Annulus and Center Location Problems

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Chapter 1

Introduction

1.1 Preface

This thesis deals with very natural and attractive topic of discrete mathematics. In spite of naturality of problems mentioned here some of them are not so „easy“ as it seems. For instance, intuitively expected equivalence does not take place always. Moreover, „well–studied“ problems can be seen in a different way that helps to construct more efficient solution. Sometimes the obtained results in various metrics are surprising. The problems have theoretical beauty as well as the practical relevance in many application fields.

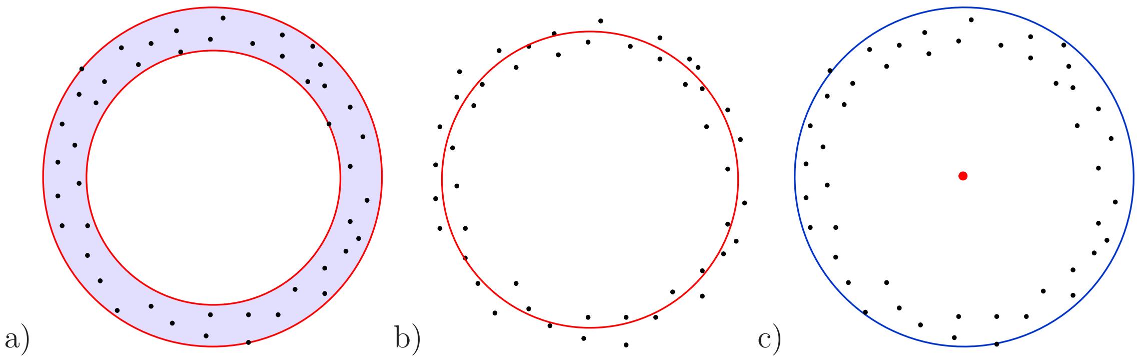


Figure 1.1: *a) MWAP; b) CLP; c) PLP.*

In this work we study and investigate the following problems, which were originally formulated and solved in Euclidean space $R^2_{l_2}$: The first is the *equity problem* where for a given finite set of existing facilities we want to find a new one so that the difference between maximal and minimal effects on existing facilities is minimized. The second is the *minimum width annulus problem* (MWAP), where we search for the most narrow ring between two concentric circles with the set of existing points between them (Figure 1.1 a)). The next problem is the *circle center location* or *circle location problem* (CLP), where the distance between existing points and circle is minimized (Figure 1.1 b)). And the last problem is the *point center location* or *point location problem* (PLP), where we look for a point with minimal maximal distance to existing points (Figure 1.1 c)).

It was conjectured by Hamacher and Schoebel (2005), confirmed and extended by Gluchshenko [29] that the equity, minimum width annulus and circle location problems are equivalent. It is shown for planar problems with respect to Euclidean, Rectilinear and Chebyshev distances. The following questions were, however, still open

- How can the minimum width annulus problem be solved in $R^2_{l_1}$ and in networks?
- How are the minimum width annulus problem and the circle location problem related in networks?
- How are the minimum width annulus problem and the point location problem related in $R^2_{l_2}$, $R^2_{l_1}$ and in networks?

These questions have triggered the research of this thesis.

1.2 Overview of the thesis

This section presents an overview of the work and explains how the material is organized.

Chapter 2 in Section 2.1 reviews the literature on equity problem, minimum width annulus problem, circle location problem and point location problem in Euclidean metric. Relations between problems for further using in the next sections are mentioned. Then in Section 2.2 the minimum width annulus problem is formulated and investigated in Rectilinear space. It is shown, that in contrast to Euclidean metric, the minimum width annulus problem and the point location problem have at least one common optimal point. It helps to find the interval containing a solution point of the minimum width annulus problem with Rectilinear metric in linear time and to solve the minimum width annulus problem in $\mathcal{O}(nlogn)$ time along this interval. The equivalence of the circle location problem to the minimum width annulus problem is proved. The obtained results are analysed and transfered to Chebyshev metric.

Chapter 3 is the main chapter of the thesis, which deals with problems in unweighted undirected networks. In Section 3.1 the notions of circle, sphere and annulus in networks are introduced. An $\mathcal{O}(mn)$ time algorithm for solving of the minimum width annulus problem is constructed and implemented. The algorithm is based on the fact that at least one middle point of edges of an unweighted undirected network solves MWAP. Obtained complexity is better than the complexity $\mathcal{O}(mn + n^2 logn)$ in unweighted case of the fastest known algorithm for minimizing of the range function, which is mathematically equivalent to MWAP. In this section we extend the minimum width annulus problem to the problem on subsets and to the restricted minimum width annulus problem, analyse and solve them. Also the p–minimum width annulus problem is formulated and explored. We have proved NP–hardness of the problem. However, the p–MWAP can be solved in polynomial $\mathcal{O}(m^2n^3p)$ time with a natural assumption, that each minimum width annulus covers all vertexes of a network having distances to the central point of annulus less than or equal to the radius of its outer circle.

In Section 3.2 the differences of planar and network circles are discussed. This differences cause nonequivalence of the circle location problem to the minimum width annulus problem in general case. However, the minimum width annulus problem is effectively used for solving of the circle location problem. The complexity of the developed and implemented algorithm is of order $\mathcal{O}(m^2n^2)$. It should be noted, that the circle location problem in networks has been formulated in this work for the first time and differs from the well–studied location of cycles in networks.

Section 3.3 focuses on the point location problem. We have not found any references on relation of the problem to the minimizing of the range function. However, the minimum width annulus problem has been very effectively used in this work for solving of the point location problem to optimality. Moreover, the developed algorithm is so simple that it can be easily applied to complex networks manually. Its theoretical complexity is $\mathcal{O}(mn + n^2 logn)$ that is not worse then the complexity of the currently best algorithms. At the same time based on our observation and a wide range of various practical experiments we expect that the theoretical complexity of the algorithm is indeed of order $\mathcal{O}(mn)$ assuming that the shortest path matrix is given. Furthermore, the lower bounds LB obtained in the solution procedure are proved to be at any case better than the Halpern's lower bound. This bound is the strongest elimination criterion in algorithms locating the center of a network. Our computational experiments shows stability of the elimination criterion deducted in the algorithm.

Chapter 4 in Section 4.1 extends the discussing problems to directed unweighted and weighted networks and explores them. Complexity $\mathcal{O}(n^2)$ of the developed algorithm for finding of the center of a minimum width annulus in the unweighted case does not depend on the number of edges in a network. However, in weighted case we have computational time of order $\mathcal{O}(mn^2)$. Section 4.2 presents the summary of this work and prospective directions for further development.

Chapter 2

Minimum Width Annulus and related problems on the plane

2.1 Euclidean space $\mathbb{R}^2_{l_2}$

The purpose of the section is to review location problems relevant to this thesis, to introduce the terminology, and to summarize the most important facts, which have served as a starting point of this research.

2.1.1 Equity and Minimum Width Annulus Problems (MWAP)

In facility location, especially in the public sector, equitable decisions are very important. The paper of Marsh and Schilling [36] was the first work which has reviewed the equity literature as it pertains to facility location. One of twenty different equity measures in [36] is minimizing of difference $\max_i Effect_{S_i} - \min_j Effect_{S_j}$, $1 \leq i, j \leq n$, between maximal and minimal effects of location decision on the group of n subjects $S_1, ..., S_n$. Mentioned equity approach was suggested by Brill et al. [8] in their water quality management study. This interpretation of equitable location was also mentioned by Erkut and Neuman [24] as a potentially useful measure in hazardous facility siting models. Introduced equity measure is mathematically equivalent to the measure $\max_{i,j} |Effect_{S_i} - Effect_{S_j}|$, $1 \leq i, j \leq n$ – so called *range function*.

Considering as an effect on existing point $Ex_i \in \mathcal{E}x = \{Ex_1, ..., Ex_n\} \subset \mathbb{R}^2$ the distance $d(Ex_i, x)$ between it and new facility x, we come to the following model, which we call an *equity problem* (EP): for n existing facilities we want to place a new one so that the difference between maximal and minimal effects on the existing facilities is minimal. For example, we have n villages and would like to place a new grocery store so its location is fair with respect to all demand locations in the sense that the difference between the longest and the shortest traveling distances is minimized. This equity problem for Euclidean distance l_2 is also known as the *minimum width annulus problem* (MWAP), where we search the narrowest ring (called an *annulus*) $\mathcal{A}(x, R, r)$ between two concentric circles $\mathcal{C}(x, R)$ and $\mathcal{C}(x, r)$ centered at the point x with the set $\mathcal{E}x$ of existing points between them. The problem has wide applications in location theory, quality control for production process, pattern recognition, etc.

The minimum width annulus problem has been well–studied in $\mathbb{R}^2_{l_2}$. Rivlin [43] first has shown that the minimum width annulus of n points is either the width of the set (the *width of a set of points* in R^2 is the minimal distance between two parallel lines that contains the set between them) or annulus with two points on the inner circle and two points on the outer circle. In the first case the radius of the annulus is infinite and in the second case is not. Ebara et al. [21] demonstrated that the center of a minimum width annulus containing the set $\mathcal{E}x$ must be at a vertex of the farthest – neighbor or the nearest – neighbor Voronoi diagrams or at an intersection point between these two diagrams and give an $\mathcal{O}(n^2)$–time algorithm for solving this problem. In [22] Ebara et al. concluded that this roundness algorithm can be improved in practical applications by introducing the deletion of unnecessary points. Garcia-Lopez et al. [27] showed that for $d = 2$ a locally minimal annulus has two points on the inner circle and two points on the outer circle that interlace anglewise as it is seen from the center of the annulus. Using this characterization Garcia–Lopez et al. [27] demonstrated that there is at most one locally minimal annulus consistent with a given circular order of the points. This annulus can be computed in $\mathcal{O}(nlogn)$ time. Furthermore, when points are in convex position the problem can be formulated as linear program and solved in linear time.

Many new solution techniques for MWAP have been developed by Agarwal et al. [1–6]. Agarwal and Sharir [4] reduced the problem of finding a minimum width annulus to the computation of a bichromatic closest pair in two given sets of lines in $\mathbb{R}^3$. They have obtained a randomized algorithm that runs in $\mathcal{O}(n^{3/2+\epsilon})$ expected time for any $\epsilon > 0$. Agarwal et al. [1] computed in $\mathcal{O}(nlogn)$ time an annulus whose width is at most twice the width of an optimal annulus and in time $\mathcal{O}(nlogn + n/\epsilon^2)$ an annulus with width $(1+\epsilon)$ of the optimal annulus width for any given parameter $\epsilon > 0$.

Chan [10] studied linear–time $(1 + \epsilon)$–factor approximation $\mathcal{O}(n + 1/\epsilon^{d^2/4})$ algorithms for minimum width annulus problems in any fixed dimension d. The idea of the algorithm is to divide the problem into two parts: for *narrow* and *wide* optimal annulus. The first case was not covered by any of the previous algorithms. This ϵ–approximation algorithm takes in the Euclidean plane $\mathcal{O}(n + 1/\epsilon)$ time. Agarwal et al. [2] improved the complexity of ϵ–approximation to $\mathcal{O}(n + 1/\epsilon^{3d})$ by using a general technique for approximating various descriptors of the extent of a set of n points in $\mathbb{R}^d$ when the dimension d is an arbitrary fixed constant. However, this bound is better for $d > 12$ only and in the Euclidean plane Chan's algorithm has lower complexity.

Simpler and faster algorithms have been created for various special cases of MWAP. Mark de Berg et al. [12] studied the problem of determining whether a manufactured disc of certain radius r is within tolerance. They presented algorithms computing the thinnest annulus with outer (or inner, or median) radius equal to r that contains all n probe points on the surface of the manufactured object. These algorithms run in $\mathcal{O}(nlogn)$ time. Duncan et al. [19] have given the more natural notion of roundness motivated from Dimensional Tolerancing and Metrology that they called *referenced roundness*. Here it is necessary to find an annulus with a given reference radius that contains a given finite set of points and has minimum width. In [19] simple deterministic and randomized methods for solving the referenced roundness problem in case of planar point sets are developed. Their running time is $\mathcal{O}(nlogn)$. Ramos [42] discussed a discrete local optimization method for solving the problem of computing the thinnest annulus containing a set of points in $\mathbb{R}^2$. He gave empirical evidence that the algorithm performs close to linear time if the input is *almost round* and explained theoretically this behavior of the algorithm. Finally, he showed that for $d = 2$ the problem can be solved in $\mathcal{O}(n)$ expected time for a fairly general family of almost round sets. Proposed algorithms give the *exact* solution for families of input sets which are specially relevant in tolerancing metrology applications.

The most recent result was published by Drezner and Drezner [14] regarding minimizing the range of the distances. The problem was solved using the global optimization technique „Big Triangle Small Triangle“. They reported that solutions of instances with 10000 demand points were determined within an accuracy of 10^{-10} in a few seconds of computer time.

2.1.2 Circle Location (CLP) or Circle Fitting Problem

Shape fitting is a fundamental problem in computational geometry, computer vision, machine lerning, data mining, and many other areas. In [5] Agarwal and Sharir reviewed efficient algorithms for various problems in geometric optimization. One of the problems is *the circle location (CLP)* or *circle fitting problem*: given a set $\mathcal{E}x = \{Ex_1, ..., Ex_n\} \subset \mathbb{R}^2$ of n points in the plane, we wish to fit a circle $\mathcal{C}(x, \rho)$ through $\mathcal{E}x$ so that the maximum distance between the points of $\mathcal{E}x$ and $\mathcal{C}(x, \rho)$ is minimized. This CLP is equivalent to finding an annulus $\mathcal{A}(x, R, r)$ of minimum width $R - r$ that contains the set $\mathcal{E}x$ [3]. In other words, the circle $\mathcal{C}(x, \rho)$ and the annulus $\mathcal{A}(x, R, r)$ for a given set of existing points are concentric and the radius of the circle $\rho = (R + r)/2$ is equal to half the sum of radii of inner and outer circles generating the annulus.

Based on the mentioned equivalence and Section 2.1.1, CLP can be considered as well–explored problem in $\mathbb{R}^2_{l_2}$. In addition to the literature reviewed in Section 2.1.1 we shall mention some papers regarding circle fitting problem. Karimaeki [34] presented a fast method for circular trajectory fitting. The method is based on an explicit solution of an nonlinear least-squares problem to fit the circle curvature, direction and position parameters. Drezner et al. [16] found a circle whose *circumference* is as close as possible to a given set of points in Euclidean space. One of the considered criteria of closeness is the minimization of the maximal distance to the circumference of the circle. This objective function called in the paper the *minimax* objective is equivalent to finding the minimum width annulus that covers all given points. They proposed an efficient gradient search algorithm for finding a local minimum of the minimax problem and reported that its run time in experiments was linear in n. Mark de Berg et al. [12] and Duncan [19] have studied the problem of fitting a circle of a given radius through $\mathcal{E}x$ so that the maximum distance between $\mathcal{E}x$ and the circle is minimized. This problem is considerably simpler and can be solved in $\mathcal{O}(nlogn)$ time.

2.1.3 Point Location Problem (PLP)

The *point location problem* (PLP) or *1-center problem* or *minimax* or *minmax location problem* is a classical combinatorial optimization problem in operations research of the facilities location type. The problem is stated as follows: given a set of n distinct demand points $\mathcal{E}x = \{Ex_1, ..., Ex_n\} \subset \mathbb{R}^2$ in the plane, find a location of the facility x which minimizes the maximum Euclidean distance $d(x, Ex_i)$, $1 \leq i \leq n$ from the point x to the set of existing points $\mathcal{E}x$. This version of the problem has a geometric interpretation of finding a circle with center x and minimum radius R so that all the given points $Ex_i \in \mathcal{E}x$ are in the circle. It is called the *minimum covering circle* or the *smallest circle problem*.

A detailed overview of the literature on the solution of PLP has been given, for instance, by Plastria [15]. We shall mention that Elzinga and Hearn [23] gave a geometric algorithm for solving the one center problem with Euclidean distances and proved the correctness of it. The theorem of Caratheodory states that to express a given point x in $\mathbb{R}^n$ as a convex combination of a given set of points at most $n+1$ of the given points are necessary. In the plane, with $n = 2$, this theorem implies that to express the center x of the minimum covering circle as a convex combination of the points $\{Ex_1, ..., Ex_n\}$ at most three of them are required. Furthermore, Megiddo [37] has constructed a linear time algorithm for the problem of finding the smallest circle enclosing the set of n given points in the plane. A simple randomized algorithm was developed by Welzl [46]. It also computes the smallest enclosing disk of a finite set of points in the plane in expected linear time.

It is important to mention that there is no equivalence of MWAP or CLP with PLP in $\mathbb{R}^2_{l_2}$.

2.2 Rectilinear space $\mathbb{R}^2_{l_1}$

The problems considered in Section 2.1 have first been solved in Euclidean metric. However, the rectilinear (also called city street, Manhattan, grid, or rectangular) distance is more appropriate for a certain class of problems than Euclidean, since for many applications this metric gives a better estimation of actual travel than the Euclidean metric. To our knowledge equity or minimum width annulus problem in $\mathbb{R}^2$ for rectilinear l_1 and Chebyshev l_∞ distances have not been studied so far. However, Dearing [13], Drezner [17], Church and ReVelle [11], Elzinga and Hearn [23], Wesolowsky [47], Halman [31], Francis et al. [25] and others have considered the *minimax*, Drezner [18], Appa and Giannikos [7], Mehrez et al. [39] and others - the *maximin* problems with rectilinear distance and have documented them well in the literature.

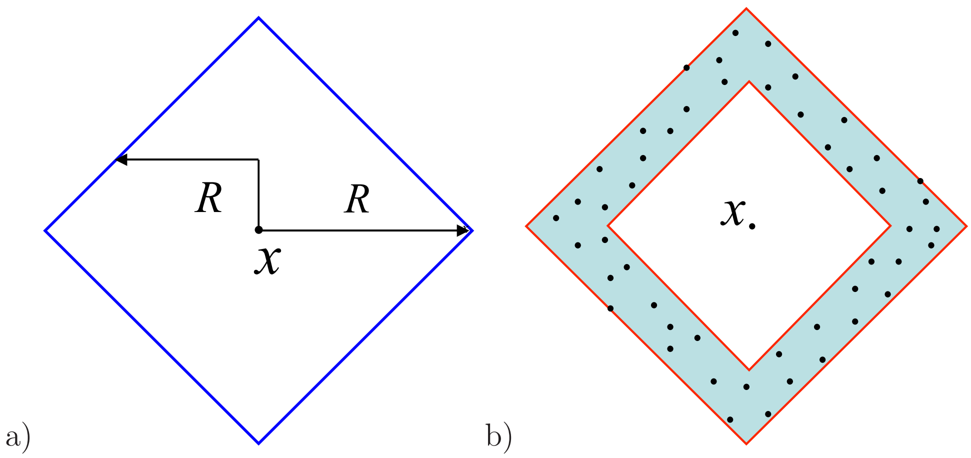


Figure 2.1: *a) Circle and b) annulus in* $\mathbb{R}^2_{l_1}$

It should be noted, that as a circle with the center point x and the radius R is a diamond in rectilinear metric (Figure 2.1 a)), an annulus in this space is a „squared ring“– the space between two concentric diamonds (Figure 2.1 b)).

2.2.1 Relation between MWAP and PLP

Let us introduce the equity problem and the problem of finding of minimum width annulus in rectilinear plane:

Definition 2.1. EP in $\mathbb{R}^2_{l_1}$

For a given finite set of existing facilities $\mathcal{E}x = \{Ex_1, \ldots, Ex_n\} \subset \mathbb{R}^2$*, where each of them is represented by its coordinates* $Ex_i = (a_{i_1}, a_{i_2})$*,* $i = 1, \ldots, n$ *on the plane (from now on without loss of generality* $a_{i_1} \geq 0$*,* $a_{i_2} \geq 0$ *and* $a_{i_1}, a_{i_2} \in \mathbb{N}$*), find a point* $x = (x_1, x_2) \in \mathbb{R}^2$*, which minimize the objective function*

$$equity_{\mathcal{E}x}(x) = \max_{Ex_i \in \mathcal{E}x} d(Ex_i, x) - \min_{Ex_i \in \mathcal{E}x} d(Ex_i, x), \tag{2.1}$$

where

$$d(Ex_i, x) \quad = \quad |a_{i_1} - x_1| + |a_{i_2} - x_2|.$$

Definition 2.2. MWAP in $\mathbb{R}^2_{l_1}$

For a given finite set of existing facilities $\mathcal{E}x = \{Ex_1, \ldots, Ex_n\} \subset \mathbb{R}^2$*, where* $Ex_i = (a_{i_1}, a_{i_2})$*,* $i = 1, \ldots, n$*, find* $x = (x_1, x_2) \in \mathbb{R}^2$ *and two concentric circles* $\mathcal{C}(x, R)$ *and* $\mathcal{C}(x, r)$ *defining the annulus* $\mathcal{A}(x, R, r)$

$$r \leq |a_{i_1} - x_1| + |a_{i_2} - x_2| \leq R, \qquad i = 1, \ldots, n$$

such that the set $\mathcal{E}x$ *is contained in* $\mathcal{A}(x, R, r)$ *and its width*

$$width_{\mathcal{A}} = R - r \tag{2.2}$$

is minimal.

Both problems are equivalent and can be formulated as the following optimization problem (OP) in the variables x_1, x_2, r, R:

$$\begin{aligned} & \min R - r \\ \text{s.t} \quad & r \leq |a_{i_1} - x_1| + |a_{i_2} - x_2| \leq R, \\ & i = 1, \ldots, n \end{aligned} \tag{2.3}$$

To illustrate all our conclusions we consider the following numerical example throughout this section:

Example 2.2.1. *Given the six points* $Ex_1 = (3, 2)$*,* $Ex_2 = (2, 4)$*,* $Ex_3 = (5, 8)$*,* $Ex_4 = (10, 7)$*,* $Ex_5 = (9, 3)$*,* $Ex_6 = (12, 4)$ *(Figure 2.2), find a minimum width annulus* $\mathcal{A}(x, R, r)$*,* $x = (x_1, x_2) \in \mathbb{R}^2$*, involving the points.*

OP – formulation of Example 2.2.1 is

$$\begin{aligned} & \min R - r && (2.4)\\ \text{s.t } r &\le |3 - x_1| + |2 - x_2| \le R,\\ r &\le |2 - x_1| + |4 - x_2| \le R,\\ r &\le |5 - x_1| + |8 - x_2| \le R,\\ r &\le |10 - x_1| + |7 - x_2| \le R,\\ r &\le |12 - x_1| + |4 - x_2| \le R,\\ r &\le |9 - x_1| + |3 - x_2| \le R. \end{aligned}$$

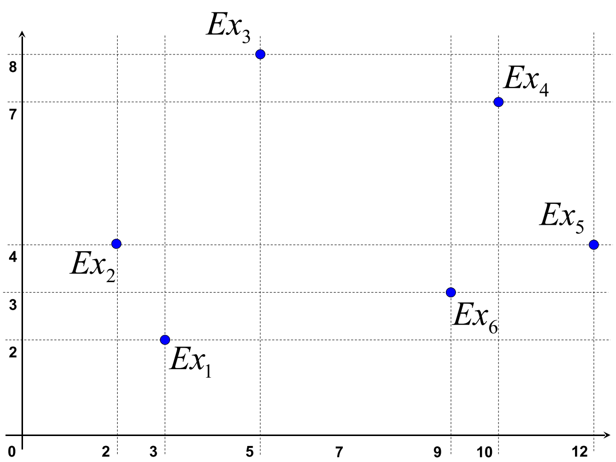


Figure 2.2: *Grid formed by points* $Ex_1, ..., Ex_6$

Of course, the problem (2.3) is linear in each cell generated by the grid of vertical and horizontal lines going through the points $Ex_i \in \mathcal{E}x$ (Figure 2.2). Thus it can be solved in each of the $(n+1)^2$ cells with linear programming methods. Megiddo [38] pointed out that the complexity of linear programming is linear in n when dimension of the space is fixed. Hence, it leads to computational time at least $\mathcal{O}(n^3)$. In order to improve the complexity, we consider properties of the objective function (2.3)

$$R - r = \max_{i=1,...,n}(|a_{i_1} - x_1| + |a_{i_2} - x_2|) - \min_{i=1,...,n}(|a_{i_1} - x_1| + |a_{i_2} - x_2|). \quad (2.5)$$

It is a continuous function at any point $x = (x_1, x_2) \in \mathbb{R}^2$, but, unfortunately, not differentiable at points of lines $x_1 = a_{i_1}$, $x_2 = a_{i_2}$ $\forall i = 1, ..., n$.

