Stochastic Optimization in Finance and Life Insurance: Applications of the Martingale Method

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To ***Ida***

Love from,

mummy

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List of research papers of my PhD studies

1. Zhang, A., (2007) *A closed-form solution to the continuous-time consumption model with endogenous labor income*, CRIEFF discussion paper, School of Economics & Finance, University of St Andrews. http://econpapers.repec.org/paper/sancrieff/0710.htm

2. Zhang, A., (2007) *Optimal Consumption, Labor Supply and Portfolio Rules in a Continuous-time Life Cycle Model*, Proceedings of the Second Conference on Game Theory and Applications, World Academic Press.

3. Zhang, A., (2007) *A secret to create a complete market from an incomplete market*, Applied Mathematics and Computation 191, 253-262.

4. Zhang, A., Korn, R., Ewald, C., (2007) *Optimal management and inflation protection for defined contribution pension plans*, Blätter der DGVFM, Volume 28 (2), 239-258. Springer.

5. Ewald, C., Zhang, A., (2006) *A New Method for the Calibration of Stochastic Volatility Models: The Malliavin Gradient Method*, Journal of Quantitative Finance, Volume 6 (2), 147-158.

Preface

This thesis is based on some of my research papers during my PhD studies in the Department of Financial Mathematics at the Fraunhofer Institut für Techno- und Wirtschaftmathematik (ITWM), Kaiserslautern in Germany. To keep it focusing on the topic of applications of the Martingale method for the optimization problems in finance and life insurance, I do not include to the thesis some of my research papers, which are of independent interest.

The continuous-time intertemporal consumption-portfolio optimization problem was pioneered by Merton (1969, 1971), using the method of dynamic programming. In the 1980s, Karatzas et al (1986), Pliska (1986) and Cox/Huang (1989) developed an alternative approach, the Martingale method, to the continuous-time problem. Certainly the economic literature is dominated by the stochastic dynamic programming approach, which has the advantage that it identifies the optimal strategy automatically as a function of the underlying observables, which is sometimes called feedback form. However, it often turns out that the corresponding Hamilton-Jacobi-Bellman equation, which in general is a second order non-linear partial differential equation, does not admit a closed-form solution. In contrast, by utilizing the

Martingale method, a closed-form solution can be obtained without solving any partial differential equation in many specific models when asset prices follow a geometric Brownian motion.

This thesis is devoted to deal with the stochastic optimization problems in various situations with the aid of the Martingale method. Chapter 2 discusses the Martingale method and its applications to the basic optimization problems, which are well addressed in the literature (for example, [15], [23] and [24]). In Chapter 3, we study the problem of maximizing expected utility of real terminal wealth in the presence of an index bond. Chapter 4, which is a modification of the original research paper joint with Korn and Ewald [39], investigates an optimization problem faced by a DC pension fund manager under inflationary risk. Although the problem is addressed in the context of a pension fund, it presents a way of how to deal with the optimization problem, in the case there is a (positive) endowment. In Chapter 5, we turn to a situation where the additional income, other than the income from returns on investment, is gained by supplying labor. Chapter 6 concerns a situation where the market considered is incomplete. A trick of completing an incomplete market is presented there. The general theory which supports the discussion followed is summarized in the first chapter.

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time although he was engaged by other commitments. Of course, any remaining errors in this thesis are my own. In particular, with an open mind, he supported and encouraged my studies in Economics, which have in return deepened my understanding of mathematics and accelerated my PhD studies. I would like to thank all of my fellow PhD students and superiors from ITWM and from the Department of Mathematics at the University of Kaiserslautern for their understanding and support. I also wish to thank the UK experts with whom I had fruitful discussions and who gave me useful comments. Those include Charles Nolan from the University of St. Andrews, Andrew Cairns and Tak Kuen (Ken) Siu from Heriot Watt University, Hassan Molana from the University of Dundee and Andy Snell from the University of Edinburgh. The supports from the Rheinland-Pfalz excellence cluster "Dependable Adaptive Systems and Mathematical Modeling" (DASMOD) and from the "Deutsche Forschungsgemeinschaft" (DFG) are greatly acknowledged with thanks. Finally, I am in debt to my little daughter whom I spent little time looking after during my very intensive studies. I am grateful to my husband who spent much of time with my daughter after her nursery, while committed to his own research and teaching. I had a lot of discussions with him and he gave me helpful hints.

Contents

Chapter 1

General theory for the continuous-time financial market

Let us consider a financial market $\mathcal{M}$, in which $m+1$ assets are traded continuously. The first asset is a riskless *bond* with price $S_0(t)$ being given by

$$\begin{aligned} \frac{dS_0(t)}{S_0(t)} &= R(t)dt, \\ S_0(0) &= s_0 \end{aligned} \tag{1.1}$$

and the remaining m assets are *stocks* with prices $S_i(t)$ satisfying

$$\begin{aligned} \frac{dS_i(t)}{S_i(t)} &= \mu_i(t)dt + \sum_{j=1}^{d} \sigma_{ij}(t)dW_j(t), \\ S_i(0) &= s_i, \text{ for } i = 1, ..., m \end{aligned} \tag{1.2}$$

Where $W(t) = (W_1(t), ..., W_d(t))^\top$ is a d-dimensional Brownian motion defined on a given complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with the component Brownian motions $W_j(t)$, $j = 1, ..., d$, being independent. The superscript $(\top)$ denotes transposition. The nominal interest rate $R(t)$, the stock appreciation rate vector $\mu(t) \equiv (\mu_1(t), ..., \mu_m(t))^\top$ and the volatility matrix $\sigma(t) \equiv \{\sigma_{ij}(t)\}_{m\times d}$ are referred to as the *coefficients* of the market $\mathcal{M}$.

It can be verified by Itô's Lemma that $S_0(t)$, $S_i(t)$, for $i = 1, ..., m$, satisfying the equations below are solutions to the differential equations (1.1) and (1.2), respectively.

$$S_0(t) = s_0 e^{\int_0^t R(s)ds} \tag{1.3}$$

and

$$S_i(t) = s_i e^{\int_0^t \left(\mu_i(s) - \frac{1}{2}\sum_{j=1}^d \sigma_{ij}^2(s)\right)ds + \int_0^t \sum_{j=1}^d \sigma_{ij}(s)dW_j(s)} \tag{1.4}$$

Definition 1.0.1. *Let $(X(t), \mathcal{F}(t))_{t\geq 0}$ be a stochastic process. $X(t)$ is called $\mathcal{F}(t)$-progressively measurable if, for all $t \geq 0$, the mapping*

$$\begin{aligned} [0,t] \times \Omega &\to \mathbb{R}^n \\ (s, \omega) &\mapsto X_s(\omega) \end{aligned} \tag{1.5}$$

is $\mathcal{B}([0,t]) \otimes \mathcal{F}(t) - \mathcal{B}(\mathbb{R}^n)$-measurable.

Obviously, every $\mathcal{F}(t)$-progressively measurable process is also adapted. The progressive measurability is for the associated stochastic integrals to be well-defined. So whenever an stochastic integral occurs in this thesis, the relevant progressive measurability is assumed either explicitly or implicitly.

Let us assume, from now on, that the filtration $\{\mathcal{F}(t)\}_t$ is generated by the driving Brownian motion $\{W(s)\}_{0\leq s\leq t}$ and is thus known as *Brownian filtration*. It is convenient to make a general assumption as following.

General Assumption 1.

(i) The coefficients of $\mathcal{M}$ are $\mathcal{F}(t)$-progressively measurable;

(ii) $m \leq d$;

(iii) The volatility matrix $\sigma(t)$ has full row rank.

Remark 1.0.1. *The assumption that $m \leq d$ is not a real restriction since otherwise the number of stocks can always be reduced by duplicating some of the additional stocks as linear combinations of others. (See Karatzas (1997) [23])*

We now assume that, in the financial market $\mathcal{M}$, a *small investor*[1.1] with an initial capital $x(\geq 0)$ can decide, at each time period $t \in [0, T]$,

- what proportion of wealth, $\pi_i(t)$, he should invest in each of the available stocks and
- what his consumption rate $C(t)$ (≥ 0) should be.

where, $\pi_i(t)$ and $C(t)$ are $\mathcal{F}(t)$-progressively measurable. Once having decided the proportions of wealth to be invested in the stocks, he then simply puts the rest of money in the bond. That is, the proportion of wealth invested in the bond is given by $1 - \sum_{i=1}^{m} \pi_i(t)$ or $1 - \pi^\top(t)\mathbf{1}_m$, where

[1.1] The term 'small investor' comes from the fact that the investor is too small to affect the market prices.

$\pi(t) \equiv (\pi_1(t), ..., \pi_m(t))^\top$ and $\mathbf{1}_m \equiv (\underbrace{1, ..., 1}_{m \text{ times}})^\top$.

Definition 1.0.2. *A pair (π, C) consisting of a portfolio π and a consumption rate C is said to be self-financing if the corresponding wealth process $X^{\pi,C}(t)$, $t \in (0, T]$, satisfies*

$$dX^{\pi,C}(t) = \sum_{i=1}^{m} \pi_i(t) X^{\pi,C}(t) \frac{dS_i(t)}{S_i(t)} + \left(1 - \sum_{i=1}^{m} \pi_i(t)\right) X^{\pi,C}(t) \frac{dS_0(t)}{S_0(t)} - C(t)dt \tag{1.6}$$

The requirement of being self-financing states that the change in wealth must equal the difference of the capital gains and infinitesimal consumption. Substituting the asset returns, Eq. (1.1)-(1.2), into Eq. (1.6), we get

$$\begin{aligned} dX^{\pi,C}(t) &= \sum_{i=1}^{m} \pi_i(t) X^{\pi,C}(t) \left(\mu_i(t)dt + \sum_{j=1}^{d} \sigma_{ij}(t) dW_j(t)\right) \\ &\quad + \left(1 - \sum_{i=1}^{m} \pi_i(t)\right) X^{\pi,C}(t) R(t)dt - C(t)dt \end{aligned}$$

Collecting terms and then writing it in terms of vector and matrix, we then have the *wealth process equation* as following

$$dX^{\pi,C}(t) = X^{\pi,C}(t) \left[R(t)dt + \pi^\top(t) \left(\mu(t) - R(t)\mathbf{1}_m\right) dt + \pi^\top(t)\sigma(t)dW(t)\right] - C(t)dt \tag{1.7}$$

The terms appearing in the braces in the last line are referred as the percentage of the *capital gains*, during a time interval of dt and are made up of three parts:

- the percentage of an average underlying gross return on the $m + 1$ assets, which is reflected by the term $R(t)dt$,

- the percentage of a risk premium for investing in the m risky asset, which is reflected by the term $\pi^\top(t)[\mu(t) - R(t)\mathbf{1}_m]dt$, and
- the volatility term proportional to the amount of the investment in the stocks, which is the term $\pi^\top(t)\sigma(t)$.

Let us introduce the *market price of risk* $\theta(t)$ by

$$\theta(t) \equiv \sigma^\top(t)\left(\sigma(t)\sigma^\top(t)\right)^{-1}[\mu(t) - R(t)\mathbf{1}_m] \tag{1.8}$$

Under the General Assumption 1, part (i), $\theta(t)$ is also $\mathcal{F}(t)$-progressively measurable. The wealth process equation (1.7) now can be rewritten as

$$dX^{\pi,C}(t) = X^{\pi,C}(t)\left[R(t)dt + \pi^\top(t)\sigma(t)\left(\theta(t)dt + dW(t)\right)\right] - C(t)dt \tag{1.9}$$

Definition 1.0.3. *A self-financing pair (π, C) of portfolio and consumption-rate process is said to be admissible for the initial capital x (≥ 0), if the corresponding wealth process $X^{\pi,C}(t)$ satisfies*

$$X^{\pi,C}(t) \geq 0, \textit{ almost surely, for all } t \in [0,T] \tag{1.10}$$

The class of admissible pairs will be denoted by $\mathcal{A}(x)$.

Let us now assume, for the moment, that there is no consumption (that is $C = 0$) and we call the corresponding wealth process as the *portfolio value process* which satisfies

$$dX^{\pi}(t) = X^{\pi}(t)\left[R(t)dt + \pi^\top(t)\sigma(t)\left(\theta(t)dt + dW(t)\right)\right] \tag{1.11}$$

Definition 1.0.4. *A portfolio π is called an arbitrage opportunity if its portfolio value process with $X^\pi(0) = 0$ satisfies*

$$\mathbb{P}[X^\pi(T) \geq 0] = 1 \textit{ and } \mathbb{P}[X^\pi(T) > 0] > 0 \tag{1.12}$$

We say a market $\mathcal{M}$ is arbitrage-free if no such portfolios exist in it.

An arbitrage opportunity is a way of trading so that one starts with zero capital and end up, at time $t = T$, with non-negative wealth for sure and furthermore has a positive probability of having made money by that time.

It will be convenient to make following notations:

$$\begin{aligned} \gamma(t) &\equiv \frac{1}{B(t)} = e^{-\int_0^t r(s)ds} \\ Z_0(t) &\equiv e^{-\int_0^t \theta^\top(s)dW(s) - \frac{1}{2}\int_0^t \|\theta(s)\|^2 ds} \\ H(t) &\equiv \gamma(t)Z_0(t) = e^{-\int_0^t R(s)ds - \frac{1}{2}\int_0^t \|\theta(s)\|^2 ds - \int_0^t \theta^\top(s)dW(s)} \end{aligned} \tag{1.13}$$

$H(t)$ is referred to as the *stochastic discount factor*. The following proposition tells us that the sum of an accumulated discounted consumption process and its corresponding discounted wealth process can be expressed as a stochastic integral with respect to the Brownian motion.

Proposition 1.0.1. *Let $X^{\pi,C}(t)$ be the wealth process of a portfolio π, then the process*

$$H(t)X^{\pi,C}(t) + \int_0^t H(s)C(s)ds$$

is a $\mathbb{P}$-local Martingale.

Proof. By Itô's formula, $H(t)$ can be written in the following differential form:

$$dH(t) = -H(t)[R(t)dt + \theta^\top(t)dW(t)] \tag{1.14}$$

An application of the product rule to $H(t)X^{\pi,C}(t)$ using Eq. (1.9) and Eq. (1.14) gives us that

$$\begin{aligned} d(H(t)X^{\pi,C}(t)) &= H(t)d(X^{\pi,C}(t)) + X^{\pi,C}(t)d(H(t)) + d(H(t))d(X^{\pi,C}(t)) \\ &= H(t)X^{\pi,C}(t)\left[R(t)dt + \pi^\top(t)\sigma(t)\left(\theta(t)dt + dW(t)\right)\right] \\ &\quad -H(t)C(t)dt - H(t)X^{\pi,C}(t)\left(R(t)dt + \theta^\top(t)dW(t)\right) \\ &\quad -H(t)X^{\pi,C}(t)\pi^\top(t)\sigma(t)\theta(t)dt \end{aligned}$$

Collecting terms results in

$$\begin{aligned} d(H(t)X^{\pi,C}(t)) &= H(t)X^{\pi,C}(t)[\pi^\top(t)\sigma - \theta^\top(t)]dW(t) - H(t)C(t)dt \\ &= H(t)X^{\pi,C}(t)[\sigma^\top(t)\pi(t) - \theta(t)]^\top dW(t) - H(t)C(t)dt \end{aligned} \tag{1.15}$$

Moving $H(t)C(t)dt$ to the left-hand side and then taking integration on both sides, we get

$$\begin{aligned} & H(t)X^{\pi,C}(t) + \int_0^t H(s)C(s)ds \\ = \; & x + \int_0^t H(s)X^{\pi,C}(s)[\sigma^\top(s)\pi(s) - \theta(s)]^\top dW(s) \end{aligned} \tag{1.16}$$

The stochastic integral on the right-hand side is a local Martingale under $\mathbb{P}$. This is to say, $H(t)X^{\pi,C}(t) + \int_0^t H(s)C(s)ds$ is a $\mathbb{P}$-local Martingale. □

If we let $C = 0$, then we have following corollary which says that any discounted portfolio value process is a $\mathbb{P}$-local Martingale.

Corollary 1.0.1. *Let $X^\pi(t)$ be the portfolio value process of a portfolio π, then we have that*

$$H(t)X^\pi(t) = x + \int_0^t H(s)X^\pi(s)[\sigma^\top(s)\pi(s) - \theta(s)]^\top dW(s) \tag{1.17}$$

or that, in the differential form,

$$d(H(t)X^{\pi}(t)) = H(t)X^{\pi}(t)[\sigma^{\top}(t)\pi(t) - \theta(t)]^{\top}dW(t) \tag{1.18}$$

In the chapters that follow, we will need the following two fundamental theorems which we refer to Karatzas (1997) [23]. The first theorem can help us decide whether a market $\mathcal{M}$ contains arbitrage opportunities or not while the second one provides us of a simple criterion to determine whether the market is complete.[1.2]

Theorem 1.0.1. *(**First Fundamental Theorem**)*
(i) If the market $\mathcal{M}$ is arbitrage-free, then there exists a market price of risk $\theta(t)$ satisfying Eq. (1.8).
(ii) Conversely, if such a market price of risk exists and satisfies

$$\int_0^T \|\theta(t)\|^2 \, dt < \infty, \ \ a.s. \tag{1.19}$$

and

$$\mathbb{E}\left[Z_0(T)\right] = 1 \ \textit{where, } Z_0(t) \textit{ is defined in Eq. (1.13).} \tag{1.20}$$

then the market is arbitrage-free.

From the Novikov's condition,[1.3] we know that if

$$\mathbb{E}\left[e^{\frac{1}{2}\int_0^T \|\theta(t)\|^2 dt}\right] < \infty \tag{1.21}$$

[1.2]These two theorems correspond to Shreve's (2004) two fundamental theorems of asset pricing which are in the context of risk-neutral probability measure. In this thesis, we only deal with the actual probability measure.

[1.3]For example, when the market price of risk $\theta(t)$ is constant for all t, the Novikov's condition holds.

then the exponential process $Z_0(t)$ in Eq. (1.13) is a martingale under the probability measure $\mathbb{P}$. In particular, the conditions of (1.19)-(1.20) are satisfied.

Before discussing the second fundamental theorem, we need to give a definition of *completeness*.

Definition 1.0.5. *A financial market $\mathcal{M}$ is called complete, if every $\mathcal{F}(T)$-measurable contingent claim B is attainable in the sense that there exists a portfolio π such that the corresponding portfolio value at time T equals the claim B with probability one, that is,*

$$X^{\pi}(T) = B, \ \ a.s.; \tag{1.22}$$

otherwise, it is called incomplete.

Theorem 1.0.2. ***(Second Fundamental Theorem)***
(i) Consider a arbitrage-free financial market $\mathcal{M}$, then $\mathcal{M}$ is complete if and only if $m = d$.
(ii) This market $\mathcal{M}$ is incomplete if and only if $m < d$.

For a market to be complete, it requires that there be exactly as many stocks as "sources of uncertainty". Incompleteness arises when the number of "sources of uncertainty" is strictly greater than the number of the stocks. We will discuss, in turn, the optimization in a complete market and an incomplete market later on.

Chapter 2

Optimization in complete markets

2.1 Introduction

The continuous-time intertemporal consumption-portfolio optimization problem was pioneered by Merton (1969, 1971), using the method of dynamic programming. In the 1980s, Karatzas et al (1986), Pliska (1986) and Cox/Huang (1989) developed an alternative approach-the *Martingale method* to solve the continuous-time problem. The main advantage of the latter over the former is that the Martingale method only involves linear partial differential equations, unlike the nonlinear partial differential equation involved by the dynamic programming. As we will see later that, in many specific models when asset prices follow a geometric Brownian motion, the optimal controls can even be obtained without solving any partial differential equation by utilizing the Martingale method.

We will start by studying the basic theory of utility functions, discussing and comparing the frequently-used utility functions in the literature. The continuous-time optimization problems are formalized in Section 2.3, where the standard Martingale method will be discussed and the procedure of implementing it is summarized. To give us a good feeling of how to use the Martingale method without having to remember the formulations of the optima, we will derive, step by step, the optima for the terminal wealth optimization problem as an example in Section 2.4.

2.2 Basics of utility theory

It is assumed throughout this thesis that the investor is *risk averse*. Therefore, his utility function must be concave.[2.1] We will give a formal definition of a utility function before looking at some examples.

Definition 2.2.1. *In the economic literature, a concave utility function is often referred to a continuous function* $u : (0, \infty) \to \mathbb{R}$ *which is (strictly) increasing, (strictly) concave, continuously differentiable (see for example [6], [11] and [14]). More rigorously, a (concave) utility function should also satisfy the* ***Inada conditions*** *that*

$$\begin{aligned} (i)\ u'(0+) \equiv \lim_{z \downarrow 0} u'(z) &= +\infty \text{ and} \\ (ii)\ \lim_{z \to \infty} u'(z) &= 0 \end{aligned} \tag{2.1}$$

[2.1] A *risk-loving* investor should have a convex utility function while a *risk-neutral* investor should have a linear utility function.

where, the prime ($'$) denotes the derivative (see [23] and [24]).

In the definition above, the requirement that a utility function be (strictly) increasing says that an increase in z (z can be, for example, consumption or wealth) increases the utility; the (strict) concavity implies a diminishing marginal utility, that is, the utility gain decreases with an increase of z. The infinite marginal utility when z approaches the origin implies that 'something is much better than nothing' and the vanishing marginal utility when z approaches ∞ suggests that, for an extremely rich investor, the utility gain from a small increase of wealth or consumption can be ignored.

A concave utility is associated with a *risk-averse* investor and the degree of curvature of the corresponding utility function determines the intensity of the investor's risk aversion. Curvature can be measured by the second derivative of the utility function, scaled by the first derivative. There are two main measures of risk aversion in economics. One is the *absolute risk aversion (ARA)* which is defined by

$$ARA(z) \equiv -\frac{u''(z)}{u'(z)} \tag{2.2}$$

where, ($'$) denotes the first derivative as before and ($''$) denotes the second derivative. The other measure is the *relative risk aversion (RRA)* defined by

$$RRA(z) \equiv -\frac{zu''(z)}{u'(z)}. \tag{2.3}$$

2.2.1 Frequently used utility functions

The following utility functions appear to be frequently used in the literature of economics and finance. Each has its own attractive and unattractive fea-

tures.

