Inverse Tension Problems and Monotropic Optimization

Çiğdem Güler[1]
Department of Mathematics, University of Kaiserslautern
D-67653 Kaiserslautern, Germany
gueler@mathematik.uni-kl.de

Abstract

Given a directed graph $G = (N, A)$, a *tension* is a function from A to $\mathbb{R}$ which satisfies Kirchhoff's law for voltages. There are two well-known tension problems on graphs. In the *minimum cost tension problem (MCT)*, a cost vector is given and a tension satisfying lower and upper bounds is seeked such that the total cost is minimum. In the *maximum tension problem (MaxT)*, the graph contains 2 special nodes and an arc between them. The aim is to find the maximum tension on this arc. In this study we assume that both problems are feasible and have finite optimal solutions and analyze their inverse versions under rectilinear and Chebyshev distances.

In the inverse minimum cost tension problem we adjust the cost parameter to make a given feasible solution the optimum, whereas in inverse maximum tension problem the bounds of the arcs are modified. We show, by extending the results of Ahuja and Orlin [3], that these inverse tension problems are in a way "dual" to the inverse network flows. We prove that the inverse minimum cost tension problem under rectilinear norm is equivalent to solving a minimum cost tension problem, while under unit weight Chebyshev norm it can be solved by finding a minimum mean cost residual cut. Moreover, inverse maximum tension problem under rectilinear norm can be solved as a maximum tension problem on the same graph with new arc bounds. Finally, we provide a generalization of the inverse problems to monotropic programming problems with linear costs.

Keywords: inverse problems, tension problems, cuts, monotropic programming

1 Introduction

Optimization problems with estimated problem parameters have lately drawn considerable attention from researchers. For this kind of problems one often knows a priori an optimal solution based on observations or experiments, but is interested in finding a set of parameters, such that the known solution is optimum (a) and the deviation from the initial estimates is minimized (b). The problem of recalculating the parameters satisfying (a) and (b) is known as *inverse optimization problem*.

Among several inverse optimization problems the inverse network flows have been intensely investigated. Ahuja and Orlin [2] and Zhang and Liu [19, 20] study inverse linear programs and derive LP formulations for several inverse network flow problems. In another paper Ahuja and Orlin [3] analyze the combinatorial aspects of inverse minimum cost flow problem under unit weight L_1 and L_∞ norms. They show that the optimum objective function value is for the former problem equal to the minimum cost of a collection of arc-disjoint cycles in residual graph, whereas the latter problem can be reduced to finding a minimum mean cycle in the residual graph. Yang *et al.* [18] study inverse maximum flow and minimum cut problems. A thorough survey study on this topic has been done by Heuberger [9] analyzing several different types of inverse and reverse problems that have been considered in the literature. As opposed to flow problems, *tension problems*, which are duals of flow problems [1], and their inverse versions have vastly been neglected. Our aim in this study is to fill this gap in the literature and extend the results of Ahuja and Orlin [3] for tensions to show that the duality relation between tensions and flows is valid for their respective inverse problems, as well. Moreover, we exploit monotropic programming (Rockafellar [14]) with linear cost functions to generalize these combinatorial results for a larger set of inverse problems including the inverse problems of generalized network flows (Ahuja *et al.* [1]).

Let $G = (N, A)$ be a connected digraph with node set N containing n nodes and arc set A containing m arcs, and a_{ij} represent an arc with tail node i and head node j. A *tension* is a function from A to $\mathbb{R}$ which

[1]The research has been partially supported by Deutsche Forschungsgemeinschaft (DFG), Grant HA 1737/7 "Algorithmik großer und komplexer Netzwerke" and by the Rhineland-Palatinate cluster of excellence "Dependable adaptive systems and mathematical modeling".

satisfies Kirchhoff's law for voltages [12]. In other words, a vector $\theta \in \mathbb{R}^A$ is a *tension* on graph G with *potential* $\pi \in \mathbb{R}^N$ such that $\forall (i,j) \in A \quad \theta_{ij} = \pi_j - \pi_i$. The basic properties of the tensions are [14]:

- For all cycles C, $\sum_{a_{ij} \in C^+} \theta_{ij} - \sum_{a_{ij} \in C^-} \theta_{ij} = 0$, where C^+ and C^- are the forward and the backward arcs of the cycle, respectively.
- Any linear combination of tensions is a tension.
- A tension is orthogonal to any circulation.

The *minimum cost tension problem (MCT)* is finding a tension θ satisfying lower ($t_{ij} \in \mathbb{R} \cup \{-\infty\}$) and upper ($T_{ij} \in \mathbb{R} \cup \{+\infty\}$) bounds on each arc such that $\sum_{a_{ij} \in A} c_{ij}\theta_{ij}$ is minimum. In the *maximum tension problem (MaxT)*, the graph G contains 2 special nodes, s and t, and an arc $a_{st} \in A$ between these two nodes with bounds $(t_{st}, T_{st}) = (-\infty, \infty)$. The maximum tension problem is finding the maximum tension on arc $a_{st} \in A$ such that the tensions on all arcs satisfy the upper and lower bounds. In this study we assume that both problems are feasible and have finite optimal solutions. Our aim is to analyze their inverse versions.

Given a feasible tension $\hat{\theta}$ for an instance of MCT, the *cost inverse minimum cost tension problem* (IMCT$_c$) is perturbing the cost vector from c to $\hat{c}$ in a way that $\hat{\theta}$ will become the optimum tension for the minimum cost tension problem with the perturbed cost vector (MCT($\hat{c}$)) while the perturbation $\|c - \hat{c}\|$ is minimized according to some norm. On the other hand, in *inverse maximum tension problem* (IMaxT) we modify the bound vectors from T to $\hat{T}$ and/or from t to $\hat{t}$ such that $\hat{\theta}_{st}$ will become the maximum tension with the perturbed bound vectors. We exploit rectilinear (L_1) and Chebyshev (L_∞) norms to measure the parameter modifications.

The theory of monotropic programming was first established by Rockafellar [14] and extended mainly by Tseng and Bertsekas [4, 15, 16, 17]. In Section 5, we provide a brief introduction to monotropic programming and refer to the book of Rockafellar [14] for details.

Monotropic programming deals with optimization problems that minimize a separable convex function subject to linear constraints. Several optimization problems such as linear and piecewise linear programs, quadratic and piecewise quadratic programs, network flows and tensions are special cases of monotropic programs. In this paper, we analyze inverse problems of monotropic programs with seperable linear cost functions and show that the combinatorial solutions of Ahuja and Orlin [3] can be extended to these problems.

The rest of this paper is organized as follows: Section 2 describes the inverse minimum cost tension problem under L_1 norm in detail and gives a combinatorial formulation of the problem. Section 3 analyzes the same problem under L_∞ norm. In Section 4 we present a solution to the inverse maximum tension problem under rectilinear norm. In Section 5, we introduce the theory of monotropic programming and study the inverse problems of monotropic programs with separable linear cost functions under L_1 and L_∞ norms. Finally, we conclude the paper by a summary of our results and a discussion of future work in this area.

