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Portfolio Optimisation and Calibration with Credit Risk

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to my family

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Abstract

This thesis covers two important fields in financial mathematics, namely the continuous time portfolio optimisation and credit risk modelling. We analyse optimisation problems of portfolios of Call and Put options on the stock and/or the zero coupon bond issued by a firm with default risk. We use the martingale approach for dynamic optimisation problems. Our findings show that the riskier the option gets, the less proportion of his wealth the investor allocates to the risky asset. Further, we analyse the Credit Default Swap (CDS) market quotes on the Eurobonds issued by Turkish sovereign for building the term structure of the sovereign credit risk. Two methods are introduced and compared for bootstrapping the risk-neutral probabilities of default (PD) in an intensity based (or reduced form) credit risk modelling approach. We compare the market-implied PDs with the actual PDs reported by credit rating agencies based on historical experience. Our results highlight the market price of the sovereign credit risk depending on the assigned rating category in the sampling period. Finally, we find an optimal leverage strategy for delivering the payments promised by a Constant Proportion Debt Obligation (CPDO). The problem is solved via the introduction and explicit solution of a stochastic control problem by transforming the related Hamilton-Jacobi-Bellman Equation into its dual. Contrary to the industry practise, the optimal leverage function we derive is a non-linear function of the CPDO asset value. The simulations show promising behaviour of the optimal leverage function compared with the one popular among practitioners.

Preface

This thesis is based on my research since I joined the Department of Financial Mathematics of Fraunhofer ITWM in November 2004, where I also had a chance to participate in the research and consulting projects for the financial industry related with my research interests.

The three chapters in this thesis are conceptually independent from each other, therefore each chapter is self-contained and has a separate introduction and a summary section. The reader may directly switch to the topic of his/her interest.

The starting point of my research was the intensity based (or reduced-form) credit risk modelling, then we decided to integrate the credit risk issues into continuous time portfolio optimisation problems. Hence, the first chapter is in line with the paper by Korn and Kraft [KK03], where they examine the portfolio optimisation problems of defaultable assets using a firm value based credit risk model. In Chapter 1, we study optimisation problems of portfolios consisting of risky options. The framework of Korn and Trautmann [KT99] is applied for the optimisation problem, where we model the credit risk with a firm-value based approach. Since the underlying in the portfolio is a European type option on the risky bond written by the firm, the compound option formula of Geske [Ges79] is adapted for pricing reasons.

The second chapter is inspired from an industry project of Fraunhofer ITWM in 2006 for a leading German bank, where we jointly with PD Dr. Marlene Müller analysed the relationship between the risk-neutral and actual default probabilities of the customers of the bank, and validate the actual default probabilities with the risk neutral ones extracted from CDS quotes. Chapter 2 takes the Turkish sovereign CDS rates for building the term structure of market implied sovereign credit risk. For that, a detailed literature survey about intensity based credit risk models is presented and two methods are introduced for extracting the default probabilities from the market CDS quotes. Furthermore, we give a detailed analysis about the linkage of the market implied default intensities and the actual default intensities estimated by a rating agency based on historical dataset.

In the third and the last chapter, we look at the problem of finding an optimal leverage strategy for delivering the payments promised by a constant proportion debt obligation (CPDO). The problem will be solved via the introduction and explicit solution by transforming the corresponding Hamilton-Jacobi-Bellman-Equation into its dual. This chapter is similar to Baydar et al. [BGK08], where we include the preliminaries about stochastic control method and provide examples in addition to our paper.

This thesis summarises the credit risk literature, including the structural and the intensity based models. Moreover, for continuous time optimisation, we present and apply both of the approaches, namely the martingale method and the stochastic control method.

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Kaiserslautern, 23 April 2008 Evren Baydar

Chapter 1

Optimal Portfolios of Options with Credit Risk

1.1 Introduction

Portfolio optimisation problems start with the pioneering work by Markowitz [Mar52], where he developed the theory in a discrete time setting. The first optimisation approach in a continuous time setting was introduced by Merton [Mer69], [Mer71] applying stochastic control methods to portfolio problems. In his work the investor is allowed to invest on the stocks and a riskless bond (or money market account).

In this chapter we are introducing portfolio optimisation problems when the portfolios are composed of a riskless bond and European options written on the stock or the bond issued by a firm, where the firm has credit risk (or default risk). Credit risk is defined as the failure of fulfilling a financial obligation by the agents determined in a contract. Credit risk problems are generally analysed in two approaches, namely the reduced form (or intensity based) and the structural (or firm-value based) models. We use the classical Merton [Mer74] approach, known to be the first firm-value based credit risk model.

In Merton model, the market value of the firm $V(t)$ follows a geometric Brownian motion, being the main source of uncertainty. The financial obligation of the firm is to return the promised face value F to the bondholders at debt maturity time T_1. Hence, the default occurs if the firm can not fulfil its obligation, i.e., the market value of the firm is less than its debt $V(T_1) < F$. The stock of the firm is valued similar to a European call option written on the market value of the firm with a strike price equal to the debt value, F. If we have another call option written on the stock price of this firm, we can consider the derivative a European call on call option, with the firm value as the underlying. Hence, we can adopt the *compound option* pricing techniques of Geske [Ges79] to our problem. We use similar techniques for the valuation of European options written on the bond of the firm.

To our knowledge, the portfolio problem with defaultable securities was first introduced by Merton [Mer71], where he used a special kind of reduced-form credit risk model for modelling the default event. A similar approach was examined by Kraft and Steffensen [KS08], where the authors proposed a model that allows for random recovery and joint default events as well in a reduced-form setting. Other papers dealing with similar problems are Bielecki and Jang [BJ07], and Lakner and Liang [LL07].

A second type of portfolio optimisation problem including credit risk, which uses the structural credit risk models was introduced by Korn and Kraft [KK03]. In this approach, the authors use the elasticity method of Kraft [Kra03], which is the generalisation of the ideas presented by Korn and Trautmann [KT99] for continuous time optimisation problem for the option portfolios. Furthermore, Kraft and Steffensen [KS06] extended the model developed in Korn and Kraft [KK03] with power utility functions delivering more reliable results. Our work can be listed in this stream of papers, as we use a structural model for modelling the credit event.

Our contributions in this chapter result from combining three ingredients:

- the Merton [Mer74] approach for modelling the credit risk,
- the optimisation method for portfolio of options from Korn and Trautmann [KT99],
- the Geske [Ges79] formula for pricing of the compound options,

in order to deal with an optimal (option) investment problem for defaultable securities.

The outline of the chapter is as follows. We analyse the structural credit risk models in Section 1.2. We give the outlines of continuous time portfolio optimisation problem in Section 1.3, where we present the martingale approach in details. We continue with Section 1.4 by introducing the Korn and Trautmann [KT99] framework for optimising option portfolios. Section 1.5 presents our findings, where we extend the results of Korn and Kraft [KK03], and add a second iteration to the problem in their paper. Here, we optimise portfolios consisting of options on options and the money market account. We finally present our findings and summarise the chapter.

1.2 Structural Credit Risk Models

In this section, we will describe the *structural* credit risk model, which is also called the *firm value based* credit risk model. This model was proposed by Merton [Mer74] and uses the option pricing techniques of Black and Scholes [BS73]. In this approach, the corporate liabilities are considered as contingent claims on the assets of the firm. This model is named as *firm-value based* since the market value of the firm is the fundamental source of uncertainty which drives the credit risk.

We can also subdivide structural models into two different approaches, namely the *classical* approach and the *first-passage* approach. In the classical Merton [Mer74] approach, the firm defaults when its market value is not sufficient to pay back its debt *at the maturity time* of the contract. This means that the default cannot be triggered before debt maturity, which is a very unrealistic assumption. However, in first-passage models, we assume that the default is triggered when the value of the firm falls below a barrier *during* the life time of the bond. This approach was pioneered by Black and Cox [BC76].

1.2.1 Classical Approach: Merton Model

We present the important results of the classical approach in this subsection. Merton [Mer74] introduces the firm value dynamics with an assumption that the firm has payouts (dividends or interest payments) to either its shareholders or liability holders. For simplicity, we assume that the firm has neither dividends nor interest payments. The dynamics for the market value of the firm V through time is described by a geometric Brownian motion:

$$\frac{dV(t)}{V(t)} = \mu_v dt + \sigma_v dW(t), \ V(0) > 0, \tag{1.1}$$

where $\mu_v \in \mathbb{R}$ is the constant drift parameter, $\sigma_v > 0$ is the constant volatility parameter and W is the one-dimensional Brownian motion under physical measure P. Here, V represents the expected discounted future cash flows of a firm. The simulated paths for the dynamics of the firm value process can be observed in Figure 1.1, where we use the algorithm described by Uğur [UŎ8]. The firm is financed by an equity (stock) $P_1(t)$ and a

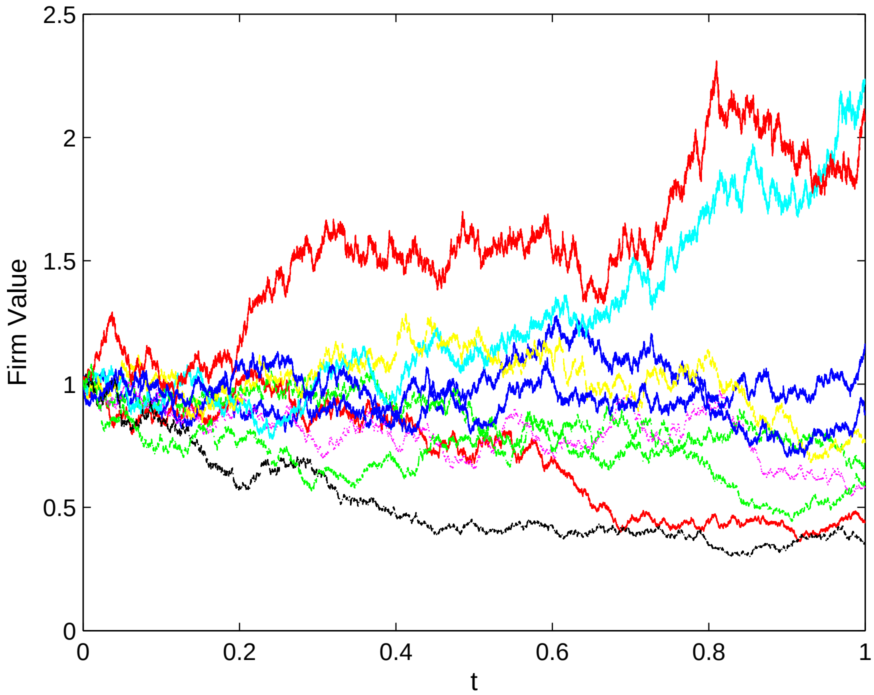

Figure 1.1: Simulated paths for the firm value process with $\mu_v = 0.1$, $\sigma_v = 0.5$, $V(0) = 1$.

risky Zero Coupon Bond (ZCB, hereafter) $\bar{B}(t, T_1)$ with face value F and maturity date T_1. The contractual obligation of the firm is to repay F to the bondholders at time T_1. We assume that if the firm cannot fulfil its payment obligation, then the bondholders will immediately take over the firm. Hence, the default time τ is a random variable with:

$$\tau = \begin{cases} T_1 & \text{if } V(T_1) < F, \\ \infty & \text{else.} \end{cases} \tag{1.2}$$

Itô's lemma implies that

$$V(t) = V(0) \exp\left(\left(\mu_v - \frac{1}{2}\sigma_v^2\right) t + \sigma_v W(t)\right).$$

Assuming that the firm can neither issue new senior debt on the firm nor repurchase

	Firm value	Bond	Stock
No default	$V(T_1) \geq F$	F	$V(T_1) - F$
Default	$V(T_1) < F$	$V(T_1)$	0

Table 1.1: Payoffs at maturity in the classical approach

shares prior to the maturity of the debt, the payoffs of the securities of the firm will be as in Table 1.1. If the firm value $V(T_1)$ exceeds or equals the face value F of the bonds, the

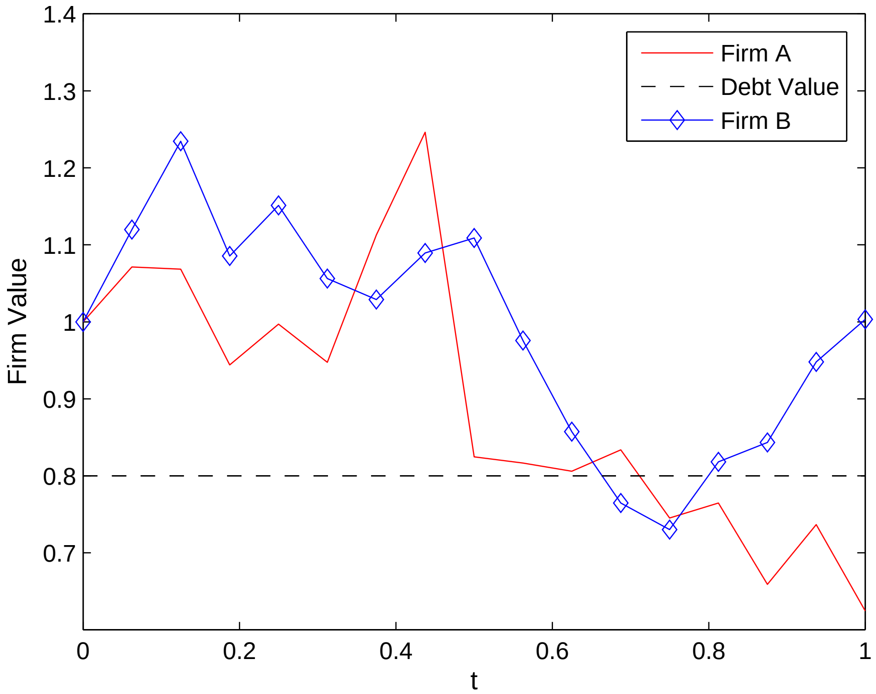

Figure 1.2: Firm A defaults, Firm B does NOT default, with $F = 0.8$.

bondholder receives the promised payment F, and the shareholder receives what remains; $V(T_1) - F$. If the value of the assets $V(T_1)$ is less than F, the firm defaults and the ownership of the firm is transferred to bondholders; and shareholders receive nothing. Therefore, the value of the ZCB at maturity time T_1 will be given by:

$$\bar{B}(T_1, T_1) = \min(F, V(T_1)) = F - (F - V(T_1))^+. \tag{1.3}$$

Now, we can relate the option pricing theory of Black and Scholes [BS73] with the following idea. The payoff given in (1.3) is the same payoff of a portfolio composed of a default-free loan with face value F maturing at T_1 and a short European put position on the value of a firm with strike F and maturity time T_1. Denoting the price of the stock with $P_1(\cdot)$, at the time T_1 we have

$$P_1(T_1) = (V(T_1) - F)^+, \tag{1.4}$$

which is equivalent to the payoff of a European call option on the firm value with strike F and maturity time T_1. Thus, valuation of the stock is the same as valuation of a European option in the classical Black-Scholes setting, where we assume the short interest rate, r, is constant and the firm value V, follows a geometric Brownian motion. The Black-Scholes call option formula gives the stock price as:

$$P_1(t) = V(t)\Phi(h_1(t)) - Fe^{-r(T_1-t)}\Phi(h_2(t)), \tag{1.5}$$

where

$$h_1(t) = \frac{\ln\left(\frac{V(t)}{F}\right) + \left(r + \frac{1}{2}\sigma_v^2\right)(T_1 - t)}{\sigma_v\sqrt{T_1 - t}} \text{ and } h_2(t) = h_1(t) - \sigma_v\sqrt{T_1 - t}.$$

The value of the risky ZCB is

$$\bar{B}(t, T_1) = Fe^{-r(T_1-t)} - X^{Put}(t, V(t)),$$

where X^{Put} is the Black-Scholes put option formula. Therefore, we will have:

$$\bar{B}(t, T_1) = V(t)\Phi(-h_1(t)) + Fe^{-r(T_1-t)}\Phi(h_2(t)), \tag{1.6}$$

which together with (1.5) proves the market value identity:

$$V(t) = P_1(t) + \bar{B}(t, T_1).$$

1.2.2 First Passage Models: Black-Cox Model

The main drawback of the Merton [Mer74] model is that the default event may occur *only* on the maturity time of the bond, which is very unrealistic. Hence, first passage models were introduced allowing the default event occur *during* the life time of the defaultable bond. The default time is the first time that the value of the firm hits a barrier, i.e.,

$$\tau = \inf\{t | V(t) = L(t)\}, \qquad t > 0 \tag{1.7}$$

where the time-dependent, deterministic barrier function is denoted with $L(\cdot)$. We can think the barrier as continuously compounded debt k with rate κ discounted to time t, i.e.,

$$L(t) = ke^{-\kappa(T_1-t)}. \tag{1.8}$$

The price of a ZCB with the face value $F \geq k$ and maturity T_1 at time $t \in [0, \min(T_1, \tau)]$ is given by

$$\begin{aligned}\bar{B}(t, T_1) &= Fe^{-r(T_1-t)}\left[\Phi(z_1(t)) - y^{2\theta-2}(t)\Phi(z_2(t))\right] \\ &\quad + V(t)\left[\Phi(-z_3(t)) + y^{2\theta}(t)\Phi(z_4(t))\right],\end{aligned}$$

with

$$\begin{aligned}z_{1;3} &= \frac{\ln\left(\frac{V(t)}{F}\right) + (r \mp \frac{1}{2}\sigma_v^2)(T_1 - t)}{\sigma_v\sqrt{T_1 - t}} \\ z_{2;4} &= \frac{\ln\left(\frac{V(t)}{F}\right) + 2\ln(y(t)) + (r \mp \frac{1}{2}\sigma_v^2)(T_1 - t)}{\sigma_v\sqrt{T_1 - t}} \\ y(t) &= \frac{ke^{-\kappa(T_1-t)}}{V(t)} \\ \theta &= \frac{r - \kappa + \frac{1}{2}\sigma_v^2}{\sigma^2}.\end{aligned}$$

Since we focus on the optimal portolio problems in classical Merton setting on this level, we refer the interested reader to the introductory paper by Giesecke [Gie04] for more information about advanced structural credit risk models. Further, Acar [Aca06] introduces an advanced firm value model including a jump component for obtaining the optimal capital structure of a firm.

1.3 Continuous Time Portfolio Optimisation Problem

The problem can be briefly defined as finding an optimal consumption and investment strategy for an investor with an initial capital of $x > 0$ in order to maximise his expected utility on terminal wealth. Hence, it is about deciding *how many* shares of *which* security one investor should hold at *which* time instant. For the general presentation in this section, we assume to be in a standard diffusion type market with d risky assets and a riskless bond (or Money Market Account). We present some definitions from Korn and Korn [KK01].

Definition 1.1. I. A **trading strategy** φ is an $\mathbb{R}^{d+1}-$ valued progressively measurable process with respect to $\{\mathcal{F}_t\}_{t\in[0,T]}$

$$\varphi := (\varphi_0(t), \varphi_1(t), \ldots, \varphi_d(t))'$$

satisfying

$$\int_0^T |\varphi_0(t)|dt < \infty \text{ a.s. }, \tag{1.9}$$

$$\sum_{j=1}^{d} \int_0^T (\varphi_i(t) \cdot P_i(t))^2 dt < \infty \text{ a.s. } \quad \text{for } i = 1 \ldots d. \tag{1.10}$$

The value $x := \sum_{i=0}^{d} \varphi_i(0) \cdot p_i$ is called the **initial value** of φ.

II. Let φ be a trading strategy with initial value $x > 0$. The process

$$X(t) := \sum_{i=0}^{d} \varphi_i(t) P_i(t)$$

is called **wealth process** corresponding to φ with initial wealth x.

III. A non-negative progressively measurable process $c(t)$ with respect to $\{\mathcal{F}_t\}_{t\in[0,T]}$ with

$$\int_0^T c(t)dt < \infty \text{ a.s.} \tag{1.11}$$

is called a **consumption rate process** (or just consumption process).

Definition 1.2. A pair (φ, c) consisting of a trading strategy φ and a consumption rate process c is called **self-financing** if the corresponding wealth process $X(t)$, $t \in [0, T]$, satisfies:

$$\begin{aligned} X(t) &= x + \sum_{i=0}^{d} \int_0^t \varphi_i(s) dP_i(s) - \int_0^t c(s) ds \text{ a.s. } . \\ \text{current wealth} &= \text{initial wealth} + \text{gains / losses} - \text{consumption} \end{aligned} \tag{1.12}$$

Definition 1.3. Let (φ, c) be a self-financing pair consisting of a trading strategy and a consumption process with corresponding wealth process $X(t) > 0$ a.s. for all $t \in [0, T]$. Then, the $\mathbb{R}^d-$valued process

$$\pi(t) := (\pi_1(t), \ldots, \pi_d(t))', t \in [0, T] \text{ with } \pi_i(t) = \frac{\varphi_i(t) \cdot P_i(t)}{X(t)}$$

is called a **self financing portolio process** corresponding to the pair (φ, c).

Remark 1.1. I. The portfolio process denotes the fraction of the total wealth invested in different stocks. Therefore, the fraction of wealth invested in the riskless bond (or MMA) is

$$(1-\pi(t)\underline{1}) = \frac{\varphi_0(t)\cdot P_0(t)}{X(t)}, \text{ where } \underline{1} := (1,\dots,1)' \in \mathbb{R}^d.$$

II. Given the knowledge of the wealth $X(t)$ and the prices $P_i(t)$, it is possible for an investor to describe his activities via a self-financing pair (π, c). More precisely, in this case, portfolio process and trading strategy are equivalent descriptions of same action.

Now, we introduce the functional J for measuring the utility of a payment stream, where large values of J should represent "good" payment streams. Therefore, the investor looks for a self-financing pair (an admissible investment strategy and consumption process) $(\pi, c) \in \mathcal{A}(x)$, which maximises the expected utility from consumption and/or terminal wealth,

$$J(x;\pi,c) = E\left[\int_0^T U_1(t,c(t))dt + U_2(X^{x,\pi,c}(T))\right], \tag{1.13}$$

where U_1, U_2 are the utility functions, $X(t)$ is the wealth process corresponding to the initial capital x and (π, c). We require that the utility functions $U_1(t,.)$ and $U_2(.)$ are C^1, strictly concave and satisfy

$$U'(0) := \lim_{x\downarrow 0} U'(x) = +\infty, \quad U'(\infty) := \lim_{x\downarrow\infty} U'(x) = 0.$$

Typical utility functions are $U(x) = \ln(x)$, $U(x) = \sqrt{x}$, or $U(x) = x^\alpha$ for $0 < \alpha < 1$. For more details on the utility functions, we refer the reader to Korn [Kor97].

Note that for an arbitrary $(\pi, c) \in \mathcal{A}(x)$, the expectation in (1.13) is not necessarily defined. Hence, we restrict the class of self-financing pairs (π, c), in which the expectation in (1.13) is finite. However, having an infinite positive expected utility would be any investor's dream if it could be reached. We can now define the problem after this restriction.

Definition 1.4. The problem

$$\max_{(\pi,c)\in\mathcal{A}'(x)} J(x;\pi,c) \tag{1.14}$$

with

$$\mathcal{A}'(x) = \left\{(\pi,c)\in\mathcal{A}(x)\middle| E\left[\int_0^T U_1(t,c(t))^- dt + U_2(X(T))^-\right] < \infty\right\}$$

is called the continuous-time portfolio problem.

Remark 1.2. I. Note that the condition in (1.14) does not exclude the strategies that will possibly lead to infinite utility. It states that the only requirement is the finiteness of the expected value over the negative parts of the utility function. Hence, by restricting to the set $\mathcal{A}'(x)$, the integral in (1.14) is always defined.

II. If $U_1(t,.) > 0$ and $U_2(.) > 0$, the equality $\mathcal{A}(x) = \mathcal{A}'(x)$ is trivially satisfied.

There are mainly two solution methods in the literature for the portfolio problem in (1.14). The first method is called the *martingale method*, which is based on the martingale theory and stochastic integration in a complete market setting. The second approach is the *stochastic control method* and it is an application of the standard methods of stochastic control theory to portolio optimisation problem. In the next subsection, we will explain the motivation of the martingale method and provide an example. We present some important results of the stochastic control theory in Chapter 3.

1.3.1 The Martingale Method

The main idea of the martingale method is to decompose the dynamic (in time) portolio problem in (1.14) into a static (in time) optimisation problem (determination of the optimal payoff profile) and a representation problem (compute the portfolio process that yields the optimal payoff profile).

Since the motivation of the approach mainly depends on the complete market assumption, we introduce the related theorem below. Remember that the number of stocks equals the dimension of the underlying Brownian motion. We use the following notation

$$\begin{aligned} \theta(t) &:= \sigma_p^{-1}(t)(\mu_p(t) - r(t)\underline{1}) \\ H(t) &:= \exp\left(-\int_0^t \theta(s)'dW(s) - \int_0^t \left(r(s) + \frac{1}{2} \parallel \theta(s) \parallel^2\right) ds\right), \end{aligned}$$

where μ_p denotes the deterministic drift process for equity dynamics, σ_p is the volatility, and r is the short rate process.

Moreover, $H(t)$ is the unique solution to the Stochastic Differential Equation (SDE)

$$\begin{aligned} dH(t) &= -H(t)[r(t)dt + \theta(t)'dW(t)], \\ H(0) &= 1. \end{aligned} \tag{1.15}$$

Theorem 1.1 (Completeness of the market). ***I.*** *Let the self-financing pair (π, c) be admissible for an initial wealth of $x \geq 0$. Then, the corresponding wealth process $X^{x,\pi,c}(t)$ satisfies*

$$E\left[H(t)X^{x,\pi,c}(t) + \int_0^t H(s)c(s)ds\right] \leq x \text{ for all } t \in [0,T].$$

II. *Let $B \geq 0$ be an $\mathcal{F}_T-$measurable random variable and $c(t)$, $t \in [0,T]$, a consumption process satisfying*

$$x := E\left[H(T)B + \int_0^T H(s)c(s)ds\right] < \infty.$$

Then, there exists a portfolio process $\pi(t)$, $t \in [0,T]$, with $(\pi, c) \in \mathcal{A}(x)$ and the corresponding wealth process $X^{x,\pi,c}(t)$ satisfies

$$X^{x,\pi,c}(T) = B \quad \text{almost surely (a.s.).}$$

Proof: See p.66 of Korn and Korn [KK01].

Motivation of the Martingale Method

We start the presentation with assuming that the portfolio problem in (1.14) does not have the consumption process, i.e., $c \equiv 0$, $U_1 \equiv 0$. Therefore, the dynamic portfolio problem reduces to

$$\max_{(\pi,0)\in\mathcal{A}'(x)} E(U_2(X^{x,\pi}(T))). \tag{1.16}$$

From the completeness of the market (Theorem 1.1), we have

$$E\left[H(T)X^{x,\pi}(T)\right] \leq x \text{ for } T \geq 0,$$

and let the final payment $B \geq 0$ be $\mathcal{F}_T-$measurable with $E[H(T)B] = x$. Furthermore, there exists a portfolio process $(\pi, 0) \in \mathcal{A}$ with $B = X^{\pi}(T)$ a.s. Define

$$\mathcal{B}(x) := \{B \mid B \geq 0,\ \mathcal{F}_T\text{-measurable},\ E[H(T)B] \leq x, E[U_2(B)^-] < \infty\},$$

representing the set of all final wealths with some initial wealth $y \in (0, x]$ and satisfying $E[U_2(B)^-] < \infty$. In order to determine the optimal final wealth, it is sufficient to solve the following problem

$$\max_{B\in\mathcal{B}(x)} E[U_2(B)]. \tag{1.17}$$

Note that we do not have any time dependent variable above, therefore, we only optimise over a set of random variables. Here, we transformed the dynamic problem in (1.16) into the static problem in (1.17). We solve the static problem (1.17) with the help of Lagrangian method (See p. 208 of Korn and Korn [KK01]).

Say, the first step results in the optimal wealth B^*, then the remaining step is to solve the "representation" problem:

$$\text{Find a } (\pi^*, 0) \in \mathcal{A}'(x) \text{ with } X^{x,\pi^*}(T) = B^* \text{ a.s. .} \tag{1.18}$$

□

Going back to general optimisation problem defined in (1.14), we introduce the function $\chi : (0, \infty) \to \mathbb{R}$:

$$\chi(y) := E\left[\int_0^T H(t) I_1(t, yH(t))dt + H(T) I_2(yH(T))\right] \quad \forall y > 0,$$

where $I_1(t, \cdot) = (U_1')^{-1}(t, .)$, is the inverse function of the partial derivative of U_1 with respect to the second component, and $I_2(\cdot) = (U_2')^{-1}(.)$. Function $\chi(y)$ is strictly decreasing, continuous and possesses an inverse function. Setting $Y(x) := \chi^{-1}(x)$ and with the help of the following theorem from Korn and Trautmann [KT99], we get the optimal terminal wealth and the optimal consumption process.

Theorem 1.2. *Let $x > 0$. Under the assumption of*

$$\chi(y) < \infty \quad y \in (0, \infty)$$

the optimal terminal wealth B^* and the optimal consumption process $c^*(t)$, $t \in [0, T]$, for problem (1.14) are given by

$$\begin{aligned} B^* &:= I_2(Y(x)H(T)), && \text{"optimal terminal wealth"} \\ c^*(t) &:= I_1(t, Y(x)H(t)), && \text{"optimal consumption"} \end{aligned}$$

Moreover, there exists a portfolio process $\pi^*(t)$, $t \in [0, T]$, such that we have

$$(\pi^*, c^*) \in \mathcal{A}'(x), \quad X^{x,\pi^*,c^*}(T) = B^* \text{ a.s.},$$

and such that (π^*, c^*) solves the problem (1.14), where $X^{x,\pi^*,c^*}(t)$ is the wealth process corresponding to the pair (π^*, c^*) and the initial wealth x.

Proof: See p. 210 of Korn and Korn [KK01].

Example 1.1. We present an example from Korn and Korn [KK01] with logarithmic utility functions for the martingale approach of portfolio optimisation. Suppose we have

$$U_1(t,x) = U_2(x) = \ln(x).$$

Note that we may have negative utilities if $x < 1$. With the utility functions given above, we have

$$\begin{aligned}
&\Rightarrow \quad I_1(t,y) = I_2(y) = \frac{1}{y} \\
&\Rightarrow \quad \chi(y) = E\left[\int_0^T H(t)\cdot\frac{1}{yH(t)}dt + H(T)\cdot\frac{1}{yH(T)}\right] = \frac{1}{y}(T+1) \\
&\Rightarrow \quad Y(x) = \chi^{-1}(x) = \frac{1}{x}(T+1).
\end{aligned}$$

With Theorem 1.2, we get the optimal consumption and wealth as

$$\begin{aligned}
B^* &:= I_2(Y(x)H(T)) = \frac{x}{T+1}\cdot\frac{1}{H(T)}, \\
c^*(t) &:= I_1(t, Y(x)H(t)) = \frac{x}{T+1}\cdot\frac{1}{H(t)}.
\end{aligned}$$

From these optimal values, we can find the the optimal portfolio process explicitly. We have

$$H(t)\cdot X^{x,\pi^*,c^*}(t) = E\left[\int_t^T H(s)c^*(s)ds + H(T)B^*\Big|\mathcal{F}_t\right] \tag{1.19}$$

$$= x\cdot\frac{1+T-t}{T+1}. \tag{1.20}$$

Then,

$$x = x\cdot\frac{T+1-t}{T+1} + x\cdot\frac{t}{T+1} = H(t)\cdot X^{x,\pi^*,c^*}(t) + \int_0^t H(s)c^*(s)ds. \tag{1.21}$$

From the self-financing pair (π^*, c^*) and the corresponding wealth process, $X := X^{x,\pi^*,c^*}$, we have the *wealth equation* as follows:

$$\begin{aligned}
dX(t) &= [r(t)X(t) - c^*(t)]dt + X(t)\pi^*(t)'(\mu_p(t) - r(t)\underline{1})dt \\
&\quad + X(t)\pi^*(t)'\sigma_p(t)dW(t) \\
X(0) &= x,
\end{aligned}$$

and $H(t)$ has the Itô representation as in (1.15). Applying Itô product rule to $H(t)\cdot X(t)$, we have

$$\begin{aligned} H(t)\cdot X(t) &= H(0)\cdot X(0) + \int_0^t H(s)dX(s) + \int_0^t X(s)dH(s) + \int_0^t d < H,X >_s \\ &= x + \int_0^t H(s)[r(s)X(s) - c^*(s)]ds + \int_0^t H(s)X(s)\pi^*(s)'(\mu_p(s) - r(s)\underline{1})ds \\ &\quad + \int_0^t H(s)X(s)\pi^*(s)'\sigma_p(s)dW(s) - \int_0^t X(s)H(s)r(s)ds - \int_0^t X(s)\theta(s)'dW(s) \\ &\quad - \int_0^t X(s)\pi^*(s)\sigma_p(s)H(s)\theta(s)'ds. \end{aligned}$$

Plugging into (1.21) we have

$$x = x + \int_0^t \underbrace{H(s)\cdot X(s)(\pi^*(s)'\sigma_p(s) - \theta(s)')}_{=:f(s)} dW(s). \tag{1.22}$$

Hence, we must have

$$f(s) = 0 \text{ a.s} \quad \text{for all } s \in [0,T].$$

As $H(s)\cdot X(s)$ is positive, we must have

$$\pi^*(t) = \left(\sigma_p(t)'\right)^{-1}\theta(t) \text{ for all } t \in [0,T].$$

Assume we have $d = 1$ and r, μ_p, σ_p are constants, then we have

$$\pi^*(t) = \frac{\mu_p - r}{\sigma_p^2}, \tag{1.23}$$

which is defined as the *local risk premium for stock investment.*

We introduce the following theorem for a general method for determining the optimal portfolio process π^*, related with the representation problem.

Theorem 1.3. *Let the portfolio problem in (1.14) be given. Suppose that $x > 0$ and assume*
$\chi(y) < \infty$ for all $y > 0$. Further, c^* and B^* is as in Theorem 1.2. If there exists a function $f \in C^{1,2}([0,T]\times\mathbb{R}^d)$ with $f(0,0,\ldots,0) = x$ and

$$\frac{1}{H(t)}\cdot E\left(\int_t^T H(s)c^*(s)ds + H(T)B^*\Big|\mathcal{F}_t\right) = f(t, W_1(t),\ldots,W_d(t)),$$

then for $t \in [0,T]$ we have

$$\pi^*(t) = \frac{1}{X^{x,\pi^*,c^*}(t)}\sigma^{-1}(t)\nabla_x f(t, W_1(t), \dots, W_d(t)),$$

where $\nabla_x f$ denotes the gradient of $f(t, x_1, \dots, x_d)$ with respect to the $x-$coordinates.

Proof: See p.214 of Korn and Korn [KK01].

1.4 Option Portfolios

In this section, we analyse a similar problem as in Section 1.3, but instead of a portfolio composed of the riskless bond and stocks, we have the riskless bond and European options written on stocks in our portfolio. Using the result that in both markets we have the same optimal terminal wealth B^*, we replicate the stock positions with the riskless bond and the options. This approach is applicable only under the assumption that the stocks and options generate the same filtration.

We provide some basic definitions and theorems of option pricing with replication approach, from Korn and Korn [KK01].

Definition 1.5. A contingent claim (g, B) consists of an $\{\mathcal{F}_t\}-$ progressively measurable payout rate process g, with $t \in [0,T]$, $g(t) \geq 0$, and an $\mathcal{F}_T-$measurable terminal payment $B \geq 0$ at time $t = T$ with

$$E\left[\left(\int_0^T g(t)dt + B\right)^\mu\right] < \infty \text{ for some } \mu > 1. \tag{1.24}$$

Definition 1.6. I. The pair (π, c) is called a **replication strategy** for the contingent claim (g, B) if we have

$$\begin{aligned} g(t) &= c(t) \text{ a.s. for all } t \in [0,T], \\ X(T) &= B \text{ a.s. }, \end{aligned}$$

where $X(t)$ is the wealth process corresponding to (π, c).

II. The set of **replication strategies of price x** is the set

$$\mathcal{D} := \mathcal{D}(x; (g, B)) := \{(\pi, c) \in \mathcal{A}(x) | (\pi, c) \text{ replication strategy for } (g, B)\}.$$

III. The **fair price** of the contingent claim (g, B) is defined as

$$\hat{p} := \inf\{p|\mathcal{D}(p) \neq \emptyset\}.$$

Remark 1.3. Since $r(t), \mu_p(t), \sigma_p(t)$ are uniformly bounded, and $\sigma_p(t)\sigma_p(t)'$ are uniformly positive definite, together with Hölder's inequality[1] and (1.24), we have

$$\tilde{x} := E\left[H(T)B + \int_0^T H(t)g(t)dt\right] < \infty.$$

From Theorem 1.1, there exists a π corresponding to (B, g) such that we have $(\pi, g) \in \mathcal{A} \cap \mathcal{D}(\tilde{x})$, which implies

$$\hat{p} \leq \tilde{x}.$$

The following theorem shows the case when $\hat{p} = \tilde{x}$.

Theorem 1.4. *Let $H(t)$ denote the stochastic deflator process. Then, the fair price $\hat{p}$ of the contingent claim (g, B) is*

$$\hat{p} = E\left[H(T)B + \int_0^T H(t)g(t)dt\right] < \infty,$$

and there exists a unique replicating strategy $(\hat{\pi}, \hat{c}) \in \mathcal{D}(\hat{p})$. Its corresponding wealth process $\hat{X}(t)$ (the valuation process for (g, B)) is

$$\hat{X}(t) = \frac{1}{H(t)} E\left[H(T)B + \int_0^T H(s)g(s)ds \Big| \mathcal{F}_t\right].$$

We can get the explicit form of the replicating strategy by imposing additional assumptions on the option price process.

Theorem 1.5. *Assume that the price of an option at time t can be written as a $C^{1,2}-$ function $f(t, p_1, \ldots, p_d)$ of time and underlying stock prices.*

1. *Then, the replicating strategy ψ^* is given by*

$$\begin{aligned} \psi_i^*(t) &= f_{p_i}(t, P_1(t), \ldots, P_d(t)), \quad i = 1, \ldots, d, \\ \psi_0^*(t) &= \frac{f(t, P_1(t), \ldots, P_d(t)) - \sum_{i=1}^n f_{p_i}(t, P_1(t), \ldots, P_d(t))P_i(t)}{P_0(t)}, \end{aligned}$$

[1]Let $1 < p < \infty$, $1 < q < \infty$, and $(1/p) + (1/q) = 1$. If $E|X|^p < \infty$ and $E|Y|^q < \infty$ then $E|XY| < \infty$ and $E|XY| \leq (E|X|^p)^{1/p}(E|Y|^q)^{1/q}$

and the function $f(t, p_1(t), \ldots, p_d(t))$ is a solution of the partial differential equation

$$f_t + \frac{1}{2}\sum_{i,j=1}^{d} a_{ij} p_i p_j f_{p_i p_j} + \sum_{i=1}^{d} r p_i f_{p_i} - rf = 0.$$

Here, we have set $a(t) := \sigma_p(t)\sigma_p(t)'$ and the subscripts $t, p_1, \ldots, p_d$ mean partial derivative with respect to the corresponding variable.

2. The price process $f(t, P_1(t), \ldots, P_d(t))$ obeys the stochastic differential equation

$$\begin{aligned} & df(t, P_1(t), \ldots, P_d(t)) \qquad (1.25) \\ = & \left(rf(t, P_1(t), \ldots, P_d(t)) + \sum_{i=1}^{d} f_{p_i}(t, P_1(t), \ldots, P_d(t)) P_i(t)(\mu_i - r)dt \right) \\ & + \left(f_{p_i}(t, P_1(t), \ldots, P_d(t)) P_i(t) \sum_{j=1}^{d} \sigma_{i,j}(t) dW_j(t) \right). \end{aligned}$$

Description of the market: We consider a financial market, where one riskless bond (or MMA), d stocks and d options are traded. Moreover, we assume that we are only allowed to hold a portfolio of the bond and the options. The options are assumed to have price processes

$$f^{(i)}(t, P_1(t), \ldots, P_d(t)), \quad i = 1, \ldots, d, \quad f \in C^{1,2}.$$

Let $\varphi(t) = (\varphi_0(t), \varphi_1(t), \ldots, \varphi_d(t))$ be an admissible trading strategy in bond and options, then the corresponding wealth process will be

$$X(t) = \varphi_0(t)P_0(t) + \sum_{i=1}^{d} \varphi_i(t) f^{(i)}(t, P_1(t), \ldots, P_d(t)),$$

where we require the assumptions that the integrals

$$\int_0^t \varphi_0(s)dP_0(s), \quad \text{and} \quad \int_0^t \varphi_i(s)df^{(i)}(s, P_1(s), \ldots, P_d(s))$$

are defined and $\varphi(t)$ is $\mathcal{F}_t$−progressively measurable.

Here, we find an optimal strategy which maximises the utility from the final wealth of the investor, who has an initial capital of $x > 0$, i.e.,

$$\max_{\varphi} E[U(X(T))]. \qquad (1.26)$$

The solution to the problem in (1.26) can be described as determining an optimal payoff B^* and the replicating strategy $\xi(t) = \xi_0(t), \xi_1(t), \ldots, \xi_d(t)$ for the bond and stock positions for optimal payoff B^*. Since we are not allowed to trade in stocks, we have to replicate the stock position with bond and options, which yields the optimal terminal wealth $X^*(T)$ of the investor. The following theorem from Korn and Trautmann [KT99] (KT framework, hereafter) is useful to understand the formulation above.

Theorem 1.6 (KT framework). *Let the Delta matrix $\Psi(t) = (\Psi_{ij}(t))_{ij}$, $i, j = 1, \ldots, d$ with*

$$\Psi_{ij} := f_{p_j}^{(i)}(t, P_1(t), \ldots, P_d(t))$$

be regular for all $t \in [0, T)$. Then, the option portfolio problem in (1.26) possesses the following explicit solution:

1. *The optimal terminal wealth B^* coincides with the optimal terminal wealth of the corresponding stock portfolio problem in (1.14).*

2. *Let $\xi(t) = (\xi_0(t), \ldots, \xi_d(t))$ be the optimal trading strategy in the corresponding basic stock portfolio problem (1.14). Then, the optimal trading strategy $\varphi(t) = (\varphi_0(t), \varphi_1(t), \ldots, \varphi_d(t))$ in the option portfolio problem in (1.26) is given by*

$$\bar{\varphi}(t) = (\Psi(t)')^{-1}\bar{\xi}(t),$$

$$\varphi_0(t) = \frac{\left(X(t) - \sum_{i=1}^{d} \varphi_i(t) f^{(i)}(t, P_1(t), \ldots, P_d(t))\right)}{P_0(t)},$$

with $\bar{\varphi}(t) := (\varphi_1(t), \ldots, \varphi_d(t))$ and $\bar{\xi}(t) := (\xi_1(t), \ldots, \xi_d(t))$.

Proof: see p.218 of Korn and Korn [KK01].