(i) ***Quadratic Utility Function***

$$u(z) = az - bz^2,\ a \geq 0,\ b > 0 \text{ and } 0 < z < \tfrac{a}{2b} \tag{2.4}$$

A quadratic utility function can make an optimization model more tractable, in particular, when uncertainty is involved. This is due to its characterization of linear marginal utility. However, quadratic utility is an implausible description of behavior toward risk as it implies an increasing absolute risk aversion in z. It is a common thought that absolute risk aversion should decrease, or at least should not increase with z (See [11]). Moreover, this utility function does not satisfy the Inada conditions of (2.1).

(ii) ***CARA-Exponential Utility Function***

$$u(z) = -e^{-\gamma z},\ \gamma > 0 \text{ and } z > 0 \tag{2.5}$$

The exponential utility function is known as a *constant absolute risk aversion*, or CARA in short, its absolute risk aversion is constant and equal to γ. Exponential utility can produce simple results if asset returns are normally distributed. The shortcoming of this function is that it implies negative consumption or wealth which is not desirable in most cases. This utility function satisfies the Inada condition (ii) but violates the Inada condition (i) of (2.1).

(iii) ***CRRA-Power Utility Function***

$$u(z) = \frac{z^{1-\gamma}}{1-\gamma},\ \gamma > 0,\ \gamma \neq 1 \text{ and } z > 0 \tag{2.6}$$

The power utility has a constant relative risk aversion of γ, and whence CRRA. This utility implies that the absolute risk aversion is declining in z and excludes negative consumption or wealth. The power utility function can produce simple results when asset returns are lognormally distributed. Furthermore, it satisfies both Inada conditions (i) and (ii) of (2.1). These are perhaps the main reasons why the CRRA utility function is so commonly employed in the literature. The coefficient $\frac{1}{\gamma}$ is referred to as the *elasticity of substitution of consumption* in economics.

As a special case when $\gamma \to 1$, the power utility function is simplified to the *logarithmic utility function* $ln(z)$. It is worth noting that the logarithmic utility function $ln(z)$ is not simply the limit of the power utility function $\frac{z^{1-\gamma}}{1-\gamma}$, but rather the limit of the power utility function subtracted by a constant $\frac{1}{1-\gamma}$ as this linear transformation does not affect an investor's preference.

From the discussion above, it appears that the CRRA utility is the most reasonable description of an investor's aversion to risk. Therefore, we will focus on the CRRA utility in what follows.

2.3 The consumption-terminal wealth optimization problem

In this section, we shall consider the financial market $\mathcal{M}$ which consists of $m+1$ assets and satisfies the General Assumption 1 with $d = m$ and the

Novikov condition of (1.21). The prices of the assets are given by

$$\begin{aligned} \frac{dS_0(t)}{S_0(t)} &= R(t)dt, \\ S_0(0) &= s_0 \end{aligned} \tag{2.7}$$

and

$$\begin{aligned} \frac{dS_i(t)}{S_i(t)} &= \mu_i(t)dt + \sum_{j=1}^{m} \sigma_{ij}(t)dW_j(t), \\ S_i(0) &= s_i, \text{ for } i = 1, ..., m \end{aligned} \tag{2.8}$$

When $m = d$, the market price of risk $\theta(t)$ defined in Eq. (1.8) becomes

$$\theta(t) = \sigma^{-1}(t)[\mu(t) - R(t)\mathbf{1}_m] \tag{2.9}$$

It is assumed that a small investor with initial capital x $(0 < x < \infty)$ invests his wealth in the market $\mathcal{M}$ of (2.7)-(2.8) and wishes to maximize his *expected utility of consumption and final wealth* for a given utility function. His optimal decision is made by observing the stock prices in the past and the present. He has no knowledge about future prices, nor has he any inside information. Mathematically, he wishes to solve the following maximization problem:

$$\max_{(\pi,C)\in\mathcal{A}_0(x)} \mathbb{E}\left[u_1(X^{\pi,C}(T)) + \int_0^T e^{-\rho t}u_2(C(t))dt\right] \tag{2.10}$$

subject to

$$\begin{aligned} dX^{\pi,C}(t) &= X^{\pi,C}(t)\left[R(t)dt + \pi^\top(t)\sigma(t)\left(\theta(t)dt + dW(t)\right)\right] - C(t)dt \\ X^{\pi,C}(0) &= x \end{aligned} \tag{2.11}$$

where, $\rho > 0$ is the *rate of time preference* and

$$\mathcal{A}_0(x) \equiv \left\{(\pi, C) \in \mathcal{A}(x) : \mathbb{E}\left[u_1^-(X^{\pi,C}(T))\right] < \infty \text{ and } \mathbb{E}\left[\int_0^T e^{-\rho t}u_2^-(C(t))dt\right] < \infty\right\} \tag{2.12}$$

with $u_i^-(\cdot) \equiv \max\{-u_i^-(\cdot), 0\}$, for $i = 1, 2$. For the convenience of the following discussions, we will name the optimization problem of (2.10)-(2.12) as the *consumption-terminal wealth optimization problem* and study how to solve this problem using Martingale method in the next section.

2.3.1 The Martingale method

In order to discuss the Martingale method, we need the following theorems.

Theorem 2.3.1. *(**Martingale Representation Theorem**)*
Let $M(t)$ be a martingale with respect to the Brownian filtration $\{\mathcal{F}(t)\}_t$ and satisfies

$$\mathbb{E}\left[M^2(t)\right] < \infty \text{ for all } t \in [0, T] \tag{2.13}$$

then there exists a $\mathbb{P}$-progressively measurable process $\psi(t)$ satisfying

$$\mathbb{E}\left[\int_0^T \|\psi(t)\|^2 \, dt\right] < \infty \tag{2.14}$$

and

$$M(t) = M(0) + \int_0^t \psi^\top(s) dW(s) \text{ a.s.} \tag{2.15}$$

Theorem 2.3.2. *(**Market Completeness**)*
(i) Given an initial wealth x (> 0), then for any admissible pair $(\pi, C) \in \mathcal{A}(x)$, the corresponding wealth process $X^{\pi,C}(t)$ satisfies

$$\mathbb{E}\left[H(t)X^{\pi,C}(t) + \int_0^t H(s)C(s)ds\right] \leq x \text{ for all } t \in [0, T]. \tag{2.16}$$

(ii) In the market $\mathcal{M}$ of (2.7)-(2.8), if a contingent claim B (≥ 0) and a consumption-rate process C (≥ 0) satisfy

$$\infty > \mathbb{E}\left[H(T)B + \int_0^T H(t)C(t)dt\right] \equiv x > 0, \tag{2.17}$$

then there exists a portfolio process π *such that* $(\pi, C) \in \mathcal{A}(x)$*, and the corresponding wealth process at the terminal date* T *satisfies*

$$X^{\pi,C}(T) = B \ \ a.s. \tag{2.18}$$

In particular, we have

$$X^{\pi,C}(t) = \frac{1}{H(t)}\mathbb{E}_t\left[H(T)B + \int_t^T H(s)C(s)dt\right], \ for \ all \ t \in [0,T] \tag{2.19}$$

where, $\mathbb{E}_t$ *denotes the expectation conditional on the information set* $\mathcal{F}(t)$*.*

Intuitively, part (i) of this theorem says that the expected discounted (i.e., discounted by the stochastic discount factor $H(t)$) wealth at any feasible time for any reasonable trading strategy cannot exceed the initial wealth. While part (ii) tells us that each desired final wealth B can be obtained by trading according to an appropriate trading strategy given that one possesses enough initial capital. In particular, Theorem 2.3.2. suggests that the consumption-terminal wealth optimization problem is equivalent to the problem of

$$\max_{B,C} \mathbb{E}\left[u_1(B) + \int_0^T e^{-\rho t}u_2(C(t))dt\right] \tag{2.20}$$

subject to the constraint that

$$\mathbb{E}\left[H(T)B + \int_0^T H(t)C(t)dt\right] = x \tag{2.21}$$

with B, C denoting all possible $\mathcal{F}(T)$-measurable contingent claims and consumption-rate processes, respectively (See for example Korn/Korn (2000) [24] and Karatzas (1997) [23] for more details).

The new problem of (2.20)-(2.21) can be solved by using the Lagrange method as follows. Write λ (> 0) for the Lagrangian multiplier, or *shadow price* in the literature of economics, and set

$$L(B, C, \lambda) \equiv \mathbb{E}\left[u_1(B) + \int_0^T e^{-\rho t} u_2(C(t))\right] + \lambda\left\{x - \left[H(T)B + \int_0^T H(t)C(t)dt\right]\right\}. \tag{2.22}$$

Equating the derivatives of the Lagrangian function L with respect to B and C respectively to zero, we obtain the *first order conditions*

$$\begin{aligned} \frac{\partial L}{\partial B} &= \mathbb{E}\left[u_1'(B) - \lambda H(T)\right] = 0 \\ \frac{\partial L}{\partial C} &= \mathbb{E}\left[\int_0^T \left(e^{-\rho t} u_2'(C(t)) - \lambda H(t)\right) dt\right] = 0 \end{aligned} \tag{2.23}$$

From the *convex dual* theory,[2.2] we know that Eq. (2.23) holds if and only if B, $C(t)$ are given by

$$\begin{aligned} B^* &\equiv (u')^{-1}(\lambda H(T)) \\ C^*(t) &\equiv (u')^{-1}(\lambda e^{\rho t} H(t)) \end{aligned} \tag{2.24}$$

where, λ can be (uniquely) obtained from the budget constraint of (2.21). B^* and C^* are the optima for the problem of (2.20)-(2.21). Having solved the equivalent problem of (2.20)-(2.21), we can then find the solution to the consumption-terminal wealth optimization problem of (2.10)-(2.12) according to the following theorem.

Theorem 2.3.3. *Let B^*, C^* be the optima of the problem of (2.20)-(2.21). Then there exists a portfolio π^* such that the pair $(\pi^*, C^*) \in \mathcal{A}_0(x)$ and*

[2.2] For a full discussion of the convex dual theory, see Karatzas (1997) [23].

(π^*, C^*) *is optimal for the consumption-terminal wealth optimization problem of (2.10)-(2.12). The corresponding wealth process satisfies*

$$X^{\pi^*,C^*}(t) = \frac{1}{H(t)}\mathbb{E}_t\left[H(T)B^* + \int_t^T H(s)C^*(s)dt\right], \text{ for all } t \in [0,T]. \tag{2.25}$$

In particular,

$$X^{\pi^*,C^*}(T) = B^* \tag{2.26}$$

The proof of Theorem 2.3.3. is done in three steps. First, we need to show that, for the optima B^* and C^* given in Eq. (2.24), there exists π^* such that $(\pi^*, C^*) \in \mathcal{A}(x)$ and the corresponding wealth process satisfies Eq. (2.25). Second, we show that $\mathbb{E}\left[u_1^-(X^{\pi^*,C^*}(T))\right] < \infty$ and $\mathbb{E}\left[\int_0^T e^{-\rho t}u_2^-(C^(t))dt\right] < \infty$, that is, to show $(\pi^*, C^*) \in \mathcal{A}_0(x)$. Finally, we verify that (π^*, C^*) is optimal for the original optimization problem of (2.10)-(2.12). To do so, we need the following Lemma which is cited from Korn and Korn (2001) [24]

Lemma 2.3.1. *Let I denote the inverse of the first derivative of a utility function u, that is $I \equiv (u')^{-1}$, then we have*

$$u(I(y)) \geq u(z) + y(I(y) - z), \text{ for } 0 < y, z < \infty \tag{2.27}$$

This lemma can easily be verified by using the Taylor expansion and the concavity of a utility function.

Proof. (**Proof of Theorem 2.3.3.**)

Step (i), since B^* and C^* are the optima of the problem of (2.20)-(2.21), they must satisfy the budget constraint (2.21), that is

$$\mathbb{E}\left[H(T)B^* + \int_0^T H(t)C^*(t)dt\right] = x \tag{2.28}$$

From Eq. (2.24), we know that $B^*, C^* \in (0, \infty)$. The existence of an admissible pair (π^*, C^*) and Eq. (2.25)-(2.26) are then followed directly from Theorem 2.3.2.

Step (ii), by Lemma 2.3.1., we know, by choosing $z = 1$ and using Eq. (2.24), that

$$u_1(B^*) \geq u_1(1) + \lambda(x)H(T)(B^* - 1) \tag{2.29}$$

Note that $u_1(\cdot)$, $H(t)$ and B^* are strictly positive and finite. If we assume that $\lambda(x)$, which is determined from the budget constraint of (2.21) and therefore is denoted by $\lambda(x)$ to indicate its dependence of the initial capital x, is strictly positive and finite, then we can get

$$\begin{aligned} u_1^-(B^*) &\leq (u_1(1) + \lambda(x)H(T)(B^* - 1))^- \\ &\leq |u_1(1) + \lambda(x)H(T)(B^* - 1)| \\ &\leq |u_1(1)| + \lambda(x)H(T)(B^* + 1) < \infty \end{aligned} \tag{2.30}$$

Since $u_1^-(\cdot)$ is nonnegative, we then have

$$\mathbb{E}\left[u_1^-(X^{\pi^*,C^*}(T))\right] = \mathbb{E}\left[u_1^-(B^*)\right] < \infty \tag{2.31}$$

Similarly, we can show that

$$\mathbb{E}\left[\int_0^T e^{-\rho t} u_2^-(C^*(t))dt\right] < \infty \tag{2.32}$$

Step (iii), let us consider an arbitrary pair $(\pi, C) \in \mathcal{A}_0(x)$ and its corresponding wealth process $X^{\pi,C}(t)$. From Lemma 2.3.1., we get that

$$u_1(B^*) \geq u_1(X^{\pi,C}(T)) + \lambda(x)H(T)(B^* - X^{\pi,C}(T)) \tag{2.33}$$

and

$$\int_0^T e^{-\rho t} u_2(C^*(t))dt \geq \int_0^T e^{-\rho t} u_2(C(t))dt + \lambda(x)H(t)(C^*(t) - C(t)) \tag{2.34}$$

Adding these two inequalities together and then taking expectation, we get that

$$\begin{aligned} & \mathbb{E}\left[u_1(B^*) + \int_0^T e^{-\rho t} u_2(C^*(t))dt\right] \\ \geq\ & \mathbb{E}\left[u_1(X^{\pi,C}(T)) + \int_0^T e^{-\rho t} u_2(C(t))dt\right] \\ +\ & \lambda(x)\left\{\mathbb{E}\left[H(T)B^* + \int_0^T H(t)C^*(t)dt] - \mathbb{E}[H(T)X^{\pi,C}(T) + \int_0^T H(t)C(t)dt\right]\right\} \end{aligned} \tag{2.35}$$

Keep in mind that B^* and C^* satisfy the constraint of (2.21) and use the conclusion from Theorem 2.3.2. part (i) that

$$\mathbb{E}\left[H(T)X^{\pi,C}(T) + \int_0^T H(t)C(t)dt\right] \leq x, \tag{2.36}$$

holds for any $(\pi, C) \in \mathcal{A}(x)$. Therefore the term in the braces on the last line of Eq. (2.35) is non-negative. We can now conclude that

$$\begin{aligned} & \mathbb{E}\left[u_1(B^*) + \int_0^T e^{-\rho t} u_2(C^*(t))dt\right] \\ \geq\ & \mathbb{E}\left[u_1(X^{\pi,C}(T)) + \int_0^T e^{-\rho t} u_2(C(t))dt\right] \end{aligned} \tag{2.37}$$

□

The portfolio π^* in Theorem 2.3.3. can be identified from the wealth process of Eq. (2.25). For typographical convenience, we denote the corresponding *optimal wealth process* $X^{\pi^*,C^*}(t)$ by $X^*(t)$.

Expanding the conditional expectation on the right-hand side of Eq. (2.25) and multiplying it by $H(t)$, we get

$$H(t)X^*(t) = \mathbb{E}_t\left[H(T)B^* + \int_0^T H(s)C^*(s)dt\right] - \mathbb{E}_t\left[\int_0^t H(s)C^*(s)dt\right] \quad (2.38)$$

The optimal consumption rate $C^*(s)$, for all $0 \leq s \leq t$, is $\mathcal{F}(t)$-measurable and so is $H(s)C^*(s)$. Therefore, the conditional expection $\mathbb{E}_t$, in the last term of Eq. (2.38), can be dropped out. It now can be written as

$$H(t)X^*(t) + \int_0^t H(s)C^*(s)dt = \underbrace{\mathbb{E}_t\left[H(T)B^* + \int_0^T H(s)C^*(s)dt\right]}_{\equiv M(t)}$$

Note that the conditional expectation is a martingale with respect to the Brownian filtration $\{\mathcal{F}(t)\}_t$[2.3]. If we denote this Martingale by $M(t)$, then it follows from the Martingale representation theorem that $M(t)$ can be represented as a stochastic integral with respect to the Brownian motion. In other words, there exists an $\mathbb{P}$-progressively measurable process $\psi(t)$ such that

$$M(t) = M(0) + \int_0^t \psi^\top(s)dt, \text{ for all t} \quad (2.39)$$

By the definition of $M(t)$, it is trivial to check that $M(0) = x$. So we now have

$$H(t)X^*(t) + \int_0^t H(s)C^*(s)dt = x + \int_0^t \psi^\top(s)dt \quad (2.40)$$

Obviously, Eq. (1.16) holds at the optimum, that is,

$$H(t)X^*(t) + \int_0^t H(s)C^*(s)ds = x + \int_0^t H(s)X^*(s)[\sigma^\top(s)\pi^*(s) - \theta(s)]^\top dW(s) \quad (2.41)$$

[2.3]This can be verified by the definition of Martingale.

Comparing Eq. (2.40) with Eq. (2.41), we get

$$H(t)X^*(t)[\sigma^\top(t)\pi^*(t) - \theta(t)]^\top = \psi^\top(t) \tag{2.42}$$

This gives us the optimal portfolio for the problem of (2.10)-(2.12)

$$\pi^*(t) = (\sigma^\top(t))^{-1}\left(\frac{\psi(t)}{H(t)X^*(t)} + \theta(t)\right) \tag{2.43}$$

For the special cases of the *terminal wealth optimization problem* and the *consumption optimization problem*, we have following corollaries which can be verified by setting $u_2 \equiv$ or $u_1 \equiv 0$ correspondingly.

Example 2.3.1. *(**Terminal wealth optimization problem**)*
Given an initial capital x $(0 < x < \infty)$ and a utility function $u_1(\cdot)$, consider the problem of maximizing expected utility from terminal wealth

$$\max_{\pi \in \mathcal{A}_1(x)} \mathbb{E}\left[u_1(X_1^\pi(T))\right] \tag{2.44}$$

subject to

$$\begin{aligned} dX_1^\pi(t) &= X_1^\pi(t)[R(t)dt + \pi^\top(t)\sigma(t)\left(\theta(t)dt + dW(t)\right)] \\ X_1^\pi(0) &= x \end{aligned} \tag{2.45}$$

with

$$\mathcal{A}_1(x) \equiv \left\{\pi \equiv (\pi, 0) \in \mathcal{A}(x) : \mathbb{E}\left[u_1^-(X_1^\pi(T))\right] < \infty\right\} \tag{2.46}$$

Corollary 2.3.1. *The terminal wealth optimization problem above is equivalent to the problem*

$$\max_B \mathbb{E}[u_1(B)] \tag{2.47}$$

subject to the constraint

$$\mathbb{E}[H(T)B] = x \tag{2.48}$$

with B denoting all possible $\mathcal{F}(T)$-measurable contingent claims.

Let B^* be the optimum of the problem of (2.47)-(2.48). There exists then a portfolio π^* such that $\pi^* \in \mathcal{A}_1(x)$ and π^* is optimal for the terminal wealth optimization problem of (2.44)-(2.46). The corresponding value of the optimal portfolio process satisfies

$$X_1^*(t) \equiv X_1^{\pi^*}(t) = \frac{1}{H(t)}\mathbb{E}_t[H(T)B^*], \text{ for all } t \in [0,T]. \tag{2.49}$$

In particular,

$$X_1^*(T) = B^* \tag{2.50}$$

Remark 2.3.1. *In the pure terminal wealth optimization problem, it is implicitly assumed that there is no consumption, that is*

$$C(t) = 0, \text{ for all } t \in [0,T] \tag{2.51}$$

Otherwise, the investor is not optimizing his utility. Let us assume that $C(t) > 0$, for some $t \in [0,T]$, then the investor could have gained more utility by saving the positive amount consumed given that there is no utility gain from any consumption (because $u_2 \equiv 0$).