2 Inverse Minimum Cost Tension Problem Under L_1 Norm

For the inverse minimum cost flow problem under unit weight L_1 norm, i.e., $w_{ij} = 1$ for all $a_{ij} \in A$, Ahuja and Orlin [3] have shown that the optimum objective function value is equal to the minimum cost of a collection of arc-disjoint cycles in residual graph. Since this collection defines a minimum cost circulation in a unit capacity network, the inverse problem can be reduced to solving a minimum cost flow problem in a unit capacity network. Similarly, by using *arc-disjoint* residual cuts, we will show that the inverse minimum cost tension problem under unit weight rectilinear norm reduces to solving a minimum cost tension problem with unit upper and lower bounds on arcs. First, we need to define arc-disjoint residual cuts and present the optimality conditions for minimum cost tensions.

A cut ω is called ***residual*** with respect to a tension $\hat{\theta}$ if

$$\forall\, a_{ij} \in \omega^+ \quad \hat{\theta}_{ij} < T_{ij} \tag{1a}$$

$$\forall\, a_{ij} \in \omega^- \quad \hat{\theta}_{ij} > t_{ij} \tag{1b}$$

The *cost* of a cut ω is

$$\text{cost}(\omega) = \sum_{a_{ij} \in \omega^+} c_{ij} - \sum_{a_{ij} \in \omega^-} c_{ij}, \tag{2}$$

and its *mean-cost* is equal to the cost divided by its cardinality. We call the residual cuts ω_1 and ω_2 to be ***arc-disjoint*** if $\omega_1^+ \cap \omega_2^+ = \emptyset$ and $\omega_1^- \cap \omega_2^- = \emptyset$. Note that arc-disjoint residual cuts might have common arcs such that $a_{ij} \in \omega_1^+ \cap \omega_2^-$ or vice versa. In this case, $t_{ij} < \hat{\theta}_{ij} < T_{ij}$ holds for these arcs.

Theorem 1. *A tension $\hat{\theta}$ is optimal if and only if all the residual cuts in G have nonnegative costs [8].*

Tensions are duals of circulations. Hence, we can also characterize the optimal tension to the minimum cost tension problem using circulations [12]. A tension $\hat{\theta}$ is optimal if there exists a circulation φ such that

$$c_{ij} - \varphi_{ij} \geq 0 \quad \text{if} \quad \hat{\theta}_{ij} = t_{ij}, \tag{3a}$$

$$c_{ij} - \varphi_{ij} = 0 \quad \text{if} \quad t_{ij} < \hat{\theta}_{ij} < T_{ij}, \tag{3b}$$

$$c_{ij} - \varphi_{ij} \leq 0 \quad \text{if} \quad \hat{\theta}_{ij} = T_{ij}. \tag{3c}$$

Theorem 2. *Suppose $\Omega^* = \{\omega_1^*, \omega_2^*, \ldots, \omega_K^*\}$ denotes a minimum cost collection of arc-disjoint residual cuts in G and let $Cost(\Omega^*)$ be its cost, which is equal to the total costs of the residual cuts in Ω^*. Then, $-Cost(\Omega^*)$ is the optimal objective function value for the inverse minimum cost tension problem under unit weight rectilinear norm.*

Proof: Suppose that $\Omega = \{\omega_1, \omega_2, \ldots, \omega_K\}$ denotes any collection of arc-disjoint residual cuts with negative costs in G and let c^* denote the optimum cost vector for the inverse problem. First, we show that for the common arcs a_{ij} of $\omega_k \in \Omega$, $c_{ij}^* = c_{ij}$ holds.

We know that by Theorem 1, the costs of the arcs in $\omega_k \in \Omega$ have to be changed so that $Cost(\omega_k) \geq 0$. Since c^* is the optimum modified cost vector, the following holds (otherwise we could find a cost vector $\bar{c}$ with $\|\bar{c} - c\|_1 \leq \|c^* - c\|_1$).

$$c_{ij}^* \geq c_{ij} \text{ for } a_{ij} \in \omega_k^+ \tag{4}$$

$$c_{ij}^* \leq c_{ij} \text{ for } a_{ij} \in \omega_k^- \tag{5}$$

For an arc a_{ij} with $a_{ij} \in \omega_{k_1}^+, \omega_{k_2}^-$, the inequalities (4) and (5) must hold with equality. By using this fact, we can show that $-Cost(\Omega)$ is a lower bound on $\|c^* - c\|_1$.

$$\|c^* - c\|_1 = \sum_{a_{ij} \in A} |c_{ij}^* - c_{ij}| \geq \sum_{k=1}^{K} \sum_{a_{ij} \in \omega_k} |c_{ij}^* - c_{ij}| \geq -\sum_{k=1}^{K} Cost(\omega_k) = -Cost(\Omega) \tag{6}$$

Now, we will prove that this lower bound is actually achieved for the minimum cost collection of arc-disjoint residual cut $\Omega^* = \{\omega_1^*, \omega_2^*, \ldots, \omega_K^*\}$. For this purpose, we need to show that there exists a circulation φ for G such that

$$c_{ij} - \varphi_{ij} \leq 0 \text{ for } a_{ij} \in \Omega^{*+} \qquad c_{ij} - \varphi_{ij} \geq 0 \text{ for } a_{ij} \in \Omega^{*-} \tag{7a}$$

$$c_{ij} - \varphi_{ij} \geq 0 \text{ for } a_{ij} \notin \Omega^* \text{ and } \hat{\theta}_{ij} < T_{ij} \qquad c_{ij} - \varphi_{ij} \leq 0 \text{ for } a_{ij} \notin \Omega^* \text{ and } \hat{\theta}_{ij} > t_{ij} \tag{7b}$$

Suppose that there exists a residual cut $\omega_k^* \in \Omega^*$ for which inequalities (7a) do not hold i.e., $c_{ij} - \varphi_{ij} \geq 0$ for $a_{ij} \in \omega_k^{*+}$ and $c_{ij} - \varphi_{ij} \leq 0$ for $a_{ij} \in \omega_k^{*-}$. Then,

$$\sum_{a_{ij} \in \omega^{*+}} (c_{ij} - \varphi_{ij}) - \sum_{a_{ij} \in \omega^{*-}} (c_{ij} - \varphi_{ij}) \geq 0$$

If we rearrange the inequality as

$$\sum_{a_{ij} \in \omega^{*+}} c_{ij} - \sum_{a_{ij} \in \omega^{*-}} c_{ij} - \left(\sum_{a_{ij} \in \omega^{*+}} \varphi_{ij} - \sum_{a_{ij} \in \omega^{*-}} \varphi_{ij} \right)$$

and use the fact that the sum of the flows on cuts is equal to 0, we come up with a contradiction that $\sum_{a_{ij} \in \omega^{*+}} c_{ij} - \sum_{a_{ij} \in \omega^{*-}} c_{ij} = Cost(\omega^*) \geq 0$. Hence, a circulation φ satisfying (7a - 7b) exists.

