin{aligned} &k_{11}\frac{u_{i+1/2,j}-u_{i,j}}{0.5h_1} + k_{12}\frac{u_{i,j+1/2}-u_{i,j}}{0.5h_2} = \\ &k_{11}^{(+11)}\frac{u_{i+1,j}-u_{i+1/2,j}}{0.5h_1} + k_{12}^{(+11)}\frac{u_{i+1,j+1/2}-u_{i+1,j}}{0.5h_2}. \end{aligned} \tag{13}$$

By subtracting equation (12) from (13), we obtain

$$
\begin{aligned}
&k_{11}\left(\frac{u_{i+1/2,j}-u_{i,j}}{0.5h_1}-\frac{\partial u}{\partial x}\Big|_{(x_{i+1/2-0},y_j)}\right)+\\
&k_{12}\left(\frac{u_{i,j+1/2}-u_{i,j}}{0.5h_2}-\frac{\partial u}{\partial y}\Big|_{(x_{i+1/2-0},y_j)}\right)=\\
&k_{11}^{(+11)}\left(\frac{u_{i+1,j}-u_{i+1/2,j}}{0.5h_1}-\frac{\partial u}{\partial x}\Big|_{(x_{i+1/2+0},y_j)}\right)+\\
&k_{12}^{(+11)}\left(\frac{u_{i+1,j+1/2}-u_{i+1,j}}{0.5h_2}-\frac{\partial u}{\partial y}\Big|_{(x_{i+1/2+0},y_j)}\right).
\end{aligned}
\tag{14}
$$

From polynomials (6) we have

$$
\begin{aligned}
\frac{u_{i+1/2,j}-u_{i,j}}{0.5h_1}&=\frac{\partial P^1}{\partial x}, &\frac{u_{i+1,j}-u_{i+1/2,j}}{0.5h_1}&=\frac{\partial P^2}{\partial x},\\
\frac{u_{i,j+1/2}-u_{i,j}}{0.5h_2}&=\frac{\partial P^1}{\partial y}, &\frac{u_{i+1,j+1/2}-u_{i+1,j}}{0.5h_2}&=\frac{\partial P^2}{\partial y}.
\end{aligned}
$$

Substituting these expressions into equation (14), we get

$$
\begin{aligned}
&k_{11}\left(\frac{\partial P^1}{\partial x}-\frac{\partial u}{\partial x}\Big|_{(x_{i+1/2-0},y_j)}\right)+k_{12}\left(\frac{\partial P^1}{\partial y}-\frac{\partial u}{\partial y}\Big|_{(x_{i+1/2-0},y_j)}\right)=\\
&k_{11}^{(+11)}\left(\frac{\partial P^2}{\partial x}-\frac{\partial u}{\partial x}\Big|_{(x_{i+1/2+0},y_j)}\right)+k_{12}^{(+11)}\left(\frac{\partial P^2}{\partial y}-\frac{\partial u}{\partial y}\Big|_{(x_{i+1/2+0},y_j)}\right).
\end{aligned}
\tag{15}
$$

Simple calculations give

$$
\frac{\partial P^2}{\partial x}=\frac{u_{i+1,j}-u_{i,j}}{0.5h_1}-\frac{\partial P^1}{\partial x}.
$$

Using the Taylor expansion at the point e, from the last expression we obtain

$$
\frac{\partial P^2}{\partial x}=\frac{\partial u}{\partial x}\Big|_{(x_{i+1/2+0},y_j)}+\frac{\partial u}{\partial x}\Big|_{(x_{i+1/2-0},y_j)}-\frac{\partial P^1}{\partial x}+O(h_1).
\tag{16}
$$

Substituting the above relation in (15), we get

$$
\begin{aligned}
&k_{11}\left(\frac{\partial P^1}{\partial x}-\frac{\partial u}{\partial x}\Big|_{(x_{i+1/2-0},y_j)}\right)+k_{12}\left(\frac{\partial P^1}{\partial y}-\frac{\partial u}{\partial y}\Big|_{(x_{i+1/2-0},y_j)}\right)=\\
&k_{11}^{(+11)}\left(\left(\frac{\partial u}{\partial x}\Big|_{(x_{i+1/2+0},y_j)}+\frac{\partial u}{\partial x}\Big|_{(x_{i+1/2-0},y_j)}-\frac{\partial P^1}{\partial x}+O(h_1)\right)-\frac{\partial u}{\partial x}\Big|_{(x_{i+1/2+0},y_j)}\right)+\\
&k_{12}^{(+11)}\left(\frac{\partial P^2}{\partial y}-\frac{\partial u}{\partial y}\Big|_{(x_{i+1/2+0},y_j)}\right).
\end{aligned}
\tag{17}
$$