Example 1.2. This example from Korn and Korn [KK01] sums up the ideas presented in this section. In Example 1.1, we calculated the optimal trading strategy in stock and MMA portolio problem with logarithmic utility, i.e., $U(x) = \ln(x)$ and get the optimal portfolio process as

$$\pi^* = \frac{\mu_p - r}{\sigma_p^2},$$

which represents the fraction of the total wealth invested to the stocks. Hence, the number of stocks will be

$$\xi_1(t) = \frac{\pi^* X(t)}{P_1(t)} = \frac{\mu_p - r}{\sigma_p^2} \cdot \frac{X(t)}{P_1(t)}.$$

The optimal trading strategy in the option portolio problem with Theorem 1.6 will be

$$\varphi_1(t) = \frac{\mu_p - r}{\sigma_p^2} \cdot \frac{X(t)}{\Psi_1(t) P_1(t)},$$

where $\Psi_1(t) = f_{p_1}^{(1)}(t, P_1(t))$. Now, if we introduce the optimal option portfolio process as π^*_{option}, which gives the fraction of total wealth invested to the option, then we will have

$$\begin{aligned}
\pi^*_{option}(t) &:= \frac{\varphi_1(t) f^{(1)}(t, P_1(t))}{X(t)} \\
&= \frac{\mu_p - r}{\sigma_p^2} \frac{X(t) f^{(1)}(t, P_1(t))}{X(t)\Psi_1(t) P_1(t)} \\
&= \pi^* \frac{f^{(1)}(t, P_1(t))}{f_{p_1}^{(1)}(t, P_1(t)) P_1(t)}.
\end{aligned}$$

Using the Black-Scholes framework, for a European type call option, we will have

$$f^{(1)}(t, P_1(t)) = P_1(t)\Phi(d_1(t)) - Ke^{-r(T-t)}\Phi(d_2(t)), \tag{1.27}$$

where

$$d_1(t) = \frac{\ln\left(\frac{P_1(t)}{K}\right) + \left(r + \frac{1}{2}\sigma_p^2\right)(T-t)}{\sigma_p\sqrt{T-t}}, \quad \text{and} \quad d_2(t) = d_1(t) - \sigma_p\sqrt{T-t}.$$

We have

$$f_{p_1}^{(1)} = \Phi(d_1(t)).$$

It is obvious that

$$f^{(1)}(t, P_1(t)) < f_{p_1}^{(1)} \cdot P_1(t) \Rightarrow \frac{f^{(1)}(t, P_1(t))}{f_{p_1}^{(1)} \cdot P_1(t)} < 1.$$

Therefore, if we compare the optimal portfolio process in stock-MMA problem with optimal process in option-MMA problem, we get

$$\pi^*_{option}(t) < \pi^* \quad \text{for all } t \in [0, T]. \tag{1.28}$$

The interpretation is that for an investor with logarithmic utility, the optimal capital that he allocates to the option in option portfolio problem is less than the capital he invests on the stock in stock portfolio problem. We present the main result in Figure 1.3 with the following parameters:

$\mu_p = 0.05$	drift term,
$\sigma_p = 0.25$	volatility
$T = 1$	maturity time for the call option in years
$r = 0$	short rate
$K = 100$	strike price

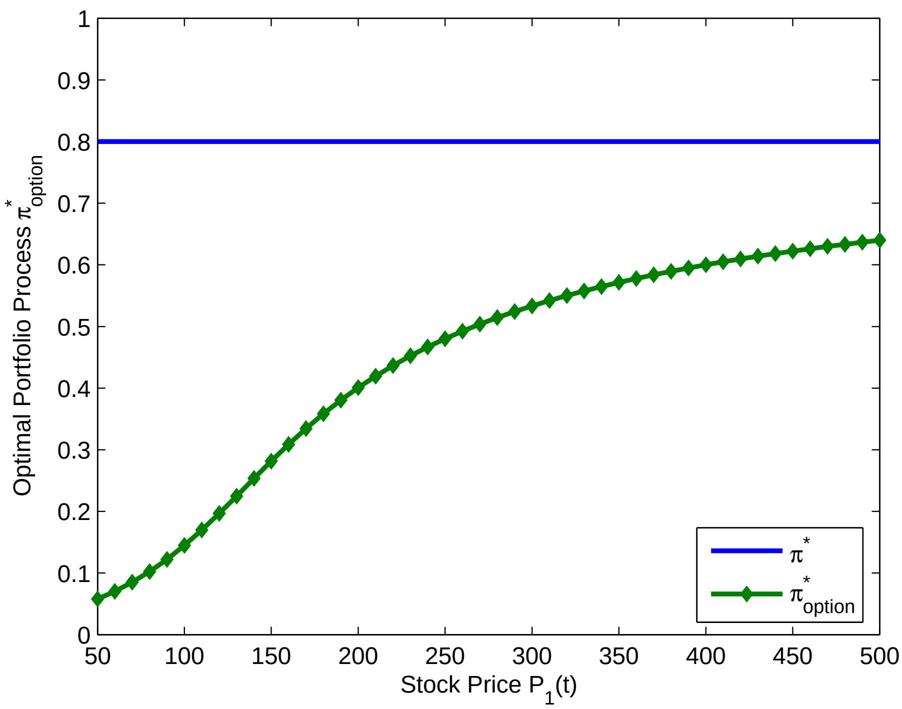

Figure 1.3: The optimal processes for call option portfolio with respect to the stock price, with the parameter set $\mu_p = 0.05$, $\sigma_p = 0.25$, $T = 1$, $r = 0$, and $K = 100$.

We observe as the call option gets riskier (as the stock price decreases, call option gets more out of the money), the optimal fraction of wealth gets smaller.

1.5 Portfolio Optimisation with a Compound Option

In this section, we introduce our problem of optimal portfolios with the money market account (MMA) and one derivative contract. In particular, these derivatives are European call and put options written on the stock of a defaultable firm and European options written on the risky bond issued by the same firm. Since we model the default risk with a firm-value based model, explained in Section 1.2, the stock price is a call option written on the firm value $V(t)$ with a strike price of F. Hence, we may consider optimal portfolios of options on options when the underlying is the firm value.

The option written on another option is called a *compound option* and to our knowledge the valuation formula was first introduced by Geske [Ges79], where the author presents a closed form formula for the call on a call option based on Merton model [Mer74]. This formula generalises the Black-Scholes option pricing formula, i.e., if the firm is unlevered[2], then the Geske formula reduces to Black-Scholes call option formula.

Optimal portfolio problems with defaultable bonds were already studied by Korn and Kraft [KK03], where the authors use firm-value approach for credit risk modelling. The authors first present the portolio problem when the portfolio consists of the firm value $V(t)$ and MMA, assuming that the firm value is traded (Merton portfolio problem), then they introduce the optimisation problem when the portfolio has a risky bond written by the firm and the MMA. This problem can be solved in two ways. One way would be with the elasticity technique of Kraft [Kra03] for optimisation as described by Korn and Kraft [KK03]. The second way is to optimise the portfolio using the methodology in KT framework [KT99]. Moreover, Kraft and Steffensen [KS06] generalised the results of Korn and Kraft [KK03] and applied the same technique when the credit risk is modelled by the Black-Cox [BC76] approach, which allows the occurrence of the default event before the debt maturity. Another approach for continuous time portfolio optimisation problem with defaultable assets is to model the credit risk within the reduced form setting. This was first studied by Merton [Mer71] and extended in a series of papers by Kraft and Steffensen (see [KS08] and [KS07]). Another example to the same problem is given by Hou and Jin [HJ02].

Our main study is applying the KT framework [KT99] for optimising portfolios of options on options and the MMA. Hence, this section provides the presentation of the compound option valuation and the proof of the call on a call option price proposed by Geske [Ges79]. Modifying the Geske [Ges79] formula, we valuate European options written on the risky ZCB. Finally, we introduce examples for presenting the main results of this chapter.

[2] This means either the firm has no debt, i.e., $M = 0$ or there is no maturity for the debt, i.e., $T = \infty$.

1.5.1 Compound Options

A compound option gives the holder the right (but not the obligation) to buy or sell an option for a pre-determined strike K at maturity time T. If we have a European type call on a call option, the holder has a right to buy the underlying European call option, which has the maturity time $T_1 > T$ and strike K_1, for strike price K. We denote the price of a compound option at time t with $X^{CC}(t, P_1(t))$, where the superscript CC indicates that the compound option is a call on a call. We denote the payoff structure at maturity of the compound option T with

$$B^{CC} = \left(X^{Call}(T, P_1(T)) - K\right)^+ .$$

Here, we first derive the pricing equation for a call on a call option, where the underlying call is written on the stock with a classical Black Scholes setting, i.e., the stock price is modelled by a geometric Brownian motion. Note that there is a critical value of the stock at the maturity of the compound option $P_1(T) = p^*$, which makes the holder indifferent between exercising or not exercising the compound option. The critical value p^* can be found as a solution to the following equation using the Black-Scholes call option formula

$$X^{Call}(T, p^*) - K = 0. \tag{1.29}$$

Since p^* is the value which makes the call option price at T equal to the strike price of the compound option K, for the values of the stock less than p^* the compound option will not be exercised. And the compound option will be exercised for the values greater than p^*.

We present the following proposition and its proof from Korn and Korn [KK01].

Proposition 1.1. ***I.*** *For a given $K > 0$, a European call with strike K_1 and maturity T_1, there exists a uniquely determined p^* for $T < T_1$ such that for $P_1(T) = p^*$ we have*

$$X^{Call}(T, p^*) = K.$$

II. *With the notations*

$$g_1(t) = \frac{\ln\left(\frac{P_1(t)}{p^*}\right) + \left(r + \frac{1}{2}\sigma_p^2\right)(T-t)}{\sigma_p\sqrt{T-t}}, \quad g_2(t) = g_1(t) - \sigma_p\sqrt{T-t},$$

$$h_1(t) = \frac{\ln\left(\frac{P_1(t)}{K_1}\right) + \left(r + \frac{1}{2}\sigma_p^2\right)(T_1 - t)}{\sigma_p\sqrt{T_1 - t}}, \quad h_2(t) = h_1(t) - \sigma_p\sqrt{T_1 - t},$$

the price of a call on a call satisfies

$$\begin{aligned} X^{CC}(t) &= P_1(t)\Phi_2^{\rho_1}(g_1(t), h_1(t)) \\ &\quad -K_1 e^{-r(T_1-t)}\Phi_2^{\rho_1}(g_2(t), h_2(t)) - Ke^{-r(T-t)}\Phi(g_2(t)) \end{aligned}$$

for $t \in [0, T]$*, where* $\Phi_2^{\rho}(x, y)$ *is the cumulative distribution function of a bivariate standard normal distribution with correlation coefficient* ρ *and*

$$\rho_1 := \sqrt{\frac{T-t}{T_1-t}}, e.g., \begin{pmatrix} X \\ Y \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & \rho_1 \\ \rho_1 & 0 \end{pmatrix}\right).$$

Proof:

I. From the explicit form of the Black-Scholes formula (see p.88 of Korn and Korn [KK01]) we obtain

$$\lim_{P_1(T)\downarrow 0} X^{Call}(T, P_1(T)) = 0, \tag{1.30}$$

$$\lim_{P_1(T)\uparrow +\infty} X^{Call}(T, P_1(T)) = +\infty \tag{1.31}$$

for $T < T_1$. Here, the first limit is a consequence of the trivial bounds 0 and $P_1(T)$ for $X^{Call}(T, P_1(T))$. For the second limit note that

$$\frac{d}{dp}X^{Call}(T, p) = \Phi(d_1(T))$$

is positive and even increasing in p. From (1.30) and (1.31), together with the intermediate value theorem we get the existence of p^* of assertion I..

II. For $t \le T$ we have

$$X^{CC}(t, P_1(t)) = E_{t,P_1(t)}\left[e^{-r(T-t)}B^{CC}\right] = E_{t,P_1(t)}\left[e^{-r(T-t)}(X^{Call}(T, P_1(T)) - K)^+\right].$$

The positive part is strictly positive if and only if $X^{Call}(T, P_1(T)) - K > 0$, hence

$$E_{t,P_1(t)}\left[e^{-r(T-t)}(X^{Call}(T, P_1(T)) - K)1_{\{X^{Call}(T,P_1(T))>K\}}\right],$$

where $1_{\{X^{Call}(T)>K\}} = 1_{\{P_1(T)>p^*\}}$. Thus, fixing t we have

$$W(T) - W(t) > \frac{1}{\sigma_p}\left(\ln\left(\frac{p^*}{P_1(t)}\right) - (r - \frac{1}{2}\sigma_p^2)(T-t)\right) = \tilde{w}. \tag{1.32}$$

Furthermore, Itô's lemma implies that

$$P_1(T) = P_1(t).\exp\left(\left(r - \frac{1}{2}\sigma_p^2\right)(T-t) + \sigma_p(W(T) - W(t))\right).$$

Since $W(T) - W(t) := x \sim \mathcal{N}(0, T-t)$, we rewrite the expectation above as

$$\frac{1}{\sqrt{2\pi(T-t)}} \int_{\tilde{w}}^{\infty} e^{-\frac{x^2}{2(T-t)}} e^{-r(T-t)} \left(X^{Call}(T, P_1(T)) - K\right) dx \tag{1.33}$$

With the help of explicit form of $X^{Call}(T, P_1(T))$ with strike K_1 and maturity T_1 we have

$$X^{Call}(T, P_1(T)) = P_1(t) e^{\left(r - \frac{1}{2}\sigma_p^2\right)(T-t) + \sigma x} \Phi(d_1(T)) - K_1 e^{-r(T_1 - T)} \Phi(d_2(T)),$$

with

$$d_1(T) = \frac{\ln\left(\frac{P_1(T)}{K_1}\right) + \left(r + \frac{1}{2}\sigma_p^2\right)(T_1 - T)}{\sigma_p \sqrt{T_1 - T}},$$

and

$$d_2(T) = d_1(T) - \sigma_p \sqrt{T_1 - T}.$$

We rewrite (1.33) as $I_1 - I_2 - I_3$, hence

$$I_1 = \frac{1}{\sqrt{2\pi(T-t)}} \int_{\tilde{w}}^{\infty} e^{-\frac{x^2}{2(T-t)}} e^{-r(T-t)} P_1(t).e^{\left(r - \frac{1}{2}\sigma_p^2\right)(T-t) + \sigma_p x} \Phi(\beta_1 + \alpha_1 x) dx, \tag{1.34}$$

where

$$\begin{aligned} \beta_1 &= \frac{\ln\left(\frac{P_1(t)}{K_1}\right) + \left(r - \frac{1}{2}\sigma_p^2\right)(T-t) + \left(r + \frac{1}{2}\sigma_p^2\right)(T_1 - T)}{\sigma_p \sqrt{T_1 - T}}, \\ \alpha_1 &= \frac{1}{\sqrt{T_1 - T}}. \end{aligned}$$

Thus,

$$\begin{aligned} I_1 &= P_1(t) \int_{\tilde{w}}^{\infty} \frac{1}{\sqrt{2\pi(T-t)}} e^{-\frac{(x - \sigma_p(T-t))^2}{2(T-t)}} \Phi(\beta_1 + \alpha_1 x) dx \\ &= P_1(t) \int_{\tilde{w}}^{\infty} \varphi_{\mu = \sigma_p(T-t), \sigma^2 = (T-t)} \Phi(\beta_1 + \alpha_1 x) dx. \end{aligned}$$

Here, φ_{μ,σ^2} is the probability density function of a normal distribution with mean μ and variance σ^2 and $\Phi()$ is a standard normal distribution function. Furthermore, we have

$$\begin{aligned} I_2 &= K_1 e^{-r(T_1 - t)} \int_{\tilde{w}}^{\infty} \frac{1}{\sqrt{2\pi(T-t)}} e^{-\frac{x^2}{2(T-t)}} \Phi(\beta_2 + \alpha_2 x) dx \\ &= K_1 e^{-r(T_1 - t)} \int_{\tilde{w}}^{\infty} \varphi_{\mu = 0, \sigma^2 = (T-t)} \Phi(\beta_2 + \alpha_2 x) dx, \end{aligned}$$

where

$$\begin{aligned}\beta_2 &= \frac{\ln\left(\frac{P_1(t)}{K_1}\right) + \left(r - \frac{1}{2}\sigma_p^2\right)(T_1 - t)}{\sigma_p\sqrt{T_1 - T}} \\ \alpha_2 &= \frac{1}{\sqrt{T_1 - T}}.\end{aligned}$$

The last component can easily be expressed as

$$\begin{aligned}I_3 &= Ke^{-r(T-t)} \int_{\tilde{w}}^{\infty} \frac{1}{\sqrt{2\pi(T-t)}} e^{-\frac{x^2}{2(T-t)}} dx \\ &= Ke^{-r(T-t)} \Phi\left(\frac{-\tilde{w}}{\sqrt{T-t}}\right) = Ke^{-r(T-t)} \Phi(g_2(t)).\end{aligned}$$

The following lemma is used for calculating I_1 and I_2

Lemma 1.1. *If X and Y are independent random variables with*

$$X \sim \mathcal{N}(\mu, \sigma^2), \quad Y \sim \mathcal{N}(0, 1),$$

then for $\tilde{x}$,α,$\beta \in \mathbb{R}$, $\alpha > 0$, we have

$$\int_{\tilde{x}}^{\infty} \varphi_{\mu,\sigma^2}(x).\Phi(\beta + \alpha x)dx = P[X \geq \tilde{x}, Y \leq \beta + \alpha X] = P[X \geq \tilde{x}, Z \leq \beta],$$

where

$$(X, Z) \sim \mathcal{N}\left(\begin{pmatrix} \mu \\ -\alpha\mu \end{pmatrix}, \begin{pmatrix} \sigma^2 & -\alpha\sigma^2 \\ -\alpha\sigma^2 & 1 + \alpha^2\sigma^2 \end{pmatrix}\right).$$

□

Note that

$$P[X \geq \tilde{x}, Z \leq \beta] = 1 - P[X \leq \tilde{x}, Z \leq \beta] - \underbrace{P[Z > \beta]}_{1 - P[Z \leq \beta]} = P[Z \leq \beta] - P[X \leq \tilde{x}, Z \leq \beta].$$

Furthermore,

$$\int_{\tilde{x}}^{\infty} \varphi_{\mu,\sigma^2}(x).\Phi(\beta + \alpha x)dx = \Phi\left(\frac{\beta - \mu_Z}{\sigma_Z}\right) - \Phi_2^{\rho}\left(\left(\frac{\tilde{x} - \mu_X}{\sigma_X}\right), \left(\frac{\beta - \mu_Z}{\sigma_Z}\right)\right).$$

Going back to the calculation of I_1, with the notation given in Lemma 1.1, we have

$$\tilde{x} = \tilde{w} \qquad \mu_X = \sigma_p(T - t) \qquad \sigma_X = \sqrt{T - t}$$

$$\mu_Z = -\frac{\sigma_p(T-t)}{\sqrt{T_1 - T}} \qquad \sigma_Z = \sqrt{\frac{T_1 - t}{T_1 - T}} \qquad \rho(X, Z) = -\sqrt{\frac{T - t}{T_1 - t}}.$$

With this setting, we rewrite I_1 as

$$\begin{aligned} I_1 &= P_1(t)\left[\Phi\left(\frac{\beta_1+\frac{\sigma_p(T-t)}{\sqrt{T_1-T}}}{\sqrt{\frac{T_1-t}{T_1-T}}}\right)-\Phi_2^{\rho}\left(\frac{\tilde{w}-\sigma_p(T-t)}{\sqrt{T-t}}\right),\left(\frac{\beta_1+\frac{\sigma_p(T-t)}{\sqrt{T_1-T}}}{\sqrt{\frac{T_1-t}{T_1-T}}}\right)\right] \\ &= P_1(t)[\Phi(h_1(t))-\Phi_2^{\rho}(-g_1(t),h_1(t))] \\ &= P_1(t)[\Phi_2^{-\rho=:\rho_1}(g_1(t),h_1(t))]. \end{aligned}$$

Calculation of I_2 is similar to I_1 but here we have $\mu_X=0$ and $\mu_Z=0$. Thus,

$$\begin{aligned} I_2 &= K_1e^{-r(T_1-t)}\left[\Phi\left(\frac{\beta_2}{\sqrt{\frac{T_1-t}{T_1-T}}}\right)-\Phi_2^{\rho}\left(\frac{\tilde{w}}{\sqrt{T-t}}\right),\left(\frac{\beta_2}{\sqrt{\frac{T_1-t}{T_1-T}}}\right)\right] \\ &= K_1e^{-r(T_1-t)}[\Phi(h_2(t))-\Phi_2^{\rho}(-g_2(t),h_2(t))] \\ &= K_1e^{-r(T_1-t)}[\Phi_2^{-\rho=:\rho_1}(g_2(t),h_2(t))]. \end{aligned}$$

□

Above we derived the compound option formula when it is a European call on a call option written on the stock. Similarly, one can rewrite the compound option formula when the underlying is the firm value with the dynamics as in (1.1). Merton [Mer74] valuates the stock of a firm as a call option written on the firm value, where Geske [Ges79] derives the compound option formula by valuating a call option on the stock price as in the Merton [Mer74] setting, i.e.,

$$X^{CC}(t,V(t))\equiv X^{Call}(t,P_1(t)),$$

when $P_1(t)\equiv X^{Call}(t,V(t))$ with $\sigma_p(t,V)=\frac{\partial P_1}{\partial V}\frac{V}{P_1}\sigma_v$ (For the proof see Geske [Ges79]).

The following proposition gives the prices of other types of the compound options, when the underlying is the firm value.

Proposition 1.2. ***I.*** *The price of a put on a call option is*

$$\begin{aligned} X^{PC}(t,V(t)) &= -V(t)\Phi_2^{\rho_2}(-g_1(t),h_1(t)) \\ &\quad +Fe^{-r(T_1-t)}\Phi_2^{\rho_2}(-g_2(t),h_2(t))+Ke^{-r(T-t)}\Phi(-g_2(t)) \end{aligned}$$

for $t\in[0,T]$ *with*

$$\rho_2:=-\sqrt{\frac{T-t}{T_1-t}}.$$

II. *If for a put with strike F and maturity T_1 the value v^* defined by*

$$X^{Put}(T, v^*) = K$$

is given by for a fixed K, then we can obtain the pricing formula for a call on this put or a put on this call in the same way as above. If we assume a strike of K and maturity $T < T_1$ for the compound options, then we obtain their prices at time $t \in [0, T]$ as

$$\begin{aligned} X^{CP}(t, V(t)) &= -V(t)\Phi^{\rho_1}(-g_1(t), -h_1(t)) \\ &\quad + Fe^{-r(T_1-t)}\Phi^{\rho_1}(-g_2(t), -h_2(t)) + Ke^{-r(T-t)}\Phi(-g_2(t)), \end{aligned}$$

and

$$\begin{aligned} X^{PP}(t, V(t)) &= V(t)\Phi^{\rho_2}(g_1(t), -h_1(t)) \\ &\quad - Fe^{-r(T_1-t)}\Phi^{\rho_2}(g_2(t), -h_2(t)) + Ke^{-r(T-t)}\Phi(g_2(t)), \end{aligned}$$

with

$$g_1(t) = \frac{\ln\left(\frac{V(t)}{v^*}\right) + \left(r + \frac{1}{2}\sigma_v^2\right)(T-t)}{\sigma_v\sqrt{T-t}}, \quad g_2(t) = g_1(t) - \sigma_v\sqrt{T-t},$$

and

$$h_1(t) = \frac{\ln\left(\frac{V(t)}{F}\right) + \left(r + \frac{1}{2}\sigma_v^2\right)(T_1-t)}{\sigma_v\sqrt{T_1-t}}, \quad h_2(t) = h_1(t) - \sigma_v\sqrt{T_1-t}.$$

1.5.2 Options on the Defaultable Zero Coupon Bonds

In this subsection, we derive the explicit formula for a European call and put option written on a defaultable ZCB. Geske [Ges77] applied the formulation in Geske [Ges79] in order to value defaultable coupon bonds. Later on, Geske and Johnson [GJ84] explain the unclear parts of the paper. Since the risky ZCB price with Merton [Mer74] setting is a linear combination of a Black Scholes put option and a deterministic payment, the formula is a modification of the one by Geske [Ges79]. The pricing of such contracts was studied by Barone et al. [BAC98] in an intensity-based framework. Reporting from Barone et al. [BAC98], risk free options on risky ZCBs (not vulnerable and usually exchange traded) have little interest in the practice. On the other hand, there is quite a number of papers in the risk literature dealing with the valuation of defaultable options (vulnerable options) on risk-free and risky assets. After presenting the derivation of the fair prices of options on risky ZCB, we analyse the portfolio optimisation problems.

Proposition 1.3. *Price of a European call option with maturity time T and strike K, with $K < Fe^{-r(T_1-T)}$ written on a risky ZCB (in Merton setting) maturing at time T_1, where $T_1 > T$, is given by*

$$\begin{aligned} X^{Call}(t, \bar{B}(t,T_1)) &= V(t)\Phi_2^{\rho_2}(g_1(t), -h_1(t)) \\ &\quad + Fe^{-r(T_1-t)}\Phi_2^{\rho_1}(g_2(t), h_2(t)) - Ke^{-r(T-t)}\Phi(g_2(t)), \end{aligned} \tag{1.35}$$

where

$$g_1(t) = \frac{\ln\left(\frac{V(t)}{v^*}\right) + \left(r + \frac{1}{2}\sigma_v^2\right)(T-t)}{\sigma_v\sqrt{T-t}}, \quad g_2(t) = g_1(t) - \sigma_v\sqrt{T-t},$$

$$h_1(t) = \frac{\ln\left(\frac{V(t)}{F}\right) + \left(r + \frac{1}{2}\sigma_v^2\right)(T_1-t)}{\sigma_v\sqrt{T_1-t}}, \quad h_2(t) = h_1(t) - \sigma_v\sqrt{T_1-t},$$

and

$$\rho_1 := \sqrt{\frac{T-t}{T_1-t}}, \qquad \rho_2 := -\rho_1,$$

e.g.,

$$\begin{pmatrix} X \\ Y \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & \rho_1 \\ \rho_1 & 0 \end{pmatrix}\right).$$

and, v^* is the value of the firm which solves the following equation

$$\bar{B}(T,T_1) - K = 0.$$

Proof: From the explicit form of the BS type formula (1.6), we obtain

$$\lim_{V(T)\downarrow 0} \bar{B}(T,T_1) = 0, \tag{1.36}$$

$$\lim_{V(T)\uparrow +\infty} \bar{B}(T,T_1) = Fe^{-r(T_1-T)} \tag{1.37}$$

for $T < T_1$. Here, the first limit is a consequence of the trivial bounds 0 and $V(T)$ for $\bar{B}(T,T_1)$. For the second limit note that

$$\frac{d}{dV}\bar{B}(T,T_1) = \Phi(-h_1(T))$$

is positive and decreasing in V. From (1.36) and (1.37), together with the intermediate value theorem, we get the existence of v^*.

Under the pricing (or risk-neutral) probability measure Q, we have the European call option formula with maturity time T and strike price K, where the underlying is the defaultable ZCB with maturity $T_1 > T$ as

$$X^{Call}(t, \bar{B}(t, T_1)) = E_{t,\bar{B}(t,T_1)}\left[e^{-r(T-t)}(\bar{B}(T, T_1) - K)^+\right]. \tag{1.38}$$

In order the payoff function to be strictly positive, we rewrite the equation above as

$$E_{t,\bar{B}(t,T_1)}\left[e^{-r(T-t)}(\bar{B}(T, T_1) - K)1_{\{\bar{B}(T,T_1)>K\}}\right]. \tag{1.39}$$

Here, we assume that there exists a critical value of the firm v^*, which makes the call option holder indifferent between exercising or not exercising it on the maturity of the call option. Hence, v^* is the value, which solves the following equation

$$\bar{B}(T, T_1) = K. \tag{1.40}$$

Using the same idea we have in the proof of the compound option formula, we have

$$1_{\{\bar{B}(T,T_1)>K\}} \equiv 1_{\{V(T)>v*\}}.$$

Hence, we rewrite (1.39) as

$$E_{t,\bar{B}(t,T_1)}[e^{-r(T-t)}(\bar{B}(T, T_1) - K)1_{\{V(T)>v^*\}}].$$

With a small modification to (1.32), we will have

$$X^{Call}(T, \bar{B}(T, T_1)) = e^{-r(T-t)}\frac{1}{\sqrt{2\pi(T-t)}}\int_w^\infty e^{-\frac{x^2}{2(T-t)}}(\bar{B}(T, T_1) - K)dx,$$

with

$$w = \frac{1}{\sigma_v}\left(\ln\left(\frac{v^*}{V(t)}\right) - \left(r - \frac{1}{2}\sigma_v^2\right)(T-t)\right)$$

Plugging the explicit formula for the ZCB price as in (1.6), we have

$$I = e^{-r(T-t)}\int_w^\infty \frac{1}{\sqrt{2\pi(T-t)}}e^{-\frac{x^2}{2(T-t)}}\left[V(T)\underbrace{\Phi(-h_1(T))}_{1-\Phi(h_1(T))} + Fe^{-r(T_1-T)}\Phi(h_2(T)) - K\right]dx, \tag{1.41}$$

with

$$h_1(T) = \frac{\ln\left(\frac{V(T)}{F}\right) + \left(r + \frac{1}{2}\sigma_v^2\right)(T_1 - T)}{\sigma_v\sqrt{T_1 - T}}$$

and

$$h_2(T) = h_1(T) - \sigma_v\sqrt{T_1 - T}.$$

We express (1.41) as $I_1 - I_2 + I_3 - I_4$, where

$$I_1 = V(t)\int_w^\infty \frac{1}{\sqrt{2\pi(T-t)}} e^{-\frac{(x-\sigma_v(T-t))^2}{2(T-t)}} dx = V(t)\Phi(g_1(t)). \tag{1.42}$$

$$I_2 = V(t)\int_w^\infty \frac{1}{\sqrt{2\pi(T-t)}} e^{-\frac{(x-\sigma_v(T-t))^2}{2(T-t)}} \Phi(\beta_2 + \alpha_2 x)dx \tag{1.43}$$

with

$$\beta_2 = \frac{\ln\left(\frac{V(t)}{F}\right) + \left(r - \frac{1}{2}\sigma_v^2\right)(T-t) + \left(r + \frac{1}{2}\sigma_v^2\right)(T_1 - T)}{\sigma_v\sqrt{T_1 - T}}$$

and

$$\alpha_2 = \frac{1}{\sqrt{T_1 - T}}$$

With Lemma 1.1, we rewrite I_2 as

$$I_2 = V(t)\left[\Phi(h_1(t)) - \Phi_2^{\rho}(-g_1(t), h_1(t))\right] = V(t)\Phi_2^{-\rho:=\rho_1}(g_1(t), h_1(t)). \tag{1.44}$$

Now, we can make a simplification

$$I_1 - I_2 = V(t)\left[\Phi(g_1(t)) - \Phi_2^{\rho_1}(g_1(t), h_1(t))\right] = \Phi_2^{-\rho_1=\rho_2}(g_1(t), -h_1(t)).$$

We calculate I_3 as

$$I_3 = Fe^{-r(T_1-t)}\int_w^\infty \frac{1}{\sqrt{2\pi(T-t)}} e^{-\frac{x^2}{2(T-t)}} \Phi(\beta_3 + \alpha_3 x)dx \tag{1.45}$$

with

$$\beta_3 = \frac{\ln\left(\frac{V(t)}{F}\right) + \left(r - \frac{1}{2}\sigma_v^2\right)(T_1 - t)}{\sigma_v\sqrt{T_1 - T}}, \qquad \alpha_3 = \frac{1}{\sqrt{T_1 - T}},$$

hence,

$$I_3 = Fe^{-r(T_1-t)}\left[\Phi(h_2(t)) - \Phi_2^{\rho}(-g_2(t), h_2(t))\right] = Fe^{-r(T_1-t)}\Phi_2^{-\rho=\rho_1}(g_2(t), h_2(t)). \tag{1.46}$$

Finally, we rewrite I_4 easily as

$$I_4 = Ke^{-r(T-t)}\int_w^\infty \frac{1}{\sqrt{2\pi(T-t)}} e^{-\frac{x^2}{2(T-t)}} dx = Ke^{-r(T-t)}\Phi\left(g_2(t)\right). \tag{1.47}$$

Proposition 1.4. *Price of a European put option with maturity time T and strike K, written on a risky ZCB (with Merton setting) maturing at time T_1, with $T_1 > T$ is given by*

$$\begin{aligned} X^{Put}(t, \bar{B}(t,T_1)) &= -V(t)\Phi_2^{\rho_1}(-g_1(t), -h_1(t)) \\ &\quad -Fe^{-r(T_1-t)}\Phi_2^{\rho_2}(-g_2(t), h_2(t)) + Ke^{-r(T-t)}\Phi(-g_2(t)) \end{aligned} \tag{1.48}$$

where

$$g_1(t) = \frac{\ln\left(\frac{V(t)}{v^*}\right) + \left(r + \frac{1}{2}\sigma_v^2\right)(T-t)}{\sigma_v\sqrt{T-t}}, \quad g_2(t) = g_1(t) - \sigma_v\sqrt{T-t},$$

$$h_1(t) = \frac{\ln\left(\frac{V(t)}{F}\right) + \left(r + \frac{1}{2}\sigma_v^2\right)(T_1-t)}{\sigma_v\sqrt{T_1-t}}, \quad h_2(t) = h_1(t) - \sigma_v\sqrt{T_1-t},$$

and

$$\rho_1 := \sqrt{\frac{T-t}{T_1-t}} \qquad \rho_2 := -\rho_1, e.g., \begin{pmatrix} X \\ Y \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & \rho_1 \\ \rho_1 & 0 \end{pmatrix}\right).$$

and v^* is the value of the firm which solves the following equation

$$K - \bar{B}(T, T_1) = 0.$$

Proof: Similar to proof of Proposition 1.3.

1.5.3 Optimal Portfolio Problem with a Compound Option

In this subsection, we will combine some results from the previous subsections in order to optimise a portfolio, consisting of a compound option and a riskless bond (or MMA). The dynamics of the MMA is

$$dP_0(t) = P_0(t)rdt, \quad P_0(0) = 1,$$

and the dynamics of the firm value with the risk-neutral probability measure Q is given by

$$\frac{dV(t)}{V(t)} = rdt + \sigma_v dW(t), \quad V(0) > 0,$$

where r is the deterministic interest rate, $\sigma_v > 0$ is the constant volatility and $W(t)$ is the Brownian motion. Assume that the investor can invest his initial wealth $x > 0$ only in the MMA $P_0(t)$ and the call on a call option $X^{CC}(t, V(t))$, where the underlying is $V(t)$. The corresponding wealth at time t, $X(t)$ can be expressed as

$$X(t) = \varphi_0(t)P_0(t) + \varphi_1(t)X^{CC}(t, V(t)), \quad X(0) = x.$$

Using the general form in Definition 1.1, the trading strategy $\varphi(t) = (\varphi_0(t), \varphi_1(t))'$ is a $\mathbb{R}^2-$ valued progressively measurable process with respect to the filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$ generated by the standard Brownian motion satisfying

$$\int_0^T |\varphi_0(t)|dt < \infty \text{ a.s. },$$

$$\int_0^T (\varphi_1(t)X^{CC}(t, V(t)))^2 dt < \infty \text{ a.s. }.$$

The corresponding portfolio process $\pi(t) = (\pi_0(t), \pi_1(t))'$ will be given as

$$\pi_1(t) := \frac{\varphi_1(t)X^{CC}(t, V(t))}{X(t)}, \tag{1.49}$$

$$\pi_0(t) := 1 - \pi_1(t) = \frac{\varphi_0(t)P_0(t)}{X(t)}. \tag{1.50}$$

With the assumption that the trading strategy is self-financing, (implies that portfolio process $\pi(t)$ is also self-financing) the corresponding wealth process can be expressed as

$$X(t) = x + \int_0^t \varphi_0(s)dP_0(s) + \int_0^t \varphi_1(s)dX^{CC}(s, V(s)). \tag{1.51}$$

Hence, our continuous time portfolio optimisation problem will be similar to (1.14), but ignoring the consumption process, i.e., $c(t) \equiv 0$, $U_1 \equiv 0$, $U_2 \equiv U$, we have:

$$\max_{\pi\in\mathcal{A}'(x)} E\left[U(X^{x,\pi}(T))\right] \tag{1.52}$$

with

$$\mathcal{A}'(x) = \left\{\pi(\cdot) \in \mathcal{A}(x) \middle| E\left[U(X(T))^-\right] < \infty\right\}.$$

The solution to our problem defined in (1.52) can be summarised in the following steps:

1. Assume that the firm value $V(t)$ is traded, and the portfolio consisting of $V(t)$ and the MMA, $P_0(t)$ is optimised (Portfolio problem by Merton [Mer69], [Mer71]).

2. Since Merton [Mer74] considers the stock of the firm, a call option, i.e., $P_1(t) = X^{Call}(t, V(t))$, we can use the KT framework [KT99] and optimise the portfolio consisting of the stock $P_1(t)$ and $P_0(t)$.

3. We use the same methodology and results of the second step, and the relation

$$X^{Call}(t, P_1(t)) \equiv X^{CC}(t, V(t)),$$

then make a second iteration for optimising the portfolio consisting of $X^{CC}(t, V(t))$ and $P_0(t)$.

Alternatively, we can skip the second step and directly solve the optimisation problem in the third step, however, we present the 2nd step in order to see that our findings are indeed in line with the results of Korn and Kraft [KK03].

1. Merton portfolio problem:

In the first step, the setting leads us to Merton's portolio problem [Mer69] and [Mer71]. Under the assumption that the firm value is tradable, and the wealth process follows the dynamics with

$$\frac{dX(t)}{X(t)} = (r + \pi_v \alpha)dt + \pi_v \sigma dW(t), \quad X(0) = x_0, \tag{1.53}$$

where we denote the constant, risk-free short rate with r, the excess return of the firm value by $\alpha = \mu_v - r$. Here, π_v stands for the proportion of the total wealth put into the firm value. The classic portfolio problem is then to solve

$$\max_{\pi} E[U(X^{\pi}(T))], \tag{1.54}$$

where T denotes the investment horizon, and U is the utility function. We present the result in the following proposition.

Proposition 1.5. *With the power utility function $U(x) = \gamma^{-1}x^{\gamma}, \gamma < 1, \gamma \neq 0$ the optimal portfolio process for the problem in (1.54) is*

$$\pi_v^*(t) = \frac{\alpha}{(1-\gamma)\sigma_v^2} \tag{1.55}$$

Note that for logarithmic utility function $U(x) = \ln(x)$ the optimal portfolio process π_v^* is obtained for $\gamma = 0$.

Proof: (see p. 236 Korn and Korn [KK01])

2. Optimal portolio problem with the stock and MMA

The problem in the second step was already studied by Korn and Kraft [KK03]. We present their result in the following proposition to compare it to our result derived within the KT framework.

Proposition 1.6. *If the investor can only invest into the MMA denoted, by $P_0(t)$ and the stocks $P_1(t)$ issued by the company, then the optimal stock portfolio process is given by*

$$\pi^*_{P_1}(t) = \frac{\pi^*_v}{\epsilon_{P_1}} = \begin{cases} \frac{\alpha}{\sigma_v^2}\frac{P_1(t)}{\Phi(h_1(t))V(t)}, & \text{for} \quad U(x) = \ln(x) \\ \frac{\alpha}{(1-\gamma)\sigma_v^2}\frac{P_1(t)}{\Phi(h_1(t))V(t)}, & \text{for} \quad U(x) = \frac{1}{\gamma}x^\gamma \end{cases}$$

where the elasticity of the stock[3] *is defined as* $\epsilon_{P_1} = \frac{\partial P_1}{\partial V} \cdot \frac{V}{P_1}$ *and* $\Phi(\cdot)$ *is a standard normal distribution, and*

$$h_1(t) = \frac{\ln\left(\frac{V(t)}{F}\right) + \left(r + \frac{1}{2}\sigma_v^2\right)(T_1 - t)}{\sigma_v\sqrt{T_1 - t}}.$$

Proof: see Korn and Kraft [KK03].

Note that in the problem above, we do not have any constraints on the number of the bonds and stocks that the firm is issuing. In fact, in Merton [Mer74] setting, the number of stocks and bonds is limited to one. Here, we use the "small investor assumption" and assume we do not have the upper bound constraint for the number of stocks and/or bonds. The optimisation problem with the constrained case is also studied by Korn and Kraft [KK03]. Their solution method to the portfolio problem in Step 2 is just the generalisation of the ideas presented in Korn and Trautmann [KT99]. Therefore, we provide a similar solution constructed within the KT framework, in particular using Theorem 1.6 with the presentation below.

Merton [Mer74] assumes that the stock of the firm is a European call option written on the market value of the firm, i.e., $P_1(t) \equiv X^{Call}(t, V(t))$. Hence, Theorem 1.6 is applicable here; further, we have the same optimal payoff B^* as in Merton portfolio problem in step 1. However, we replicate the firm value position with the stock and MMA positions since

[3] We refer the interested reader to Kraft [Kra03] for more details on the elasticity approach.

the firm value is not a tradable asset. With Theorem 1.6 and from (1.5) we have the replicating strategy as

$$\Psi_1(t) = \frac{\partial P_1(t)}{\partial V} = \Phi(h_1(t)). \tag{1.56}$$

From step 1, we have the optimal trading strategies $\xi(t) = (\xi_0(t), \xi_1(t))$ as

$$\xi_1(t) = \frac{\pi_v^* X(t)}{V(t)},$$

and

$$\xi_0(t) = \frac{(1-\pi_v^*)X(t)}{P_0(t)}.$$

Hence, the optimal trading strategy for the second step using Theorem 1.6 will be

$$\varphi_1(t) = \Psi(t)^{-1} \cdot \xi_1(t) = \frac{1}{\Phi(h_1(t))} \cdot \frac{\pi_v^* X(t)}{V(t)} = \frac{1}{\Phi(h_1(t))} \cdot \frac{\alpha X(t)}{\sigma_v^2 V(t)}, \tag{1.57}$$

where for the MMA we have the optimal trading strategy as

$$\varphi_0(t) = \frac{X(t) - \varphi_1(t) P_1(t)}{P_0(t)}. \tag{1.58}$$

Now we can derive the optimal portolio process

$$\pi_{P_1}^*(t) = \frac{\varphi_1(t) P_1(t)}{X(t)} = \frac{1}{\Phi(h_1(t))} \cdot \frac{\alpha X(t) P_1(t)}{\sigma_v^2 V(t) X(t)} = \frac{\alpha}{\sigma_v^2} \frac{P_1(t)}{\Phi(h_1(t)) V(t)}, \tag{1.59}$$

where we have the same result for $U(x) = \ln(x)$ as in Proposition 1.6. Note that for $U(x) = \frac{1}{\gamma} x^\gamma$, we have

$$\pi_{P_1}^*(t) = \frac{\alpha}{(1-\gamma)\sigma_v^2} \frac{P_1(t)}{\Phi(h_1(t)) V(t)}.$$

3. Optimal portolio with the compound option and MMA

Here, we imitate our calculations from step 2 and solve the problem we defined in (1.52). From the first part of Theorem 1.6, we have the optimal payoff B^*. Hence, we search for the optimal strategies for replicating the position on stocks with the positions in the option and the MMA.