Let $c^*_{ij} = \varphi_{ij}$ for $a_{ij} \in \Omega^*$ and $c^*_{ij} = c_{ij}$ otherwise. Clearly, this cost vector satisfies the optimality conditions (3a - 3c), hence it is a feasible solution to the inverse problem. Moreover,

$$\begin{aligned}
\|c^* - c\|_1 &= \sum_{a_{ij} \in \Omega^{*-}} (c_{ij} - \varphi_{ij}) - \sum_{a_{ij} \in \Omega^{*+}} (c_{ij} - \varphi_{ij}) \\
&= \sum_{k=1}^{K} \sum_{a_{ij} \in \omega_k^{*-}} (c_{ij} - \varphi_{ij}) - \sum_{a_{ij} \in \omega_k^{*+}} (c_{ij} - \varphi_{ij}) \\
&= -\sum_{k=1}^{K} \left(\sum_{a_{ij} \in \omega^{*+}} c_{ij} - \sum_{a_{ij} \in \omega^{*-}} c_{ij} \right) = -Cost(\Omega^*)
\end{aligned}$$

Thus, the result of the theorem follows.

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We next show that the minimum cost collection of arc-disjoint residual cuts can be found by solving a minimum cost tension problem by deriving a linear programming formulation of the inverse problem using the ideas of Ahuja and Orlin [2]. Under unit weight rectilinear norm, the objective function of the inverse problem would be

$$\text{Minimize} \quad \sum_{a_{ij} \in A} |c_{ij} - \hat{c}_{ij}| \tag{8}$$

and the constraints of the inverse problem are derived from the flow optimality conditions (3a), (3b) and (3c). If we dualize the corresponding linear program, we obtain

$$\text{Minimize} \quad \sum_{a_{ij} \in A} c_{ij}(\pi_j - \pi_i) \tag{9a}$$

subject to

$$-1 \leq \pi_j - \pi_i \leq 1 \qquad \text{for} \quad a_{ij} \in K \tag{9b}$$

$$0 \leq \pi_j - \pi_i \leq 1 \qquad \text{for} \quad a_{ij} \in L \tag{9c}$$

$$-1 \leq \pi_j - \pi_i \leq 0 \qquad \text{for} \quad a_{ij} \in U \tag{9d}$$

$$\pi \gtreqless 0$$

where

$$\begin{aligned}
K &:= \{a_{ij} \in A : t_{ij} < \hat{\theta}_{ij} < T_{ij}\}, \\
L &:= \{a_{ij} \in A : \hat{\theta}_{ij} = t_{ij}\}, \\
U &:= \{a_{ij} \in A : \hat{\theta}_{ij} = T_{ij}\}.
\end{aligned}$$

Obviously, this LP is actually the formulation of a minimum cost tension problem with lower and upper bounds on the tensions given by the inequalities (9b), (9c) and (9d).

If we are given positive weights $w_{ij} > 0 \; \forall a_{ij} \in A$, the only change that occurs in the LP formulation of inverse problem is the objective function (8), which now looks like

$$\text{Minimize} \quad \sum_{a_{ij} \in A} w_{ij}(|c_{ij} - \hat{c}_{ij}|) \tag{10}$$

This modification does not influence the outcome of the dualization of the inverse LP. The dual LP of the inverse LP remains to be a minimum cost tension problem but with new bounds for the tension. Hence, the new bound inequalities are

$$\begin{aligned}
-w_{ij} \leq \pi_j &- \pi_i \leq w_{ij} \quad \text{for } a_{ij} \in K \\
0 \leq \pi_j &- \pi_i \leq w_{ij} \quad \text{for } a_{ij} \in L \\
-w_{ij} \leq \pi_j &- \pi_i \leq 0 \quad \text{for } a_{ij} \in U
\end{aligned}$$

3 Inverse Minimum Cost Tension Problem Under L_∞ Norm

Ahuja and Orlin [3] showed that the inverse minimum cost flow problem under unit weight L_∞ norm can be reduced to solving a minimum mean cycle problem in the residual graph. Similarly we will show that the inverse minimum cost tension problem under Chebyshev norm reduces to solving a minimum mean residual cut problem.

As mentioned in Section 2, a given tension $\hat{\theta}$ is optimal if and only if the graph does not contain any negative cost residual cuts with respect to $\hat{\theta}$. Since in the inverse problem we are given a non-optimal tension, the graph contains residual cuts with negative costs. Our aim is to modify the cost vector of the arcs c to $\hat{c}$ such that none of the residual cuts have negative costs and $\max_{a_{ij}\in A}|\hat{c}_{ij}-c_{ij}|$ is minimum.

Let ω^* be a minimum mean (cost) residual cut in G w.r.t. $\hat{\theta}$, i.e., ω^* is a residual cut with $\mu^* = MCost(\omega^*) = cost(\omega^*)/|\omega^*|$ is minimum among all residual cuts where $|\omega^*|$ denotes the number of arcs in cut ω^*. We adopt an idea of Hadjiat and Maurras [7] who define ϵ-*optimality* and show that $\epsilon = -\mu^*$ is the smallest positive real number for which $\hat{\theta}$ is ϵ-optimal.

Definition 3. For an $\epsilon \geq 0$, a tension $\hat{\theta}$ is ϵ***-optimal*** if there exists a circulation φ such that

$$\forall\, a_{ij} \in A : [(\hat{\theta}_{ij} < T_{ij}) \Longrightarrow (\varphi_{ij} \leq c_{ij} + \epsilon)] \quad \text{and} \quad [(\hat{\theta}_{ij} > t_{ij}) \Longrightarrow (\varphi_{ij} \geq c_{ij} - \epsilon)] \tag{11}$$

Theorem 4. *Tension $\hat{\theta}$ is ϵ-optimal if and only if every cut ω residual w.r.t. $\hat{\theta}$ satisfies $MCost(\omega) \geq -\epsilon$.*

The definition of ϵ-optimality (11) and the given results imply the following property of the tensions.