Consider new variables, z_1, z_2, which are the approximation error for x-derivative at the point $(x_{i+1/2-0}, y_j)$, and for y-derivative of the solution at point $(x_i, y_{j+1/2-0})$, respectively:

$$
z_1=\left(\frac{\partial P^1}{\partial x}-\frac{\partial u}{\partial x}\right)\Big|_{(x_{i+1/2-0},y_j)},\quad z_2=\left(\frac{\partial P^1}{\partial y}-\frac{\partial u}{\partial y}\right)\Big|_{(x_i,y_{j+1/2-0})}.
$$

Now, we assume that the solution is piecewise smooth within the cells, where the coefficients are piecewise constants. Under this assumption, we have

$$\left(\frac{\partial u}{\partial y}\right)\Big|_{(x_{i+1/2-0},y_j)} = \left(\frac{\partial u}{\partial y}\right)\Big|_{(x_i,y_{j+1/2-0})} + O(h)$$

and respectively,

$$\left(\frac{\partial P^1}{\partial y} - \frac{\partial u}{\partial y}\right)\Big|_{(x_{i+1/2-0},y_j)} = \left(\frac{\partial P^1}{\partial y} - \frac{\partial u}{\partial y}\right)\Big|_{(x_i,y_{j+1/2-0})} + O(h) = z_2 + O(h)$$

Thus, equation (17) can be rewritten as

$$\left(k_{11} + k_{11}^{(+11)}\right) z_1 + k_{12} z_2 = O(h_1) + k_{12}^{(+11)} \left(\frac{\partial P^2}{\partial y} - \frac{\partial u}{\partial y}\Big|_{(x_{i+1/2+0},y_j)}\right). \tag{18}$$

We leave for a while the above equation in this unfinished form, and continue with relations at the point $(x_i, y_{j+1/2})$. Similarly to the above derivations for the point $(x_{i+1/2}, y_j)$, we consider the flux continuity condition at the point $(x_i, y_{j+1/2})$:

$$\begin{aligned}
&k_{12}\left(\frac{u_{i+1/2,j} - u_{i,j}}{0.5h_1} - \frac{\partial u}{\partial x}\Big|_{(x_i,y_{j+1/2-0})}\right) + \\
&k_{22}\left(\frac{u_{i,j+1/2} - u_{i,j}}{0.5h_2} - \frac{\partial u}{\partial y}\Big|_{(x_i,y_{j+1/2-0})}\right) = \\
&k_{12}^{(+12)}\left(\frac{u_{i+1/2,j+1} - u_{i,j+1}}{0.5h_1} - \frac{\partial u}{\partial x}\Big|_{(x_i,y_{j+1/2+0})}\right) + \\
&k_{22}^{(+12)}\left(\frac{u_{i,j+1} - u_{i,j+1/2}}{0.5h_2} - \frac{\partial u}{\partial y}\Big|_{(x_i,y_{j+1/2+0})}\right).
\end{aligned} \tag{19}$$

Recall that derivatives of the polynomial and P^4 (6) are given by

$$\frac{u_{i+1/2,j+1} - u_{i,j+1}}{0.5h_1} = \frac{\partial P^4}{\partial x}, \quad \frac{u_{i,j+1} - u_{i,j+1/2}}{0.5h_2} = \frac{\partial P^4}{\partial y}.$$

Substituting these expressions together with similar expressions for P^1 into equation (19), we get

$$\begin{aligned}
&k_{12}\left(\frac{\partial P^1}{\partial x} - \frac{\partial u}{\partial x}\Big|_{(x_i,y_{j+1/2-0})}\right) + k_{22}\left(\frac{\partial P^1}{\partial y} - \frac{\partial u}{\partial y}\Big|_{(x_i,y_{j+1/2-0})}\right) = \\
&k_{12}^{(+12)}\left(\frac{\partial P^4}{\partial x} - \frac{\partial u}{\partial x}\Big|_{(x_i,y_{j+1/2+0})}\right) + k_{22}^{(+12)}\left(\frac{\partial P^4}{\partial y} - \frac{\partial u}{\partial y}\Big|_{(x_i,y_{j+1/2+0})}\right).
\end{aligned} \tag{20}$$

It is easy to see that

$$\frac{\partial P^4}{\partial y} = \frac{u_{i,j+1} - u_{i,j}}{0.5h_2} - \frac{\partial P^1}{\partial y} = \frac{\partial u}{\partial y}\Big|_{(x_i,y_{j+1/2+0})} + \frac{\partial u}{\partial y}\Big|_{(x_i,y_{j+1/2-0})} - \frac{\partial P^1}{\partial y} + O(h_2).$$

Substituting this into (20), and using the expansion

$$\left(\frac{\partial P^1}{\partial x} - \frac{\partial u}{\partial x}\right)\Big|_{(x_i,y_{j+1/2-0})} = \left(\frac{\partial P^1}{\partial x} - \frac{\partial u}{\partial x}\right)\Big|_{(x_{i+1/2-0},y_j)} + O(h) = z_1 + O(h),$$

we get

$$k_{12} z_1 + \left(k_{22} + k_{22}^{(+12)}\right) z_2 = k_{12}^{(+12)} \left(\frac{\partial P^4}{\partial x} - \frac{\partial u}{\partial x}\Big|_{(x_i,y_{j+1/2+0})}\right) + O(h). \tag{21}$$