Using Theorem 1.6, and (1.5) we have the replicating strategy for our problem for $U(x) = \ln(x)$ as follows:

$$\Psi_1(t) = \frac{\partial X^{Call}(t, P_1(t))}{\partial P_1} = \frac{\partial X^{CC}(t, V(t))}{\partial P_1} = \frac{\partial X^{CC}(t, V(t))}{\partial V} \Big/ \frac{\partial P_1}{\partial V} = \frac{\Phi_2^{\rho_1}(g_1(t), h_1(t))}{\Phi(h_1(t))}$$

with

$$g_1(t) = \frac{\ln\left(\frac{V(t)}{v^*}\right) + \left(r + \frac{1}{2}\sigma_v^2\right)(T-t)}{\sigma_v\sqrt{T-t}},$$

$$h_1(t) = \frac{\ln\left(\frac{V(t)}{F}\right) + \left(r + \frac{1}{2}\sigma_v^2\right)(T_1-t)}{\sigma_v\sqrt{T_1-t}},$$

and with the correlation coefficient

$$\rho_1 := \sqrt{\frac{T-t}{T_1-t}}.$$

The optimal trading strategies from the previous step are given by

$$\xi_1(t) = \frac{\pi_{P_1}^* X(t)}{P_1(t)} = \frac{\alpha}{\sigma_v^2}\frac{X(t)}{\Phi(h_1(t))V(t)},$$

$$\xi_0(t) = \frac{(1-\pi_{P_1}^*)X(t)}{P_0(t)}.$$

With Theorem 1.6, the optimal trading strategies of our problem defined in (1.52) will be

$$\varphi_1(t) = (\Psi_1(t))^{-1} \cdot \xi_1(t) = \frac{\alpha}{\sigma_v^2}\frac{X(t)}{\Phi_2^{\rho_1}(g_1(t), h_1(t))V(t)},$$

$$\varphi_0(t) = \frac{X(t) - \varphi_1(t)X^{CC}(t, V(t))}{P_0(t)}.$$

Now, we have the optimal portfolio process for our call on a call option π_{CC}^* as

$$\pi_{CC}^*(t) = \frac{\varphi_1(t)X^{CC}(t, V(t))}{X(t)} = \frac{\alpha}{\sigma_v^2}\frac{X^{CC}(t, V(t))}{\Phi_2^{\rho_1}(g_1(t), h_1(t))V(t)}, \tag{1.60}$$

which is the fraction of total wealth optimally invested to the compound option.

Proposition 1.7. *For a portfolio consisting of a MMA and the compound option of call on a call type, written on the market value of a firm, the optimal portfolio process, giving the optimal proportion of the total wealth invested to the compound option is*

$$\pi_{CC}^*(t) = \begin{cases} \frac{\alpha}{\sigma_v^2}\frac{X^{CC}(t,V(t))}{\Phi_2^{\rho_1}(g_1(t),h_1(t))V(t)} & \text{for} \quad U(x) = \ln(x) \\ \\ \frac{\alpha}{(1-\gamma)\sigma_v^2}\frac{X^{CC}(t,V(t))}{\Phi_2^{\rho_1}(g_1(t),h_1(t))V(t)} & \text{for} \quad U(x) = \frac{1}{\gamma}x^\gamma \end{cases}$$

Let us compare the optimal portfolio processes π_{CC}^* and $\pi_{P_1}^*$ for a log-utility investor. We have the property that

$$\Phi_2^{\rho}(g_1(t), h_1(t)) \geq \Phi(g_1(t))\Phi(h_1(t))$$

for a positive correlation. Since $\rho_1 = \frac{T-t}{T_1-t}$ is always positive, we can write

$$\frac{X^{CC}(t, V(t))}{\Phi_2^{\rho_1}(g_1(t), h_1(t))V(t)} \leq \frac{X^{CC}(t, V(t))}{\Phi(g_1(t))\Phi(h_1(t))V(t)}.$$

Moreover, we know that

$$X^{CC}(t, V(t)) \leq X^{Call}(t, V(t)) \equiv P_1(t).$$

Therefore,

$$\frac{\alpha}{\sigma_v^2} \cdot \frac{X^{CC}(t, V(t))}{\Phi(g_1(t))\Phi(h_1(t))V(t)} \leq \frac{\alpha}{\sigma_v^2} \cdot \frac{P_1(t)}{\Phi(g_1(t))\Phi(h_1(t))V(t)} < \frac{\alpha}{\sigma_v^2} \cdot \frac{P_1(t)}{\Phi(h_1(t))V(t)} = \pi_{P_1}^*(t).$$

Remark 1.4. The interpretation of the result we have in Proposition 1.7 is that for an investor with logarithmic and/or power utility function, we will have the optimal portfolio processes in the following order

$$\pi_{CC}^*(t) < \pi_{P_1}^*(t) < \pi_v^* \qquad \text{for all } t \in [0, T]. \tag{1.61}$$

Example 1.3. Let us present the results in an example. Consider the case when we have the following parameters:

$\mu_v =$	0.05	drift term
$\sigma_v =$	0.25	volatility
$T =$	0.8	maturity time for the compound option
$T_1 =$	1.5	maturity time for the underlying call option
$r =$	0	short rate
$K =$	20	strike price for the compound option
$K_1 =$	100	strike price for the underlying call option

We observe from Figure 1.4 that an investor with log-utility will invest less in the compound option than he invests in the call option, as expected, since the call on call option is a riskier product than a European call option. The deeper the call option and call on call option are in the money, the closer π_{Call}^* and π_{CC}^* get to the optimal value in stock-MMA problem, denoted by π_v^*. □

Alternatively, we can solve the problem in the third step in a direct way using KT framework, by skipping the second step. Say, we have the solution in the Merton portfolio problem as π_v^* from step 1 for $U(x) = \ln(x)$, then the replication strategy for the call on call option $X^{CC}(t, V(t))$ is

$$\Psi_1^{CC}(t) = \frac{\partial X^{CC}(t, V(t))}{\partial V(t)} = \Phi_2^{\rho_1}(g_1(t), h_1(t)).$$

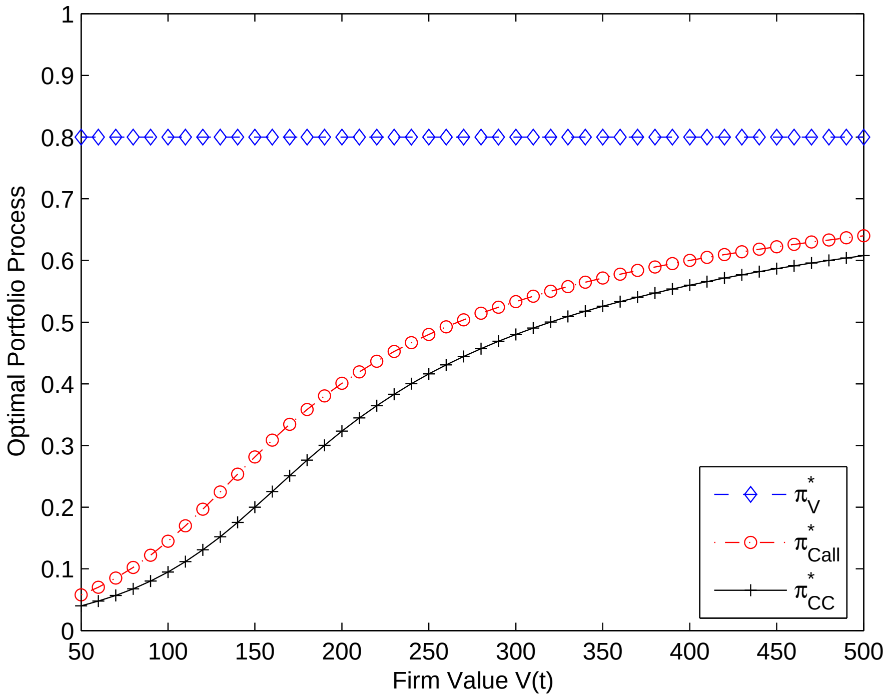

Figure 1.4: The optimal portfolio processes of stock, call option, call on call option vs. value of the firm, with the parameter set $\mu_v = 0.05$, $\sigma_v = 0.25$, $T = 0.8$, $T_1 = 1.5$, $r = 0$, $K = 20$, and $K_1 = 100$.

Having the same payoff as the Merton problem, with Theorem 1.6, the optimal trading strategies from the Merton problem are

$$\xi_1(t) = \frac{\pi_v^* X(t)}{V(t)},$$

and

$$\xi_0(t) = \frac{(1-\pi_v^*)X(t)}{P_0(t)}.$$

Now, again with Theorem 1.6, the optimal trading strategy for the compound option is

$$\varphi_1^{CC}(t) = \frac{\xi_1(t)}{\Psi_1^{CC}(t)} = \frac{\alpha}{\sigma_v^2} \frac{X(t)}{\Phi_2^{\rho_1}(g_1(t), h_1(t))V(t)}.$$

And the optimal portfolio process is

$$\pi_{CC}^*(t) = \frac{\varphi_1^{CC}(t) X^{CC}(t, V(t))}{X(t)} = \frac{\alpha}{\sigma_v^2} \frac{X^{CC}(t, V(t))}{\Phi_2^{\rho_1}(g_1(t), h_1(t))V(t)}. \tag{1.62}$$

Comparing the results in Proposition 1.7 and (1.62), we see that they are exactly the same, therefore, we can apply the same formulation above in order to have the optimal portfolio strategies to portfolios of MMA and call on put option $X^{CP}(t, V(t))$, put on a put option $X^{PP}(t, V(t))$ or put on a call option $X^{PC}(t, V(t))$.

Proposition 1.8. *Using the KT framework in Theorem 1.6, the optimal portfolio processes for the call on put, put on call and put on put options, where the underlying is the market value of the firm, are as follows*

$$
\begin{aligned}
\pi^*_{CP}(t) &= -\frac{\alpha}{\sigma_v^2} \frac{X^{CP}(t, V(t))}{\Phi_2^{\rho_1}(-g_1(t), -h_1(t))V(t)}, \\
\pi^*_{PC}(t) &= -\frac{\alpha}{\sigma_v^2} \frac{X^{PC}(t, V(t))}{\Phi_2^{\rho_2}(-g_1(t), h_1(t))V(t)}, \\
\pi^*_{PP}(t) &= \frac{\alpha}{\sigma_v^2} \frac{X^{PP}(t, V(t))}{\Phi_2^{\rho_2}(g_1(t), -h_1(t))V(t)},
\end{aligned}
$$

with the notation given as in Proposition 1.2 and assuming that the investor (with logarithmic and power utility functions) can only trade in these options and the MMA without an upper bound on the number of securities issued by the firm.

Example 1.4. Let us analyse the problem when the compound option is a put on the call type. On Figure 1.5, we observe the optimal portfolio process for the put on call option. Negative portfolio process in Figure 1.5 implies short selling of the put on call option in the portfolio. Note that the optimal strategy (not the optimal portfolio process) attains the maximum expected utility.

1.5.4 Optimal Portfolio Problem with an Option on the Defaultable ZCB

In this subsection, we analyse the optimisation problem of a portfolio consisting of the MMA and European call or put option written on a risky zero coupon bond with face value F and maturity T_1. Assuming that $T_1 > T$, during the investment period $(0, T]$ we can not have a default event since the Merton [Mer74] model has the restriction that a default event can only occur *at the maturity of the ZCB*. However, a low firm value indicates a high probability of default and a low bond value. We also do not have a constraint on the number of bonds issued by this firm.

As before, the problem will be solved in a three step procedure,

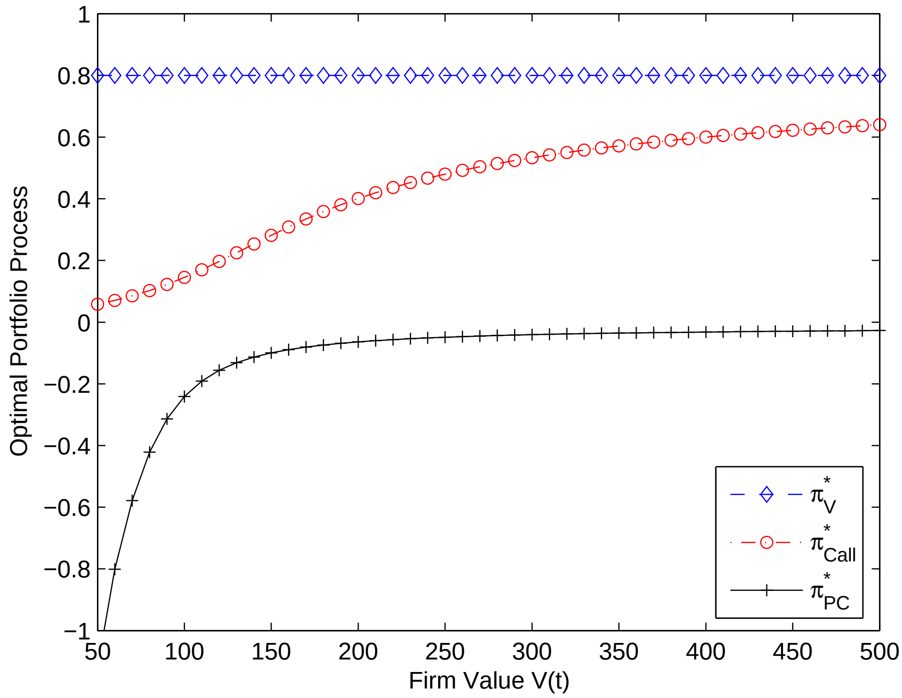

Figure 1.5: The optimal portfolio processes of stock, call option, put on call option vs. value of the firm, with the parameter set $\mu_v = 0.05$, $\sigma_v = 0.25$, $T = 0.8$, $T_1 = 1.5$, $r = 0$, $K = 20$, and $K_1 = 100$.

1. The optimisation of a portfolio with the firm value and the MMA, where the firm value is traded (Merton portfolio problem).

2. The optimisation of a portfolio consisting of the defaultable bond issued by the firm and the MMA.

3. Using the results of the second step, we optimise a portfolio with the European call and/or the put option written on the defaultable bond and the MMA within KT framework.

1. Merton portfolio problem

The result is given in Proposition 1.5

2. Optimal portfolio with the risky ZCB and MMA

In the Merton [Mer74] model, the value of a risky ZCB is given by (1.6). We observe the risky bond price when $K = 100$ in Figure 1.6. Our aim is to find the optimal portfolio

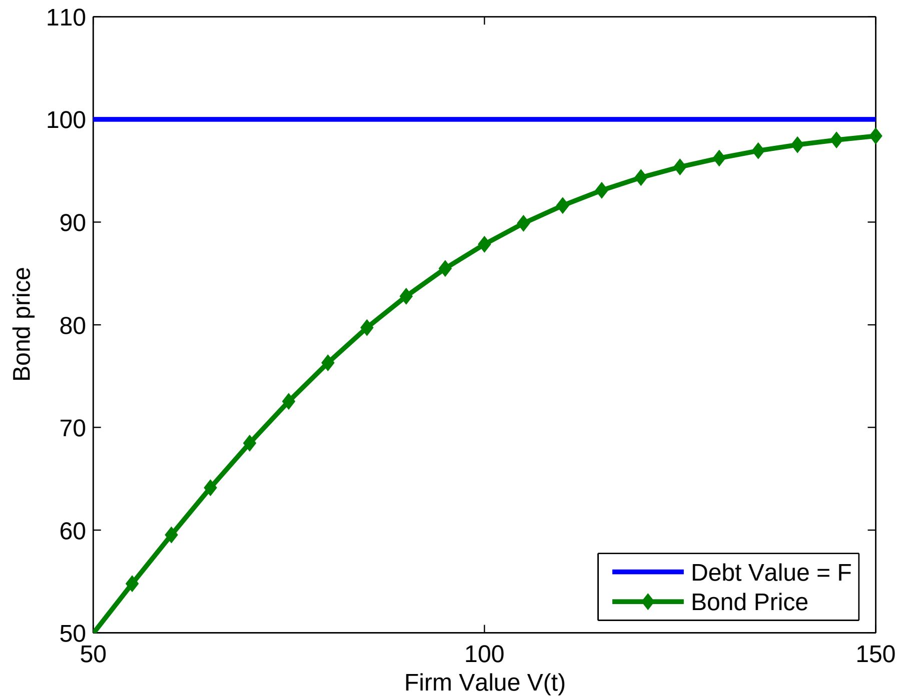

Figure 1.6: The price of the risky bond in the Merton setting with respect to the market value of the firm.

process that maximises the final wealth of the investor, i.e.,

$$\max_{\pi} E(U(X^{\pi}(T))), \tag{1.63}$$

when the wealth of the investor equals

$$X(t) = \varphi_0(t)P_0(t) + \varphi_1(t)\bar{B}(t, T_1).$$

Korn and Kraft [KK03] present the solution of the problem in (1.63) with the following proposition.

Proposition 1.9. *If the investor can only invest in the MMA $P_0(t)$ and the risky bond $\bar{B}(t, T_1)$ with $T_1 > T$ issued by the company, then the optimal bond portfolio process is given by*

$$\pi_B^* = \frac{\pi_v^*}{\epsilon_B} = \begin{cases} \frac{\alpha}{\sigma_v^2} \frac{\bar{B}(t,T_1)}{\Phi(-h_1(t))V(t)} & \textit{for} \quad U(x) = \ln(x) \\ \\ \frac{\alpha}{(1-\gamma)\sigma_v^2} \frac{\bar{B}(t,T_1)}{\Phi(-h_1(t))V(t)} & \textit{for} \quad U(x) = \frac{1}{\gamma}x^{\gamma} \end{cases}$$

where the elasticity of the bond is defined as $\epsilon_B = \frac{\partial \bar{B}(t,T_1)}{\partial V(t)} \cdot \frac{V(t)}{\bar{B}(t,T_1)}$.

Proof: (see Korn and Kraft [KK03])

We present the solution within the KT framework. From the first part of Theorem 1.6, we have the same optimal payoff as in Merton portfolio problem from step 1. Replicating the firm value position with the defaultable bond and MMA positions, from second part of the Theorem 1.6 and from (1.6), we find the replicating strategy as

$$\Psi_1(t) = \frac{\partial \bar{B}(t,T_1)}{\partial V(t)} = \Phi(-h_1(t)). \tag{1.64}$$

From step 1, we have the optimal trading strategy as

$$\xi_1(t) = \frac{\pi_v^* X(t)}{V(t)}.$$

Hence, the optimal trading strategy for the risky ZCB will be

$$\varphi_1(t) = \Psi(t)^{-1} \cdot \xi_1(t) = \frac{1}{\Phi(-h_1(t))} \cdot \frac{\pi_v^*(t) X(t)}{V(t)} = \frac{1}{\Phi(-h_1(t))} \cdot \frac{\alpha X(t)}{\sigma_v^2 V(t)}, \tag{1.65}$$

where the optimal trading strategy for the MMA, is

$$\varphi_0(t) = \frac{X(t) - \varphi_1(t)\bar{B}(t,T_1)}{P_0(t)}. \tag{1.66}$$

Now, we can derive the optimal portolio process as

$$\pi_B^*(t) = \frac{\varphi_1(t)\bar{B}(t,T_1)}{X(t)} = \frac{1}{\Phi(-h_1(t))} \cdot \frac{\alpha X(t)\bar{B}(t,T_1)}{\sigma_v^2 V(t) X(t)} = \frac{\alpha}{\sigma_v^2} \frac{\bar{B}(t,T_1)}{\Phi(-h_1(t))V(t)}. \tag{1.67}$$

Comparing the result for $U(x) = \ln(x)$ in Proposition 1.9, we have the same finding. Note that for $U(x) = \frac{1}{\gamma}x^\gamma$ we have

$$\pi_B^*(t) = \frac{\alpha}{(1-\gamma)\sigma_v^2} \frac{\bar{B}(t,T_1)}{\Phi(-h_1(t))V(t)}. \tag{1.68}$$

3. Optimal portolio with the option on risky ZCB and MMA

In this step, we optimise the portfolio of European call and/or put option written on the risky ZCB and the MMA. Using the same methodology as in Proposition 1.7, we present the results with the following proposition.

Proposition 1.10. *If the investor is allowed to invest only in the European call option written on the risky ZCB and the MMA, we will have the optimal portfolio process $\pi^*_{CallBond}$ for the call option with maturity time T and strike price K as*

$$\pi^*_{CallBond}(t) = \begin{cases} \frac{\alpha}{\sigma_v^2} \frac{X^{Call}(t,\bar{B}(t,T_1))}{\Phi_2^{\rho 2}(g_1(t),-h_1(t))V(t)} & for \quad U(x) = \ln(x) \\ \\ \frac{\alpha}{(1-\gamma)\sigma_v^2} \frac{X^{Call}(t,\bar{B}(t,T_1))}{\Phi_2^{\rho 2}(g_1(t),-h_1(t))V(t)} & for \quad U(x) = \frac{1}{\gamma}x^\gamma \end{cases} \tag{1.69}$$

with maturity of the underlying ZCB is denoted by T_1, with $T_1 > T$ and ρ^2 is the correlation coefficient given as in Proposition 1.3.

Example 1.5. Consider the case when we have a call option written on the risky bond with the following parameters:

$\mu_v =$	0.05	drift term
$\sigma_v =$	0.25	volatility
$T =$	0.8	maturity time for the call option in years
$T_1 =$	1.5	maturity time for the underlying call option
$r =$	0	short rate
$K =$	80	strike price for the call option
$F =$	100	debt value for the underlying defaultable bond

We can observe the optimal portfolio process with the logarithmic utility function for the call on the risky ZCB with respect to the firm value in Figure 1.7. The interpretation is that with increasing firm value, the probability of the default of the ZCB decreases. This implies an increase in the price of the ZCB and the call option on this ZCB. For an investor with logarithmic utility, the fraction of the wealth that he invests on the risky ZCB and call option on the ZCB increase as well.

Proposition 1.11. *If the investor is allowed to invest only in the European put option written on the risky ZCB and the MMA, we will have the optimal portfolio process for a European put option with maturity time T and strike K on the risky bond as*

$$\pi^*_{PutBond}(t) = \begin{cases} -\frac{\alpha}{\sigma_v^2} \frac{X^{Put}(t,\bar{B}(t,T_1))}{\Phi_2^{\rho 1}(-g_1(t),-h_1(t))V(t)} & for \quad U(x) = \ln(x) \\ \\ -\frac{\alpha}{(1-\gamma)\sigma_v^2} \frac{X^{Put}(t,\bar{B}(t,T_1))}{\Phi_2^{\rho 1}(-g_1(t),-h_1(t))V(t)} & for \quad U(x) = \frac{1}{\gamma}x^\gamma \end{cases} \tag{1.70}$$

with maturity of the underlying ZCB is denoted by T_1, with $T_1 > T$ and ρ^1 is the correlation coefficient as given in Proposition 1.3.

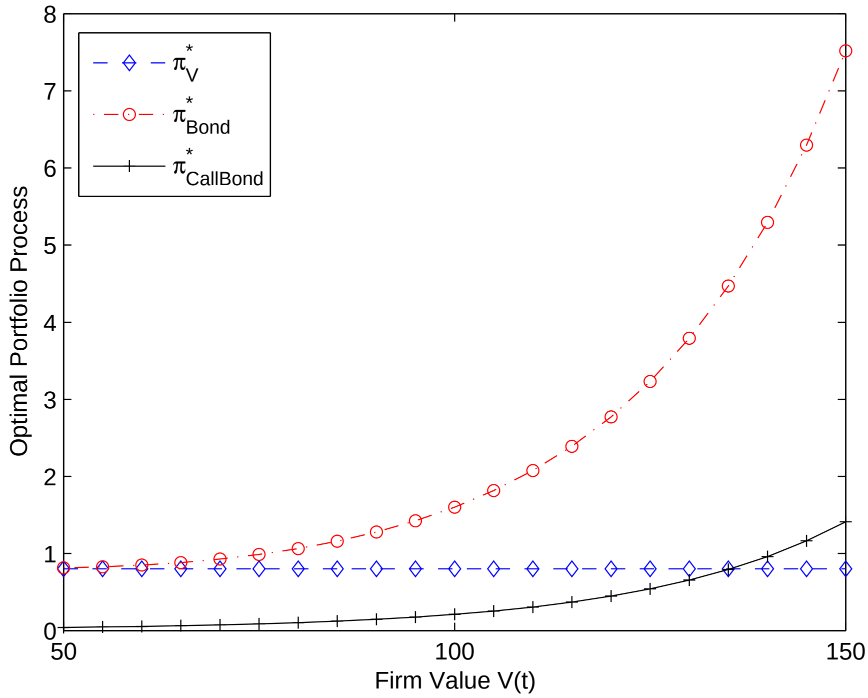

Figure 1.7: The optimal portfolio processes of stock, defaultable bond, call option on defaultable bond vs. value of the firm with the parameter set $\mu_v = 0.05$, $\sigma_v = 0.25$, $T = 0.8$, $T_1 = 1.5$, $r = 0$, $K = 80$, and $F = 100$.

Example 1.6. Let us give an example when we have a put option written on the risky ZCB in our portfolio. Consider the case when the derivative has the same paramater set as in Example 1.5. We can observe the optimal portfolio process for the put option written on the risky bond with respect to the firm value in Figure 1.8, where we use the logarithmic utility function. Note that the negativity of portfolio process is interpreted as short selling of the put option on the ZCB in the portfolio. This optimal portfolio process has a similar behaviour as in the example on the put on call option.

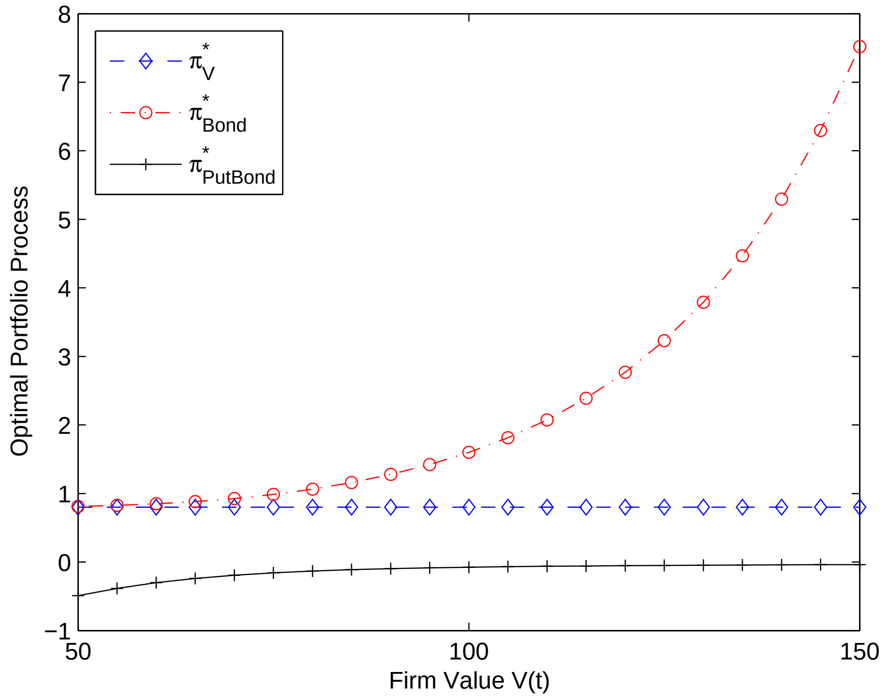

Figure 1.8: The optimal portfolio processes of stock, defaultable bond, put option on defaultable bond vs. value of the firm, with the parameter set $\mu_v = 0.05$, $\sigma_v = 0.25$, $T = 0.8$, $T_1 = 1.5$, $r = 0$, $K = 20$, and $K_1 = 100$.

1.6 Summary

In this chapter, we derived optimal portfolios including compound options, when the compound options has the market value of a firm as underlying. Modifying the compound option valuation of Geske [Ges79], we priced European options written on the risky ZCBs. Further, we optimised the portfolios consisting of the risky ZCB and a MMA, and of European options written on the defaultable ZCB. For that, we first supplied the necessary information about the ingredients of our optimisation problem, namely the firm value based credit risk models, continuous-time portfolio optimisation with the martingale approach, and the methodology for optimising portfolios of options, named as *Korn-Trautmann framework* during this work.

Our main findings show that, for the investors with logarithmic and power utility functions, the riskier the option gets, the less proportion of wealth they invest in the risky product in the portfolio. For the portfolios consisting of put options written on the call option and on the risky ZCB, we calculated negative optimal portfolio processes implying shortselling of the assets.

There are of course many shortcomings of our modelling approach. Among those, we can comment on two important ones. Firstly, we use the classical structural model by Merton [Mer74] for credit risk, where the occurrence of the credit event is allowed only on the maturity of the debt, i.e., T_1. Hence, we do not allow the credit event happen during the investment horizon, i.e., $[0, T]$ via assuming that $T_1 > T$. Second, the number of bonds and/or stocks issued by the firm is not restricted.

The first shortcoming can be handled by using the Black-Cox credit risk model [BC76], where an intermediate default is possible during the investment period $[0, T]$. Optimising the portfolio of a risky bond in the Black-Cox model was studied by Korn and Kraft [KK03]. To our knowledge, the optimisation problem with an *option* on the defaultable bond, where the credit risk is modelled in Black-Cox framework is still not studied. We leave this for a future research problem.

The second shortcoming is the "small investor assumption", which omits the upper bounds on the number of bonds and stocks issued by the corporate firm. This problem might also be handled using the accounting equation, i.e., the market value of the firm equals the sum of the risky bond price and the equity price of the firm, so this can be extended to a constrained problem in a future research topic as well. Further extensions to our problem can be done, making the problem applicable in practice via optimising portfolios of vulnerable options on the risky ZCB, or even coupon paying bonds.

Chapter 2

Sovereign CDS and Market-implied Credit Risk of Turkey

2.1 Introduction

Sovereign Credit Default Swap (CDS) contracts are being actively traded in emerging markets with increasing volumes and these are typically the most liquid credit derivative instruments in the related countries. As the credit literature documents[1], CDS contracts are better proxies for credit risk modelling than the risky bonds due to two main reasons. Firstly, the CDS contracts are typically more liquid than the underlying reference assets. Second, being unfunded contracts, they are not influenced by the tax effects. This chapter analyses the market implied (or risk-neutral) probabilities of default extracted from the market quotes of the Turkish sovereign CDS contracts.

The sovereign CDS's have very similar features to corporate CDS contracts but there are some differences that stem from the reference asset, premium payment interval, and the credit event definitions. The reference asset in a sovereign CDS contract is the sovereign debt, which usually requires a different modelling framework than the corporate debt, since sovereign credit risk is driven mainly by economical and political factors. In general, sovereign CDS contracts have semi-annually premium payments guaranteeing the physical delivery of the underlying reference upon a credit event. The credit event definitions in sovereign CDS include obligation acceleration, failure to pay, restructuring/renegotiation , and repudiation/moratorium of the sovereign. Note that the "default" is excluded since

[1] See Berndt et al. [BDD$^+$05] and Hull et al. [HPW05].

there is not an international bankruptcy court regulating the sovereign issuers. However, we use the term “default probability” as a measure of the arrival risk of the credit event. Further, the outright default of a sovereign is a very rare event and it is rather a political decision.

In this chapter, we model the credit risk in a setting that allows us to extract the term structure of the market implied default probabilities. We are interested in practical methodologies for extracting the probabilities, rather than explaining the economical and/or political factors that might trigger the credit event in a sovereign. In order to do so, we first bootstrap the term structure of the market implied intensity rates implicit in the market prices of the sovereign CDS contracts. A second method, where we minimise an objective function with respect to the risk neutral forward conditional default probabilities, is also presented for comparing the methods. Furthermore, we explore the risk premium for the Turkish sovereign in depth. We use the credit risk model introduced by Jarrow and Turnbull [JT95] (JT model hereafter), which is a pioneer work in reduced form models, due to its simplicity in the calibration.

The JT model assumes a constant, deterministic intensity rate allowing independence from the expected recovery rate and the short rate process. The exogenous intensity process may of course depend on some macroeconomic variables[2] but this is not in the scope of our analysis. The constant intensity process assumption provides easiness in numerics but do not significantly explain the market rates as documented by Frühwirth and Sögner [FS06], where the authors examine the German corporate bond market. Their findings show that the intensity should be modelled within a stochastic framework as in the Lando [Lan98] model, or the Duffie and Singleton [DS99] model. In this sense, we provide a parallel analysis to Frühwirth and Sögner [FS06], keeping in mind that instead of the corporate bonds, the CDS market rates are used for extracting the market implied intensities of the credit risk.

We fix the expected recovery rate under risk-neutral measure a priori, hence, CDS spreads are forced to be driven only by risk neutral intensity of default. A similar paper analysing the credit risk parameters of Japanese government and major Japan banks is by Ueno and Baba [UB06], where the authors use the Duffie and Singleton [DS99] credit risk model,

[2]See Duffie et al. [DPS03], and Pan and Singleton [PS07a].

allowing a joint estimation of intensity and the recovery rate. In contrast, Frühwirth and Sögner [FS06] report that joint estimation is numerically unstable. Moreover, Rocha and Garcia [RG04] illustrate the calibration of a structural credit risk model for pricing the sovereign CDS including an analysis with Turkish sovereign CDS, hence we compare our results with those by Rocha and Garcia [RG04] for a certain date in the sampling period.

There are many papers in the credit literature about corporate CDS valuation and their standardisation is documented by International Swaps and Derivatives Association (ISDA) in 2003. A detailed literature survey is done by Das and Hanoua [DH06], where they present the CDS spreads with structural and reduced form credit risk models. Pricing of corporate and sovereign CDS is quite similar, but for the exact formulation and a list of references, we refer the reader to the paper of Realdon [Rea07], where the author extends the one factor model of Pan and Singleton [PS07a] with a two factor modelling approach. Moreover, Pan and Singleton [PS07a] give detailed analysis about the time series properties of the risk neutral intensity rates of three sovereigns, namely Mexican, Turkish, and Korean. The authors use the risk-adjusted short rate modelling approach introduced by Duffie and Singleton [DS99], where they claim the CDS prices reveal not only the market-implied hazard rates but also the loss rates ($Loss\ rate = 1 - Recovery\ rate$). Papers about sovereign CDS market are Ranciere [Ran91], Packer and Suthiphongchai [PS03]. Another reference is Keller et al. [KKS07a], where the authors analyse the sovereign risk of Turkey, with contingent claims approach. Furthermore, an empirical work on Turkish CDS contracts is done by Baklaci and Arslan [BA06], where their findings show that the sovereign CDSs of Turkey with 10 year maturity are overpriced using the valuation methodology introduced by Ranciere [Ran91].

The remainder of this chapter is as follows. In Section 2.2 we present a detailed survey about the intensity based (or reduced-form) models and supply the mathematical background necessary for a better understanding of the risk models. Since these models are also used for pricing the credit risk derivatives, we focus especially on methodologies for constructing the term structures of the risk-neutral PDs for pricing the sovereign CDS in Section 2.3. In Section 2.4 we run empirical analysis on the sovereign CDS contracts of Turkey and present the results. Section 2.5 highlights the linkage between the actual and the risk neutral intensities. The last section summarises and gives our main conclusions.

2.2 Reduced-form Credit Risk Models

In this subsection, we present widely accepted reduced form models in corporate credit risk literature as well as in the financial industry. The general idea of reduced-form model is to model the default arrival time with a Poisson arrival process. These models accept the default event as a sudden "surprise", implying an inaccessible stopping time for the credit event in contrast to structural models with predictable stopping times, e.g., Merton [Mer74] model. The pioneers of the reduced form modelling are Jarrow and Turnbull [JT95], taking a term structure of default free interest rates and a maturity specific credit-risk spread as given. Given these two term structures, the arbitrage free pricing of risky bonds can be done using the martingale measure technique. Then, Jarrow et al. [JLT97] introduce a Markovian model for the term structures of credit risk spreads. The authors extend the model by Jarrow and Turnbull [JT95] via including the credit rating information into the risky bond pricing methodology. Lando [Lan98] generalises the model proposed by Jarrow et al. [JLT97] with a Cox process for the default probability, providing randomness of the intensities and credit spreads. Furthermore, Lando [Lan98] allows the dependence between risk-free term structure and the default process via a common state variable. The model proposed by Duffie and Singleton [DS99] allows us to use the standard term structure models by parameterising the *risk-adjusted short rate*, instead of the standard risk-free short rate process.

2.2.1 Preliminaries for Reduced-form Models

In this subsection, we present the mathematics behind the reduced-form credit risk models and give necessary definitions, mainly from Schönbucher [Sch03], Bielecki and Rutkowski [BR02], Durrett [Dur99] and Lando [Lan02].

Stopping Time

In order to model the arrival risk of a credit event, which is the uncertainty whether a default will occur or not, we need to model an unknown, random point in time $\tau \in \mathbb{R}^+$. Since there is a possibility that the default will not occur, ∞ is also included in the set of realisations of τ. The connection between stopping times and the filtration $(\mathcal{F}_t)_{(t\geq 0)}$

is that if τ is the time of some event, we know that this event has occurred or not from the information contained in $\mathcal{F}_t$. Mathematically, we can define the random time τ as a stopping time with the following property:

$$\{\tau \leq t\} \in \mathcal{F}_t \quad \forall t \geq 0. \tag{2.1}$$

Furthermore, the stochastic representation of a stopping time is possible with an *indicator process* which jumps from zero to one at the stopping time:

$$N_\tau(t) := 1_{\{\tau \leq t\}}. \tag{2.2}$$

The property, which determines whether the stopping time is predictable or totally inaccessible, set the reduced form models apart from the structural models of credit risk (see Chapter 1). If it is a *predictable* stopping time, then the indicator process of the stopping time is a predictable process as well. A predictable stopping time has an *announcing sequence* of stopping times $\tau_1 \leq \tau_2 \leq \ldots$ with

$$\tau_n < \tau \text{ and } \lim_{n\to\infty} \tau_n = \tau \text{ for all } \omega \in \Omega \text{ with } \{\tau(\omega) > 0\}. \tag{2.3}$$

This implies the existence of a sequence of early warning signals τ_n that occur before τ and announce the predictable stopping time. In classical firm value based credit risk model, the default time is predictable and this makes sense in economical interpretation of the credit event, since the firm might give bad signals before it defaults.

For the *totally inaccessible* stopping time τ, there is no predictable stopping time that gives information, i.e., for all predictable stopping times τ' we have :

$$P[\tau = \tau' < \infty] = 0. \tag{2.4}$$

In reduced form models, the default time is totally inaccessible, implying that the default event is a sudden surprise. However, as it is highlighted by Jarrow and Protter [JP04], the main distinction point in the debate between these two types of credit risk modelling is the *information set* available to the modeller and not the *type of the stopping time*. If we are a manager of a firm, then we will have full access to all the information about the firm's assets and liabilities. Thus, we rather use a structural model, which implies a predictable default time. On the other hand, if we do not have full access to the information set, then we use only what is available in the financial market. Hence, we use a reduced form approach, which implies a totally inaccessible stopping time. Furthermore, the link between the reduced form and structural credit risk models based on the information set is studied by Guo et al. [GJZ05]. Moreover, structural modelling approaches with incomplete information are presented by Giesecke [Gie06].

Hazard Rate

The hazard rate (also known as failure rate, or default intensity) is the ratio of the probability density function to the survival function, with the following definition.

Definition 2.1. Let τ be a stopping time and $F(T) := P[\tau \leq T]$ be its cumulative distribution function. Further, assume that $F(T) < 1$ for all T and that $F(T)$ has a probability density function $f(T)$. The hazard rate function h of τ is defined as:

$$h(T) := \frac{f(T)}{1 - F(T)} = \frac{f(T)}{S(T)}. \tag{2.5}$$

where $S(t)$ is called the *survival function*, $S(t) = P[\tau > t]$. Hence, another representation will be

$$h(T) = \frac{-d \ln S(T)}{dt} = -\frac{S'(T)}{S(T)}$$

Solving the differential equation above, we will have

$$S(T) = \exp\left(-\int_0^T h(s)ds\right) \tag{2.6}$$

The hazard rate $h(t)$ can be interpreted as the local arrival probability of the stopping time per time unit:

$$h(t) = \lim_{dt \to 0} \frac{P[t \leq \tau \leq t + dt | \tau > t]}{dt}. \tag{2.7}$$

Forward Default Probability and Intensity Process

The probability of default between time interval $(t, T]$ with $T \geq t$ is $S(t) - S(T)$. By Bayes' rule, the probability of surviving to time T, given survival to time t but no other information about the issuer or the economy is

$$ps(t, T) = \frac{S(T)}{S(t)}. \tag{2.8}$$

Hence, if we define the *forward default probability* as

$$pd(t,T) = 1 - ps(t,T),$$

which gives the probability of default between time points t and T given survival to time t (no other information). Moreover, from (2.8) and (2.6) in terms of hazard function, we can express it as

$$pd(t,T) = 1 - \exp\left(-\int_t^T h(u)du\right), \quad \text{with } T \geq t \geq 0. \tag{2.9}$$

The reduced form models are also called the *intensity based models*, therefore, we give here the notion of the link between the intensity and the hazard rate. The hazard rate function $h(t)$ is used to characterise the distribution of the survival time, hence it is also called the *credit curve* giving the term structure of the default probabilities. If h is continuous, then for small dt we have

$$h(t)dt \approx P[t \leq \tau \leq t + dt \mid \tau > t].$$

In the intensity based approach, we model the first arrival time of a default event τ as a Poisson arrival time. Hence, we have a constant mean arrival rate h and it is called the *intensity*. In general, λ is used for denoting the intensity of the default. As Bluhm et al. [BOW03] indicate, some authors explicitly distinguish between the intensity $\lambda(t)$ as the arrival rate of default at t conditional on *all* the information available at t and the forward default rate (or hazard rate) $h(t)$ as the arrival rate of default at t, conditional *only* on survival until t. Of course, if the available information is only the "survival", then the hazard rate and the intensity are identical. In this chapter, assuming that the survival is given as the whole information set, we denote the hazard rate (or intensity interchangeably) with λ. Hence, the forward conditional PD in (2.9) can be written as

$$pd(t,T) = 1 - \exp\left(-\int_t^T \lambda(u)du\right), \quad \text{with } T \geq t \geq 0. \tag{2.10}$$

Formulation of the conditional forward PD depends on whether the intensity process is deterministicly or randomly varying. If we have a deterministic intensity process, then the intensity coincides with the forward default rate given that the only information relevant is the survival up to that date. Whereas, in a random intensity setting, (2.9) modifies to

$$pd(t,T) = 1 - E\left[\exp\left(-\int_t^T \lambda(u)du\right) \middle| \mathcal{F}_t\right], \quad \text{with } T \geq t \geq 0 \tag{2.11}$$

where $\mathcal{F}_t$ represents all information available at time t.