Example 2.3.2. *(**Consumption optimization problem**)*
Given an initial capital x $(0 < x < \infty)$ and a utility function $u_2(\cdot)$, consider the problem of maximizing expected utility from consumption

$$\max_{(\pi,C)\in\mathcal{A}_2(x)} \mathbb{E}\left[\int_0^T e^{-\rho t}u_2(C(t))dt\right] \tag{2.52}$$

subject to

$$\begin{aligned} dX^{\pi,C}(t) &= X^{\pi,C}(t)\left[R(t)dt + \pi^{\top}(t)\sigma(t)\left(\theta(t)dt + dW(t)\right)\right] - C(t)dt \\ X^{\pi,C}(0) &= x \end{aligned} \tag{2.53}$$

with

$$\mathcal{A}_2(x) \equiv \left\{(\pi, C) \in \mathcal{A}(x) : \mathbb{E}\left[\int_0^T e^{-\rho t} u_2^-(C(t))dt\right] < \infty\right\} \tag{2.54}$$

Corollary 2.3.2. *The consumption optimization problem above is equivalent to the problem*

$$\max_{\pi,C} \mathbb{E}\left[\int_0^T e^{-\rho t} u_2(C(t))dt\right] \tag{2.55}$$

subject to the constraint

$$\mathbb{E}\left[\int_0^T H(t)C(t)dt\right] = x \tag{2.56}$$

Let C^* be the optimum of problem of (2.55)-(2.56). Then there exists a portfolio π^* such that the pair $(\pi^*, C^*) \in \mathcal{A}_2(x)$ and (π^*, C^*) is optimal for the consumption optimization problem The corresponding optimal wealth process satisfies

$$X_2^*(t) \equiv X_2^{\pi^*,C^*}(t) = \frac{1}{H(t)}\mathbb{E}_t\left[\int_t^T H(s)C^*(s)dt\right], \text{ for all } t \in [0,T]. \tag{2.57}$$

Remark 2.3.2. *In the pure consumption optimization problem, it is implicitly assumed that the terminal wealth is zero, that is*

$$X^{\pi,C}(T) = 0 \tag{2.58}$$

Otherwise, the consumer is not optimizing his utility. More specifically, if we assume that $X^{\pi,C}(T) > 0$, then the consumer could have gained more utility by consuming the positive amount wealth of $X^{\pi,C}(T)$ before the terminal date given that there is no utility gain from having a positive wealth (because $u_1 \equiv 0$).

Let us summarize the Martingale method below.

Procedure of implementing the Martingale method:

- *Step 1*: Identify the equivalent optimization problem;
- *Step 2*: Solve the equivalent optimization problem of Step 1 to obtain B^* and/or C^*;
- *Step 3*: Substitute B^* and/or C^* to the corresponding optimal wealth process $X^*(t)$;
- *Step 4*: Find the optimal portfolio π^* that produces this optimal wealth process $X^*(t)$.

Following the steps above, we will have obtained the optimal solution π^* (and C^*) to the original optimization problem. To familiarize ourselves with the procedure of using the Martingale method, we will look at the terminal wealth optimization problem with the CRRA utility function in the next section.

2.4 The terminal wealth optimization problem

We adopt the market $\mathcal{M}$ of (2.7)-(2.8). For simplicity, we assume the market coefficients are *deterministic*. Let us look at the terminal wealth optimization problem with the CRRA utility function defined in Eq. (2.6).

$$\max_{\pi \in \mathcal{A}_1(x)} \mathbb{E}\left[u_1(X_1^\pi(T))\right] \tag{2.59}$$

subject to

$$\begin{aligned} dX_1^\pi(t) &= X_1^\pi(t)\left[R(t)dt + \pi^\top(t)\sigma(t)\left(\theta(t)dt + dW(t)\right)\right] \\ X_1^\pi(0) &= x \end{aligned} \tag{2.60}$$

with

$$\mathcal{A}_1(x) \equiv \left\{\pi \equiv (\pi, 0) \in \mathcal{A}(x) : \mathbb{E}\left[u_1^-(X_1^\pi(T))\right] < \infty\right\} \tag{2.61}$$

where

$$u_1(z) = \frac{z^{1-\gamma}}{1-\gamma}, \quad \gamma > 0. \tag{2.62}$$

According to the Corollary 2.3.2., the problem of (2.59)-(2.62) is equivalent to the following *static* optimization problem.

$$\max_B \mathbb{E}[u_1(B)] \tag{2.63}$$

subject to

$$\mathbb{E}[H(T)B] = x. \tag{2.64}$$

The Lagrangian function of this problem is written as

$$L(B,\lambda) \equiv \mathbb{E}\left[u_1(B) + \lambda\left(x - H(T)B\right)\right] \tag{2.65}$$

Equating the derivatives of the Lagrangian L with respect to B to zero, we obtain:

$$\frac{\partial L}{\partial B} \;=\; \mathbb{E}\left[u_1'\left(B\right) - \lambda H(T)\right] = 0 \tag{2.66}$$

This gives us the optimum

$$B^* = (u_1')^{-1}(\lambda H(T)). \tag{2.67}$$

For the choice of CRRA utility function, we have

$$(u_1')^{-1}(z) = z^{-\frac{1}{\gamma}}. \tag{2.68}$$

Substituting it from Eq. (2.67) gives us

$$B^* = \lambda^{-\frac{1}{\gamma}}(H(T))^{-\frac{1}{\gamma}} \tag{2.69}$$

while the Lagrange multiplier λ is determined by the constraint

$$\mathbb{E}\left[\lambda^{-\frac{1}{\gamma}}(H(T))^{\frac{\gamma-1}{\gamma}}\right] = x$$

Solving it for λ, we get

$$\lambda^{-\frac{1}{\gamma}} = \frac{x}{\mathbb{E}\left[(H(T))^{\frac{\gamma-1}{\gamma}}\right]} \tag{2.70}$$

A substitution of $\lambda^{-\frac{1}{\gamma}}$ of Eq. (2.70) from Eq. (2.69) gives us the optimal terminal value via the following formula

$$B^* = x\frac{(H(T))^{-\frac{1}{\gamma}}}{\mathbb{E}\left[(H(T))^{\frac{\gamma-1}{\gamma}}\right]} \tag{2.71}$$

Substituting B^* in Eq. (2.71) to the optimal wealth process $X_1^*(t)$ in Eq. (2.49), we get

$$\begin{aligned} X_1^*(t) &= \frac{1}{H(t)}\mathbb{E}_t[H(T)B^*] \\ &= x\frac{1}{H(t)}\frac{\mathbb{E}_t\left[(H(T))^{\frac{\gamma-1}{\gamma}}\right]}{\mathbb{E}\left[(H(T))^{\frac{\gamma-1}{\gamma}}\right]} \end{aligned} \tag{2.72}$$

Multiply both sides of Eq. (2.72) by $H(t)$

$$H(t)X_1^*(t) = x\frac{\mathbb{E}_t\left[(H(T))^{\frac{\gamma-1}{\gamma}}\right]}{\mathbb{E}\left[(H(T))^{\frac{\gamma-1}{\gamma}}\right]} \tag{2.73}$$

By introducing the *exponential martingale*

$$Z(t) \equiv e^{\frac{1-\gamma}{\gamma}\int_0^t \theta^\top(s)W(s) - \frac{1}{2}\left(\frac{1-\gamma}{\gamma}\right)^2\int_0^t \|\theta(s)\|^2 ds}, \tag{2.74}$$

and defining that

$$f(t) \equiv e^{\frac{1-\gamma}{\gamma}\int_0^t\left(R(s)+\frac{1}{2\gamma}\|\theta(s)\|^2\right)ds}, \tag{2.75}$$

we obtain

$$(H(t))^{\frac{\gamma-1}{\gamma}} = f(t)Z(t) \tag{2.76}$$

Noting that the Martingale $Z(t)$ has expectation of one and $f(t)$ is non-random, we can get

$$\begin{aligned} \frac{\mathbb{E}_t\left[(H(T))^{\frac{\gamma-1}{\gamma}}\right]}{\mathbb{E}\left[(H(T))^{\frac{\gamma-1}{\gamma}}\right]} &= \frac{\mathbb{E}_t[f(T)Z(T)]}{\mathbb{E}[f(T)Z(T)]} \\ &= \frac{\mathbb{E}_t[Z(T)]}{\mathbb{E}[Z(T)]} \\ &= \frac{Z(t)}{Z(0)} \\ &= Z(t) \end{aligned} \tag{2.77}$$

Substituting back into Eq. (2.73), we then arrive at that

$$H(t)X_1^*(t) = xZ(t) \tag{2.78}$$

Taking differentials on both sides of Eq. (2.77) and using Eq. (2.74) results in

$$\begin{aligned} d(H(t)X_1^*(t)) &= xd(Z(t)) \\ &= -xZ(t)\frac{\gamma-1}{\gamma}\theta^\top(t)dW(t). \end{aligned}$$

A use of Eq. (2.77) gives[2.4]

$$d(H(t)X_1^*(t)) = H(t)X_1^*(t)\frac{1-\gamma}{\gamma}\theta^\top(t)dW(t) \tag{2.79}$$

And $H(t)X_1^*(t)$ also satisfies Eq. (1.18), i.e.,

$$d(H(t)X_1^*(t)) = H(t)X_1^*(t)[\sigma^\top(t)\pi^*(t) - \theta(t)]^\top dW(t) \tag{2.80}$$

A comparison of Eq. (2.78) with Eq. (2.79) gives us the *optimal portfolio* $\pi^*(t)$ for the problem of (2.59)-(2.62)

$$\pi^*(t) = \frac{1}{\gamma}(\sigma^{-1}(t))^\top\theta(t). \tag{2.81}$$

The elements of the optimal portfolio vector is given then by

$$\pi_i^*(t) = \frac{1}{\gamma}\frac{\mu_i(t) - R(t)}{\sigma_i^2(t)}, \text{ for i=1, ..., m} \tag{2.82}$$

The optimal rule of (2.81) or (2.82), referred to as the *classical Merton rule*, tells us that, when the small investor has a constant-relative-risk-aversion utility, then the optimal share of wealth invested in each of the risky assets is

[2.4]Note, in this example, that $\psi(t)$ in Eq. (2.39) equals $H(t)X_1^*(t)\frac{1-\gamma}{\gamma}\theta$.

constant over time. By definition, we know that the optimal share of wealth invested in the risk-free bond, denoted by $\pi_0^*(t)$, is determined as

$$\pi_0^*(t) = 1 - \sum_{i=1}^{m} \pi_i^*(t) = 1 - \frac{1}{\gamma} \sum_{i=1}^{m} \frac{\mu_i(t) - R(t)}{\sigma_i^2(t)} \tag{2.83}$$

Let $\alpha_i \equiv \frac{\mu_i(t) - R(t)}{\sigma_i^2(t)}$, we can further get

$$(\pi_0^*(t), \pi_1^*(t), ..., \pi_m^*(t))^\top = (1, 0, ..., 0)^\top - \frac{1}{\gamma}(\sum_{i=1}^{m} \alpha_i, \alpha_1, ..., \alpha_m)^\top \tag{2.84}$$

By writing the vector $(\pi_0^*(t), \pi_1^*(t), ..., \pi_m^*(t))^\top$ of fractions of wealth invested in the $m+1$ assets as the linear combination of two independent vectors $(1, 0, ..., 0)^\top$ and $(\sum_{i=1}^{m} \alpha_i, \alpha_1, ..., \alpha_m)^\top$. Eq. (2.83) is interpreted as that the optimal portfolio, including the optimal share invested in the risk-free bond, can be formed from a linear combination of two *mutual funds*.

The *optimal expected utility of final wealth* is then obtained by substituting B^* of Eq. (2.71) into the objective function of (2.59) and (2.62),

$$\begin{aligned}
\max_{\pi \in \mathcal{A}_1(x)} \mathbb{E}\left[u_1(X^\pi(T))\right] &= \mathbb{E}[u_1(B^*)] \\
&= \frac{x^{1-\gamma}}{1-\gamma} \mathbb{E}\left[\left(\frac{(H(T))^{-\frac{1}{\gamma}}}{\mathbb{E}\left[(H(T))^{\frac{\gamma-1}{\gamma}}\right]} \right)^{1-\gamma} \right] \\
&= \frac{x^{1-\gamma}}{1-\gamma} \left(\mathbb{E}\left[(H(T))^{\frac{\gamma-1}{\gamma}}\right] \right)^{\gamma}
\end{aligned}$$

Using Eqs. (2.74)-(2.76), we then obtain

$$\max_{\pi \in \mathcal{A}_1(x)} \mathbb{E}\left[u_1(X^\pi(T))\right] = \frac{x^{1-\gamma}}{1-\gamma} e^{(1-\gamma)\int_0^T \left(R(t) + \frac{1}{2\gamma}\|\theta(t)\|^2\right) dt} \tag{2.85}$$

Chapter 3

Index bond

3.1 Introduction

Inflation-linked bonds were introduced more than 30 years ago in a number of Latin American and European countries and have enjoyed an increasing popularity in the UK, Canada, and some continental European states (Zhang/Korn/Ewald (2007)). Since the inflation-linked bonds are risky, they could be sold at a real interest rate below that on the risk-free bond. In the work conducted by Korn and Kruse (2004), a fair price for an inflation-linked bond has been derived with the Black-Scholes argument.

In this chapter, we investigate an optimization problem in a market where there exists an inflation-linked bond. When an inflation-linked bond presents, it should make more sense to study the optimization problem maximizing the expect utility of real terminal wealth or/and from real consumption. We adopt the basic price dynamics of the inflation-linked bond that has been

used in the paper by Fischer (1975), where the inflation-linked bond is referred to as the *index bond*. We provide a way to transform the problem of maximizing the expected utility of real terminal wealth to the problem of maximizing the expected utility of nominal terminal wealth, to which the Martingale method can be directly applied. This is summarized in Theorem 3.4.1. We find that the optimal portfolio rule offers both optimal expected utility of real and nominal terminal wealth. This would be desirable for insurance agents that bear the risk due to unexpected inflation. We will turn to discuss the issue related to pension insurance in the next chapter. Let us now begin by studying the price dynamics of the price level of the market.

It is assumed that the behavior of the price level $P(t)$ is described by the geometric Brownian motion

$$\begin{aligned} \frac{dP(t)}{P(t)} &= i(t)dt + \sigma_1(t)dW_1(t) \\ P(0) &= p > 0 \end{aligned} \tag{3.1}$$

where, $i(t)$ is the expected rate of inflation and $W_1(t)$ is the source of uncertainty which causes the price level to fluctuate around the expected inflation with an instantaneous intensity of fluctuation $\sigma_1(t)(\neq 0)$.

3.2 Asset price dynamics

We consider a market $\mathcal{M}$ consisting of a risk-free bond, an index bond and a stock.

(1) As before, the risk-free bond pays a rate of nominal return of $R(t)$ and follows the differential equation

$$\frac{dB(t)}{B(t)} = R(t)dt \tag{3.2}$$

(2) The *index bond* offers a rate of real return of $r(t)$ and its price process $I(t)$ is given by

$$\begin{aligned} \frac{dI(t)}{I(t)} &= r(t)dt + \frac{dP(t)}{P(t)} \\ &= (r(t) + i(t))dt + \sigma_1(t)dW_1(t) \end{aligned} \tag{3.3}$$

So the index bond pays an expected rate of nominal return equal to the sum of the rate of real return and the expected rate of inflation, that is $r(t)+i(t)$.

(3) The price process of the stock $S(t)$ is given by

$$\frac{dS(t)}{S(t)} = \mu(t)dt + \sigma_2(t)dW_2(t) \tag{3.4}$$

where, $\mu(t)$ is the expected rate of return on the stock and $\sigma_2(t)(\neq 0)$ is the volatility caused by the source of risk $W_2(t)$. We assume that $W_2(t)$ is independent of $W_1(t)$.

If we denote by $1-\pi_1(t)-\pi_2(t)$, $\pi_1(t)$, $\pi_2(t)$ the shares of portfolio value invested in the risk-free bond, the index bond and the stock, respectively, then the corresponding portfolio value process $X^\pi(t)$ satisfies

$$\frac{dX^\pi(t)}{X^\pi(t)} = R(t)dt + \pi^\top(t)\sigma(t)[\theta(t)dt + dW(t)] \tag{3.5}$$

where $\pi(t) = (\pi_1(t), \pi_2(t))^\top$, $W(t) = (W_1(t), W_2(t))^\top$,

$$\sigma(t) = \begin{pmatrix} \sigma_1(t) & 0 \\ 0 & \sigma_2(t) \end{pmatrix} \tag{3.6}$$

and

$$\theta(t) = \begin{pmatrix} \frac{r(t)+i(t)-R(t)}{\sigma_1(t)} \\ \frac{\mu(t)-R(t)}{\sigma_2(t)} \end{pmatrix} \equiv \begin{pmatrix} \theta_1(t) \\ \theta_2(t) \end{pmatrix}. \tag{3.7}$$

3.3 Asset real returns

The real value of an asset is defined by the quotient of its nominal value divided by the price level. Dividing the prices of risk-free bond, index bond and stock by the price level and applying Itô's lemma, we can express the returns on the assets in real terms as follows.

(1) The real return on the risk-free bond satisfies

$$\frac{d(B(t)/P(t))}{B(t)/P(t)} = r_1(t)dt - \sigma_1(t)dW_1(t) \tag{3.8}$$

where $r_1(t) \equiv R(t) - i(t) + \sigma_1^2(t)$. This says that the expected rate of real return on the risk-free bond is the difference between the rate of nominal return and the rate of expected inflation $R(t) - i(t)$ plus the variance of the price level $\sigma_1^2(t)$.

(2) The real return on the index bond is given by

$$\frac{d(I(t)/P(t))}{I(t)/P(t)} = r(t)dt \tag{3.9}$$

This confirms that the index bond pays a rate of real return of $r(t)$.

(3) The real return on the stock satisfies

$$\frac{d(S(t)/P(t))}{S(t)/P(t)} = r_2(t)dt - \sigma_1(t)dW_1(t) + \sigma_2(t)dW_2(t) \tag{3.10}$$

where, $r_2(t) \equiv \mu(t) - i(t) + \sigma_1^2(t)$. This says that the expected rate of real return on the stock is the difference between the rate of expected nominal return and the rate of expected inflation $\mu(t) - i(t)$ plus the variance of the price level $\sigma_1^2(t)$.

3.4 The real terminal wealth optimization problem

We are now in a position to consider an optimization problem in the market $\mathcal{M}$ of (3.2)-(3.4). It is assumed that a small agent with an initial wealth of x $(0 < x < \infty)$ invests $1 - \pi_1(t) - \pi_2(t)$, $\pi_1(t)$, $\pi_2(t)$ shares of wealth in the risk-free bond, the index bond and the stock, respectively. He wishes to maximize his *expected utility of terminal real value of portfolio*. In other words, he faces a maximization problem as

$$\max_{\pi \in \mathcal{A}_1(x)} \mathbb{E}\left[u\left(\frac{X^\pi(T)}{P(T)}\right)\right] \tag{3.11}$$

subject to

$$\begin{aligned} \frac{dX^\pi(t)}{X^\pi(t)} &= R(t)dt + \pi^\top(t)\sigma(t)[\theta(t)dt + dW(t)] \\ X^\pi(0) &= x \end{aligned} \tag{3.12}$$

where,

$$\mathcal{A}_1(x) \equiv \left\{\pi \in \mathcal{A}(x) : \mathbb{E}\left[u^-\left(\frac{X^\pi(T)}{P(T)}\right)\right] < \infty\right\}. \tag{3.13}$$

We will demonstrate below that the maximization problem of (3.11)-(3.13) is equivalent to maximizing the expected utility of terminal value of

the portfolio on the assets of (3.8)-(3.10). Let we construct a market $\tilde{\mathcal{M}}$, in which there are a risk-free bond and two risky assets as in (3.8)-(3.10).

(1) The risk-free bond offers a rate of nominal return equal to $r(t)$ and its price process, denoted by $\tilde{I}(t)$, satisfies

$$\begin{aligned} \frac{d\tilde{I}(t)}{\tilde{I}(t)} &= r(t)dt \\ \tilde{I}(0) &= \frac{I(0)}{P(0)} \end{aligned} \tag{3.14}$$

That is, the rate of nominal return on this risk-free bond is equal to the rate of real return on the index bond.

(2) One of the risky assets' price process, denoted by $\tilde{B}(t)$, is defined by

$$\begin{aligned} \frac{d\tilde{B}(t)}{\tilde{B}(t)} &= r_1(t)dt - \sigma_1(t)dW_1(t) \\ \tilde{B}(0) &= \frac{B(0)}{P(0)} \end{aligned} \tag{3.15}$$

where, $r_1(t)$ is treated as the expected rate of nominal return on the first risky asset and $W_1(t)$ is the Brownian motion associated with the price level of (3.1).

(3) The price dynamics of the other risky asset, denoted by $\tilde{S}(t)$, is given by

$$\frac{d\tilde{S}(t)}{\tilde{S}(t)} = r_2(t)dt - \sigma_1(t)dW_1(t) + \sigma_2(t)dW_2(t) \tag{3.16}$$

where, $r_2(t)$ is treated as the expected rate of nominal return on the second risky asset and $W_2(t)$ is the Brownian motion associated with the stock price of (3.4) in the market $\mathcal{M}$.

Therefore, in the constructed market $\tilde{\mathcal{M}}$ of (3.14)-(3.16), the volatility matrix and the market price of risk are given by

$$\tilde{\sigma}(t) \equiv \begin{pmatrix} -\sigma_1(t) & 0 \\ -\sigma_1(t) & \sigma_2(t) \end{pmatrix} \tag{3.17}$$

and

$$\begin{aligned} \tilde{\theta}(t) &\equiv \begin{pmatrix} \tilde{\theta}_1(t) \\ \tilde{\theta}_2(t) \end{pmatrix} \equiv \tilde{\sigma}^{-1}(t) \begin{pmatrix} r_1(t) - r(t) \\ r_2(t) - r(t) \end{pmatrix} \\ &= \begin{pmatrix} -\frac{1}{\sigma_1(t)} & 0 \\ -\frac{1}{\sigma_2(t)} & \frac{1}{\sigma_2(t)} \end{pmatrix} \begin{pmatrix} r_1(t) - r(t) \\ r_2(t) - r(t) \end{pmatrix} \\ &= \begin{pmatrix} \frac{r(t)-r_1(t)}{\sigma_1(t)} \\ \frac{r_2(t)-r_1(t)}{\sigma_2(t)} \end{pmatrix} \end{aligned} \tag{3.18}$$

respectively. A simple computation can show that

$$\begin{aligned} \tilde{\theta}_1(t) &= \theta_1(t) - \sigma_1(t) \\ \tilde{\theta}_2(t) &= \theta_2(t) \end{aligned} \tag{3.19}$$

If we denote by $1 - \tilde{\pi}_1(t) - \tilde{\pi}_2(t)$, $\tilde{\pi}_1(t)$, $\tilde{\pi}_2(t)$ the shares of portfolio value invested in the risk-free bond, the first risky asset and the second risky asset in the market $\tilde{\mathcal{M}}$, respectively, then the corresponding portfolio value process $\tilde{X}^{\tilde{\pi}}(t)$ with an initial value of $\frac{x}{p}$ satisfies

$$\frac{d\tilde{X}^{\tilde{\pi}}(t)}{\tilde{X}^{\tilde{\pi}}(t)} = r(t)dt + \tilde{\pi}^{\top}(t)\tilde{\sigma}(t)[\tilde{\theta}(t)dt + dW(t)] \tag{3.20}$$

where $\tilde{\pi}(t) = (\tilde{\pi}_1(t), \tilde{\pi}_2(t))^{\top}$, $W(t) = (W_1(t), W_2(t))^{\top}$.