Let us enumerate all first and second coordinates of the points $Ex_i \in \mathcal{E}x$ so that $-\infty = \bar{a}_0 < \bar{a}_1 \le \bar{a}_2 \le ... \le \bar{a}_i \le ... \le \bar{a}_n < \bar{a}_{(n+1)} = +\infty$ and $-\infty = \bar{\bar{a}}_0 < \bar{\bar{a}}_1 \le \bar{\bar{a}}_2 \le ... \le \bar{\bar{a}}_j \le ... \le \bar{\bar{a}}_n < \bar{\bar{a}}_{(n+1)} = +\infty$. Then at any box $[\bar{a}_i, \bar{a}_{(i+1)}] \times [\bar{\bar{a}}_j, \bar{\bar{a}}_{(j+1)}]$, $i, j = 0, ..., n+1$ the optimization problem (2.3) can be written as linear program

(LP)

$$
\begin{aligned}
&\min \quad R - r && (2.6)\\
\text{s.t}\\
&(A)\begin{cases} r \le a_{i_1} - x_1 + a_{i_2} - x_2 \le R,\\ where\ \ a_{i1} \ge \bar{a}_{(i+1)}, a_{i2} \ge \bar{\bar{a}}_{(j+1)},\end{cases}\\
&(B)\begin{cases} r \le a_{i_1} - x_1 - a_{i_2} + x_2 \le R,\\ where\ \ a_{i1} \ge \bar{a}_{(i+1)}, -a_{i2} \ge -\bar{\bar{a}}_j,\end{cases}\\
&(C)\begin{cases} r \le -a_{i_1} + x_1 - a_{i_2} + x_2 \le R,\\ where\ \ -a_{i1} \ge -\bar{a}_i, -a_{i2} \ge -\bar{\bar{a}}_j,\end{cases}\\
&(D)\begin{cases} r \le -a_{i_1} + x_1 + a_{i_2} - x_2 \le R,\\ where\ \ -a_{i1} \ge -\bar{a}_i, a_{i2} \ge \bar{\bar{a}}_{(j+1)}.\end{cases}
\end{aligned}
$$

and the total number of constraines is equal to n. Each group of inequalities (A)-(D) is a family of parallel lines. Since the objective function (2.5) is the difference of *max* and *min* functions the number of the inequalities in each group (A)-(D) can be reduced to two with minimal and maximal values of corresponding sum $\pm a_{i_1} \pm a_{i_2}$.

Without loss of generality, we assume the following numbering of points $Ex_i \in \mathcal{E}x$ with respect to the fixed cell $[\bar{a}_i, \bar{a}_{(i+1)}] \times [\bar{\bar{a}}_j, \bar{\bar{a}}_{(j+1)}]$. The points $(a_{i_1}, a_{i_2}) \in \mathcal{E}x$ with the maximal values of $a_{i_1} + a_{i_2}$, $a_{i_1} - a_{i_2}$, $-a_{i_1} - a_{i_2}$, $-a_{i_1} + a_{i_2}$ are denoted as $({a^1}_1, {a^1}_2) = Ex^1$, $({a^3}_1, {a^3}_2) = Ex^3$, $({a^5}_1, {a^5}_2) = Ex^5$ and $({a^7}_1, {a^7}_2) = Ex^7$, respectively. They are common for all cells. However, the points with corresponding minimal values depend on the box boundaries. Therefore,

$$({a^2}_1, {a^2}_2) = Ex^2 = argmin(a_{i_1} + a_{i_2}),\ where\ \ a_{i1} \ge \bar{a}_{(i+1)}, a_{i2} \ge \bar{\bar{a}}_{(j+1)},$$

$$({a^4}, {a^4}_2) = Ex^4 = argmin(a_{i_1} - a_{i_2}),\ where\ \ a_{i1} \ge \bar{a}_{(i+1)}, -a_{i2} \ge -\bar{\bar{a}}_j,$$

$$({a^6}_1, {a^6}_2) = Ex^6 = argmin(-a_{i_1} - a_{i_2}),\ where\ \ -a_{i1} \ge -\bar{a}_i, -a_{i2} \ge -\bar{\bar{a}}_j,$$

$$({a^8}_1, {a^8}_2) = Ex^8 = argmin(-a_{i_1} + a_{i_2}),\ where\ \ -a_{i1} \ge -\bar{a}_i, a_{i2} \ge \bar{\bar{a}}_{(j+1)}.$$

If there is more than one of such points we take any one of them. For instance, in the numerical Example 2.2.1 for the cell $[5, 9] \times [4, 7]$ the correspnding points are $Ex^1 = Ex^6 = Ex_4$, $Ex^3 = Ex^8 = Ex_6$, $Ex^5 = Ex^2 = Ex_1$, $Ex^7 = Ex^4 = Ex_3$ and the points Ex^1, Ex^3, Ex^5, Ex^7 are shown in Figure 2.3.

Hence, the problem (2.6) has the following feasible region

$$\begin{aligned} &\min \quad R-r \\ \text{s.t} \quad &1) \quad r \le a^1{}_1 - x_1 + a^1{}_2 - x_2 \le R, \\ &2) \quad r \le a^2{}_1 - x_1 + a^2{}_2 - x_2 \le R, \\ &3) \quad r \le a^3{}_1 - x_1 - a^3{}_2 + x_2 \le R, \\ &4) \quad r \le a^4{}_1 - x_1 - a^4{}_2 + x_2 \le R, \\ &5) \quad r \le -a^5{}_1 + x_1 - a^5{}_2 + x_2 \le R, \\ &6) \quad r \le -a^6{}_1 + x_1 - a^6{}_2 + x_2 \le R, \\ &7) \quad r \le -a^7{}_1 + x_1 + a^7{}_2 - x_2 \le R, \\ &8) \quad r \le -a^8{}_1 + x_1 + a^8{}_2 - x_2 \le R \end{aligned} \tag{2.7}$$

on the fixed cell $[\bar{a}_i, \bar{a}_{(i+1)}] \times [\bar{\bar{a}}_j, \bar{\bar{a}}_{(j+1)}]$. Therefore, for any number n of given points Ex_i, $i = 1, ..., n$ at most 8 of them $Ex^1, ..., Ex^8$ (if they do not coincide) have influence on the objective function value (2.3) in each box $[\bar{a}_i, \bar{a}_{(i+1)}] \times [\bar{\bar{a}}_j, \bar{\bar{a}}_{(j+1)}]$, $i, j = 0, ..., n+1$. The intersection lines of the planes $1) - 8)$ in (2.7) are

$$x_1 = ((\pm a^{\cdot}{}_1 \pm a^{\cdot}{}_2) - (\pm a^{\cdot}{}_1 \pm a^{\cdot}{}_2))/2 = const, \tag{2.8}$$

or

$$x_2 = ((\pm a^{\cdot}{}_1 \pm a^{\cdot}{}_2) - (\pm a^{\cdot}{}_1 \pm a^{\cdot}{}_2))/2 = const, \tag{2.9}$$

or

$$x_1 + x_2 = ((a^{\cdot}{}_1 + a^{\cdot}{}_2) - (-a^{\cdot}{}_1 - a^{\cdot}{}_2))/2 = const, \tag{2.10}$$

or

$$x_1 - x_2 = ((a^{\cdot}{}_1 - a^{\cdot}{}_2) - (-a^{\cdot}{}_1 + a^{\cdot}{}_2))/2 = const. \tag{2.11}$$

These lines divide the box $[\bar{a}_i, \bar{a}_{(i+1)}] \times [\bar{\bar{a}}_j, \bar{\bar{a}}_{(j+1)}]$ (if they intersect it) into regions where the objective function (2.7) is constant, or monotonically decreasing, or monotonically increasing.

Let us go from $-\infty$ to $+\infty$ along $x_1 = const$, i.e. $-\infty < x_2 < +\infty$ and $x_1 = const$, considering the functions $\min_{i=1,...,n}(|a_{i_1} - x_1| + |a_{i_2} - x_2|)$ and $\max_{i=1,...,n}(|a_{i_1} - x_1| + |a_{i_2} - x_2|)$ at points of this line (conclusions will be valid for any line $x_2 = const$, $-\infty < x_1 < +\infty$). Then for any fixed point $Ex_i = (a_{i_1}, a_{i_2}) \in \mathcal{E}x$

$$\begin{aligned} &|a_{i_1} - x_1| + |a_{i_2} - x_2| = |a_{i_1} - const| + |a_{i_2} - x_2| = \\ &const + |a_{i_2} - x_2| = \begin{cases} const - x_2, & a_{i_2} \le x_2 \\ const + x_2, & a_{i_2} > x_2. \end{cases} \end{aligned} \tag{2.12}$$

Therefore, the function $\min_{i=1,...,n}(|a_{i_1} - x_1| + |a_{i_2} - x_2|)$ has at most $2n-1$ breakpoints denoted in Figure 2.4 as $p_1, p_2, ...$ and slopes between these points alternating as follows $-1, +1, -1, ..., -1, +1,$ where the first and last slopes are -1 and $+1$,

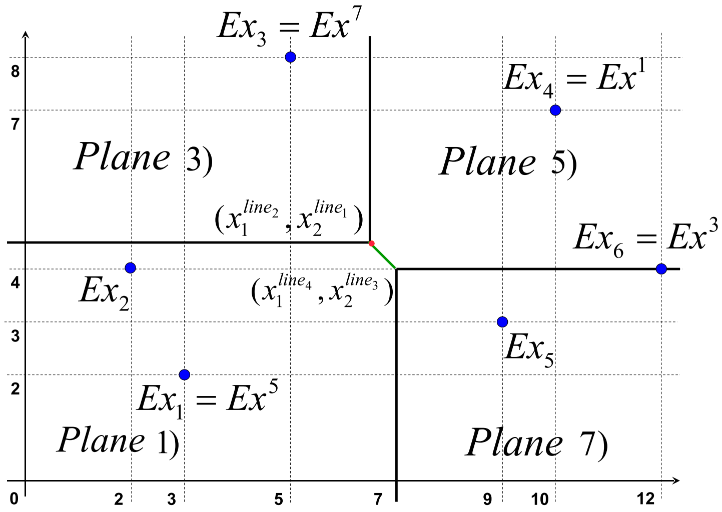


Figure 2.3: *Intersection lines for function max in Example 2.2.1*

respectively. The function $\max_{i=1,\dots,n}(|a_{i_1} - x_1| + |a_{i_2} - x_2|)$ has one breakpoint P (Figure 2.4) and two slopes -1 and $+1$ only. Hence, it would be interesting to investigate relation of solution sets of point location and minimum width annulus problems.

Definition 2.3. PLP in $\mathbb{R}^2_{l_1}$

For a given finite set of existing facilities $\mathcal{E}x = \{Ex_1, \dots, Ex_n\} \subset \mathbb{R}^2$, where $Ex_i = (a_{i_1}, a_{i_2})$, $i = 1, \dots, n$, find a point $x = (x_1, x_2) \in \mathbb{R}^2$, so that the maximal distance from the point x to the set $\mathcal{E}x$

$$\max_{i=1,\dots,n} (|a_{i_1} - x_1| + |a_{i_2} - x_2|) \tag{2.13}$$

is minimal.

Theorem 2.4. *Along any direction $x_1 = const$ or $x_2 = const$ the minimum for PLP objective function*

$$\max_{i=1,\dots,n} (|a_{i_1} - x_1| + |a_{i_2} - x_2|)$$

is attained among the points minimizing the MWAP objective function

$$\max_{i=1,\dots,n} (|a_{i_1} - x_1| + |a_{i_2} - x_2|) - \min_{i=1,\dots,n} (|a_{i_1} - x_1| + |a_{i_2} - x_2|). \tag{2.14}$$

Moreover, along $x_1 = const$ or $x_2 = const$ the minimum of (2.14) is global.

Proof. Let us consider the function (2.14) along $x_1 = const$, where points $P, p_1, p_2, \dots, p_{2k}, p_{2k+1}, p_{2k+2}, \dots$ are breakpoints of functions $\max_{i=1,\dots,n}(|a_{i_1} - x_1| + |a_{i_2} - x_2|)$ and $\min_{i=1,\dots,n}(|a_{i_1} - x_1| + |a_{i_2} - x_2|)$ (Figure 2.4):

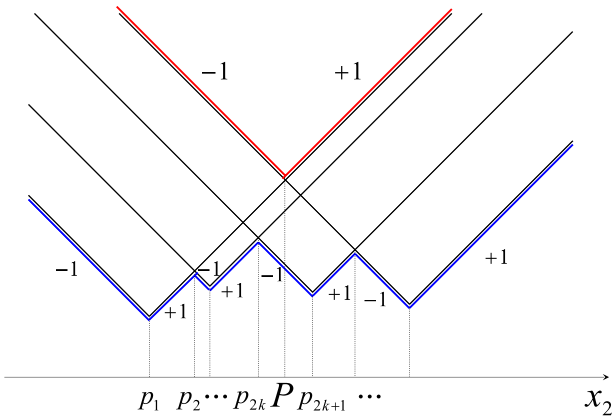


Figure 2.4: *Functions max and min along $x_1 = const$*

$$\max_{i=1,\dots,n}(|a_{i_1} - x_1| + |a_{i_2} - x_2|) - \min_{i=1,\dots,n}(|a_{i_1} - x_1| + |a_{i_2} - x_2|) =$$

$$= \begin{cases} const - x_2, & x_2 \le P, \\ const + x_2, & x_2 > P \end{cases} - \begin{cases} const - x_2, & -\infty < x_2 < p_1, \\ const + x_2, & p_1 \le x_2 < p_2, \\ \dots & \\ const + x_2, & \dots \le x_2 < p_{2k}, \\ const - x_2, & p_{2k} \le x_2 < p_{2k+1}, \\ const + x_2, & p_{2k+1} \le x_2 < p_{2k+2}, \\ const - x_2, & p_{2k+2} \le x_2 < \dots, \\ \dots & \\ const - x_2, & \dots, \\ const + x_2, & \dots \le x_2 < +\infty \end{cases}$$

This difference can be equal to:

Case 1: $p_{2k+1} < P < p_{2k+2}$ (i.e. the point of minimum for function *max* lies in interval on which the function *min* has slope +1)

$$= \begin{cases} const, & -\infty < x_2 < p_1, \\ const - 2x_2, & p_1 < x_2 < p_2, \\ \dots & \\ const - 2x_2, & \dots \le x_2 < p_{2k}, \\ const, & p_{2k} \le x_2 < p_{2k+1}, \\ const - 2x_2, & p_{2k+1} \le x_2 < P, \\ const, & P \le x_2 < p_{2k+2}, \\ const + 2x_2, & p_{2k+2} \le x_2 < \dots, \\ \dots & \\ const + 2x_2, & \dots, \\ const, & \dots \le x_2 < +\infty \end{cases}$$

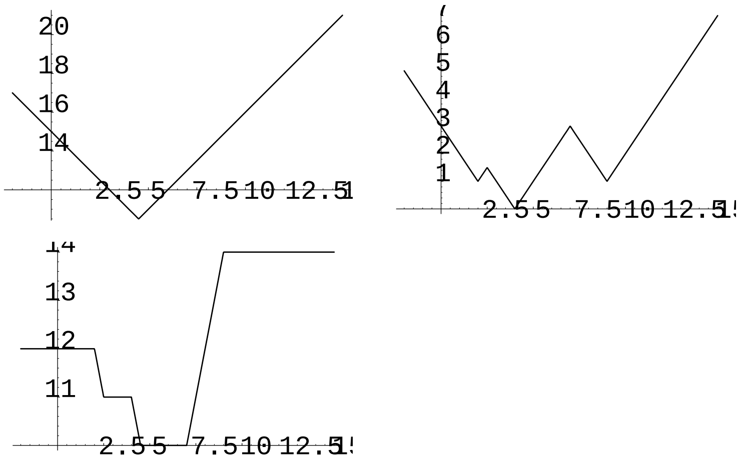


Figure 2.5: *Case 1: Functions max, min and max − min along* $x_1 = const$ *or* $x_2 = const$

Case 2: $p_{2k} < P < p_{2k+1}$ (i.e. the point of minimum for function *max* lies in interval on which the function *min* has slope -1)

$$= \begin{cases} const, & -\infty < x_2 < p_1, \\ const - 2x_2, & p_1 < x_2 < p_2, \\ \dots & \\ const - 2x_2, & ... \le x_2 < p_{2k}, \\ const, & p_{2k} \le x_2 < P, \\ const + 2x_2, & P \le x_2 < p_{2k+1}, \\ const, & p_{2k+1} \le x_2 < p_{2k+2}, \\ const + 2x_2, & p_{2k+2} \le x_2 < ..., \\ \dots & \\ const + 2x_2, & ..., \\ const, & ... \le x_2 < +\infty \end{cases}$$

Case 3: $P = p_{2k+1}$ (i.e. the point of minimum for function *max* is a point of

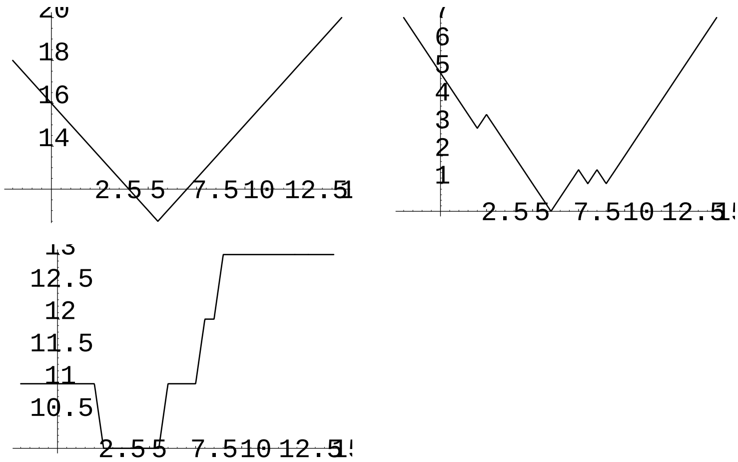


Figure 2.6: *Case 2: Functions max, min and max − min along* $x_1 = const$ *or* $x_2 = const$

local minimum for the function *min*)

$$
= \begin{cases}
const, & -\infty < x_2 < p_1, \\
const - 2x_2, & p_1 < x_2 < p_2, \\
\dots & \\
const - 2x_2, & ... \le x_2 < p_{2k}, \\
const, & p_{2k} \le x_2 < P, \\
const, & P \le x_2 < p_{2k+2}, \\
const + 2x_2, & p_{2k+2} \le x_2 < ..., \\
\dots & \\
const + 2x_2, & ..., \\
const, & ... \le x_2 < +\infty,
\end{cases}
$$

Case 4: $P = p_{2k+2}$ (i.e. the point of minimum for function *max* is a point of local maximum for the function *min*)

$$
= \begin{cases}
const, & -\infty < x_2 < p_1, \\
const - 2x_2, & p_1 < x_2 < p_2, \\
\dots & \\
const - 2x_2, & ... \le x_2 < p_{2k}, \\
const, & p_{2k} \le x_2 < p_{2k+1}, \\
const - 2x_2, & p_{2k+1} \le x_2 < P, \\
const + 2x_2, & P \le x_2 < ..., \\
\dots & \\
const + 2x_2, & ..., \\
const, & ... \le x_2 < +\infty
\end{cases}
$$

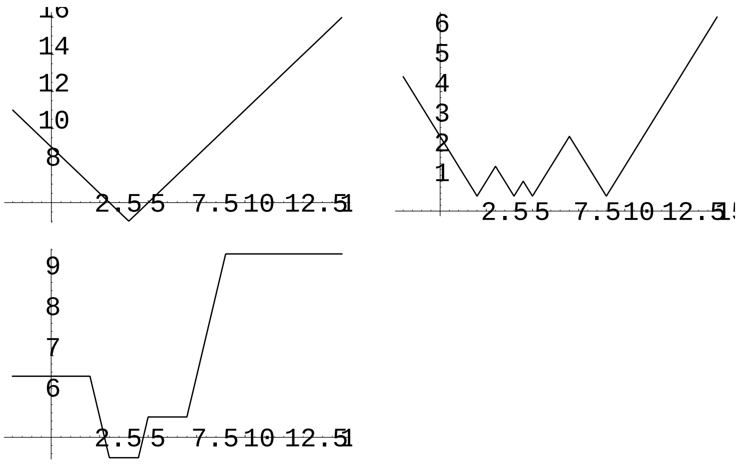


Figure 2.7: *Case 3: Functions max, min and max − min along* $x_1 = const$ *or* $x_2 = const$

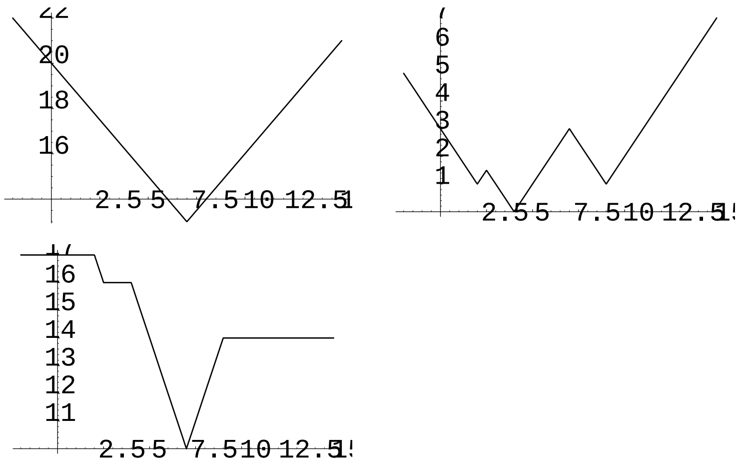


Figure 2.8: *Case 4: Functions max, min and max − min along* $x_1 = const$ *or* $x_2 = const$

So in each case along a line $x_1 = const$ for the function (2.14) a local minimum is global. Moreover, along this line the point P of minimum for the PLP objective function (2.13) is among points of minimum for the MWAP objective function (2.14). A geometrical illustration of these cases is shown in Figures 2.5 – 2.8.