Generally, as time passes we gather more information about the obligor, which bears on the credit quality. Any additional information during time implies the intensity process to be randomly varying. We will see how the intensity is modelled with an underlying state variable (such as credit rating, distance to default, business cycle or equity price of the obligor) in Subsection 2.2.2. Before we present the models for the intensity process, we recall the definition and properties of the exponential distribution.

Definition 2.2. A random variable T has an exponential distribution with rate λ (or $T \sim exponential(\lambda)$), if

$$P[T \leq t] = 1 - e^{-\lambda t} \text{ for all } t \geq 0,$$

with $E[T] = \frac{1}{\lambda}$.

An important property of the exponential distribution is the *lack of memory property*. Mathematically,

$$P[T > t + s \mid T > t] = P[T > s], \tag{2.12}$$

implying that the conditional probability of "failure" in a given interval is the same regardless of when the observation is made. Moreover, the exponential distribution has a constant hazard rate, i.e.,

$$h(t) = \frac{\lambda e^{-\lambda t}}{e^{-\lambda t}} = \lambda, \tag{2.13}$$

reflecting the lack of memory property.

Further modelling approaches for the distributions of survival times are summarised in Table 2.2.1. These distributions are generally used in the reliability literature. Andritzky [And06] uses these distributions in order to model the default intensity of the sovereign debt.

We may observe the behaviours of the intensity processes and the term structures of the corresponding survival probabilities in Figures 2.1, 2.2, 2.3, 2.4, and 2.5 for exponential, Weibull, loglogistic, lognormal, and Nelson-Siegel type of survival modelling, respectively.

Distribution	Hazard function, $h(t)$	Survival function, $S(t)$
Exponential	λ	$\exp(-\lambda t)$
Weibull	$\lambda\gamma(\lambda t)^{\gamma-1}$	$\exp(-(\lambda t)^{\gamma})$
Lognormal	$(\gamma/t)\phi(\gamma\ln(\lambda t))$	$\Phi(-\gamma\ln(\lambda t))$
Log-logistic	$\lambda\gamma(\lambda t)^{\gamma-1}/[1+(\lambda t)^{\gamma}]$	$1/[1+(\lambda t)^{\gamma}]$
Nelson-Siegel	$\beta_0+\beta_1\exp(-t/\lambda)$ $+\beta_2(t/\lambda)\exp(-t/\lambda)$	$\exp\Big[-\beta_0 t-\beta_1 t\frac{1-\exp(-t/\lambda)}{t/\lambda}$ $-\beta_2 t\Big(\frac{1-\exp(-t/\lambda)}{t/\lambda}-\exp(-t/\lambda)\Big)\Big]$

Table 2.1: Survival distributions, where $\phi(u)=\varphi(u)/[1-\Phi(u)]$, with $\varphi()$ denoting the density function of a standard normal distribution and $\Phi()$ its cumulative distribution function.

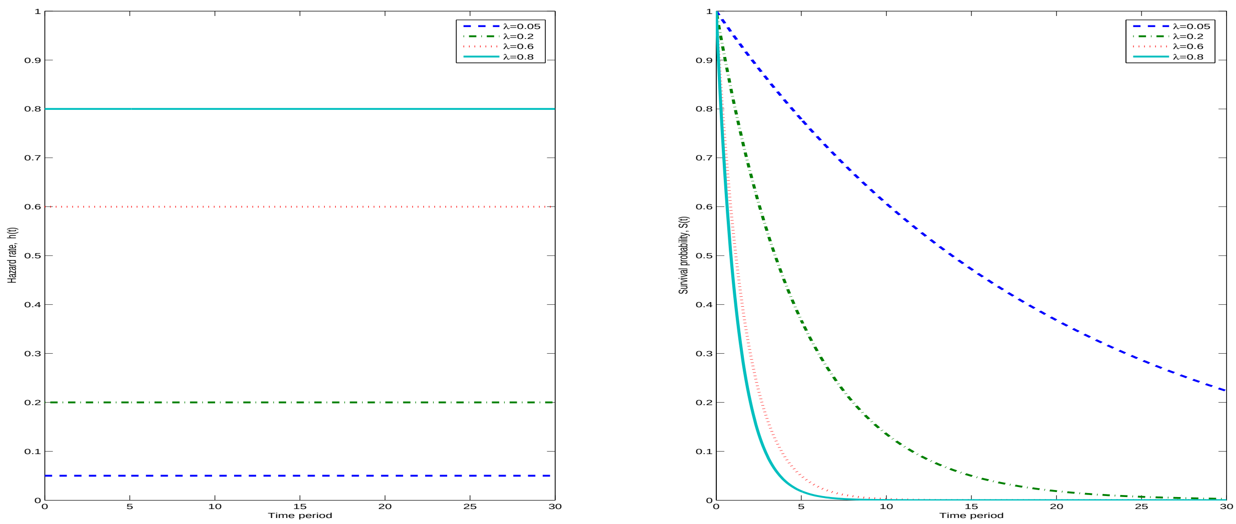

Figure 2.1: Exponential distribution

Point Processes

Mathematically, we can describe the occurrence of one event with a stopping time (default time of a single obligor) and for a generalisation to multiple events (default times of several obligors), we should rather use the point processes. A point process can be defined as some collection of points in time, i.e.,

$$\{\tau_i, i\in\mathbb{N}\}=\{\tau_1,\tau_2,\dots\}. \tag{2.14}$$

Under the assumptions that the stopping times are indexed by ascending order, $(\tau_i<\tau_{i+1})$, and that they are all different, we can transform this collection of points in time to a

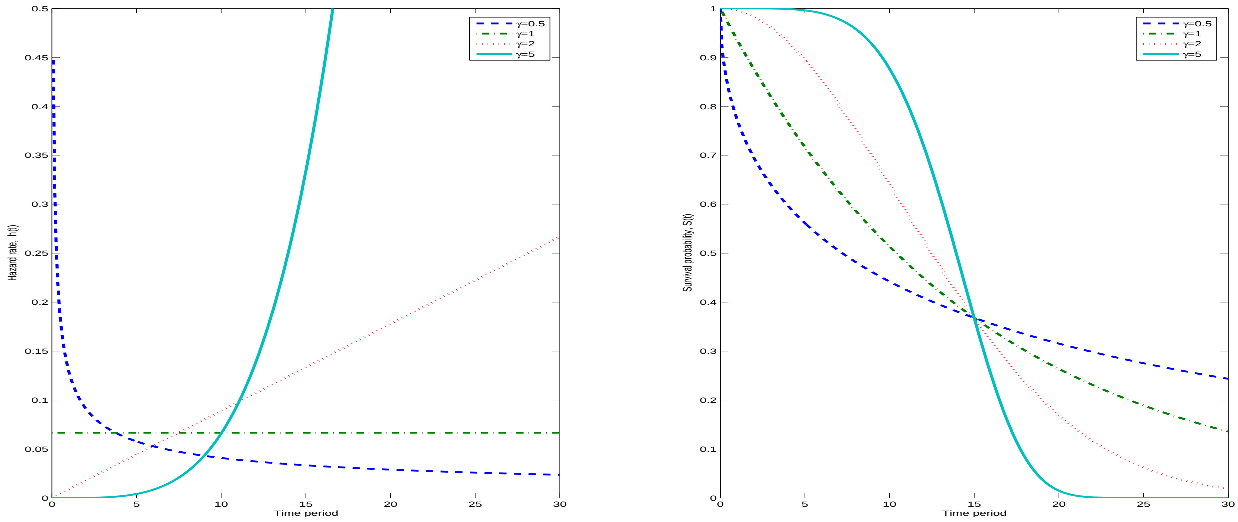

Figure 2.2: Weibull distribution, $\lambda = 0,0067$

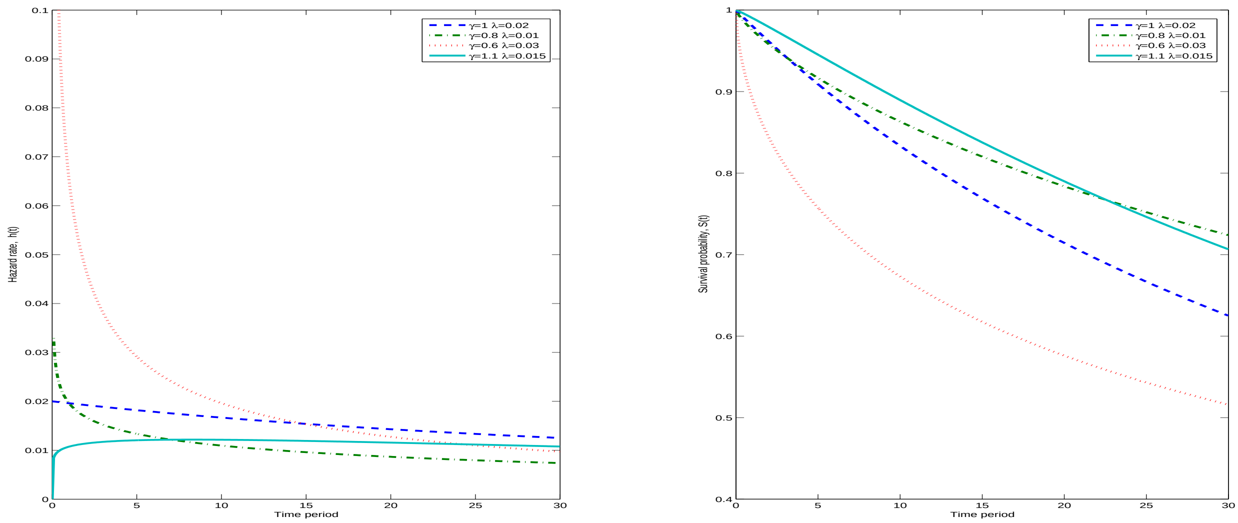

Figure 2.3: Loglogistic distribution

stochastic process by introducing the *counting process*:

$$N(t) := \sum_i 1_{\{\tau_i \leq t\}}, \tag{2.15}$$

which gives the number of stopping times before time t.

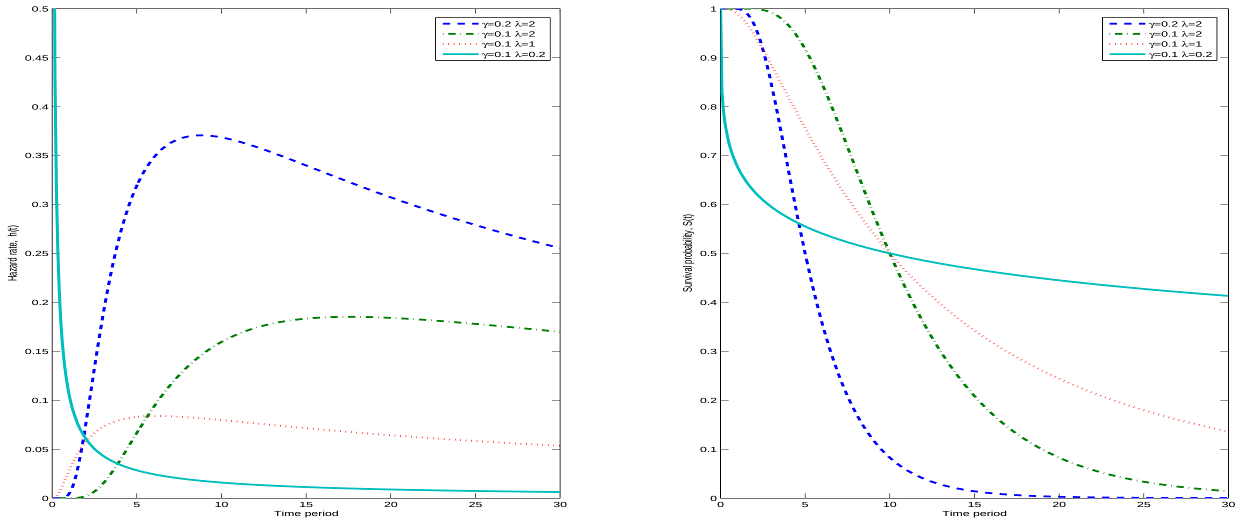

Figure 2.4: Lognormal distribution

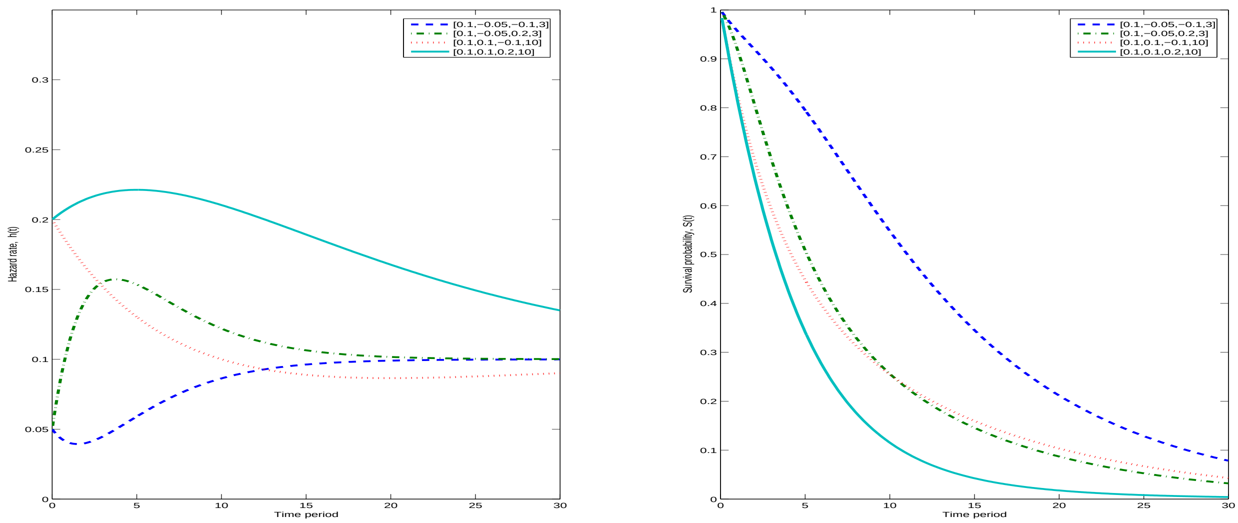

Figure 2.5: Nelson-Siegel

Poisson Process

Now, let us define the (homogeneous) Poisson process with constant rate λ.

Definition 2.3. Let $t_1, t_2, \ldots$ be independent, exponentially distributed random variables (with rate (λ)). Let $T_n = t_1 + \cdots + t_n$ for $n \geq 1$ and define $N(s) = \max\{n : T_n \leq s\}$.

The following definition relates the intensity with the Poisson process.

Definition 2.4. The **homogenous Poisson process** with constant intensity λ is a counting process with

$$P[N(t) - N(s) = k] = \frac{1}{k!}(\lambda(t-s))^k e^{-\lambda(t-s)},$$

where $s < t$ and $k = 0, 1, \ldots$.

Lemma 2.1. *$N(t+s) - N(s)$, $t \geq 0$ is a Poisson process with rate λ and independent of $N(r)$, $0 \leq r \leq s$.*

Proof: See p.132 of Durrett [Dur99].

Theorem 2.1. *If $\{N(s), s \geq 0\}$ is a Poisson process, then*

1. *$N(0) = 0$*
2. *$N(t+s) - N(s) = Poisson(\lambda t)$ and*
3. *$N(t)$ has independent increments.*

Conversely, if (1), (2) and (3) hold, then $\{N(s), s \geq 0\}$ is a Poisson process.

Definition 2.5. The **inhomogeneous Poisson process** is a generalisation of a homogenous Poisson process with a time-varying intensity. We call N an inhomogeneous process with deterministic intensity process $\lambda(t)$, if the increments $N(t) - N(s)$ are independent for $s < t$ and we have

$$P[N(t) - N(s) = k] = \frac{1}{k!}\left(\int_s^t \lambda(u)du\right)^k e^{-\int_s^t \lambda(u)du}.$$

Definition 2.6. The **Cox process** $N(t)$ with intensity $\lambda = \{\lambda(t)\}_{t \geq 0}$ is a generalisation of the inhomogeneous Poisson process in which the intensity is random, but with the restriction that conditional on the realisation of λ, $N(t)$ is an inhomogeneous Poisson process. Therefore, the Cox process is also called conditional Poisson process, or doubly-stochastic Poisson process.

Continuous-time Markov Chains

Let η_t, $t \in \mathbb{R}^+$, be a right-continuous stochastic process on the probability space $(\Omega, \mathcal{G}, P)$ with values in the finite set $\mathcal{K}$ and let $\mathbb{F}^\eta$ be the filtration generated by this process. Also, let $\mathbb{G}$ be some filtration such that $\mathbb{F}^\eta \subseteq \mathbb{G}$.

Definition 2.7. A process η is a *continuous-time* $\mathbb{G}$-Markov chain if for any arbitrary function $f : \mathcal{K} \to \mathbb{R}$ and any $s, t \in \mathbb{N}^+$ we have

$$E^P[f(\eta_{t+s}) \mid \mathcal{G}_t] = E^P[f(\eta_{t+s}) \mid \eta_t].$$

A continuous-time $\mathbb{G}$-Markov chain η is said to be *time-homogenous* if, in addition, for any $s, t, u \in \mathbb{N}^+$ we have

$$E^P[f(\eta_{t+s}) \mid \eta_t] = E^P[f(\eta_{u+s}) \mid \eta_u].$$

Definition 2.8. A two-parameter family $\mathcal{P}(t, s)$, t, $s \in \mathbb{R}^+$, $t \leq s$, of stochastic matrices is called the family of *transition probability matrices* for the $\mathbb{G}$-Markov chain η under P if for every $t, s \in \mathbb{R}^+$, $s \leq t$,

$$P[\eta_t = j \mid \eta_s = i] = p_{ij}(s, t), \quad \forall i, j \in \mathcal{K}.$$

In particular, the equality $\mathcal{P}(t, t) = I$ is satisfied for every $t \in \mathbb{R}^+$.

Definition 2.9. The one-parameter family $\mathcal{P}(t)$, $t \in \mathbb{R}^+$, of stochastic matrices is called the family of transition probability matrices for the time-homogeneous $\mathbb{G}$-Markov chain η under P if for every $t, s \in \mathbb{R}^+$,

$$P[\eta_{s+t} = j \mid \eta_s = i] = p_{ij}(t), \quad \forall i, j \in \mathcal{K}. \tag{2.16}$$

Let us now introduce an important assumption on the family $\mathcal{P}(t)$, namely that this family is right-continuous at $t = 0$, implying that

$$\lim_{t \downarrow 0} \mathcal{P}(t) = \mathcal{P}(0).$$

With the Chapman-Kolmogorov equation[3], we have

$$\lim_{s \to 0} \mathcal{P}(t + s) = \mathcal{P}(t), \quad \forall t > 0,$$

[3] $\mathcal{P}(t + s) = \mathcal{P}(t)\mathcal{P}(s) = \mathcal{P}(s)\mathcal{P}(t), \quad \forall s, t, \in \mathbb{R}^+$

hence

$$\lim_{s\to 0} P[\eta_{t+s} = j \mid \eta_t = i] := \delta_{ij}, \quad \forall i,j \in \mathcal{K}, t > 0.$$

Furthermore, $\mathcal{P}(t)$ is right-continuous, implying that it is right-differentiable, the following limit exists for every $i, j \in \mathcal{K}$,

$$\lambda_{ij} := \lim_{t\downarrow 0} \frac{p_{ij}(t) - p_{ij}(0)}{t} = \lim_{t\downarrow 0} \frac{p_{ij}(t) - \delta_{ij}}{t}. \tag{2.17}$$

Note that for every $i \neq j$ we have $\lambda_{ij} \geq 0$ and $\lambda_{ii} = -\sum_{j=1,j\neq i}^{K} \lambda_{ij}$. We call the matrix $\mathbf{\Lambda} := [\lambda_{ij}]_{1\leq i,j\leq K}$ the *infinitesimal generator matrix* for a Markov chain associated with the family $\mathcal{P}(\cdot)$ via expression (2.16). This matrix is also called the *intensity matrix* since each entry λ_{ij} represents the intensity of transition from state i to state j.

We can derive the *backward Kolmogorov equation*

$$\frac{d\mathcal{P}(t)}{dt} = \mathbf{\Lambda}\mathcal{P}(t), \quad \mathcal{P}(0) = \mathbf{I}, \tag{2.18}$$

and the *forward Kolmogorov equation*

$$\frac{d\mathcal{P}(t)}{dt} = \mathcal{P}(t)\mathbf{\Lambda}, \quad \mathcal{P}(0) = \mathbf{I}, \tag{2.19}$$

where at $t = 0$, we take right-hand side derivatives. Both equations have the same unique solution:

$$\mathcal{P}(t) = \exp(t\mathbf{\Lambda}) := \sum_{n=0}^{\infty} \frac{\Lambda^n t^n}{n!}, \quad t \in \mathbb{R}^+. \tag{2.20}$$

Definition 2.10. A state $K \in \mathcal{K}$ is called *absorbing* for time-homogeneous Markov chain $\eta_t, t \in \mathbb{R}^+$, if the following equation holds:

$$P[\eta_s = K \mid \eta_t = K] = 1, \quad \forall t, s \in \mathbb{R}^+, s \geq t. \tag{2.21}$$

2.2.2 Intensity Models and Valuation of the Corporate Bonds

In this subsection, we present the well-known approaches for the intensity based credit risk models and provide the corresponding risky corporate bond formulas. The intensity based models assume that the default arrival time τ is the first jump time of a Poisson arrival process. However, depending on whether the intensity of the Poisson process is deterministic or stochastic, these models can also be subdivided into categories.

An example for a deterministic intensity is the model by Jarrow and Turnbull [JT95](hereafter, JT model), where the authors assume a constant intensity, i.e., $\lambda(t) = \lambda$. This assumption brings easiness in calibration to the market data, however, it is not very realistic in real world. In this setting, a constant intensity rate of 5% will indicate a mean arrival rate of 5 defaults per 100 obligors, conditioning on all current information available. Expected time to default of an obligor is $1/\lambda = 20$ years, where the cumulative probability of default in one-year equals $1 - \exp(-0.05) = 4.88\%$.

In practice, generally the intensity is assumed to be time-dependent, e.g., it can be described with a linear function,

$$\lambda(t) = a + bt, \tag{2.22}$$

or with a piecewise constant function

$$\lambda(t) = a_1 + a_2 1_{\{t \geq t_1\}} + a_3 1_{\{t \geq t_2\}} + \ldots \tag{2.23}$$

An innovative default intensity model is proposed by Jarrow et al. [JLT97](hereafter, JLT model), where the authors include the *credit rating* information to the risk pricing. In JLT model, the authors characterise the default with a finite state Markov process in the credit rating of the firm. Markovian credit migration process has the state space

$$\mathcal{K} = \{1, 2, \ldots, K\},$$

where 1 represents the highest credit rating class and K represents the default state. The intensities $\lambda_{i,j}$ $i = 1, \ldots, K-1$, and $j = 1, \ldots K$ are the transition rates of jumping from credit class i to credit class j, where these intensities are the off-diagonal elements for the generator matrix of the Markov migration process.

Lando, [Lan98] generalises JLT model and instead of constant intensities, he assumes stochastic intensities which are driven by some state variable X. Therefore, the author uses a Cox process in order to model the default event. Moreover, assuming that the state variable X is a Markov process, we have

$$\lambda_{i,j}(t) = \Lambda_{i,j}(X_t),$$

where $\Lambda_{i,j}$ is a continuous non-negative function on $\mathbb{R}^d$, which maps the risk factors X into the transition intensity.

Mathematically, the relationship between the risk-neutral short rate process and the risk-free ZCB price corresponds to the relationship between the risk neutral intensity process and the survival probability. Therefore, this analogy allows us to model the stochastic intensity with the term-structure models for short rate. Duffie and Singleton [DS03] present these models in the third chapter of their book. Examples to this kind of intensity modelling approaches are well known from the interest rate literature, namely the Cox-Ingersoll-Ross (CIR, hereafter) [CIR85] and Heath-Jarrow-Morton (HJM, hereafter) [HJM92] frameworks. A recent application of the HJM framework using Cheyette type specification for capturing the stochasticity of the credit spreads is introduced by Acar et al. [AAK07]. Moreover, Duffie and Singleton contribute to credit risk modelling with affine processes, adopting the Cox process approach of Lando [Lan98] model. Hence, they assume that the state process X and the non-negative function Λ are affine, implying closed form solutions for the PDs.

Due to rapid growth in the credit derivative markets, pricing of multi name credit products (e.g., CDO, CDO^2) bring new modelling approaches to the stochastic intensity. The recent papers by Chapovsky et al. [CRT06], and Papageorgiou and Sircar [PS07b] propose multiscale intensities, where the authors present a review of the stochastic models in the latter. Using a Markov chain is introduced by Kraft and Steffensen [KS06], extended by De Kock et al. [KKS07b] for the CDO pricing. Another paper to valuation of multi-name credit derivative contracts in a Markovian framework is by Di Graziano and Rogers [GR06].

Jarrow and Turnbull Model

Jarrow and Turnbull [JT95] assume a constant intensity λ, implying statistical independence of the default event and the short rate process.

Now, let us remember some bond-pricing mathematics. We have

$$b(t) = \exp\left(\int_0^t r(s)ds\right) \text{ and} \tag{2.24}$$

$$B(t,T) = E_t^Q\left[\frac{b(t)}{b(T)}\right], \tag{2.25}$$

where $r(t)$ is the risk-free short rate and the conditional expectation under the martingale measure Q is denoted with $E_t^Q[\cdot] \equiv E^Q[\cdot \mid \mathcal{F}_t]$. The risk-free MMA is represented by $b(t)$ and $B(t,T)$ is the price of a risk-free ZCB at time t, with maturity time T, $T \geq t \geq 0$.

The JT model gives the price of a risky ZCB at time t with maturity time T, $\bar{B}(t,T)$ as

$$\bar{B}(t,T) = E_t^Q\left[\frac{b(t)}{b(T)}\left(R 1_{\{\tau \leq T\}} + 1_{\{\tau > T\}}\right)\right], \tag{2.26}$$

where R is the exogenously given, constant recovery rate $R \in [0,1]$ and τ is the random default time. Assuming that the short rate process $r(t)$ and the default process are statistically independent under Q and that at default time τ the claim holders receive a fraction of the equivalent risk-free ZCB, i.e., $\bar{B}(\tau,T) = RB(\tau,T)$ (Recovery of treasury or equivalent recovery assumption), we may rewrite (2.26) as:

$$\begin{aligned} \bar{B}(t,T) &= E_t^Q\left[\frac{b(t)}{b(T)}\right] \cdot E_t^Q\left[\left(R 1_{\{\tau \leq T\}} + 1_{\{\tau > T\}}\right)\right] \\ &= B(t,T)\left[R + (1-R) ps^Q(t,T)\right]. \end{aligned} \tag{2.27}$$

Here $ps^Q(t,T)$ represents the martingale probability of survival until T, conditional on survival to time t. Note that with the constant intensity λ, it is given by

$$ps^Q(t,T) = e^{-\lambda(T-t)}. \tag{2.28}$$

For a detailed analyses of the JT model, we refer the reader to Baydar [Bay04].

Jarrow, Lando and Turnbull Model

Jarrow et al. [JLT97] extend JT model via including the credit rating information into the risky bond price. Since credit rating is a crude measure of credit quality and a rough aggregation of credit information, it is an important ingredient both to credit risk models and to risk management issues. The popular credit rating classifications are the ones published by credit rating agencies like Moody's (highest rate:Aaa, lowest rate: C) and Standard & Poor's (S&P hereafter, with highest rate: AAA lowest rate: CCC), and those by Fitch. In JLT model 1 represents the highest rating grade and K represents the default state (the absorbing state in Markovian setting). Within this framework, we define the default time as follows:

Definition 2.11. Suppose the default time of a firm is the first time that the firm credit migration (or credit transition) process $\eta(t)$ hits the absorbing (default) state, e.g., K.

Considering a continuous time framework, we define the *default time* as follows:

$$\tau = \inf\{t \geq s : \eta(t) = K\}, \forall s \in \mathbb{R}^+$$

Let us assume that the Kth state is absorbing, then we will have the following generator matrix under the physical probability measure as follows:

$$\mathbf{\Lambda} = \begin{pmatrix} -\lambda_1 & \dots & \lambda_{1,K-1} & \lambda_{1,K} \\ . & \dots & . & . \\ \lambda_{K-1,1} & \dots & -\lambda_{K-1} & \lambda_{K-1,K} \\ 0 & \dots & 0 & 0 \end{pmatrix} \tag{2.29}$$

where $\lambda_{ij} \geq 0$ for all i, j and

$$\lambda_i = \sum_{\substack{j=1 \\ j\neq i}}^{K} \lambda_{ij} \quad \text{for } i = 1, \dots, K.$$

Proposition 2.1. *The generator matrix under the equivalent martingale measure is given by:*

$$\mathbf{\Lambda}^Q(t) = \mathbf{U}(t)\mathbf{\Lambda}, \tag{2.30}$$

where $\mathbf{U}(t) = \text{diag}(\mu_1(t), \dots, \mu_{K-1}(t), 1)$ *is a* $K \times K$ *diagonal matrix, whose first* $K-1$ *entries are strictly positive deterministic functions of* t *satisfying*

$$\int_0^T \mu_i(t)dt < +\infty \quad \textit{for } i = 1, \dots, K-1.$$

The entries $(\mu_1(t), \dots, \mu_{K-1}(t), 1)$ can be interpreted as *risk premiums*, which are adjusting the actual probabilities into the probabilities used in valuation process. These risk premiums will be analysed in detail in Subsection 2.5 for Turkish sovereign.

Let us denote the transition matrix under EMM from time t to T with $\mathbf{Q}(t,T)$ whose (i,j)th entry is $q_{ij}(t,T) = Q[\eta(T) = j \mid \eta(t) = i]$, $0 \leq t \leq T$. We will get $\mathbf{Q}(t,T)$ from the solutions to the Kolmogorov differential equations below:

$$\frac{\partial \mathbf{Q}(t,T)}{\partial t} = -\mathbf{\Lambda}^Q(t)\mathbf{Q}(t,T) \text{ and} \tag{2.31}$$

$$\frac{\partial \mathbf{Q}(t,T)}{\partial T} = \mathbf{Q}(t,T)\mathbf{\Lambda}^Q(T), \text{ with the initial condition } \mathbf{Q}(t,t) = \mathbf{I}. \tag{2.32}$$

The credit rating process is still Markovian under the assumption with (2.30) but *time inhomogeneous* here. The process is time homogeneous only when the following equation holds:

$$\mathbf{\Lambda}^Q = \mathrm{diag}(\mu_1, \dots, \mu_{K-1}, 1)\mathbf{\Lambda} \tag{2.33}$$

where $\mu_1, \dots, \mu_{K-1}$ are strictly positive constants. In this case, the solution to Kolmogorov equations are easy to calculate and the solution is

$$\mathbf{Q}(t, T) = \exp(\mathrm{diag}(\mu_1, \dots, \mu_{K-1}, 1)\mathbf{\Lambda}(T - t)).$$

Proposition 2.2. *Let the firm have rating i at time t, $\eta_t = i$ and define the default time with $\tau = \inf\{s \geq t : \eta_s = K\}$, then the probability of survival until T, given survival to time t is*

$$ps_i^Q(t, T) = \sum_{j \neq K} q_{ij}(t, T) = 1 - q_{iK}(t, T)$$

Hence, we can write the price of a risky ZCB, which has the rating $i \in \{1, \dots, K-1\}$ with Recovery of Treasury convention as:

$$\bar{B}^i(t, T) = B(t, T)[R + (1 - R)(1 - q_{iK}(t, T))]. \tag{2.34}$$

The estimation techniques of the transition probability matrices are explained by Lando [Lan02] for corporate debt, whereas Hu et al. [HKP02] introduce a sovereign credit risk specific estimation methodology.

Lando Model

Lando [Lan98] generalises the JLT model and uses doubly stochastic Poisson process for modelling the default time. With this setting, one may relax the assumption that the default process and risk-free term structure are independent. This generalisation also allows the credit spreads to fluctuate randomly even between rating transitions. We introduce the state variable X and randomise the default intensities depending on X, where X reflects the changes in economic conditions determining the rating transition intensities. Let us define the generator matrix

$$\mathbf{\Lambda_X}(t) = \begin{pmatrix} -\lambda_1(X_t) & \dots & \lambda_{1,K-1}(X_t) & \lambda_{1,K}(X_t) \\ \vdots & . & \vdots & \vdots \\ \lambda_{K-1,1}(X_t) & \dots & -\lambda_{K-1}(X_t) & \lambda_{K-1,K}(X_t) \\ 0 & \dots & 0 & 0 \end{pmatrix} \tag{2.35}$$

and assume that

$$\lambda_i(X_t) = \sum_{\substack{j=1 \\ j\neq i}}^{K} \lambda_{ij}(X_t), \quad i = 1, \dots, K-1 \quad \lambda_{i,j} \geq 0.$$

With the construction above, the probability that the firm will start from rating class 1 and jump to a different class or default within the small time interval dt is $\lambda_1(X_t)dt$. Further, conditional on the evolution of the state variables, we obtain a non-homogeneous Markov chain with the transition probabilities satisfying

$$\frac{\partial \mathbf{Q}_X(t,T)}{\partial t} = -\mathbf{\Lambda}_X(t)\mathbf{Q}_X(t,T).$$

Unfortunately, we can *not* say the solution to the differential equation is

$$\mathbf{Q}_X(t,T) = \exp\left(\int_t^T \mathbf{\Lambda}_X(u)du\right), \tag{2.36}$$

since for only square matrices $\mathbf{A}$ and $\mathbf{B}$ which *commute* we can write

$$\exp(\mathbf{A}+\mathbf{B}) = \exp(\mathbf{A})\exp(\mathbf{B}).$$

In order to ensure that the intensity measures for different intervals commute, we assume that they have a common basis of eigenvectors. Hence, let us assume that $K \times K$ generator matrix $\mathbf{\Lambda}$ is given and it permits a diagonalisation

$$\mathbf{\Lambda} = \mathbf{B}\mathbf{D}\mathbf{B}^{-1},$$

with $\mathbf{D} = \text{diag}(d_1, \dots, d_{K-1}, 0)$ is the diagonal matrix of eigenvalues. Let μ be a *scalar-valued* positive function defined on the state space of the state variable X and the local intensity is defined as

$$\mathbf{\Lambda}_X(t) = \mathbf{\Lambda}\mu(X_t) = \mathbf{B}\mathbf{D}\mu(X_t)\mathbf{B}^{-1},$$

which corresponds to considering a one-dimensional scalar multiple of the generator. Moreover, we define the $K \times K$ diagonal matrix

$$\mathbf{E}_X(t,T) = \begin{pmatrix} \exp(d_1 \int_t^T \mu_1(X_u)du) & 0 & \dots & 0 \\ 0 & \cdot & \dots & 0 \\ \vdots & \dots & \exp(d_{K-1} \int_t^T \mu_{K-1}(X_u)du) & 0 \\ 0 & \dots & & 0 \end{pmatrix}.$$

Then, we will have $\mathbf{Q}_X(t,T) = \mathbf{B}\mathbf{E}_X(t,T)\mathbf{B}^{-1}$, hence we can compute the unconditional migration matrix $\mathbf{Q}(t,T)$ as the expected value of $\mathbf{Q}_X(t,T)$. The survival probability conditionally starting with the rating i will be

$$1 - q_X(t,T)_{iK} = \sum_{j=1}^{K-1} \beta_{ij} \exp\left(d_j \int_t^T \mu(X_u)du \right),$$

where

$$\beta_{ij} = -b_{ij} b_{jK}^{-1},$$

and b_{ij} is the (i,j)th value of the matrix $\mathbf{B}$ and $q_X(t,T)_{iK}$ is the (i,K)th entry of the transition probability matrix $\mathbf{Q}_X(t,T)$.

Now, consider the price of a defaultable ZCB at time t maturing at time T, issued by a firm with credit class i using zero recovery assumption

$$\begin{aligned}
\bar{B}^i(t,T) &= E_t^Q\left[\exp\left(-\int_t^T r(X_s)ds \right) (1 - q_X(t,T)_{i,K}) \right] \\
&= \sum_{j=1}^{K-1} \beta_{ij} E_t^Q \left[\exp\left(\int_t^T (d_j \mu(X_s) - r(X_s))ds \right) \right],
\end{aligned}$$

where we denote the short rate depending on the state process with $r(X_s)$. If $\mu(X_s)$ is an affine process, we can compute it easily.

2.3 Valuation of the Sovereign Credit Default Swaps

In Section 2.2, we illustrated the intensity based credit risk models, which are mainly used for valuation of risky corporate bonds as well as extracting the risk-neutral PD for the financial obligors. In this section, our aim is to introduce the state of the art in valuation of the CDS contract, when the reference asset is the sovereign debt. Although modelling the corporate and the sovereign debt should be treated distinctly (see Duffie et al. [DPS03] and Andritzky [And06]), the valuation of the sovereign and corporate CDS contracts is quite similar.

The CDS contract (also called *credit swap* or *default swap* in different sources) transfers the potential loss on the reference asset that can result from specific credit events. Depending on the reference asset, a CDS is named the corporate CDS, or sovereign CDS. Since we are analysing the CDS contracts written on the Turkish Eurobonds maturing in 2030 and denominated in the USD, we explain the valuation of the sovereign CDS in a simple modelling approach by O'Kane and Turnbull [OT03].

The contract consists of two parties; the protection buyer (**B**) and the protection seller (**S**). Moreover, CDS has two legs; namely the *premium leg* and the *protection leg*. The premium leg stands for the payments transfered by the **B** to the **S**. The premium leg is the periodic payments[4], as percentages of the notional on the issue date until whichever occurs first: the reference asset defaults and the CDS contract terminates or CDS contract matures without any credit event. Alternatively, the investor may decide to make an up-front premium payment. With 2003 ISDA definitions, the premiums are paid on dates 20th March, June, September, and December (if quarterly based), independent from the inception date for the corporate CDS contracts. If the contract is made between those dates, the premium is adjusted accordingly, whereas if the contract starts on those dates then the first premium is paid on the next payment date. Upon a default between these payment dates, **B** requires to pay the part of the premium payment that has accrued since the last payment, which is called the *accrued premium payment*.

The protection leg refers to the potential payment (upon the credit event of the reference asset) is done to the **B** by the **S**. At the inception date, the *default payment* is unknown and generally specified as physical delivery of the reference asset (Turkish sovereign bond) against repayment at par. In Figure 2.6, we see the pay off structure of the product.

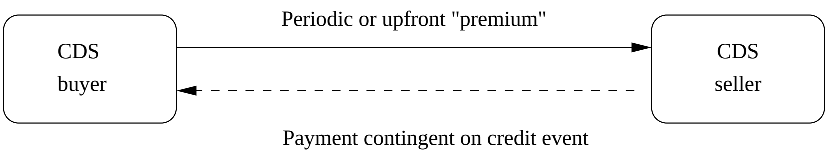

Figure 2.6: CDS payoff shema

Example 2.1. We consider a sovereign CDS with the following characteristics:

- Swap parties: **B** (protection buyer) and **S** (protection seller)
- Inception[5]: 20th March, 2007

[4]This periodic payment is also called swap rate, swap spread or swap premium.
[5]This is the date/time where the coverage under the insurance contract takes effect.

- Maturity: 5 years

- Reference asset: Eurobond of Turkish government maturing in 2030, denominated in the USD

- Notional amount: 100 million USD

- Credit event: obligation acceleration, failure to pay, restructuring/renegotiation, repudiation/moratorium

- Swap rate: 90 BPS (= 900000 USD) per annum, first payment on 20 September 2007

If the Turkish government does not suffer from a credit event until 20th March 2012: **B** pays $5 \times 2 \times 450000$ USD to **S** at the respective premium payment dates (first premium payment is on 20th September 2007) and receives nothing from **S**. If there is a credit event on 3rd March 2010: **B** pays to **S** $[(2\times 2)+1]\times 450000$ USD at the respective coupon dates and a fraction of the premium accrued from 21th September 2009 until 3rd March 2010. In return, **B** delivers the defaulted Eurobond to **S**, who pays 100 million USD (notional value of the bond) as described in the physical settlement feature. Hence, **B** does not suffer a loss due to credit event of Turkish government. This protection of course requires a fair pricing formula, which will be explained in details in the next subsection.

We have two pricing problems here:

- when making markets, we are interested in the fair swap rate at the inception of the contract, i.e., $C_{DS}(t_0, t_N)$.

- when hedging or marking-to-market[6], we are interested in the market value of the swap, i.e., $C_{DS}(t_v, t_N)$, which need not to be the same with the contractual rate, i.e., $C_{DS}(t_0, t_N)$ due to changing interest rates and credit quality of the reference asset.

[6]Recording the price or value of a security, portfolio, or account on a daily basis and calculate profits and losses or to confirm that margin requirements are being met.

We are focusing on the first problem in this section. Generally, counterparty risk is not taken into account when determining the deal prices. A good reference about determining the corporate CDS rate would be Hull and White [HW00], where the authors value a binary CDS and a plain vanilla CDS under the assumption that there is no counterparty risk. Duffie [Duf99] uses the Floating Rate Note (FRN) as reference entity to create synthetic CDS cash flows. Moreover, Brigo and Alfonsi [BA05] use a two-dimensional shifted square root diffusion model with a stochastic intensity framework. Jarrow and Yildirim [JY02] provide a simple analytic formula for valuation of the CDS when the market and credit risk are correlated. Some papers about empirical studies of corporate CDSs are Cossin and Nerin [CN02], Houweling and Vorst [HV05], and Skinner and Diaz [SD03].

In this chapter, we use the following notations:

- t_v : date of valuation of the CDS
- $n = 1, \ldots, N$: number of payments and $t_1, \ldots, t_N$: the dates for CDS premium payments, where t_N is the maturity date of the CDS
- $C_{DS}(t_0, T)$: the contractual swap rate on time t_0, when the maturity of the CDS is T, (in a new contract $t_v = t_0$)
- $ps(t_v, T)$: the forward probability of survival from t_v until T, given survival to t_v
- $pd(t_v, T) := 1 - ps(t_v, T)$ the forward probability of default at time T, given survival to t_v
- $PS(t_n)$: the cumulative probability of survival until t_n
- $PD(t_n) := 1 - PD(t_n)$ the cumulative probability of default by time t_n
- R: expected recovery rate under the risk-neutral measure
- $r(t)$: short interest rate process (LIBOR for USD)
- $D(t_v, T)$: the discount factor on t_v for time T
- $\lambda(t)$: the intensity rate (or hazard rate) of the credit event

- $PV_{Protection\ Leg}(t_v, T)$: present value of the protection leg with CDS maturity time T
- $PV_{Premium\ Leg}(t_v, T)$: present value of the premium leg with CDS maturity time T
- $\Delta(t_{n-1}, t_n, C)$: the day count fraction between dates t_{n-1} and t_n using chosen convention C (e.g., 30/360, meaning 30 days in a month and 360 days in a year, for the details see ISDA definitions.)