Property 5. *Let ω^* be a minimum mean residual cut in G w.r.t. $\hat{\theta}$ and μ^* be the mean cost of it. There exists a circulation φ such that $c_{ij} - \varphi_{ij} = \mu^*$ for the outgoing and $c_{ij} - \varphi_{ij} = -\mu^*$ for the incoming arcs of the cut ω^*. The outgoing and incoming arcs of all other residual cuts satisfy $c_{ij} - \varphi_{ij} \geq \mu^*$ and $c_{ij} - \varphi_{ij} \leq -\mu^*$, respectively.*

Theorem 6. *Let μ^* denote the mean cost of a minimum mean residual cut in G w.r.t. $\hat{\theta}$. Then, the optimal objective function value for the inverse minimum cost tension problem under L_∞ norm is $max(0, -\mu^*)$.*

Proof: We can solve the minimum mean residual cut problem in G w.r.t. $\hat{\theta}$ in strongly polynomial time by using the method of Hadjiat and Maurras [7]. Moreover, we choose φ as in Property 5. If $\mu^* \geq 0$, then $\hat{\theta}$ is an optimum tension and the theorem is true. Suppose that $\mu^* < 0$ and ω^* is the minimum mean residual cut in G w.r.t. $\hat{\theta}$. Let z^* be the optimum solution to the inverse minimum tension problem under Chebyshev norm. We first claim that $z^* \geq -\mu^*$. Recall

$$\text{cost}(\omega^*) = \sum_{a_{ij}\in\omega^{*+}} c_{ij} - \sum_{a_{ij}\in\omega^{*-}} c_{ij} = |\omega^*|\mu^*$$

If $z^* < -\mu^*$, then, in order to make $\hat{\theta}$ the optimal solution, it would be sufficient to increase the costs of $a_{ij} \in \omega^{*+}$ by an amount z^* and decrease the costs of $a_{ij} \in \omega^{*-}$ by z^*. The resulting cost of the cut ω^* is $|\omega^*|\mu^* + |\omega^*|z^* < 0$, which is a contradiction to the optimality of $\hat{\theta}$. Hence, $z^* \geq -\mu^*$.

Now we prove that there exists a vector c^* with $\|c^* - c\| = -\mu^*$ such that $\hat{\theta}$ is optimal w.r.t. c^*. Define c^* as follows:

$$c^*_{ij} = \begin{cases} c_{ij} - \mu^* & \text{if } \hat{\theta}_{ij} < T_{ij} \text{ and } c_{ij} - \varphi_{ij} < 0 \\ c_{ij} + \mu^* & \text{if } \hat{\theta}_{ij} > t_{ij} \text{ and } c_{ij} - \varphi_{ij} > 0 \\ c_{ij} & \text{otherwise} \end{cases} \tag{12}$$

It is obvious that $\|c^* - c\| \leq -\mu^*$. Moreover, by Property 5

$$\begin{aligned} c^*_{ij} - \varphi_{ij} = c_{ij} - \mu^* - \varphi_{ij} \quad &\geq \quad \mu^* - \mu^* = 0 \qquad \text{for } \hat{\theta}_{ij} < T_{ij} \\ c^*_{ij} - \varphi_{ij} = c_{ij} + \mu^* - \varphi_{ij} \quad &\leq \quad \mu^* - \mu^* = 0 \qquad \text{for } \hat{\theta}_{ij} > t_{ij} \end{aligned}$$

Hence, $\hat{\theta}$ satisfies the optimality conditions and c^* is an optimal solution of the inverse minimum cost tension problem under Chebyshev norm.

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Hadjias and Maurras [7] provide a Newton type algorithm to solve the minimum mean residual cut problem. Using their algorithm we can find an optimum solution for the inverse problem in strongly polynomial time. McCormick and Ervolina [11] study max mean cuts and mention that a direct method of calculating max mean cuts as Karp [10] does for minimum mean cycles has not yet been found. Radzik [13] improves the best known running time bound of Newton's method for maximum mean weight cut problem and proves that Newton's method runs in strongly polynomial number of iterations for all linear fractional optimization problems. He also shows that the maximum mean weight cut problem, *parametric flow problem* and *minimum maximum arc cost flow problem* are closely related to each other. Here, we revise Radzik's result [13] to include the inverse minimum cost tension problem under Chebyshev distance.

An instance of the parametric flow problem (PF) consists of a network G with arc capacities u and supplies/demands on nodes, and a weight function $w : A \to \mathbb{R}$. The goal is to find minimum nonnegative δ such that $G_{u+w\delta}$, network G with capacity function $u + w\delta$, is feasible. Minimum maximum arc cost problem (MMAC) is defined on a network G with a nonnegative cost function $c : A \to \mathbb{R}$. The goal is to find a flow satisfying the demands on nodes while minimizing the maximum arc cost i.e., minimizing $\max_{a_{ij}\in A} f_{ij}c_{ij}$. In the uniform versions of the problems all weights and costs equal to 1, respectively.

The relationship between IMCT$_c$ under L_∞ norm and PF is more straightforward to justify. In IMCT$_c$, we are given a tension $\hat{\theta}$, which is feasible to MCT with cost vector c but not optimal. Hence, the dual circulation problem of the given MCT problem is infeasible, i.e., there does not exist a circulation φ to satisfy (3a), (3b) and (3c). Our aim is to find the minimum $|\mu|$ such that the circulation problem on G with arc capacities $c + \mu$ is feasible.

In order to show the relationship between IMCT$_c$ under L_∞ norm and MMAC problem we exploit LP duality. We apply the linear programming methods of Ahuja and Orlin [2] to obtain the following LP formulation for IMCT$_c$ under L_∞ norm.

$$\text{Minimize} \quad \sum_{a_{ij}\in A} c_{ij}(\pi_j - \pi_i) \tag{13}$$

subject to

$$\begin{aligned}
\sum_{a_{ij}\in A} \eta_{ij} &= 1 \\
-\eta_{ij} \le \pi_j - \pi_i &\le \eta_{ij} \quad \text{for } a_{ij} \in K \\
0 \le \pi_j - \pi_i &\le \eta_{ij} \quad \text{for } a_{ij} \in L \\
-\eta_{ij} \le \pi_j - \pi_i &\le 0 \quad \text{for } a_{ij} \in U \\
\eta \ge 0 \quad \pi &\gtreqless 0
\end{aligned}$$

By Theorem 6 we know that (13) is the LP formulation for finding minimum mean cost residual cut in G with respect to $\hat{\theta}$. Let us consider its dual.

$$\text{Maximize} \quad \lambda \tag{14}$$

subject to

$$\begin{aligned}
&\left(\sum_{j\in N, a_{ij}\in L} \varphi^1_{ij} + \sum_{j\in N, a_{ij}\in U} -\varphi^2_{ij} + \sum_{j\in N, a_{ij}\in K} (\varphi^1_{ij} - \varphi^2_{ij})\right) \quad - \\
&\left(\sum_{j\in N, a_{ji}\in L} \varphi^1_{ji} + \sum_{j\in N, a_{ji}\in U} -\varphi^2_{ji} + \sum_{j\in N, a_{ji}\in K} (\varphi^1_{ji} - \varphi^2_{ji})\right) \quad = \quad \sum_{j\in N} c_{ji} - \sum_{j\in N} c_{ij} \quad \forall i \in N \\
&\lambda \le -(\varphi^1_{ij} + \varphi^2_{ij}) \qquad \forall a_{ij} \in A \\
&\varphi^1_{ij}, \varphi^2_{ij} \ge 0
\end{aligned}$$

Obviously, (14) is an instance of the uniform MMAC problem on a graph $G' = (N, A')$ with $A' := \{a_{ij} : a_{ij} \in A \text{ and } a_{ij} \in L \cup K\} \cup \{a_{ji} : a_{ij} \in A \text{ and } a_{ij} \in U \cup K\}$. The demands/supplies on the nodes are

$\sum_{j\in N} c_{ji} - \sum_{j\in N} c_{ij} = -Cost(\omega(i)) \quad \forall i \in N$ and the flow capacities of the arcs are $[0,\infty)$. This result establishes the fact that IMCT$_c$ under L_∞ norm and MMAC problems are dual to each other.