For the case $x_2 = const$ the proof is analogous.

□

2.2.2 Solution of PLP

The Theorem 2.4 implies that it is useful to consider the objective function

$$\max_{i=1,\ldots,n}(|a_{i_1}-x_1|+|a_{i_2}-x_2|) \tag{2.15}$$

of the point location problem described in Definition 2.3 more detailed. PLP in $\mathbb{R}^2_{l_1}$ was first introduced and solved by Elzinga and Hearn in [23]. However, they call it *the rectilinear delivery boy problem*. Based on the paper [23] we state that

Theorem 2.5. *For any number n of existing points $Ex_i=(a_{i_1},a_{i_2})\in\mathcal{E}x$ and for any point $x=(x_1,x_2)\in\mathbb{R}^2$ the objective function (2.15) of PLP has the following representation*

$$\begin{gathered}\max\nolimits_{i=1,\ldots,n}(|a_{i_1}-x_1|+|a_{i_2}-x_2|)= \\ \max\{\max(a_{i_1}+a_{i_2})-x_1-x_2,\max(a_{i_1}-a_{i_2})-x_1+x_2, \\ \max(-a_{i_1}-a_{i_2})+x_1+x_2,\max(-a_{i_1}+a_{i_2})+x_1-x_2\}= \\ \max\{{a^1}_1+{a^1}_2-x_1-x_2,{a^3}_1-{a^3}_2-x_1+x_2, \\ -{a^5}_1-{a^5}_2+x_1+x_2,-{a^7}_1+{a^7}_2+x_1-x_2\},\end{gathered} \tag{2.16}$$

i.e. points $Ex^1,Ex^3,Ex^5,Ex^7\in\mathcal{E}x$ only (see formulation (2.7)) have influence on the function value (see Figure 2.3).

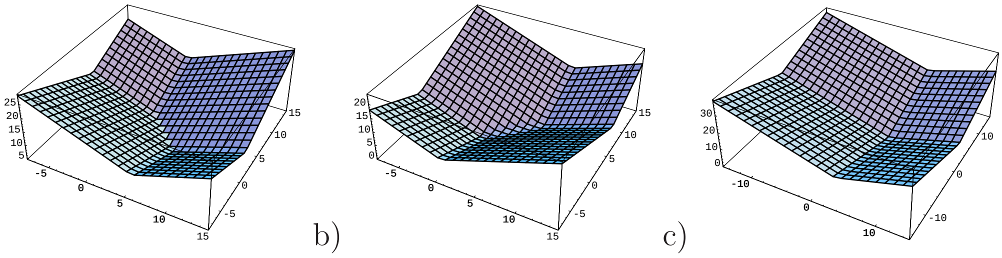
The intersection lines of the four planes given in (2.16) are the following:

$$\begin{array}{lrcl}
line_1: & x_2^{line_1} & = & (max(a_{i1}+a_{i2})-max(a_{i1}-a_{i2}))/2 \\
line_2: & x_1^{line_2} & = & (max(a_{i1}-a_{i2})-max(-a_{i1}-a_{i2}))/2= \\
 & & & (max(a_{i1}-a_{i2})+min(a_{i1}+a_{i2}))/2 \\
line_3: & x_2^{line_3} & = & (max(-a_{i1}+a_{i2})-max(-a_{i1}-a_{i2}))/2= \\
 & & & (max(-a_{i1}+a_{i2})+min(a_{i1}+a_{i2}))/2 \\
line_4: & x_1^{line_4} & = & (max(a_{i1}+a_{i2})-max(-a_{i1}+a_{i2}))/2 \\
line_5: & x_1+x_2 & = & (max(a_{i1}+a_{i2})-max(-a_{i1}-a_{i2}))/2= \\
 & & & (max(a_{i1}+a_{i2})+min(a_{i1}+a_{i2}))/2 \\
line_6: & x_1-x_2 & = & (max(a_{i1}-a_{i2})-max(-a_{i1}+a_{i2}))/2.
\end{array}$$

Based on discussions in the previous section and on Theorem 2.5, we can describe the solution set of PLP in the rectilinear plane:

Optimal Set 2.1. *PLP in $\mathbb{R}^2_{l_1}$*

For the given set of existing points $Ex_i=(a_{i1},a_{i2})$, $1\le i\le n$ in case when



a) b) c)

Figure 2.9: *Function max for: a)* $x_1^{line_2} < x_1^{line_4}$*; b)* $x_1^{line_2} > x_1^{line_4}$*; c)* $x_1^{line_2} = x_1^{line_4}$.

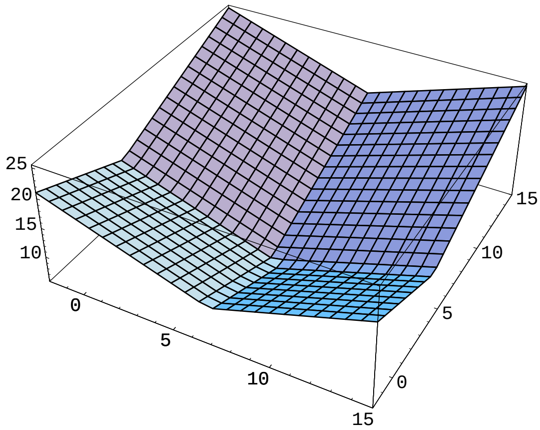


Figure 2.10: *Function max for points of Example 2.2.1*

- $x_1^{line_2} < x_1^{line_4}$ (Figures 2.3, 2.9 a)) any point on the interval I_1

$$x_1 + x_2 = (max(a_{i1} + a_{i2}) + min(a_{i1} + a_{i2}))/2, \tag{2.17}$$

where

$$x_1 \in (max(a_{i1} - a_{i2}) + min(a_{i1} + a_{i2}))/2, (max(a_{i1} + a_{i2}) - max(-a_{i1} + a_{i2}))/2),$$

- $x_1^{line_2} > x_1^{line_4}$ (Figure 2.9 b)) any point on the interval I_2

$$x_1 - x_2 = (max(a_{i1} - a_{i2}) - max(-a_{i1} + a_{i2}))/2, \tag{2.18}$$

where

$$x_1 \in (max(a_{i1} + a_{i2}) - max(-a_{i1} + a_{i2}))/2, (max(a_{i1} - a_{i2}) + min(a_{i1} + a_{i2}))/2$$

- $x_1^{line_2} = x_1^{line_4}$ (Figure 2.9 c)) the point interval I_3 with coordinates

$$(max(a_{i1} - a_{i2}) + min(a_{i1} + a_{i2}))/2, \\ (max(a_{i1} + a_{i2}) - max(a_{i1} - a_{i2}))/2) \tag{2.19}$$

gives minimum of the objective function $max_{i=1,\dots,n}(|a_{i_1} - x_1| + |a_{i_2} - x_2|)$ in PLP. A geometrical illustration of the possible cases is shown in Figure 2.9.

For six points of Example 2.2.1 the value of $x_1^{line_2}$ is less than $x_1^{line_4}$ and function $max_{i=1,\dots,6}\,(|a_{i_1} - x_1| + |a_{i_2} - x_2|)$ for the mentioned points is illustrated in Figures 2.10 and 2.3.

2.2.3 Algorithm for solving of MWAP

It is known from Theorem 2.4 that at least one solution of PLP solves MWAP for the given set $\mathcal{E}x$ of existing points. In order to find it we should solve the following *restricted to the solution interval I of PLP obnoxious problem*:

Definition 2.6. Restricted obnoxious problem

Let the interval $I \subseteq \mathbb{R}^2$, which can be in one of forms (2.17)–(2.19), solve PLP for a given finite set of existing facilities $\mathcal{E}x = \{Ex_1, \dots, Ex_n\} \subset \mathbb{R}^2$, where $Ex_i = (a_{i_1}, a_{i_2})$, $i = 1, \dots, n$ (see Optimal Set 2.1). Find a point $x = (x_1, x_2) \in I$, so that the minimal distance from the point x to the set $\mathcal{E}x$

$$\min_{i=1,\dots,n}(|a_{i_1} - x_1| + |a_{i_2} - x_2|) \tag{2.20}$$

is maximal.

Let us assume the solution interval I is of the form (2.17) that is equal to $I_1 = x_1 + x_2 = (max(a_{i1} + a_{i2}) + min(a_{i1} + a_{i2}))/2$, where $x_1 \in (max(a_{i1} - a_{i2}) + min(a_{i1} + a_{i2}))/2, (max(a_{i1} + a_{i2}) - max(-a_{i1} + a_{i2}))/2)$. Then the restricted obnoxious problem with respect to the interval I is

$$\max\min_{i=1,\dots,n}(|a_{i_1} - x_1| + |a_{i_2} - x_2|) \tag{2.21}$$
$$\text{s.t} \quad I \equiv \begin{cases} x_1 + x_2 = (max(a_{i1} + a_{i2}) + min(a_{i1} + a_{i2}))/2, \\ x_1 \geq (max(a_{i1} - a_{i2}) - min(-a_{i1} - a_{i2}))/2 \\ x_1 \leq (max(a_{i1} + a_{i2}) - max(-a_{i1} + a_{i2}))/2 \end{cases}$$
$$i = 1, \dots, n.$$

As it has been shown in Section 2.2.1 the minimum width annulus problem can be reformulated at each box $[\bar{a}_i, \bar{a}_{(i+1)}] \times [\bar{\bar{a}}_j, \bar{\bar{a}}_{(j+1)}]$, $i, j = 0, \dots, n+1$ in the form (2.7).

Hence, in box $[\bar{a}_i, \bar{a}_{(i+1)}] \times [\bar{\bar{a}}_j, \bar{\bar{a}}_{(j+1)}]$ the function $min_{i=1,...,n}(|a_{i_1} - x_1| + |a_{i_2} - x_2|)$ is defined by parts of planes

$$\begin{aligned}
&2) \;\; min(a_{i_1} + a_{i_2}) - x_1 - x_2, \; where \;\; a_{i1} \geq \bar{a}_{(i+1)}, a_{i2} \geq \bar{\bar{a}}_{(j+1)}, \\
&4) \;\; min(a_{i_1} - a_{i_2}) - x_1 + x_2, \; where \;\; a_{i1} \geq \bar{a}_{(i+1)}, -a_{i2} \geq -\bar{\bar{a}}_j, \\
&6) \;\; min(-a_{i_1} - a_{i_2}) + x_1 + x_2, \; where \;\; -a_{i1} \geq -\bar{a}_i, -a_{i2} \geq -\bar{\bar{a}}_j, \\
&8) \;\; min(-a_{i_1} + a_{i_2}) + x_1 - x_2, \; where \;\; -a_{i1} \geq -\bar{a}_i, a_{i2} \geq \bar{\bar{a}}_{(j+1)}, \\
& \;\; i = 1, ..., n.
\end{aligned}$$

The optimal set I of the center problem can lie either in one box $[\bar{a}_i, \bar{a}_{(i+1)}] \times [\bar{\bar{a}}_j, \bar{\bar{a}}_{(j+1)}]$ or in some boxes which number is not greater than n. In box $[\bar{a}_i, \bar{a}_{(i+1)}] \times [\bar{\bar{a}}_j, \bar{\bar{a}}_{(j+1)}]$ the problem (2.21) is equivalent to

$$\begin{aligned}
\max\min \quad & \{\min_{a_{i1} \geq \bar{a}_{(i+1)}, a_{i2} \geq \bar{\bar{a}}_{(j+1)}} (a_{i_1} + a_{i_2}) - x_1 - x_2, \\
& \min_{a_{i1} \geq \bar{a}_{(i+1)}, -a_{i2} \geq -\bar{\bar{a}}_j} (a_{i_1} - a_{i_2}) - x_1 + x_2, \\
& \min_{-a_{i1} \geq -\bar{a}_i, -a_{i2} \geq -\bar{\bar{a}}_j} (-a_{i_1} - a_{i_2}) + x_1 + x_2, \\
& \min_{-a_{i1} \geq -\bar{a}_i, a_{i2} \geq \bar{\bar{a}}_{(j+1)}} (-a_{i_1} + a_{i_2}) + x_1 - x_2\} \\
\text{s.t} \quad & x_1 + x_2 = (max(a_{i1} + a_{i2}) + min(a_{i1} + a_{i2}))/2 \equiv C \\
& max(\bar{a}_i, x_1^{line_2}) \leq x_1 \leq min(x_1^{line_4}, \bar{a}_{(i+1)})
\end{aligned}$$

or in one variable to

$$\begin{aligned}
\max\min\{f_1 \;\; &\equiv \;\; min\{\min_{a_{i1} \geq \bar{a}_{(i+1)}, a_{i2} \geq \bar{\bar{a}}_{(j+1)}} (a_{i_1} + a_{i_2}) - C, \\
& \qquad \min_{-a_{i1} \geq -\bar{a}_i, -a_{i2} \geq -\bar{\bar{a}}_j} (-a_{i_1} - a_{i_2}) + C\}, \\
f_2 \;\; &\equiv \;\; \min_{a_{i1} \geq \bar{a}_{(i+1)}, -a_{i2} \geq -\bar{\bar{a}}_j} (a_{i_1} - a_{i_2}) + C - 2x_1, \\
f_3 \;\; &\equiv \;\; \min_{-a_{i1} \geq -\bar{a}_i, a_{i2} \geq \bar{\bar{a}}_{(j+1)}} (-a_{i_1} + a_{i_2}) - C + 2x_1\} \\
\text{s.t} \quad & \qquad max(\bar{a}_i, x_1^{line_2}) \leq x_1 \leq min(x_1^{line_4}, \bar{a}_{(i+1)}).
\end{aligned} \tag{2.22}$$

To solve the problem (2.22) we should find the first coordinate of an intersection point of the functions f_2 and f_3. Depending on its arrangement concerning the interval $(max(\bar{a}_i, x_1^{line_2}), min(x_1^{line_4}, \bar{a}_{(i+1)}))$ and the function f_1, a solution of the problem (2.22) in the box $[\bar{a}_i, \bar{a}_{(i+1)}] \times [\bar{\bar{a}}_j, \bar{\bar{a}}_{(j+1)}]$ is either one point or a part of the interval $I \cap [\bar{a}_i, \bar{a}_{(i+1)}] \times [\bar{\bar{a}}_j, \bar{\bar{a}}_{(j+1)}]$.

Now Algorithm 2.1 for finding an optimal solution of MWAP can be stated.

Applying this algorithm to Example 2.2.1 (Figure 2.3) we get:

Algorithm 2.1 *Optimal solution of MWAP in* $\mathbb{R}^2_{l_1}$

Input: Set of existing points $Ex_i = (a_{i1}, a_{i2})$, $i = 1, ..., n$;

1. Find $max(a_{i1} + a_{i2})$, $max(a_{i1} - a_{i2})$, $max(-a_{i1} - a_{i2})$, $max(-a_{i1} + a_{i2})$, $i = 1, ..., n$
– complexity $\mathcal{O}(n)$

2. Calculate

$$\begin{aligned} x_2^{line_1} &= max(a_{i1} + a_{i2}) - max(a_{i1} - a_{i2})/2, \\ x_1^{line_2} &= max(a_{i1} - a_{i2}) - max(-a_{i1} - a_{i2})/2, \\ x_2^{line_3} &= (max(-a_{i1} + a_{i2}) - max(-a_{i1} - a_{i2}))/2, \\ x_1^{line_4} &= (max(a_{i1} + a_{i2}) - max(-a_{i1} + a_{i2}))/2 \end{aligned}$$

and solution interval I for PLP;
if $x_1^{line_2} \neq x_1^{line_4}$ **then**
in each box $[\bar{a}_i, \bar{a}_{(i+1)}] \times [\bar{\bar{a}}_j, \bar{\bar{a}}_{(j+1)}]$, for which $[\bar{a}_i, \bar{a}_{(i+1)}] \times [\bar{\bar{a}}_j, \bar{\bar{a}}_{(j+1)}] \cap I \neq \emptyset$ solve restricted to the interval I obnoxious problem (2.20) and find maximum of the solutions, which is equal to r over whole interval I
– complexity $\mathcal{O}(nlogn)$
Output: optimal annulus $\mathcal{A}(x, R, r)$, where x is solution of obnoxious problem, $R = max_{i=1,...,n} d(Ex_i, x)$, $width_{\mathcal{A}} = R - r$;
else
Output: optimal annulus $\mathcal{A}(x, R, r)$, where $x = (x_1^{line_2}, x_2^{line_1})$, $r = min_{i=1,...,n} d(Ex_i, x)$, $R = max_{i=1,...,n} d(Ex_i, x)$, $width_{\mathcal{A}} = R - r$;
end if

1. $max(a_{i1} + a_{i2}) = 17$, $max(a_{i1} - a_{i2}) = 8$,
$max(-a_{i1} - a_{i2}) = -5$, $max(-a_{i1} + a_{i2}) = 3$.
2. $x_2^{line_1} = 4.5$,
$x_1^{line_2} = 6.5$,
$x_2^{line_3} = 4$,
$x_1^{line_4} = 7$.
on interval $I = \{x_1 + x_2 = 11, x_1 \in [6.5, 7]\}$
a solution of restricted obnoxious problem is the point $(6.5, 4.5)$;
Output: optimal annulus $\mathcal{A}((6.5, 4.5), 6, 4)$ of $width_{\mathcal{A}} = 2$.

Moreover, all solutions Example 2.2.1 belong to the polyhedron with corner points $(6, 4.5)$, $(6.5, 4.5)$, $(6.5, 5)$. It is region containing solution of restricted obnoxious problem and over it functions *max* and *min* grow equally. The objective function of the example is illustrated in Figure 2.11.

Let us show some examples of optimal sets for equity problem in $\mathbb{R}^2_{l_1}$. In Fig-

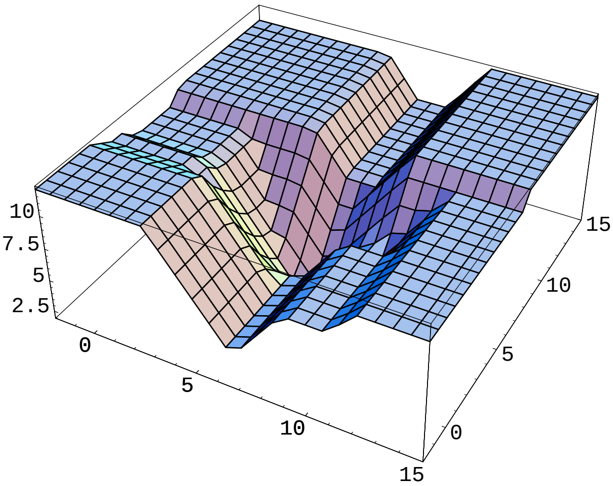


Figure 2.11: *Objective function of Example 2.2.1*

ure 2.12 a) the objective function of MWAP for six existing points $(1,1)$, $(2,2)$, $(3,3)$, $(4,4)$, $(5,5)$, $(6,6)$ is illustrated. Its optimal region is the union of two unbounded convex sets which are shown in Figure 2.12 b). In the following Figure 2.13

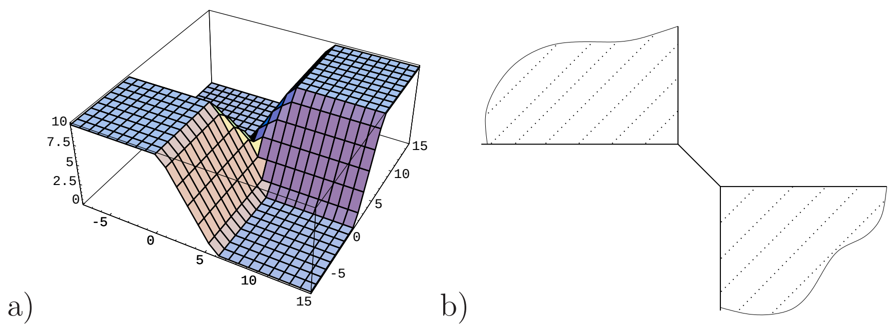


Figure 2.12: *MWAP for points* $(1,1)$, $(2,2)$, $(3,3)$, $(4,4)$, $(5,5)$, $(6,6)$*: a) objective function; b) set of optimal solutions on* $\mathbb{R}^2_{l_1}$.

MWAP is solved with respect to seven given points. An optimal region for them is the halfline. Next case illustrate the objective function of MWAP for two points $(2,2)$ and $(4,5)$. The optimal set here is union of two halflines and the optimal interval for the center problem (Figure 2.14). A degenerate case of MWAP for one point is shown in Figure 2.15. As optimal set we have the union of quarters, which compose the whole plane.

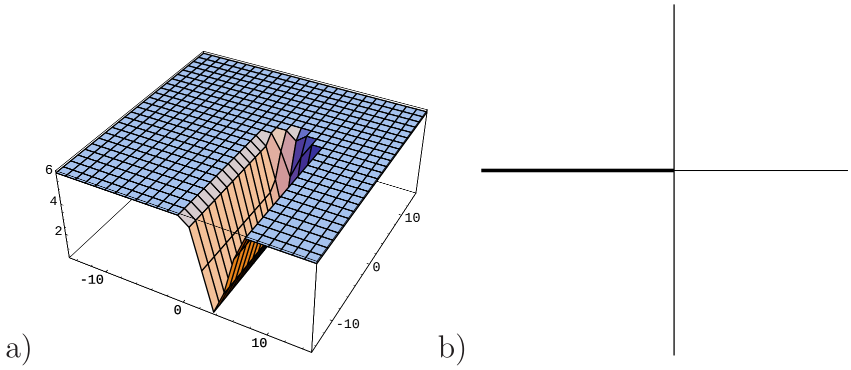


Figure 2.13: *MWAP for points* $(1,1)$, $(2,2)$, $(3,3)$, $(4,4)$, $(5,3)$, $(6,2)$, $(7,1)$*: a) objective function; b) set of optimal solutions on* $\mathbb{R}^2_{l_1}$.

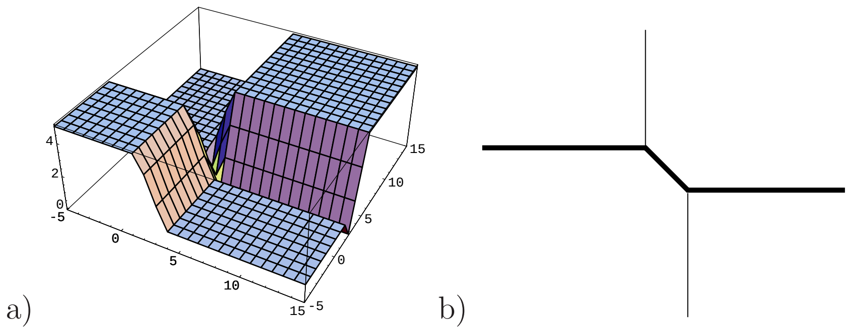


Figure 2.14: *MWAP for points* $(2,2)$, $(4,5)$*: a) objective function; b) set of optimal solutions on* $\mathbb{R}^2_{l_1}$.