Now we proceed with the remaining two edges, $(x_{i+1/2}, y_{j+1})$ and $(x_{i+1}, y_{j+1/2-0})$, as well as with remaining terms in the equations at $(x_{i+1/2+0}, y_j)$ and $(x_i, y_{j+1/2+0})$. First, we introduce the variables

$$z_3 = \left(\frac{\partial P^3}{\partial x} - \frac{\partial u}{\partial x} \right) \Big|_{(x_{i+1/2+0}, y_{j+1})}, \quad z_4 = \left(\frac{\partial P^3}{\partial y} - \frac{\partial u}{\partial y} \right) \Big|_{(x_{i+1}, y_{j+1/2+0})},$$

By simple calculations we get

$$\frac{\partial P^2}{\partial y} = \frac{\partial u}{\partial y}\Big|_{(x_{i+1}, y_{j+1/2+0})} + \frac{\partial u}{\partial y}\Big|_{(x_{i+1}, y_{j+1/2-0})} - \frac{\partial P^3}{\partial y} + O(h_2), \tag{22}$$

$$\frac{\partial P^4}{\partial x} = \frac{\partial u}{\partial x}\Big|_{(x_{i+1/2+0}, y_{j+1})} + \frac{\partial u}{\partial x}\Big|_{(x_{i+1/2-0}, y_{j+1})} - \frac{\partial P^3}{\partial x} + O(h_1). \tag{23}$$

Next, from the assumption that the solution is piecewise smooth, we have

$$\frac{\partial u}{\partial x}\Big|_{(x_{i+1}, y_{j+1/2-0})} = \frac{\partial u}{\partial x}\Big|_{(x_{i+1/2+0}, y_j)} + O(h),$$

$$\frac{\partial u}{\partial y}\Big|_{(x_{i+1/2+0}, y_j)} = \frac{\partial u}{\partial y}\Big|_{(x_{i+1}, y_{j+1/2-0})} + O(h).$$

Then using the above formula together with formula (22), we obtain

$$\begin{aligned} \frac{\partial P^2}{\partial y} - \frac{\partial u}{\partial y}\Big|_{(x_{i+1/2+0}, y_j)} &= \frac{\partial P^2}{\partial y} - \frac{\partial u}{\partial y}\Big|_{(x_{i+1}, y_{j+1/2-0})} + O(h) = \\ &- \left(\frac{\partial P^3}{\partial y} - \frac{\partial u}{\partial y}\Big|_{(x_{i+1}, y_{j+1/2+0})} \right) + O(h) = -z_4 + O(h). \end{aligned}$$

Thus, equation (18) can be written as

$$\left(k_{11} + k_{11}^{(+11)} \right) z_1 + k_{12} z_2 + k_{12}^{(+11)} z_4 = O(h_1).$$

In the same way we obtain

$$\begin{aligned} \frac{\partial P^4}{\partial x} - \frac{\partial u}{\partial x}\Big|_{(x_i, y_{j+1/2+0})} &= \frac{\partial P^4}{\partial x} - \frac{\partial u}{\partial x}\Big|_{(x_{i+1/2-0}, y_{j+1})} + O(h) = \\ &- \left(\frac{\partial P^3}{\partial x} - \frac{\partial u}{\partial x}\Big|_{(x_{i+1/2+0}, y_{j+1})} \right) + O(h) = -z_3 + O(h), \end{aligned}$$

and rewrite equation (21) in the following way

$$k_{12} z_1 + \left(k_{22} + k_{22}^{(+12)} \right) z_2 + k_{12}^{(+12)} z_3 = O(h).$$

The last what we need before finalizing the derivations, are the continuity conditions at the points $(x_{i+1/2}, y_{j+1})$ and $(x_{i+1}, y_{j+1/2-0})$:

$$\begin{aligned} &k_{11}^{(+12)} \left(\frac{\partial P^4}{\partial x} - \frac{\partial u}{\partial x}\Big|_{(x_{i+1/2-0}, y_{j+1})} \right) + k_{12}^{(+12)} \left(\frac{\partial P^4}{\partial y} - \frac{\partial u}{\partial y}\Big|_{(x_i, y_{j+1/2+0})} \right) = \\ &k_{11}^{(+11,+12)} \left(\frac{\partial P^3}{\partial x} - \frac{\partial u}{\partial x}\Big|_{(x_{i+1/2+0}, y_{j+1})} \right) + k_{12}^{(+11,+12)} \left(\frac{\partial P^3}{\partial y} - \frac{\partial u}{\partial y}\Big|_{(x_{i+1}, y_{j+1/2+0})} \right). \end{aligned} \tag{24}$$