If we are given positive weights $w_{ij} > 0 \ \forall a_{ij} \in A$, the objective function of IMCT$_c$ under Chebyshev distance would be

$$\text{Minimize} \quad \max_{a_{ij}\in A} w_{ij}(|c_{ij} - \hat{c}_{ij}|) \tag{15}$$

In this case, the inverse problem reduces to finding a minimum mean-weight residual cut on graph G.

4 Inverse Maximum Tension Problem (IMaxT) under L_1 Norm

Yang *et al.* [18] study inverse maximum flow problem and show that for unit weight case this problem can be reduced to solving a maximum flow problem. In this section we will show a similar result for inverse maximum tension problem under L_1 norm.

Given a weight vector w for changing the bounds of the arcs, the inverse maximum tension problem under L_1-norm is

$$\begin{aligned} &\min \sum_{a_{ij}\in A} w_{ij}(|\hat{T}_{ij} - T_{ij}| + |\hat{t}_{ij} - t_{ij}|) \\ &\text{subject to} \\ &\qquad \hat{t}_{ij} \le \hat{\theta}_{ij} \le \hat{T}_{ij} \quad \forall a_{ij} \in A \\ &\qquad \hat{\theta}_{st} \quad \text{is the maximum tension} \end{aligned} \tag{16}$$

The maximum tension problem is the dual of the maximum flow problem, and so is the optimality condition [14].

Theorem 7. ***(Maximum Tension Minimum Path Theorem)*** *Suppose there is at least one tension satisfying the upper and lower bounds. Then, the maximum in max tension problem is equal to the minimum in min path problem. Both of the problems are unbounded if there is an $s-t$ cut ω with an unlimited span i.e., all forward arcs have infinite upper bounds and all backward arcs have infinite lower bounds.*

By Theorem 7 we know that there exists a minimum path, which has a length equal to the maximum tension. Moreover, for this minimum path the following property holds.

Property 8. *If P denotes the minimum path between s and t on graph G and P^+ and P^- are the corresponding sets of forward and backward arcs in P, then $\theta^*_{ij} = T_{ij}$ for all $a_{ij} \in P^+$ and $\theta^*_{ij} = t_{ij}$ for all $a_{ij} \in P^-$ for the maximum tension θ^*.*

Lemma 9. *If problem (16) has an optimal solution (t^*, T^*) and P^* is the minimum $s-t$ path in network $G = (N, A, t^*, T^*)$, then*

1. *$T^* \le T$ and $t^* \ge t$*
2. *$T^*_{ij} = T_{ij}$ and $t^*_{ij} = t_{ij}$ for each arc $a_{ij} \notin P^*$. Moreover, $t^*_{ij} = t_{ij}$ for arcs $a_{ij} \in P^{*+}$ and $T^*_{ij} = T_{ij}$ for arcs $a_{ij} \in P^{*-}$.*

Proof:

1. As $\hat{\theta}$ is the maximum tension in $G(t^*, T^*)$, $\hat{\theta}_{ij} = T^*_{ij}$ for $a_{ij} \in P^{*+}$ and $\hat{\theta}_{ij} = t^*_{ij}$ for $a_{ij} \in P^{*-}$ by Property 8. If there is an arc $a_{kl} \in A$ with $T^*_{kl} > T_{kl}$ (or $t^*_{kl} < t_{kl}$), then obviously $a_{kl} \notin P^*$ since otherwise $\hat{\theta}$ cannot be a feasible tension in $G(t, T)$. We define the new bound vectors as follows:

$$\bar{T}_{ij} = \begin{cases} T^*_{ij} & \text{if } a_{ij} \ne a_{kl} \\ T_{ij} & \text{if } a_{ij} = a_{kl} \end{cases} \qquad \bar{t}_{ij} = \begin{cases} t^*_{ij} & \text{if } a_{ij} \ne a_{kl} \\ t_{ij} & \text{if } a_{ij} = a_{kl} \end{cases}$$

By Property 8, it is easy to verify that $\hat{\theta}$ is a maximum tension under $(\bar{t}, \bar{T})$. Moreover,

$$\sum_{a_{ij}\in A} w_{ij}(|\bar{T}_{ij} - T_{ij}| + |\bar{t}_{ij} - t_{ij}|) < \sum_{a_{ij}\in A} w_{ij}(|T^*_{ij} - T_{ij}| + |t^*_{ij} - t_{ij}|)$$

which is a contradiction to the optimality of (t^*, T^*). Hence, the result follows.

2. Let us define the bound vectors $(\bar{t}, \bar{T})$ as follows:

$$\bar{T}_{ij} = \begin{cases} T^*_{ij} & \text{if } a_{ij} \in P^{*+} \\ T_{ij} & \text{otherwise} \end{cases} \qquad \bar{t}_{ij} = \begin{cases} t^*_{ij} & \text{if } a_{ij} \in P^{*-} \\ t_{ij} & \text{otherwise} \end{cases}$$

By Property 8, $\hat{\theta}$ remains to be a maximum tension under $(\bar{t}, \bar{T})$. Since

$$\sum_{a_{ij} \in A} w_{ij}(|\bar{T}_{ij} - T_{ij}| + |\bar{t}_{ij} - t_{ij}|) \leq \sum_{a_{ij} \in A} w_{ij}(|T^*_{ij} - T_{ij}| + |t^*_{ij} - t_{ij}|) \tag{17}$$

and (t^*, T^*) is an optimum solution of the inverse max tension problem, the inequality (17) holds with equality and the conclusion is true.

∎

Recall that the tension $\hat{\theta}$, which we want to be maximum, is a feasible tension for $G(t, T)$, thus $\hat{\theta} < \theta^*$ where θ^* is the optimum tension for $G(t, T)$. By using this fact and Lemma 9 we can reformulate IMaxT as follows.

Lemma 10. *The inverse maximum tension problem under L_1 norm is equivalent to finding a path P from s to t in $G = (N, A)$ such that $\sum_{a_{ij} \in P^+} w_{ij}(T_{ij} - \hat{\theta}_{ij}) + \sum_{a_{ij} \in P^-} w_{ij}(\hat{\theta}_{ij} - t_{ij})$ is minimum.*

Theorem 11. *Suppose P^* is the minimum path corresponding to the maximum tension problem in $G(t, T)$. The optimum solution of the inverse maximum tension problem w.r.t. unit weight L_1-norm is*

$$T^*_{ij} = \begin{cases} \hat{\theta}_{ij} & \textit{if } a_{ij} \in P^{*+} \\ T_{ij} & \textit{otherwise} \end{cases} \qquad t^*_{ij} = \begin{cases} \hat{\theta}_{ij} & \textit{if } a_{ij} \in P^{*-} \\ t_{ij} & \textit{otherwise} \end{cases}$$

Hence, solving the inverse problem is equivalent to solving a maximum tension problem on $G(t, T)$.