2.2.4 CLP and its equivalence to MWAP

Recall the *circle location problem* in rectilinear metric:

Definition 2.7. CLP in $\mathbb{R}^2_{l_1}$

For a given finite set of existing facilities $\mathcal{E}x = \{Ex_1, \dots, Ex_n\} \subset \mathbb{R}^2$*, where* $Ex_i = (a_{i_1}, a_{i_2})$*,* $i = 1, \dots, n$*, find a circle* $\mathcal{C}(x,\rho)$*,* $x = (x_1, x_2) \in \mathbb{R}^2$*, so that the maximal distance from the existing points to the circle* $\mathcal{C}(x,\rho)$

$$\max_{i=1,\dots,n} d(Ex_i, \mathcal{C}(x,\rho)) = \max_{i=1,\dots,n} \min_{y\in\mathcal{C}(x,\rho)} d(Ex_i, y) = \max_{i=1,\dots,n} \min_{y\in\mathcal{C}(x,\rho)} (|a_{i_1} - y_1| + |a_{i_2} - y_2|) \tag{2.23}$$

is minimal.

CLP is more complicated than MWAP. However, it has been proved by the author [29] that in $\mathbb{R}^2_{l_1}$ the minimum width annulus problem and the circle location

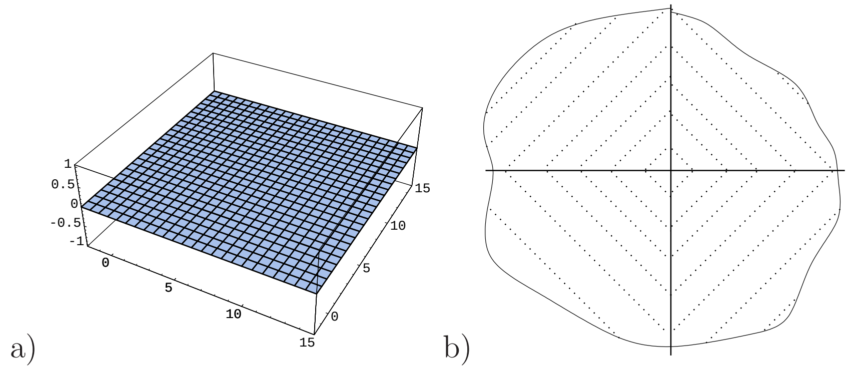


Figure 2.15: *MWAP for point* $(3,2)$*: a) objective function; b) set of optimal solutions on* $\mathbb{R}^2_{l_1}$.

problem are equivalent. Optimal annulus $\mathcal{A}(x,R,r)$ and optimal circle $\mathcal{C}(x,\rho)$, which are obtained by solving of MWAP and CLP on the same set of existing points, have identical sets of center points x. For the fixed center point x the radius ρ of the circle $\mathcal{C}(x,\rho)$ is equal to the half the sum $(R+r)/2$ of radii of outer and inner circles bounding the annulus $\mathcal{A}(x,R,r)$. For this reason we can build effective solution procedure for CLP based on Algorithm 2.1 for solving MWAP in $\mathbb{R}^2_{l_1}$:

Algorithm 2.2 *Optimal solution of CLP in* $\mathbb{R}^2_{l_1}$

Input: Set of existing points $Ex_i = (a_{i1}, a_{i2})$, $i = 1, ..., n$;

Using Algorithm 2.1 find optimal annuli $\mathcal{A}(x, R_x, r_x)$, $x \in X \subset \mathbb{R}^2$;

Output: optimal circles $\mathcal{C}(x,\rho_x)$, where $x \in X$ and $\rho_x = (R_x + r_x)/2$.

2.2.5 Optimal solution of MWAP, CLP, PLP in $\mathbb{R}^2_{l_\infty}$

Let us formulate MWAP, CLP and PLP in Chebyshev metric:

Definition 2.8. MWAP in $\mathbb{R}^2_{l_\infty}$

For a given finite set of existing facilities $\mathcal{E}x = \{Ex_1, \ldots, Ex_n\} \subset \mathbb{R}^2$*, where* $Ex_i = (a_{i_1}, a_{i_2})$*,* $i = 1, \ldots, n$*, find* $x = (x_1, x_2) \in \mathbb{R}^2$ *and two concentric circles* $\mathcal{C}(x,R)$ *and* $\mathcal{C}(x,r)$ *defining the annulus* $\mathcal{A}(x,R,r)$

$$r \leq max(|a_{i_1} - x_1|, |a_{i_2} - x_2|) \leq R, \qquad i = 1, \ldots, n$$

such that the set $\mathcal{E}x$ *is contained in* $\mathcal{A}(x,R,r)$ *and its width*

$$width_{\mathcal{A}} = R - r \tag{2.24}$$

is minimal.

Definition 2.9. CLP in $\mathbb{R}^2_{l_\infty}$

For a given finite set of existing facilities $\mathcal{E}x = \{Ex_1, \dots, Ex_n\} \subset \mathbb{R}^2$, *where* $Ex_i = (a_{i_1}, a_{i_2})$, $i = 1, \dots, n$, *find a circle* $\mathcal{C}(x, \rho)$, $x = (x_1, x_2) \in \mathbb{R}^2$, *so that the maximal distance from the existing points to the circle* $\mathcal{C}(x, \rho)$

$$\max_{i=1,\dots,n} d(Ex_i, \mathcal{C}(x,\rho)) = \max_{i=1,\dots,n} \min_{y \in \mathcal{C}(x,\rho)} d(Ex_i, y) = \max_{i=1,\dots,n} \min_{y \in \mathcal{C}(x,\rho)} max(|a_{i_1} - y_1|, |a_{i_2} - y_2|) \tag{2.25}$$

is minimal.

Definition 2.10. PLP in $\mathbb{R}^2_{l_\infty}$

For a given finite set of existing facilities $\mathcal{E}x = \{Ex_1, \dots, Ex_n\} \subset \mathbb{R}^2$, *where* $Ex_i = (a_{i_1}, a_{i_2})$, $i = 1, \dots, n$, *find a point* $x = (x_1, x_2) \in \mathbb{R}^2$, *so that the maximal distance from the point* x *to the set* $\mathcal{E}x$

$$\max_{i=1,\dots,n} max(|a_{i_1} - x_1|, |a_{i_2} - x_2|) \tag{2.26}$$

is minimal.

To solve these problems we use result of the following Lemma:

Lemma 2.11. *[33] Let* $a = (a_1, a_2), b = (b_1, b_2) \in R^2$. *Then*

$$l_\infty(a, b) = l_1(T(a), T(b)),$$

where

$$T = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}.$$

For the proof see [33].

Therefore, a point $x_{l_1} = (x^1_{l_1}, x^2_{l_1})$ on rectilinear plane has the following coordinates on Chebyshev plane:

$$x_{l_\infty} = T^{-1}(x_{l_1}) = (x^1_{l_1} - x^2_{l_1}, x^1_{l_1} + x^2_{l_1}).$$

Hence, in order to solve MWAP, CLP and PLP in $\mathbb{R}^2_{l_\infty}$ we first apply corresponding algorithms developed for rectilinear metric and after that find coordinates of center points on Chebyshev plane by Lemma 2.11.

2.2.6 Interpretation of obtained results

As it was pointed out before PLP and MWAP are not equivalent in Euclidean space. It should be noted that a solution of PLP with Euclidean metric is unique. In contrast, this problem with rectilinear (or Chebyshev) distance usually has a set of solutions. For instance, in Example 2.2.1 every covering diamond which is obtained by shifting the circle $D1$ centered at the point $(x_1^{line_4}, x_2^{line_3})$ along the line interval $I_1 = \{x_1 + x_2 = max(a_{i1} + a_{i2}) + min(a_{i1} + a_{i2}))/2, x_1^{line_2} \leq x_1 \leq x_1^{line_4}\}$ till the circle $D2$ with the center point $(x_1^{line_2}, x_2^{line_1})$ is optimal (Figure 2.16). The distances

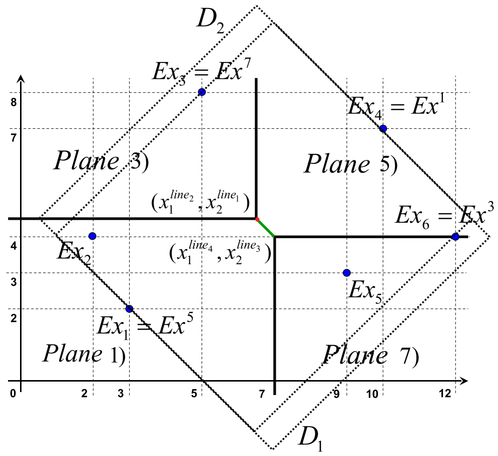


Figure 2.16: *Geometrical solution of PLP for Example 2.2.1 in* $\mathbb{R}^2_{l_1}$

$d(Ex^1, I_1)$ and $d(Ex^5, I_1)$ are always equal to minimum of maximal distance between existing points and the points on the interval I_1. Due to this fact the points Ex^1, Ex^5 (Ex^3, Ex^7 in the case $x_1^{line_2} > x_1^{line_4}$) are always on the optimal circle. As it can be seen the optimal circle of PLP in $\mathbb{R}^2_{l_1}$ always contains a so-called diametral pair of points. That is not always the case on Euclidean plane. The circles $D1$ and $D2$ contain additionally one more point Ex^7 and Ex^3, respectively. When $x_1^{line_2} = x_1^{line_4}$ a solution of PLP is unique and there are two diamertal pairs Ex^1, Ex^5 and Ex^3, Ex^7 on the optimal circle. In other words, a solution of PLP with rectilinear (Chebyshev) distance is unique if and only if on each site of covering diamond (square) lies at least one existing point.

In this work it was shown that PLP in $\mathbb{R}^2_{l_1}$ does not increase along directions d_1, d_2, d_3, d_4 (Figure 2.17). That is why MWAP and PLP have at least one common optimal point x. In this point the second part of objective function in MWAP $min_{i=1,\ldots,n}(|a_{i_1} - x_1| + |a_{i_2} - x_2|)$ has its maximal value. Thus, each of all common optimal points of MWAP and PLP in $\mathbb{R}^2_{l_1}$ (or in $\mathbb{R}^2_{l_\infty}$) is

- center of covering circle which contains at least one diametral pair $\{Ex^1, Ex^5\}$

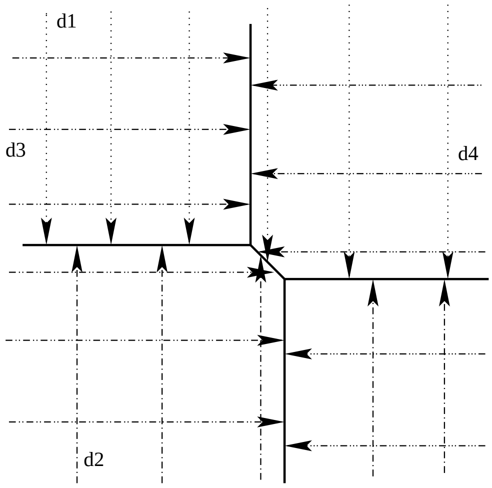


Figure 2.17: *Four types of directions along which PLP does not increase in* $\mathbb{R}^2_{l_1}$

or $\{Ex^3, Ex^7\}$ (diametral pair is the same for all common optimal points);

- center of annulus where this diametral pair lies on the outer circle.

This is not true in Euclidean space. There is either a diametral pair or three points on the covering circle in the optimal solution of PLP. Moreover, Rivlin [43] showed that the minimum width annulus of n points must have two points on the inner circle and two points on the outer circle of the annulus which are not necessary diametral. Therefore, the equivalence of PLP and MWAP which was achieved for $\mathbb{R}^2_{l_1}$ and $\mathbb{R}^2_{l_\infty}$ can not be stated in Euclidean space.

Chapter 3

MWAP and related problems in unweighted undirected networks $G(V, E)$

In this chapter we introduce the notions of circle, sphere and annulus in networks and show how to solve the resulting minimum width annulus problem in networks with an $\mathcal{O}(nm)$ algorithm, where n and m is the number of nodes and edges in the given graph, respectively. The chapter discusses the relation of the minimum width annulus problem to the circle center problem which is surprisingly different from the planar one. Finally, we use the MWAP to improve lower and upper bounds in the currently best center point location algorithm. It reduces the number of candidate arcs, convergence of the solution procedure, and consequently the complexity in practice.

3.1 MWAP in $G(V, E)$

3.1.1 Circle and annulus in networks

Let $G(V, E)$ be an undirected network, where $V = \{v_1, \ldots, v_n\}$ is the set of nodes and $E = \{e_1, \ldots, e_m\}$ is the set of edges with cardinality $|V| = n$ and $|E| = m$, respectively. The edges are usually written as $e = [v_i, v_j]$ with end nodes v_i, v_j. We denote with $\mathbb{P}(G) := \{x : x = (e, t), e \in E, 0 \leq t \leq 1\}$ the set of all points in the network G, where $x = (e, t)$ is a *point* in edge $e = [v_i, v_j]$ if it has the distance $d(i, x) := tl_e$ to node v_i and $d(j, x) := (1 - t)l_e$ to node v_j. If $d(k, i)$ and $d(k, j)$ is the length of a shortest path between nodes v_k and v_i, and v_k and v_j, respectively, then the distance between point $x = (e, t) \in \mathbb{P}(G)$ and node $v_k \in V$ is

$$d(x, k) = \min\{d(k, i) + tl_e, d(k, j) + (1 - t)l_e\}. \tag{3.1}$$

Consequently, the distance between two points $x = (e, t)$ and $y = (e', t')$ in G with $e = [v_i, v_j] \neq e' = [v_{i'}, v_{j'}]$ is

$$d(x, y) = \min\{d(x, i') + t'l'_e, d(x, j') + (1 - t')l_{e'}\} \tag{3.2}$$

$$= \min\{d(y, i) + tl_e, d(y, j) + (1 - t)l_e\} \tag{3.3}$$

Definition 3.1. A *circle, sphere*, and *open sphere* with *center* $x \in \mathbb{P}(G)$ and *radius* ρ in the network $G(V, E)$ is the set of points

$$\mathcal{C}(x, \rho) = \{y \in \mathbb{P}(G) : d(x, y) = \rho\}, \tag{3.4}$$

$$\mathcal{S}(x, \rho) = \{y \in \mathbb{P}(G) : d(x, y) \leq \rho\}, \quad \text{and} \tag{3.5}$$

$$\mathcal{S}^o(x, \rho) = \{y \in \mathbb{P}(G) : d(x, y) < \rho\}, \quad \text{respectively.} \tag{3.6}$$

It should be noted that the notion of a *circle* in a network is different from the notion of a *cycle*, where the latter consists of a sequence of nodes with coinciding first and last node and where consecutive nodes in the sequence are connected by an edge. The location of cycles has been studied quite extensively (see, for example, Rodrigues [44]). Literature on the location of circles in networks is, on the other hand, virtually non-existent. In contrast to circles in the Euclidean plane, circles in networks consist of a finite number of points. In fact, it is easy to see that any circle may intersect each edge at most twice, such that the circle satisfies

$$|\mathcal{C}(x, \rho)| \leq 2m. \tag{3.7}$$

With the assumption that the number of edges in networks is greater or equal to 3 this bound can be replaced by $2m - 4$.

Example 3.1.1. *The example of Figure 3.1 shows a network with lengths l_e on the edges, nodes $\{v_1, \ldots, v_7\}$, and point $x = (e, \frac{1}{6}) \in \mathbb{P}(G)$ where $e = [v_4, v_7]$ (i.e., the distance between v_4 and x is $\frac{1}{6}\ l_{47} = 1$ and the distance between x and v_7 is $(1 - \frac{1}{6})\ l_{47} = 5$). In this example, the circle $\mathcal{C}(x, 4)$ consists of the three points $\mathcal{C}(x, 4) = \{y_1, y_2, y_3\}$. The shortest paths with lengths 4 between x and $y_i, i = 1, 2, 3$ are indicated in Figure 3.1 by dashed arrows. The sphere $\mathcal{S}(x, 4)$ is the set of points in G shown in solid, thick lines. It should be noted that the triangle built by the nodes v_4, v_5, v_6 is a subset of the sphere $\mathcal{S}(x, 4)$, but none of the points is contained on any shortest path from the center x to some point y in the circle $\mathcal{C}(x, 4)$. This is a decisive difference to circles in the plane, which will become important when we discuss the relation between minimum width annulus and circle location problems in networks. This difference becomes even more obvious, when we extend the radius of the cycle to 7. Then $\mathcal{C}(x, 7) = \emptyset$ while $\mathcal{S}(x, 7) = \mathbb{P}(G)$ is the set of all points in G.*

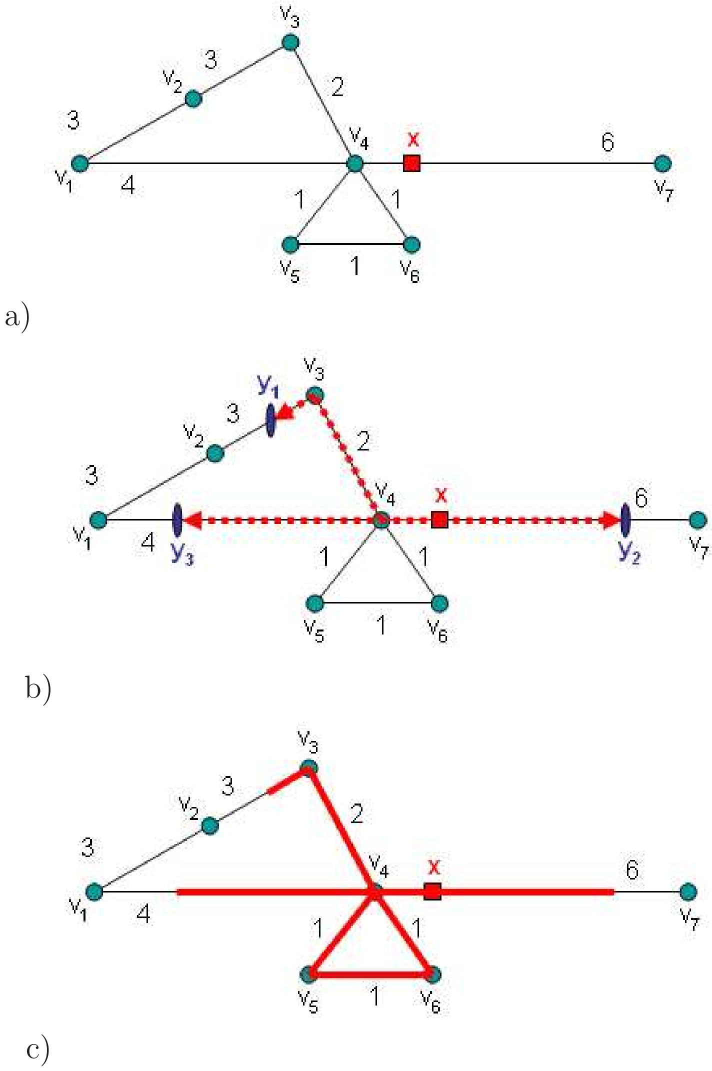


Figure 3.1: *a). Network G with edge lengths l_e; b). circle $\mathcal{C}(x, 4) = \{y_1, y_2, y_3\}$ (ellipsoidal points); c). sphere $\mathcal{S}(x, 4)$ (thick lines).*

Definition 3.2. An *annulus* $\mathcal{A}(x, R, r)$ centered at a point $x \in \mathbb{P}(G)$ in network $G(V, E)$ is defined by two concentric circles $\mathcal{C}(x, R)$ and $\mathcal{C}(x, r)$ such that

$$\mathcal{A}(x, R, r) = \mathcal{S}(x, R) \setminus \mathcal{S}^o(x, r). \tag{3.8}$$

If $V \subseteq \mathcal{A}(x, R, r)$, then $\mathcal{A}(x, R, r)$ is called a *node covering annulus*.

Definition 3.3. MWAP in $G(V, E)$

For a given network $G(V, E)$ with cardinality $|V| = n$ and $|E| = m$ find a node covering annulus $\mathcal{A}(x, R, r)$, $x \in \mathbb{P}(G)$ of minimum width

$$width_{\mathcal{A}} = R - r. \tag{3.9}$$

The solution of MWAP depends on the choice of $x \in \mathbb{P}(G)$ only, since the node covering property and the minimality of $width_{\mathcal{A}} = R - r$ is obtained by choosing

$$r = \min_{v_k \in V} d(x, v_k) \quad \text{and} \quad R = \max_{v_k \in V} d(x, v_k). \tag{3.10}$$

Hence, MWAP can be rewritten as equity problem in network G:

$$\min f(x)_{x \in \mathbb{P}(G)} := \max_{k=1,\dots,n} d(x, v_k) - \min_{k=1,\dots,n} d(x, v_k). \tag{3.11}$$

Mesa et al. [40] have reviewed known algorithmic results and present improved algorithms for some of single facility network location problems with equity measures. One of the discussed models is the model with the range measure, where minimizing the objective function results to minimizing the difference between the maximum and the minimum weighted distances. The authors have concluded that an $\mathcal{O}(mnlogn)$ algorithm for solving this model on general networks can be obtained by applying the algorithm in Burkard and Dollani [9]. This algorithm was constructed for the pos/neg 1-center problem, which requires to minimize a linear combination of the maximum weighted distances of the center to the vertexes with positive weights and to the vertexes with negative weights, respectively. However, finding of positive 1-center is performed using Kariv and Hakimi's algorithm [35]. This algorithm in unweighted case requires $\mathcal{O}(mn)$ time with the assumption that for each vertex its distances to all other vertexes are already sorted. Therefore, minimizing the range in unweighted case by the algorithm in [40] can be achieved in $\mathcal{O}(mn + n^2 logn)$ time only when the distance matrix is already given. We have not found any references on location of annuli in networks.

3.1.2 Algorithm for solving of MWAP

We first consider the objective function value of MWAP in its equity form (3.11) for a given point $x = (e, t)$ with $e = [v_i, v_j] = [i, j]$. We call any minimizing point $x = (e, t)$ in edge e a *local optimum* for MWAP.

According to (3.3) the distance between the point $x \in e$ and any point $y \in \mathbb{P}(G) \setminus e$ is either a linear increasing (iff $d(y, i) + l_e = d(y, j)$), a linear decreasing (iff $d(y, j) + l_e = d(y, i)$), or a piecewise linear, concave, and continuous function in t with breakpoint $t^b = \frac{1}{2} + \frac{d(j,y) - d(y,i)}{2l_e}$ (see Figure 3.2). The first part $\max_{k=1,\dots,n} d(x, v_k)$ of the objective function (3.11) is the well-known piecewise linear objective function of the point center location problem (PLP) in networks (see Figure 3.3 a)).