$$
\begin{aligned}
&k_{12}^{(+11)}\left(\frac{\partial P^2}{\partial x}-\frac{\partial u}{\partial x}\Big|_{(x_{i+1/2+0},y_j)}\right)+k_{22}^{(+11)}\left(\frac{\partial P^2}{\partial y}-\frac{\partial u}{\partial y}\Big|_{(x_{i+1},y_{j+1/2-0})}\right)=\\
&k_{12}^{(+11,+12)}\left(\frac{\partial P^3}{\partial x}-\frac{\partial u}{\partial x}\Big|_{(x_{i+1/2+0},y_{j+1})}\right)+k_{12}^{(+11,+12)}\left(\frac{\partial P^3}{\partial y}-\frac{\partial u}{\partial y}\Big|_{(x_{i+1},y_{j+1/2+0})}\right).
\end{aligned}
\tag{25}
$$

Using similar approach and taking into account that

$$
\frac{u_{i+1,j+1}-u_{i+1/2,j+1}}{0.5h_1}=\frac{\partial P^3}{\partial x},\quad \frac{u_{i+1,j+1}-u_{i+1,j+1/2}}{0.5h_2}=\frac{\partial P^3}{\partial y},
$$

$$
\frac{\partial u}{\partial x}\Big|_{(x_{i+1},y_{j+1/2-0})}=\frac{\partial u}{\partial x}\Big|_{(x_{i+1/2+0},y_j)}+O(h),\quad \frac{\partial u}{\partial y}\Big|_{(x_{i+1/2-0},y_{j+1})}=\frac{\partial u}{\partial y}\Big|_{(x_i,y_{j+1/2+0})}+O(h),
$$

$$
\left(\frac{\partial P^3}{\partial x}-\frac{\partial u}{\partial x}\right)\Big|_{(x_{i+1},y_{j+1/2+0})}=\left(\frac{\partial P^3}{\partial x}-\frac{\partial u}{\partial x}\right)\Big|_{(x_{i+1/2+0},y_{j+1})}+O(h)=z_3+O(h),
$$

$$
\left(\frac{\partial P^3}{\partial y}-\frac{\partial u}{\partial y}\right)\Big|_{(x_{i+1/2+0},y_{j+1})}=\left(\frac{\partial P^3}{\partial y}-\frac{\partial u}{\partial y}\right)\Big|_{(x_{i+1},y_{j+1/2+0})}+O(h)=q_3+O(h),
$$

we rewrite equations (24), (25) in the following form

$$
k_{12}^{(+12)}z_1+\left(k_{11}^{(+12)}+k_{11}^{(+11,+12)}\right)z_3+k_{12}^{(+11,+12)}z_4=O(h),
$$

$$
k_{12}^{(+11)}z_1+k_{12}^{(+11,+12)}z_3+\left(k_{22}^{(+11)}+k_{22}^{(+11,+12)}\right)z_4=O(h).
$$

So, system (15), (20), (24), (25) can be written as

$$
Mz=r,
$$

where $M=\tilde{A}$, $z=(z_1,z_2,z_3,z_4)^T$, $r=(r_1,r_2,r_3,r_4)^T$, and $r_i=O(h)$. It is easy to see that matrix M is symmetric. To show that it is positive definite, consider the scalar product

$$
\begin{aligned}
(Mv,v)=&k_{11}^{(+11)}v_1^2+k_{11}v_1^2+2k_{12}v_1v_2+k_{22}^{(+12)}v_2^2+k_{22}v_2^2+\\
&2k_{12}^{(+12)}v_2v_3+k_{11}^{(+12)}v_3^2+k_{11}^{(+11,+12)}v_3^2+2k_{12}^{(+11)}v_1v_4+\\
&2k_{12}^{(+11,+12)}v_3v_4+k_{22}^{(+11)}v_4^2+k_{22}^{(+11,+12)}v_4^2,
\end{aligned}
\tag{26}
$$

where $v=(v_1,v_2,v_3,v_4)^T$. Since K is positive definite in every point, than taking into account the ellipticity condition (3) valid for any nonzero $\xi=(\xi_1,\xi_2)$, $|\xi|=\xi_1^2+\xi_2^2\neq 0$, and using $\xi=(v_1,v_2)$, $\xi=(v_1,v_4)$, $\xi=(v_3,v_2)$, $\xi=(v_3,v_4)$, from (26) we get

$$
\begin{aligned}
(Mv,v)=&\left(k_{11}v_1^2+2k_{12}v_1v_2+k_{22}v_2^2\right)+\left(k_{11}^{(+11)}v_1^2+2k_{12}^{(+11)}v_1v_4+k_{22}^{(+11)}v_4^2\right)+\\
&\left(k_{11}^{(+12)}v_3^2+2k_{12}^{(+12)}v_2v_3+k_{22}^{(+12)}v_2^2\right)+\\
&\left(k_{11}^{(+11,+12)}v_3^2+2k_{12}^{(+11,+12)}v_3v_4+k_{22}^{(+11,+12)}v_4^2\right)\geq\\
&c_1\left(v_1^2+v_2^2\right)+c_1\left(v_1^2+v_4^2\right)+c_1\left(v_3^2+v_2^2\right)+c_1\left(v_3^2+v_4^2\right)=\\
&2c_1\left(v_1^2+v_2^2+v_3^2+v_4^2\right)>0.
\end{aligned}
$$

Hence, there exists M^{-1} such that $z=\tilde{r}$, where $\tilde{r}=M^{-1}r$, and $\tilde{r}_i=O(h)$. Thus

$$
z_i=O(h),\quad q_i=O(h),\quad i=\overline{1,4},
$$

and we have first order approximation for the normal components of the fluxes at the midpoints of the edges.