The following theorem tells us that the optimal portfolio $\pi(t)$ of the problem of (3.11)-(3.13) in the market $\mathcal{M}$ can be obtained by solving the corresponding maximization problem in the market $\tilde{\mathcal{M}}$.

Theorem 3.4.1. *Consider the market $\tilde{\mathcal{M}}$ of (3.14)-(3.16), the maximization problem of (3.11)-(3.13) is equivalent to*

$$\max_{\tilde{\pi}\in\mathcal{A}_1(x/p)} \mathbb{E}\left[u\left(\tilde{X}^{\tilde{\pi}}(T)\right)\right] \tag{3.21}$$

subject to

$$\begin{aligned} \frac{d\tilde{X}^{\tilde{\pi}}(t)}{\tilde{X}^{\tilde{\pi}}(t)} &= r(t)dt + \tilde{\pi}^{\top}(t)\tilde{\sigma}(t)[\tilde{\theta}(t)dt + dW(t)] \\ \tilde{X}^{\tilde{\pi}}(0) &= \frac{x}{p} \end{aligned} \tag{3.22}$$

where, $\tilde{\sigma}(t)$ and $\tilde{\theta}(t)$ satisfy Eq. (3.17) and Eq. (3.18), respectively, and

$$\mathcal{A}_1(x/p) \equiv \left\{\tilde{\pi}\in\mathcal{A}(x/p) : \mathbb{E}\left[u^{-}\left(\tilde{X}^{\tilde{\pi}}(T)\right)\right] < \infty\right\} \tag{3.23}$$

Moreover, we have that

$$\begin{aligned} \pi(t) &= (\pi_1(t), \pi_2(t))^{\top} = (\tilde{\pi}_0(t), \tilde{\pi}_2(t))^{\top} \\ \pi_0(t) &= \tilde{\pi}_1(t) \end{aligned} \tag{3.24}$$

Proof. First, we show that $\frac{X^{\pi}(t)}{P(t)} = \tilde{X}^{\tilde{\pi}}(t)$.

An application of Itô's Lemma to the differential equation of the price level, Eq. (3.1), gives us that

$$d\left(\frac{1}{P(t)}\right) = -\frac{1}{P(t)}[(i(t) - \sigma_1^2(t))dt + \sigma_1(t)dW_1(t)] \tag{3.25}$$

Applying then the stochastic product rule to $\frac{X^\pi(t)}{P(t)}$ and using Eqs. (3.5) and (3.25), we have

$$\begin{aligned} d\left(\frac{X^\pi(t)}{P(t)}\right) &= \frac{1}{P(t)}dX^\pi(t) + X^\pi(t)d\left(\frac{1}{P(t)}\right) + dX(t)d\left(\frac{1}{P(t)}\right) \\ &= \frac{X^\pi(t)}{P(t)}[(\underbrace{R(t) - i(t) + \sigma_1^2(t)}_{=r_1(t)})dt \\ &\quad +\pi_1(t)\sigma_1(t)(\theta_1(t) - \sigma_1(t))dt + \pi_2(t)\sigma_2(t)\theta_2(t)dt \\ &\quad -(1-\pi_1(t))\sigma_1(t)dW_1(t) + \pi_2(t)\sigma_2(t)dW_2(t)] \end{aligned}$$

Rearranging gives

$$\begin{aligned} d\left(\frac{X^\pi(t)}{P(t)}\right) &= \frac{X^\pi(t)}{P(t)}[r(t)dt - (\underbrace{r(t) - r_1(t)}_{=\sigma_1(t)\tilde{\theta}_1(t)})dt \\ &\quad +\pi_1(t)\sigma_1(t)(\underbrace{\theta_1(t) - \sigma_1(t)}_{=\tilde{\theta}_1(t)})dt + \pi_2(t)\sigma_2(t)\underbrace{\theta_2(t)}_{=\tilde{\theta}_2(t)}dt \\ &\quad -(1-\pi_1(t))\sigma_1(t)dW_1(t) + \pi_2(t)\sigma_2(t)dW_2(t)] \end{aligned} \tag{3.26}$$

Using Eq. (3.18)-(3.19) and collecting terms, we get

$$\begin{aligned} d\left(\frac{X^\pi(t)}{P(t)}\right) &= \frac{X^\pi(t)}{P(t)}[r(t)dt - (1-\pi_1(t))\sigma_1(t)\tilde{\theta}_1(t)dt + \pi_2(t)\sigma_2(t)\tilde{\theta}_2(t)dt \\ &\quad -(1-\pi_1(t))\sigma_1(t)dW_1(t) + \pi_2(t)\sigma_2(t)dW_2(t)] \end{aligned} \tag{3.27}$$

Noting that $1 - \pi_1(t) = \pi_0(t) + \pi_2(t)$ and dividing both sides of Eq. (3.27) by $\frac{X^\pi(t)}{P(t)}$, we obtain

$$\begin{aligned} \frac{d\left(X^\pi(t)/P(t)\right)}{X^\pi(t)/P(t)} &= r(t)dt - (\pi_0(t) + \pi_2(t))\sigma_1(t)\tilde{\theta}_1(t)dt + \pi_2(t)\sigma_2(t)\tilde{\theta}_2(t)dt \\ &\quad -(\pi_0(t) + \pi_2(t))\sigma_1(t)dW_1(t) + \pi_2(t)\sigma_2(t)dW_2(t) \end{aligned} \tag{3.28}$$

If we write $\tilde{\pi}(t) \equiv (\tilde{\pi}_1(t), \tilde{\pi}_2(t))^\top$, set that

$$\tilde{\pi}_1(t) = \pi_0(t),\ \tilde{\pi}_2(t) = \pi_2(t) \tag{3.29}$$

and recall that

$$\tilde{\sigma}(t) = \begin{pmatrix} -\sigma_1(t) & 0 \\ -\sigma_1(t) & \sigma_2(t) \end{pmatrix},$$

then Eq. (3.28) can be rewritten as

$$\frac{d\left(X^\pi(t)/P(t)\right)}{X^\pi(t)/P(t)} = r(t)dt + \tilde{\pi}^\top(t)\tilde{\sigma}(t)\left[\tilde{\theta}(t)dt + dW(t)\right]. \tag{3.30}$$

At the initial time, we have

$$\frac{X^\pi(0)}{P(0)} = \frac{x}{p}. \tag{3.31}$$

So we can conclude that

$$\frac{X^\pi(t)}{P(t)} = \tilde{X}^{\tilde{\pi}}(t), \text{for all } t \in [0, T]. \tag{3.32}$$

In particular, we have that

$$\frac{X^\pi(T)}{P(T)} = \tilde{X}^{\tilde{\pi}}(T). \tag{3.33}$$

Secondly, we show that the constraint of (3.22) is equivalent to the constraint of (3.12).

We have actually proved the necessity in the first step above. We only need to show the sufficiency. Assume

$$\frac{d\tilde{X}^{\tilde{\pi}}(t)}{\tilde{X}^{\tilde{\pi}}(t)} = r(t)dt + \tilde{\pi}^\top(t)\tilde{\sigma}(t)[\tilde{\theta}(t)dt + dW(t)] \tag{3.34}$$

Write the vectors appearing on the right-hand side of Eq. (3.34) in components as

$$\begin{aligned}\frac{d\tilde{X}^{\tilde{\pi}}(t)}{\tilde{X}^{\tilde{\pi}}(t)} &= r(t)dt - (\tilde{\pi}_1(t)+\tilde{\pi}_2(t))\sigma_1(t)\tilde{\theta}_1(t)dt + \tilde{\pi}_2(t)\sigma_2(t)\tilde{\theta}_2(t)dt \\ &\quad -(\tilde{\pi}_1(t)+\tilde{\pi}_2(t))\sigma_1(t)dW_1(t) + \tilde{\pi}_2(t)\sigma_2(t)\tilde{\theta}_2(t)dW_2(t)\end{aligned} \tag{3.35}$$

By using the fact that $\tilde{\pi}_1(t)+\tilde{\pi}_2(t) = 1-\tilde{\pi}_0(t)$, Eq. (3.35) can be further written as

$$\begin{aligned}\frac{d\tilde{X}^{\tilde{\pi}}(t)}{\tilde{X}^{\tilde{\pi}}(t)} &= r(t)dt - \underbrace{\sigma_1(t)\tilde{\theta}_1(t)}_{=r(t)-r_1(t)}dt + \tilde{\pi}_0(t)\sigma_1(t)\tilde{\theta}_1(t)dt + \tilde{\pi}_2(t)\sigma_2(t)\tilde{\theta}_2(t)dt \\ &\quad -\sigma_1(t)dW_1(t) + \tilde{\pi}_0(t)\sigma_1(t)dW_1(t) + \tilde{\pi}_2(t)\sigma_2(t)dW_2(t)\end{aligned} \tag{3.36}$$

After canceling the terms of $r(t)dt$ on the right-hand side of Eq. (3.36), we arrive at

$$\begin{aligned}\frac{d\tilde{X}^{\tilde{\pi}}(t)}{\tilde{X}^{\tilde{\pi}}(t)} &= r_1(t)dt + \tilde{\pi}_0(t)\sigma_1(t)\tilde{\theta}_1(t)dt + \tilde{\pi}_2(t)\sigma_2(t)\tilde{\theta}_2(t)dt \\ &\quad -\sigma_1(t)dW_1(t) + \tilde{\pi}_0(t)\sigma_1(t)dW_1(t) + \tilde{\pi}_2(t)\sigma_2(t)dW_2(t)\end{aligned} \tag{3.37}$$

Now using Eq. (3.32), we get

$$X^{\pi}(t) = P(t)\tilde{X}^{\tilde{\pi}}(t). \tag{3.38}$$

In particular, we have

$$X^{\pi}(0) = P(0)\tilde{X}^{\tilde{\pi}}(0) = p\frac{x}{p} = x. \tag{3.39}$$

Differentiating on the both sides of Eq. (3.38), then applying the stochastic product rule to $P(t)X^{\tilde{\pi}}(t)$ and using Eqs. (3.37) and (3.1), we can get

$$\begin{aligned} dX^{\pi}(t) &= d(P(t)\tilde{X}^{\tilde{\pi}}(t)) \\ &= P(t)d\tilde{X}^{\tilde{\pi}}(t) + \tilde{X}^{\tilde{\pi}}(t)dP(t) + dP(t)d\tilde{X}^{\tilde{\pi}}(t) \\ &= \underbrace{P(t)\tilde{X}^{\tilde{\pi}}(t)}_{=X^{\pi}(t)}[(\underbrace{r_1(t) + i(t) - \sigma_1^2(t)}_{=R(t)})dt \\ &\quad +\tilde{\pi}_0(t)\sigma_1(t)(\underbrace{\tilde{\theta}_1(t) + \sigma_1(t)}_{=\theta_1(t)})dt + \tilde{\pi}_2(t)\sigma_2(t)\underbrace{\tilde{\theta}_2(t)}_{=\theta_2(t)}dt \\ &\quad +\tilde{\pi}_0(t)\sigma_1(t)dW_1(t) + \tilde{\pi}_2(t)\sigma_2(t)dW_2(t)] \end{aligned} \tag{3.40}$$

If we set $\tilde{\pi}_0(t) = \pi_1(t)$ and $\tilde{\pi}_2(t) = \pi_2(t)$, we then get

$$\begin{aligned} dX^{\pi}(t) &= X^{\pi}(t)[R(t)dt + \pi_1(t)\sigma_1(t)\theta_1(t)dt + \pi_2(t)\sigma_2(t)\theta_2(t)dt \\ &\quad +\pi_1(t)\sigma_1(t)dW_1(t) + \pi_2(t)\sigma_2(t)dW_2(t)] \end{aligned} \tag{3.41}$$

which is the same as

$$\frac{dX^{\pi}(t)}{X^{\pi}(t)} = R(t)dt + \pi^{\top}(t)\sigma(t)\theta(t)dt + \pi^{\top}(t)\sigma(t)dW(t) \tag{3.42}$$

Finally, we show that $\pi \in \mathcal{A}_1(x)$ if and only if $\tilde{\pi} \in \mathcal{A}_1(x/p)$.

Let $\pi \in \mathcal{A}(x)$, then by the definition of admissibility we know that

$$X^{\pi}(t) \geq 0, \text{ a.s., for all } t \in [0, T]. \tag{3.43}$$

Since $P(t) > 0$, we have

$$\tilde{X}^{\tilde{\pi}}(t) = \frac{X^{\pi}(t)}{P(t)} \geq 0, \text{ a.s., for all } t \in [0, T]. \tag{3.44}$$

Using the result concluded in Eq. (3.32), we know that if $\mathbb{E}\left[u^{-}\left(\frac{X^{\pi}(T)}{P(T)}\right)\right] < \infty$, then $\mathbb{E}\left[u^{-}\left(\tilde{X}^{\tilde{\pi}}(T)\right)\right] < \infty$. Similarly, let $\tilde{\pi} \in \mathcal{A}_1(x/p)$, we then have $\pi \in \mathcal{A}_1(x)$. $\square$

Now we can solve the problem of maximizing real terminal wealth of (3.11)-(3.13) by solving a typical terminal (nominal) wealth optimization problem as in (3.21)-(3.23). Let us consider the CRRA utility function. Directly applying the result obtained in the last chapter, Section 2.4 Eqs. (2.81), to the problem of (3.21)-(3.23), we can get the optimal portfolio $\tilde{\pi}^*$ as follows

$$\begin{aligned}
\tilde{\pi}^*(t) &= \begin{pmatrix} \tilde{\pi}_1^*(t) \\ \tilde{\pi}_2^*(t) \end{pmatrix} \\
&= \frac{1}{\gamma} \left(\tilde{\sigma}^{-1}(t)\right)^\top \tilde{\theta}(t) \\
&= \frac{1}{\gamma} \begin{pmatrix} -\frac{1}{\sigma_1(t)} & -\frac{1}{\sigma_2(t)} \\ 0 & \frac{1}{\sigma_2(t)} \end{pmatrix} \begin{pmatrix} \theta_1(t) - \sigma_1(t) \\ \theta_2(t) \end{pmatrix} \\
&= \frac{1}{\gamma} \begin{pmatrix} 1 - \frac{\theta_1(t)}{\sigma_1(t)} - \frac{\theta_2(t)}{\sigma_2(t)} \\ \frac{\theta_2(t)}{\sigma_2(t)} \end{pmatrix}
\end{aligned} \tag{3.45}$$

and

$$\tilde{\pi}_0^*(t) = 1 - \tilde{\pi}_1^*(t) - \tilde{\pi}_2^*(t) = \frac{1}{\gamma} \frac{\theta_1(t)}{\sigma_1(t)} \tag{3.46}$$

According to the Theorem 3.4.1., we have the relationship between the optimal portfolio π^* for the problem of (3.11)-(3.13) and the optimal portfolio $\tilde{\pi}^*$ for the problem of (3.21)-(3.23) as

$$\pi^*(t) = \begin{pmatrix} \pi_1^*(t) \\ \pi_2^*(t) \end{pmatrix} = \begin{pmatrix} \tilde{\pi}_0^*(t) \\ \tilde{\pi}_2^*(t) \end{pmatrix} \tag{3.47}$$

Therefore, the optimal portfolio π^* for the problem of (3.11)-(3.13) is given

by

$$\pi^*(t) = \frac{1}{\gamma} \begin{pmatrix} \frac{\theta_1(t)}{\sigma_1(t)} \\ \frac{\theta_2(t)}{\sigma_2(t)} \end{pmatrix} = \frac{1}{\gamma} \left(\sigma^{-1}(t)\right)^\top \theta(t) \tag{3.48}$$

and

$$\pi_0^*(t) = 1 - \pi_1^*(t) - \pi_2^*(t) = \frac{1}{\gamma} \left(1 - \frac{\theta_1(t)}{\sigma_1(t)} - \frac{\theta_2(t)}{\sigma_2(t)}\right) = \tilde{\pi}_1^*(t) \tag{3.49}$$

The optimal portfolio given in Eq. (3.48) is exactly the classical Merton's portfolio rule of maximizing the expected utility of (nominal) terminal wealth in the market $\mathcal{M}$ of (3.2)-(3.4). That suggests that the optimal portfolio of maximizing the expected utility of real terminal wealth, with a CRRA utility function, can also be achieved by solving the problem of maximizing the expected utility of (nominal) terminal wealth instead. However, this in general is not true for markets where the index bond is absent.

3.5 Conclusion

The optimal portfolio of maximizing the expected utility of terminal *real* wealth has appeared to coincide with that of maximizing the expected utility of terminal *nominal* wealth. Put it differently, by following the trading strategy of (3.48)-(3.49), one can achieve the optimal expected utilities of both terminal nominal wealth and terminal real wealth, at least for the CRRA utility. That actually is a good news for the investor who invests in the index bond because it indeed protects himself against risk due to the unanticipated inflation.

Chapter 4

Optimal investment for a pension fund under inflation risk

4.1 Introduction

There are two basic types of pension schemes: *defined benefit* (DB) and *defined contribution* (DC). In a DB plan, the plan sponsor promises to the plan beneficiaries a final level of pension benefits. This level is usually defined according to a benefit formula, as a function of a member's (or employee's) final salary (or average salary) and/or years of service in the company. Benefits are usually paid as a life annuity rather than as a lump sum. The main advantage of a DB plan is that it offers stable income replacement rates (i.e. pension as a proportion of final salary) to retired beneficiaries and is subsequently indexed to inflation. The financial risks associated with a pure DB

plan are borne by the plan sponsor, usually a large company, rather than the plan member. The sponsor is obliged to provide adequate funds to cover the plan liabilities. The major drawbacks include the lack of benefit portability when changing jobs and the complex valuation of plan liabilities. In a DB plan, when a worker moves jobs, he can end up with a much lower pension in retirement. For example, a typical UK worker moving jobs six times in a career could end up with a pension of only 71-75% that of a worker with the same salary experience who remains in the same job for his whole career (Blake and Orszag (1997)). In a defined benefit pension plan, the risk associated with future returns on a fund's assets is carried out by the employer or sponsor and the contribution rate varies through time as the level of the fund fluctuates above and below its target level. This fluctuation can be dealt with through the plan's *investment policy* (including *asset allocation decision, investment manager selection* and *performance measurement*). Cairns (2000) has considered an investment problem, in which the sponsor minimises the discounted expected loss by selecting a contribution rate and an asset-allocation strategy.

A defined contribution (DC) plan has a defined amount of contribution payable by both employee and employer, often as a fixed percentage of salary. The employee's retirement benefit is determined by the size of the accumulation at retirement. The benefit ultimately paid to the member is not known for certain until retirement. The benefit formula is not defined either, as opposed to the DB pension plans. At retirement, the beneficiaries can usually take the money as a life annuity, a phased withdrawal plan, a lump sum

payment, or some combination of these. As the value of the pension benefits is simply determined as the market value of the backing assets, the pension benefits are easily transferable between jobs. In a pure DC plan, plan members have extensive control over their accounts' investment strategy (usually subject to the investment menu offered). While the employer or sponsor is only obliged to make regular contributions, the employees bear a range of risks. In particular, they bear *asset price risk* (the risk of losses in the value of their pension fund due to falls in asset values) at retirement and *inflation risk* (the risk of losses in the real value of pensions due to unanticipated inflation). Generally speaking, a pure DC pension plan is more costly for employees than a pure DB plan.

Nevertheless, Pension plans, in the world, have been undergoing a transition from DB plans toward DC plans, which involves enormous transfers of risks from taxpayers and corporate DB sponsors to the individual members of DC plans (see, for example, Winklevoss 1993 and Blake/Cairns/Dowd (2001)). So it is of interest to study a DC plan's investment policy, under which the plan members can protect themselves against both asset price risk and inflation risk. We have seen in the last chapter that one can protect himself against the risk due to unexpected inflation by investing some share of wealth in the index bond. The way to reduce these risks to the minimum would then be to trade the DC pension funds in the index bond by following the optimal portfolio rule which we will address in the sections that follow.

4.2 The investment problem for a DC pension fund

It is assumed that a representative member of a DC pension plan makes contributions continuously to the pension fund during a fixed time horizon $[0, T]$. The contribution rate is fixed as a percentage c of his salary. We will consider the investment problem, from the perspective of a representative DC plan member, in which the investment decision is made through an insurance company or a pension manager. The objective is to maximize the expected utility of terminal value of his pension fund.

Let us define the stochastic price level as

$$\begin{aligned} \frac{dP(t)}{P(t)} &= idt + \sigma_1 dW_1(t) \\ P(0) &= p > 0 \end{aligned} \tag{4.1}$$

where, the constant i is the expected rate of inflation and $W_1(t)$ is the source of uncertainty which causes the price level to fluctuate around the expected inflation with an instantaneous intensity of fluctuation σ_1.

Assume there is a market $\mathcal{M}$ consisting of three assets which are of interest for the pension fund manager. These assets are a risk-free bond, an index bond and a stock.