The value $\min_{k=1,\dots,n} d(x, v_k)$ of the second part in the objective function of the

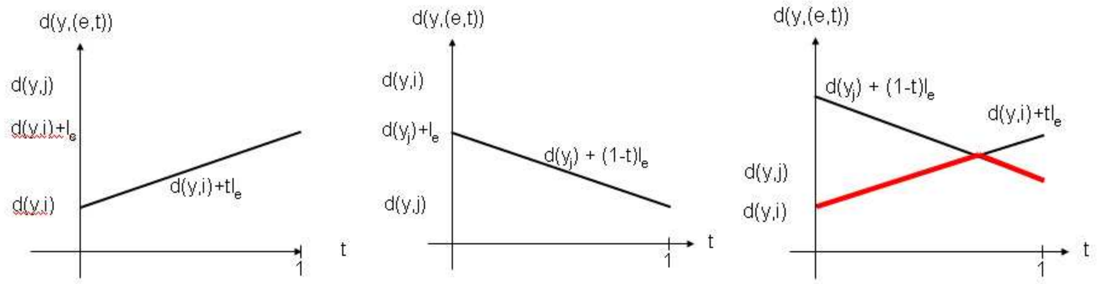


Figure 3.2: *Three possible alternatives for the network distance between point y and (e, t) as function of t.*

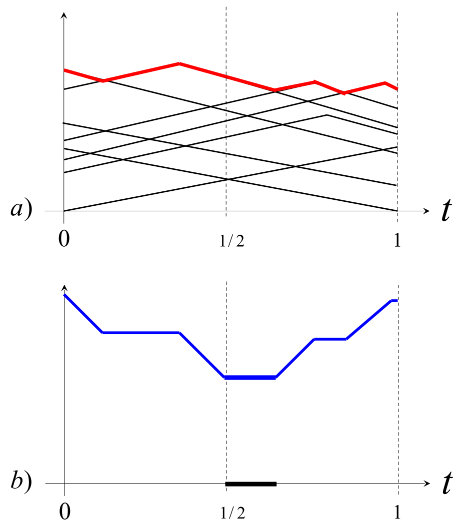


Figure 3.3: *The objective function (a) of the local point center problem and of the local annulus problem (b). The thicker part in (b) is the set of center points for the locally optimal annulus.*

local MWAP is obviously attained in v_i or v_j, such that

$$\min_{v_k \in V} d(x, v_k) = \min\{tl_e, (1-t)l_e\} = \begin{cases} tl_e & \text{if } 0 \leq t \leq \frac{1}{2}, \\ (1-t)l_e & \text{if } \frac{1}{2} < t \leq 1. \end{cases}$$

Hence, the objective value (3.11) of the local MWAP is a piecewise linear function which is non-increasing left of $\frac{1}{2}$ and non-decreasing right of $\frac{1}{2}$. More precisely, we can make the following statements (see Figure 3.3).

- The knot points of $f(x)$, $x = (e, t)$ coincide with the knot points of the point center objective $\max_{k=1,\ldots,n} \; d(x, v_k)$ and with the point $(e, \frac{1}{2})$ (in some cases),
- a linear piece including $t \leq \frac{1}{2}$ has a slope of $-2l_e$ and 0, if the corresponding

linear piece of $\max_{k=1,\ldots,n}\ d(x,v_k)$ has a negative and positive slope, respectively,

- a piece including $t \geq \frac{1}{2}$ has a slope of 0 and $2l_e$, if the corresponding linear piece of $\max_{k=1,\ldots,n}\ d(x,v_k)$ has a negative and positive slope, respectively.

We have thus proved the following result.

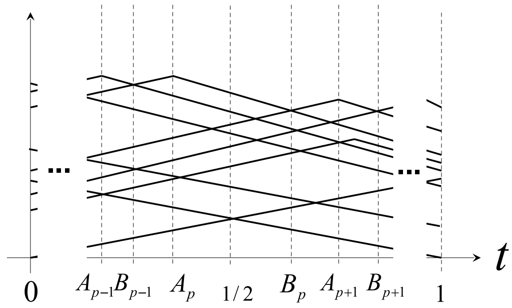


Figure 3.4: *Knots of function* $\max_{k=1,\ldots,n}\ d(x,v_k)$.

Theorem 3.4. *Let A_p and B_p be the knots of the point center location objective* $\max_{k=1,\ldots,n}\ d(x,v_k)$ *where the slope changes from positive to negative and from negative to positive, respectively (see Figure 3.4).*

The set of local optimisers for MWAP is

$$LocOpt = \begin{cases} [\frac{1}{2}, B_p], & if\ A_p < \frac{1}{2} < B_p, \\ \{\frac{1}{2}\}, & if\ \frac{1}{2} = B_p, \\ [B_{p-1}, \frac{1}{2}], & if\ B_{p-1} < \frac{1}{2} < A_p, and \\ [B_{p-1}, B_p], & if\ \frac{1}{2} = A_p. \end{cases} \tag{3.12}$$

In particular, the middle point $(e, 1/2)$ is for any edge $e \in E$ an optimal solution of the local MWAP.

Theorem 3.4 allows us to construct an efficient exact algorithm for solving of MWAP in unweighted networks with the following frame (assuming the shortest distance matrix is given): first, we search edges $e' \in E$, middle point of which solves MWAP. Using (3.12), we find for each of these edges e' point B_{p-1} or B_p or both of them as endpoint(s) of the optimal interval with center points of MWA. These points are intersection points of distance plots $d(v_k, x)$, $v_k \in V$, $x \in e'$ belonging to first farthest vertex from the middle point of the edge e' and to second farthest vertex from the middle point $(e', \frac{1}{2})$ nondominated by the first farthest point (see Figures 3.5 and 3.3).

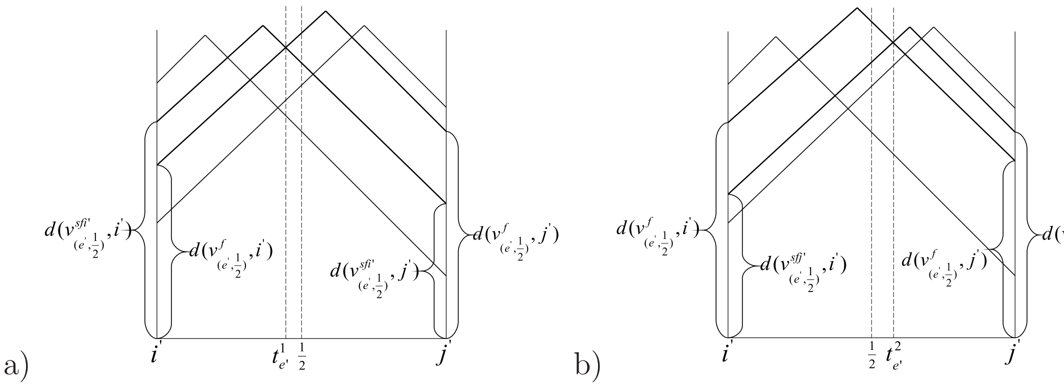


Figure 3.5: *Finding of optimal interval for center points of MWA.*

Definition 3.5. A vertex $v_x^f \in V$ is the *first farthest vertex* from a point $x \in e = [i, j] \in E$, iff $d(v_x^f, x) = \max_{v_k \in V} d(v_k, x)$ and, in the case of not uniqueness of the first farthest point, the distance from v_x^f to the endpoint of e not contained in the shortest path $v_x^f \to x$ is maximal.

Definition 3.6. A vertex $v_x^{sf} \in V$ is the *second farthest vertex* from a point $x \in e = [i, j] \in E$, iff the distance from v_x^{sf} to the endpoint of e not contained in the shortest path $v_x^{sf} \to x$ is strictly greater than the distance to this endpoint from v_x^f and $d(v_x^{sf}, x)$ is maximal. In the case of not uniqueness of the second farthest point, v_x^{sf} is a point with the maximal distance to the endpoint of e not contained in the shortest path $v_x^{sf} \to x$.

Hence, the farthest points are defined in a lexicographic sense. Thus we have proved Algorithm 3.1 for solving MWAP in undirected networks which has not been studied yet. The only known algorithm for minimizing the range function in general case obtained by Mesa et al. [40] has a complexity $\mathcal{O}(mnlogn)$. For unweighted case its complexity reduces to $\mathcal{O}(mn + n^2logn)$. The overall running time of Algorithm 3.1 is of order $\mathcal{O}(mn)$.

3.1.3 Computational results

Algorithm 3.1 was implemented in C++ and compiled with g++ v.3.3.3. All computations were done at the University of Kaiserslautern on a server equipped with Dual Intel Xeon 3.2 GHz CPUs, 4 GB RAM running on Linux Kernel 2.6.5. Computational results for 23 problems are summarized in Table 3.1. Each problem was solved 16 times: 1 time for constructed network, which is connected and has the fixed lengths of edges, and 15 times for the network with the same edges connectivity but with random lengths of edges. The randomly generated problems have 100 nodes and a number of edges corresponding to decreasing average node degrees from 99 to 1 (columns (1) – (2)). Our goal is to estimate the influence of the density of a

Table 3.1: Summary of computational results for MWAP (368 problems).

n=100, m= (1)	average node degree (2)	centers of MWA range (No. of edges) (3)	centers of MWA average (No. of edges) (4)
4950	99	1 – 20	4.625
4783	95	1 – 72	8.0625
4549	90	1 – 90	9.875
4224	84	1 – 16	4.125
4018	80	1 – 16	3.125
3730	74	1 – 19	7.4375
3468	69	1 – 5	1.6875
3218	64	1 – 16	3.375
2991	59	1 – 24	5.125
2784	55	1 – 8	2.4375
2539	50	1 – 6	1.6875
2279	45	1 – 5	3.5
2033	40	1 – 77	13.1875
1757	35	1 – 6	1.9375
1548	30	1 – 15	3.6875
1282	25	1 – 11	2.75
1008	20	1 – 4	1.75
772	15	1 – 4	1.75
509	10	1 – 16	3.5
275	5	1 – 9	3.3125
117	2	1 – 2	1.1875
100	2	2	2
99	1	1 – 99	43.8125
all		1 – 99	5.823

Algorithm 3.1 *Optimal solution of MWAP in* $G(V, E)$

Input: $G(V, E)$, $|E| = m$, $|V| = n$, distance matrix $D = (d_{ij})$;

Step 1. $\forall e = [i, j] \in E$ find vertex $v^f_{(e,\frac{1}{2})} \in V$ which is the first farthest from the point $(e, \frac{1}{2})$ – complexity $\mathcal{O}(nm)$;

Step 2. find optimal width of annuli

$$width_{\mathcal{A}^*} = min_{e \in E} min(d(v^f_{(e,\frac{1}{2})}, i), d(v^f_{(e,\frac{1}{2})}, j))$$

and put edges e with $min(d(v^f_{(e,\frac{1}{2})}, i), d(v^f_{(e,\frac{1}{2})}, j)) = width_{\mathcal{A}^*}$ into the set E' – complexity $\mathcal{O}(m)$;

Step 3. $\forall e' = [i', j'] \in E'$ set initially endpoints of optimal interval $t^1_{e'} = t^2_{e'} = \frac{1}{2}$ and

if $d(v^f_{(e',\frac{1}{2})}, i') \leq d(v^f_{(e',\frac{1}{2})}, j')$ (see Figure 3.5a)) **then**

find nondominated by $v^f_{(e',\frac{1}{2})}$ second farthest from $(e', \frac{1}{2})$ vertex $v^{sfi'}_{(e',\frac{1}{2})} \in V$, i.e. $v^{sfi'}_{(e',\frac{1}{2})}$ is the point $v_k \in V$ with $d(v_k, i') > d(v^f_{(e',\frac{1}{2})}, i')$ and with maximal value $d(v_k, j')$;

end if

if $d(v^f_{(e',\frac{1}{2})}, j') \leq d(v^f_{(e',\frac{1}{2})}, i')$ (see Figure 3.5b)) **then**

find nondominated by $v^f_{(e',\frac{1}{2})}$ second farthest from $(e', \frac{1}{2})$ vertex $v^{sfj'}_{(e',\frac{1}{2})} \in V$, i.e. $v^{sfj'}_{(e',\frac{1}{2})}$ is the point $v_k \in V$ with $d(v_k, j') > d(v^f_{(e',\frac{1}{2})}, j')$ and with maximal value $d(v_k, i')$;

end if

if $v^{sfi'}_{(e',\frac{1}{2})}$ *does not exist* ($v^{sfj'}_{(e',\frac{1}{2})}$ *does not exist*) **then**

$t^1_{e'} = 0$ ($t^2_{e'} = 1$);

else

$t^1_{e'} = \frac{l_{e'} - d(v^f_{(e',\frac{1}{2})}, i') + d(v^{sfi'}_{(e',\frac{1}{2})}, j')}{2l_{e'}}$ ($t^2_{e'} = \frac{l_{e'} + d(v^f_{(e',\frac{1}{2})}, j') - d(v^{sfj'}_{(e',\frac{1}{2})}, i')}{2l_{e'}}$);

end if

– complexity $\mathcal{O}(n|E'|) \leq \mathcal{O}(nm)$.

Output: $\forall e' \in E'$ optimal annulus $\mathcal{A}^*(x, R, r)$ of $width_{\mathcal{A}^*}$, where $x = tl_{e'}$, $t \in [t^1_{e'}, t^2_{e'}]$, $r = min\{tl_{e'}, (1-t)l_{e'}\}$, $R = width_{\mathcal{A}^*} + r$.

network on the quantity of solutions of MWAP. Columns (3) – (4) show the range and average number of edges in the networks containing centers of minimum width annuli in this networks. It can be easily seen that the number of such edges is independent on density of a network, even on trees.

3.1.4 MWAP on subsets

In this section we consider a problem when only a subset V' of the set V needs to be covered by the annulus.

Definition 3.7. MWAP on subset $V' \subseteq V$ in $G(V, E)$

For a given network $G(V, E)$ with cardinality $|V| = n$ and $|E| = m$ find an annulus $\mathcal{A}(x, R, r)$, $x \in \mathbb{P}(G)$ of minimum width

$$width_{\mathcal{A}} = R - r, \tag{3.13}$$

which covers the set $V' \subseteq V$, where $|V'| = n_1$, $n_1 \leq n$. We assume without loss of generality $V' = \{v_1, ..., v_{n_1}\}$.

As it was shown in Section 3.1.2 the function $f(x) = \max_{k=1,...,n} \ d(x, v_k) - \min_{k=1,...,n} \ d(x, v_k)$, $x = (e, t)$ is piecewise linear, non-increasing left of $(e, \frac{1}{2})$, and non-decreasing right of $(e, \frac{1}{2})$. A local minimum of $f(x)$ on the edge $e \in E$ is attained at point of maximum for function $\min_{k=1,...,n} \ d(x, v_k)$, which is equal to $(e, \frac{1}{2})$. Unlike the function $\min_{k=1,...,n} \ d(x, v_k)$, function $\min_{k=1,...,n_1} \ d(x, v_k)$, $v_k \in V'$, $x = (e, t)$, $e = [v_i, v_j]$ reaches the maximum in the breakpoint point $t^b \in [0, 1]$

$$t^b = \frac{\min_{v_k \in V'} \ d(j, v_k) - \min_{v_k \in V'} \ d(i, v_k) + l_e}{2l_e}. \tag{3.14}$$

Therefore, a local minimum for the function

$$f(x) = \max_{v_k \in V'} \ d(x, v_k) - \min_{v_k \in V'} d(x, v_k) \tag{3.15}$$

on the edge $e \in E$ is attained

- at the point v_i, if $t^b = 0$,
- at the point v_j, if $t^b = 1$,
- at the point (e, t^b) (3.14) otherwise.

Hence, to find all solutions of the annulus problem on subsets we use the preceding arguments to modify Algorithm 3.1 and obtain the solution procedure for MWAP on subsets described in Algorithm 3.2.

3.1.5 Restricted MWAP

Here for choosing the center point x of an annulus $\mathcal{A}(x, R, r)$ only subsets of the network are allowed:

Algorithm 3.2 *Optimal solution of MWAP on subset $V' \subseteq V$ in $G(V,E)$*

Input: $G(V,E)$, $|E| = m$, $|V| = n$, subset $V' \subseteq V$ with $|V'| = n_1$, distance matrix $D = (d_{ij})$;

Step 1. $\forall e \in E$, $e = [i,j]$ find breakpoint t^b (3.14) and farthest and nearest from the point (e,t^b) vertexes $v^f_{(e,t^b)} \in V'$ and $v^n_{(e,t^b)} \in V'$, respectively
– complexity $\mathcal{O}(n_1 m)$;

Step 2. find optimal width of annuli $width_{\mathcal{A}^*} = \min_{e \in E}\{d(v^f_{(e,t^b)}, (e,t^b)) - d(v^n_{(e,t^b)}, (e,t^b))\}$ and put edges e of $width_{\mathcal{A}^*}$ at the point (e,t^b) into the set E'
– complexity $\mathcal{O}(m)$;

Step 3. *finding of all MWA centers:*
$\forall e' = [i',j'] \in E'$ set initially endpoints of optimal interval $t^1_{e'} = t^2_{e'} = t^b$ and

if $d(v^f_{(e',t^b)}, i') + t^b l_e \le (1-t^b) l_e + d(v^f_{(e',t^b)}, j')$ **then**
find the second farthest from (e',t^b) vertex $v^{sfi'}_{(e',t^b)} \in V'$, i.e. $v^{sfi'}_{(e',t^b)}$ is the point $v_k \in V'$ with $d(v_k, i') > d(v^f_{(e',t^b)}, i')$ and with maximal value $d(v_k, j')$;
end if
if $(1-t^b) l_e + d(v^f_{(e',t^b)}, j') \le d(v^f_{(e',t^b)}, i') + t^b l_e$ **then**
find the second farthest from (e',t^b) vertex $v^{sfj'}_{(e',t^b)} \in V$, i.e. $v^{sfj'}_{(e',t^b)}$ is the point $v_k \in V'$ with $d(v_k, j') > d(v^f_{(e',t^b)}, j')$ and with maximal value $d(v_k, i')$;
end if

if $v^{sfi'}_{(e',t^b)}$ *does not exist* ($v^{sfj'}_{(e',t^b)}$ *does not exist*) **then**
$t^1_{e'} = 0$ ($t^2_{e'} = 1$);
else
$t^1_{e'} = \frac{l_{e'} - d(v^f_{(e',t^b)}, i') + d(v^{sfi'}_{(e',t^b)}, j')}{2 l_{e'}}$ $\left(t^2_{e'} = \frac{l_{e'} + d(v^f_{(e',t^b)}, j') - d(v^{sfj'}_{(e',t^b)}, i')}{2 l_{e'}}\right)$;
end if
– complexity $\mathcal{O}(n_1 |E'|) \le \mathcal{O}(n_1 m)$.

Output: $\forall e' \in E'$ optimal annulus $\mathcal{A}^*(x,R,r)$ of $width_{\mathcal{A}^*}$, where $x = t l_{e'}$, $t \in [t^1_{e'}, t^2_{e'}]$, $r = \min_{v_k \in V'} d(v_k, x)$, $R = width_{\mathcal{A}^*} + r$.

Definition 3.8. Restricted MWAP in $G(V,E)$

For a given network $G(V,E)$ with cardinality $|V| = n$ and $|E| = m$ and a given subset $\mathbb{P}'(G) \subset \mathbb{P}(G)$ find an annulus $\mathcal{A}(x,R,r)$, $x \in \mathbb{P}'(G)$ of minimum width

$$width_{\mathcal{A}} = R - r, \tag{3.16}$$

which covers the set V.

Recall the properties of the objective function $f(x) = \max_{k=1,\dots,n} d(x,v_k) - \min_{k=1,\dots,n} d(x,v_k)$, $x = (e,t)$ of MWAP on the edge $e \in E$. It is non–increasing left of $(e, \frac{1}{2})$ and non-decreasing right of $(e, \frac{1}{2})$ function. Therefore, if the point $(e, \frac{1}{2}) \in \mathbb{P}'(G)$ and $[t_e^1 l_e, t_e^2 l_e]$ is the set of local optimizers of MWAP on the edge $e \in E$, the set $[t_e^1 l_e, t_e^2 l_e] \cap \mathbb{P}'(G)$ is the set of local optimizers for the restricted MWAP. If $(e, \frac{1}{2}) \notin \mathbb{P}'(G)$ then according to the properties of the function $f(x)$ the local minimum of restricted MWAP will be attained in at least one of the nearest to $(e, \frac{1}{2})$ from left and right sites points of the set $\mathbb{P}'(G)$. Hence, in the case $(e, \frac{1}{2}) \notin \mathbb{P}'(G)$ one of local optimizers of the restricted MWAP lies on the boundary of restricted region of the edge $e \in E$. Thus, finding at least one central point of an optimal annulus in the restricted MWAP can be performed by Algorithm 3.3.

Algorithm 3.3 *Optimal solution of restricted MWAP in $G(V, E)$*

Input: $G(V, E)$, $|E| = m$, $|V| = n$, subset $\mathbb{P}'(G) \subset \mathbb{P}(G)$, distance matrix $D = (d_{ij})$;

Step 1. $\forall e \in E$, $e = [i, j]$ find points $x_1 = (e, t_{left})$ and $x_2 = (e, t_{right})$, where $t_{left} \leq \frac{1}{2}$ is the maximal and $t_{right} \geq \frac{1}{2}$ is the minimal possible value of t for points $(e, t) \in e \cap \mathbb{P}'(G)$;
if $x_1 = x_2$ **then**
set $x = x_1$ and calculate $width_{\mathcal{A}(x,R,r)}$;
else
calculate $width_{\mathcal{A}(x_1,R,r)}$ and $width_{\mathcal{A}(x_2,R,r)}$;
end if
– complexity $\mathcal{O}(nm)$;

Step 2. find minimal width of annuli $width_{\mathcal{A}^*}$
– complexity $\mathcal{O}(m)$;

Output: points x or (x_1 or/and x_2) on $e \in E' \subseteq E$, where $e \in E'$ if width of the corresponding annulus at these points is equal to $width_{\mathcal{A}^*}$.

In order to find the complete set of all solutions of MWAP we check the optimal points computed with Algorithm 3.3. If there is at least one middle point x of some edge $e \in E$ among them, to calculate all optimal points it is enough to solve MWAP on edges e from the set E' in Algorithm 3.3. Hence, in this case we perform Algorithm 3.1 on edges $e \in E'$ and intersect the solution set obtained in it with points of the set $\mathbb{P}'(G)$ to choose all feasible optimal solutions.

Otherwise, if there are no middle points in the optimal set obtained from Algorithm 3.3, we should find interval of the constancy of the function $f(x)$ left from each point x_1, which exists if the first part of the function $f(x)$ increases directly left from x_1, and right from each point x_2, which exists if the first part of $f(x)$ decreases directly right from x_2. These intervals settle the set of solutions of the restricted MWAP in $G(V, E)$.

3.1.6 p–Minimum Width Annulus Problem

The *p–Minimum Width Annulus Problem* (p–MWAP) is the problem of finding p node covering annuli with minimal possible maximal width.