3.2. **Convergence of MPFA.** In this section we prove convergence of MPFA knowing that the flux is approximated with $O(h)$.

In the domain $\bar{\Omega}$ we consider staggered mesh $\bar{\omega}_0 = \omega_0 \cup \gamma_0$, $\bar{\omega}_1 = \omega_1 \cup \gamma_1$, $\bar{\omega}_2 = \omega_2 \cup \gamma_2$. Here ω_0 defines the cell centers where the pressure is considered, while ω_1 and ω_2 define the midpoints of the edges, where the velocity components (i.e., fluxes) are defined.

$$\begin{aligned}
\bar{\omega}_0 =& \{((i+1/2)\,h_1, (j+1/2)\,h_2)\,, i = -1/2, 0, 1, ..., N_1, N_1+1/2, \\
& \quad j = -1/2, 0, 1, ..., N_2, N_2+1/2\}, \\
\bar{\omega}_1 =& \left\{(ih_1, (j+1/2)h_2)\right), i = 0, 1, ..., N_1,\ j = 0, 1, ..., N_2\}\,, \\
\bar{\omega}_2 =& \left\{((i+1/2)h_1, jh_2)\right), i = 0, 1, ..., N_1,\ j = 0, 1, ..., N_2\}\,,
\end{aligned}$$

where $h_1 = 1/N_1$, $h_2 = 1/N_2$, and N_1, N_2 are positive integers. Note that on the boundary there are nodes belonging to ω_0 and ω_1, or ω_0 and ω_2.

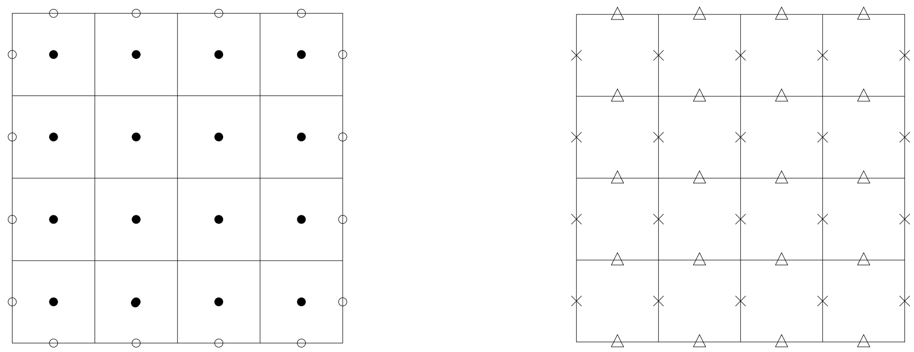

FIGURE 2. Grids for pressure u (left picture) and fluxes W_1, W_2 (right picture)

Following notations for the spaces of grid functions defined on ω_0, ω_1, and ω_2 are introduced:

$$H_y = \{y(x),\ x \in \bar{\omega}_0\}, \quad H_y^0 = \{y(x),\ x \in \bar{\omega}_0,\ y(x) = 0 \text{ for } x \in \gamma_0\},$$
$$H_1 = \{w_1(x),\ x \in \bar{\omega}_1\}, \quad H_2 = \{w_2(x),\ x \in \bar{\omega}_2\},$$
$$\mathbf{H_w} = H_1 \times H_2,\ \mathbf{w} = (w_1, w_2).$$

The inner products are defined in the usual way

$$(y, \tilde{y})_{\omega_0} = \sum_{i=1}^{N_1}\sum_{j=1}^{N_2} y_{i,j}(x)\tilde{y}_{i,j}(x)h_1h_2, \quad \|y\|_{\omega_0}\sqrt{(y,y)_{\omega_0}}.$$

$$(\mathbf{w},\tilde{\mathbf{w}})_{\bar{\omega}_1\times\bar{\omega}_2} = (w_1, \tilde{w}_1)_{\bar{\omega}_1} + (w_2, \tilde{w}_2)_{\bar{\omega}_2} = \sum_{i=0}^{N_1}\sum_{j=1}^{N_2-1} w_1(x)\tilde{w}_1(x)h_ih_j + \sum_{i=1}^{N_1-1}\sum_{j=0}^{N_2} w_2(x)\tilde{w}_2(x)h_ih_j,$$

where $h_i = h_1,\ i = \overline{1, N_1-1},\ h_0 = h_{N_1} = h_1/2,\ h_j = h_2,\ j = \overline{1, N_2-1},\ h_0 = h_{N_2} = h_2/2$.