Determining the fair price of a CDS contract, requires the following algorithm:

1. Choose an appropriate credit risk model for determining the term structure of PDs.
2. Construct the zero curve (for discount factors).
3. Set the CDS contract details (accrued payment assumptions, the delivery type on default, day count conventions, etc.).
4. Fix the expected recovery rate under risk neutral measure.
5. Construct the hazard rate term structure (ideally from market CDS rates).
6. Determine the present values of the protection leg and the premium leg.
7. Calculate the fair value of the CDS.

2.3.1 Sovereign CDS Valuation with Deterministic Intensity

As mentioned before, pricing of sovereign CDS is similar to corporate CDS, hence we may imitate the pricing techniques for a corporate CDS presented by O'Kane and Turnbull [OT03]. Hence, with a deterministic intensity as in the JT model, the forward PS is given by

$$ps(t_v, T) = \exp\left(-\int_{t_v}^{T} \lambda(s)ds\right). \tag{2.37}$$

The general pricing rule of the swap contracts tells the present values of the premium leg and the protection leg should be equal to each other on the valuation date. Thus, we have

$$PV_{Protection\ Leg}(t_v, t_N) = PV_{Premium\ Leg}(t_v, t_N), \tag{2.38}$$

where the expected present value of the premium leg is given as

$$PV_{Premium\ Leg}(t_v, t_N) = C_{DS}(t_0, t_N) \sum_{n=1}^{N} \Delta(t_{n-1}, t_n, C) D(t_v, t_n) ps(t_v, t_n), \tag{2.39}$$

and the expected present value of the protection leg is

$$PV_{Protection\ Leg}(t_v, t_N) = (1 - R) \int_{t_v}^{t_N} D(t_v, s) ps(t_v, s) \lambda(s) ds. \tag{2.40}$$

Note that the accrued payment upon a default between two premium dates is ignored in (2.39). Considering the protection fee, that has accrued from the last premium date to the time of default, calculated as the sum over all premium periods from $n = 1$ to the final one $n = N$; (2.39) modifies to

$$C_{DS}(t_0, t_N) \sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} \Delta(t_{n-1}, s, C) D(t_v, s) ps(t_v, s) \lambda(s) ds. \tag{2.41}$$

Here, probability of surviving from t_v to each time s and defaulting in the next small time interval ds is given by $ps(t_v, s)\lambda(s)ds$. This integral should be discretised daily since the premiums are calculated on a daily basis. Since this brings complexity in numerics, we assume that it is continuous and that if the default occurs between two premium dates, then the premium accrued is the half of the full premium to be paid at the end of the premium payment interval. Hence, we approximate (2.41) with

$$\frac{C_{DS}(t_0, t_N)}{2} \sum_{n=1}^{N} \Delta(t_{n-1}, t_n, C) D(t_v, t_n) [ps(t_v, t_{n-1}) - ps(t_v, t_n)]. \tag{2.42}$$

The term $[ps(t_v, t_{n-1}) - ps(t_v, t_n)]$ stands for the probability that the obligor will default between the dates t_{n-1} and t_n. Summing this difference per each time interval $[t_{n-1}, t_n]$, $n = 1, \ldots, N$, we will have the obligors default probability during the life of the CDS. Since we assume that the accrued premium is the half of the full premium, division by two and discounting it from the end of each accrued payment period explains the formulation of (2.42).

Thus, it follows from (2.39) and (2.42) that the present value of the premium leg including the accrued payment can be approximated by

$$\begin{aligned} PV_{PremiumLeg}(t_v, t_N) \;&=\; C_{DS}(t_0, t_N) \sum_{n=1}^{N} \Delta(t_{n-1}, t_n, C) D(t_v, t_n) \\ &\quad \cdot \left[ps(t_v, t_n) + \frac{1_{PA}}{2} [ps(t_v, t_{n-1}) - ps(t_v, t_n)] \right], \end{aligned} \tag{2.43}$$

where 1_{PA} equals

$$1_{PA} = \begin{cases} 1 & \text{if accrued payment is agreed in CDS contract} \\ 0 & \text{otherwise} \end{cases}$$

The value of the protection leg is calculated with the assumption that the transaction to the protection buyer is made immediately after the notification of the credit event.

We approximate the integral in (2.40) assuming that the default can occur only on a finite number of discrete points, i.e., M, per year. Hence, we will have $M \times t_N$ discrete times labeled as $m = 1, \ldots, M \times t_N$. We approximate (2.40) with

$$(1 - R) \sum_{m=1}^{M \times t_N} D(t_v, t_m)[ps(t_v, t_{m-1}) - ps(t_v, t_m)] \tag{2.44}$$

By decreasing the value of M, we will have less calculations but also less accuracy. When $M = 12$, we will have a monthly discretisation frequency.

In order to have the market implied PS, we now relate these formulas for premium and protection leg to the market quoted swap spreads. For an appropriate fair spread[7] with $t_v = t_0$, the value of the CDS should be 0, hence we have

$$0 = PV_{Protection\ Leg}(t_v, t_N) - PV_{Premium\ Leg}(t_v, t_N)$$

such that

$$\widehat{C_{DS}}(t_v, t_N) = \tag{2.45}$$
$$\frac{(1 - R) \sum_{m=1}^{M \times t_N} D(t_v, t_m)[ps(t_v, t_{m-1}) - ps(t_v, t_m)]}{\sum_{n=1}^{N} \Delta(t_{n-1}, t_n, C) D(t_v, t_n) \left[ps(t_v, t_n) + \frac{1_{PA}}{2}[ps(t_v, t_{n-1}) - ps(t_v, t_n)]\right]}.$$

To illustrate this with an example; say we have a 1Y CDS which has a mid market quote of 75 bp. With semi-annual premium payments and assuming that we do not have the accrued premium payment, we have

$$0.0075 = \frac{(1 - R) \sum_{m=1}^{12} D(0, t_m)(PS(t_{m-1}) - PS(t_m))}{\sum_{n=6,12} \Delta(t_{n-6}, t_n, C) D(0, t_n) PS(t_n)}, \tag{2.46}$$

where we assume that the expected recovery rate and LIBOR discount factors are given, i.e., $R = 0.25$ and assuming a flat zero curve with

$$r = 0.05 \quad \Rightarrow \quad D(0, t_m) = \exp(-0.05 \times (t_m)),$$

[7] The price at which a securities transaction produces neither a gain nor a loss.

we are left only with the unknown $12 + 2$ PS[8]. We plug in (2.37) to calculate the PS to (2.46), note that for $t_v = t_0 = 0$, we have $ps(0, t_n) = PS(t_n)$. We see that it is not possible to extract unknown PS for every time point, hence we must have a simplifying assumption about term structure of the hazard rates. At this point, the need for the bootstrapping methodology, which we explain in the next subsection, shows up.

2.3.2 Generating Hazard Curves with the Bootstrapping Method

In this subsection, we explain how we construct the term structure of the risk neutral intensity rates for pricing a CDS, namely the *bootstrapping* methodology. The fair CDS rate formula in (2.45) with deterministic intensity is already standard in financial industry but the approximation methods to the integrals may imply different results.

Although bootstrapping is a practical method, it has also disadvantages, which are listed by Martin et al. [MTB01] as;

- it is an iterative method, an unreliable CDS market rate, i.e., $\widehat{C_{DS}}(0, j)$ will affect *not only* the extracted intensity λ_j but also the other subsequent intensities λ_{j+1}, $\lambda_{j+2}\ldots$.
- We can have intensities as many as the market swap rates. Typically, the CDS rates for different maturities may not be available. Here, we have to use an interpolation method for the maturities which are not traded. Different interpolation methods may imply different results.
- With the bootstrapping we may even have negative intensities, that are totally nonsense.

Empirical facts[9] show the recovery rate should be modelled in a stochastical framework, due to the relationship between the expected recovery rate and the intensity rate process. Unfortunately, the bootstrapping method separates the recovery and default risk, while we fix the recovery rate under risk neutral measure a priori, then extract the intensities.

[8] This is the upper bound of the unknown terms, when premium dates and the determined default dates in protection leg do not coincide.

[9] Interested reader may see the papers by Bakshi et al. [BMZ04], Das and Hanouna [DH06], Pan and Singleton [PS07a], and Christensen [Chr07] for a stochastic recovery approach.

This is similar to the fractional recovery of face value convention of Duffie [Duf98], and Duffie and Singleton [DS99], where the authors propose a fair swap rate as

$$C_{DS}(t) = (1 - R) f(\lambda^Q(t)). \tag{2.47}$$

However, the fractional recovery of market value convention introduced by Duffie and Singleton [DS99] delivers a CDS pricing formula as follows

$$C_{DS}(t) = f((1 - R)\lambda^Q(t)). \tag{2.48}$$

This implies that the recovery and intensity processes can not be separately identified from the market CDS rates. Leaving the discussion about the recovery rate conventions for a future research problem, we use a valuation formula, that is similar to (2.47). We use a constant recovery rate, i.e., $R = 0.25$, as it is proposed in Pan and Singleton [PS07a].

In our dataset, we have the mid-market quotes of CDSs for the maturities of 1, 2, 3, 5, 7, and 10 years. From each market rate, we can extract only one piece of information. As O'Kane and Turnbull [OT03] indicate, the widely used methodology is assuming the hazard rate term structure as a piecewise constant function of the maturity time. We may also construct it with a piecewise linear hazard rate function, but this typically will not create a big difference, unless we have spreads for many CDS maturities.

Our aim is to find the market-implied (or risk-neutral) constant hazard rates λ_1^Q, λ_2^Q, λ_3^Q, λ_4^Q, λ_5^Q and λ_6^Q via bootstrapping method. Suppose we have the stepwise constant intensity function as follows

$$\lambda^Q(t) := \begin{cases} \lambda_1^Q & \text{if } t \le 1 \\ \lambda_2^Q & \text{if } 1 < t \le 2 \\ \lambda_3^Q & \text{if } 2 < t \le 3 \\ \lambda_4^Q & \text{if } 3 < t \le 5 \\ \lambda_5^Q & \text{if } 5 < t \le 7 \\ \lambda_6^Q & \text{if } t > 7. \end{cases} \tag{2.49}$$

First, we will use the 1Y CDS market spread in order to calculate λ_1^Q, then we use it to calculate λ_2^Q. The iterative method will continue until we have the complete term structure of the intensities.

With semi-annual premium payments and assuming that there is no accrued premium ($1_{PA} = 0$ in (2.43)) and plugging the PS formula given with (2.37) in, we get λ_1^Q by solving

$$\frac{\widehat{C_{DS}}(t_v, t_v + 1Y)}{1 - R} \sum_{n=6,12} \Delta(t_{n-6}, t_n, C) D(t_v, t_n) e^{-\lambda_1^Q \tau_n}$$
$$= \sum_{m=1}^{12} D(t_v, t_m)[e^{-\lambda_1^Q \tau_{m-1}} - e^{-\lambda_1^Q \tau_m}],$$

where with a monthly discretisation frequency ($M = 12$), we have

$$\tau_0 = 0, \ \tau_1 = 0.0833, \ldots, \tau_{12} = 1.$$

This equation can be solved with bisection or gradient-based methods such as Newton-Raphson algorithm. Given λ_1^Q, this can be redone to solve for λ_2^Q using the market rate $\widehat{C_{DS}}(t_v, t_v + 2Y)$. Define τ as time to maturity, i.e., $\tau = T - t_v$ and assume that the hazard rate is constant beyond 10Y maturity, then we have

$$ps(t_v, t_v + \tau)$$
$$= \begin{cases} \exp(-\lambda_1^Q \tau) & \text{if } 0 < \tau \leq 1 \\ \exp(-\lambda_1^Q - \lambda_2^Q(\tau - 1)) & \text{if } 1 < \tau \leq 2 \\ \exp(-\lambda_1^Q - \lambda_2^Q - \lambda_3^Q(\tau - 2)) & \text{if } 2 < \tau \leq 3 \\ \exp(-\lambda_1^Q - \lambda_2^Q - \lambda_3^Q - \lambda_4^Q(\tau - 3)) & \text{if } 3 < \tau \leq 5 \\ \exp(-\lambda_1^Q - \lambda_2^Q - \lambda_3^Q - 2\lambda_4^Q - \lambda_5^Q(\tau - 5)) & \text{if } 5 < \tau \leq 7 \\ \exp(-\lambda_1^Q - \lambda_2^Q - \lambda_3^Q - 2\lambda_4^Q - 2\lambda_5^Q - \lambda_6^Q(\tau - 7)) & \text{if } \tau > 7 \end{cases}$$

Note that these are the risk-neutral probabilities, which include other non-default factors such as liquidity risk premium, spread risk premium and market supply-demand effects. These are generally bigger than the hazard rates implied by historical data. In Section 2.5, we will explain the relationship between historical and risk-neutral default intensities.

Since the market demand and supply play a role in determining the CDS quotes, there is a possibility that the CDS rates may not be monotonously increasing with respect to the maturity of the contract. Therefore, an inverted credit curve may imply negative hazard rates which has no sense and reflects an arbitrage possibility, which can be model dependent (or not). The optimisation method introduced by Martin et al. [MTB01] solve the problem of having negative intensities. We explain this method in the following subsection.

2.3.3 Generating Hazard Curves with the Optimisation Method

In this subsection, we present the method introduced by Martin et al. [MTB01]. The method focuses on extracting the forward conditional default probabilities, i.e., $pd(t_{m-1}, t_m)$ directly from the market CDS rates. Once we have the forward default probabilities, we may construct the term structure of the intensities via the approximation i.e., if $\Delta(t_{m-1}, t_m, C) \rightarrow 0$, then $\frac{pd(t_{m-1},t_m)}{\Delta} \rightarrow \lambda^Q_{m-1}$. Furthermore, the cumulative survival (or default) probabilities, $PS(\cdot)$[10] can be calculated via the recursion:

$$\begin{aligned} PS(t_m) &= PS(t_{m-1}) - [PS(t_{m-1})pd(t_{m-1}, t_m)] \quad m = 1, \ldots, M \times t_N \\ PS(0) &= 1. \end{aligned}$$

Remember that pd is given by

$$pd(t_{m-1}, t_m) = 1 - \exp\left(-\int_{t_{m-1}}^{t_m} \lambda(t)dt\right). \tag{2.50}$$

We further assume that for each time period, we approximate the discount factor by an average, i.e.,

$$D(t_v, t) \approx \frac{1}{2}\left[D(t_v, t_{m-1}) + D(t_v, t_m)\right], \qquad \text{where } t_{m-1} < t < t_m.$$

With this setting, we approach the integral in (2.40) via assuming that the default can occur only on a finite number of discrete points, i.e., M, per year. In a semi-annual discretisation we have $M = 2$. And we label the discrete time points for the CDS with maturity t_N as $m = 0, \ldots, M \times t_N$. Hence, we approximate (2.40) with a sum of $P := M \times t_N$ integrals as we previously did in the bootstrapping method. Using the recursive relation

$$PS(t_{m-1}) - PS(t_m) = PS(t_{m-1})pd(t_{m-1}, t_m),$$

and assuming there is no accrued premium, the market quote for maturity t_N on date $t_v = t_0 = 0$ should hold

$$\begin{aligned} \widehat{C_{DS}}(t_0, t_N) &= \frac{(1-R)\sum_{m=1}^{P} \frac{1}{2}[D(t_0, t_{m-1}) + D(t_0, t_m)]PS(t_{m-1})pd(t_{m-1}, t_m)}{\sum_{n=6,12,\cdots}^{N} \Delta(t_{n-6}, t_n, C)D(t_0, t_n)PS(t_n)} \\ &\equiv C_{DS}(0, t_N; pd_0, pd_1, \ldots, pd_{P-1}). \end{aligned}$$

[10]Note that the probabilities and intensities are the risk-neutral ones, we drop the superscript Q here.

We denote the model price calculated with the extracted pds with $C_{DS}(0, t_N; pd_0, \ldots)$. In order to find the unknown P forward default probabilities, which are labelled as $pd(t_{m-1}, t_m) = pd_{m-1}$, with $m = 1, 2, \ldots, P$, we minimise the objective function given by

$$G(pd_0, pd_1, \ldots, pd_m) = v \sum_{m=1}^{P} d(pd_m; pd_{m-1})^2 + \frac{1}{2} \sum_{j=1}^{K} \left(\frac{\widehat{C_{DS}}(0, j) - C_{DS}(0, j; pd_0, pd_1, \ldots, pd_{P-1})}{\sigma} \right)^2, \tag{2.51}$$

where K denotes the number of CDS contracts with different maturities. Moreover, we assume that the market CDS rates are subject to a Gaussian error. The distance function $d(\cdot)$ in (2.51) is defined by

$$d(q'; q) = \sqrt{(pd' - pd) \ln \frac{pd'}{pd} + (pd - pd') \ln \frac{1 - pd'}{1 - pd}}. \tag{2.52}$$

Note that this function is non-negative

$$d(pd', pd) = 0 \text{ if and only if } pd' = pd.$$

Setting the parameters $v = 10$ and $\sigma = 0.001$ in (2.51) provides a better fit[11] to the market rates.

The interpretation of the objective function defined in (2.51) is that, if the successive pds differ significantly, then the first term will assign a penalty, whereas the second term assigns a penalty for not fitting the market CDS rates. With this setting, v controls the balance between two penalties. The main advantage of this method is unlike the bootstrapping method, we do not have the possibility to have negative hazard rates.

Once we minimise the function in (2.51), we will get the forward probabilities. Then, we can approximate the market-implied intensities via division by the discretisation length, i.e., if $\Delta \to 0$, then $pd(t_m)/\Delta \to \lambda(t_m)$. The number of parameters to be estimated depends on the discretisation frequency. If we have a semi-annual discretisation, i.e., $\Delta \approx 0.5$ for a CDS with 10 year maturity, then we minimise the objective function subject to 20 unknown parameters. Decreasing the length of the discretisation interval

[11] Increasing v results in higher deviations from market rates, see Martin et al. [MTB01]

will lead to precise estimations but this will typically increase the computational costs, e.g., for monthly discretisation we have to perform the optimisation algorithm for 120 parameters for a CDS with 10 years maturity time.

2.4 Data Description and Empirical Analysis

Our data consists of daily bid, ask quotes for sovereign CDS contracts[12], which are available in the maturities of 1, 2, 3, 5, 7, and 10 years. The reference asset is the Eurobond of Turkish sovereign, which is maturing in 2030 and denominated in the USD. For the analyses, we use the mid-market quotes, i.e., $mid\ market := (bid + ask)/2$. The time series of CDS spreads cover the time period from 20 April 2004 to 29 January 2008, which counts for 985 trading days.

The descriptive statistics of the CDS mid-market quotes are given in Table 2.2. During the sampling interval, the average mid-market quote for the CDS with 1 year maturity is 75.2 bp, ranging from 21.8 up to 425 bp. Comparing this with the average market spread for 1 year maturity CDS in Pan and Singleton [PS07a]; calculated as 378.4 bp, we can conclude that the traders were adding larger risk premiums before April 2004, where their sample covers the rates from March 2001 until August 2006. The difference between these two averages is quite high, (approximately 3%) and it indicates that the economical measures get better for Turkey as it had a very high inflation and a volatile interest rate structure in the near past.

maturity	1	2	3	5	7	10
min	21.8	44.9	70	116.5	146.8	176.8
max	425	543.7	612.5	687.5	710	722.3
stdev	60.5	80.8	94.9	99.8	97.1	92.1
median	56.7	90.1	129.5	197.4	240.5	276.7
mean	75.2	120.1	162.9	231.6	270.9	303.5

Table 2.2: Summary statistics for the mid-market quotes of the Turkish CDS rates (in bp).

The CDS spreads show interesting patterns due to the local political (and economical)

[12] The data is downloaded from Bloomberg. Ticker for the CDS contract is CTURK1U, where 1 indicates the maturity of the CDS.

crises as well as the global ones, which had influenced the behaviours of the local and foreign investors in Turkish CDS market. In Figure 2.7, we may observe that there is generally a high comovement among the term structure of CDS spreads. Since exploring the nature and the degree of the comovement (by fitting a factor model) is not our objective, we do not perform a principal component analysis. However, Pan and Singleton [PS07a] find out that the first principal explains over 96% of the variation in Turkish sovereign CDSs.

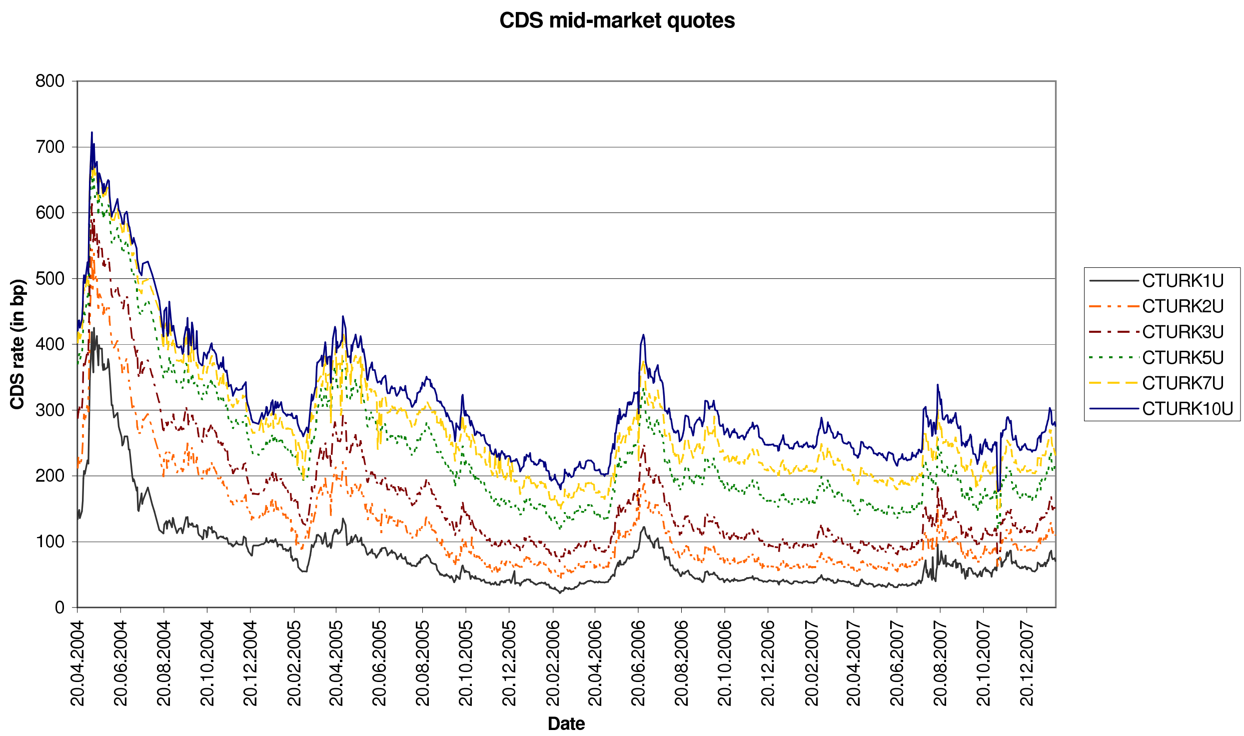

Figure 2.7: The mid-market quotes for different CDS maturities.

Generally, the term structure for the CDS spreads has a positive slope with respect to the increasing maturity. On the other hand, there are some dates that spreads were inverted due to the demand-supply effects in turbulence periods during the local and/or global crises. A recent example would be the subprime mortgage crisis in the USA, which actually started in the last quarter of 2006 and show its enormous effects in 2007 and 2008. The subprime crisis caused the Dow Jones indexes drop to record levels especially in July and August 2007. Turkish markets were also affected by the subprime crises.

Eventually, there were large declines in Istanbul Stock Exchange and in the Turkish Derivatives Exchange Market. Some of the global investment banks and Turkish banks had their biggest losses in their history. These losses had also created high volatility in the Turkish CDS rates as it can be observed on Figure 2.7. Some other important events which had influenced the Turkish markets were the parliamentary elections of Turkey on July 2007 and the presidential elections afterwards. The conflicts between the Turkish government and the USA about the terrorist group PKK located in northern Iraq had also played big role in the volatile structure of the Turkish financial markets, e.g., on 8 November 2007 when the cross border operation of the Turkish army was on discussion, we observe that the CDS prices dropped by up to 80 bp. Furthermore, the CDS rates were affected by the political issues mainly connected during the negotiations between the EU commision and the Turkish government about the conflicts between Cyprus and Turkish Republic of Northern Cyprus.

In Figure 2.8 we observe the mid market quote for the reference asset in the sovereign CDS, namely the Turkish Eurobond with 2030 maturity with respect to the mid market quote for the CDS with one year maturity. Note that the y-axis on the left hand side is for the CDS mid-market quote. As we can observe, they are negatively correlated, where we calculated a correlation coefficient of -86,9% based on 1013 dates in the sampling period.

We illustrate the behaviour of the ask-bid spreads during the sample period in Figure 2.9, where we simply take the difference between the two quotes, i.e., $\widehat{C_{DS}}^{ask} - \widehat{C_{DS}}^{bid}$. In general, bid and ask quotes show the demand-supply effects in the market. As we can observe in Figure 2.9, the biggest spread widening is observed in the second quarter of 2004 on the CDSs with 1 year maturity, which had reached levels up to more than 70 bp. This typically indicates that the supply for the CTURK1U is larger then the market demand on that period, indicating a potential decrease in the corresponding CDS prices. In our sampling period, e.g., on 09th August 2007, the bid quotes for the short term maturities of CDSs (1 year, 2 years and 3 years) are significantly larger than the ask quotes, showing the high market demand for the short term insurance of sovereign risk. This also indicates that default probability of Turkish sovereign is likely to increase, implying the potential rise of the CDS premiums. There are negative ask-bid spreads on the days following 9th August 2007, for the CDS contracts with 10 year maturity corresponding to the

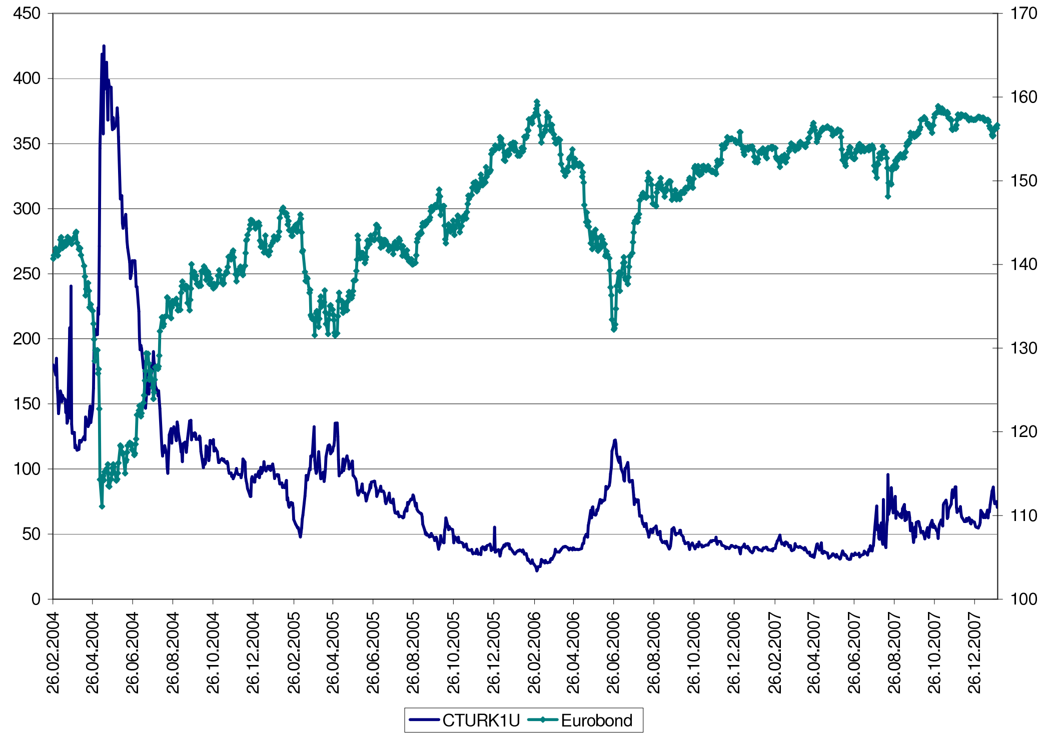

Figure 2.8: The mid-market quotes for the underlying Eurobond vs. CDS with 1 year maturity.

date when the New York Stock Exchange, and eventually, the Istanbul Stock Exchange experienced declines due to the subprime mortgage crisis. The interpretation is that the market expectations for longer terms were not very optimistic on those dates. Table 2.3 gives the descriptive statistics of the bid-ask spreads of the Turkish CDS.

maturity	1	2	3	5	7	10
min	-46.3	-35.3	-22.7	-1	1.7	-5.7
max	73.3	50	55	45	55	50
std	15.9	11	10.9	8.9	11.7	9.8
med	8.5	6	7.3	6.2	6.7	6.7
mean	14.9	11.2	12.2	10.2	12.4	10.9

Table 2.3: Summary statistics for the ask-bid spread of the CDS quotes (in bp).

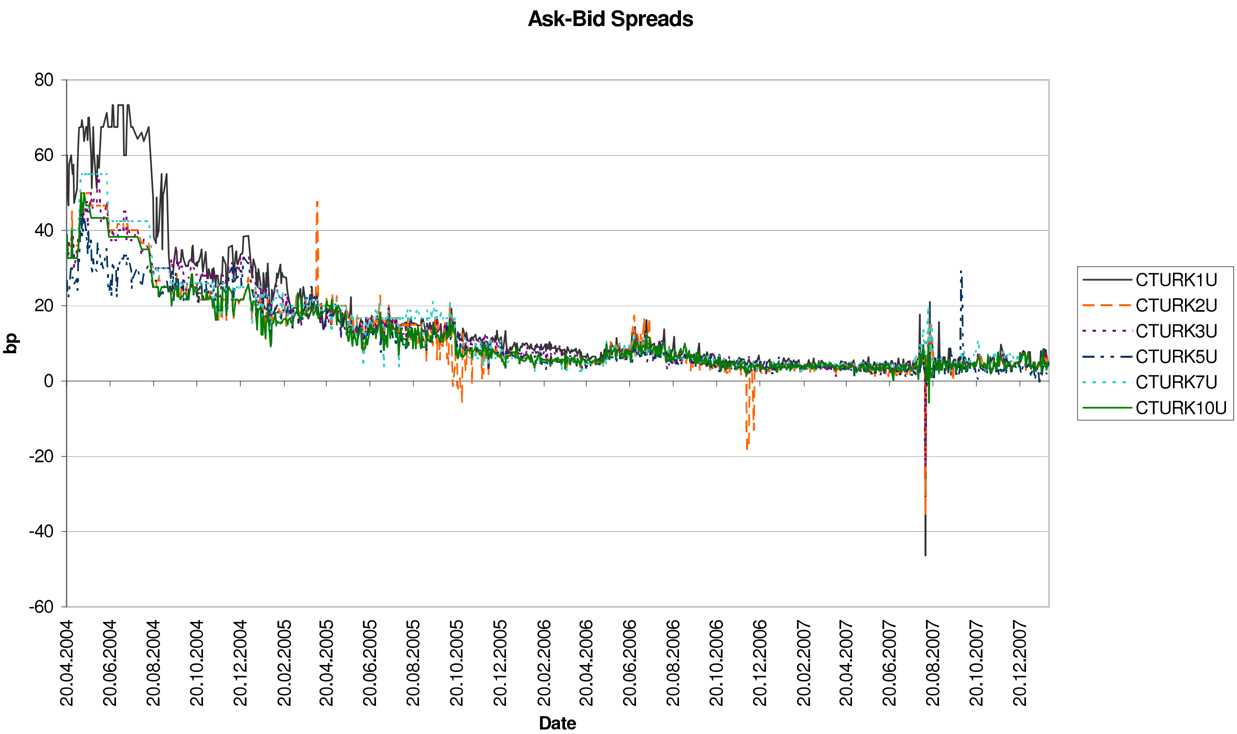

Figure 2.9: The ask-bid spreads of the CDS ($:= ask - bid$ in bp).

2.4.1 Results with the Bootstrapping Method

We explained in details the bootstrapping method in Subsection 2.3.2. Therefore, skipping the technical part, we present the results in this section. Remember that, we first fix the expected recovery rate under the risk neutral measure, i.e., $R = 0.25$, afterwards extract the intensities for each trading day in the sampling interval. Further, we assume a flat zero curve for the discount factors, i.e., $r = 0.05$. Assuming a stochastic short rate model would be more realistic but this does not affect the results significantly[13]. Note that the stepwise constant risk-neutral intensities[14] $\lambda_1, \lambda_2, \ldots, \lambda_6$ are defined in (2.49).

We present the corresponding risk-neutral intensities in Figure 2.10. The bootstrapping method brings many easiness in numerics while constructing the term structure of default probabilities, but there might be instabilities described by Martin et al. [MTB01] as well. We can see in Figure 2.10 that for some dates, e.g., 20th June 2006 and in the time period

[13]See Pan and Singleton [PS07a], Ueno and Baba [UB06], O'Kane and Turnbull [OT03].
[14]We drop the risk neutral measure superscript Q for easiness of notation.

All		n: 985				(in bp)
λ	1	2	3	4	5	6
min	28.6	69.5	159.8	250.2	116.3	299.9
max	550.8	881.4	1003.2	1089.3	1090.3	1099
std	78.5	136.4	165.6	150.8	129.8	112.6
med	74.4	166.1	278	427	503.6	532.3
mean	98.5	219.6	337.8	467.8	529	556.2

Table 2.4: The summary statistics for the risk-neutral default intensities, for all CDS in sample

between 20th February and 20th June 2005, the change in λ_5^Q is quite big. On those dates, inverted term structure of CDS rates might imply unstable intensities. One can typically have negative intensities as well, which make no sense at all. During our sampling period, we did not have any negative intensities.

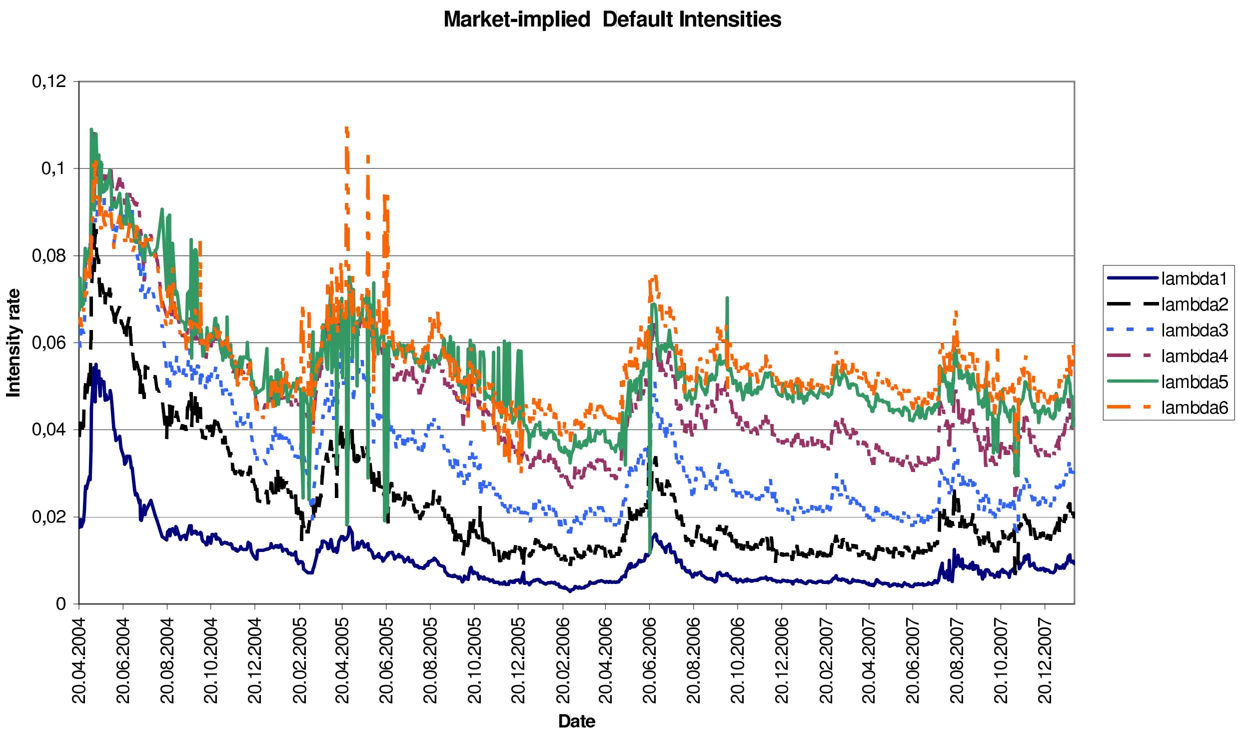

Figure 2.10: The default intensities bootstrapped from daily CDS mid-market quotes (in bp).

Fitch ratings had upgraded the rating of the long-term Turkish sovereign debt in foreign currency to $B+$ on 09 February 2004. Later on, it was upgraded on 13 January 2005 to $BB-$ (See Parker [Par06]). Since the last upgrade it remained in the same rating category in our sampling period. Hence, we run a rating-based analysis only based on these two rating classes, where the major rating classes BB and B mean *speculative* and *highly speculative* credit quality, respectively. The + and − signs are suffixes to show the relative status within the major rating category, e.g., + indicates a better credit quality. We present the results based on rating categories of Fitch $B+$ and $BB-$ in Table 2.5 and in Table 2.6, respectively.

Rating	B+	n: 153				(in bp)
λ	1	2	3	4	5	6
min	103.4	232.4	319.7	447.7	463.1	429
max	550.8	881.4	1003.2	1089.3	1090.3	1014.8
std	124.6	150.6	194.5	177.2	161.6	145.2
med	165.1	421.9	540.9	644.2	708.1	645.5
mean	226.5	468.8	624.8	711.5	727.4	690.6

Table 2.5: The summary statistics for the risk-neutral default intensities, for the Fitch rating category $B+$

Rating	BB-	n: 832				(in bp)
λ	1	2	3	4	5	6
min	28.6	69.5	159.8	250.2	116.3	299.9
max	176.8	413.9	586.3	695.4	751.1	1099
std	29.8	65.9	87.2	90.8	81.3	84.9
med	64.8	149.1	259.4	405.5	489.9	519.7
mean	74.9	173.8	285	422.9	492.5	531.5

Table 2.6: The summary statistics for the risk-neutral default intensities, for the Fitch rating category $BB-$

The average intensity of default for $0 < t \leq 1$ is $\lambda_1^Q = 226.5bp$, with the rating $B+$, whereas $\lambda_1^Q = 74.9bp$, if the rating is $BB-$. This result is expected since the default intensity decreases with increasing credit quality. Another expected result is that with the increasing maturity time, the corresponding default intensities should increase as well. If we look at Table 2.4, where we present the summary statistics of the intensities for the whole sample, we can observe this. This is also the case when the Turkish sovereign have the $BB-$ rating from Fitch agency. On the other hand, in Table 2.5, we see the inconsistency for the maturities more than 7 years. Figure 2.11 illustrates the average intensities with respect to rating categories of Fitch.

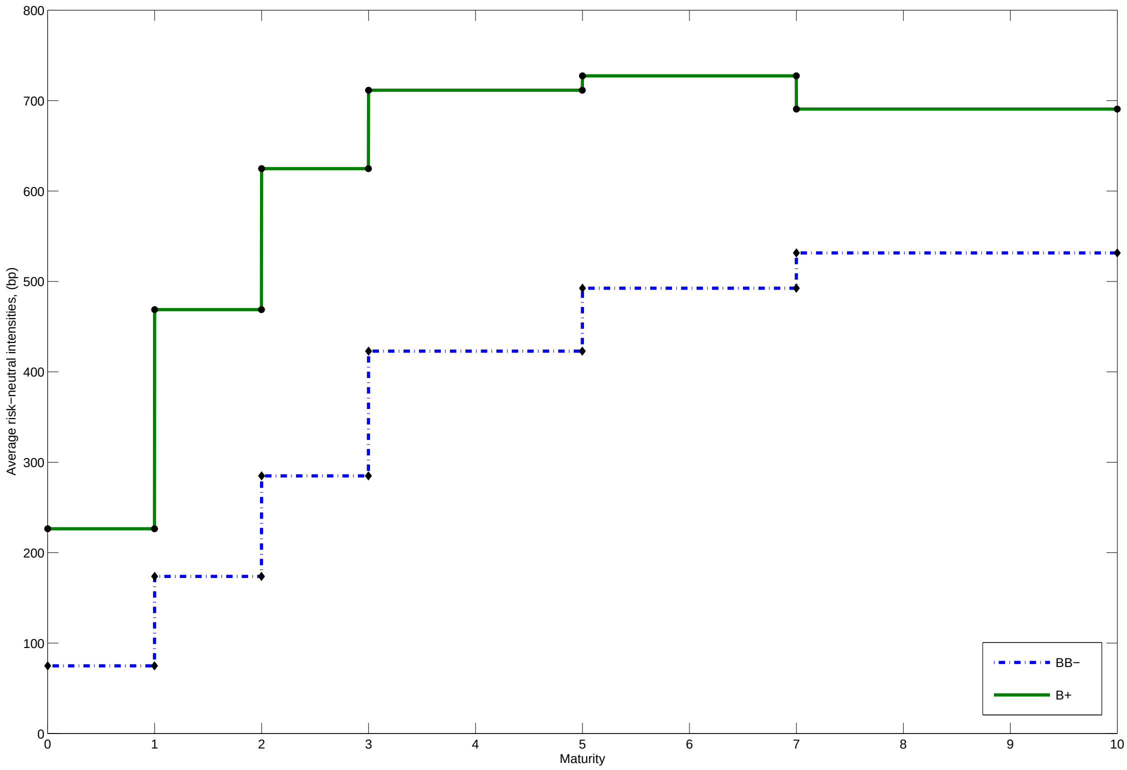

Figure 2.11: Average risk neutral intensity rates based on the rating category of the Turkish foreign currency long term debt by Fitch.

Using the stepwise constant intensity process, we calculate the cumulative default probabilities as described in Subsection 2.3.2. We can observe the default probabilities for 1, 3, and 5 years on each date for the sampling period in Figure 2.12.

The difference between the market rates and the CDS rates that are calculated with the extracted survival probabilities (the model price), $\widehat{C_{DS}} - C_{DS}$ gives a measure for the modelling error. We observe the errors for the CDSs with the maturities 1, 5, and 10 years in Figure 2.13. As the figure illustrates, the modelling error is not very significant. We further observe that the largest deviation between market and model prices is for the CDS with 1 year maturity.