Definition 3.9. p–MWAP in $G(V, E)$

For a given network $G(V, E)$ with cardinality $|V| = n$ and $|E| = m$ and for a given natural number p find sets $V_1, ..., V_p \subset V$, $V \subset V_1 \cup ... \cup V_p$, and center points $x_1, ..., x_p \in \mathbb{P}(G)$, $1 \leq p < n - 1$, of annuli $\mathcal{A}_1(x_1, R_1, r_1)$, ... , $\mathcal{A}_p(x_p, R_p, r_p)$ so that the maximal width

$$\begin{aligned} width_p &= \max_{1 \leq s \leq p} (R_s - r_s) \qquad (3.17) \\ &= \max_{1 \leq s \leq p} \Big(\max_{v_k \in V_s, k=1,...,|V_s|} d(v_k, x_s) - \min_{v_k \in V_s, k=1,...,|V_s|} d(v_k, x_s) \Big) \end{aligned}$$

of covering annuli is minimal.

Note, that deleting one or more points from a set of points leads to nonincreasment of width of minimal annulus involving the set. Therefore, the problem reduces to finding of p disjoint subsets $V_1, ..., V_p \subset V$, i.e. $V_1 \cap ... \cap V_p = \emptyset$ and $V = V_1 \cup ... \cup V_p$ and p centers $x_1, ..., x_p \in \mathbb{P}(G)$, $1 \leq p < n-1$, which minimize the objective function (3.17).

For each fixed order of points in the set of vertexes the number of different partitions of V into disjoint sets $V_1, ..., V_p$ is equal to the number of solutions of Diofants Equation $y_1 + ... + y_p = n$, where $y_1, ..., y_p$ are natural numbers, which can be found as

$$C_{n-1}^{p-1} = \frac{(n-1)!}{(p-1)!(n-p)!}.$$

Hence, finding all partitions of the set V with fixed order of points in it into p disjoint sets takes $\mathcal{O}(\frac{n^{p-1}}{(p-1)!})$ time. Applying of Algorithm 3.2 for optimal solution of MWAP on subsets $V' \subseteq V$ in $G(V, E)$ to a partition $V_1, ..., V_p$ can be performed in

$$\mathcal{O}(|V_1|m + ... + |V_p|m) = \mathcal{O}(p \max_{1 \leq s \leq p} |V_s|m) = \mathcal{O}(pnm)$$

time. Hence, the worst case complexity of the resulting algorithm for solving of p–MWAP is $\mathcal{O}(\frac{n^{p-1}}{(p-1)!} pnm) = \mathcal{O}(\frac{pmn^p}{(p-1)!})$.

The problem of finding in a given graph $G(V, E)$ a *dominating set* of cardinality $\leq p$ is NP–complete even in the case when G is a planar graph of maximum vertex degree 3 (see Garey and Johnson [28]). In other words, to determine whether there is a subset $\tilde{V} \subset V$ of size less than or equal to p such that every vertex in $V - \tilde{V}$ is joined to at least one member of $\tilde{V}$ by an edge in E is NP–complete problem.

Theorem 3.10. *The problem of finding p minimum width annuli is NP–hard even in the case when the network is vertex–unweighted planar graph of maximum degree 3 and all its edges are of length 1.*

Proof. To prove NP–completeness of p–MWAP we observe that the problem of finding a dominating set of cardinality $\leq p$ is polynomial time reducible to the p–MWAP under conditions of Theorem 3.10. Let us suppose that we can find p–annuli with centers in the sets $X_1, ..., X_p \subset \mathbb{P}(G)$ and the maximal width of them is equal to $width^*$. Then there exists a dominating set of cardinality $\leq p$ if and only if $width^* \leq 1$ and each set $X_1, ..., X_p$ includes at least one point of the vertex set V. Namely, there is a dominating set of cardinality $\leq p$ in the set V if and only if $width^* \leq 1$ and at least a halfedge of each edge which middle point solves p–MWAP is the set of central points for one of p annuli. Clearly, if $width^* > 1$ or(and) at least one of the sets $X_1, ..., X_p$ does not contain one vertex point, then there exists no dominating set of cardinality $\leq p$ in the graph. Thus, the p–MWAP is NP–hard. $\square$

However, if we assume, that each annulus $\mathcal{A}_s(x_s, r_s, R_s)$, $1 \leq s \leq p$ covers all points of the set V for which the distance to the center point x_s is less than or equal to radius R_s, this problem can be solved in polynomial time.

Definition 3.11. p–nearest–MWAP in $G(V, E)$

For a given network $G(V, E)$ with cardinality $|V| = n$ and $|E| = m$ and for a given natural number p find sets $V_1, ..., V_p \subset V$, $V \subset V_1 \cup ... \cup V_p$, and center points $x_1, ..., x_p \in \mathbb{P}(G)$, $1 \leq p < n-1$, of annuli $\mathcal{A}_1(x_1, r_1, R_1), \ldots, \mathcal{A}_p(x_p, r_p, R_p)$ so that every point $v_k \in V$ belongs to each subset $V_s \subset V$ for which $d(v_k, x_s) \leq R_s$ and the maximal width

$$\begin{aligned} width_p &= \max_{1 \leq s \leq p} (R_s - r_s) \\ &= \max_{1 \leq s \leq p} \Big(\max_{v_k \in V_s, k=1,...,|V_s|} d(v_k, x_s) - \min_{v_k \in V_s, k=1,...,|V_s|} d(v_k, x_s) \Big) \end{aligned} \tag{3.18}$$

of covering annuli is minimal.

The difference between optimal solutions of p–MWAP and p–nearest–MWAP is shown in Figure 3.6. In this example $p = 3$ and the points y_1^n, y_2^n, y_3^n are the centers in an optimal solution for 3–nearest–MWAP, where the first annulus covers the set $V_1 = \{v_1, v_2\}$, the second contains the vertexes $V_2 = \{v_2, v_3, v_4\}$ and the last covers the points of the set $V_3 = \{v_5, v_6, v_7, v_8\}$. The width of 3–nearest–MWA is equal to 2. However, the width of 3–MWA is equal to 1. The points y_1, y_2, y_3 are the centers of the optimal annuli in 3–MWAP. The corresponding covered sets of vertexes are $V_1 = \{v_1, v_2\}$, $V_2 = \{v_3, v_4, v_7, v_8\}$, $V_3 = \{v_5, v_6\}$, respectively.

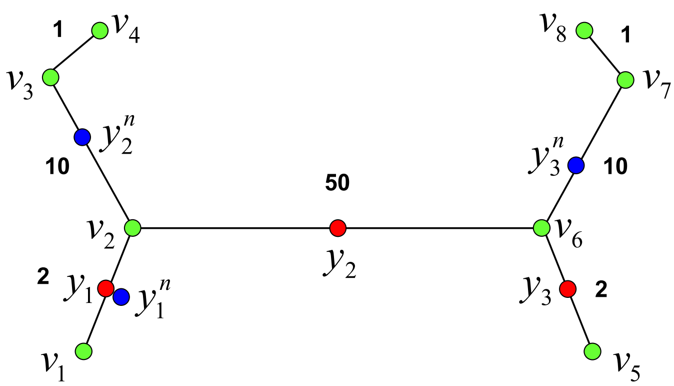


Figure 3.6: *Difference in solutions of* 3–*MWAP and* 3–*nearest*–*MWAP.*

Polynomial time solution of p–nearest–MWAP is possible because for the center point $x_s \in e = [i, j] \in E$ the nearest vertex is either v_i or v_j. Therefore, as it follows from Theorem 3.4, the middle point $(e, 1/2)$ is for any edge $e \in E$ an optimal placement of the center point for a local annulus which should cover at least N, $1 \leq N \leq n$ points of the set V.

Let us denote as

$$d(v_k, e) = \min_{v_k \in V}\{d(v_k, i), d(v_k, j)\} \tag{3.19}$$

the minimal distance between the point $v_k \in V$ and the edge $e = [i, j] \in E$. Accordingly to Section 3.1.2, the value $\max_{v_k \in V} d(v_k, e)$ is the width of a local MWA centered on the edge e. The values (3.19) are calculated for each vertex $v_k \in V$ and for each edge $e \in E$ and sorted in nondecreasing order with respect to edges $e \in E$:

$$\begin{aligned} e_1 \quad &: \quad d(v_{(1)_1}, e_1) \leq d(v_{(2)_1}, e_1) \leq ... \leq d(v_{(n)_1}, e_1), \\ &... \\ e_m \quad &: \quad d(v_{(1)_m}, e_m) \leq d(v_{(2)_m}, e_m) \leq ... \leq d(v_{(n)_m}, e_m). \end{aligned} \tag{3.20}$$

The entries $d(v_{(1)_q}, e_q)$ and $d(v_{(2)_q}, e_q)$, $1 \leq q \leq m$ are equal to zero. The value $\min_{1 \leq q \leq m} d(v_{(n)_q}, e_q)$ is the width of an minimum width annulus in the network $G(V, E)$. Clearly, the optimal value of the function (3.18) is equal to at least one of mn distance values in (3.20).

The main idea of the algorithm for p–nearest–MWAP is to fix the number of points N_1, $1 \leq N_1 \leq n - p + 1$ necessary covered by the first annulus $\mathcal{A}_1$ and to set the value $width_{\mathcal{A}_1}$ as local optimal value for $width_p$ (3.18). First, an upper bound UB for maximal width $width_p$ of p –nearest– MWA can be defined. We assume that each of $p - 1$ annuli covers one vertex point only and one annulus contains the rest $n - p + 1$ points of the set V. Therefore, the upper bound is equal to

$$UB = \min_{1 \leq q \leq m} d(v_{(n-p+1)_q}, e_q). \tag{3.21}$$

From here the expression an annulus $\mathcal{A}_s$ covers exactly N_s points of the set V means that the annulus covers at least this points and rest $n - N_s$ vertexes are covered by annuli $\mathcal{A}_1, \dots, \mathcal{A}_{s-1}, \mathcal{A}_{s+1}, \dots, \mathcal{A}_p$.

Next, we fix that the annulus $\mathcal{A}_1$ covers exactly $N_1 = n - p$ points of the set V and calculate the value

$$M = \min_{1 \leq q \leq m} d(v_{(n-p)_q}, e_q). \tag{3.22}$$

If $M > UB$ then there are no solutions of p–nearest–MWAP where $n - p$ points of the set V are covered by one annulus and other p points are in the remaining $p - 1$ annuli. Otherwise, one feasible solution with respect to the $width_p = M$ should be found or its nonexistence must be shown. For this purpose let us assume without loss of generality $M = d(v_{(n-p)_1}, e_1)$. Hence, the annulus $\mathcal{A}_1$ of the $width_{\mathcal{A}_1} = M$ covers the points $v_{(1)_1}, \dots, v_{(n-p)_1}$. Therefore, all entries for points $v_{(1)_1}, \dots, v_{(n-p)_1}$ in sequences (3.20) can be eliminated. After this all inequalities in (3.20) contain exactly p distances. The distances form $m \times p$ matrix where a minimal distance value in each column must be calculated. Since the less points of the set are covered by an minimum width annulus the smaller is its width, we should choose a column with minimal entry in it less or equal to M more to the right in the $m \times p$ matrix. Points which correspond to all distances from the left side of the minimal entry in the row including the minimal entry itself are covered by the second annulus $\mathcal{A}_2$. We eliminate all distances of covered points in the matrix and iterate until either all points of the set V are covered or the number of constructed annuli is equal to p. If the number of obtained annuli is less than or equal to p and all points of the set V are covered then the constructed solution is feasible. Otherwise, it is unfeasible. Therefore, the new minimal distance value without taking into account already considered must be explored. In our assumption it is $M = \min_{2 \leq q \leq m} d(v_{(n-p)_q}, e_q)$. We iterate until either a feasible solution is found or there are no more values of M in the column $n - p$ of the sequences (3.20) which are less than or equal to UB. Described steps are performed for each column in (3.20) from the column $(n - p)$ to the column (1).

Since the optimal value of $width_p$ is among the values (3.20), the constructed algorithm finds at least one optimal solution of p–nearest–MWAP.

So, the preprocessing step of building of the sequences (3.20) take $\mathcal{O}(mnlogn)$ time. There are $n - p + 1$ columns to explore in (3.20). In each of them we can choose at most m values for M. Deleting from the matrix of distance values for points covered by previous annulus is the most costly in time. It is of order $\mathcal{O}(mn^2)$. And finally, this deletions are performed at most p times. Hence, the worst case complexity of the algorithm for solving of p–nearest–MWAP is at most of order $\mathcal{O}(m^2n^3p)$.

The solution of p–nearest–MWAP give an upper bound for p–MWAP and can be used, for instance, in Branch and Bound procedure for solving of p–MWAP.

3.2 CLP in $G(V, E)$

This section contains an introduction and a study of the circle location problem in networks.

3.2.1 Problem definition and its nonequivalence with MWAP

Definition 3.12. CLP in $G(V, E)$

For a given network $G(V, E)$ with cardinality $|V| = n$ and $|E| = m$ find a circle $\mathcal{C}(x, \rho)$, $x \in \mathbb{P}(G)$ so that the maximal distance

$$\max_{v_k \in V} d(v_k, \mathcal{C}(x, \rho)) \tag{3.23}$$

is minimal.

Let us return to the distinction between planar and network circles mentioned before in Example 3.1.1 and recall the circle location problem on the plane. The former is the problem of finding a planar circle minimizing maximal distance from a finite set of giving points to the circle. The equivalence of CLP and MWAP on the plane can be established easily.

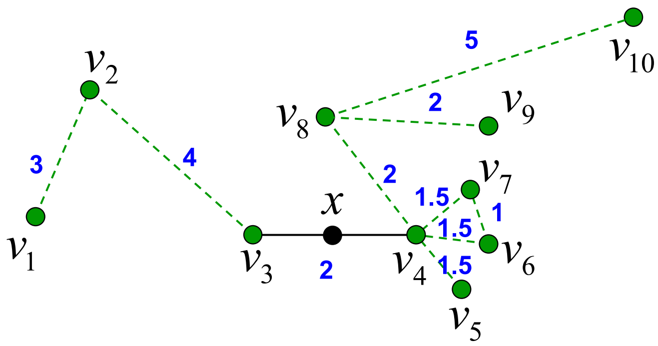


Figure 3.7: *MWA in the network G_1. The numbers denote the length of edges.*

In contrast to the planar case the circle $\mathcal{C}(x, \rho)$ in networks has some special features. It is finite and can be empty. The circle $\mathcal{C}(x, \rho)$ is not empty if and only if its radius satisfies the condition $\rho \leq max_{v \in \mathbb{P}(G)} d(x, v)$. It is known, that for any point v and for any circle $\mathcal{C}(x, \rho)$, $\rho > 0$, $x \neq v$ in the Euclidean plane there always is the unique closest point of the circle in the unique direction $x \to v$. This direction is the halfline going from x through the point v (see Figure 3.9). Moreover, in the Euclidean space for all points v with the distance $d(x, v) < \rho$ we can find a unique

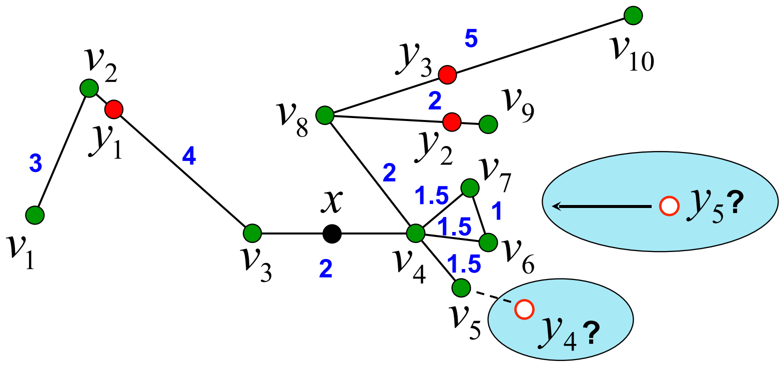


Figure 3.8: *Phenomena of „missing“ directions and points in the network G_1.*

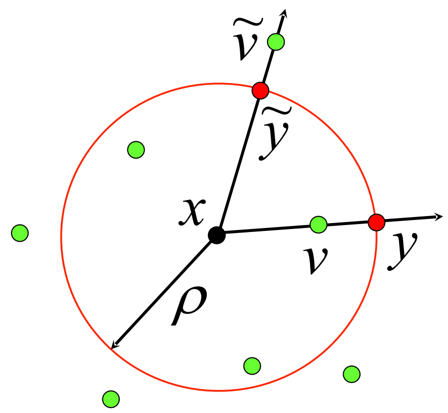


Figure 3.9: *Two directions in the Euclidean plane: $x \to v$ and $x \to \tilde{v}$.*

point y of the circle $\mathcal{C}(x, \rho)$ such that the shortest path $x \to y$ contains the shortest path $x \to v$. This is not longer true in network circles (see Figure 3.8). We call this phenomena the „missing“ directions and points properties. The last distinction of planar and network circles gives a basis of nonequivalence of MWAP and CLP in networks.

Considered differences are illustrated in Figures 3.7 and 3.8. For instance, in the network $G_1(V, E)$, $V = \{v_1, ..., v_{10}\}$ (see Figure 3.7) the minimum width annulus $\mathcal{A}(x, r, R)$, where $r = 1$ and $R = 8$ is the union of dashed edges and all vertexes. The circle $\mathcal{C}(x, (r + R)/2) = \mathcal{C}(x, 4.5)$ in the network G_1 (Figure 3.8) consists of 3 points y_1, y_2, y_3 only. Two points y_4, y_5 are „missing“. Hence, for example, $d(v_5, \mathcal{C}(x, \rho)) = d(v_5, y_2) = 5$ is greater than the maximal distance from the set of points V to the circle in planar case, which is equal to $(R - r)/2 = 3.5$. As a consequence, the circle $\mathcal{C}(x, 3.75)$ attains a better value of maximal distance $\max_{v_k \in V} d(v_k, \mathcal{C}(x, 3.75)) = 4.25$. Therefore, the circle $\mathcal{C}(x, 4.5)$ is not optimal for CLP and MWAP and CLP are in general not equivalent in networks. Despite of this fact we can effectively use

MWAP for solving CLP in networks. Let us note also, that we have not found any references on location of circles in networks.

3.2.2 Solution of CLP: Theory

CLP for a set $\mathcal{E}x$ of existing points on the plane can be directly solved using the solution (solutions) $\mathcal{A}^*(x, R, r)$ of MWAP for this set $\mathcal{E}x$. The center points of annulus $\mathcal{A}^*(x, R, r)$ and fitting circle $\mathcal{C}^*(x, \rho)$ coincide and value of radius ρ is equal to $(R + r)/2$. In networks our algorithm is based on using solutions of MWAP as good lower and upper bounds in CLP.

Let us assume that an annulus $\mathcal{A}^*(x^*, R^*, r^*)$ with fixed point $x^* \in e \in E$ solves MWAP in the network $G(V, E)$. Then, because of the minimality of the width $R^* - r^*$, the concentric with the annulus circle $\mathcal{C}(x^*, \rho^*)$, $\rho^* = (R^* + r^*)/2$ gives the minimal possible in the network $G(V, E)$ value

$$dist_{\mathcal{A}^*} = \frac{R^* - r^*}{2} \tag{3.24}$$

of the maximal distance from points of the set V to the circles in the network. The value (3.24) is a lower bound for maximal distances between the set V and circles $\mathcal{C}(x, \rho)$ with $x \in \mathbb{P}(G)$ and $\rho \geq 0$.

There are two cases. First, the lower bound $dist_{\mathcal{A}^*}$ is tight. More specially, for any point $v_k \in V$ the distance $d(v_k, \mathcal{C}(x^*, \rho^*)) \leq dist_{\mathcal{A}^*}$. Hence, the circle $\mathcal{C}(x^*, \rho^*)$ solves CLP. In the opposite case the lower bound $dist_{\mathcal{A}^*}$ is not tight. Therefore, there are „missing“ points in the circle $\mathcal{C}(x^*, \rho^*)$ and, as a consequence, a subset $\tilde{V} \subset V$, where for any vertex $\tilde{v_k} \in \tilde{V}$ the distance $d(\tilde{v_k}, \mathcal{C}(x^*, \rho^*))$ is strictly greater than $dist_{\mathcal{A}^*}$, but

$$d(\tilde{v_k}, x^*) < \rho^*. \tag{3.25}$$

We refer to the points of the set $\tilde{V}$ as violating vertexes.

Proposition 3.13. *Let us assume, that the distance from the violating vertex $\tilde{v_k} \in \tilde{V}$ to the circle $\mathcal{C}(x, \rho)$, concentric with an annulus $\mathcal{A}(x, R, r)$, $x \in e = [v_i, v_j]$, is equal to $d(\tilde{v_k}, y)$, where $y \in \mathcal{C}(x, \rho)$. Then the shortest paths $P_1 : x \mapsto y$ from the point x to the point y and $P_2 : \tilde{v_k} \mapsto y$ from the vertex $\tilde{v_k}$ to the point y begin to coincide from some vertex $v_k \in V$ to the point y.*

Proof. Since $\tilde{v_k} \in \tilde{V}$, then $d(\tilde{v_k}, x) < \rho = (R + r)/2$ and $d(\tilde{v_k}, \mathcal{C}(x, \rho)) = d(\tilde{v_k}, y) > (R - r)/2$.

Let us assume that the shortest path P_2 does not contain any vertex $v_k \in V$, belonging to the path P_1. Because $\tilde{v_k} \in \tilde{V}$, the length of the path P_2 is greater than $(R - r)/2$. Hence, the length of the path $P_3 : x \mapsto \tilde{v_k} \mapsto y$ is greater than

$\min\{d(x, v_i), d(x, v_j)\} + d(\tilde{v_k}, y) = r + (R - r)/2 = (R + r)/2 = \rho$. Therefore, there is an other point $y^{'}$ of the circle $\mathcal{C}(x, \rho)$ on the path P_3. The distance $d(x, y^{'}) = \rho$, $d(x, \tilde{v_k}) > r$ and, as follows, $d(\tilde{v_k}, y^{'}) < (R + r)/2 - r = (R - r)/2$. Hence, our assumption leads to a contradiction. □

Proposition 3.13 gives vise to decrease the radius ρ^* of the circle $\mathcal{C}(x^*, \rho^*)$ in order to achieve a better value of maximal distance from a circle with the center point x^* to the set V. Decreasing the radius ρ by $\delta_{\tilde{v_k}} > 0$ we want to equalize the maximal distance from the set $V - \tilde{V}$ to the circle $\mathcal{C}(x^*, \rho)$ and the distance from the violating vertex $\tilde{v_k} \in \tilde{V}$ to it

$$\frac{R^* - r^*}{2} + \delta_{\tilde{v_k}} = d(\tilde{v_k}, \mathcal{C}(x^*, \rho^*)) - \delta_{\tilde{v_k}}.$$

Hence, in order to balance two distances,

$$\delta_{\tilde{v_k}} = \frac{d(\tilde{v_k}, \mathcal{C}(x^*, \rho^*)) - (R^* - r^*)/2}{2} \tag{3.26}$$

and

$$\rho_{\tilde{v_k}} = \rho^* - \delta_{\tilde{v_k}} \tag{3.27}$$

is the radius of balance with respect to the point $\tilde{v_k} \in \tilde{V}$. Therefore, when

$$\delta = \frac{\max_{\tilde{v_k} \in \tilde{V}} d(\tilde{v_k}, \mathcal{C}(x^*, \rho^*)) - (R^* - r^*)/2}{2}$$

and $\rho_\delta = \rho^* - \delta$, all points $\tilde{v_k} \in \tilde{V}$ are balanced and

$$d(V, \mathcal{C}(x^*, \rho_\delta)) = (R^* - r^*)/2 + \delta. \tag{3.28}$$

It is best possible „balanced“ distance to the circle with central point at x^*.