(1) The risk-free bond pays a constant rate of nominal return of R and

its price dynamics is given by

$$\frac{dB(t)}{B(t)} = Rdt. \tag{4.2}$$

(2) The index bond offers a constant rate of real return of r and its price process is given by

$$\begin{aligned} \frac{dI(t)}{I(t)} &= rdt + \frac{dP(t)}{P(t)} \\ &= (r+i)dt + \sigma_1 dW_1(t) \end{aligned} \tag{4.3}$$

(3) The price process of the stock is given by

$$\frac{dS(t)}{S(t)} = \mu dt + \sigma_2 dW_2(t) \tag{4.4}$$

where, μ is the expected rate of return on the stock and σ_2 is the volatility caused by the source of risk W_2. As before, $W_2(t)$ is assumed to be independent on $W_1(t)$.

Let us assume that $\sigma_1 \neq 0$ and $\sigma_2 \neq 0$. Then the volatility matrix

$$\sigma \equiv \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix} \tag{4.5}$$

is nonsingular. As a consequence, there exists a (unique) market price of risk θ satisfying

$$\theta = \sigma^{-1} \begin{pmatrix} r+i-R \\ \mu - R \end{pmatrix} \equiv \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}. \tag{4.6}$$

The market is therefore arbitrage-free and complete. We further assume that the salary of the pension plan member follows the dynamics:

$$\begin{aligned} \frac{dY(t)}{Y(t)} &= \kappa dt + \sigma_3 dW_1(t) \\ Y(0) &= y > 0 \end{aligned} \tag{4.7}$$

where κ is the expected growth rate of salary and σ_1 is the volatility of salary which is driven by the source of uncertainty of inflation. Both κ and σ_3 are constants. It can be verified by Itô's Lemma that the following process is the solution to the stochastic differential equation (4.7)

$$Y(t) = ye^{\left(\kappa - \frac{1}{2}\sigma_3^2\right)t + \sigma_3 W_1(t)} \tag{4.8}$$

If we write $\sigma_Y \equiv (\sigma_3, 0)^\top$, then we can rewrite Eq. (4.8) as [4.1]

$$Y(t) = ye^{\left(\kappa - \frac{1}{2}\|\sigma_Y\|^2\right)t + \sigma_Y^\top W(t)} \tag{4.9}$$

If the plan member contributes continuously to his DC pension fund with a fixed *contribution rate* (i.e. the percentage of the member's salary) of c (> 0) and $1 - \pi_1(t) - \pi_2(t)$, $\pi_1(t)$, $\pi_2(t)$ shares of the pension fund are invested in the riskless bond, the index bond and the stock, respectively. Then the corresponding wealth process with an initial value of x ($0 < x < \infty$), which we denote by $X^\pi(t)$, is governed by the following equation

$$dX^\pi(t) = X^\pi(t)[Rdt + \pi^\top(t)\sigma\left(\theta dt + dW(t)\right)] + cY(t)dt \tag{4.10}$$

where, $cY(t)$ is the amount of money contributed to the pension fund at time t and $\pi(t) = (\pi_1(t),\ \pi_2(t))^\top$. Note that the contributions are assumed to be invested continuously over time. The contribution at time t, $cY(t)$, can be viewed as the rate of a random endowment and is strictly positive.

Definition 4.2.1. *A portfolio process π is said to be admissible if the corresponding wealth process $X^\pi(t)$ in (4.10) satisfies*

$$X^\pi(t) + \mathbb{E}_t\left[\int_t^T \frac{H(s)}{H(t)} cY(s)ds\right] \geq 0, \text{ for all } t \in [0, T] \tag{4.11}$$

[4.1] So it is straightforward to allow the salary process to be correlated with the stock price.

almost surely. We denote the class of admissible portfolio processes by $\mathcal{A}_Y$.

Note that, in the presence of a positive random endowment stream, the wealth is allowed to become negative, so long as the present value of future endowments is large enough to offset such a negative value.

We are now in a position to formally describe the objective of the plan member. The objective is written mathematically as

$$\max_{\pi \in \bar{\mathcal{A}}(x)} \mathbb{E}\left[u\left(X^{\pi}(T)\right)\right] \tag{4.12}$$

subject to

$$\begin{aligned} dX^{\pi}(t) &= X^{\pi}(t)[Rdt + \pi^{\top}(t)\sigma\left(\theta dt + dW(t)\right)] + cY(t)dt \\ X^{\pi}(0) &= x \end{aligned} \tag{4.13}$$

where,

$$\bar{\mathcal{A}}(x) \equiv \left\{\pi \in \mathcal{A}_Y(x) : \mathbb{E}\left[u^{-}\left(X^{\pi}(T)\right)\right] < \infty\right\}. \tag{4.14}$$

The utility function is assumed to be of CRRA form

$$u(z) = \frac{z^{1-\gamma}}{1-\gamma} \tag{4.15}$$

By the comparison of the constraint of (4.13) with the constraints for the terminal wealth optimization problems, with which we have dealt so far, we notice that there is an additional term $cY(t)dt$, which is not proportional to the corresponding wealth. We have to get rid of this term before we can follow the procedure of the Martingale method. We will treat this in the next section.

4.3 How to solve it

In order to express the first equality in the budget constraint of (4.13) as a form linear in the corresponding wealth, let us examine the expectation of the plan member's future contribution which is defined below.

Definition 4.3.1. *The present value of expected future contribution process is defined as*

$$D(t) \;=\; \mathbb{E}_t\left[\int_t^T \frac{H(s)}{H(t)} cY(s)ds\right] \tag{4.16}$$

where, $\mathbb{E}_t$ is the conditional expectation with respect to the Brownian filtration $\{\mathcal{F}(t)\}_t$ and

$$H(t) \equiv e^{-Rt-\frac{1}{2}\|\theta\|^2 t-\theta^\top W(t)} \tag{4.17}$$

is the stochastic discount factor which adjusts for nominal interest rate and market price of risk.

By inspecting the Markovian structure of the expression on the right-hand side of Eq. (4.16), we note that it should be possible to express $D(t)$ in terms of the instantaneous contribution $cY(t)$. The following proposition shows this possibility.

Proposition 4.3.1. *The present value of expected future contribution process $D(t)$ is proportional to the instantaneous contribution process $cY(\cdot)$, that is,*

$$D(t) = \frac{1}{\beta}\left(e^{\beta(T-t)}-1\right) cY(t), \text{ for all } t \in [0,T] \tag{4.18}$$

with $\beta \equiv \kappa - R - \sigma_3\theta_1$. In particular,

$$\begin{aligned} d \equiv D(0) &= \frac{1}{\beta}\left(e^{\beta T}-1\right) cy \\ D(T) &= 0 \end{aligned} \tag{4.19}$$

Proof. By definition we have

$$\begin{aligned} D(t) &= \mathbb{E}_t\left[\int_t^T \frac{H(s)}{H(t)} cY(s)ds\right] \\ &= cY(t)\mathbb{E}_t\left[\int_t^T \frac{H(s)}{H(t)}\frac{Y(s)}{Y(t)}ds\right] \end{aligned} \tag{4.20}$$

Both processes $H(\cdot)$ and $Y(\cdot)$ are geometric Brownian motions and therefore it follows easily that $\frac{H(s)}{H(t)}\frac{Y(s)}{Y(t)}$ is independent of $\mathcal{F}(t)$ for $s \geq t$. As a consequence, the conditional expectation collapses to an unconditional expectation and we obtain

$$D(t) = cY(t)g(t,T) \tag{4.21}$$

with the deterministic function $g(t,T)$ being defined by

$$g(t,T) \equiv \mathbb{E}\left[\int_0^{T-t} H(s)\frac{Y(s)}{Y(0)}ds\right] \tag{4.22}$$

Noting that

$$\begin{aligned} H(s)\frac{Y(s)}{Y(0)} &= e^{(\kappa-R)s}e^{(\sigma_3-\theta_1)W_1(s)-\theta_2 W_2(s)-\frac{1}{2}\left(\|\theta\|^2+\sigma_3^2\right)s} \\ &= e^{(\kappa-R)s}e^{(\sigma_Y-\theta)^\top W(s)-\frac{1}{2}\left(\|\theta\|^2+\|\sigma_Y\|^2\right)s} \\ &= e^{\beta s}e^{(\sigma_Y-\theta)^\top W(s)-\frac{1}{2}\left(\|\sigma_Y-\theta\|^2\right)s}, \end{aligned} \tag{4.23}$$

we obtain

$$\begin{aligned} \mathbb{E}\left[H(s)\frac{Y(s)}{Y(0)}\right] &= \mathbb{E}\left[e^{\beta s}\right]\cdot\mathbb{E}\left[e^{(\sigma_Y-\theta)^\top W(s)-\frac{1}{2}\left(\|\sigma_Y-\theta\|^2\right)s}\right] \\ &= e^{\beta s} \end{aligned} \tag{4.24}$$

The last equality is obtained by the fact that a stochastic exponential Martingale has expectation of one. Integrating both sides of Eq. (4.24) gives

$$\begin{aligned} \int_0^{T-t}\mathbb{E}\left[H(s)\frac{Y(s)}{Y(0)}\right]ds &= \int_0^{T-t} e^{\beta s}ds \\ &= \frac{1}{\beta}\left(e^{\beta(T-t)}-1\right) \end{aligned} \tag{4.25}$$

The left-hand side of Eq. (4.25) is equal to $g(t,T)$ by the Fubini theorem. $\square$

Differentiating both sides of Eq. (4.18) and using Eq. (4.7), we get

$$\begin{aligned} dD(t) &= d\left(\frac{1}{\beta}\left(e^{\beta(T-t)}-1\right)cY(t)\right) \\ &= \frac{1}{\beta}\left(e^{\beta(T-t)}-1\right)cdY(t)+\frac{1}{\beta}cY(t)d\left(e^{\beta(T-t)}-1\right) \\ &= \frac{1}{\beta}\left(e^{\beta(T-t)}-1\right)cY(t)\left(\kappa dt+\sigma_Y^\top dW(t)\right)-\frac{1}{\beta}cY(t)e^{\beta(T-t)}\beta dt \end{aligned}$$

Collecting terms, we obtain

$$dD(t)=\frac{1}{\beta}\left(e^{\beta(T-t)}-1\right)cY(t)\left((\kappa-\beta)dt+\sigma_Y^\top dW(t)\right)-cY(t)dt \tag{4.26}$$

Using the equality of (4.18) and the definition of β in Proposition 4.3.1, we then have

$$dD(t)=D(t)\left[(R+\sigma_3\theta_1)dt+\sigma_Y^\top dW(t)\right]-cY(t)dt \tag{4.27}$$

If we add Eq. (4.27) and the first equality in Eq. (4.13) together, the term $cY(t)$ will be canceled out. We will define a process based on this observation below.

Definition 4.3.2. *Let us define a process*

$$V(t)\equiv X^\pi(t)+D(t) \tag{4.28}$$

where $X^\pi(t)$ and $D(t)$ satisfy Eqs. (4.13) and (4.16), respectively.

Taking differentials on both sides of Eq. (4.28) and using Eq. (4.13) and

Eq. (4.27), we have

$$\begin{aligned} dV(t) &= dX^{\pi}(t) + dD(t) \\ &= X^{\pi}(t)\left[Rdt + \pi^{\top}(t)\sigma\left(\theta dt + dW(t)\right)\right] + cY(t)dt \\ &\quad + D(t)\left[(R + \sigma_3\theta_1)dt + \sigma_Y^{\top}dW(t)\right] - cY(t)dt \end{aligned}$$

Collecting terms gives us

$$dV(t) = V(t)\left[Rdt + \frac{X^{\pi}(t)\pi^{\top}(t)\sigma + D(t)\sigma_Y^{\top}}{V(t)}\left(\theta dt + dW(t)\right)\right] \tag{4.29}$$

From Eq. (4.29), we can see that $dV(t)$ is proportional to $V(t)$. Next, we check whether the discounted process of $V(t)$ is a $\mathbb{P}$-local Martingale. Multiplying $V(t)$ by $H(t)$ in (4.17) and taking diffeentials, we can get

$$\begin{aligned} d\left(H(t)V(t)\right) &= H(t)dV(t) + V(t)dH(t) + dH(t)dV(t) \\ &= H(t)V(t)\left[Rdt + \frac{X^{\pi}(t)\pi^{\top}(t)\sigma + D(t)\sigma_Y^{\top}}{V(t)}\left(\theta dt + dW(t)\right)\right] \\ &\quad -H(t)V(t)\left[Rdt + \theta^{\top}dW(t)\right] \\ &\quad -H(t)V(t)\frac{X^{\pi}(t)\pi^{\top}(t)\sigma + D(t)\sigma_Y^{\top}}{V(t)}\theta dt \end{aligned}$$

After canceling out terms, we obtain

$$\begin{aligned} d\left(H(t)V(t)\right) &= H(t)V(t)\left[\frac{X^{\pi}(t)\pi^{\top}(t)\sigma + D(t)\sigma_Y^{\top}}{V(t)} - \theta^{\top}\right]dW(t) \\ &= H(t)V(t)\left[\frac{X^{\pi}(t)\sigma^{\top}\pi(t) + D(t)\sigma_Y}{V(t)} - \theta\right]^{\top}dW(t) \end{aligned} \tag{4.30}$$

This shows that $H(t)V(t)$ is a $\mathbb{P}$- local martingale as it can be written as a stochastic integral with respect to the Brownian motion $W(t)$. Moreover, we

know that

$$V(T) = X^{\pi}(T) + D(T) = X^{\pi}(T) \tag{4.31}$$

and

$$V(0) = X^{\pi}(0) + D(0) = x + d \tag{4.32}$$

So we can conclude that the plan member's optimization problem of (4.12)-(4.14) is equivalent to maximizing $\mathbb{E}[u(V(T))]$ over a class of admissible portfolio process π, subject to the constraint of (4.29) and (4.32) (or (4.31)). We will discuss this formally in the next section.

4.4 Optimal management of the pension fund

We have just discussed in the previous section that the plan member's optimization problem of (4.12)-(4.14) can be solved by solving the problem of

$$\max_{\pi \in \mathcal{A}_1(x+d)} \mathbb{E}[u(V(T))] \tag{4.33}$$

subject to

$$\begin{aligned} dV(t) &= V(t)\left[Rdt + \frac{X^{\pi}(t)\pi^{\top}(t)\sigma + D(t)\sigma_Y^{\top}}{V(t)}\left(\theta dt + dW(t)\right)\right] \\ V(0) &= x + d \end{aligned} \tag{4.34}$$

where,

$$\mathcal{A}_1(x+d) \equiv \left\{\pi \in \mathcal{A}(x+d) : \mathbb{E}\left[u^{-}\left(V(T)\right)\right] < \infty\right\} \tag{4.35}$$

and $\mathcal{A}(x+d)$ is the class of admissible portfolio processes with an initial value $x+d$, such that the corresponding portfolio value process satisfies

$$V(t) = X^{\pi}(t) + D(t) \geq 0, \text{ for all } t \in [0,T] \tag{4.36}$$

It is easy to check that $\pi \in \bar{\mathcal{A}}(x)$ if and only if $\pi \in \mathcal{A}_1(x+d)$.

A direct application of the results obtained in Section 2.4 to the problem of (4.33)-(4.35) with the CRRA utility in (4.15), we can get

$$B^* = (x+d)\frac{(H(T))^{-\frac{1}{\gamma}}}{\mathbb{E}\left[(H(T))^{\frac{\gamma-1}{\gamma}}\right]} \tag{4.37}$$

The corresponding optimum wealth process is then given by

$$V^*(t) = \frac{(x+d)}{H(t)}\frac{\mathbb{E}_t\left[(H(T))^{\frac{\gamma-1}{\gamma}}\right]}{\mathbb{E}\left[(H(T))^{\frac{\gamma-1}{\gamma}}\right]} \tag{4.38}$$

Let us write

$$Z_1(t) \equiv e^{\frac{1-\gamma}{\gamma}\theta^{\top}W(t) - \frac{1}{2}\left(\frac{1-\gamma}{\gamma}\right)^2\|\theta\|^2 t} \tag{4.39}$$

and

$$f_1(t) \equiv e^{\frac{1-\gamma}{\gamma}\left(R + \frac{1}{2\gamma}\|\theta\|^2\right)t}. \tag{4.40}$$

We then obtain

$$(H(t))^{\frac{\gamma-1}{\gamma}} = f_1(t)Z_1(t) \tag{4.41}$$

Eq. (4.38) now becomes

$$V^*(t) = \frac{(x+d)}{H(t)}Z_1(t) \tag{4.42}$$

Multiplying $H(t)$ on both sides of Eq. (4.42) and then differentiating both sides, we get

$$d\left(H(t)V^*(t)\right) = H(t)V^*(t)\frac{1-\gamma}{\gamma}\theta^\top dW(t) \tag{4.43}$$

As Eq. (4.30) should also hold at the optimum, we have

$$d\left(H(t)V^*(t)\right) = H(t)V^*(t)\left[\frac{X^*(t)\sigma^\top\pi^*(t) + D(t)\sigma_Y}{V^*(t)} - \theta\right]^\top dW(t), \tag{4.44}$$

where, $X^*(t) \equiv X^{\pi^*}(t)$. A comparison of Eq.(4.44) with Eq. (4.43) leads to

$$\frac{X^*(t)\sigma^\top\pi^*(t) + D(t)\sigma_Y}{V^*(t)} - \theta = \frac{1-\gamma}{\gamma}\theta, \tag{4.45}$$

from which we can solve for $\pi^*(t)$

$$\pi^*(t) = \frac{1}{\gamma}(\sigma^\top)^{-1}\theta\frac{V^*(t)}{X^*(t)} - (\sigma^\top)^{-1}\sigma_Y\frac{D(t)}{X^*(t)}. \tag{4.46}$$

This formula depends on the optimal portfolio value $V^*(t)$, which consists of the optimal pension fund level $X^*(t)$ and the expected future contributions $D(t)$. We have seen in Proposition 4.3.1 that the expected future contributions of the plan member is observable given the member's current salary. So it will be more convenient for the fund manager to implement the optimal strategy if we express it in terms of $D(t)$. Substituting $V^*(t)$ in Eq. (4.46) by $X^*(t) + D(t)$, we can get

$$\pi^*(t) = \frac{1}{\gamma}(\sigma^\top)^{-1}\theta + (\sigma^\top)^{-1}\left(\frac{1}{\gamma}\theta - \sigma_Y\right)\frac{D(t)}{X^*(t)} \tag{4.47}$$

In particular, at the initial date $t = 0$, we have

$$\pi^*(0) = \frac{1}{\gamma}(\sigma^\top)^{-1}\theta + (\sigma^\top)^{-1}\left(\frac{1}{\gamma}\theta - \sigma_Y\right)\frac{d}{x}, \text{ for } x > 0 \tag{4.48}$$

Remark 1. *The optimal investment strategy of (4.47) is made up of two parts:*

- *one is the classical optimal portfolio rule we saw in Section 2.4, and*
- *the other is proportional to the ratio of the present value of expected future contribution to the optimal portfolio value-to-date.*

We can express the optimal portfolio strategy $\pi^*(t)$ obtained in Eq. (4.47) in terms of the asset prices at time t ($I(t)$ and $S(t)$) and the plan member's current salary $Y(t)$. Since $X^*(t) = V^*(t) - D(t)$ and $D(t)$ is proportional to $Y(t)$ (recall Eq. (4.18)), we only need to write $V^*(t)$ in terms of the observable variables $I(t)$, $S(t)$ and $Y(t)$.[4.2]

Dividing both sides of Eq. (4.41) by $H(t)f_1(t)$ gives us

$$\frac{1}{f_1(t)}(H(t))^{-\frac{1}{\gamma}} = \frac{1}{H(t)}Z_1(t) \tag{4.49}$$

Eq. (4.42) then can be rewitten as

$$V^*(t) = \frac{(x+d)}{f_1(t)}(H(t))^{-\frac{1}{\gamma}} \tag{4.50}$$

So we only need to write $H(t)$ in terms of the observable variables. It can be shown that

$$H(t) = e^{\alpha t}\left(\frac{I(t)}{I(0)}\right)^a\left(\frac{S(t)}{S(0)}\right)^b, \tag{4.51}$$

[4.2] In fact, it will be sufficient to express the optimal investment strategy in terms of only two variables from the combination of $S(t)$ and any one of the variables $Y(t)$, $I(t)$ and the current price level $P(t)$ (see Zhang/Korn/Ewald (2007) for details).

where,

$$\begin{aligned}
\alpha &\equiv (r+i)\left(\frac{r+i-R}{\sigma_1}-\frac{1}{2}\right)+\mu\left(\frac{\mu-R}{\sigma_2}-\frac{1}{2}\right)-\frac{1}{2}\|\theta\|^2 \\
a &\equiv -\frac{r+i-R}{\sigma_1} \\
b &\equiv -\frac{\mu-R}{\sigma_2}
\end{aligned} \tag{4.52}$$

Therefore, we have

$$V^*(t)=\frac{(x+d)}{f_1(t)}e^{-\frac{\alpha}{\gamma}t}\left(\frac{I(t)}{I(0)}\right)^{-\frac{a}{\gamma}}\left(\frac{S(t)}{S(0)}\right)^{-\frac{b}{\gamma}} \tag{4.53}$$

Chapter 5

Optimal decisions in a labor market

5.1 Introduction

We have dealt with a terminal wealth optimization problem with an exogenous *income stream* (contributions) being added to the budget constraint in Chapter 4. In this chapter, we will turn to the intertemporal consumption optimization problem with a *labor-income stream* which is determined endogenously within the model.

A closed-form solution to the intertemporal consumption problem, in which both asset return and labor income uncertainty are considered simultaneously, appears to be *absent* in the literature of economics. Much of the existing literature either provides an approximate solution or relies on restrictive assumptions to obtain analytical results. The former includes Ma-

son/Wright (2001), Chan/Viceira (2000) and Chamberlain/Wilson (2000). For the latter, much of the published work assumes that the asset return is non-stochastic. This is reflected, for example, in the book by Blanchard and Fischer (1989) and many others. Some of the literature takes the advantage of the quadratic utility function that has the characterization of linear marginal utility. This can be seen, for example, in the work by Blanchard and Fischer (1989) and the some references therein. However, as having been discussed in Section 2.2 of Chapter 2, the quadratic utility is an unattractive description of behavior toward risk as it implies increasing absolute risk aversion. Some assumptions are also restricted to the nature of the uncertainty of labor income. For example, Toche (2005) assumes that the uncertainty is about the timing of the income loss in addition to the assumption of non-stochastic asset return. Similarly, Pitchford (1991) takes the form of uncertainty as the timing of the reversal of an income shock.