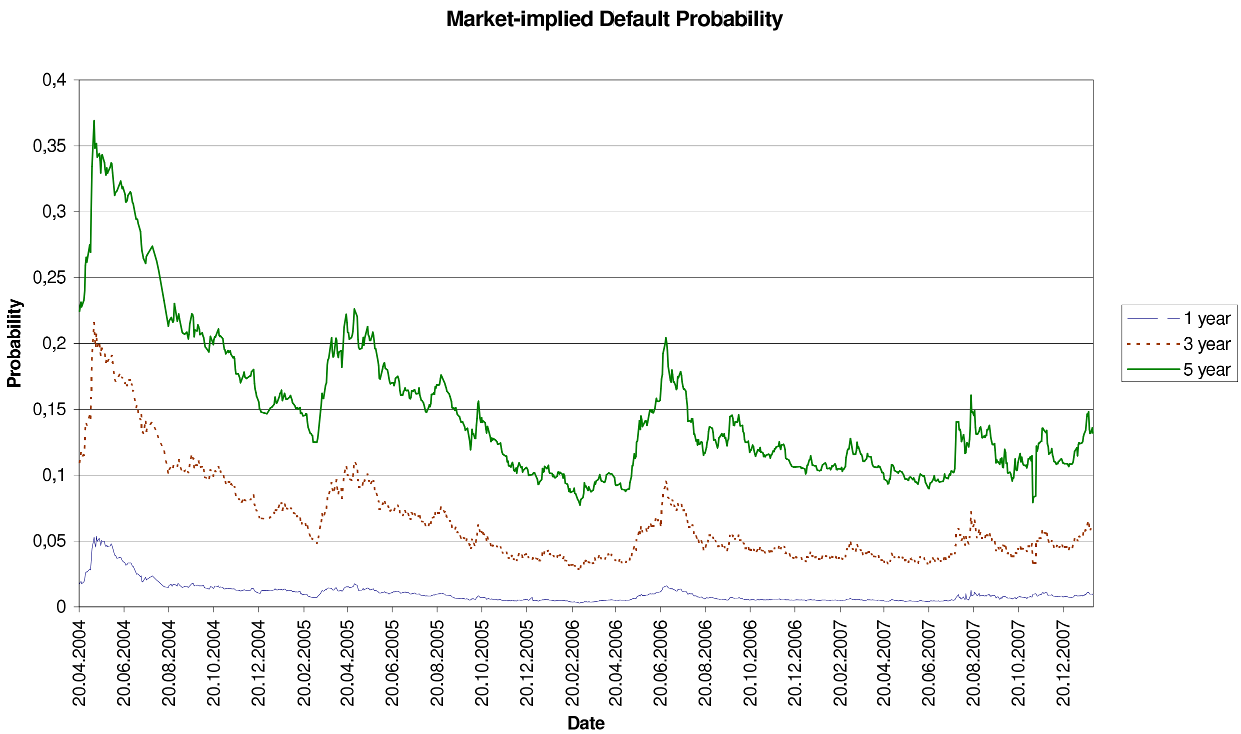

Figure 2.12: Market-implied cumulative default probabilities for 1, 3, and 5 years with bootstrapping method.

2.4.2 Results with the Optimisation Method

In this subsection we present the results of the optimisation method described in Section 2.3.3. We use the same expected recovery, i.e., $R = 0.25$ and the LIBOR, i.e., $r = 0.05$ as in the bootstrapping method, for a comparison of the market implied probabilities. The number of extracted intensities of optimisation method by Martin et al. [MTB01] depends on the discretisation interval. In our case, we have the semi-annual premium payments (no accrued premiums), and we further assume that the credit event can occur only on those dates. With this setting, we have 20 forward default probabilities, which minimises the objective function in (2.51) for each trading day in the sample. Moreover, we calculated the corresponding intensities, i.e., $\lambda_0, \lambda_1, \ldots, \lambda_{19}$ and the corresponding cumulative default probabilities. For the presentation we chose $\lambda_1, \lambda_3, \ldots, \lambda_{19}$, note that

$$pd(t_1, t_2) = 1 - \exp\left(-\int_{t_1}^{t_2} \lambda_1(t)dt\right) \Rightarrow \lambda_1 \approx \frac{pd(t_1, t_2)}{t_2 - t_1}.$$

Figure 2.13: The modelling errors for CDS contracts with 1, 5, and 10 years of maturity with bootstrapping method.

With the semi annual discretisation and the maturity of 10 years, we have the discrete time points as $t_0 = 0$, $t_1 = 0.5, \ldots, t_N = 10$. Moreover, the cumulative probability of default, e.g., $t_1 = 0.5$ is calculated with

$$PD(0) = 0 \Rightarrow PD(t_1) = PD(0) + PS(0)pd(0, t_1), \tag{2.53}$$

where we continue the recursion until we have the complete term structure of the cumulative PD's.

For a precise estimation, using the CDS mid-market quotes we have, we generated the CDS rates for 1, 2, ..., 10 year maturities with the linear interpolation method.

In Figure 2.14, we observe the paths of the intensities for our sampling period we had with the optimisation method. The intensities show similar behaviour compared to those boostrapped in the previous subsection. The main observation is that the optimisation method delivers higher intensities than the bootstrapping method comparing Figure (2.14) with Figure (2.10). In Table, 2.8 we have the intensities when the Fitch had rated Turkish sovereign with B+, where Table 2.9 presents the case when it was upgraded to $BB-$. When $0.5 < t \leq 1$ we have $\lambda_1 = 113.7$ bp for the whole dataset. For Fitch rating category $B+$, the average λ_1 is 261.7 bp, and 86.5 bp for $BB-$.

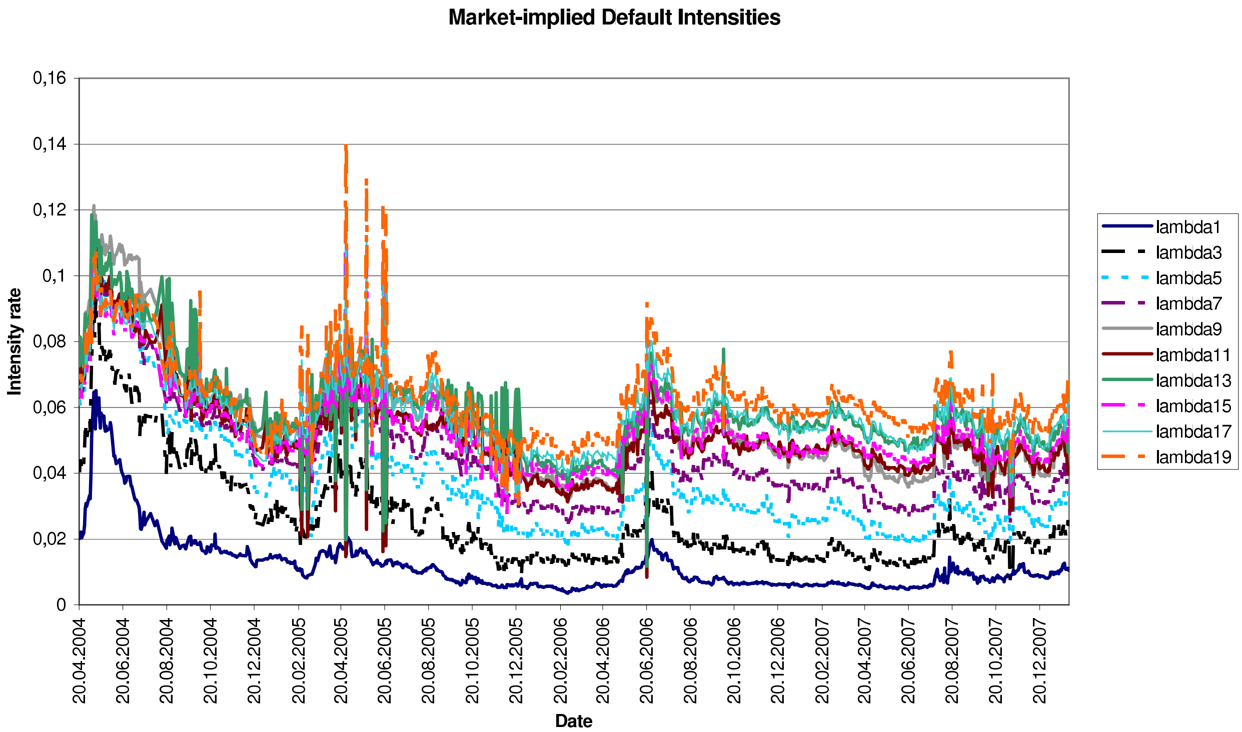

Figure 2.14: Market-implied default intensities, calculated with the optimisation method.

We observe the averages of the intensities in Figure 2.15 with the optimisation method with respect to the rating categories of S&P. As expected, for the rating category $BB-$, we have lower default intensities than the $B+$. The intensities tend to have an upward slope with respect to increasing maturity.

We construct the term structure of the risk neutral cumulative default probabilities via the recursive formula in (2.53). We illustrate the probabilities on each date in Figure 2.16.

Figure 2.17 illustrates the modelling error when we valuate the CDSs each day in the sampling period using the term structure of PDs extracted with optimisation method. Since the error is the difference between market and model price, CDS with 1 year maturity is overpriced with the optimisation method, whereas for 5 and 10 year maturities are

All					n: 985					(in bp)
λ	1	3	5	7	9	11	13	15	17	19
min	35.6	76.3	185.6	245.5	297.5	83.7	117.5	280.5	304.3	307.5
max	650.9	953	1062.5	1079	1212.6	1084.4	1185.1	1152.8	1200.7	1400.9
std	90	143.9	170	148	164	133.6	135.8	119.7	116.9	121.2
med	84.9	195.5	310.5	403.6	504.8	489.4	561.1	501.6	561	610.4
mean	113.7	247.0	363.9	441.3	544.7	512.9	584.5	528.7	579.9	626.8

Table 2.7: The summary statistics for the risk-neutral default intensities, for all CDS in sample calculated with the optimisation method.

Rating	B+				n: 153					(in bp)
λ	1	3	5	7	9	11	13	15	17	19
min	116.3	243.6	330.3	412.7	508.8	451.5	498.2	420.4	437.5	449.9
max	650.9	953	1062.5	1079	1212.6	1084.4	1185.1	1016.3	1043.3	1077.4
std	141.6	161.4	204.7	179.2	195.3	167	172.3	151.1	149.2	149.8
med	193.7	450.4	560.5	612.7	722.5	691.4	768.3	634.5	661.6	694.2
mean	261.7	504.1	653	679.9	803.2	719.2	784.8	681.4	711	740.7

Table 2.8: The summary statistics for the risk-neutral default intensities calculated with the optimisation method, for the Fitch rating category $B+$.

Rating	BB-				n: 832					(in bp)
λ	1	3	5	7	9	11	13	15	17	19
min	35.6	76.3	185.6	245.5	297.5	83.7	117.5	280.5	304.3	307.5
max	204.7	497.4	626.9	673.9	790.3	731.9	818.4	1152.8	1200.7	1400.9
std	34	73	91.4	87.6	101.5	82.3	87.6	87.7	91.4	102.3
med	74.3	183.5	293.4	388	484.3	475.2	547.3	489.3	549.3	597.5
mean	86.5	199.7	310.8	397.4	497.2	475	547.6	500.6	555.8	605.9

Table 2.9: The summary statistics for the risk-neutral default intensities calculated with the optimisation method, for the Fitch rating category $BB-$.

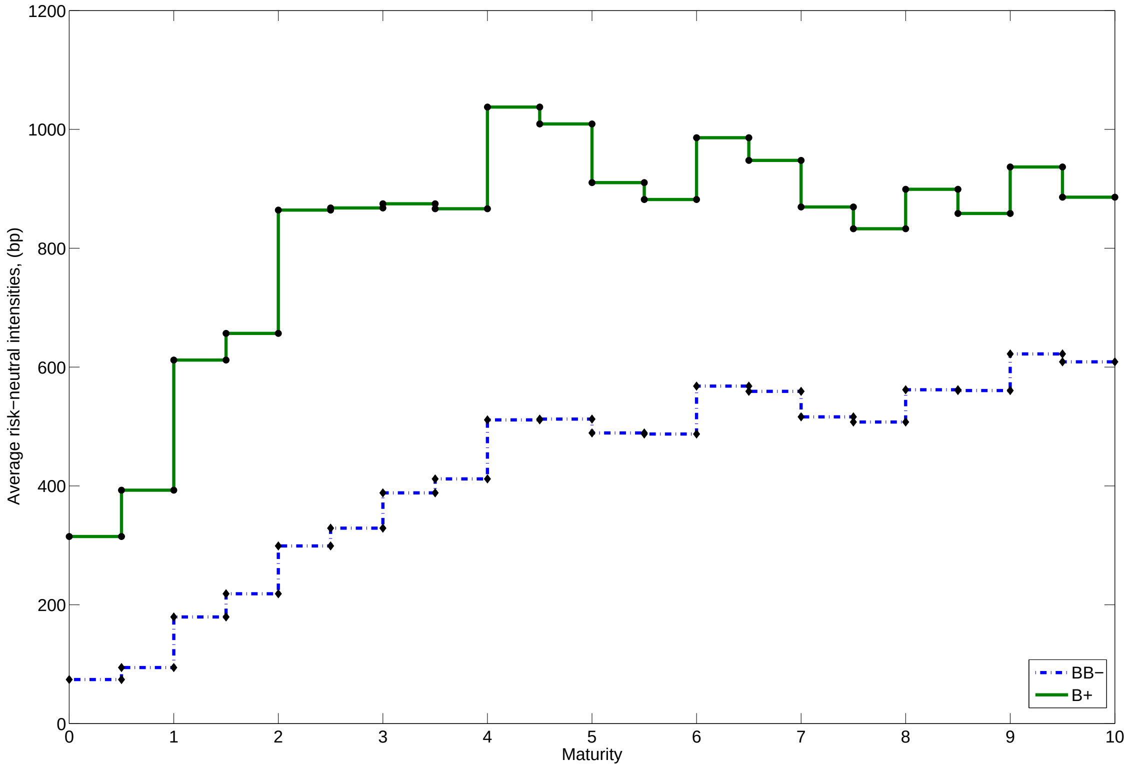

Figure 2.15: Risk neutral intensities with the optimisation method, based on S&P's ratings.

underpriced. We observe that the largest deviation of error is observed in the CDS with 5 years maturity.

2.4.3 Comparison of the Results

As the reliability of the calculated CDS prices heavily depends on the realism of the assumptions in the valuation model, we find it useful to comment on the *marking-to-model* issue. Marking-to-model is the valuation of a position or a portfolio of securities at prices depending on a financial model. In CDS market, where the illiquidity risk does not significantly exist, "marking-to-market" is more reliable. However, suppose we are pricing a new issued security, implying the "illiquidity problem". In this case marking-to-market might be misleading due to the scarcity in market prices. Therefore, marking-to-model is an important issue for exotic instruments, especially in new structured credit products. If the financial model is realistic, implying insignificant modelling errors, then it is sufficient

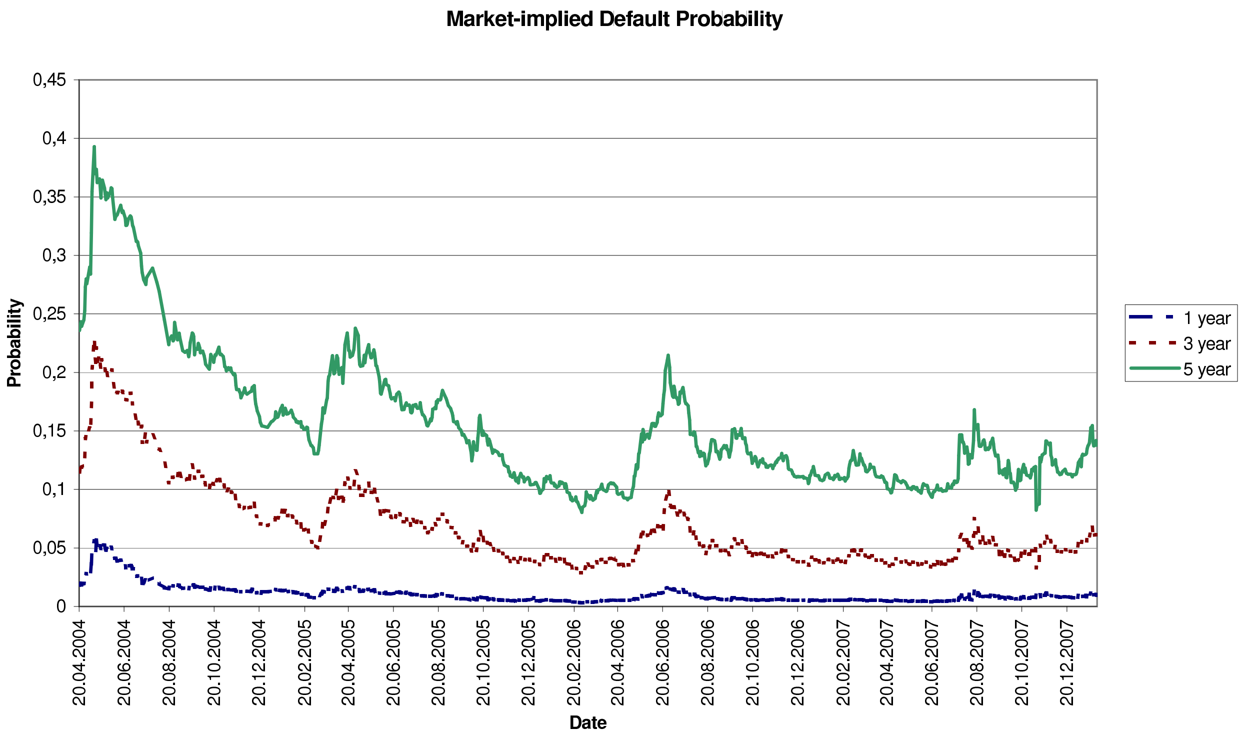

Figure 2.16: Market-implied cumulative default probabilities in 1, 3, and 5 years, calculated with the optimisation method.

for us to show the approximation and estimation errors are the main source of the total error between the model and the actual prices when we take the total error as a sum of modelling, approximation and estimation errors.

If we compare Figures 2.13 and 2.17, we see that both pricing models have insignificant deviations from the market price, where optimisation method delivers in general higher default intensities and probabilities, consequently.

For a comparison, we take the paper by Rocha and Garcia [RG04], where the authors use a structural credit risk model for extracting the market implied sovereign credit risk. The authors take the real YTL / USD exchange rate, which follows a pure diffusion process, as a proxy for modelling the source of uncertainty. In Table 2.10, we present the cumulative risk neutral default probabilities of Rocha and Garcia (RG model) and those implied by CDS rates on 15th July 2004 using the bootstrapping method and the optimisation method. The cumulative probabilities with optimisation and bootstrapping are similar for the short maturities, whereas the difference rises up to 2% with increasing maturity.

Figure 2.17: Error between the market and model prices calculated with the optimisation method.

Comparing the RG model for maturities of 1 and 2 years, the bootstrapping method delivers closer results but, for the maturities between 3 and 6 years, the optimisation method have closer probabilities. However, boostrapping and RG model have similar probabilities after maturities of 7 years.

Maturity	1	2	3	4	5	6	7	8	9	10
RG model	0.75	6.68	14.57	21.99	28.45	34.01	38.80	42.96	46.60	49.81
Bootstrapping	1.90	6.86	13.24	20.27	26.73	32.52	37.86	42.76	47.29	51.45
Optimisation	1.98	7.17	13.91	20.64	28.23	34.04	39.89	44.73	49.42	53.98

Table 2.10: Comparison of the market implied cumulative PDs on 15 July 2004.

2.5 Relationship between the Risk-neutral and the Actual Default Probabilities

In this section, we first present the literature survey about modelling the relationship between the actual and risk neutral default intensities for the corporate debt, then introduce our results. As the risk premium maps the actual intensities to risk neutral intensities, one has to first estimate the actual default intensities using the historical default experience. Berndt et al. [BDD+05] use Moody's Estimated Default Frequency as a proxy for the actual default intensities. Hull et al. [HPW05] calculates the intensities from actual cumulative default probabilities, where Driessen [Dri05] uses a similar methodology. Since we have the rating history of the Turkish sovereign foreign currency debt, we calculated the actual intensities using the cumulative default rates published by S&P using the methodology by Hull et al. [HPW05]. Since credit event in a sovereign occurs rarely, estimation of these rates is rather a difficult task.

2.5.1 Risk Neutral and Actual Intensities

As mentioned before, credit risk models are mainly used for two reasons, firstly they are used in the prediction of the PDs and they are tools for pricing and hedging of credit sensitive instruments. Serving both purposes, one selects different probability measures. For the prediction of the PDs, we need the actual probabilities, whereas the risk-neutral probabilities are used for pricing and hedging reasons. Therefore, a good credit risk model must fulfil both needs. In this context, the importance of *default risk premium* comes into play, which we try to explain in this section.

The risk-neutral probabilities are available under weak no-arbitrage conditions. Incomplete markets imply many alternative choices of risk neutral probabilities consistent with pricing of the traded assets. However, independent from the market being complete or not, knowledge of *only* the risk-neutral probabilities is not enough to fit the credit risk models to the historical default experience.

A typical example, which can be found in each credit risk book[15] is as follows: Suppose

[15] See Bluhm et al. [BOW03], or Duffie and Singleton [DS03].

we have a 1 year, risky par bond with a promised face value of 100 YTL and a coupon payment of 10%. Hence, the bondholder receives 110 YTL after one year if there is no default, or the recovery of the face value, which is $R = 50\%$. The historical experience tells us $PD^P = 0.02$ in the corresponding rating category of the risky bond. With a short rate of 4%, the expected simple discounted bond value under P is given by

$$\frac{1}{1.04}(0.98 \times 110 + 0.02 \times 50) = 104.62,$$

which overprices the actual market price ($Face\ value = 100$) of par bond by 4.62 since the risk-premium is not considered. However, under the risk neutral pricing framework, we have

$$100 = \frac{1}{1.04}[(1 - PD^Q) \times 110 + PD^Q \times 50].$$

Hence, $PD^Q = 0.10$. Assuming the deterministic intensity is constant, we have

$$\lambda^Q = -\ln(1 - PD^Q) = 0.10 \quad \lambda^P = -\ln(1 - PD^P) = 0.02.$$

We see that $\lambda^Q > \lambda^P$, reflecting the risk premium. Note that there is not any change in the intensity rate or uncertainty of recovery here, so that the market implied PD^Q is unique. As we can see in the example above, it is documented that the RN default intensities are generally greater than the actual ones (See Hull et al. [HPW05], Driessen [Dri05], Berndt et al. [BDD+05], O'Kane and Turnbull [OT03]), as the traders do not price the risky securities only based on the APDs. For the compensation of the risks that they are bearing, they build in an extra return. Hence, the difference between the risk neutral and actual intensities shows up.

As Duffie states, "a common but naive measure of probability of default for a firm or sovereign that is rated by an agency such as Moody's or S&P, is the average frequency with which obligors of the same rating have defaulted".

In reduced form approach, remember from the JLT model that the actual intensities are mapped with some scalar μ (risk premium) to risk neutral intensities, i.e.,

$$\lambda^Q = \mu\lambda^P, \quad \text{with } \mu \geq 1. \tag{2.54}$$

One can choose μ in order to have a good match both to the historical data and the market credit spreads, which still remains as an empirical issue to be explored that we present in this section. With scaling as in (2.54), Driessen [Dri05] finds out an average ratio of $\lambda^Q/\lambda^P = 1.89$, backing out RNPD from U.S. corporate bond prices. Another study by Berndt et al. [BDD+05] gives similar results to Driessen [Dri05] using market CDS rates for bootstrapping the PD^Q. Giesecke and Goldberg [GG07] present some references of empirical work about risk premium, where the authors propose a structural model for analysing determinants of the risk premium. We illustrate some of the modelling approaches for the default risk premium in the next subsections.

As mentioned before, RNPD are good for pricing and hedging issues. But, suppose we are pricing a new security and the market prices are scarce, then we need to use the historical information about the obligor (implying APD) and transfer the APD to RNPD. On the other hand, we use the APDs in risk managemet, trading and credit allocation issues. One may need to use the credit spreads in the market in order to estimate the APDs. The problem is that market-implied RNPDs may be very pessimistic and this can cause unnecessary burdens on business (excessive regularity capital). Hence, a tool that maps the RNPD to APD (and vice versa) is important and needed by the practitioners.

The literature survey we are going to present in the next subsections are based on corporate default risk. Note that the rating methodology and the corresponding term structure of actual PDs differ when we are dealing with the sovereign credit risk. For an illustration, we borrow Figure 2.18 from Hamilton et al. [HVOC06].

2.5.2 Method from Berndt et al.

Here, we give an overview to the paper of Berndt et al. [BDD+05], where the authors undertook a panel regression analysis of the corporate CDS market rates and Moody's estimated default frequency (EDF) data[16]. This analysis is for obtaining a simple and robust measure of the sensitivity of CDS rates to actual PDs. The authors regress the CDS observations for 5 year maturities and the 5 year EDF with an Ordinary Least Squares (OLS) and have an $R^2 = 73\%$. However, linearity of the CDS-EDF relationship is placed in doubt by the authors. Moreover, they tried a log-log specification on the same dataset in order to mitigate the non-linearity and heteroscedasticity effects, where the resulting

[16]EDF is a measure of default probability used by Moody's KMV based on a database of historical default frequencies.

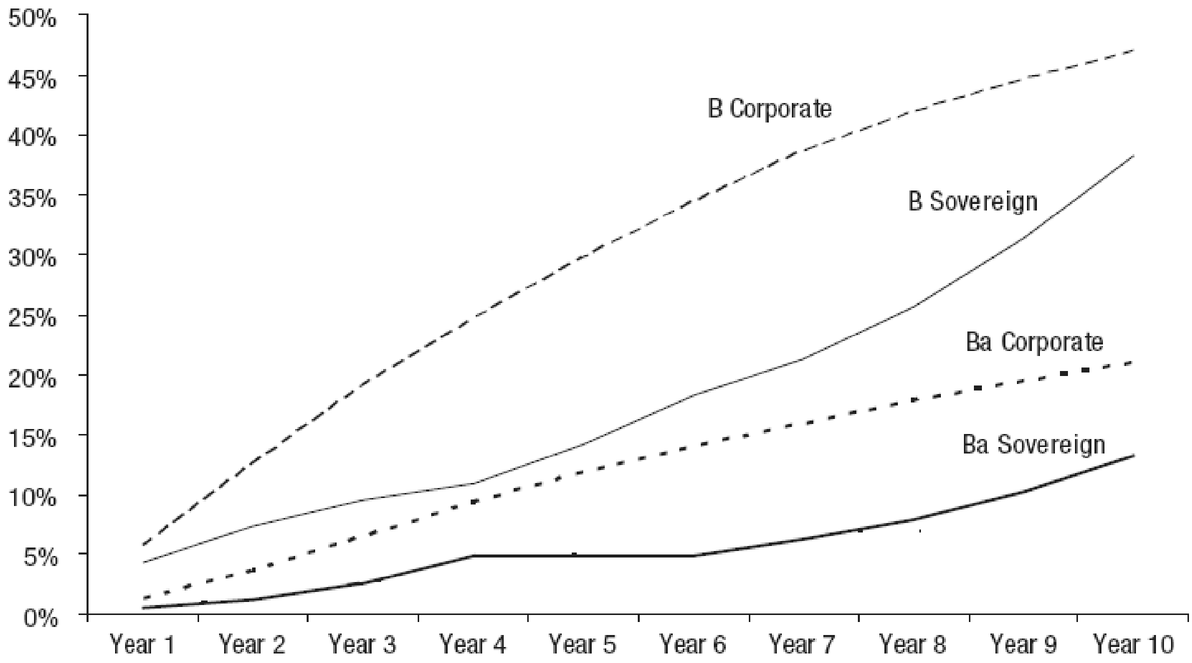

Figure 2.18: Source: Moody's, Average cumulative default rates: Sovereign vs. Corporates, 1983-2005

R^2 is equal to 69%. Adding some dummy variables (month and sector specific) to the log-log regression equation increased the R^2 to 74.4%.

In the last sections of their paper, the authors focus on modelling the relationship between the actual and risk neutral default intensities, where we explain the details below.

Time-series model for Default Intensity

The authors claim that the logarithm of the default intensity under the actual probability measure $X(t) = \log(\lambda^P(t))$ satisfies the Ornstein-Uhlenbeck equation

$$dX(t) = \kappa(\theta - X(t))dt + \sigma dW(t), \tag{2.55}$$

where W is a standard Brownian motion and κ, θ, σ are some constant values. The unknown parameter set $\Theta = (\theta, \kappa, \sigma)$ is estimated from available monthly EDF observations. The authors used a maximum likelihood technique for estimating the parameter vector Θ.

Further, the authors introduce a flat cross-firm correlation structure, within the sector[17]

[17]The available observations are separated into three sectors, namely Oil and Gas, Healthcare, Broad-

by generalising (2.55). Hence, assuming that $X_i(t) = \log \lambda_i^P(t)$ for firm i, the logarithm of the intensity satisfies

$$dX_i(t) = \kappa(\theta_i - X_i(t))dt + \sigma(\sqrt{\rho}dW_c(t) + \sqrt{1-\rho}dW_i(t)), \tag{2.56}$$

where W_c and W_i are independent standard Brownian motions, independent of $\{W_j\}_{j\neq i}$ and ρ is the within sector pairwise constant correlation coefficient.

Risk-Neutral Intensity from CDS and EDF

Here, we explain the joint model of actual and risk-neutral default intensities. The model contains the risk-neutral default intensity of a given firm as a function of its own default intensity, a measure of aggregate default risk in the sector and a latent variable capturing the variation in default risk premium, which is not captured by the first two variables.

The model is specified as follows: Let us denote the risk-neutral default intensity and the actual intensity process of any given firm i by λ_i^Q and λ_i^P , respectively . Suppose

$$\log \lambda_i^Q(t) = \beta_0 + \beta_1 \log(\lambda_i^P(t)) + \beta_2 \log v(t) + u_i(t), \tag{2.57}$$

where β_0, β_1, and β_2 are constants, $X_i = \log \lambda_i^P$ is specified by (2.56) and v is the geometric average of the default intensities $\{\lambda_i^P\}_{i\in J}$, over a benchmark subset J of large liquid firms in the same sector, i.e.,

$$\log v(t) = \frac{1}{|J|}\sum_{i\in J} X^i(t).$$

Moreover, suppose that

$$du_i(t) = \kappa^u(\theta^u - u_i(t))dt + \sigma^u\sqrt{\rho^u}d\xi_c(t) + \sigma^u\sqrt{1-\rho^u}d\xi_i(t), \tag{2.58}$$

where θ^u, κ^u, and σ^u are constants, ρ^u is a constant correlation parameter and ξ_c, ξ_i are independent (under P) standard Brownian motions, independent from W_c and W_j in (2.56).

After fitting the model and estimating the parameters, for the healthcare sector, Berndt et al. have

$$\log \lambda^Q(t) = 0.576 + 0.522 \log \lambda^P(t) + 0.628 \log v(t) + u(t),$$

casting and Entertainment

where for $u(t) = 0$, a geometric average of all default intensities in the sector of 100 bp and an actual intensity of $100bp$, we get a risk-neutral intensity of roughly $355bp$.

The averages of the ratios of (λ^Q/λ^P) are 3.30, 2.17, and 2.04 for the oil-and-gas, health care, and broadcasting-and-entertainment sector, respectively. For the whole dataset, they estimate an average ratio of 2.757, where intensities are given in basis points per year.

2.5.3 Method from Hull et al.

Here, the authors estimate the actual default intensity, λ^P from statistics on average cumulative default probabilities of corporate bonds published by Moody's between 1970-2003. As reported by Hull et al. [HPW05], the cumulative default rate is $PD(T)$ for T years and λ^P denotes the average historical default intensity over T years. The survival probability of the corporate bond for T years, given there is no previous default, is given by

$$\exp(-\lambda^P T) = 1 - PD^P(T). \tag{2.59}$$

It follows that the actual intensity is

$$\lambda^P = -\frac{1}{T}\log(1 - PD^P(T))$$

The authors approximate the risk-neutral default intensity for a defaultable corporate bond per year with

$$\lambda^Q \approx \frac{y - r}{1 - R}, \tag{2.60}$$

where y is the bond's yield, r is the yield on a similar risk-free bond. Taking the common market recovery rate assumption that $R = 0,40$, the authors give a table of estimated actual and risk-neutral intensities dependent on the rating of the bonds. Table 2.11 shows that the ratio of the risk neutral to actual default intensity decreases as the credit quality declines. However, the difference between them increases as the credit quality declines. This is referred as the "credit spread puzzle".

Rating	λ^Q	λ^P	$\frac{\lambda^Q}{\lambda^P}$	$\lambda^Q - \lambda^P$
Aaa	67	4	16.8	63
Aa	78	6	13	72
A	128	13	9.8	115
Baa	238	47	5.1	191
Ba	507	240	2.1	267
B	902	749	1.2	153
Caa and lower	2130	1690	1.3	440

Table 2.11: The risk premiums (in bp), depending on Moody's ratings (Source: Hull et al. [HPW05])

2.5.4 Method from Driessen

Driessen [Dri05] estimates the sources of risk that cause corporate bonds to earn an excess return over default-free bonds. Moreover, the author estimates a risk premium associated with a default event. Denoting the risk premium with μ on default jump, we have

$$\lambda_i^Q(t) = \mu \lambda_i^P(t)$$

for the ith name. If the default risk is priced, μ should be greater than 1.

Denoting the actual probability that a firm defaults in T years from $t_0 = 0$ given there was no default before with $PD^P(T;\mu)$, we have

$$PD^P(T;\mu) = 1 - E^P\left[\exp\left(-\int_0^T \lambda^P(s)ds\right)\right] = 1 - E^P\left[\exp\left(-\int_0^T \frac{\lambda^Q(s)}{\mu}ds\right)\right].$$

Given the affine process for $\lambda^Q(t)$, this probability is an explicit function of the risk premium μ. Driessen calculates the actual PDs depending on the rating of the firm, hence the actual PDs are the same for the firms having the same rating, i.e., PD^P_{Rating}. Yearly risk-neutral conditional default probabilities can be calculated with

$$PD^Q_{Rating}(T;\mu) \equiv 1 - \frac{1 - PD^P_{Rating}(T+1;\mu)}{1 - PD^P_{Rating}(T;\mu)}$$

By confronting the above equation with actual default rates, μ can be estimated.

2.5.5 Our Results

After the literature survey about estimating the risk premium in corporate debt, we present our findings in this subsection. Note that, we use the sovereign default rates from historical data published by S&P for calculating the actual default intensities. The S&P foreign currency long term debt note for Turkish sovereign was $B+$ on 8th March 2004 and was upgraded to $BB-$ on 17th August 2004. Since this date, the rating has been in the same category (See Soussa and Faulks [SF07]). In our sampling period, we will have 64 corresponding dates when the sovereign debt was rated with $B+$ and 921 dates with the rating $BB-$. The details of sovereign rating methodologies are explained by Beers and Cavanaugh [BC06], and by Klaar and Rawkins [KR07].

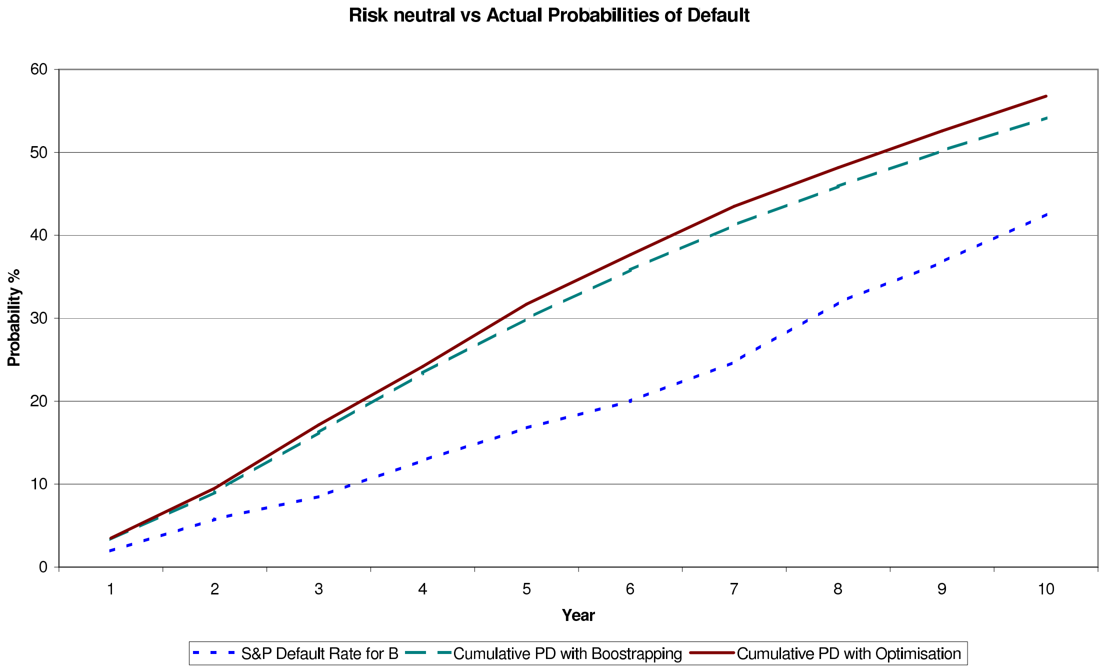

Figure 2.19: Average of the risk neutral cumulative default probabilities vs actual default probabilities for S&P rating, B

Figures 2.19 and 2.20 show the term structure of the cumulative risk neutral PDs with the optimisation and bootstrapping methodologies versus the actual cumulative PDs published by S&P based on the historical experience. Note that for both Figures, we use the rating categories B and BB (without modifiers $+$, $-$) for the illustration. The figures show that the risk neutral PDs are greater than the actual PDs, as one could expect due to market price of default risk that the traders add on. However, when we use the S&P estimations for the rating classes with modifiers, which are subclasses of the main rating categories, we have a different picture. As Figures 2.21 and 2.22 illustrate.

We may see in Figure 2.21 the average of the market implied PDs stripped out from

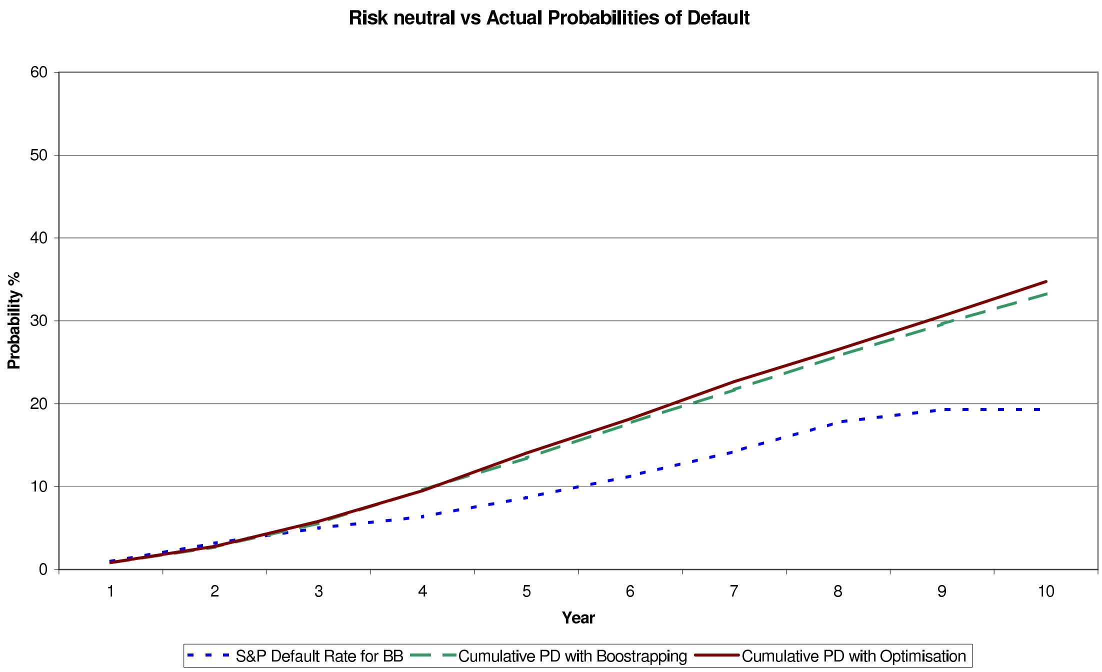

Figure 2.20: Average of the risk neutral cumulative default probabilities vs actual default probabilities for S&P rating, BB

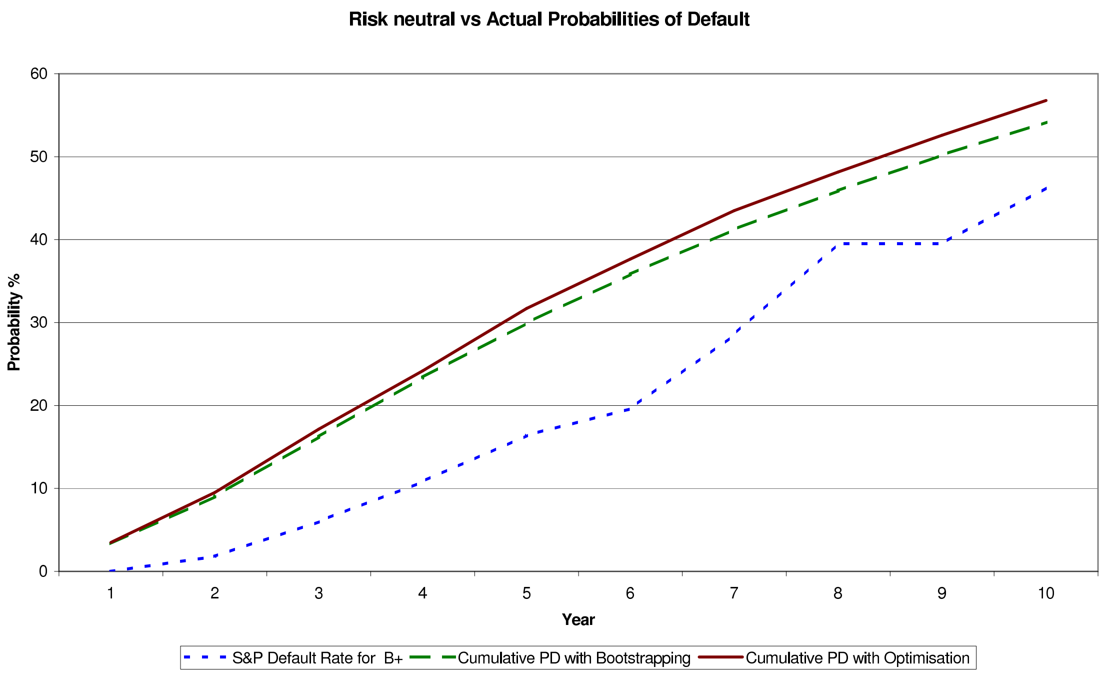

Figure 2.21: Average of the risk neutral cumulative default probabilities vs actual default probabilities for S&P rating, $B+$

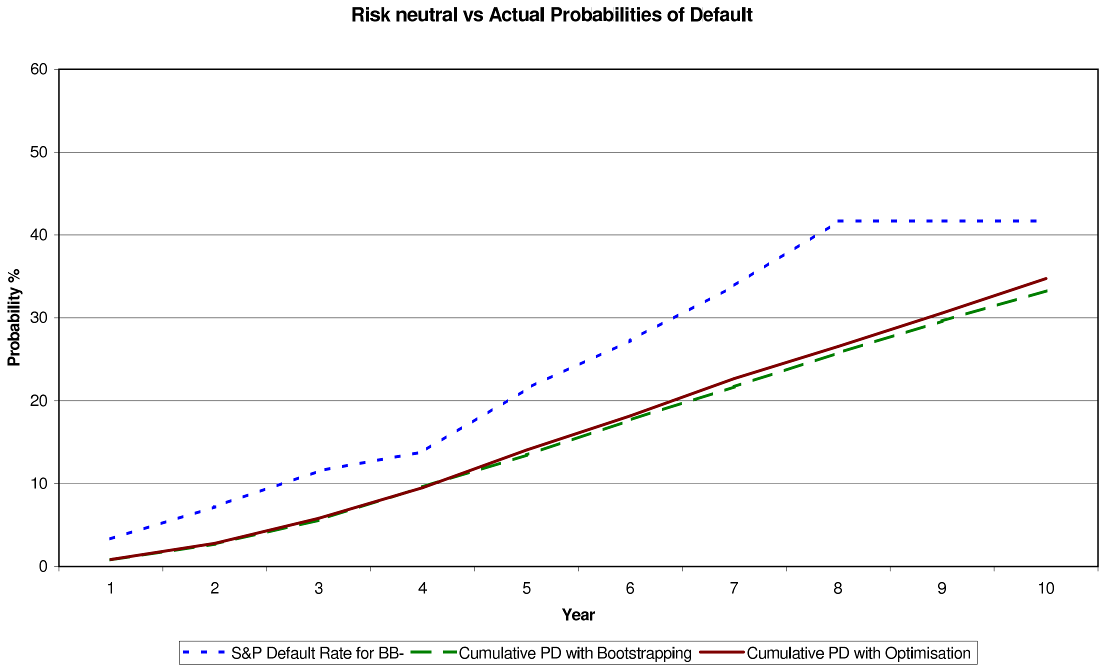

Figure 2.22: Average of the risk neutral cumulative default probabilities vs actual default probabilities for S&P rating, $BB-$

the CDS rates with bootstrapping and optimisation methods, versus the default rates for rating category $B+$ of S&P's. The key observation is that we have higher risk neutral PDs with both methods than the actual probabilities as expected. In Figure 2.22 we take the average of the PDs for the dates when the rating is $BB-$ for the market implied PDs with respect to rates published by S&P for category $BB-$. In contrast, as Figure 2.22 show, the actual PDs are higher than what the CDS rates imply, which is interestingly an unexpected result.