Let us now define discrete operators associated with divergence and gradient operators. Discrete divergence operator $D : \mathbf{H_w} \to H_y$ is defined in the following way

$$(D\mathbf{w})_{i,j} = \frac{(w_1)_{i+1,j} - (w_1)_{i,j}}{h_1} + \frac{(w_2)_{i,j+1} - (w_2)_{i,j}}{h_2},\ 1 < i \leq N_1,\ 1 < j \leq N_2.$$

Discrete gradient operator $G : H_y \to \mathbf{H_w}$ is defined as $Gy = (G^1y, G^2y)$, where

$$G^1y = \begin{cases} \frac{y_{i+1,j}-y_{i,j}}{h_1}, & 1 \le i \le N_1 - 1, \\ \frac{y_1-y_0}{h_1/2}, & i = 0, \\ \frac{y_{N_1+1}-y_{N_1}}{h_1/2}, & i = N_1. \end{cases}$$

$$G^2y = \begin{cases} \frac{y_{i,j+1}-y_{i,j}}{h_2}, & 1 \le j \le N_2 - 1, \\ \frac{y_1-y_0}{h_2/2}, & j = 0, \\ \frac{y_{N_2+1}-y_{N_2}}{h_2/2}, & j = N_2. \end{cases}$$

Note that the above introduced operators satisfy $D = -G^*$ in the respective scalar products:

$$(Dw, \tilde{y}) = (DGy, \tilde{y})_{\omega_0} = -(Gy, G\tilde{y})_{\omega_1 \times \omega_2}.$$

We will need some auxiliary inequalities, norms, etc., which we list here for completeness.

Lemma 3.1. *For any grid function $y \in H_y^0$ the following inequality (Friedrichs' inequality) holds*

$$\|y\|_{\bar{\omega}_0} \le c_1 \|Gy\|_{\bar{\omega}_1 \times \bar{\omega}_2}.$$

Discrete W_2^1 norm is defined as

$$\|y\|_{W_2^1} = (y, y)_{\bar{\omega}_0} + (Gy, Gy)_{\bar{\omega}_1 \times \bar{\omega}_2}.$$

From Friedrichs' inequality we get

$$\|y\|_{W_2^1} \le c_2 \|Gy\|_{\bar{\omega}_1 \times \bar{\omega}_2};$$

Let us denote by z the error, $z = y - u$. We will need to write in divergence form the equation for the error. Suppose there exists $\bar{K}$ such that

$$c_1(w, \tilde{w})_{\omega_1 \times \omega_2} \le (\bar{K}w, \tilde{w})_{\omega_1 \times \omega_2} \le c_2(w, \tilde{w})_{\omega_1 \times \omega_2},$$

and such that we can write our discrete equation for the error as

$$-D\bar{K}Gz = -D\psi.$$

Note: Later we will show that for MPFA there exist such $\bar{K}$ (although we can not write it in explicit form). So, from the inequalities above we have:

$$-(D\bar{K}Gz, z)_{\omega_u} = -(D\psi, z)_{\omega_u} \Rightarrow$$

$$c_1 \|Gz\|^2_{\omega_1 \times \omega_2} \le (\bar{K}Gz, Gz)_{\omega_1 \times \omega_2} = (\psi, Gz)_{\omega_1 \times \omega_2} \le \|\psi\|_{\omega_1 \times \omega_2} \|Gz\|_{\omega_1 \times \omega_2}.$$

Next, we can use the approximation property of MPFA, shown above, to get

$$c_1 \|Gz\|_{\omega_1 \times \omega_2} \le \|\psi\|_{\omega_1 \times \omega_2} = O(h).$$

This is the end of the proof, because from Friedrichs inequality applied to z we have the desired result.

It still remains to show that there exist $\bar{K}$ with the desired properties. From equation $\tilde{A}\tilde{v} = \tilde{w}$ with $\tilde{v} = (a^1, b^1, a^3, b^3)^T$, where $\tilde{A}$ is given by (9), we see that $\tilde{\omega}_1$, $\tilde{\omega}_2$, $\tilde{\omega}_3$, $\tilde{\omega}_4$ contain different components of the gradient Gy, and ONLY components of Gy. If we denote $\tilde{B} = \tilde{A}^{-1}$, we will have

$$a_1 = b_{11}\tilde{w}_1 + b_{12}\tilde{w}_2 + b_{13}\tilde{w}_3 + b_{14}\tilde{w}_4,$$

i.e., a_1 will also contain only components of Gy, multiplied by some coefficients independent on the solution. Similar conclusion can be drawn for b_1, a_2, b_2, and other similar expressions which we use to write our finite difference scheme 11. Note, that 11 is already written in divergence form, thus we are able to write MPFA as $D\bar{K}Gy$, with some $\bar{K}$ which can not be explicitely written. Next, we have to clarify what are the properties of $\bar{K}$. From [1] we know that on rectangular grid MPFA can be written as $My = f$ with $M = M^* > 0$. Thus we have

$$My = f, \quad -D\bar{K}Gy = f, \Rightarrow M = D\bar{K}G.$$

It follows from here that

$$(My, y) = -(D\bar{K}Gy, y) = -(\bar{K}Gy, Gy).$$

Now we can conclude: i) Because M is symmetric, it follows $\bar{K}$ is also symmetric.