In this chapter, we investigate the continuous-time consumption model with stochastic asset returns and stochastic labor income, while the latter is caused by the stochastic labor supply which is to be determined within the model upon the available market information. The utility function is assumed to be a linear combination of two CRRA utility functions with respect to consumption and labor supply, respectively. Our model appears to have some similarity with the model considered by Bodie, Merton and Samuelson in that, both consider the objective of maximizing expected discounted lifetime utility and both assume that the utility function has two arguments

(consumption and labor/leisure).[5.1] Nonetheless, our work differs from that studied by Bodie, Merton and Samuelson in the following main aspects:

(i) The first distinction, which is the main finding in this work, is that we derive analytically a closed-form solution for the consumption, labor supply and portfolio, other than the conclusion drawn also by Bodie, Merton and Samuelson that the labor income induces the individual to invest an additional amount of wealth to the risky asset. The paper by Bodie, Merton and Samuelson appears to make much effort to show that labor income and investment choices are related, while leaving the optimal consumption unanalyzed. Moreover, the optimal consumption and leisure appears to rely on the indirect utility function, when using the dynamic programming approach, which in general does not admit a closed-form solution. By contrast, the Martingale method enables us to obtain a closed-form solution (even without solving any partial differential equation). We also establish the Euler equation under uncertainty, finding that the uncertainty gives rise to an additional term (corresponding to the *market price of risk*) in the Euler equation under certainty. This is represented in Eq. (5.38). The finding is also supported by the results concluded by Toche (2005) and Mason/Wright (2001). In Toche (2005), the inclusion of an additional term to the Euler equation is due to the risk of permanent income loss while, in Mason/Wright (2001), the conclusion is drawn based on the approximation of a discrete-time problem.

[5.1] I wish to take this opportunity to thank an anonymous referee for the comments and criticism of the very early version of this model and for bringing the work by Bodie, Merton and Samuelson and others to my mind.

(ii) In their work, consumption and leisure are treated as a 'composite' good. This makes their model essentially equivalent to the basic consumption model with the consumption or leisure being the only argument of the utility function. Indeed, from the constraint of (4) on page 430 [7], it is easy to see that the consumption or the leisure can be expressed as the other. See also the step 3 on page 431 [7] for the confirmation that they determine the optimal level of the total spending on the composite good in order to solve a static problem, in which the optimal consumption is determined as if it would be in any static decision. By contrast, our decisions of optimal consumption and labor supply are made individually. Moreover, the optimal investment in the risky asset, in our work, is directly determined as the proportion of the financial wealth, rather than as the sum of the financial wealth and uncertain future labor income as done in the work of Bodie, Merton and Samuelson. In their case, the optimal portfolio is determined by subtracting the total investment in the risky asset by the implicit investment of the expected future labor income in this asset.

(iii) In addition, our work shows a way of modeling the situation that people do not work after retirement by introducing a dummy variable. This model also covers the general case, in which it is commonly assumed that people work for their whole time horizon (see also Romer [32] on the Baseline Real-Business-Cycle model and Walsh [35] on the basic New Keynesian model). To see this, we simply need to redefine the dummy variable as equal to one for all time t. That is the same as to simply omitting the dummy

variable and then following the same techniques used in this model. [5.2]

In Section 5.2, we will specify the model which is solved by using the martingale method in Section 5.3. The closed forms of consumption, labor supply and portfolio are given in Eqs. (5.34), (5.46) and (5.61) in Section 5.3, where the economic interpretations of the results are provided. Conclusions follow in Section 5.4.

5.2 Description of the model

We assume that an infinitely-lived small individual with zero initial capital works only before retirement age T while consuming continuously during his time horizon.[5.3] Given his wage rate $w(t)$ with an initial wage $w(0)$ and its growth rate a being fixed, he earns a labor income of $w(t)L(t)$ by supplying an amount of labor $L(t)$ at the time t when $t \leq T$. The labor income is invested into a risk-free bond offering a gross return R and a risky asset offering an instantaneous expected gross return μ. After the individual retires, his post-retirement consumption is financed by his savings when young and the

[5.2] In the very early version of this work, it was introduced two such dummy variables. The other of which was used to capture a situation, where people may not need to consume of their financial wealth and labor income before retirement as they may have rich families or huge bequest to finance their living expense before they retire. That was criticized immediately as non-realistic. However, dropping this dummy variable even saves our effort on computation, while not affecting the basic structure of the optimal solution.

[5.3] In what we have discussed before, the time horizon is finite. An infinite time horizon can be viewed as the limit of a finite time horizon.

capital gains from the investment. His objective is to maximize his expected lifetime utility by choosing the optimal consumption, the amount of work (when young) and portfolio. The individual has a utility function which is a linear combination of two CRRA utilities with respect to consumption and labor supply, respectively.

5.2.1 The dynamics of the asset prices

(1) The price of the risk-free bond $B(t)$ satisfies

$$\frac{dB(t)}{B(t)} = Rdt \tag{5.1}$$

where, R is the nominal interest rate.

(2) The price of the risky asset, or stock, follows the geometric Brownian motion

$$\frac{dS(t)}{S(t)} = \mu dt + \sigma dW(t), \tag{5.2}$$

Where, μ is the expected nominal return on the risky asset per unit time, σ ($\sigma \neq 0$) is the volatility of the asset price and $W(t)$ is a Brownian motion defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

As before, the *stochastic discount factor* $H(t)$ is given by

$$H(t) \equiv e^{-Rt - \frac{1}{2}\theta^2 t - \theta W(t)} \tag{5.3}$$

where, θ is the *market price of risk* defined by

$$\theta \equiv \frac{\mu - R}{\sigma} \tag{5.4}$$

5.2.2 The utility function

The (instantaneous) utility function is defined by

$$u(C(t), L(t)) = \frac{(C(t))^{1-\gamma}}{1-\gamma} - b\frac{(L(t))^{1+\eta}}{1+\eta} \tag{5.5}$$

where, $\gamma > 1$ [5.4] and $\eta > 0$ [5.5]. The utility function has two arguments. One is the consumption per unit time $C(t)$, and the other is the labor supply (or the amount of work) per unit time $L(t)$. The coefficient b is positive, together with the negative sign, indicating disutility gained from working. The parameters γ and η regulate the curvature of the utility function with respect to consumption and labor supply, respectively.

5.2.3 The wealth process

By assumption, the individual works only before age T. In other words, he no longer works after reaching his retirement age T. In order to capture this fact in the lifetime horizon, we introduce a *dummy variable* as follows:

$$\mathbf{1}(t) = \begin{cases} 0 & , \quad t > T \\ 1 & , \quad t \leq T \end{cases} \tag{5.6}$$

If we assume that the individual with no initial capital invests proportion $\pi(t)$ of his wealth into the risky asset at time t, $t > 0$, and $1-\pi(t)$ fraction of his wealth into the risk-free bond, then his *wealth process* $X(t) \equiv X^{\pi,C,L}(t)$

[5.4] So that the utility function with respect to the consumption is bounded

[5.5] So that the utility function with respect to the labor supply without the negative sign is convex

satisfies

$$\begin{aligned} dX(t) &= X(t)\left[Rdt + \pi(t)\sigma\left(\theta dt + dW(t)\right)\right] - C(t)dt + w(t)L(t)\mathbf{1}(t)dt \\ X(0) &= 0 \end{aligned} \tag{5.7}$$

It says that the change in wealth must equal the capital gains less infinitesimal consumption plus infinitesimal labor income when $t \leq T$ and equal the difference between the capital gains and the infinitesimal consumption when $t > T$. The labor income before retirement is equal to the amount of work $L(t)$ times a wage rate $w(t)$. The wage rate grows exponentially at a rate of a with a strictly positive initial wage $w(0)$, that is

$$w(t) = w(0)e^{at} \tag{5.8}$$

The financial market under consideration is complete (because the number of the risky asset is the same as the number of the driving Brownian motion) and is free of arbitrage (because $\sigma \neq 0$). As a consequence, the individual's current wealth must equal the expected present value of his future consumption less the expected present value of his labor income. In other words, the resources for his expected future consumption come from the current value of his accumulated financial wealth through investment plus the expected present value of his future labor income if he is still working. Formally, the wealth process $X(t)$ must satisfy that

$$X(t) = \mathbb{E}_t\left[\int_t^\infty \frac{H(s)}{H(t)}C(s)ds\right] - \mathbb{E}_t\left[\int_t^\infty \frac{H(s)}{H(t)}w(t)L(s)\mathbf{1}(s)ds\right] \tag{5.9}$$

where, $\mathbb{E}_t$ is the expectation conditional on the Brownian filtration $\{\mathcal{F}(t)\}_t$ and $\mathcal{F}(t) \subseteq \mathcal{F}$. By the definition of the dummy variable $\mathbf{1}(t)$, Eq. (5.9) can

be further expressed as

$$X(t) = \mathbb{E}_t \left[\int_t^{\infty} \frac{H(s)}{H(t)} C(s) ds \right] - \mathbb{E}_t \left[\int_t^{T} \frac{H(s)}{H(t)} w(t) L(s) ds \right] \tag{5.10}$$

The wealth vanishes when lifetime comes to end, that is

$$\lim_{t \to \infty} X(t) = 0 \tag{5.11}$$

Furthermore, at time zero,[5.6] we have

$$\mathbb{E}_0 \left[\int_0^{\infty} H(s) C(s) ds \right] = \mathbb{E}_0 \left[\int_0^{T} H(s) w(t) L(s) ds \right] \tag{5.12}$$

where $\mathbb{E}_0$ is the conditional expectation conditional on the trivial information set $\mathcal{F}(0)$, which is actually equal to the unconditional expectation $\mathbb{E}$. We will drop the subscript 0 when it appears in what follows. Intuitively, Eq. (5.12) says that the expected discounted future consumption when old (i.e. after retirement age T) is financed by the expected discounted labor income when young (i.e. before retirement).

5.2.4 The consumption-labor supply-portfolio problem

We start by defining the *admissibility* which is eqivalent to the constraint of non-negative values of the present expected future consumption.

Definition 5.2.1. *A consumption-labor supply-portfolio process set* $(C(t), L(t), \pi(t))$ *is said to be admissible if*

$$X(t) + \mathbb{E}_t \left[\int_t^{\infty} \frac{H(s)}{H(t)} w(t) L(s) \mathbf{1}(s) ds \right] \geq 0, \text{ for all } t \geq 0 \tag{5.13}$$

with probability one. The resulting class of admissible sets is denoted by $\mathcal{A}_L$.

[5.6] Note that $H(0) = 1$ and $X(0) = 0$

Similar to the case with a positive endowment of Definition 4.2.1., the wealth before age T is allowed to become negative so long as that the present value of future labor income is large enough to offset such a negative value .

The individual wishes to maximize the expected total discounted utility by choosing an optimal consumption-labor supply-portfolio set over the class that

$$\mathcal{A}_2 \equiv \left\{ (C(t), L(t), \pi(t)) \in \mathcal{A}_L : \mathbb{E}\left[\int_0^\infty e^{-\rho t} u^-(C(t), L(t)) dt \right] < \infty \right\} \tag{5.14}$$

Namely, his optimization problem is given by

$$\max_{(C(t),L(t),\pi(t))\in\mathcal{A}_2} \mathbb{E}\left[\int_0^\infty e^{-\rho t} u(C(t), L(t)) dt \right] \tag{5.15}$$

subject to

$$\begin{aligned} dX(t) &= X(t)\left[Rdt + \pi(t)\sigma\left(\theta dt + dW(t)\right)\right] - C(t)dt + w(t)L(t)\mathbf{1}(t)dt \\ X(0) &= 0 \end{aligned} \tag{5.16}$$

where, the discount rate satisfies $\rho > 0$.

5.3 Solving the optimization problem

It follows from the martingale method that the (dynamic) maximization problem (5.15)-(5.16) is equivalent to the following problem

$$\max_{C(t),L(t)} \mathbb{E}\left[\int_0^\infty e^{-\rho t} u(C(t), L(t)) dt \right] \tag{5.17}$$

subject to

$$\mathbb{E}\left[\int_0^\infty H(t)C(t)dt\right] = \mathbb{E}\left[\int_0^T H(t)w(t)L(t)dt\right] \tag{5.18}$$

This budget constraint of (5.18) is the same as

$$\mathbb{E}\left[\int_0^\infty H(t)\left(C(t) - w(t)L(t)\mathbf{1}(t)\right)dt\right] = 0 \tag{5.19}$$

We see that, in the maximization problem above, the portfolio $\pi(t)$ has disappeared from the control variables.

5.3.1 Optimal consumption and labor supply

We now apply the Lagrangian method to the static problem just described. The Lagrangian function $L \equiv L(\lambda; C, L)$ is written as

$$L = \mathbb{E}\left[\int_0^\infty e^{-\rho t}u(C(t), L(t))dt\right] + \lambda\left(0 - \mathbb{E}\left[\int_0^\infty H(t)(C(t) - w_t L_t \mathbf{1}(t))dt\right]\right) \tag{5.20}$$

where, λ is the Lagrangian multiplier. The first order conditions are

$$\begin{aligned}\frac{\partial u}{\partial C(t)} &= \lambda e^{\rho t}H(t) \\ \frac{\partial u}{\partial L(t)} &= -\lambda e^{\rho t}H(t)w(t)\mathbf{1}(t)\end{aligned} \tag{5.21}$$

From the utility function given in Eq. (5.5), we know that

$$\begin{aligned}\frac{\partial u}{\partial C(t)} &= C(t)^{-\gamma} \\ \frac{\partial u}{\partial L(t)} &= -bL(t)^{\eta}\end{aligned} \tag{5.22}$$

Substituting them back into the first order conditions (5.21) leads to

$$\begin{aligned}C^*(t) &= \lambda^{-\frac{1}{\gamma}}e^{-\frac{\rho}{\gamma}t}H(t)^{-\frac{1}{\gamma}} \\ L^*(t) &= \lambda^{\frac{1}{\eta}}e^{\frac{\rho}{\eta}t}H(t)^{\frac{1}{\eta}}\left(\frac{w(t)}{b}\right)^{\frac{1}{\eta}}\mathbf{1}(t)\end{aligned} \tag{5.23}$$

The multiplier λ can then be obtained from the budget constraint: Substituting $C^*(t)$ and $L^*(t)$ into the budget constraint of (5.18), we get

$$\lambda^{-\frac{1}{\gamma}}\mathbb{E}\left[\int_0^\infty e^{-\frac{\rho}{\gamma}t}H(t)^{\frac{\gamma-1}{\gamma}}dt\right] = \lambda^{\frac{1}{\eta}}b^{-\frac{1}{\eta}}\mathbb{E}\left[\int_0^T e^{\frac{\rho}{\eta}t}(H(t)w(t))^{\frac{\eta+1}{\eta}}dt\right]$$

According to the Fubini theorem, we can interchange the order of the expectation and integration as

$$\lambda^{-\frac{1}{\gamma}}\int_0^\infty e^{-\frac{\rho}{\gamma}t}\mathbb{E}\left[H(t)^{\frac{\gamma-1}{\gamma}}\right]dt = \lambda^{\frac{1}{\eta}}b^{-\frac{1}{\eta}}\int_0^T e^{\frac{\rho}{\eta}t}\mathbb{E}\left[(H(t)w(t))^{\frac{\eta+1}{\eta}}\right]dt \quad (5.24)$$

From the definition of $H(t)$ in (5.3), we have that

$$\begin{aligned} H(t)^{\frac{\gamma-1}{\gamma}} &= e^{-\frac{\gamma-1}{\gamma}(r+\frac{\theta^2}{2})t-\frac{\gamma-1}{\gamma}\theta W(t)} \\ &= e^{-\frac{\gamma-1}{\gamma}\theta W(t)-\frac{1}{2}(\frac{\gamma-1}{\gamma})^2\theta^2 t}\cdot e^{-\frac{\gamma-1}{\gamma}(r+\frac{\theta^2}{2\gamma})t} \end{aligned} \quad (5.25)$$

Noting that $e^{-\frac{\gamma-1}{\gamma}\theta W(t)-\frac{1}{2}(\frac{\gamma-1}{\gamma})^2\theta^2 t}$ is a martingale and thus has unit expectation, we obtain

$$\mathbb{E}\left[H(t)^{\frac{\gamma-1}{\gamma}}\right] = e^{-yt}, \text{ with } y \equiv \tfrac{\gamma-1}{\gamma}(r+\tfrac{\theta^2}{2\gamma}). \quad (5.26)$$

Similarly, we can get that

$$\mathbb{E}\left[(H(t)w(t))^{\frac{\eta+1}{\eta}}\right] = w_0^{\frac{\eta+1}{\eta}}e^{-zt}, \text{ with } z \equiv \tfrac{\eta+1}{\eta}(r-a-\tfrac{\theta^2}{2\eta}) \quad (5.27)$$

The substitution of $\mathbb{E}\left[H(t)^{\frac{\gamma-1}{\gamma}}\right]$ (5.26) and $\mathbb{E}\left[(H(t)w(t))^{\frac{\eta+1}{\eta}}\right]$ (5.27) from Eq. (5.24) gives us that

$$\lambda^{-\frac{1}{\gamma}}\int_0^\infty e^{-\frac{\rho}{\gamma}t}e^{-yt}dt = \lambda^{\frac{1}{\eta}}b^{-\frac{1}{\eta}}w_0^{\frac{\eta+1}{\eta}}\int_0^T e^{\frac{\rho}{\eta}t}e^{-zt}dt \quad (5.28)$$

Making the following denotations

$$\begin{aligned} \bar{y} &\equiv y+\frac{\rho}{\gamma} \\ \bar{z} &\equiv z-\frac{\rho}{\eta} \end{aligned} \quad (5.29)$$

we can rewrite Eq. (5.28) as

$$\lambda^{-\frac{1}{\gamma}} \int_0^\infty e^{-\bar{y}t} dt = \lambda^{\frac{1}{\eta}} b^{-\frac{1}{\eta}} w_0^{\frac{\eta+1}{\eta}} \int_0^T e^{-\bar{z}t} dt \tag{5.30}$$

A simple calculation leads to[5.7]

$$\lambda^{-\frac{1}{\gamma}} \frac{1}{\bar{y}} = \lambda^{\frac{1}{\eta}} b^{-\frac{1}{\eta}} w_0^{\frac{\eta+1}{\eta}} \frac{1}{\bar{z}} (1 - e^{-\bar{z}T}) \tag{5.31}$$

provided that $\rho \neq (\eta + 1)(r - a - \frac{\theta^2}{2\eta})$.[5.8] Multiplying both sides by $\lambda^{-\frac{1}{\eta}} \bar{y}$ results in[5.9]

$$\lambda^{-\frac{\gamma+\eta}{\gamma\eta}} = A(T), \text{ with } A(T) \equiv \bar{y} b^{-\frac{1}{\eta}} w_0^{\frac{\eta+1}{\eta}} \frac{1}{\bar{z}} (1 - e^{-\bar{z}T}) \tag{5.32}$$

So we have

$$\begin{aligned} \lambda^{-\frac{1}{\gamma}} &= (A(T))^{\frac{\eta}{\gamma+\eta}} \\ \lambda^{\frac{1}{\eta}} &= (A(T))^{-\frac{\gamma}{\gamma+\eta}} \end{aligned} \tag{5.33}$$

Replacing $\lambda^{-\frac{1}{\gamma}}$ and $\lambda^{\frac{1}{\eta}}$ in (5.31) by (5.33), we obtain the optimal consumption and optimal labor supply as follows

$$\begin{aligned} C^*(t) &= (A(T))^{\frac{\eta}{\gamma+\eta}} e^{-\frac{\rho}{\gamma}t} H(t)^{-\frac{1}{\gamma}}, \\ L^*(t) &= (A(T))^{-\frac{\gamma}{\gamma+\eta}} e^{\frac{\rho}{\eta}t} H(t)^{\frac{1}{\eta}} \left(\frac{w(t)}{b}\right)^{\frac{1}{\eta}} \mathbf{1}(t) \end{aligned} \tag{5.34}$$

[5.7] $\gamma > 1$, $\rho > 0$ and $r > 0$, so $\bar{y} > 0$

[5.8] This condition is to ensure that $\bar{z} \neq 0$.

[5.9] The variable T indicates that A depends on T

5.3.2 Economic interpretation and the Euler equation

In the last subsection, we have obtained the optimal consumption and labor supply policies in Eq. (5.34). By inspecting Eq. (5.34), it is clear that the individual will work more when his wage rate $w(t)$ rises but work less when the relative weight of the disutility from working b is bigger. With constant risk aversions (i.e. γ and η are held fixed), the individual will work more and consume less when the rate of time preference ρ becomes larger. Moreover, we can see from Eq. (5.34) that C_t^* is decreasing in $H(t)$ and $L^*(t)$ is increasing in $H(t)$. If we refer to the inverse of the stochastic discount factor $\frac{1}{H(t)}$ as the *market deflater*, these phenomena can then be interpreted as that the individual is allowed to consume more if the market as a whole performs well but has to work harder if the market develops bad as he has to compensate for the market's bad performance.