Further, we illustrate the behaviour of the actual intensity rates that we calculated from the cumulative average default rate table for the sovereign foreign currency (See Table 17 of Chambers [Cha07]) in Figures 2.23 and 2.24 for categories $B+$ and $BB-$ of S&P's respectively. We construct the actual intensities via formulation as described by Bluhm et al. [BOW03] as follows:

$$\lambda_m^P = -\frac{1}{t_{m+1} - t_m} \ln\left(\frac{1 - PD_{m+1}^P}{1 - PD_m^P}\right), \quad \text{where } m = 0, 1, 2, \ldots, 9. \tag{2.61}$$

and PD^P corresponds to the cumulative default rate calculated by S&P based on historical data covering a period of 1975-2006.

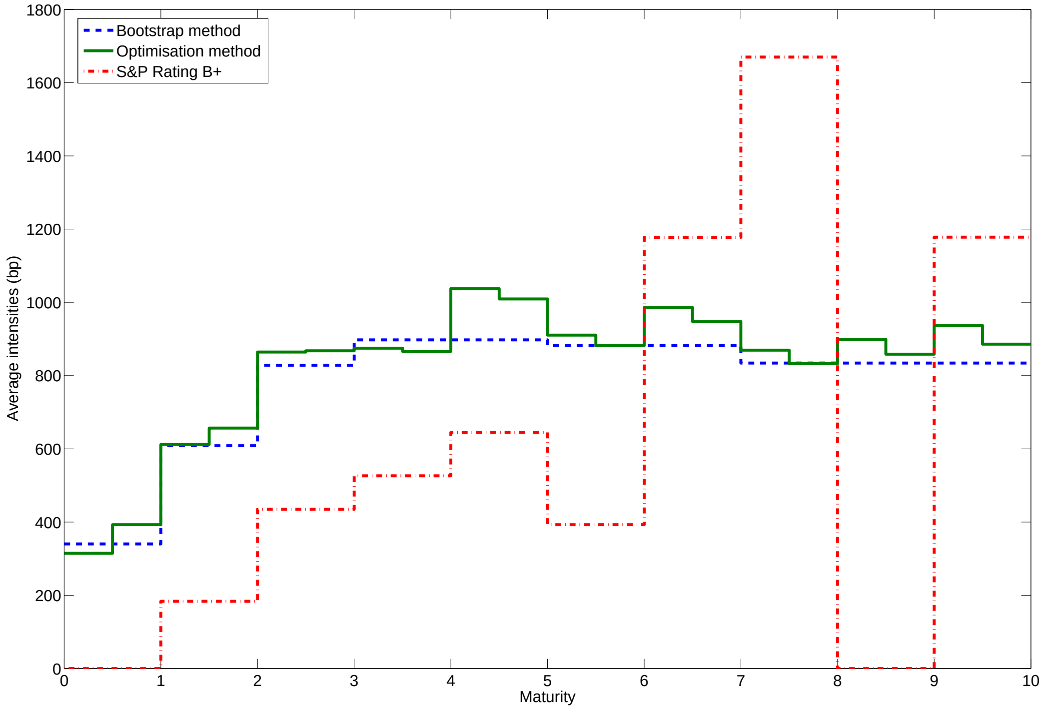

Figure 2.23: Average of the risk neutral default intensities calculated with two methods vs. actual default intensity for S&P rating, $B+$

2.6 Summary

In this chapter, we presented the well known intensity based credit risk models in the literature. After building the necessary mathematical background, we introduced the state of the art of pricing the sovereign CDS contracts. Since we mainly focus on the market implied sovereign risk of Turkey, we presented two methodologies for extracting the risk neutral default intensities out. Further, we presented the risk premium modelling approaches from corporate credit risk literature and provide an analyses of the risk neutral and actual intensities calculated from the studies by S&P.

Our sample is composed of the Turkish sovereign CDSs including the maturities of 1, 2, 3, 5, 7, and 10 years, where the reference asset is the USD denominated Eurobond maturing in 2030. We captured interesting patterns of the risk neutral default intensities during our sampling period due to the recent global and local financial and political crises. Fixing the expected recovery rate a priori, i.e., $R = 0.25$ and taking a flat zero curve when constructing the discount factors, our pricing models provided good fits to the market

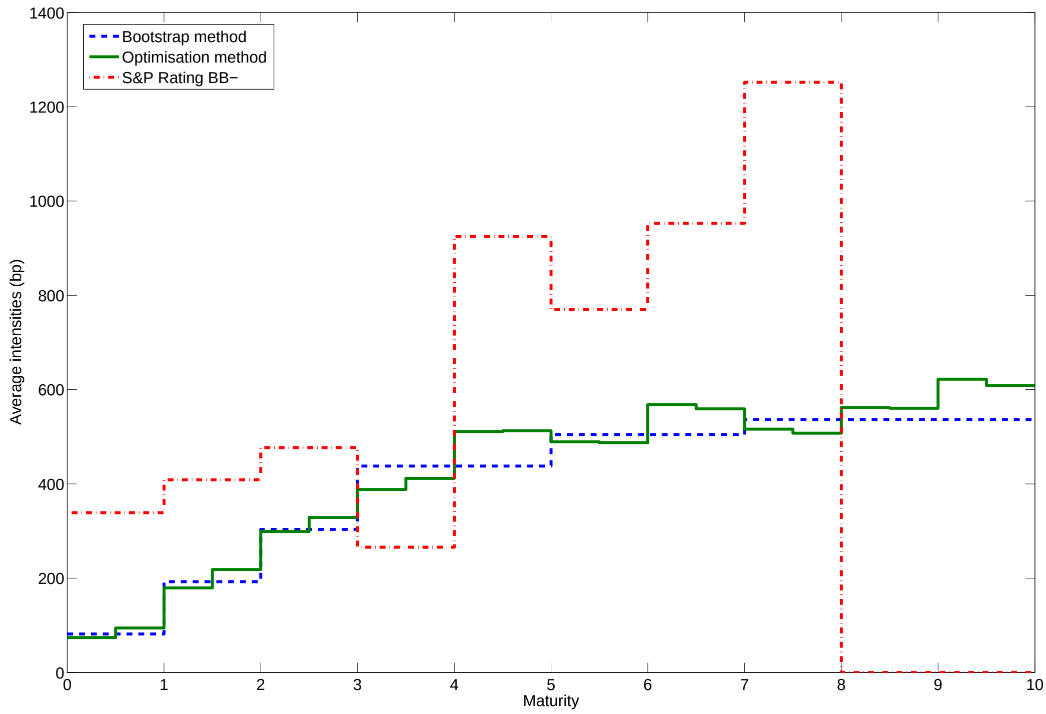

Figure 2.24: Average of the risk neutral default intensities calculated with two methods vs. actual default intensity for S&P rating, $BB-$

CDS rates. Our main finding is that the optimisation method of Martin et al. [MTB01] delivers higher default intensities and consequently higher default probabilities.

Furthermore, using the default probabilities based on historical experience reported by the rating agency S&P (based on the dataset between the years 1975-2006), we calculated the actual default intensities. We concluded that, the relationship of the risk neutral and actual PDs are as expected in rating categories B and BB without the modifiers. On the other hand, when we take the rating categories with modifiers, the results for the rating category $B+$ are as expected, as it is well documented in the literature that the actual default probabilities are less than the risk neutral ones. However, this was not the case for the rating category $BB-$. Interestingly, the term structure of cumulative PDs of S&P is larger than the risk-neutral PDs we calculated. Moreover, the analyses of the actual and risk neutral intensities based on the rating categories show that, due to the scarcity in the data concerning the default experience of the sovereigns, to present a conclusion is rather a difficult task. One might alternatively calculate the actual default rates or try to model the actual intensity process in a time-series framework, but we leave this for a future research topic.

Chapter 3

Optimal Leverage in CPDOs

3.1 Introduction

The volume of the credit derivative contracts traded at the corresponding market has increased considerably over the last few years. Sophisticated products have been introduced into the market. The Constant Proportion Debt Obligation (CPDO, hereafter) is one of such products and offers a sizeable spread over LIBOR and returns the initial investment minus the losses at the maturity. This spread is generated by taking a dynamic leveraged position on a portfolio of credit indices (e.g. ITRAXX and CDX.NA.IG). Hence, both coupons and the principal is at risk. CPDOs have generated a lot of interest among the investor community as they pay a relatively high spread for their credit rating.

The leverage function so far used in CPDO products implicitly assumes that credit spreads are constant and defaults in the underlying credit index are known in advance. To the best of our knowledge, no attempt has been made to obtain an "optimal" leverage for dynamic investment in the underlying index/portfolio, given some objective function for the investor.

In this chapter, we introduce an "optimal“ leverage function for the CPDO based on some simple dynamics for the credit investment process. The optimal leverage function is derived using the stochastic control technique. In particular, we assume that the objective of the investor is to minimise the losses due to the leveraged risky position, or equivalently to maximise the expected redemption at maturity given a stream of mandatory coupon payments.

The control variable of the problem is the leverage function, i.e., the notional exposure at any time to the portfolio of the credit indices. The return from investment in these indices, which includes mark-to-market spreads as well as losses stemming from defaults in the underlying credit portfolio of the index, is modelled via an arithmetic Brownian motion.

The control problem involves solving a highly non linear PDE. It turns out that the dual problem is much easier to solve and give rise to Black and Scholes type formulas.

Differently from the Constant Proportion Portfolio Insurance (CPPI) instruments, the leverage of a CPDO decreases in favourable market conditions (spread tightening, no defaults in index portfolio) and vice-versa. However, contrary to the industry practise, the optimal leverage function we derive is a non linear function of the Net Asset Value (NAV) of the note and for low levels of NAV the leverage behaves similarly to a CPPI.

The rest of the chapter is organised as follows: Section 3.2 explains the concept and the terminology of a CPDO. We supply the mathematical background in Section 3.3. We develop the model, introduce and solve our control problem in Section 3.4 and present the numerical results in Section 3.5. Finally, we summarise and comment on the results.

3.2 Terminology and Product Description

Before we present the terminology and the details about CPDOs, we find it useful to describe the underlying portfolio of CDS Index (CDX, hereafter) contracts that the CPDO strategy invests in. A CDX contract provides protection against a standardised *basket* of reference entities. Therefore, it is different from the CDS, which provides protection against losses on default of a *single* reference asset. As we know from Chapter 2, the premium payment is cut off in a CDS upon the credit event of the reference, whereas in the CDX contract, the premium payment continue to be made, but based on a reduced notional in case there are defaulted names in the basket.

The most actively traded securities are the CDX.NA.IG and the ITRAXX Europe Index. The CDX.NA.IG includes 125 North American Investment Grade companies, where the latter covers 125 investment grade European companies. Both indexes are available with

3, 5, 7, and 10 year maturities of protection and each company in both indexes are equally weighted.

On the *roll dates* (20th September and 20th March or the following business days), the new version of CDX starts after the composition of the reference entities, which is determined by the votes of participating dealers. A new version CDX will be called *on the run* for the next six months. The defaulted references are excluded from the index on each roll date, however the composition stays static if there are no defaulted entities in the CDX. The popular indexes we mentioned above are unfunded, hence they can be thought as a CDS on a basket of names generally using the physical settlement upon the credit events. The CDX contracts are standardised and transparent products having the advantages of being efficient and diversified.

A CPDO is a relatively new structured credit product that entered the market in 2006. A CPDO seems to be attractive for the investors due to its both high rated (normally AAA/Aaa of S&P and Moody's) principal repayment and fixed coupon payments. In its most typical form, a CPDO is simply an investment vehicle (Special Purpose Vehicle or SPV) paying a periodic coupon of Libor plus a constant spread s as well as the initial investment at the maturity, unless a *default event* occurs. In this chapter we shall define the CPDO *default event* as the failure to pay the stated periodic coupons and/or to repay the principal investment at the maturity. Although it is still an open question, whether they had deserved the top ratings of rating companies when the first generation of CPDOs were launched, in general the CPDO aims to return high yield coupons to investors by taking a leveraged exposure to a basket of credit indexes (typically 50% CDX.NA.IG and 50 % ITRAXX Europe).

The cash flow obligations of the CPDO are backed from the exposure to the CDS indexes, often called *leverage* and varies according to the performance of the underlying indexes, where the leverage simply can be thought as:

$$leverage = \frac{credit\ exposure}{initial\ investment}. \tag{3.1}$$

We denote the exposure to the underlying index by $\alpha(t)$, which is chosen in such a way to yield a relatively low default probability of the CPDO. More specifically, the industry standard choice of $\alpha(t)$ is given by

$$\alpha(t) \equiv \frac{PVL(t) - V(t)}{\mu(t)DV01(t)}, \tag{3.2}$$

where $PVL(t)$ and $V(t)$ are the present value of the CPDO liabilities, and assets, respectively, $\mu(t)$ is the spread paid by the credit index at time t and $DV01(t)$ is the present value of a stream of periodic risk-less payments equal to 1 per annum.

A poor performance of the indices will imply a high leverage level, while a good performance of the indices will decrease the leveraged exposure. As the CPDO targets a credit exposure, which is *sufficient* to pay the promised coupons and the principal, the returns are capped at the stated coupon rate. Therefore, the leverage is controlled dynamically in order to reach the target portfolio size on each roll date and is limited with a maximum portfolio size. Additionally, the CPDO also has the advantage of not being directly affected by the market implied correlation risk, in contrast to the Collateral Debt Obligation (CDO) instrument.

Another favourable feature of the product is that if the credit indices' performances are well enough to guarantee the future promised payments, then the investor benefits from the "cash-in" feature, i.e., as soon as all the promised payments can be made with certainty, the risky investment is reduced to zero. In this case, until the CPDO expires, the investor is only exposed to a risk-free asset but still receives high coupons. On the other hand, a CPDO does not guarantee the repayment of the initial capital invested. The investor can therefore lose 100% of his initial capital. If the accumulated losses from the risky exposure reach a pre-determined threshold for the note value, (typically 10 % of the notional amount invested), then the investor meets a 'cash-out' event, i.e., the loss is locked in and the risky investment is stopped. With this setting, the investors are protected from any losses exceeding the notional invested by banks. The risk that the bank will suffer such a loss is called the *gap risk*.

The following definitions are used in the rest of the chapter.

- **Net Asset Value (NAV)**: NAV is the current market value of the CPDO that is the present value of all outstanding positions including the cash deposit and any other unrealised gains/losses. We denote the NAV at time t with $V(t)$.

- **Cash Deposit Account**: This account holds the proceeds from the investor, interest, premiums and any Mark-to-Market (MtM) gains achieved. Losses are also settled from this account. Hence, NAV is actually what the cash deposit account holds.

- **Target Redemption Value**: This is the present value of all promised liabilities (coupons and principal). We denote the target redemption value at time t with $PVL(t)$.

- **Shortfall** $:= PVL - NAV$, it represents the value that still has to be gained from the CPDO strategy to enable it to *cash-in*. The aim of the CPDO strategy is to make the shortfall equal to zero before the maturity of the contract.

We observe the flows of the product in Figure 3.1.

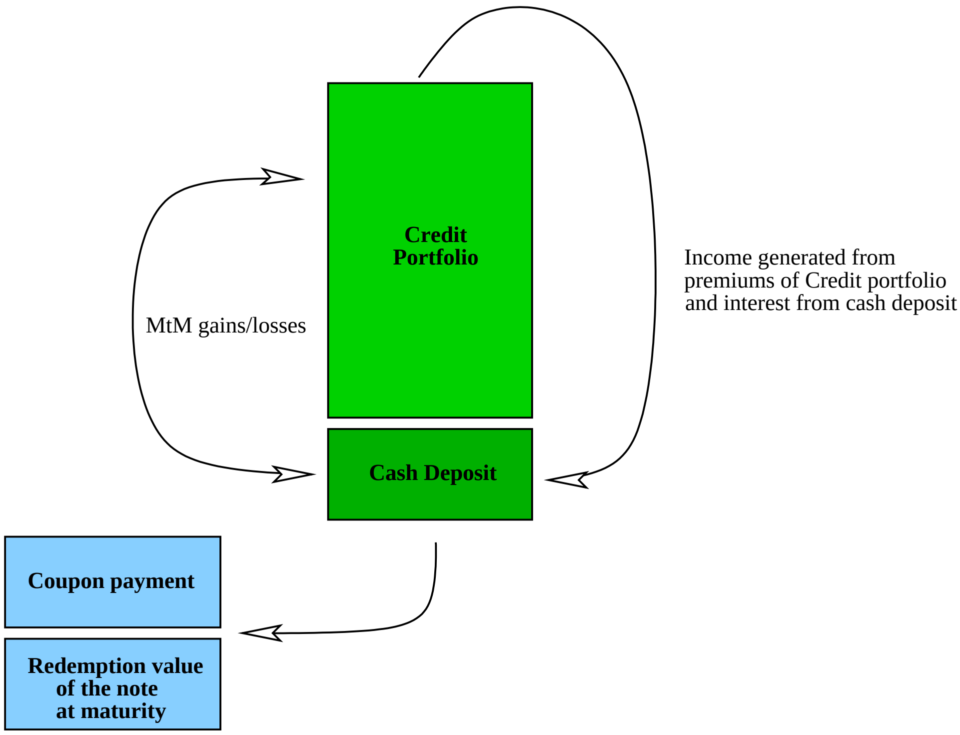

Figure 3.1: CPDO transactions

How the ratings are assigned to CPDOs is not a question that we are going to answer in this chapter. However, interested readers might see the technical reports about rating issues, which can be listed as Torresetti and Pallavicini [TP08], Wong and Chandler [WC06]. Another stream of articles, e.g., Linden et al. [LNB07], Toutain et al. [TTM06], Formica et al. [FMS+06], Varloot et al. [VCC06], [VCC07]) analyse the mechanics and risks that CPDO products are exposed to, providing scenario analyses. In this chapter, we are rather interested in *optimality* of the leverage function introduced in product mechanics. Therefore, we take a different approach and derive an *optimal* leverage function using the stochastic optimal control techniques. We show that the standard leverage function (3.2) is *optimal* when the index spreads are constant, interest rates are zero and defaults are deterministic. We analyse the behaviour of the optimal leverage in the more general case of stochastic defaults and spreads.

3.3 Preliminaries

Generally, stochastic control is used as an alternative solution technique to the martingale approach, which we have introduced in Chapter 1 for continuous time portfolio optimisation problems. The application of the stochastic control methods in portfolio problems is pioneered by Merton [Mer69], [Mer71]. In this section, we first introduce the stochastic control method and the dual approach via Legendre transformation to related Hamilton-Jacobi-Bellmann equation. Finally, we give an example where we apply the stochastic control method for the Merton portfolio problem.

Different from the portfolio optimisation problems in the literature, this chapter introduces a new problem, where we apply the stochastic control technique for minimising the losses in a strategy subject to credit risk. In order to do so, we model the net asset value of the CPDO with a controlled stochastic equation. This section presents mainly from Korn and Korn [KK01], and Johsson and Sircar [JS02].

3.3.1 Stochastic Control

Let $V^\alpha(t)$ be a one dimensional Itô process. A *controlled* stochastic differential equation (CSDE) with an initial value $V(0) = v$ has the form

$$dV^\alpha(t) = \mu(t, V^\alpha(t), \alpha(t))dt + \sigma(t, V^\alpha(t), \alpha(t))dW(t), \tag{3.3}$$

where we denote the one dimensional Brownian motion with $W(t)$ and one dimensional stochastic process we are *free* to choose with $\alpha(t)$. In our problem defined in (3.21), $\alpha(t)$ is the control process, and V denotes the wealth process (or Net Asset Value of the CPDO). Main task is to find an *optimal control process* with respect to a certain cost functional. Translated back to our problem in (3.21), we try to find an *optimal leverage function*, i.e., $\alpha^*(t)$, that maximise the redemption received at maturity (or equivalently minimise the losses) of the CPDO strategy due to long position in credit-risky portolio.

In general, we want to solve the following problem

$$\max_{\alpha(\cdot)\in\mathcal{A}(v,I)} E_{0,v}[F(V^{\alpha}(T))], \tag{3.4}$$

where I is the time set (e.g., $I=[0,T]$, or $I=\{0,1,\dots\}$) and $\mathcal{A}(v,I)$ is the set of *admissible controls*. The control is admissible if $\alpha\in A\subset\mathbb{R}$ and all α are progressively measurable with respect to the filtration $\mathcal{F}_t=\sigma\{W(s);s\leq t\}$ generated by the one dimensional Brownian motion, and additionally if $V^{\alpha}(t)$ is the unique solution to CSDE in (3.3).

Further, let the coefficient functions in (3.3)

$$\begin{aligned}\mu &: [0,T]\times\mathbb{R}\times A\rightarrow\mathbb{R}\\ \sigma &: [0,T]\times\mathbb{R}\times A\rightarrow\mathbb{R}\end{aligned}$$

be continuous and Lipschitz-continuous in v uniformly on $[0,T]\times\mathbb{R}$. Now, let us introduce the *value function* of the problem defined in (3.4) as

$$\sup_{\alpha(\cdot)\in\mathcal{A}(v,I(t))} E_{t,v}[F(V^{\alpha}(T))] =: \phi(t,v), \tag{3.5}$$

where $I(t)=[t,T]\wedge I$. Note that, we assume implicitly that the controlled stochastic process is Markovian. We obtain the characterisations of the value function with the following theorem:

Theorem 3.1. *(Martingale Optimality Principle) Let $\alpha^*()$ be an admissible control, such that, for a function F we have:*

$$H(t,v) := E_{t,v}[F(V^{\alpha^*}(T))] \tag{3.6}$$

we have

$$H(t, V^{\alpha^*}(t)) \text{ is a martingale}$$

$$H(t, V^{\alpha}(t)) \text{ is a supermartingale}$$

for all admissible controls $\alpha(\cdot)$. *Then, we have*

1. $\alpha^*(\cdot)$ *is an optimal control,*

2. $H(t, v) = \phi(t, v)$ *for all* $t \in I$.

Proof: (see p. 230 of [KK01])

3.3.2 Hamilton-Jocobi-Bellman Equation of Stochastic Control

In this subsection, we apply the Theoem 3.1 for the problem defined as

$$\max_{\alpha(\cdot)\in\mathcal{A}(v,[0,T])} E_{0,v}[F(V^{\alpha}(T)), \tag{3.7}$$

where we assume that

$$h(t, v) := E_{t,v}[F(V^{\alpha}(T))]$$

is a $C^{1,2}$ function. Applying Itô's formula, we have

$$\begin{aligned} h(t, V^{\alpha}(t)) &= h(0, v) + \int_0^t h_v(s, V^{\alpha}(s))\sigma(s, V^{\alpha}(s), \alpha(s))dW(s) \\ &+ \int_0^t \Big[h_t(s, V^{\alpha}(s)) + h_v(s, V^{\alpha}(s))\mu(s, V^{\alpha}(s), \alpha(s)) \\ &+ \frac{1}{2}h_{vv}(s, V^{\alpha}(s))\sigma^2(s, V^{\alpha}(s), \alpha(s))\Big]ds. \end{aligned}$$

Note that, $h(t, V^{\alpha}(t))$ is a martingale if the ds integrand equal to 0, given sufficient growth conditions for the integrand of the Itô integral. From the Theorem 3.1, following martingale optimality principle, we can write the HJB-Equation with the theorem below.

Theorem 3.2. *(Verification Theorem for the HJB-Equation) Let* $A \subset \mathbb{R}$ *be bounded and assume that there exists a polynomially bounded* $C^{1,2}$ *solution* $h(\cdot)$ *to the HJB-Equation*

$$\sup_{\alpha\in A}\left\{h_t(t, v) + h_v(t, v)\mu(t, v, \alpha) + \frac{1}{2}h_{vv}(t, v)\sigma^2(t, v, \alpha)\right\} = 0 \tag{3.8}$$

$$h(T, v) = F(v) \tag{3.9}$$

for $(t, v) \in [0, T] \times \mathbb{R}$*,* $v \in \mathbb{R}$*. Then, we have*

$$h(t, v) \geq \phi(t, v).$$

If there exists an admissible control $\alpha^*(t)$ *with*

$$\alpha^*(t) \in \arg\max_{\alpha \in A}\{\dots\}, \tag{3.10}$$

then, we have even

$$h(t, v) = \phi(t, v), \text{ and } \alpha^*(\cdot) \text{ is an optimal control.}$$

With the help of two theorems in this subsection, we apply the following algorithm in order to solve a stochastic control problem:

1. Solve (formally) the optimisation problem in the HJB-equation (3.8) and replace α with the optimal control α^*.

2. Substitute α in (3.8) by α^* obtained in Step 1, omit the supremum operator, and solve the resulting (non-linear) PDE with the boundary condition defined in (3.9).

3. Check if the assumptions made in previous steps are indeed satisfied (concavity of $h(t, v)$, existence of a maximum).

Example 3.1. Merton Portfolio Problem
In this example, we solve the Merton portfolio problem with dual approach using the Legendre transform. Note that we solve the same problem defined in (1.16) with the martingale approach in Chapter 1. There, the control variable is denoted with π, which determines the fraction of wealth ($X(t)$ denotes the wealth process) invested in the stock. We continue the presentation with the notation we introduced in this section. Hence, we want to find the optimal control α^* for the following problem:

$$\sup_{\alpha} E_{t,v}[U(V^\alpha(T))] = \phi(t, v), \tag{3.11}$$

where we use a power utility function of the form

$$U(v) = \frac{v^\gamma}{\gamma}, \quad 0 < \gamma < 1.$$

After applying Itô's formula, we get the related Bellman equation as

$$\phi_t + \sup_{\alpha} \left(\frac{1}{2} \sigma^2 \alpha^2 \phi_{vv} + \mu \alpha \phi_v \right) = 0. \tag{3.12}$$

With $\phi_{vv} < 0$, the maximum of (3.11) attained at

$$\alpha^* = -\frac{\mu \phi_v}{\sigma^2 \phi_{vv}}.$$

Substituting α in (3.12) with α^*, and dropping the supremum operator, we rewrite the Bellman equation as

$$\phi_t - \frac{\mu^2}{2\sigma^2} \frac{\phi_v^2}{\phi_{vv}} = 0. \tag{3.13}$$

Note that at terminal time T we have the boundary condition

$$\phi(T, v) = U(v) = \frac{v^\gamma}{\gamma}. \tag{3.14}$$

In order to solve the non-linear PDE in (3.13) with terminal condition (3.14), we apply a dual approach. Denoting the dual variable to v with $z > 0$ and with assumed convexity of ϕ, we define the Legendre transform of the value function ϕ as

$$\hat{\phi}(t, z) = \sup_{v>0} \{ \phi(t, v) - zv \}. \tag{3.15}$$

We denote the value of v where the optimum is attained with $g(t, z)$, therefore we have

$$g(t, z) = \inf\{ v > 0 | \phi(t, v) \geq zv + \hat{\phi}(t, z) \}.$$

We get the relation between g and $\hat{\phi}$ from (3.15), i.e.,

$$g = -\hat{\phi}_z.$$

Further, with the assumption that ϕ is strictly concave and smooth in v, we have

$$\phi_v(t, g(t, z)) = z \quad \text{or equivalently} \quad g = \phi_v^{-1}.$$

Differentiating with respect to t and z, we get:

$$\phi_{tv} = -\frac{g_t}{g_z} \quad \phi_{vv} = \frac{1}{g_z} \quad \phi_{vvv} = -\frac{g_{zz}}{g_z^3}.$$

Now, differentiating (3.12) with respect to v and substituting the partial derivatives with the ones we have above, we transform the non-linear PDE in (3.13) to a linear PDE as we have

$$\begin{aligned} g_t + \frac{\mu^2}{2\sigma^2} z^2 g_{zz} + \frac{\mu^2}{\sigma^2} z g_z &= 0, \\ g(T, z) &= z^{\frac{1}{\gamma-1}} \text{ (with the power utility function).} \end{aligned}$$

In this case, we may solve the linear PDE in (3.16) with separation of variables as

$$g(t, z) = z^{\frac{1}{\gamma-1}} u(t)$$

for function $u(t)$. For a given (t, v), we have the relation

$$g(t, z) = v$$

and the optimal strategy $\alpha^*(t)$ is

$$\alpha^*(t) = -\frac{\mu}{\sigma^2} z g_z = -\frac{\mu}{\sigma^2} \frac{1}{(\gamma - 1)} g = \frac{\mu}{\sigma^2(1-\gamma)} v.$$

The interpretation is that we hold the fraction $\frac{\mu}{\sigma^2(1-\gamma)}$ of wealth in stocks and the rest in the riskless bond (money market account). Note that, we arrive at the same solution in Example 1.1 defined as the *local risk premium for stock investment*, where for $\gamma = 0$ we have the solution for the logarithmic utility, i.e., $U(v) = \log(v)$.

In the following section, we present the dynamics of the model and using the techniques we introduced so far, we find the "optimal" leverage function used in the CPDO.

3.4 Model Proposal

We denote the initial wealth at time $t_0 = 0$ with $V(0) = 1$, which represents the initial notional of the note (NAV). Suppose CPDO pays a continuous coupon of

$$r + s,$$

where r is the risk-free short interest rate and s is the agreed spread. These coupons are paid from the cash deposit account, which holds the assets of the note. In order to generate the coupon spread s, the CPDO engages in a dynamic investment strategy in an underlying, unfunded index. The cash return of the investment strategy in any infinitesimal interval $(t; t+dt]$ is given by $\alpha(t)dB(t)$, with

$$dB(t) = \mu dt + \sigma dW(t), \tag{3.16}$$

where W is a standard Brownian motion, μ and σ are the suitably chosen constant drift and volatility terms, with $B(0) = 0$. These dynamics in (3.16), where we illustrate some simulated paths in Figure 3.2, will allow us to find the optimal leverage function by applying stochastic control approach techniques below. Note that in our simple model $dB(t)$

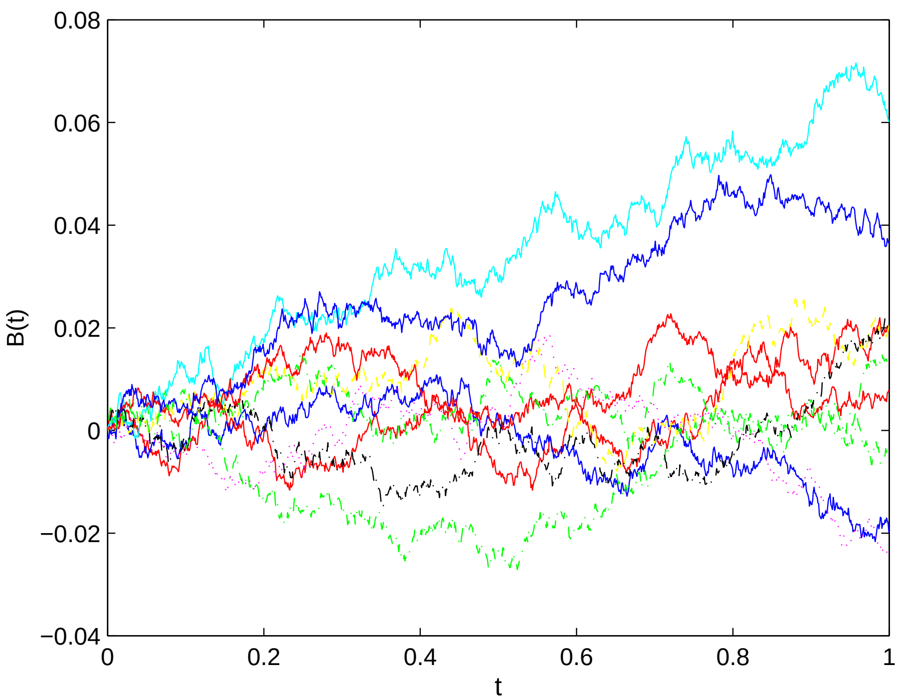

Figure 3.2: Simulated paths for the Brownian motion with drift (or arithmetic Brownian motion).

incorporates the carry generated by the index spread, the mark to market losses/gains deriving from changes in the index spread due to the changes in the default probability of the underlying portfolio. Since the index is rolled over into a new series on a continuous basis, the default risk in index can be negleted, at least to the first order. Of course, one could think of the index spread dynamics being mean-reverting or including a jump component. However in order to obtain a semi-analytic solution for the optimal leverage problem, we assumed the simple dynamics for the gain and losses linked to the underlying credit index investment.

Building the dynamics of the note value

We denote the leverage function, i.e., the notional exposure to the risky investment at time t with $\alpha(t)$. In order to construct the dynamics of the wealth process, we define the discrete time points $0 = t_0 < t_1 < \ldots < t_n = T$ with $\Delta t = t_i - t_{i-1}$, $i = 1, 2, \ldots, n$ and where $t_n = T$ is the maturity of the CPDO. We have our initial wealth, or the NAV ($t_0 = 0$) as $V(0) = 1$. The NAV holds the cash deposit account and the MtM gains/losses from the risky credit position, which forms the asset side of the CPDO. On the other hand, we have the promised coupon payments $r + s$ and repayment of the principal on the liability side. We assume the note pays coupons continuously and the gains/losses are *only* due to the interest that the cash deposit account earns and the MtM gains/losses due to the long position in the CDS index portfolio. Moreover, the leverage $\alpha(0)$ in the credit index portfolio is chosen at the inception of the CPDO and changed dynamically at the beginning of each infinitesimal period. Hence, the main idea of the product is covered, which is basically "betting" on the performance of the CDS index portfolio on each roll date. We express the discrete version of (3.16) as

$$\Delta B = \mu \Delta t + \sigma \Delta W,$$

with $\Delta W = \epsilon\sqrt{\Delta t}$, where ϵ is a standard normal random variable.

In the next time point, i.e., $t_1 = t_0 + \Delta t$ we observe the following flows:

1. We take the notional exposure with the function α to $B(t_1)$; this earns $\alpha(t_0)\Delta B(t_1)$, which has zero cost, since it is a swap contract with zero value at inception.

2. We pay $(r + s)\Delta t$ in the form of a coupon.

3. The initial wealth in the cash deposit account $V(0) = 1$ earns the constant interest rate r until t_1, so we get $rV(0)\Delta t$,

so the change in our wealth (or NAV) can be described as

$$\Delta V(t_1) = V(t_1) - V(0) = \alpha(t_0)\Delta B(t_1) + rV(0)\Delta t - (r+s)\Delta t,$$

where $\Delta B(t_1) = B(t_1) - B(0)$ and $\Delta t = t_1 - t_0$.

Until the next time point, i.e., $t_2 = t_1 + \Delta t$, the cash deposit account earns the constant interest r over $V(t_1)$. If we had losses due to $\Delta B(t_1)$ being negative, then $V(t_1) < V(0)$, otherwise we have $V(t_1) \geq V(0)$. Hence, at the next time point, i.e., $t_2 = t_1 + \Delta t$, we will have similar flows:

1. We take notional exposure with the function α to $B(t_2)$, earning $\alpha(t_1)\Delta B(t_2)$, which has again zero cost.

2. We pay the coupon $(r+s)\Delta t$

3. We put the $V(t_1)$ in the cash deposit account at time t_1, which earns the constant interest rate r until t_2, so we get $rV(t_1)\Delta t$,

implying the change in NAV as

$$\Delta V(t_2) = V(t_2) - V(t_1) = \alpha(t_1)\Delta B(t_2) + rV(t_1)\Delta t - (r+s)\Delta t,$$

where $\Delta B(t_2) = B(t_2) - B(t_1)$, and $\Delta t = t_2 - t_1$. The process continues until the maturity time of the contract, i.e., $t_n = T$. If $\Delta t \to 0$, the dynamics of the NAV can be expressed as

$$dV(t) = \alpha(t)dB(t) + rV(t)dt - (r+s)dt, \tag{3.17}$$

with $V(0) = 1$, and $B(0) = 0$.

We shall impose $V(t) \geq K \geq 0$ for all $t \in [0, T]$, where we define K as the cash-out threshold.

If the wealth process falls below the threshold K at any time, a cash-out event will occur and any risky investment is unwind. Denote by τ, the first time the wealth hits the Target Redemption Value denoted by $PVL(t)$, i.e.,

$$\tau = \inf\{t : V(t) \geq PVL(t)\} \tag{3.18}$$

where

$$PVL(t) \equiv e^{-r(T-t)} + \frac{(r+s)}{r}(1 - e^{-r(T-t)}). \tag{3.19}$$

After this event, we must have that $\alpha(t) = 0$ for all $t \geq \tau$, since NAV is enough to pay for the principal at maturity and for the coupon payments of $(r + s)$.

Problem definition: Obtaining the optimal leverage

Our goal is to choose $\alpha(t)$ optimally in such a way to minimise any shortfall between the CPDO liability and assets. This imply the maximisation of the capital we have at maturity. We define the loss as $1 - V(T)$, due to the risky investment and promised coupon payment $r + s$. More formally, we need to solve the following stochastic optimal control problem

$$\phi^0(t, v) = \sup_{\alpha} E\left[1 - (1 - V(T))^+ \mid V(t) = v\right], \tag{3.20}$$

subject to (3.17). We shall impose that the value of the asset of the CPDO stays positive at any point in time, by setting the *cash-out* boundary condition $\phi^0(t, 0) = 0$ for $0 \leq t \leq T$. Note that by specifying the asset dynamics as in (3.17) and imposing the non-negative asset constraint, we are implicitly assuming that $V(t)$ is always greater than the present value of all future coupon payments, which is a reasonable assumption.

Following Jonsson and Sircar [JS02], we can smooth out the investor's utility function and transform the original optimisation problem (3.20) into

$$\phi(t, v) = \sup_{\alpha} E_{t,v}\left[U(V(T))\right], \tag{3.21}$$

where

$$U(v) = \frac{1}{p}\left[1 - \left((1 - v)^+\right)^p\right], \tag{3.22}$$

is the investor's utility function and we have used of the notation $E_{t,v}[\cdot] = E[\cdot|V(t) = v]$. We shall assume that $p > 1$. Note in the limit of $p \to 1$, the two formulations of the problem yield the same result.

In order to simplify the calculations, it is convenient to work with the discounted wealth process

$$\tilde{V}(t) = e^{-rt}V(t),$$

with the corresponding dynamics of

$$d\tilde{V}(t) = \underbrace{e^{-rt}\alpha(t)}_{:=\tilde{\alpha}(t)} dB(t) - e^{-rt}(r+s)dt \ . \tag{3.23}$$

This, together with $\tilde{v} = e^{-rt}v$, leads to the following modification of problem (3.21):

$$\phi(t,\tilde{v}) = \sup_{\tilde{\alpha}} E_{t,\tilde{v}} \left[\frac{1}{p} \left[1 - \left((1 - e^{rT}\tilde{V}_T)^+ \right)^p \right] \right] \ , \tag{3.24}$$

where $\tilde{v} = e^{-rt}v$ and $\tilde{\alpha} = e^{-rt}\alpha$.

Using the principle of the stochastic optimal control, we formally arrive at the corresponding Hamilton-Jacobi-Bellman equation[1] of

$$\phi_t + \sup_{\tilde{\alpha}} \left[\phi_{\tilde{v}} \left(\tilde{\alpha}\mu - e^{-rt}(r+s) \right) + \frac{1}{2}\phi_{\tilde{v}\tilde{v}}\tilde{\alpha}^2\sigma^2 \right] = 0, \tag{3.25}$$

with the boundary condition

$$\phi(T,\tilde{v}) = \frac{1}{p}\left(1 - \left((1 - \tilde{v}e^{rT})^+ \right)^p \right) \ .$$

Before we are going to solve this equation, we have to point out that actually we would need two more boundary conditions present on the whole time interval, one describing the cash-out and one describing the cash-in event in the transformed variable $\tilde{v}$:

$$\phi\left(t, e^{-rT}K\right) = \frac{1}{p}\left(1 - (1-K)^p\right), \tag{3.26}$$

$$\phi\left(t, e^{-rT} + \frac{r+s}{r}\left(e^{-rt} - e^{-rT}\right)\right) = \frac{1}{p} \ . \tag{3.27}$$

However, to be able to obtain explicit solutions to our optimal leverage problems, we leave those two constraints aside and comment on their relevance in Section 3.5.