ii) Because $(My, y) \geq 0, \Rightarrow (\bar{K}Gy, Gy) \geq 0$. Moreover, $(My, y) > 0$ for $y \neq 0$. It remains to answer the question is $(\bar{K}Gy, Gy) > 0$ for $Gy \neq 0$? In our case, $y = const \Rightarrow$ is possible only for $y \equiv 0$, and with this we have shown that there exist $\bar{K}$ with the desired properties.

4. Numerical results

In this section, we validate the developed difference scheme by performing several numerical experiments for variable continuous and discontinuous full tensor coefficients.

4.1. **Validation of the discretization for continuous coefficients.** Consider the example from [3]. In this experiment, the permeability tensor is full symmetrical with variable continuous components

$$K = \begin{pmatrix} (x+2)^2 + y^2 & \sin(xy) \\ \sin(xy) & 1 \end{pmatrix} \tag{27}$$

The exact solution is given by the following formula

$$u = x^3 y + y^4 + \sin(x)\cos(y),$$

with f and g^D defined accordingly by (1), (2).

Convergence rate for the pressure is established by running cases for seven levels of grid refinement, starting with 5×5 grid cells on level one and refining the grid steps by a factor of two for each successive level. Further, we will use the following notations

$$\|\delta_u\|_C = \frac{\|u - u_h\|_C}{\|u\|_C}, \quad \|\delta_u\|_{L_2} = \frac{\|u - u_h\|_{L_2}}{\|u\|_{L_2}},$$

where u is the solution of the differential problem (1), (2), and u_h is the solution of the difference scheme (11).

From Tab. 1 we see that if we refine the grid step by the factor of two, then the error reduces by the factor of four. Hence, the approximate solution u_h converges to the exact solution u with the second order both in the maximum and L_2 norms.

In the same way we calculate the convergence for the fluxes W_1 and W_2. In Tab. 2 the following notations are used

$$\|\delta_1\|_C = \frac{\|W_1 - W_{1,h}\|_C}{\|W_1\|_C}, \quad \|\delta_1\|_{L_2} = \frac{\|W_1 - W_{1,h}\|_{L_2}}{\|W_1\|_{L_2}},$$

$$\|\delta_2\|_C = \frac{\|W_2 - W_{2,h}\|_C}{\|W_2\|_C}, \quad \|\delta_2\|_{L_2} = \frac{\|W_2 - W_{2,h}\|_{L_2}}{\|W_2\|_{L_2}}.$$

Grid	$\|\delta_u\|_C$	$\|\delta_u\|_{L_2}$
5×5	1.97×10^{-2}	1.73×10^{-2}
10×10	6.02×10^{-3}	4.87×10^{-3}
20×20	1.56×10^{-3}	1.23×10^{-3}
40×40	3.84×10^{-4}	3.05×10^{-4}
80×80	9.49×10^{-5}	7.56×10^{-5}
160×160	2.36×10^{-5}	1.86×10^{-5}
320×320	5.90×10^{-6}	4.64×10^{-6}

TABLE 1. Convergence of the pressure for problem (1), (2) with permeability tensor (27)

Grid	$\|\delta_1\|_C$	$\|\delta_1\|_{L_2}$	$\|\delta_2\|_C$	$\|\delta_2\|_{L_2}$
5×5	7.78×10^{-3}	7.76×10^{-3}	7.47×10^{-3}	6.83×10^{-3}
10×10	4.69×10^{-3}	2.98×10^{-3}	4.76×10^{-3}	4.18×10^{-3}
20×20	2.21×10^{-3}	9.93×10^{-4}	2.20×10^{-3}	1.73×10^{-3}
40×40	1.00×10^{-3}	2.88×10^{-4}	9.23×10^{-4}	5.40×10^{-4}
80×80	4.72×10^{-4}	8.08×10^{-5}	4.29×10^{-4}	1.56×10^{-4}
160×160	2.27×10^{-4}	2.23×10^{-5}	2.09×10^{-4}	4.39×10^{-5}
320×320	1.11×10^{-4}	6.09×10^{-6}	1.03×10^{-4}	1.21×10^{-5}

TABLE 2. Convergence of the fluxes for problem (1), (2) with permeability tensor (27)

From Tab. 2 we can see that for the fluxes we have the first order convergence in the maximum norm and the second order convergence in the L_2 norm.