On the other hand, decreasing the radius ρ of the circle $\mathcal{C}(x^*, \rho)$ from ρ^* to ρ_δ, points of $\mathcal{C}(x^*, \rho)$ can appear in directions, where existence of points of $\mathcal{C}(x^*, \rho^*)$ was impossible. As a consequence, some points from the set $\tilde{V}$ become nonviolating. Hence, we should find appearing points to define largest value of radius ρ at which the set $\tilde{V}$ is empty because the value of distance $d(V, \mathcal{C}(x^*, \rho))$ will be the smallest possible with respect to the point x^*. By doing this, for each edge $e = [v_{k_1}, v_{k_2}] \in E$ with at least one endpoint in the set $\tilde{V}$ the value $\rho_e = \frac{d(x^*, v_{k_1}) + d(x^*, v_{k_2}) + l_e}{2} = \max_{0 \leq t \leq 1} d(x^*, tl_e)$ must be calculated. If $\rho_\delta < \rho_e < \rho^*$, we save the received value ρ_e of radius and appearing by it new point of the circle $\mathcal{C}(x^*, \rho)$ for each endpoint of the edge e, which belongs to the set $\tilde{V}$. In case the value of radius for this endpoint has already been calculated, we take greatest of them with corresponding new point of the circle. At the end of the procedure no more than $|\tilde{V}|$ different new points $\mathcal{C}^{new}(x^*, \rho)$ of the circle $\mathcal{C}(x^*, \rho)$, $\rho_\delta < \rho < \rho^*$, can appear. Points $y \in \mathcal{C}^{new}(x^*, \rho)$

should be sorted on reduction of the radius value $\rho_e = d(x^*, y)$. If for each point $\tilde{v_k} \in \tilde{V}$ and for the sorted set $\mathcal{C}^{new}(x^*, \rho) = \{y_1, ..., y_{|\mathcal{C}^{new}(x^*,\rho)|}\}$ there is the smallest value of $p = p(\tilde{v_k})$, $1 \leq p \leq |\mathcal{C}^{new}(x^*, \rho)|$ so that

$$d(\tilde{v_k}, y_p) \leq \frac{R^* - r^*}{2} + (\rho^* - d(x^*, y_p)),$$

then the circle $\mathcal{C}(x^*, \rho_{p^{max}})$, where $p^{max} = max_{\tilde{v_k} \in \tilde{V}} p(\tilde{v_k})$ has the best possible value of the distance with respect to the point x^*

$$d(V, \mathcal{C}(x^*, \rho_{p^{max}})) = \frac{R^* - r^*}{2} + (\rho^* - d(x^*, y_{p^{max}})). \tag{3.29}$$

Otherwise, the distance of balance formulated in (3.28) remains the best and we set $d(V, \mathcal{C}(x^*, \rho_{p^{max}})) = \infty$.

The value

$$UB = \min_{x^* \ solves \ MWAP} \min\{d(V, \mathcal{C}(x^*, \rho_{p^{max}})), d(V, \mathcal{C}(x^*, \rho_\delta))\} \tag{3.30}$$

is an upper bound for the solution of CLP in the network $G(V,E)$.

The objective value of the local MWAP with respect to the edge e is a piecewise linear function, which is non-increasing from left of 1/2 of the edge and non-decreasing from right of 1/2 (Figure 3.3). Moreover, for each point $x \in \mathbb{P}(G)$ and centered at this point annulus $\mathcal{A}(x, R, r)$ the value $(R - r)/2$ is a lower bound for the minimal maximal distance between circles with the center point x and the set V. This implies the following very powerful elimination criterion:

Lemma 3.14. *If the width of local MWA $\mathcal{A}$ on the edge $e \in E$ is strictly greater than double value (3.30)*

$$width_{\mathcal{A}} > 2 * UB,$$

then there are no solutions of CLP on the edge e.

Furthermore, the edge $e \in E$ can be divided into at most three sets of intervals, where objective function of MWAP is constant, linear increasing, or linear decreasing.

Let $\mathcal{A}(x^c, R^c, r^c)$ be the minimum width annulus on an interval I_c of a constant value of the objective function on the edge e. The center point x^c of the annulus is either $\frac{1}{2}l_e$, if $\frac{1}{2}l_e \in I_c$, or the closest to $\frac{1}{2}l_e$ endpoint of I_c. The circle, giving us a lover bound for maximal distance from all circles centered at the point x^c to the set V is $\mathcal{C}(x^c, \rho^c)$, $\rho^c = (R^c + r^c)/2$. If we move the central point x of the circle $\mathcal{C}(x, \rho)$ within the interval I_c from the point x^c to its closest endpoint of interval e by $\Delta > 0$, then the radius of the circle having the smallest value of distance to the set V

$$\rho = \frac{(R^c - \Delta) + (r^c - \Delta)}{2} = \frac{R^c + r^c}{2} - \Delta \tag{3.31}$$

decreases by Δ compare to the radius ρ^c. Moreover, if $\tilde{V}$ is the set of violating vertexes for the circle $\mathcal{C}(x^c, \rho^c)$, then, by moving to the nearest endpoint of the edge e, some of points in the set $\tilde{V}$ can become non violating. The set $\tilde{V}$ reduces along the interval I_c with constant distance from non violating vertexes $V - \tilde{V}$ to the circle $\mathcal{C}(x, \rho)$, which is equal to $(R^c - r^c)/2$. Hence, the minimal possible value of the distance between the set V and circles with the center point on an interval I_c can be found, when the center point of a circle is farthest from $\frac{1}{2}l_e$ endpoint of I_c. For an interval I_c, where $\frac{1}{2}l_e$ lies in interior of it, both endpoints must be considered.

On an interval $I_i \subseteq e$ with increasing value of the objective function of MWAP $\mathcal{A}(x^i, R^i, r^i)$ and $\mathcal{C}(x^i, \rho^i)$, $\rho^i = (R^i + r^i)/2$ are the minimum width annulus and the corresponding circle with the center point at the left endpoint of the interval I_i. By moving the central point x of $\mathcal{A}(x, R, r)$ and $\mathcal{C}(x, \rho)$ within the interval I_i from left to right by $\Delta > 0$ the value of R increases by Δ, the radius r decreases by Δ, the distance from non violating vertexes to the circle increases by Δ, but

$$\rho = \frac{(R^i + \Delta) + (r^i - \Delta)}{2} = \frac{R^i + r^i}{2} \tag{3.32}$$

stays constant compare to corresponding values for $\mathcal{A}(x^i, R^i, r^i)$ and $\mathcal{C}(x^i, \rho^i)$. Hence, the more to the right on the interval I_i is the central point x, the more points become the set of violating vertexes $\tilde{V}$. Therefore, $max_{v_k \in V} d(v_k, C(x^i, \rho^i - \Delta)) \leq max_{v_k \in V} d(v_k, C(x^i + \Delta, \rho)) = max_{v_k \in V} d(v_k, C(x^i + \Delta, \rho^i))$, where $x^i + \Delta \in I_i$. It follows that the smallest possible value of the distance from the set V to a circle with center point on an interval I_i can be achieved at the left endpoint of the interval.

Similarly, on an interval $I_d \subseteq e$ with decreasing value of the objective function of MWAP the right endpoint of the interval is the center point of the circle $\mathcal{C}(x, \rho)$ giving local optimum for CLP on I_i. We note (see Figures 3.3 and 3.10), that these points x^i and x^d are intersection points of distance functions forming the shape of the max-function $max_{v_k \in V, x \in e} d(v_k, x)$. The width of the minimal annulus at each such point is equal to the width of the minimal annulus at one of the closest to it breakpoint of the max-function.

3.2.3 Solution of CLP: Algorithm

Section 3.2.2 proves Algorithm 3.4 described below.

The circle location problem is more complicated on $G(V, E)$ than finding of an annulus of minimal width on it. Because of nonequivalence of CLP and MWAP in networks the complexity of Algorithm 3.4 is of order $\mathcal{O}(n^2 m^2)$. Most heavy in the sense of computational time is the last step of the algorithm.

Algorithm 3.4 *Optimal solution of CLP in $G(V, E)$*

Input: $G(V, E)$, $|E| = m$, $|V| = n$, $E^{cand} = E$, distance matrix $D = (d_{ij})$;

Step 1:

- use Algorithm 3.1 to find $width_{\mathcal{A}}$ of local MWA $\mathcal{A}(x, R, r)$ and intervals $[t_e^1, t_e^2]$, $e \in E^{cand}$, involving centers $x^* = tl_e$, $t \in [t_e^1, t_e^2]$ of MWA $\mathcal{A}^*(x^*, R^*, r^*)$, where $r^* = min\{tl_e, (1-t)l_e\}$ and $R^* = width_{\mathcal{A}^*} + r^*$;
- set $LB = width_{\mathcal{A}^*}/2$;

– complexity $\mathcal{O}(nm)$;

Step 2: on each such interval $[t_e^1 l_e, t_e^2 l_e] \subset e \in E^{cand}$

if $t_e^1 \neq t_e^2$ **then**

take each endpoint x^* which is not equal to $1/2l_e$ as center point of circle $\mathcal{C}(x^*, \rho^*)$, $\rho^* = (R^* + r^*)/2$;

else

take the point $1/2l_e$ as center point of circle $\mathcal{C}(x^*, \rho^*)$, $\rho^* = (R^* + r^*)/2$;

end if

- for the circle $\mathcal{C}(x^*, \rho^*)$ choose a subset $\widetilde{V} \subset V$ of violating vertexes ($v_k \in \widetilde{V}$ iff $d(x^*, v_k) < \rho^*$ and $d(v_k, \mathcal{C}(x^*, \rho^*)) > LB$);

if $\widetilde{V} = \emptyset$ for at least one circle $\mathcal{C}(x^*, \rho^*)$ **then**

set $UB = LB$ – **STOP**: the solution point x^* of MWAP solves CLP and there are no solutions of CLP in $G(V, E)$, which does not solve MWAP;

else

GO TO Step 3;

end if

– complexity $\mathcal{O}(nm^2)$ – number of intervals $[t_e^1 l_e, t_e^2 l_e]$ can be equal to m, number of points in $\mathcal{C}(x^*, \rho^*)$ is of order $\mathcal{O}(m)$ and calculation of $d(v_k, \mathcal{C}(x^*, \rho^*))$ takes $\mathcal{O}(nm)$ time;

Step 3:

- for each circle $\mathcal{C}(x^*, \rho^*)$ find $d(V, \mathcal{C}(x^*, \rho_\delta))$ in $\mathcal{O}(n)$ time and $d(V, \mathcal{C}(x^*, \rho_{p^{max}}))$ in $\mathcal{O}(n \log n + n^2) = \mathcal{O}(n^2)$ time (see (3.28) and (3.29));
- set $UB = \min_{x^*} \min\{d(V, \mathcal{C}(x^*, \rho_\delta)), d(V, \mathcal{C}(x^*, \rho_{p^{max}}))\}$ – $\mathcal{O}(m)$;

– complexity $\mathcal{O}(n^2 m + m) = \mathcal{O}(n^2 m)$;

Algorithm 3.4 *Optimal solution of CLP in* $G(V,E)$ *(continue)*

Step 4:

- eliminate from set E^{cand} all edges e on which $width_{\mathcal{A}} > UB$;
- **if** $E^{cand} = \emptyset$ **then STOP**: CLP is solved by MWAP by decreasing of ρ^*;

– complexity $\mathcal{O}(m)$;

Step 5:
repeat

- for each $e \in E^{cand}$ find values $\{t_1^{x^d}, t_2^{x^d}, ...\} = T^d$ and $\{t_1^{x^i}, t_2^{x^i}, ...\} = T^i$ on e (see Figure 3.10, procedure is similar to finding endpoints of optimal interval for MWAP in Algorithm 3.1) – $\mathcal{O}(n^2 m)$;
- take $x = tl_e$, where the value t from sets T^d and T^i by turns, as center point of circle $\mathcal{C}(x,\rho)$ with corresponding to annulus $\mathcal{A}(x,R,r)$ value of $\rho = (R+r)/2$;
- for the circle $\mathcal{C}(x,\rho)$ choose a subset $\widetilde{V} \subset V$ of violating vertexes;
- **if** $\widetilde{V} = \emptyset$ and $UB > (R-r)/2$ **then**
  update UB, T^d, T^i and E^{cand};
  else find ρ_δ, $\rho_{p^{max}}$ and if $UB > \min_x \min\{d(V,\mathcal{C}(x,\rho_\delta)), d(V,\mathcal{C}(x,\rho_{p^{max}}))\}$
  update UB, T^d, T^i and E^{cand};
  end if

until $E^{cand} = \emptyset$
– complexity $\mathcal{O}(n^2 m^2)$;

Output: circles $\mathcal{C}(x,\rho)$ with minimal distance $d(V,\mathcal{C}(x,\rho)) = UB$.

3.2.4 Computational results

Algorithm 3.4 was implemented in C++ and compiled with g++ v.3.3.3. All computations were done at the University of Kaiserslautern on a server equipped with Dual Intel Xeon 3.2 GHz CPUs, 4 GB RAM running on Linux Kernel 2.6.5. Computational results for 23 problems are summarized in Table 3.2. The design of the testing networks for Algorithm 3.1 and Algorithm 3.4 coincides. Each problem was solved 16 times – 1 time for constructed network, which is connected and has the fixed lengths of edges, and 15 times for the network with the same edges connectivity but with random lengths of edges. Therefore, the testing problems for both algorithms differ in the lengths of edges only (except basis networks, which are identical). The problems have 100 nodes and number of edges that corresponds

Table 3.2: Summary of computational results for CLP.

n= 100, m= (1)	aver node degr (2)	No. nets (3)	No. sol. edg. range (4)	No. sol. edg. average (5)	No. viol. vtxs min (6)	No. viol. vtxs max (7)	No. solved by MWAP (8)	No. solved by MWAP decr. (9)	No. solved by other edge (10)
4950	99	16	1 – 10	3	0	0	16	0	0
4783	95	16	1 – 35	6.4	0	0	16	0	0
4549	90	16	1 – 90	9.6	0	0	16	0	0
4224	84	16	1 – 16	4.2	0	0	16	0	0
4018	80	16	1 – 6	1.8	0	0	16	0	0
3730	74	16	1 – 18	4.8	0	0	16	0	0
3468	69	16	1 – 28	5.6	0	0	16	0	0
3218	64	16	1 – 90	8.2	0	0	16	0	0
2991	59	16	1 – 13	3.3	0	0	16	0	0
2784	55	16	1 – 18	3.4	0	0	16	0	0
2539	50	16	1 – 13	2.9	0	0	16	0	0
2279	45	16	1 – 5	2.3	0	0	16	0	0
2033	40	16	1 – 35	4.6	0	0	16	0	0
1757	35	16	1 – 8	2.2	0	0	16	0	0
1548	30	16	1 – 10	2.5	0	0	16	0	0
1282	25	16	1 – 5	1.8	0	0	16	0	0
1008	20	16	1 – 10	2.7	0	1	15	1	0
772	15	16	1 – 11	1.9	0	0	16	0	0
509	10	16	1 – 7	2.1	0	4	14	1	1
275	5	16	1 – 6	1.5	1	25	0	4	12
117	2	16	1 – 2	1.1	0	4	13	0	3
100	2	16	2	2	0	0	16	0	0
99	1	16	1	1	0	0	16	0	0
all		368	1 – 90	3.4	0	25	346	6	16

Table 3.3: Percentage of CLP solved by MWAP.

	No. nets	%
Solved by MWAP	346	94.02
Solved by MWAP with decreasing ρ	6	1.63
Solved by point on other edge	16	4.35
All solved problems	368	100

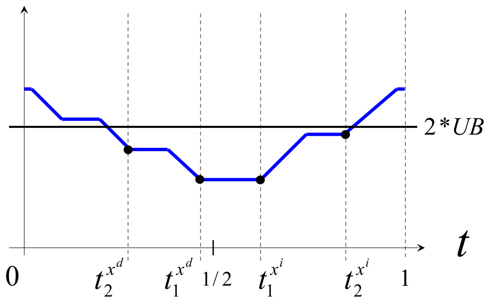


Figure 3.10: *Breakpoints of MWA-objective function, which must be explored for finding of CLP-solution on the edge e with* $width_{\mathcal{A}_e} < 2UB$.

to decreasing average node degree from 99 to 1 (columns (1) – (2)). In columns (4) – (5) is shown that density of a network has no influence on quantity of solutions of CLP. The greatest part of problems – 346 or about 94 % of all networks – has been solved by means of MWAP (Table 3.2 column (8) and Table 3.3). In other words, Algorithm 3.4 stops on Step 2 at their solution. In the remained networks Algorithm 3.4 has stopped on Step 4 and Step 5 in 1.63 % and 4.35 % of cases, respectively (Table 3.2 columns (9) – (10) and Table 3.3).

3.3 Relation between MWAP and PLP in $G(V, E)$

3.3.1 Review of solution methods for PLP

The point location problem on a network is a problem of finding the absolute center of the network:

Definition 3.15. PLP in $G(V, E)$

For a given network $G(V, E)$, $|V| = n$ *and* $|E| = m$ *find a point* $x \in \mathbb{P}(G)$ *so that the maximal distance*

$$\max_{v_k \in V} d(x, v_k) \tag{3.33}$$

is minimal.

An approach to finding the absolute center of a network was first presented in the paper of Hakimi [30] in 1964. Kariv and Hakimi [35] in 1979 constructed an $\mathcal{O}(mnlogn)$ algorithm for finding an absolute 1–center of a vertex–weighted network and an $\mathcal{O}(mn + n^2 logn)$ algorithm for finding an absolute 1–center of a vertex–unweighted network assuming that the distance matrix is given. In 1981 Minieka [41] demonstrated that determining of local center on an edge e does not require explicit knowledge of the distance function.

Proposed by Sforza [45] in 1990 solution procedure requires only the knowledge of the shortest distance matrix between all pairs of vertexes. An initial upper bound of the absolute radius in the algorithm is the value of the vertex radius. Each edge is examined to find a local absolute center smaller than or equal to the current one. Therefore, the algorithm finds the absolute center or all the equivalent absolute centers of a network. It requires in unweighted networks $\mathcal{O}(mnlogn)$ time. Sforza [45] reported that the algorithm was applied to a sample of small and medium size randomly generated networks and compared with the algorithm of Minieka [41] for unweighted networks and gave good experimental results in terms of computer time.

The most recent and currently best center point location algorithm for unweighted case was constructed by Dvir and Handler [20] in 2004. It uses the concept of minimum–diameter trees. The algorithm finds local centers and their corresponding radii by a monotonically increasing sequence of lower bounds on the radii. The computational time of the algorithm is of order $\mathcal{O}(mn + n^2logn)$.

3.3.2 MWAP solves PLP: Sufficient conditions

It can be easily shown that MWAP and PLP are nonequivalent on general networks. For example, on the network in Figure 3.11 all intervals indicated by thick lines solve MWAP and width of optimal annuli on them is equal to 8. The minimal value of the maximal distance from points of the intervals to the vertexes $v_1, ..., v_5$ of the network is achieved at the point x_1 and equal to 10. However, $\max_{k=1,...,5} d(x_2, v_k) = 9.5$ and width of the optimal annulus centered at the point x_2 is equal to 9. The example proves nonequivalence of the problems. Nevertheless, in some cases MWAP can solve PLP, that reduces complexity of finding of an absolute center on this networks to $\mathcal{O}(mn)$ assuming the shortest path matrix is given.

Let $x = tl_e$, $t \in [t_e^1, t_e^2]$ be the set of optimisers for local minimum width annulus $\mathcal{A}_e(x, R, r)$ on the edge $e \in E$. When t goes from $\frac{1}{2}$ to t_e^1 or to t_e^2 the radii of outer circle $\mathcal{C}_e(x, R)$ and inner circle $\mathcal{C}_e(x, r)$ decrease. Hence, they have the minimal local value either at point $x = t_e^1 l_e$ or at point $x = t_e^2 l_e$. It is necessary to point out that minimum width annulus $\mathcal{A}^*(x, R, r)$ and local minimum width annulus $\mathcal{A}_e(x, R, r)$ with minimal radius R of outer circle $\mathcal{C}_e(x, R)$ can be different annuli.

Let us denote with $R^*_{\mathcal{A}_e} = argmin_{\{e \in E,\, R\}} \mathcal{A}_e(x, R, r)$ the minimal value of the outer radius of local minimum width annuli $\mathcal{A}_e(x, R, r)$, $e \in E$. In order to find on the edge $e \in E$ the interval $x = tl_e$, $t \in [t_e^1, t_e^2]$ of centers for local minimum width annulus we search for one or two second farthest points $v^{sfi}_{(e,\frac{1}{2})}$ and $v^{sfj}_{(e,\frac{1}{2})}$ (see Step 3 in Algorithm 3.1). Then points tl_e on the edge e, where value of local radius R_e of the network can be less or equal to $R^*_{\mathcal{A}_e}$, should be searched on intervals $t \in [0, \frac{b_i - R^*_{\mathcal{A}_e}}{l_e}]$

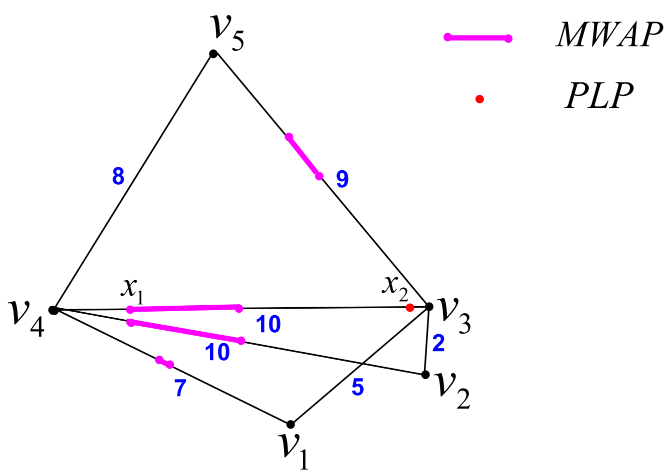


Figure 3.11: *Nonequivalence of MWAP and PLP in* $G(V, E)$

and $t \in [1 - \frac{b_j - R^*_{\mathcal{A}_e}}{l_e}, 1]$ only (see Figure 3.12), where

$$b_i = \begin{cases} d(v^{sfi}_{(e,\frac{1}{2})}, i), & if\ v^{sfi}_{(e,\frac{1}{2})}\ exists, \\ d(v^{f}_{(e,\frac{1}{2})}, i), & if\ v^{sfi}_{(e,\frac{1}{2})}\ does\ not\ exist \end{cases} \tag{3.34}$$

and

$$b_j = \begin{cases} d(v^{sfj}_{(e,\frac{1}{2})}, j), & if\ v^{sfj}_{(e,\frac{1}{2})}\ exists, \\ d(v^{f}_{(e,\frac{1}{2})}, j), & if\ v^{sfj}_{(e,\frac{1}{2})}\ does\ not\ exist. \end{cases} \tag{3.35}$$

Therefore, we can use values of b_i and b_j as lower bounds for the local radius R_e in PLP on half edges $[i, (e, \frac{1}{2})]$ and $[(e, \frac{1}{2}), j]$, respectively. In the case, when the values of lower bounds b_i and b_j on $e \in E$ are greater or equal than $R^*_{\mathcal{A}_e}$, the interior of the interval set $e - [t^1_e l_e, t^2_e l_e]$ cannot contain solutions of PLP. Therefore, we have proved the sufficient condition at which MWA $\mathcal{A}^*(x, R, r)$ solves PLP to optimality:

Theorem 3.16. *(sufficient condition: MWAP solves PLP)*
Let x^* be the central point of minimum width annulus $\mathcal{A}^*(x, R, r)$ with minimal radius

$$R_{x^*} = argmin_{\{e \in E,\ R\}} \mathcal{A}_e(x, R, r) = argmin_{\{R\}} \mathcal{A}^*(x, R, r)$$

of the outer circle among all local minimum width annuli in the network $G(V, E)$. If for any edge $e = [i, j] \in E$ lower bounds b_i and b_j are greater or equal to R_{x^*}, then the point x^* solves PLP in $G(V, E)$.