Proof: Result follows from Lemma 10 and the fact that P^* is the minimum path in graph $G(t, T)$.

∎

Theorem 12. *The solution to the inverse maximum tension problem under L_1 norm with a positive weight function w can be found by solving a maximum tension problem in graph G with respect to upper and lower bounds $w_{ij}(T_{ij} - \hat{\theta}_{ij})$ and $w_{ij}(t_{ij} - \hat{\theta}_{ij})$ on arcs $a_{ij} \in A \backslash \{a_{st}\}$, respectively.*

Proof: The maximum tension problem on G with upper bounds $w_{ij}(T_{ij} - \hat{\theta}_{ij})$ and lower bounds $w_{ij}(t_{ij} - \hat{\theta}_{ij})$ for $a_{ij} \in A \backslash \{a_{st}\}$ is feasible since $w_{ij}(t_{ij} - \hat{\theta}_{ij}) \leq 0 \leq w_{ij}(T_{ij} - \hat{\theta}_{ij})$. Moreover, the length of the minimum path P is

$$\sum_{a_{ij} \in P^+} w_{ij}(T_{ij} - \hat{\theta}_{ij}) - \sum_{a_{ij} \in P^-} w_{ij}(t_{ij} - \hat{\theta}_{ij}) \tag{18}$$

which is by Lemma 10 a solution to the inverse maximum tension problem.

5 Generalization to Monotropic Optimization

In this section, we generalize the results of inverse network flows and tensions to monotropic programs with separable linear cost functions. First, we provide a brief introduction to monotropic programming.

Monotropic programming deals with optimization problems that minimize a separable convex function subject to linear constraints written in the following form

$$\begin{aligned} \text{Minimize} \quad \Phi(x) &= \sum_{j \in J} f_j(x_j) \\ y_i &= \sum_{j \in J} e(i, j) x_j = b_i \quad \forall i \in I \\ x_j &\in C_j \quad \forall j \in J \end{aligned} \tag{P}$$

Here, $E = e(i, j)$ is an arbitrary real matrix expressed in terms of nonempty and finite index sets I and J. Each $f_j : \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ is a closed, proper convex function and C_j is the interval where f_j is finite. We call (P) the *primal problem*.

We denote the left and right derivatives of f_j at ξ with $f_j^-(\xi)$ and $f_j^+(\xi)$, respectively, and extend these functions from $C_j = [c_j^-, c_j^+]$ to $\mathbb{R}$ by defining

$$\begin{array}{llll} f_j^-(\xi) = f_j^+(\xi) = +\infty & \text{if } \xi > c_j^+ & \text{and} & f_j^+(\xi) = +\infty \quad \text{if } \xi = c_j^+ \\ f_j^-(\xi) = f_j^+(\xi) = -\infty & \text{if } \xi < c_j^- & \text{and} & f_j^-(\xi) = -\infty \quad \text{if } \xi = c_j^- \end{array}$$

The *dual problem* of (P) is of the form

$$\begin{aligned} \text{Maximize} \quad \Psi(u) &= -\sum_{i\in I} b_i u_i - \sum_{j\in J} g_j(v_j) \\ v_j &= -\sum_{i\in I} u_i e(i,j) \quad \forall j \in J \\ v_j &\in D_j \quad \forall j \in J \end{aligned} \tag{D}$$

where g_j denotes the conjugate function of f_j, i.e.,

$$g_j(v_j) = \sup_{\xi\in\mathbb{R}} \{v_j \xi - f_j(\xi)\}$$

and D_j is the interval in which g_j is finite. By definition, the respective subspaces of the primal and dual problems,

$$\begin{aligned} \mathcal{C} &= \{x : Ex = 0\} \\ \mathcal{D} &= \{v : \exists\, u \,\text{with} \, -uE = v\}, \end{aligned}$$

are orthogonally complementary to each other. Graphically, this means that (x_j, v_j) is on the *characteristic curve* Γ_j, i.e., $(x_j, v_j) \in \Gamma_j$ where

$$\Gamma_j = \{(\xi, \eta) \in \mathbb{R}^2 : f_j^-(\xi) \le \eta \le f_j^+(\xi)\} \quad \forall j \in J.$$

In this paper, we will assume that there exists a feasible solution x to the primal problem (P), satisfying

$$f_j^-(x_j) < \infty \quad \text{and} \quad f_j^+(x_j) > -\infty \quad \forall j \in J.$$

Such an x is called *regularly feasible solution* of (P). Moreover, we will consider only the special case where the cost function of (P) is separable linear, i.e.,

$$f_j(x_j) = \begin{cases} d_j x_j & \text{if } c_j^- \le x_j \le c_j^+ \\ \infty & \text{otherwise} \end{cases} \tag{20}$$

5.1 Inverse Primal Problem with Linear Costs under L_1 Norm

In the inverse problem of (P), we are given a regularly feasible solution $\tilde{x}$, which is not optimal. Our aim is to modify the cost functions f_j such that the given solution $\tilde{x}$ will be optimum for the new cost functions while the perturbation of the cost is minimized according to some norm. Under the rectilinear norm, we would like to perturb d_j to $\tilde{d}_j$ for which $\tilde{x}$ is an optimum solution to (P) and $\sum_{j\in J} |\tilde{d}_j - d_j|$ is minimum.

First of all, we repeat some of the basic definitions and results on the optimality of monotropic programs where we refer again to Rockafellar [14] for details.

Definition 13. A signed subset P of J is called a ***support*** of $\mathcal{C}$, or a *primal support*, if there is a vector $x \in \mathcal{C}$ such that

$$P^+ = \{j \in J : x_j > 0\} \quad \text{and} \quad P^- = \{j \in J : x_j < 0\}.$$

A primal support P is *elementary* if it is nonempty and does not properly include any other primal support. For an elementary support P, we define an ***elementary vector*** e_P to be the unique elementary $x \in \mathcal{C}$ having P as its support and satisfying

$$|e_P(j)| \leq 1. \tag{21}$$

Hence,

$$\sum_{j \in P^+} e_P(j) - \sum_{j \in P^-} e_P(j) \leq |P|. \tag{22}$$

Note that this definition of the elementary vector e_P is different from the definition given in Rockafellar [14] where he normalizes $x \in \mathcal{C}$ to get e_P such that the inequality (22) holds with equality. However, in this paper we normalize $x \in \mathcal{C}$ to get e_P such that (21) holds. This new normalization of elementary primal support vector is necessary for the future discussions.