Let us now concentrate on the simplified problem: assuming sufficient smoothness of the value function, existence of the optimal leverage strategy, and that $\phi_{\tilde{v}\tilde{v}} < 0$, the first order conditions imply

$$\tilde{\alpha}^*(t) = -\frac{\mu\phi_{\tilde{v}}}{\sigma^2\phi_{\tilde{v}\tilde{v}}}. \tag{3.28}$$

Substituting (3.28) back into (3.25), we are left with the non-linear PDE

$$\phi_t - e^{-rt}(r+s)\phi_{\tilde{v}} - \frac{\mu^2\phi_{\tilde{v}}^2}{2\sigma^2\phi_{\tilde{v}\tilde{v}}} = 0 \tag{3.29}$$

[1] For a better insight of the stochastic control approach, we refer the interested reader to Korn [Kor97].

In order to solve the PDE in (3.29), we transform it in a linear PDE similar to the Black Scholes equation. Assuming the concavity of $\phi(t, \tilde{v})$ and defining the Legendre transform as

$$\hat{\phi}(t, z) = \sup_{\tilde{v}>0}\{\phi(t, \tilde{v}) - z\tilde{v}\}, \tag{3.30}$$

where $z > 0$ denotes the dual variable[2] to $\tilde{v}$. We denote the value of $\tilde{v}$ where the optimum is attained with $g(t, z)$, so that

$$g(t, z) = \inf\{\tilde{v} > 0 | \phi(t, \tilde{v}) \geq z\tilde{v} + \hat{\phi}(t, z)\}.$$

Using the relation

$$\phi_{\tilde{v}}(t, \tilde{v}^*) = \phi_{\tilde{v}}(t, g(t, z)) = z,$$

and differentiating with respect to z, we have for $(t, g(t, z))$ as argument

$$\frac{\partial}{\partial z} z = 1 = \frac{\partial}{\partial z}\left(\phi_{\tilde{v}}(t, g(t, z))\right) = \phi_{\tilde{v}\tilde{v}} g_z$$

$$\Rightarrow \phi_{\tilde{v}\tilde{v}} = \frac{1}{g_z}. \tag{3.31}$$

Differentiating with respect to t,

$$\frac{\partial}{\partial t} z = 0 = \frac{\partial}{\partial t}\left(\phi_{\tilde{v}}(t, g(t, z))\right) = \phi_{\tilde{v}t} + \phi_{\tilde{v}\tilde{v}} g_t$$

$$\Rightarrow \phi_{\tilde{v}t} = -\frac{g_t}{g_z} \tag{3.32}$$

and with respect to z again, we arrive at

$$\frac{\partial^2}{\partial z^2} z = 0 = \frac{\partial^2}{\partial z^2}\left(\phi_{\tilde{v}}(t, g(t, z))\right) = \frac{\partial}{\partial z}\left(\phi_{\tilde{v}\tilde{v}} g_z\right) = \phi_{\tilde{v}\tilde{v}\tilde{v}} g_z^2 + \phi_{\tilde{v}\tilde{v}} g_{zz}$$

$$\Rightarrow \phi_{\tilde{v}\tilde{v}\tilde{v}} = -\frac{g_{zz}}{g_z^3}. \tag{3.33}$$

Now, differentiate (3.29) with respect to $\tilde{v}$, implying

$$\phi_{t\tilde{v}} - e^{-rt}(r+s)\phi_{\tilde{v}\tilde{v}} - \frac{\mu^2}{2\sigma^2}\frac{2\phi_{\tilde{v}\tilde{v}}^2\phi_{\tilde{v}} - \phi_{\tilde{v}}^2\phi_{\tilde{v}\tilde{v}\tilde{v}}}{\phi_{\tilde{v}\tilde{v}}^2} = 0. \tag{3.34}$$

Substituting (3.31), (3.32), and (3.33) back in (3.34), we have the linear PDE only along $(t, g(t, z))$ as

$$g_t + \frac{\mu^2}{\sigma^2} z g_z + \frac{\mu^2}{2\sigma^2} z^2 g_{zz} + e^{-rt}(r+s) = 0\ . \tag{3.35}$$

[2]See Jonsson and Sircar [JS02] and p. 134 of Korn [Kor97] for the details of the dual approach.

With the modified terminal condition

$$\phi(T,\tilde{v}) = \frac{1}{p}\left(1 - \left((1-\tilde{v}e^{rT})^{+}\right)^{p}\right), \tag{3.36}$$

where for our problem, we have $p > 1$. We derive the terminal condition for $g(t,z)$ as follows

$$\begin{aligned} \phi_{\tilde{v}}(T,\tilde{v};p) &= e^{rT}\left(1-e^{rT}\tilde{v}\right)^{p-1} 1_{\tilde{v}<e^{-rT}} = z \\ \Rightarrow \quad & \tilde{v} = e^{-rT}\left(1 - z^{\frac{1}{p-1}} e^{\frac{-rT}{p-1}}\right)^{+}. \end{aligned}$$

Therefore, the problem is now to solve the following parabolic, linear PDE

$$\frac{\partial g}{\partial t} + \frac{\mu^2}{\sigma^2} z \frac{\partial g}{\partial z} + \frac{\mu^2}{2\sigma^2} z^2 \frac{\partial^2 g}{\partial z^2} + e^{-rt}(r+s) = 0, \tag{3.37}$$

with the terminal condition

$$g(T,z) = c\left(1 - z^{\beta} c^{\beta}\right)^{+}, \tag{3.38}$$

where

$$c := e^{-rT} \quad \text{and} \quad \beta := \frac{1}{p-1}.$$

Furthermore, the discounted optimal leverage function $\tilde{\alpha}^*(t)$ can be written in terms of the dual function only

$$\tilde{\alpha}^*(t) = -\frac{\mu}{\sigma^2} z \frac{\partial g}{\partial z}, \tag{3.39}$$

and it is related to the wealth $\tilde{v}$ via the equality

$$\tilde{v} = g(t,z) \equiv \phi_{\tilde{v}}^{-1}(t,z). \tag{3.40}$$

Equation (3.37) can be reduced to a standard heat equation by means of some simple standard change of variable. Define

$$a = \frac{\mu^2}{\sigma^2}, \qquad \tau(t,z) = T - t, \text{ and } \qquad y(t,z) = \ln z.$$

We rewrite (3.37) with the definition below

$$\tilde{g}(\tau(t,z), y(t,z)) := g(t,z)$$

Hence, we have the partial derivatives

$$\frac{\partial g}{\partial t} = \frac{\partial \tilde{g}}{\partial \tau}\frac{\partial \tau}{\partial t} + \frac{\partial \tilde{g}}{\partial y}\frac{\partial y}{\partial t} = \frac{\partial \tilde{g}}{\partial \tau}(-1),$$

$$\frac{\partial g}{\partial z} = \frac{\partial \tilde{g}}{\partial \tau}\frac{\partial \tau}{\partial z} + \frac{\partial \tilde{g}}{\partial y}\frac{\partial y}{\partial z} = \frac{\partial \tilde{g}}{\partial y}\frac{1}{z},$$

and

$$\frac{\partial^2 g}{\partial z^2} = -\frac{1}{z^2}\frac{\partial \tilde{g}}{\partial y} + \frac{1}{z}\left(\frac{\partial^2 \tilde{g}}{\partial y^2}\frac{1}{z}\right) = \frac{1}{z^2}\left(\frac{\partial^2 \tilde{g}}{\partial y^2} - \frac{\partial \tilde{g}}{\partial y}\right).$$

Plugging these partial derivatives into (3.37), the PDE will have constant coefficients as

$$\frac{\partial \tilde{g}}{\partial \tau} = \frac{a}{2}\left(\frac{\partial^2 \tilde{g}}{\partial y^2} + \frac{\partial \tilde{g}}{\partial y}\right) + (r+s)e^{-r(T-\tau)}. \tag{3.41}$$

Now, we want to represent (3.41) in an inhomogeneous heat equation form as defined to be

$$\frac{\partial \hat{g}}{\partial \hat{\tau}} = \frac{\partial \hat{g}}{\partial x^2} + \Theta(\hat{\tau}), \tag{3.42}$$

where $-\infty < x < \infty$, $\hat{\tau} > 0$ and $\Theta(\cdot)$ is a continuous, bounded function on $\mathbb{R} \times (0, \infty)$. Using the following change of variables,

$$x(\tau, y) = y + \frac{a}{2}\tau, \text{ and } \qquad \hat{\tau}(\tau, y) = \frac{a}{2}\tau,$$

rewriting (3.41) by using

$$\hat{g}(\hat{\tau}(\tau, y), x(\tau, y)) := \tilde{g}(\tau, y),$$

we obtain

$$\frac{\partial \hat{g}}{\partial \hat{\tau}} = \frac{\partial^2 \hat{g}}{\partial x^2} + \underbrace{\frac{2(r+s)}{a}e^{-r\left(T-\frac{2\hat{\tau}}{a}\right)}}_{=:\Theta(\hat{\tau})}, \tag{3.43}$$

which is the standard heat equation with the inhomogeneous term $\Theta(\cdot)$ depending only on the time variable. The expression in (3.43) has a unique solution for $-\infty < x < \infty$ and $\hat{\tau} > 0$, where $\Theta(\hat{\tau})$ is bounded and continuous on $\mathbb{R} \times (0, \infty)$ as

$$\hat{g}(\hat{\tau}, x) = \underbrace{\int_{-\infty}^{\infty} G(x, \xi, \hat{\tau}) f(\xi) d\xi}_{=:I_1} + \underbrace{\int_0^{\hat{\tau}} \int_{-\infty}^{\infty} G(x, \xi, \hat{\tau} - t')\Theta(t') d\xi dt'}_{=:I_2}, \tag{3.44}$$

where the Green function denoted by $G(\cdot)$ is defined as

$$G(x, \xi, \tau) = \frac{1}{2\sqrt{\pi\tau}} \exp\left(-\frac{(x-\xi)^2}{4\tau}\right), \tag{3.45}$$

and $f(\cdot)$ denotes the initial condition for the heat equation defined for $t = T \Rightarrow \hat{\tau} = 0$, and $x = y$ as

$$f(\xi) := \hat{g}(0, \xi) = c(1 - e^{\beta\xi}c^{\beta})^+. \tag{3.46}$$

First, we calculate the integral I_1, which is similar to a Black and Scholes Put option as,

$$I_1 = \int_{-\infty}^{\infty} c(1 - e^{\beta\xi} c^{\beta})^{+} \frac{1}{2\sqrt{\pi\hat{\tau}}} \exp\left(-\frac{(x-\xi)^2}{4\hat{\tau}}\right) d\xi. \tag{3.47}$$

Now, changing the variable with

$$-\frac{(x-\xi)}{\sqrt{2\hat{\tau}}} = u \Leftrightarrow d\xi = \sqrt{2\hat{\tau}}du \text{ and } \xi = x + u\sqrt{2\hat{\tau}}$$

we have

$$I_1 \neq 0 \Leftrightarrow u < \frac{\ln(\frac{1}{c^{\beta}}) - x\beta}{\beta\sqrt{2\hat{\tau}}} = -\frac{\ln(c) + x}{\sqrt{2\hat{\tau}}},$$

therefore,

$$\begin{aligned} I_1 &= \Bigg[c\Phi\left(-\frac{\ln(c)+x}{\sqrt{2\hat{\tau}}}\right) \\ &\quad - c^{\beta+1} \int_{-\infty}^{-\frac{\ln(c)+x}{\sqrt{2\hat{\tau}}}} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(u-\sqrt{2\hat{\tau}}\beta)^2}{2}\right) \exp\left(\hat{\tau}\beta^2 + \beta x\right) du \Bigg]. \end{aligned}$$

Again, changing the variable

$$u - \sqrt{2\hat{\tau}}\beta = v \Leftrightarrow du = dv,$$

and the upper bound for the integral is

$$-\frac{\ln(c)+x}{\sqrt{2\hat{\tau}}} - \beta\sqrt{2\hat{\tau}}.$$

Hence, we have

$$I_1 = c\Phi\left(-\frac{\ln(c)+x}{\sqrt{2\hat{\tau}}}\right) - c^{\beta+1}\exp\left(\hat{\tau}\beta^2 + \beta x\right)\Phi\left(-\frac{\ln(c)+x}{\sqrt{2\hat{\tau}}} - \beta\sqrt{2\hat{\tau}}\right), \tag{3.48}$$

where $\Phi(\cdot)$ denotes the standard normal distribution function.

Furthermore, the double integral I_2 in (3.44) can be expressed as

$$\begin{aligned} I_2 &= \int_0^{\hat{\tau}} \frac{2(r+s)}{a} \exp\left(-r\left(T - \frac{2t'}{a}\right)\right) dt' \underbrace{\int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi(\hat{\tau}-t')}} \exp\left(-\frac{(x-\xi)^2}{4(\hat{\tau}-t')}\right) d\xi}_{=1} \\ &= \frac{2(r+s)}{a} \exp(-rT) \int_0^{\hat{\tau}} \exp\left(\frac{2rt'}{a}\right) dt' \\ &= \frac{e^{-rT}(r+s)}{r}\left(e^{\frac{2r\hat{\tau}}{a}} - 1\right) \end{aligned}$$

Transforming the variables back, and using the fact that

$$\hat{g}(\hat{\tau}(\tau, y), x(\tau, y)) = \tilde{g}(\tau, y) = g(t, z)$$

we arrive at the unique solution to (3.37) given by

$$\begin{aligned} g(t,z) &= e^{-rT}\Phi[d_1(t,z)] - z^{\beta} e^{(\beta+1)\left(\beta\frac{\mu^2}{2\sigma^2}(T-t)-rT\right)}\Phi[d_2(t,z)] \\ &\quad + \frac{r+s}{r}\left(e^{-rt} - e^{-rT}\right), \end{aligned}$$

where

$$d_1(t,z) = \frac{rT - \ln z - \frac{\mu^2}{2\sigma^2}(T-t)}{\sqrt{\frac{\mu^2}{\sigma^2}(T-t)}},$$

and

$$d_2(t,z) = d_1(t,z) - \beta\sqrt{\frac{\mu^2}{\sigma^2}(T-t)}.$$

Remember the optimal discounted leverage function $\tilde{\alpha}^*(t)$ is given by

$$\tilde{\alpha}^*(t) = -\frac{\mu}{\sigma^2} z \frac{\partial g}{\partial z} = \frac{\mu}{\sigma^2}\kappa\beta z^{\beta}\Phi(d_2(t,z)) \tag{3.49}$$

As we have

$$\begin{aligned} g_z &= e^{-rT}\varphi[d_1(t,z)]d_z^1 - \kappa\left(\beta z^{\beta-1}\Phi[d_2(t,z)] + z^{\beta}\varphi[d_2(t,z)]d_z^2\right) \\ &= -\kappa\beta z^{\beta-1}\Phi(d_2(t,z)) \end{aligned}$$

with

$$\kappa := e^{(\beta+1)\left(\beta\frac{\mu^2}{2\sigma^2}(T-t)-rT\right)},$$

and

$$d_z^1 = d_z^2 := -\frac{1}{\sqrt{\frac{\mu^2}{\sigma^2}(T-t)}}\frac{1}{z},$$

denoting the standard normal density function with $\varphi(\cdot)$, we have an explicit formula of the optimal leverage strategy. Note that, $Z(t)$ (the optimal dual variable) is related to the asset value by

$$V(t) = e^{rt} g(t, Z(t)), \tag{3.50}$$

which can in general only be solved numerically. In total, we have shown that the HJB-Equation of our stochastic control problem thus possesses the desired solution.

3.5 Numerical Results

In this section, we present some graphs and simulations for illustrating the behaviour of the optimal leverage function with respect to the NAV. We further examine its sensitivity with respect to the duration of the contract, and to the volatility of the relative return of the risky asset. Moreover, we analyse the behaviour of the optimal leverage with different offered spreads s and its sensitivity to the exponent p (risk-aversion parameter) characterising our loss function. Also, we compare our optimal leverage function with the one that is popular among practitioners.
After having fixed the present time variable t, we use the following algorithm for determining the optimal leverage function:

1. For given values of NAV $v \in [0, 1.5]$ introduce $\tilde{v} = \exp(-rt)v$.
2. Determine the optimal dual value z^* which solves the equation $g(t, z^*) = \tilde{v}$ by a root finding method (such as a Newton type method).
3. With the value z^* obtained in the previous step, calculate the optimal leverage function in (3.49) for fixed t, i.e., $\tilde{\alpha}^*(t, z^*)$.
4. Find $\alpha^*(t)$, i.e., $\alpha^*(t) = \exp(rt)\tilde{\alpha}^*(t)$

Figures 3.3 and 3.4 show that unlike the standard leverage function commonly employed in the industry, α_t does not decrease linearly in $V(t)$ but exhibits a non-monotone behaviour. For $V(t)$ equal to $PVL(t)$ (marked by the red diamond on x-axis), where

$$PVL(t) \equiv (r + s)\frac{1 - e^{-r(T-t)}}{r} + e^{-r(T-t)} \tag{3.51}$$

is the present value of the outstanding liabilities of the CPDO, then $\alpha(t) = 0$.

When the level of NAV is equal to the liabilities of the CPDO, no further risky investment is required in order to pay the outstanding coupons and repay the principal investment. The *cash-in* feature is endogenous in the specification of the investor's utility function (3.22) as no benefit is associated with a redemption value higher than the initial investment. As soon as $V(t) = PVL(t)$, the CPDO becomes in effect a risk-less coupon paying bond which can be unwound at market prices or held by investors until maturity. When

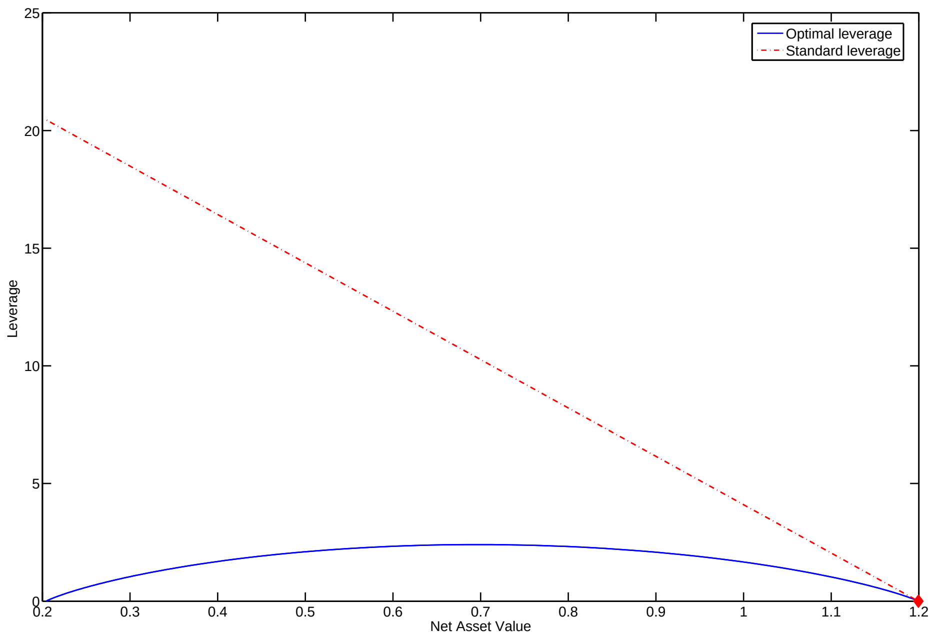

Figure 3.3: Optimal and standard leverage as a function of $V(t)$. Parameters' set: $\mu = 0.005$, $\sigma = 0.05$, $r = 0.0005$, $s = 0.02$, $T = 10$, and $p = 1.1$

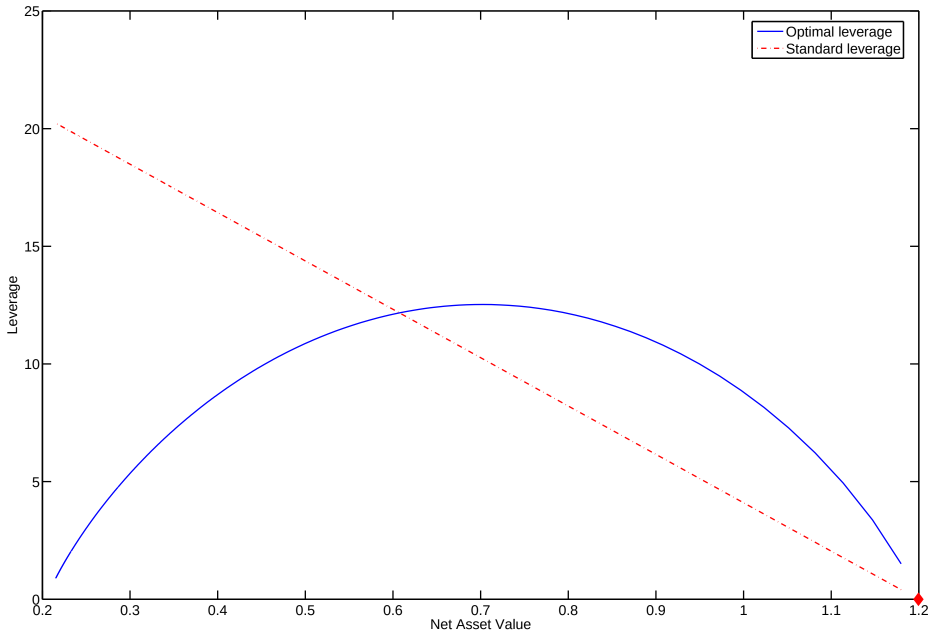

Figure 3.4: Optimal and standard leverage as a function of $V(t)$. Parameters' set: $\mu = 0.005$, $\sigma = 0.01$, $r = 0.0005$, $s = 0.02$, $T = 10$, and $p = 1.1$

$V(t)$ decreases as a consequence of losses, due to defaults or adverse spread movements, the optimal leverage $\alpha(t)$ increases up to a maximum level, which depends on the specification of the model parameters, and then decreases to 0 when

$$V(t) = \left[(r+s)\frac{1-e^{-r(T-t)}}{r} \right], \tag{3.52}$$

i.e., the present value of the remaining coupon payments. This behaviour is related to our specification of the asset dynamics (3.23) as well as the positivity constraint on $V(t)$. In the current formulation of the problem coupons are always paid by the SPV and the credit risk affects only the principal repayment at maturity. If the present value of the stated coupon payments is lower than the initial investment, the SPV can always use the cash account to pay for coupons. Our problem specification imposes that $V(t)$ is always greater than the present value of all outstanding coupon payments so that the value of the assets is always non negative. The bell shaped functional form of the optimal leverage function is hence explained. As the value of the firm approaches $PVL(t)$ the value of the risky investment must be reduced.

Note also that in our formulation of the problem, the *gap risk*, i.e. the risk of jumps in the asset values which would make $V(t)$ be negative, is equal to zero. The gap risk is usually underwritten by the sponsor of the SPV for a fee. Allowing for the possibility of jumps and negative assets would conceivably change the shape of the optimal leverage as investors would have an incentive to increase their leverage for small levels of $V(t)$, since the sponsor of the CPDO would bear a considerable portion of the potential losses. Investors on the other side would retain the upside. In order to control the gap risk, it is common practise in the industry to cap the maximum leverage. Also, CPDOs are usually unwound if the asset value $V(t)$ falls below a strictly positive threshold (cash-out event).

We consider the time point $t = 0$ which starts the period $[0, T]$, where T denotes the maturity of the CPDO. We observe the leverage function with respect to the NAV in Figure 3.5 with different values of T, and the rest of the parameters do not change. We observe that the leverage function gets lower with increasing maturity. This is plausible, on one hand, the cash-in point moves to the right with increasing maturity, and on the other hand, one has to take a higher risk (i.e. a higher leverage), if he wants to succeed in a shorter time.

Using different values of the volatility of the relative return of the risky asset resulted in Figure 3.6. We observe that with decreasing σ, the leverage function increases. This

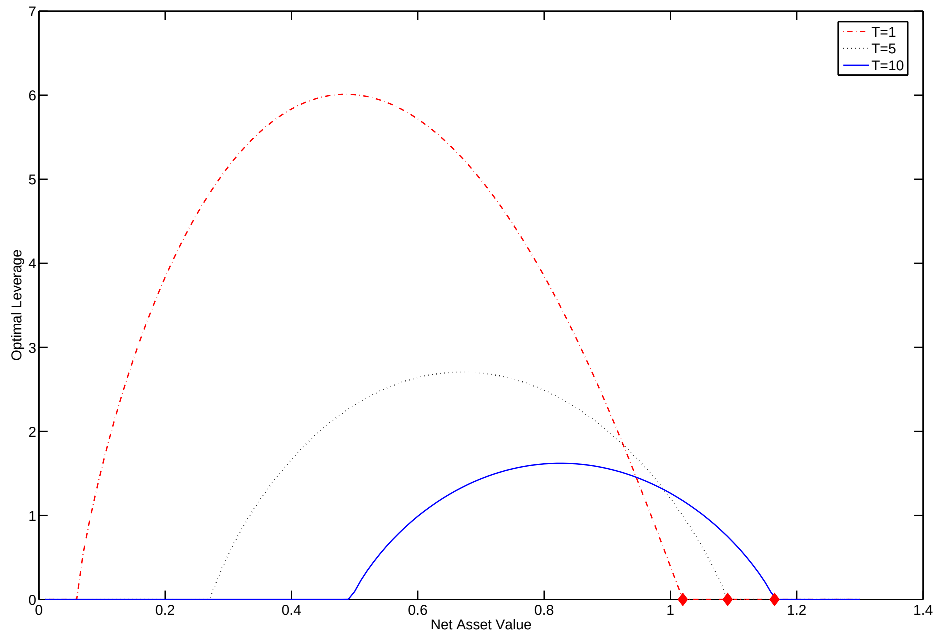

Figure 3.5: The optimal leverage function with respect to the NAV for different CPDO maturities, where the parameter set is $\mu = 0.005$, $\sigma = 0.05$, $r = 0.04$, $s = 0.02$, $T = 1/5/10$, and $p = 1.1$

behaviour can be explained by the option type final utility function of our problem formulation. Further, it is clear that one needs some level of volatility to have a chance to succeed in generating the necessary payoffs, if one is below the cash-in point. So with a lower volatility in the underlying, one has to take a higher leveraged position to reach such a level of volatility. In the case of $\sigma = 0$ and $r = 0$, if we interpret μ as the spread paid by the index investment, then the optimal leverage function $\alpha(t)$ is linear in $V(t)$,

$$\alpha(t) = \frac{(1+s)(T-t) - V(t)}{\mu t} = \frac{PVL(t) - V(t)}{\mu DV01(t)} \tag{3.53}$$

and the optimal leverage function derived in this paper coincides with the leverage function commonly used in the industry.

The sensitivity of the optimal leverage function to the offered spread s is explained in Figure 3.7. As the figure illustrates, with increasing s, the leverage function shifts to the right on the x-axis. This behaviour can be explained by the linear increase of the required payments.

Figure 3.8 demonstrates the sensitivity of the optimal leverage function with respect to variations of the risk aversion parameter p. There is the obvious tendency that the closer

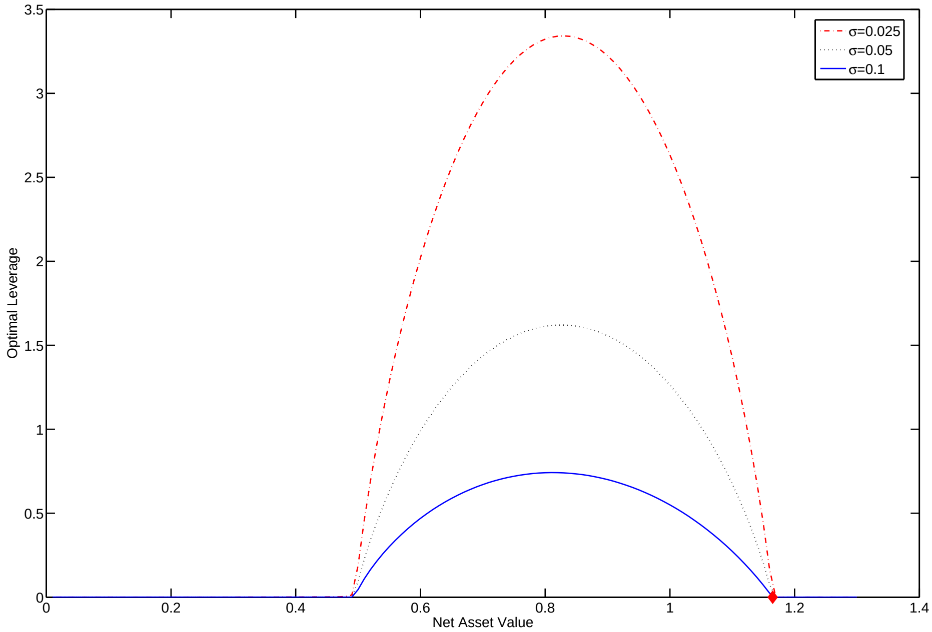

Figure 3.6: The optimal leverage function with respect to the NAV for different volatilities, where the parameter set is $\mu = 0.005$, $\sigma = 0.025/0.05/0.1$, $r = 0.04$, $s = 0.02$, $T = 10$, and $p = 1.1$

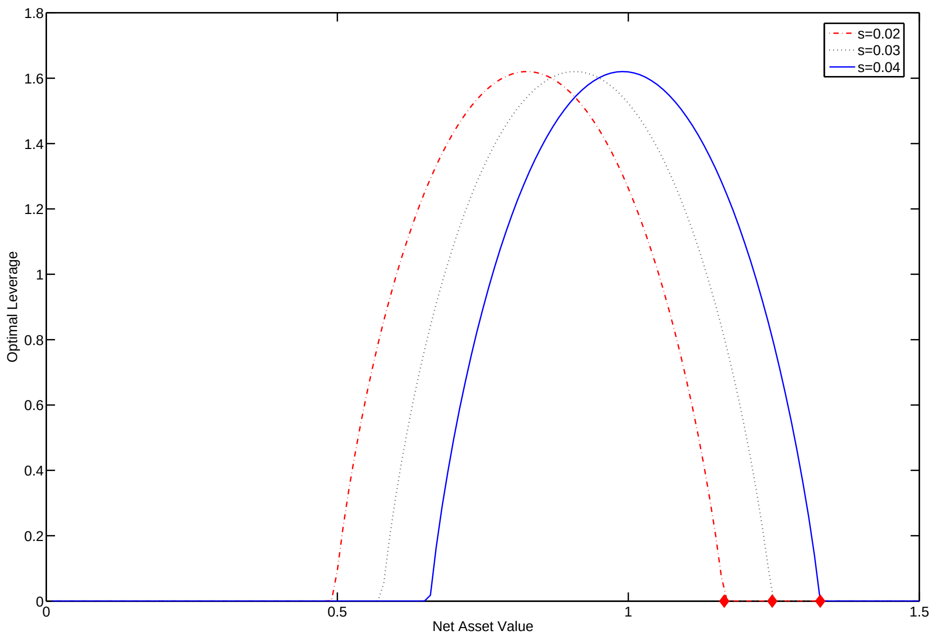

Figure 3.7: The optimal leverage function with respect to the NAV for different offered spreads, where the parameters' set is $\mu = 0.005$, $\sigma = 0.05$, $r = 0.04$, $s = 0.02/0.03/0.04$, $T = 10$, and $p = 1.1$

p gets to 1, the higher the leverage in the optimal strategy is. However, we observe that the leverage is decreasing when NAV reaches small values. The reason for this is that, for those values, the investors becomes very risk averse. They seem to have accepted the losses for small values of NAV and tries to avoid even bigger losses by following a strategy of only a small leverage. It seems that there is a kind of *automatic cash-out behaviour*. This is similar to the behaviour of hedging strategies that one can observe in the area of quantile hedging of stock options (e.g., see Föllmer and Leukert [FL99]). Further, if we look at our computed optimal leverage strategies, they are quite similar to strategies used in the industry (see below when analysing the dynamic behaviour of our strategy), although they implement a linear leverage that decreases with increasing wealth. However, the cash-out feature in the industry strategy limits the risky behaviour of the investor. This can be compared with our built-in automatic cash-out feature as mentioned above.

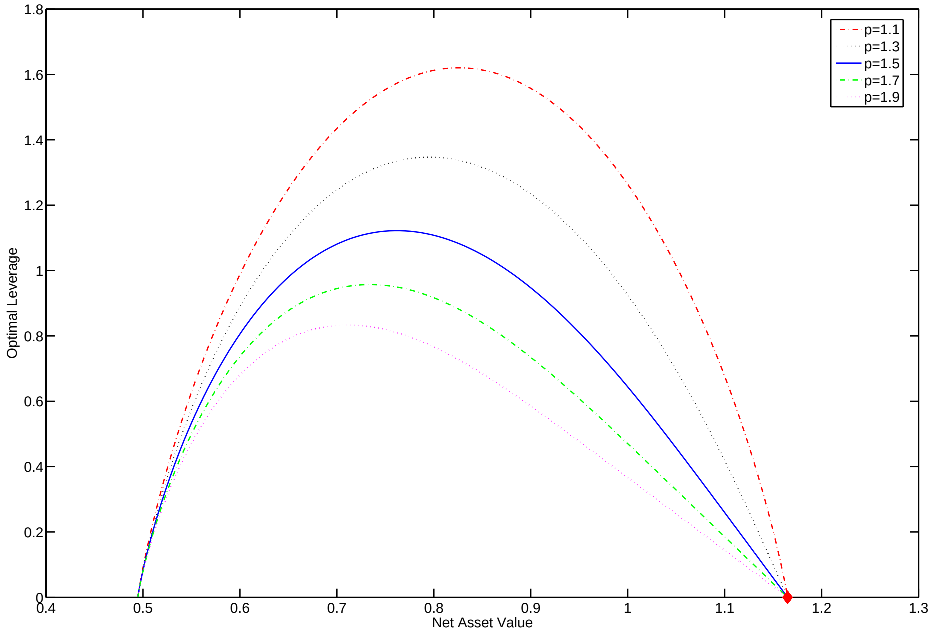

Figure 3.8: Optimal leverage as a function of V_t for different levels of p. Parameters' set: $\mu = 0.005$, $\sigma = 0.05$, $r = 0.04$, $s = 0.02$, $T = 10$, and $p = 1.1/1.3/1.5/1.7/1.9$

After having analysed the static behaviour of the optimal leverage strategy, we are now illustrating its dynamic performance in dependence on the underlying NAV process. Therefore, we simulated independent paths of the NAV via discretising the $B(t)$-process in (3.16) starting with $B(0) = 0$. Remember that the paths of $B(t)$ explain the gains/losses process, and initially we have $V(0) = 1$. The maturity of the CPDO in the three simulations is $T = 10$ and we fix $p = 1.1$.

The first simulation demonstrates the cash-in feature of the CPDO strategy. We observe in Figure 3.9 that for the simulated path of NAV, the optimal leverage drops to 0 when the NAV reaches the PVL (plotted by dashed red line), i.e., $\alpha^*(\tau) = 0$ when $PVL(\tau) = V(\tau)$.

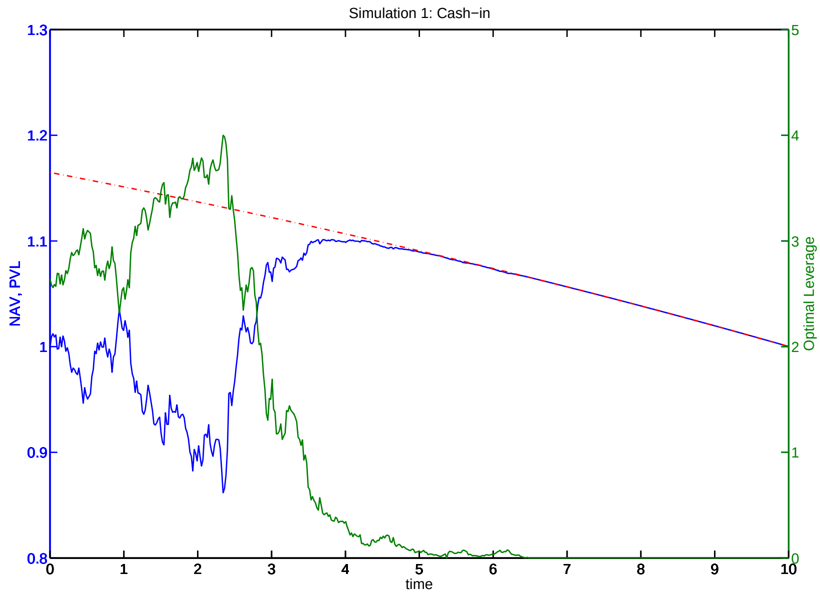

Figure 3.9: Sample path with cash-in event. Parameters' set: $\mu = 0.015$, $\sigma = 0.025$, $r = 0.04$, $s = 0.02$, $T = 10$, and $p = 1.1$

The second simulation in Figure 3.10 considers the case when the cash-in feature of the product is not achieved and the principal redemption at the maturity is less then the initial investment, hence the strategy defaults.

In the last simulation, we compare the behaviour of the NAV dynamics of both the optimal and the linear leverage functions with respect to the same simulated gain/loss process $B(t)$. The key observation in Figure 3.11 is that, using the proposed optimal leverage function, the CPDO cashes-in approximately after 6 years, whereas with the linear leverage function, the strategy cashes-in approximately after 8 years. Remembering that the major aim of the CPDO strategy is to cash-in and offer the investors (quasi risk-free) high coupon rates, we may conclude that usage of the optimal leverage function we

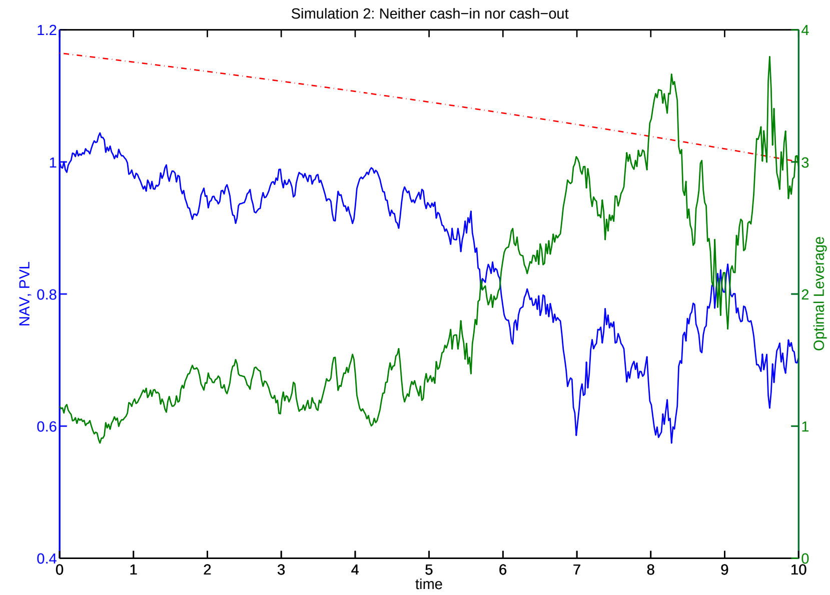

Figure 3.10: Sample path with default at maturity. Parameters' set $\mu = 0.0025$, $\sigma = 0.05$, $r = 0.04$, $s = 0.02$, $T = 10$, and $p = 1.1$

propose can help on catching an earlier cash-in feature.

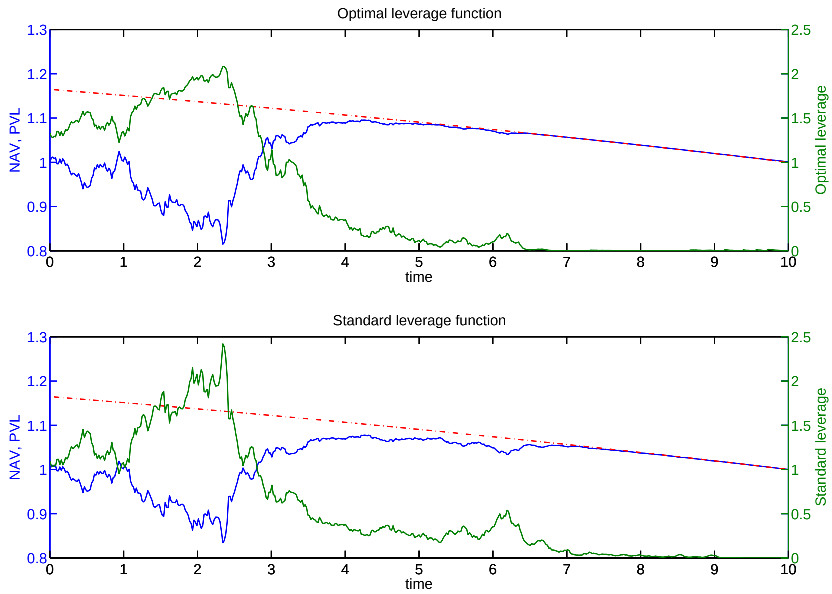

Figure 3.11: The simulation of the NAV dynamics with the standard and the optimal leverage function with the parameters' set $\mu = 0.02$, $\sigma = 0.05$, $r = 0.04$, $s = 0.02$, $T = 10$, and $p = 1.1$

3.6 Summary

A CPDO is a very recent financial product that is generated on the basis of a simple linear leverage strategy as described in (3.2). Thus, it still contains the possibility of ending up with a (bounded) loss, an event that even our optimal strategy cannot exclude. However, by setting up a dynamic optimisation problem that focuses on minimising a possible loss at maturity, we have computed a leverage strategy that possesses an optimality property and coincides with the linear leverage function used among practitioners only for the case of zero interest rate and volatility. If one constrains the CPDO's assets V_t to stay positive at any time and coupons are assumed to be always paid, the optimal leverage exhibits a bell shaped form on the SPV asset value. Some numerical examples have shown promising behaviour of our strategy. It is particularly satisfying that we seem to have an automatic cash-out like behaviour when the NAV has become so small that the probability of being able to pay out all our promised payments is too low. Further, the choice of the risk-aversion parameter p also leaves the investor some freedom to specify his attitude towards risks as seen in Figure 3.8.

Of course, one should consider more realistic gain/loss processes as the Brownian motion with drift type one that we looked at. Also, the inclusion of the cash-out feature is desirable aspect. Including this will not be a big problem from the numerical point of view, but it probably will not allow to solve the corresponding dynamic programming problem. For practical purposes, one could also use our computed optimal strategy and modify it in such a way that this feature is treated.

Our work should be seen as a starting point and it has already demonstrated that an optimised strategy can perform better than an adhoc strategy. In order to derive a closed form solution for the leverage function, we had to resort to a set of simplifying assumptions for the dynamics of the risky investment. In particular, we refrained from modelling the dynamics of the credit index spread and losses arising from the defaults in the underlying portfolio separately, but instead condensed the returns of this two components into a single random process. Also, in our framework we have not considered the possibility of the negative SPV assets, which in presence of jumps in the dynamics of V_t, would give rise to a contingent payment by the SPV sponsor (typically a bank). Both extensions would offer a valid contribution to the understanding of the problem. Finally, one should note that we, indeed, solved a stochastic control problem (nearly) explicitly that has not been dealt with in the literature before.

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