4.2. **Validation of the discretization for discontinuous coefficients.** Consider jump discontinuity on the interfaces $x_\xi = 0.4$ and $y_\eta = 0.4$. The permeability tensor is given by

$$K_1 = K_4 = \begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix}, \quad K_2 = K_3 = \begin{pmatrix} \lambda & \lambda/2 \\ \lambda/2 & \lambda \end{pmatrix}, \tag{28}$$

where λ is called the contrast of discontinuity. We choose the exact solution which satisfies the conditions of the pressure and fluxes continuity across the interfaces

$$u = (x - x_\xi)^2 (y - y_\eta)^2 \sin(\pi(x + y)).$$

The right hand side and the boundary conditions can be found from the initial differential problem (1), (2) for known exact solution. The experiments are performed for $\lambda = 10$ and $\lambda = 1000$.

From Tab. 3 it is easy to see that the approximate solution u_h converges to the exact solution u with the second order both in the maximum and L_2 norms. For the fluxes we have first order convergence in the C norm and the second order convergence in the L_2 norm (see Tab. 4).

Note, that for this artificial smooth solution the accuracy does not depend on jump discontinuity.

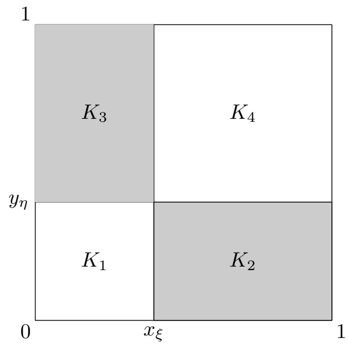

FIGURE 3. The surface of discontinuity

Grid	$\|\delta_u\|_C$		$\|\delta_u\ {}_{L_2}$	
	$\lambda = 10$	$\lambda = 1000$	$\lambda = 10$	$\lambda = 10000$
5×5	1.56×10^{-1}	1.56×10^{-1}	1.61×10^{-1}	1.63×10^{-1}
10×10	5.77×10^{-2}	5.78×10^{-2}	4.58×10^{-2}	4.64×10^{-2}
20×20	1.72×10^{-2}	1.73×10^{-2}	1.18×10^{-2}	1.20×10^{-2}
40×40	4.91×10^{-3}	4.91×10^{-3}	3.02×10^{-3}	3.06×10^{-3}
80×80	1.30×10^{-3}	1.30×10^{-3}	7.67×10^{-4}	7.77×10^{-4}
160×160	3.37×10^{-4}	3.37×10^{-4}	1.93×10^{-4}	1.96×10^{-4}
320×320	8.66×10^{-5}	8.66×10^{-5}	4.86×10^{-5}	4.92×10^{-5}

TABLE 3. Convergence of the pressure for problem (1), (2) with permeability tensor (28)

Grid	$\|\delta_1\|_C = \|\delta_2\|_C$		$\|\delta_1\ {}_{L_2} = \|\delta_2\|_{L_2}$	
	$\lambda = 10$	$\lambda = 1000$	$\lambda = 10$	$\lambda = 10000$
5×5	1.63×10^{-1}	1.72×10^{-1}	1.92×10^{-1}	2.29×10^{-1}
10×10	1.51×10^{-2}	1.60×10^{-2}	2.92×10^{-2}	3.06×10^{-2}
20×20	1.70×10^{-2}	1.70×10^{-2}	1.05×10^{-2}	1.08×10^{-2}
40×40	1.04×10^{-2}	1.04×10^{-2}	4.32×10^{-3}	4.39×10^{-3}
80×80	5.59×10^{-3}	5.59×10^{-3}	1.52×10^{-3}	1.54×10^{-3}
160×160	2.87×10^{-3}	2.87×10^{-3}	4.89×10^{-4}	4.95×10^{-4}
320×320	1.45×10^{-3}	1.45×10^{-3}	1.47×10^{-4}	1.49×10^{-4}

TABLE 4. Convergence of the fluxes for problem (1), (2) with permeability tensor (28)

5. Concluding remarks

The purpose of the paper was to study the approximation properties of the multipoint flux approximation discretization for two-dimensional second order elliptic equations in the case of discontinuous tensor coefficients. The discretization is based on the finite volume method and multipoint flux approximation approach. Specific derivation of MPFA is written and it is shown that the components of the discrete flux approximate the components of the continuous flux with the first order in the midpoints of the edges. The error of the finite difference scheme is written i divergence form, so this is used to show first order convergence for the fluxes in discrete W_2^1 norm. The analysis is also valid for the problems with continuous tensor coefficients. Results from numerical experiments are presented and discussed.

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Fraunhofer Institut fuer Techno- und Wirtschaftsmathematik, Fraunhofer-Platz 1, 67663 Kaiserslautern, Germany, and Institute of Mathematics, Bulgarian Academy of Science, Acad. G.Bonchev str., bl.8, BG-1113, Sofia, Bulgaria, iliev@itwm.fhg.de

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