Definition 14. An elementary primal support P gives an ***elementary direction of descent at*** $\tilde{x}$ if and only if

$$Cost(P) = \sum_{j \in P^+} f_j^+(\tilde{x}_j) e_P(j) + \sum_{j \in P^-} f_j^-(\tilde{x}_j) e_P(j) < 0. \tag{23}$$

Theorem 15. *A feasible solution $\tilde{x}$ to the primal problem is optimal if and only if there is no elementary direction of descent for Φ at $\tilde{x}$ (Rockafellar [14]).*

In the primal problem, the given regularly feasible solution $\tilde{x}$ is not optimum. Hence, there exists at least one elementary direction of descent for Φ at $\tilde{x}$. By using the definition of f_j (20) and the existence conditions of left and right derivatives of f_j, we can conclude that there exists an elementary vector e_P such that

$$\text{for } j \in P^+ \Rightarrow \tilde{x}_j < c_j^+ \quad \text{and} \quad \text{for } j \in P^- \Rightarrow \tilde{x}_j > c_j^- \tag{24}$$

$$\text{and } Cost(P) = \sum_{j \in P} d_j e_P(j) < 0. \tag{25}$$

Following the denotations of previous sections, we call two elementary primal supports P_1 and P_2 *disjoint* if $P_1^+ \cup P_2^+ = \emptyset$ and $P_1^- \cup P_2^- = \emptyset$.

Theorem 16. *Let $\mathcal{P} = \{P_1, \ldots, P_K\}$ be a minimum cost collection of disjoint elementary primal supports defining descent direction at $\tilde{x}$. The objective function value of the inverse primal problem under unit weight L_1 norm is $-Cost(\mathcal{P}) = -\sum_{k=1}^{K} Cost(P_k)$.*

Proof: First of all, we will show that $-Cost(\mathcal{P})$ is a lower bound on the objective function value. By the definition of e_P, we know that

$$\text{for } j \in P^+ \begin{cases} e_P(j) d_j \leq d_j & \text{if } d_j \geq 0 \\ e_P(j) d_j \geq d_j & \text{if } d_j \leq 0 \end{cases}, \quad \text{for } j \in P^- \begin{cases} e_P(j) d_j \geq -d_j & \text{if } d_j \geq 0 \\ e_P(j) d_j \leq -d_j & \text{if } d_j \leq 0 \end{cases} \tag{26}$$

Then,

$$\begin{aligned} \sum_{j \in J} |\tilde{d}_j - d_j| \geq \sum_{k=1}^{K} \sum_{j \in P_k} |\tilde{d}_j - d_j| &= \sum_{k=1}^{K} \left(\sum_{j \in P_k^+} (\tilde{d}_j - d_j)(+1) + \sum_{j \in P_k^-} (\tilde{d}_j - d_j)(-1) \right) \\ &\geq \sum_{k=1}^{K} \left(\sum_{j \in P_k^+} (\tilde{d}_j - d_j)(e_P(j)) + \sum_{j \in P_k^-} (\tilde{d}_j - d_j)(e_P(j)) \right) \\ &\geq \sum_{k=1}^{K} -Cost(P_k) = -Cost(\mathcal{P}) \end{aligned}$$

Here, the first inequality holds because the elementary primal supports are disjoint and the second inequality holds by (26).

In order to complete the proof, we need to show that this lower bound is indeed achievable. Rockafellar [14] mentions that solving the primal optimality problem is equivalent to solving the dual feasibility problem with respect to the dual spans $D_x(j) = [d_x^-(j), d_x^+(j)]$ where

$$d_x^+(j) = f_j^+(\tilde{x}_j) = \begin{cases} \infty & \text{if } \tilde{x}_j = c_j^+ \\ d_j & \text{if } \tilde{x}_j < c_j^+ \end{cases}, \quad d_x^-(j) = f_j^-(\tilde{x}_j) = \begin{cases} d_j & \text{if } \tilde{x}_j > c_j^- \\ -\infty & \text{if } \tilde{x}_j = c_j^- \end{cases} \tag{27}$$

for the linear cost function $f_j(x_j)$ defined by (20). According to our assumption $\tilde{x}$ is a feasible nonoptimal solution. Hence, the dual problem with respect to the spans $D_x(j)$ for $j \in J$ is infeasible and the following property holds.

Property 17. *For the elementary primal supports in $\mathcal{P}$ there exists a $v \in \mathcal{D}$ such that*

$$\text{For } j \in \mathcal{P} \begin{cases} v_j \geq d_j & \text{if } j \in \mathcal{P}^+ \\ v_j \leq d_j & \text{if } j \in \mathcal{P}^- \end{cases}, \text{ and for } j \notin \mathcal{P} \begin{cases} v_j \leq d_j & \text{if } \tilde{x}_j < c_j^+ \\ v_j \geq d_j & \text{if } \tilde{x}_j > c_j^- \end{cases}$$

Moreover, we can find a v for which the Property 17 holds and $v_j = d_j$ for all $|e_P(j)| \neq 1$. Here, we will not prove the Property 17 since it is a straightforward extension of flow and tension cases. We will show only that the last claim is true.

Suppose that there exists a $P_k \in \mathcal{P}$ for which the claim does not hold, i.e., there does not exist v such that for $j \in P_k$ with $|e_{P_k}(j)| \neq 1$ the equality $v_j = d_j$ holds. Assume without loss of generality that $0 < e_{P_k}(j) < 1$ and d_j is nonnegative. Since $v \in \mathcal{D}$,

$$Sum = \sum_{l \in \{t \in P_k : |e_{P_k}(t)| \neq 1\}} e_{P_k}(l) d_l + \sum_{l \in \{t \in P_k : |e_{P_k}(t)| = 1\}} e_{P_k}(l) v_l + e_{P_k}(j) v_j = 0$$

and $v_j > d_j$ by Property 17. Suppose we set $v_j = d_j$. As the elementary primal supports are disjoint, the effect of this change will only be on Sum, i.e., $Sum < 0$. In order to achieve $Sum = 0$, we need to increase either v_l for $l \in \{j \in P_k : e_{P_k}(j) = 1\}$ or decrease v_l for $l \in \{j \in P_k : e_{P_k}(j) = -1\}$. In either case the new v satisfies $v \in \mathcal{D}$ and the Property 17 holds with $v_j = d_j$ for all $|e_P(j)| \neq 1$. Hence, the claim is true.

Now we are ready to define our new cost function $\tilde{d}$. We set $\tilde{d}_j = v_j$ for all $j \in \mathcal{P}$ and $\tilde{d}_j = d_j$ otherwise. Then,

$$\begin{aligned}
\|\tilde{d} - d\|_1 &= \sum_{j \in J} |\tilde{d}_j - d_j| = \sum_{j \in \mathcal{P}} v_j - d_j \\
&= -\left(\sum_{k=1}^{K} \sum_{j \in P_k^+} (d_j - v_j)(+1) + \sum_{j \in P_k^-} (d_j - v_j)(-1) \right) \\
&= -\left(\sum_{k=1}^{K} \sum_{j \in P_k^+} (d_j - v_j)(e_{P_k}(j)) + \sum_{j \in P_k^-} (d_j - v_j)(e_{P_k}(j)) \right) \\
&= -Cost(\mathcal{P})
\end{aligned}$$

Here, the third equality holds since $v_j = d_j$ holds for $j \in \mathcal{P}$ and $|e_P(j)| \neq 1$ as shown previously. Hence, the proof of the theorem is complete.