As we also see from Eq. (5.34), both of the optimal consumption and labor supply are stochastic and depend on the market prices through the market deflater $H(t)$. It is thus more convenient to study their growth in terms of expectation. Similar to (5.25)-(5.26), we have

$$\begin{aligned} H(t)^{-\frac{1}{\gamma}} &= e^{\frac{1}{\gamma}(r+\frac{\theta^2}{2})t+\frac{\theta}{\gamma}W(t)} \\ &= e^{\frac{\theta}{\gamma}W(t)-\frac{\theta^2}{2\gamma^2}t} \cdot e^{\frac{1}{\gamma}(r+\frac{\gamma+1}{2\gamma}\theta^2)t} \end{aligned} \tag{5.35}$$

The expectation of $e^{\frac{\theta}{\gamma}W(t)-\frac{\theta^2}{2\gamma^2}t}$ equals one, so

$$\mathbb{E}\left[H(t)^{-\frac{1}{\gamma}}\right] = e^{\frac{1}{\gamma}(r+\frac{\gamma+1}{2\gamma}\theta^2)t} \tag{5.36}$$

And consequently, the expected optimal consumption is given by

$$\begin{aligned} \bar{C}_t^* \equiv \mathbb{E}\left[C^*(t)\right] &= (A(T))^{\frac{\eta}{\gamma+\eta}} e^{-\frac{\rho}{\gamma}t} \mathbb{E}\left[H(t)^{-\frac{1}{\gamma}}\right] \\ &= (A(T))^{\frac{\eta}{\gamma+\eta}} e^{\frac{1}{\gamma}(r-\rho+\frac{\gamma+1}{2\gamma}\theta^2)t} \end{aligned} \tag{5.37}$$

The *growth rate* of the expected optimal consumption then equals

$$\frac{1}{\bar{C}_t^*}\frac{d(\bar{C}_t^*)}{dt} = \frac{1}{\gamma}\left(r-\rho+\frac{\gamma+1}{2\gamma}\theta^2\right) \tag{5.38}$$

This is referred to the *Euler equation* for the intertemporal maximization above (under uncertainty). The positive term θ^2 captures the uncertainty of the financial market. A risky financial market induces the consumer to shift consumption over time. It can be seen that the growth rate is decreasing in γ or increasing in the elasticity of substitution between consumptions $\frac{1}{\gamma}$: when γ is smaller, the less marginal utility changes as consumption changes, the more the individual is willing to substitute consumption between periods. When the difference $r-\rho$ is fixed, a higher market price of risk θ leads to a steeper slope of the expected consumption, thus a more prudent behavior. If the risk premium $\mu - r$ equals nothing, then the market price of risk becomes zero and therefore all the wealth will be optimally invested into the risk-free bond. The Euler equation (under uncertainty) will then coincide with the well-known Euler equation for the case of certainty and becomes.

$$\frac{1}{\bar{C}_t^*}\frac{d(\bar{C}_t^*)}{dt} = \frac{r-\rho}{\gamma} \tag{5.39}$$

It states that, when the nominal interest rate exceeds the discount rate, the expected consumption of the individual with a constant relative risk aversion (with respect to the consumption) is rising and falling if the reverse holds.

From (5.38), it is easy to see that the growth rate of the expected consumption is strictly positive when $\rho < r + \frac{\gamma+1}{2\gamma}\theta^2$, strictly negative if $\rho > r + \frac{\gamma+1}{2\gamma}\theta^2$ and constant if $\rho = r + \frac{\gamma+1}{2\gamma}\theta^2$. Intuitively, as the discount rate captures the consumer's preference over time, a smaller discount rate implies that the consumer is more patient and therefore is more willing to shift consumption between different periods (that is, consumption is rising). Similarly, he will be less patient if ρ is larger, in particular, when the discount rate exceed the critical value $r + \frac{\gamma+1}{2\gamma}\theta^2$, he will prefer to consume more earlier than later (that is, consumption is falling).

Parallel to the analysis of the optimal consumption, we have

$$\begin{aligned} H(t)^{\frac{1}{\eta}} &= e^{-\frac{1}{\eta}(r+\frac{\theta^2}{2})t - \frac{\theta}{\eta}W(t)} \\ &= e^{-\frac{\theta}{\eta}W(t) - \frac{\theta^2}{2\eta^2}t} \cdot e^{-\frac{1}{\eta}(r+\frac{\eta-1}{2\eta}\theta^2)t} \end{aligned} \tag{5.40}$$

So the expectation is equal to

$$\mathbb{E}\left[H(t)^{\frac{1}{\eta}}\right] = e^{-\frac{1}{\eta}(r+\frac{\eta-1}{2\eta}\theta^2)t} \tag{5.41}$$

The expected optimal labor supply for $t \leq T$ is then computed as

$$\begin{aligned} \bar{L}_t^* \equiv \mathbb{E}\left[L^*(t)\right] &= (A(T))^{-\frac{\gamma}{\gamma+\eta}} \left(\frac{w(t)}{b}\right)^{\frac{1}{\eta}} e^{\frac{\rho}{\eta}t} \mathbb{E}\left[H(t)^{\frac{1}{\eta}}\right] \\ &= (A(T))^{-\frac{\gamma}{\gamma+\eta}} \left(\frac{w(t)}{b}\right)^{\frac{1}{\eta}} e^{-\frac{1}{\eta}(r-\rho+\frac{\eta-1}{2\eta}\theta^2)t} \end{aligned} \tag{5.42}$$

The growth rate of the expected labor supply equals

$$\frac{1}{\bar{L}_t^*}\frac{d(\bar{L}_t^*)}{dt} = -\frac{1}{\eta}\left(r - \rho + \frac{\eta-1}{2\eta}\theta^2\right) \tag{5.43}$$

When $\theta = 0$, it becomes

$$\frac{1}{\bar{L}_t^*}\frac{d(\bar{L}_t^*)}{dt} = -\frac{r-\rho}{\eta} \tag{5.44}$$

It can be included from (5.43) that the expected labor supply is constant when $\rho = r + \frac{\eta-1}{2\eta}\theta^2$. And it is strictly decreasing in time when $\rho < r + \frac{\eta-1}{2\eta}\theta^2$ while strictly increasing when $\rho > r + \frac{\eta-1}{2\eta}\theta^2$.

Using the first order conditions (5.21) and the marginal utility functions (5.22), we can obtain the *tradeoff* between consumption and labor supply

$$\frac{bL_t^\eta}{C(t)^{-\gamma}} = w(t)\mathbf{1}(t) \tag{5.45}$$

This implies that labor supply when $t \leq T$ is decreasing in consumption (due to the diminishing marginal utility with respect to consumption) and increasing in the wage rate.

5.3.3 Optimal wealth and portfolio rule

By following the procedure of the Martingale method discussed in Chapter 2, it is easy to check that the optimal portfolio for the consumption-portfolio problem with CRRA utility is constant over time and given by

$$\pi^*(t) = \frac{1}{\gamma}\frac{\mu - r}{\sigma^2} \tag{5.46}$$

Since there is no labor supply after retirement, the intertemporal *consumption-labor supply-portfolio* problem for $t > T$ collapses to the intertemporal *consumption-portfolio* problem starting from the time point T with a constant relative risk

aversion (CRRA) utility.[5.10] That is to say, the optimal portfolio rule for our problem when $t > T$ is given by (5.46). We now focus on the case when $t \leq T$.

Clearly, the optimally invested wealth $X^*(t)$, $t \leq T$, satisfies Eq. (5.10) at the optimum, that is

$$X^*(t) = \mathbb{E}_t \left[\int_t^\infty \frac{H(s)}{H(t)} C^*(s) ds \right] - \mathbb{E}_t \left[\int_t^T \frac{H(s)}{H(t)} w(t) L(s)^* ds \right] \tag{5.47}$$

We compute the first term on the right-hand side of Eq. (5.47) below. The second term can be computed in a similar manner. Multiplying and dividing the integrand of the first term by $C^*(t)$ and noting that $C^*(t)$ is $\mathcal{F}(t)$ measurable and therefore can be taken out from the conditional expectation[5.11]

$$\mathbb{E}_t \left[\int_t^\infty \frac{H(s)}{H(t)} C^*(s) ds \right] = C^*(t) \mathbb{E}_t \left[\int_t^\infty \frac{H(s) C^*(s)}{H(t) C^*(t)} ds \right] \tag{5.48}$$

[5.10] As the horizon is infinite, it does not matter for the optimal investment strategy whether it starts at time zero or at a positive time point T.

[5.11] The reason $C^*(t)$ is $\mathcal{F}(t)$ measurable is simply because $C^*(t)$ is a function of $H(t)$ which is $\mathcal{F}(t)$ measurable. The fact that $C^*(t)$ can be taken out from the conditional expectation is due to the property of '*Taking out what is known*' of conditional expectation.

Substituting the optimal consumption obtained in (5.34) gives us

$$\begin{aligned}
\mathbb{E}_t\left[\int_t^\infty \frac{H(s)}{H(t)}C^*(s)ds\right] &= C^*(t)\mathbb{E}_t\left[\int_t^\infty e^{-\frac{\rho}{\gamma}(s-t)}\left(\frac{H(s)}{H(t)}\right)^{\frac{\gamma-1}{\gamma}}ds\right] \\
&= C^*(t)\mathbb{E}\left[\int_t^\infty e^{-\frac{\rho}{\gamma}(s-t)}\left(\frac{H(s)}{H(t)}\right)^{\frac{\gamma-1}{\gamma}}ds\right] \\
&= C^*(t)\int_0^\infty e^{-\frac{\rho}{\gamma}s}\mathbb{E}\left[H(s)^{\frac{\gamma-1}{\gamma}}\right]ds \\
&= C^*(t)\int_0^\infty e^{-\frac{\rho}{\gamma}s}e^{-ys}ds \\
&= C^*(t)\int_0^\infty e^{-\bar{y}s}ds \\
&= C^*(t)\frac{1}{\bar{y}} \qquad (5.49)
\end{aligned}$$

where, the conditional expectation is replaced by the unconditional expectation (the second equality) since the increment of a Brownian motion $W(s)-W(t)$ is independent of $\mathcal{F}(t)$ for $s \geq t$. The third equality is obtained by relabeling $s-t$ as s for the reason that $\{W(s)-W(t)\}_{s\geq t}$ is again a Brownian motion. We have used the result obtained in Eq. (5.26) to get the fourth equality. Similarly, we can get that

$$\mathbb{E}_t\left[\int_t^T \frac{H(s)}{H(t)}w(t)L^*(s)ds\right] = \frac{1}{\bar{z}}(1-e^{-\bar{z}(T-t)})w(t)L^*(t) \qquad (5.50)$$

We now have obtained that

$$X^*(t) = \frac{1}{\bar{y}}C^*(t) - \frac{1}{\bar{z}}(1-e^{-\bar{z}(T-t)})w(t)L^*(t) \qquad (5.51)$$

Rearranging it, we get

$$C_t^* = \bar{y}X^*(t) + \frac{\bar{y}}{\bar{z}}(1-e^{-\bar{z}(T-t)})w(t)L^*(t) \qquad (5.52)$$

Noting that both $\bar{y}$ and $\frac{\bar{y}}{\bar{z}}(1-e^{-\bar{z}(T-t)})$ are strictly positive (so long as $\bar{z} \neq 0$), we can say that the consumption before retirement, at the optimum, is both

proportional to the financial wealth-to-date and proportional to the labor income.

Discounting the optimally invested wealth in Eq. (5.51) by the stochastic discount factor $H(t)$ and then taking differentials, we can get[5.12]

$$\begin{aligned} d(H(t)X^*(t)) &= -H(t)\left(\frac{1}{\bar{y}}C^*(t)\frac{\gamma-1}{\gamma} - \frac{1}{\bar{z}}(1-e^{-\bar{z}(T-t)})w(t)L^*(t)\frac{\eta+1}{\eta}\right)\theta dW(t) \\ &\quad -H(t)C^*(t)dt + H(t)w(t)L^*(t)dt \end{aligned} \tag{5.53}$$

On the other hand, we know, by applying Itô's lemma to the stochastic discount factor $H(t)$, that

$$dH(t) = H(t)(Rdt + \theta dW(t)) \tag{5.54}$$

and, by further applying the stochastic product rule to $H(t)X(t)$, that

$$\begin{aligned} d(H(t)X(t)) &= H(t)dX(t) + X(t)dH(t) + dH(t)dX(t) \\ &= H(t)X(t)(\pi(t)\sigma - \theta)dW(t) - H(t)C(t)dt + H(t)w(t)L(t)\mathbf{1}(t)dt \end{aligned} \tag{5.55}$$

From the definition of the dummy variable, we have $\mathbf{1}(t) = 0$ when $t > T$ and $\mathbf{1}(t) = 1$ when $t \leq T$. So the last equation becomes

$$d(H(t)X(t)) = H(t)X(t)(\pi(t)\sigma - \theta)dW(t) - H(t)C(t)dt + H(t)w(t)L(t)dt, \tag{5.56}$$

for $t \leq T$. This also holds at the optimum as

$$d(H(t)X^*(t)) = H(t)X^*(t)(\pi^*(t)\sigma - \theta)dW(t) - H(t)C^*(t)dt + H(t)w(t)L^*(t)dt \tag{5.57}$$

[5.12] The details of the derivation are given in the Appendix

A Comparison of Eq. (5.53) with Eq. (5.57) gives us the optimal portfolio rule for the case when $t \leq T$

$$\pi^*(t) = \left(1 - \frac{\frac{1}{\bar{y}} C^*(t)^{\frac{\gamma-1}{\gamma}} - \frac{1}{\bar{z}}(1 - e^{-\bar{z}(T-t)}) w(t) L^*(t)^{\frac{\eta+1}{\eta}}}{X^*(t)}\right) \frac{\theta}{\sigma} \tag{5.58}$$

with

$$\theta = \frac{\mu - R}{\sigma} \tag{5.59}$$

and $C^*(t)$, $L^*(t)$ satisfy Eq. (5.34). By comparing the numerator in the parentheses in Eq. (5.58) with $X^*(t)$ in Eq. (5.51), it is trivial to conclude that

$$\pi^*(t) \to 0, \text{ when } \gamma \to \infty \text{ and } \eta \to \infty \tag{5.60}$$

In words, when the investor is extremely risk-averse, he will invest almost all of his wealth in the risk-free bond for the reason of safety.

The optimal portfolio Eq. (5.58) can be further written as

$$\pi^*(t) = \frac{1}{\gamma} \frac{\mu - r}{\sigma^2} + (\frac{1}{\gamma} + \frac{1}{\eta}) \frac{\mu - r}{\sigma^2} \frac{1}{\bar{z}} (1 - e^{-\bar{z}(T-t)}) \frac{w(t) L^*(t)}{X^*(t)} \tag{5.61}$$

Similar to the case with endowments (contributions) discussed in the previous chapter, the share of the wealth optimally invested into the risky asset is made up of two parts:

- the classical Merton portfolio rule for the consumption-portfolio problem $\frac{1}{\gamma} \frac{\mu - r}{\sigma^2}$ plus
- the correction term which is proportional to the optimal labor income relative to the optimally invested wealth-to-date, $(\frac{1}{\gamma} + \frac{1}{\eta}) \frac{\mu - r}{\sigma^2} \frac{1}{\bar{z}} (1 - e^{-\bar{z}(T-t)}) \frac{w(t) L^*(t)}{X^*(t)}$.

Remark 5.3.1. *The fact that the solution* $(C^*(t), L^*(t), \pi^*(t))$ *obtained in Eqs. (5.34), (5.46) and (5.61) is optimal for the consumption-labor supply-portfolio problem of (5.15)-(5.16) with the utility function in (5.5) can be verified in a similar way to the proof of Theorem 2.3.3.*

5.4 Conclusion

It has been found that, at the optimum, a smaller discount rate implies lower expected labor supply before retirement and higher expected consumption during retirement. When the discount rate is below some critical values (for example, when $\rho < \min\{r + \frac{\gamma+1}{2\gamma}\theta^2, r + \frac{\eta-1}{2\eta}\theta^2\}$), the expected labor supply will drop during the working period and reach its minimum at the retirement age; while the consumption is expected to grow with no limit. When the discount rate exceeds these critical values, the expected labor supply will increase and reaches its maximum at the retirement age; but the consumption is expected to decrease and converge to zero. We have derived that when labor income supplements total wealth, the classical Merton portfolio rule needs to be adjusted by adding an additional share of the total wealth which is proportional to the labor income.

Chapter 6

Optimization in incomplete markets

6.1 Introduction

In this chapter, we consider a small investor with an initial capital x ($0 < x < \infty$) who invests his wealth into the market $\mathcal{M}$ of (1.1)-(1.2) which satisfies the General Assumption 1 and $m < d$. As before, he wishes to maximize his *expected utility* for a given utility function but has no knowledge about future prices and has no inside information either. His optimal decision is made only by observing the past and the present stock prices. For a complete market, we have demonstrated that such an optimization problem and its modifications can be solved using the Martingale method. However, as we have seen, the traditional Martingale method is based on the fact that,in a complete financial market, every $\mathcal{F}(T)$-measurable contingent claim B can be obtained by trading following an appropriate portfolio strategy given enough

initial wealth. According to the second fundamental theorem, we know that incompleteness arises when the number of stocks is strictly less than the dimension of the underlying Brownian motion. In such a situation, the traditional Martingale method can't solve the investor's maximization problem directly. To overcome the problem of incompleteness, Karatzas/Shreve/Xu (1991) have developed a way to complete the market by introducing additional *fictitious* stocks and then making them uninteresting to the investor so that the optimal proportions of his wealth invested into such stocks are actually equal nothing (For the details of this approach, see for example, Karatzas/Shreve/Xu (1991)). In this chapter, we will introduce an easier way to transform an incomplete market to a complete one, in which the traditional Martingale method can be applied. Our principal is to reduce the dimension of the driving Brownian motion by summing up the normalized Itô integrals to get new Brownian motions. However, the created new Brownian motions are no longer independent when the number of new Brownian motions needed is more than one. We then need to recreate independent Brownian motions from correlated ones.[6.1]

[6.1]This method is drawn form Zhang (2007a). I believe that the result presented in this chapter was known by many authors. But, to the best of my knowledge, it had never been published explicitly in the literature.

6.2 Transformation from an incomplete market to a complete one

Let us assume that the market is incomplete and the market price of risk $\theta(t)$ is given by

$$\theta(t) \equiv \sigma^\top(t) \left(\sigma(t)\sigma^\top(t)\right)^{-1} [\mu(t) - R(t)\mathbf{1}_m] \tag{6.1}$$

Given an initial capital x $(0 < x < \infty)$ and a CRRA utility function $u(\cdot)$, we consider the problem of maximizing expected utility from terminal wealth

$$\max_{\pi \in \mathcal{A}_1(x)} \mathbb{E}\left[u(X^\pi(T))\right] \tag{6.2}$$

subject to

$$\begin{aligned} dX^\pi(t) &= X^\pi(t)\left[R(t)dt + \pi^\top(t)\sigma(t)\left(\theta(t) + dW(t)\right)\right] \\ X^\pi(0) &= x \end{aligned} \tag{6.3}$$

with

$$\mathcal{A}_1(x) \equiv \left\{\pi \in \mathcal{A}(x) : \mathbb{E}\left[u^-(X^\pi(T))\right] < \infty\right\} \tag{6.4}$$

6.2.1 One stock

We start by looking at a simple case where, except one riskless bond, there is only one stock whose price is driven by a d-dimensional Brownian motion and $d > 1$. Their price dynamics are given by

$$\begin{aligned} \frac{dS_0(t)}{S_0(t)} &= R(t)dt, \\ S_0(0) &= s_0 \end{aligned} \tag{6.5}$$

and

$$\begin{aligned} \frac{dS(t)}{S(t)} &= \mu(t)dt + \sum_{j=1}^{d} \sigma_j(t) dW_j(t) \\ S(0) &= s \end{aligned} \tag{6.6}$$

Denoting by $\sigma(t) \equiv (\sigma_1(t), \sigma_2(t), ..., \sigma_d(t))^\top$ the volatility vector, we define a process by

$$B(t) \equiv \sum_{j=1}^{d} \int_0^t \frac{\sigma_j(s)}{\|\sigma(s)\|} dW_j(s), \tag{6.7}$$

Then $B(t)$ is a Brownian motion. To see this, let us refer to Lévy's Theorem. Roughly speaking, the theorem says that a Martingale starting at origin, with continuous paths and quadratic variation of $< W(t), W(t) >= t$ is a Brownian motion. For a full story, see for example Shreve (2004), page 168-171. Being a sum of stochastic integrals, $B(t)$ is a continuous Martingale with $B(0) = 0$ and

$$dB(t)dB(t) = \sum_{j=1}^{d} \frac{\sigma_j^2(t)}{\|\sigma(t)\|^2} dt = dt \tag{6.8}$$

So $B(t)$ is a Brownian motion according to Lévy's Theorem. We can then write the stock price in terms of $B(t)$, with volatility $\|\sigma(t)\|$, as

$$\frac{dS(t)}{S(t)} = \mu(t)dt + \|\sigma(t)\| \, dB(t) \tag{6.9}$$

As we see, the dimension of the Brownian motion has been reduced to one. Consequently, the market now becomes complete. The Martingale method can now be applied in the completed market. The market coefficients, i.e., the interest rate $R(t)$, the stock appreciation rate $\mu(t)$ and the volatility vector $\sigma(t)$, are assumed to be $\mathcal{F}(t)$-measurable and $\mathcal{F}(t)$ is the Brownian filtration.

If $\pi^{old}(t)$, $\pi^{new}(t)$ denote the proportion of wealth invested into the stock in old market and new market, respectively. The optimal portfolio $\pi^*(t)$ in both markets should coincide under our assumption that the optimal decision is made via observing the stock prices in the past and the present only. So the maximization problem in the incomplete market can be replicated in the complete market using the Martingale method. A direct application of the optimal portfolio process $\pi^*(t)$ of the Section 2.4 in Chapter 2 for the completed market gives us that

$$\begin{aligned} \pi^*(t) &= \frac{1}{\gamma}\frac{\mu(t)-R(t)}{\|\sigma(t)\|^2} \\ &= \frac{1}{\gamma}\frac{\mu(t)-R(t)}{\sigma(t)\sigma^\top(t)} \\ &= \frac{1}{\gamma}\frac{\mu(t)-R(t)}{\sum_{j=1}^d \sigma_j^2(t)} \end{aligned} \tag{6.10}$$

6.2.2 More than one stock

When there is more than one stock, the created new Brownian motions will be no longer independent. We then need to recreate independent Brownian motions from correlated ones. We will treat this below.