5.2 Inverse Primal Problem with Linear Costs under L_∞ Norm

Under Chebyshev norm, we would like to perturb d_j to $\tilde{d}_j$ for which $\tilde{x}$ is an optimum solution to (P) and $\max_{j \in J} |\tilde{d}_j - d_j|$ is minimum.

Following Tseng and Bertsekas [15], we say that an $x \in \mathbb{R}^{|J|}$ and a $v \in \mathbb{R}^{|J|}$ satisfy ***ϵ-complementary slackness***, where ϵ is any positive scalar, if

$$f_j(x_j) < \infty \quad \text{and} \quad f_j^-(x_j) - \epsilon \leq v_j \leq f_j^+(x_j) + \epsilon, \text{ for } j \in J. \tag{28}$$

Graphically, this means that (x_j, v_j) is within ϵ vertical distance of the characteristic curve Γ_j. We call x an *ϵ-optimal solution* if x satisfies the ϵ-complementary slackness conditions (28).

Theorem 18. *A given feasible solution $\tilde{x}$ to (P) is an ϵ-optimal solution if and only if all the elementary primal supports defining a descent direction with respect to $\tilde{x}$ have a mean cost ($MCost(P)$), which is greater than $-\epsilon$, i.e.,*

$$MCost(P) = \frac{Cost(P)}{|P|} = \frac{\sum_{j\in P^+} f_j^+(\tilde{x}_j)e_P(j) + \sum_{j\in P^-} f_j^-(\tilde{x}_j)e_P(j)}{|P|} \geq -\epsilon$$

Proof: "⇒" Suppose that the given feasible solution $\tilde{x}$ is an ϵ-optimal solution but there exists an elementary primal support P defining a descent direction at $\tilde{x}$ with $MCost < -\epsilon$. Without loss of generality we consider the worst case, i.e., the most negative cost case where $f_j^+(\tilde{x}_j) < 0$ for $j \in P^+$ and $f_j^-(\tilde{x}_j) > 0$ for $j \in P^-$. Since $e_P v = 0$ for $v \in \mathcal{D}$, the following holds.

$$\begin{aligned} Cost(P) &= Cost(P) - e_P v \\ &= \sum_{j\in P^+} f_j^+(\tilde{x}_j)e_P(j) + \sum_{j\in P^-} f_j^-(\tilde{x}_j)e_P(j) - \sum_{j\in P} e_P(j)v_j \\ &= \sum_{j\in P^+} (f_j^+(\tilde{x}_j) - v_j)e_P(j) + \sum_{j\in P^-} (f_j^-(\tilde{x}_j) - v_j)e_P(j) \\ &\geq \sum_{j\in P^+} (f_j^+(\tilde{x}_j) - v_j)(+1) + \sum_{j\in P^-} (f_j^-(\tilde{x}_j) - v_j)(-1) \\ &\geq -\epsilon|P| \end{aligned}$$

Here, the first inequality holds by the definition of the elementary vector e_P and by (26). The second inequality holds by ϵ-complementary slackness (28). As it can be concluded $MCost(P) \geq -\epsilon$, and we get a contradiction to the assumption.
"⇐" Suppose that all the elementary primal supports have $MCost(P) \geq -\epsilon$ but the solution $\tilde{x}$ is not ϵ-optimal,i.e., $f_j^-(x_j) - v_j > \epsilon$ and $f_j^+(x_j) - v_j < -\epsilon$ for $j \in J$. Then,

$$\begin{aligned} Cost(P) &= Cost(P) - e_P v \\ &= \sum_{j\in P^+} (f_j^+(\tilde{x}_j) - v_j)e_P(j) + \sum_{j\in P^-} (f_j^-(\tilde{x}_j) - v_j)e_P(j) \\ &< -\epsilon|P| \end{aligned}$$

which is a contradiction.

■

Theorem 19. *Let P^* be the minimum mean cost elementary primal support defining a descent direction at $\tilde{x}$ and μ^* be its mean cost. The optimum objective function value of inverse primal problem with linear costs under unit weight Chebyshev norm is* $\max(0, -\mu^*)$.

Proof: By using similar arguments as in Theorem 6, it is easy to show that $-\mu^*$ is a lower bound on the optimal objective function value. Moreover, by Theorem 18, we know that there exists $v \in \mathcal{D}$ satisfying the ϵ-complementary slackness conditions (28) with $\epsilon = -\mu^*$. Thus, we define the new cost function to be

$$f_j(x_j) = \begin{cases} d_j^* x_j & \text{if } c_j^- \leq x_j \leq c_j^+ \\ \infty & \text{otherwise} \end{cases}$$

where

$$d_j^* = \begin{cases} d_j - \mu_j^* & \text{if } x_j < c_j^+ \text{ and } d_j - v_j < 0 \\ d_j + \mu_j^* & \text{if } x_j > c_j^- \text{ and } d_j - v_j > 0 \\ d_j & \text{otherwise} \end{cases} \tag{29}$$

Obviously, d^* is the optimum solution to the inverse primal problem with linear costs under unit weight Chebyshev norm.

6 Conclusion and Future Work

For the inverse minimum cost and maximum value tension problems under rectilinear and Chebyshev norms we showed that the results of Ahuja and Orlin [3] and Yang *et al.* [18] for inverse network flows can be extended. We proved that the inverse minimum cost tension problem under rectilinear norm is equivalent to solving a minimum cost tension problem, while under unit weight Chebyshev norm it can be solved by finding a minimum mean cost residual cut. Moreover, inverse maximum tension problem under rectilinear norm can be solved as a maximum tension problem on the same graph with new arc bounds.

In this paper we also presented a generalization of the inverse problems for monotropic programming with linear costs. This generalization certifies the validity of the given combinatorial results for network flows and tensions even if they do not possess totally unimodularity, i.e., generalized flows and tensions. Another generalization of inverse network flows and tensions would be inverse flows in regular matroids [5], which is currently investigated.

In Güler and Hamacher [6], we have studied the capacity inverse minimum cost flow problem and shown that under L_1 norm this problem is NP-Hard. A similar problem in tensions is the bound inverse minimum cost tension problem where we perturb the upper and lower bounds instead of costs. Analyzing this problem would complete the comparison of inverse network flow and tension problems. Moreover, it seems that inverse tension problems may have potential for practical applications, especially in scheduling problems. These are currently explored, as well.

Acknowledgment

I would like to thank to my supervisor Prof. Dr. Horst W. Hamacher for his valuable ideas that founded this research paper and his encouraging support.

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