Let us consider an incomplete market $\mathcal{M}$, where the number of independent Brownian motions is strictly greater than the number of stocks and strictly greater than one, that is $2 \leq m < d$. The asset prices of this market are given by

$$\begin{aligned} \frac{dS_0(t)}{S_0(t)} &= R(t)dt, \\ S_0(0) &= s_0 \end{aligned} \tag{6.11}$$

and

$$\begin{aligned} \frac{dS_i(t)}{S_i(t)} &= \mu_i(t)dt + \sum_{j=1}^{d} \sigma_{ij}(t)dW_j(t), \\ S_i(0) &= s_i, \text{ for } i = 1, ..., m, \end{aligned} \tag{6.12}$$

where $W(t) = (W_1(t), ..., W_d(t))^\top$ is a d-dimensional Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with the component Brownian motions $W_j(t)$, $j = 1, ..., d$, being independent. When the market has more than one stock available, we need two main procedures to 'complete' the incomplete market:

Step 1: **Reducing the dimension of the Brownian motion**

In this step, the principle of reducing the dimension of the driving Brownian motion is the same as in the case with one stock. Denote by $\sigma_i(t) \equiv (\sigma_{i1}(t), \sigma_{i2}(t), ..., \sigma_{id}(t))$, for $i = 1, ..., m$, row vectors. Then the volatility matrix in the incomplete market can be written as $\sigma(t) = (\sigma_1(t), \sigma_2(t), ..., \sigma_m(t))^\top$ which is a matrix of size $m \times d$. we define processes by

$$B_i(t) \equiv \sum_{j=1}^{d} \int_0^t \frac{\sigma_{ij}(s)}{\|\sigma_i(s)\|} dW_j(s), \quad i = 1, ..., m \tag{6.13}$$

Expressing the stock prices in terms of $B_i(t)$ gives that

$$\frac{dS_i(t)}{S_i(t)} = \mu_i(t)dt + \|\sigma_i(t)\| \, dB_i(t), \quad i = 1, ..., m \tag{6.14}$$

Where $(B_1(t), ..., B_m(t))^\top$ is an m-dimensional Brownian motion. But the component Brownian motions are not independent. Specifically, we have (see Shreve (2004) Exercise 4.15, Page 199)

$$dB_i(t)dB_k(t) = \rho_{ik}(t)dt, \text{ for all } i \neq k \tag{6.15}$$

with

$$\begin{aligned}\rho_{ik}(t) &= \frac{1}{\|\sigma_i(t)\| \cdot \|\sigma_k(t)\|} \sum_{j=1}^{d} \sigma_{ij}(t)\sigma_{kj}(t) \\ &= \frac{\langle \sigma_i(t), \sigma_k(t) \rangle}{\|\sigma_i(t)\| \cdot \|\sigma_k(t)\|} \end{aligned} \tag{6.16}$$

From the Cauchy-Schwarz inequality, it follows that $|\rho_{ik}(t)| \leq 1$. Under the assumption that the volatility matrix has full row rank, i.e., row vectors $\sigma_1(t), ..., \sigma_m(t)$ are linearly independent, we have

$$|\rho_{ik}(t)| < 1 \text{ for } i \neq k \tag{6.17}$$

Step 2: **Creating independent component Brownian motions from correlated ones**

Denote by $\Psi(t)$ the matrix generated by the correlation coefficients of the correlated m-dimensional Brownian motion, $(B_1(t), ..., B_m(t))^\top$

$$\Psi(t) \equiv \begin{bmatrix} \rho_{11}(t) & \rho_{12}(t) & \cdots & \rho_{1m}(t) \\ \rho_{21}(t) & \rho_{22}(t) & \cdots & \rho_{2m}(t) \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{m1}(t) & \rho_{m2}(t) & \cdots & \rho_{mm}(t) \end{bmatrix} \tag{6.18}$$

with

$$\rho_{ik}(t) \begin{cases} = 1, \text{ if } i = k \\ < 1, \text{ if } i \neq k \end{cases}$$

The matrix $\Psi(t)$ is nonsingular, symmetric and positively semi-definite for all t. So there exists a nonsingular matrix $A(t) \equiv (a_{ij}(t))_{i,j=1,...,m}$ such that

$$\Psi(t) = A(t)A^\top(t) \tag{6.19}$$

It can be shown that there exist m independent Brownian motions $\tilde{W}_1(t), ..., \tilde{W}_m(t)$ such that (see Shreve (2004) Exercise 4.16, Page 200)

$$B_i(t) = \sum_{j=1}^{m} \int_0^t a_{ij}(s) d\tilde{W}_j(s), \text{for all } i = 1, ..., m \tag{6.20}$$

So far we have arrived at a complete market with m stocks and m independent component Brownian motions. The m stocks in the incomplete market can now be rewritten as

$$\frac{dS_i(t)}{S_i(t)} = \mu_i(t)dt + \|\sigma_i(t)\| \sum_{j=1}^{m} a_{ij}(t) d\tilde{W}_j(t), \text{ for all } i = 1, ..., m \tag{6.21}$$

The volatility matrix, denoted by $\tilde{\sigma}(t)$, under the completed market is given by

$$\begin{aligned}
\tilde{\sigma}(t) &= \begin{bmatrix} \|\sigma_1(t)\| a_{11}(t) & \|\sigma_1(t)\| a_{12}(t) & \cdots & \|\sigma_1(t)\| a_{1m}(t) \\ \|\sigma_2(t)\| a_{21}(t) & \|\sigma_2(t)\| a_{22}(t) & \cdots & \|\sigma_2(t)\| a_{2m}(t) \\ \vdots & \vdots & \ddots & \vdots \\ \|\sigma_m(t)\| a_{m1}(t) & \|\sigma_m(t)\| a_{m2}(t) & \cdots & \|\sigma_m(t)\| a_{mm}(t) \end{bmatrix} \\
&= \begin{bmatrix} \|\sigma_1(t)\| & 0 & \cdots & 0 \\ 0 & \|\sigma_2(t)\| & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \|\sigma_m(t)\| \end{bmatrix} \cdot A(t)
\end{aligned} \tag{6.22}$$

Let $\Sigma(t)$ denote the diagonal matrix

$$\Sigma(t) \equiv \begin{bmatrix} \|\sigma_1(t)\| & 0 & \cdots & 0 \\ 0 & \|\sigma_2(t)\| & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \|\sigma_m(t)\| \end{bmatrix} \tag{6.23}$$

Since both of the matrices $\Sigma(t)$ and $A(t)$ are nonsingular, the inverse of the volatility matrix reads

$$\tilde{\sigma}^{-1}(t) = (\Sigma(t)A(t))^{-1} = A^{-1}(t)\Sigma^{-1}(t) \tag{6.24}$$

Applying the Martingale method in the completed market of (6.11) and (6.21) as before, we get the optimal portfolio process as follows

$$\begin{aligned}
\pi^*(t) &= \frac{1}{\gamma} \cdot (\tilde{\sigma}^{-1}(t))^\top \cdot (\tilde{\sigma}^{-1}(t)) \cdot (\mu(t) - R(t)\mathbf{1}_m) \\
&= \frac{1}{\gamma} \cdot \left(A^{-1}(t)\Sigma^{-1}(t)\right)^\top \cdot \left(A^{-1}(t)\Sigma^{-1}(t)\right) \cdot (\mu(t) - R(t)\mathbf{1}_m) \\
&= \frac{1}{\gamma} \cdot \Sigma^{-1}(t) \cdot \left(A(t)A^\top(t)\right)^{-1} \cdot \Sigma^{-1}(t) \cdot (\mu(t) - R(t)\mathbf{1}_m)
\end{aligned}$$

On the other side, we know from (6.19) that $A(t)A^\top(t) = \Psi(t)$, so we get

$$\begin{aligned}
\pi^*(t) &= \frac{1}{\gamma} \cdot \Sigma^{-1}(t) \cdot \Psi^{-1}(t) \cdot \Sigma^{-1}(t) \cdot (\mu(t) - R(t)\mathbf{1}_m) \\
&= \frac{1}{\gamma} \cdot (\Sigma(t)\Psi(t)\Sigma(t))^{-1} \cdot (\mu(t) - R(t)\mathbf{1}_m)
\end{aligned} \tag{6.25}$$

where $\Psi(t)$ and $\Sigma(t)$ are defined by (6.18) and (6.23), respectively.

Keep in mind our assumptions that the optimal decision is made on the basis of observation of the past and present stock prices and that the coefficients of stocks are either deterministic or functions of stock prices. The optimal portfolio process for the completed market coincides with that for the original incomplete market. We summarize the main results for the maximization problem in the incomplete market in the following proposition.

Proposition 6.2.1. *Consider the maximization problem of (6.2)-(6.4) with a CRRA utility function. The market coefficients are assumed to be adapted*

to the Brownian filtration. We then have the following results:

(1) the optimal portfolio $\pi^*(t)$, $t \in [0,T]$, *is given by*

$$\pi^*(t) = \frac{1}{\gamma} \cdot \left(\sigma(t)\sigma^\top(t)\right)^{-1} \cdot (\mu(t) - R(t)\,\mathbf{1}_m) \tag{6.26}$$

with the corresponding optimal terminal wealth $X^{\pi^*}(T)$ *satisfying*

$$X^{\pi^*}(T) = B^* = x\frac{(H(T))^{-\frac{1}{\gamma}}}{\mathbb{E}\left[(H(T))^{\frac{\gamma-1}{\gamma}}\right]} \tag{6.27}$$

where,

$$H(T) = e^{-\int_0^T R(t)dt - \frac{1}{2}\int_0^T \left\|\tilde{\theta}(t)\right\|^2 dt - \int_0^T \theta^\top(t)dW(t)} \tag{6.28}$$

with $\tilde{\theta}(t)$ *satisfying*

$$\tilde{\theta}(t) = A^{-1}(t) \cdot \Sigma^{-1}(t) \cdot (\mu(t) - R(t)\,\mathbf{1}_m), \tag{6.29}$$

(2) the maximal expected utility of final wealth is given by

$$\begin{aligned} \mathbb{E}[u(X^\pi(T))] &= \mathbb{E}\left[\frac{1}{1-\gamma}(B^*)^{1-\gamma}\right] \\ &= \frac{x^{1-\gamma}}{1-\gamma} e^{(1-\gamma)\int_0^T \left(R(t) + \frac{1}{2(1-\gamma)}\left\|\tilde{\theta}(t)\right\|^2\right)dt} \end{aligned} \tag{6.30}$$

with

$$\left\|\tilde{\theta}(t)\right\|^2 = (\mu(t) - R(t)\,\mathbf{1}_m)^\top \cdot \left(\sigma(t)\sigma^\top(t)\right)^{-1} \cdot (\mu(t) - R(t)\,\mathbf{1}_m) \tag{6.31}$$

Proof. (*Sketch*). From Eqs. (6.16), (6.18) and (6.23), it is clear that

$$\Sigma(t)\Psi(t)\Sigma(t) = \sigma(t)\sigma^\top(t) \tag{6.32}$$

(1) $\pi^*(t)$ in (6.26) is obtained by substituting (6.32) into (6.25)

When the market is complete, the market price of risk $\tilde{\theta}(t)$ then satisfies

$$\tilde{\theta}(t) = \tilde{\sigma}^{-1}(t) \cdot (\mu(t) - R(t)\mathbf{1}_m) \tag{6.33}$$

where, $\tilde{\sigma}^{-1}(t)$ satisfies Eq. (6.24). That is the equality of (6.29).

(2) From (6.29), (6.19) and (6.32), we have

$$\begin{aligned}
&\left\|\tilde{\theta}(t)\right\|^2 \\
=\ & \theta^\top(t) \cdot \tilde{\theta}(t) \\
=\ & (\mu(t) - R(t)\mathbf{1}_m)^\top \cdot (\Sigma^{-1}(t))^\top \cdot (A^{-1}(t))^\top \cdot A^{-1}(t) \cdot \Sigma^{-1}(t) \cdot (\mu(t) - R(t)\mathbf{1}_m) \\
=\ & (\mu(t) - R(t)\mathbf{1}_m)^\top \cdot (\Sigma^{-1}(t))^\top \cdot \left(A(t)A^\top(t)\right)^{-1} \cdot \Sigma^{-1}(t) \cdot (\mu(t) - R(t)\mathbf{1}_m) \\
=\ & (\mu(t) - R(t)\mathbf{1}_m)^\top \cdot (\Sigma^{-1}(t))^\top \cdot (\Psi(t))^{-1} \cdot \Sigma^{-1}(t) \cdot (\mu(t) - R(t)\mathbf{1}_m) \\
=\ & (\mu(t) - R(t)\mathbf{1}_m)^\top \cdot (\Sigma(t)\Psi(t)\Sigma(t))^{-1} \cdot (\mu(t) - R(t)\mathbf{1}_m) \\
=\ & (\mu(t) - R(t)\mathbf{1}_m)^\top \cdot \left(\sigma(t)\sigma^\top(t)\right)^{-1} \cdot (\mu(t) - R(t)\mathbf{1}_m)
\end{aligned} \tag{6.34}$$

□

Remark 6.2.1. *In Proposition 6.2.1., since the matrix $A(t)$ appears in Eq. (6.29), the optimal final wealth $X^{\pi^*}(T)$ is not unique although both the optimal portfolio process $\pi^*(t)$ and the optimal expected utility are unique. The non-uniqueness of $X^{\pi^*}(T)$ is a result of the non-uniqueness of the decomposition of the matrix of correlation coefficients $\Psi(t)$.*

Remark 6.2.2. *When the optimal decision is made based on the observation of the stock prices in the past and the present, the optimal solution to the maximization problem of (6.2)-(6.4) in an incomplete market appears to be analogue to that in the complete market.*

Chapter 7

Appendix

The derivation of Eq. (5.53)

Multiplying Eq. (5.51) by $H(t)$ and then substituting $C^*(t)$ and $L^*(t)$ obtained in Eq. (5.34) gives us

$$\begin{aligned} & H(t)X^*(t) \\ = \; & \frac{1}{\bar{y}}H(t)C^*(t) - \frac{1}{\bar{z}}\left(1 - e^{-\bar{z}(T-t)}\right)H(t)wL^*(t) \\ = \; & \frac{1}{\bar{y}}(A(T))^{\frac{\eta}{\gamma+\eta}}e^{-\frac{\rho}{\gamma}t}H(t)^{\frac{\gamma-1}{\gamma}} - \frac{1}{\bar{z}}\left(1 - e^{-\bar{z}(T-t)}\right)(A(T))^{-\frac{\gamma}{\gamma+\eta}}b^{-\frac{1}{\eta}}w(t)^{\frac{\eta+1}{\eta}}e^{\frac{\rho}{\eta}t}H(t)^{\frac{\eta+1}{\eta}} \end{aligned} \tag{7.1}$$

Taking differentials

$$\begin{aligned} & d(H(t)X^*(t)) \\ = \; & \frac{1}{\bar{y}}(A(T))^{\frac{\eta}{\gamma+\eta}}d\left(e^{-\frac{\rho}{\gamma}t}H(t)^{\frac{\gamma-1}{\gamma}}\right) \\ & -\frac{1}{\bar{z}}(A(T))^{-\frac{\gamma}{\gamma+\eta}}b^{-\frac{1}{\eta}}w_0^{\frac{\eta+1}{\eta}}d\left(e^{\left(\frac{\rho}{\eta}+a\frac{\eta+1}{\eta}\right)t}\left(1 - e^{-\bar{z}(T-t)}\right)H(t)^{\frac{\eta+1}{\eta}}\right) \end{aligned} \tag{7.2}$$

By applying Itô's lemma and then using Eq. (5.54), we can get

$$d\left(H(t)^{\frac{\gamma-1}{\gamma}}\right) = -H(t)^{\frac{\gamma-1}{\gamma}}\left(\frac{\gamma-1}{\gamma}\theta dW(t) + ydt\right) \tag{7.3}$$

and

$$d\left(H(t)^{\frac{\eta+1}{\eta}}\right) = -H(t)^{\frac{\eta+1}{\eta}}\left(\frac{\eta+1}{\eta}\theta dW(t) + \left(z + a\frac{\eta+1}{\eta}\right)dt\right) \tag{7.4}$$

Applying the product rule to $e^{-\frac{\rho}{\gamma}t}H(t)^{\frac{\gamma-1}{\gamma}}$ and $e^{\left(\frac{\rho}{\eta}+a\frac{\eta+1}{\eta}\right)t}(1-e^{-\bar{z}(T-t)})H(t)^{\frac{\eta+1}{\eta}}$, respectively, and using (7.3), (7.4) and (5.29), we get

$$\begin{aligned}
& d\left(e^{-\frac{\rho}{\gamma}t}H(t)^{\frac{\gamma-1}{\gamma}}\right) \\
= \;& H(t)^{\frac{\gamma-1}{\gamma}}d\left(e^{-\frac{\rho}{\gamma}t}\right) + e^{-\frac{\rho}{\gamma}t}d\left(H(t)^{\frac{\gamma-1}{\gamma}}\right) \\
= \;& -\frac{\rho}{\gamma}e^{-\frac{\rho}{\gamma}t}H(t)^{\frac{\gamma-1}{\gamma}}dt - e^{-\frac{\rho}{\gamma}t}H(t)^{\frac{\gamma-1}{\gamma}}\left(\frac{\gamma-1}{\gamma}\theta dW(t) + ydt\right) \\
= \;& -e^{-\frac{\rho}{\gamma}t}H(t)^{\frac{\gamma-1}{\gamma}}\left(\frac{\gamma-1}{\gamma}\theta dW(t) + \bar{y}dt\right)
\end{aligned} \tag{7.5}$$

and

$$\begin{aligned}
& d\left(e^{\left(\frac{\rho}{\eta}+a\frac{\eta+1}{\eta}\right)t}\left(1-e^{-\bar{z}(T-t)}\right)H(t)^{\frac{\eta+1}{\eta}}\right) \\
= \;& d\left(\left(e^{\left(\frac{\rho}{\eta}+a\frac{\eta+1}{\eta}\right)t} - e^{-\bar{z}T}e^{\left(z+a\frac{\eta+1}{\eta}\right)t}\right)H(t)^{\frac{\eta+1}{\eta}}\right) \\
= \;& H(t)^{\frac{\eta+1}{\eta}}d\left(e^{\left(\frac{\rho}{\eta}+a\frac{\eta+1}{\eta}\right)t} - e^{-\bar{z}T}e^{\left(z+a\frac{\eta+1}{\eta}\right)t}\right) \\
& + \left(e^{\left(\frac{\rho}{\eta}+a\frac{\eta+1}{\eta}\right)t} - e^{-\bar{z}T}e^{\left(z+a\frac{\eta+1}{\eta}\right)t}\right)d\left(H(t)^{\frac{\eta+1}{\eta}}\right) \\
= \;& H(t)^{\frac{\eta+1}{\eta}}\left(\left(\frac{\rho}{\eta}+a\frac{\eta+1}{\eta}\right)e^{\left(\frac{\rho}{\eta}+a\frac{\eta+1}{\eta}\right)t} - \left(z+a\frac{\eta+1}{\eta}\right)e^{-\bar{z}T}e^{\left(z+a\frac{\eta+1}{\eta}\right)t}\right)dt \\
& - \left(e^{\left(\frac{\rho}{\eta}+a\frac{\eta+1}{\eta}\right)t} - e^{-\bar{z}T}e^{\left(z+a\frac{\eta+1}{\eta}\right)t}\right)H(t)^{\frac{\eta+1}{\eta}}\left(\frac{\eta+1}{\eta}\theta dW(t) + \left(z+a\frac{\eta+1}{\eta}\right)dt\right) \\
= \;& -e^{\left(\frac{\rho}{\eta}+a\frac{\eta+1}{\eta}\right)t}\left(1-e^{-\bar{z}(T-t)}\right)H(t)^{\frac{\eta+1}{\eta}}\frac{\eta+1}{\eta}\theta dW(t) - \bar{z}e^{\left(\frac{\rho}{\eta}+a\frac{\eta+1}{\eta}\right)t}H(t)^{\frac{\eta+1}{\eta}}dt
\end{aligned} \tag{7.6}$$

Substituting (7.5) and (7.6) back to (7.2) and then collecting terms and using (5.8), (5.29) and (5.34) results in

$$\begin{aligned}
& d(H(t)X^*(t)) \\
= \; & -\frac{1}{\bar{y}}(A(T))^{\frac{\eta}{\gamma+\eta}} e^{-\frac{\rho}{\gamma}t} H(t)^{\frac{\gamma-1}{\gamma}} \left(\frac{\gamma-1}{\gamma}\theta dW(t) + \bar{y}dt \right) \\
& + \frac{1}{\bar{z}}(A(T))^{-\frac{\gamma}{\gamma+\eta}} b^{-\frac{1}{\eta}} w_0^{\frac{\eta+1}{\eta}} \left(e^{\left(\frac{\rho}{\eta}+a\frac{\eta+1}{\eta}\right)t}(1-e^{-\bar{z}(T-t)}) H(t)^{\frac{\eta+1}{\eta}} \frac{\eta+1}{\eta}\theta dW(t) \right. \\
& \left. + \bar{z} e^{\left(\frac{\rho}{\eta}+a\frac{\eta+1}{\eta}\right)t} H(t)^{\frac{\eta+1}{\eta}} dt \right) \\
= \; & -H(t)\left(\frac{1}{\bar{y}} C^*(t)\frac{\gamma-1}{\gamma} - \frac{1}{\bar{z}}(1-e^{-\bar{z}(T-t)}) w(t) L^*(t) \frac{\eta+1}{\eta} \right) \theta dW(t) \\
& - H(t)C^*(t)dt + H(t)w(t)L^*(t)dt
\end{aligned} \tag{7.7}$$

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