Since the minimal radius of the outer circle for solutions of MWAP is greater or equal to the minimal radius of the outer circle for all local MWA in the network we state

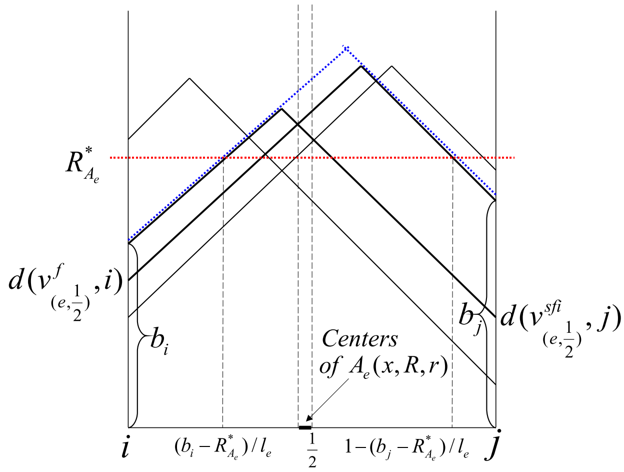


Figure 3.12: *Candidate intervals for local center on $e = [i, j]$.*

Theorem 3.17. *(sufficient condition 1: local MWAP solves PLP)*
Let x^* be the central point of local minimum width annulus $\mathcal{A}_e(x, R, r)$ with minimal radius

$$R_{x^*} = argmin_{\{e \in E,\ R\}} \mathcal{A}_e(x, R, r)$$

of the outer circle in the network $G(V, E)$. If for any edge $e = [i, j] \in E$ lower bounds b_i and b_j are greater or equal to R_{x^*}, then the point x^* solves PLP in $G(V, E)$.

Furthermore, as an upper bound for the absolute radius R^* in the network can be taken

$$R^{ub} = \min\{R^*_{\mathcal{A}_e}, R_{VC}\}, \tag{3.36}$$

where $R_{VC} = \min_{v_k \in V} \max_{v_p \in V} d(v_k, v_p)$ is the radius of vertex center of the network. Let us denote as v^f_i and v^f_j farthest vertexes from the points v_i and v_j, respectively. To obtain a better than (3.34) and (3.35) lower bounds for the radius of PLP on the edge $e \in E$ two values $B_i = d(v^f_i, j) + l_e$ and $B_j = d(v^f_j, i) + l_e$ are calculated (see Figure 3.13). Then lower bounds for the local radius R_e on half edges $[i, (e, \frac{1}{2})]$ and $[(e, \frac{1}{2}), j]$ are equal to

$$LB^i_{ij} = \frac{B_i + b_i}{2} \tag{3.37}$$

at the point $(e, (B_i - b_i)/2l_e)$ and to

$$LB^j_{ij} = \frac{B_j + b_j}{2} \tag{3.38}$$

at the point $(e, (l_e - (B_j - b_j))/2l_e)$. Hence, a sufficient condition which is stronger than one in Theorem 3.17 can be formulated:

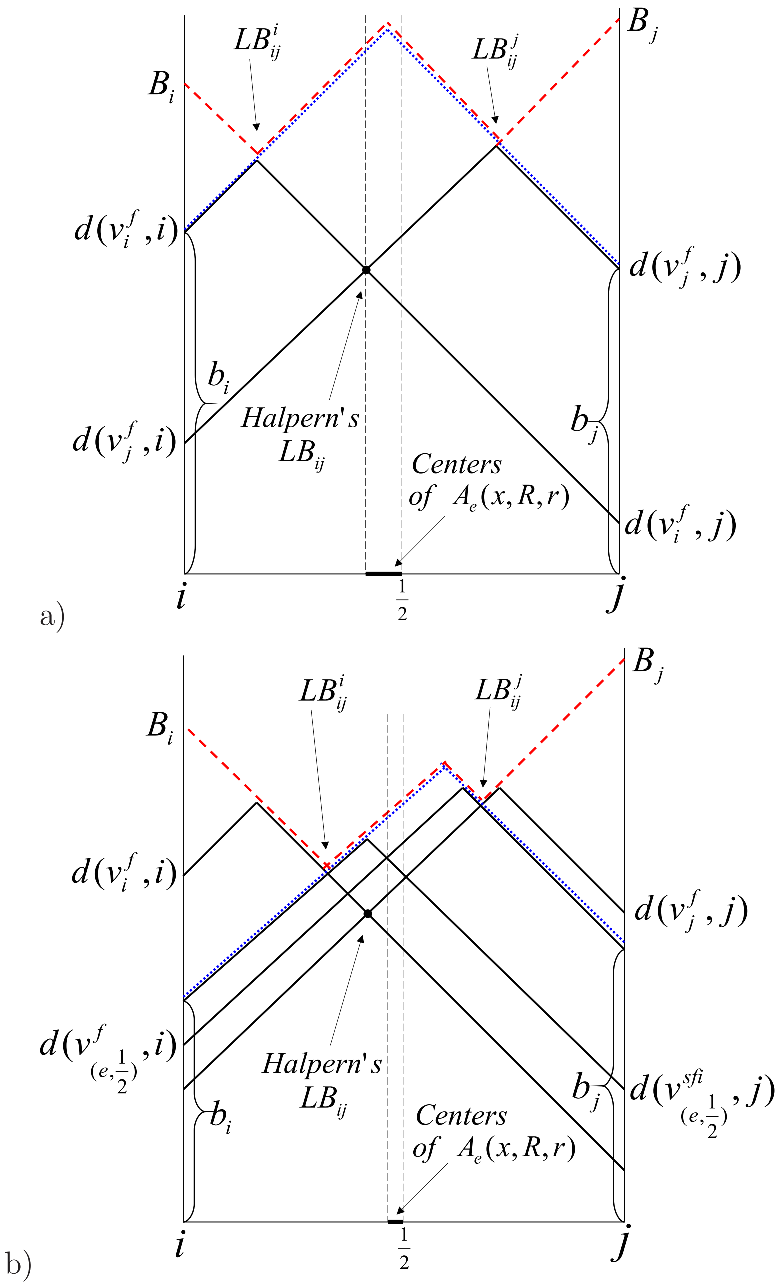


Figure 3.13: *Lower bounds on edge* $e = [i, j]$.

Theorem 3.18. *(sufficient condition 2: local MWAP solves PLP)*
Let x^* be the central point of local minimum width annulus $\mathcal{A}_e(x, R, r)$ with minimal radius

$$R_{x^*} = argmin_{\{e \in E,\, R\}} \mathcal{A}_e(x, R, r) \leq R_{VC}$$

of the outer circle in the network $G(V, E)$. If for any edge $e = [i, j] \in E$ lower bounds LB^i_{ij} and LB^j_{ij} are greater or equal to R_{x^*}, then the point x^* solves PLP in $G(V, E)$.

3.3.3 Comparison with Halpern's lower bound

So far, the strongest bound, which is an edge elimination criterion in algorithms locating the center of a graph, was devised by Halpern [32]. It states that the radius of the network at a local center on the edge $e = [i, j] \in E$ is greater or equal to

$$LB_{ij} = \frac{d(v^f_i, j) + d(v^f_j, i) + l_e}{2}, \tag{3.39}$$

where v^f_i, v^f_j are farthest nodes from i, j, respectively. Halpern [32] presented empirical evidence that this criterion reduces about 95% of edges, which have to be checked thoroughly, by comparing LB_{ij} to the vertex radius R_{VC} of the network. It should be mentioned, that calculations of both bounds LB_{ij} and LB^i_{ij}, LB^j_{ij} require the information easily received from the distance matrix.

Theorem 3.19. *The lower bounds LB^i_{ij} and LB^j_{ij} are stronger than Halpern's lower bound LB_{ij} for any edge $e = [i, j] \in E$ in the network $G(V, E)$, that is $LB^i_{ij} > LB_{ij}$ and $LB^j_{ij} > LB_{ij}$.*

Proof. There are two possible cases with respect to each edge $e = [i, j] \in E$: either Halpern's point $(e, t_{LB_{ij}})$, which is intersection point of lines $d(v^f_i, j) + (1 - t)l_e$ and $d(v^f_j, i) + tl_e$, solves local MWAP or not.

Let us assume that the point $(e, t_{LB_{ij}})$ is optimal for local MWAP. Consequently, without loss of generality $v^f_{(e,\frac{1}{2})} = v^f_j$ and $v^{sfi}_{(e,\frac{1}{2})} = v^f_i$ (see Figure 3.13 a)). Then

$$\begin{aligned} b_i &= d(v^f_i, i) > d(v^f_j, i), & b_j &= d(v^f_j, j) > d(v^f_i, j), \\ B_i &= d(v^f_i, j) + l_e, & B_j &= d(v^f_j, i) + l_e. \end{aligned}$$

Therefore

$$LB^i_{ij} = \frac{B_i + b_i}{2} = \frac{d(v^f_i, j) + l_e + d(v^f_i, i)}{2} > \frac{d(v^f_i, j) + l_e + d(v^f_j, i)}{2} = LB_{ij}$$

and

$$LB_{ij}^j = \frac{B_j + b_j}{2} = \frac{d(v_j^f, i) + l_e + d(v_j^f, j)}{2} > \frac{d(v_j^f, i) + l_e + d(v_i^f, j)}{2} = LB_{ij}.$$

In the opposite case, at the point $(e, t_{LB_{ij}})$ the radius of outer circle for local MWAP is greater than LB_{ij} (see Figure 3.13 b)). Therefore, without loss of generality, $b_i = d(v^{sfi}_{(e,\frac{1}{2})}, i) > d(v_j^f, i)$, $b_j = d(v^f_{(e,\frac{1}{2})}, j) > d(v_i^f, j)$ and $B_i = d(v_i^f, j) + l_e$, $B_j = d(v_j^f, i) + l_e$. Hence,

$$LB_{ij}^i = \frac{B_i + b_i}{2} = \frac{d(v_i^f, j) + l_e + d(v^{sfi}_{(e,\frac{1}{2})}, i)}{2} > \frac{d(v_i^f, j) + l_e + d(v_j^f, i)}{2} = LB_{ij}$$

and

$$LB_{ij}^j = \frac{B_j + b_j}{2} = \frac{d(v_j^f, i) + l_e + d(v^f_{(e,\frac{1}{2})}, j)}{2} > \frac{d(v_j^f, i) + l_e + d(v_i^f, j)}{2} = LB_{ij}.$$

So, at any case the lower bounds LB_{ij}^i and LB_{ij}^j are better than Halpern's lower bound LB_{ij} for the local radius of the network with respect to the edge $e = [i, j] \in E$. □

We have obtained lower bounds which are stronger than Halpern's . Moreover, the upper bound (3.36) is less or equal to R_{VC}. Therefore, applying LB_{ij}^i and LB_{ij}^j in solution of PLP we expect a much larger percentage of eliminated arcs than from Halpern's bound.

3.3.4 Solution of PLP: Theory

The main idea of the solution procedure is to begin with solving local MWAP on each edge of the network. In this case two lower bounds LB_{ij}^i and LB_{ij}^i for the local radius of the network on each halfedge $[i, (e, \frac{1}{2})]$ and $[(e, \frac{1}{2}), j]$ of the edge $e = [i, j] \in E$ and an upper bound UB for the value of absolute radius can be calculated. The candidate solution set for PLP consists of halfedges with lower bound less than or equal to UB. A central point and an absolute radius of the network will be identified by a monotonically increasing sequence of lower bounds and a monotonically decreasing sequence of upper bounds. These sequences are obtained by consecutive by exploring a part of halfedges from the candidate set having minimal value of the lower bound and updating the candidate set until this set is empty.

Let us first show that some, sometimes very big, parts of edges can be eliminated from consideration even before calculating of lower bounds. There are the parts on which the objective function coincides with the distance function of one vertex of the network. The elimination criterion formulated in the following lemma is very powerful in some types of networks.

Lemma 3.20. (*Interior Elimination Criterion*)
Let v_i^f and v_j^f be farthest points from the endpoints i and j of the edge $e = [i, j] \in E$, respectively. If equality $d(v_i^f, i) = d(v_j^f, i)$ (or $d(v_i^f, j) = d(v_j^f, j)$) is satisfied then for all points tl_e, $0 < t < 1$ in interior of the edge e

$$\max_{v_k \in V} d(v_k, tl_e) > \min\{d(v_i^f, i), d(v_j^f, j)\}.$$

Proof. By condition $d(v_i^f, i) = d(v_j^f, i)$ (or $d(v_i^f, j) = d(v_j^f, j)$) the objective function value $\max_{v_k \in V} d(v_k, tl_e)$ is equal to $d(v_j^f, tl_e)$ (or $d(v_i^f, tl_e)$) for $0 \le t \le 1$. According to (3.1)

$$d(v_j^f, tl_e) = \min\{d(v_j^f, i) + tl_e, d(v_j^f, j) + (1 - t)l_e\}$$

for $0 \le t \le 1$. The function has its minimal value at $t = 0$ or (and) at $t = 1$ only. Therefore, there are no points in the interior of the edge e with value of $\max_{v_k \in V} d(v_k, tl_e)$ less or equal to $\min\{d(v_i^f, i), d(v_j^f, j)\}$. $\square$

The Interior Elimination Criterion is applied to the set $E^{cand} = E$. Then bounds (3.36) – (3.38) are calculated for the remaining edges. After that, the arcs in E^{cand} are checked with the help of an elimination criterion described next.

Lemma 3.21. (*LB Elimination Criterion*)
Halfedges $[i, (e, \frac{1}{2})]$ with $LB_{ij}^i > R^{ub}$ and $[(e, \frac{1}{2}), j]$ with $LB_{ij}^j > R^{ub}$ do not contain points with value of radius less or equal to R^{ub}, except those which we already considered while finding R^{ub}.

Therefore, halfedges satisfying LB Elimination Criterion can be deleted from the candidate set E^{cand}. If after this step the set $E^{cand} = \emptyset$ then PLP is solved by local MWAP.

Otherwise, among all halfedges $[i, (e, \frac{1}{2})]$ and $[(e, \frac{1}{2}), j]$ of the set E^{cand} the minimal value $\min\{LB_{ij}^i, LB_{ij}^j\} = LB^{min}$ is found. Let us assume without loss of generality that $LB^{min} = LB_{ij}^i$ for the halfedge $[i, (e, \frac{1}{2})]$. Then an objective function value $R^{curr} = \max_{v_k \in V} d(v_k, tl_e)$ at the point $t = (B_i - b_i)/2l_e \equiv t_{LB_{ij}^i}$ of LB^{min} should be calculated. If $R^{curr} = LB^{min}$, i.e. LB is achieved, then PLP is solved by lower bound. In the case $R^{curr} > LB^{min}$ this lower bound is tested for "effectiveness" from geometrical point of view:

Lemma 3.22. (*Ineffective LB*)
Let LB_{ij}^i be the lower bound on the halfedge $[i, (e, \frac{1}{2})]$ and R^{ub} be a current upper bound. If functional value $R^{curr} = \max_{v_k \in V} d(v_k, t_{LB_{ij}^i} l_e)$ is strictly greater than LB_{ij}^i

and $R^{curr} - R^{ub} > R^{ub} - LB^i_{ij}$ then halfedge $[i,(e,\frac{1}{2})]$ does not contain new points with value of radius less or equal to R^{ub}. The same statement is true for a halfedge $[(e,\frac{1}{2}),j]$.

If LB^i_{ij} of the halfedge $[i,(e,\frac{1}{2})]$ is effective, i.e. $R^{curr} - R^{ub} \leq R^{ub} - LB^i_{ij}$, then points at which radius of the network may be less or equal to R^{ub} can be on the halfedge in interval tl_e, $t_{LB^i_{ij}} - \frac{R^{ub}-LB^i_{ij}}{l_e} \leq t \leq t_{LB^i_{ij}} + \frac{R^{ub}-LB^i_{ij}}{l_e}$ only. To find these points we introduce in the network $G(V,E)$ an artificial edge $e_{atf} \equiv tl_e$, $[t_{LB^i_{ij}} - \frac{R^{ub}-LB^i_{ij}}{l_e}, t_{LB^i_{ij}} + \frac{R^{ub}-LB^i_{ij}}{l_e}]$ and explore it. If the examination is stopped by Interior Elimination Criterion then in at least one endpoint of artificial edge the network radius is equal to current value of R^{ub}. Therefore, the interval $[i,(e,\frac{1}{2})] \in E^{cand}$ is out of consideration. Otherwise, the local MWAP should be solved on the artificial interval e_{atf}. Then R^{ub} is updated and two lower bounds $LB^1_{e_{atf}}$ and $LB^2_{e_{atf}}$ are calculated. The interval $[i,(e,\frac{1}{2})]$ must be eliminated from E^{cand} and, if a new lower bound is less than the current R^{ub}, the corresponding halfedge of the artificial interval e_{atf} is added to the set E^{cand}. After that we update the set E^{cand} eliminating halfedges with lower bound which is greater than R^{ub}. We iterate until the set $E^{cand} = \emptyset$. Then solution of PLP is found.

3.3.5 Solution of PLP: Algorithm and its complexity

In Section 3.3.4 we have established the ideas for an algorithm to solve PLP. The steps of the solution procedure are summarized in Algorithm 3.5. An worked example can be found in Section 3.3.6.

Next, we calculate the worst–case performance of the algorithm assuming that the shortest path matrix is given.

- **Step 1** : Finding the farthest points v_i^f and v_j^f requires $2*n$ operations with respect to each edge $e=[i,j]\in E$. Hence, the step has complexity $\mathcal{O}(nm)$.
- **Step 2** : The check of equality $d(v_i^f,i) = d(v_j^f,i)$ $\forall e \in E^{cand}$ takes $\mathcal{O}(m)$ time.
- **Step 3** : Solution of local MWAP and calculation of points, radii and bounds on the edge $e \in E^{cand}$ requires maximal $3*n+13$ operations. Therefore, complexity of the step is $\mathcal{O}(n|E^{cand}|) \leq \mathcal{O}(nm)$.
- **Step 4** : Calculation of the current upper bound takes $\mathcal{O}(|E^{cand}|) \leq \mathcal{O}(m)$ time.
- **Step 5** : Elimination of halfedges with the lower bound greater than the current upper bound has complexity $\mathcal{O}(2*|E^{cand}|) \leq \mathcal{O}(m)$.

Algorithm 3.5 *Optimal solution of PLP in $G(V, E)$*

Input: $G(V, E)$, $|E| = m$, $|V| = n$, $E^{cand} = E$, distance matrix $D = (d_{ij})$;

Step 1: $\forall e = [i, j] \in E^{cand}$ find farthest points v_i^f and v_j^f from the points i and j, respectively, such that distances $d(v_i^f, j)$ and $d(v_j^f, i)$ are maximal, and R^{VC};

Step 2: *Interior Elimination Criterion*
if $d(v_i^f, i) = d(v_j^f, i)$ **then**
eliminate edge e from E^{cand};
end if

Step 3: *Solution of local MWAP and calculation LBs*

- $\forall e \in E^{cand}$ find $v^f_{(e,\frac{1}{2})}$, $v^{sfi}_{(e,\frac{1}{2})}$, $v^{sfj}_{(e,\frac{1}{2})}$ and optimal interval $[t_e^1, t_e^2]$ for local MWAP (see Algorithm 3.1) and values R_e^1 and R_e^2 of network radius at the endpoints of the interval;
- calculate $R_e = \min\{R_e^1, R_e^2\}$, b_i, b_j, B_i, B_j, LB_{ij}^i, LB_{ij}^j;

Step 4: *Current upper bound*
$R^{ub} = \min\{\min_{e=[i,j]\in E^{cand}}\{R_e\}, R^{VC}\}$;

Step 5: *LB Elimination Criterion*
$\forall$ halfedges $[i, (e, \frac{1}{2})]$ and $[(e, \frac{1}{2}), j] \in E^{cand}$
if $LB_{ij}^i > R^{ub}$ **then** eliminate $[i, (e, \frac{1}{2})]$ from E^{cand};
if $LB_{ij}^j > R^{ub}$ **then** eliminate $[(e, \frac{1}{2}), j]$ from E^{cand};

Step 6:
if $E^{cand} = \emptyset$ **then**
STOP: PLP is solved;
else
$LB^{min} = \min_{E^{cand}} LB_{ij}^{i,j}$, $R^{curr} = \max_{v_k \in V} d(v_k, t_{LB^{min}} l_e)$;
Step 7: if $LB^{min} = R^{curr}$ **then STOP**: PLP is solved;

Step 8: *Ineffective LB*
if $R^{curr} - R^{ub} > R^{ub} - LB^{min}$ **then** eliminate the halfedge from E^{cand};

Step 9: *Artificial edge*
if $R^{curr} - R^{ub} \leq R^{ub} - LB^{min}$ **then** introduce artificial edge $e_{art} = tl_e$, $[t_{LB^{min}} - \frac{R^{ub} - LB^{min}}{l_e}, t_{LB^{min}} + \frac{R^{ub} - LB^{min}}{l_e}]$ into E^{cand} instead the halfedge having LB^{min}, perform on e_{art} **Step 1** – **Step 3**, update R^{ub} and go to **Step 5**;

end if

Output: absolute radius $R^* = R^{ub}$ and absolute center(s) x^* of the network $G(V, E)$.

- **Steps 6 − 8** : $\mathcal{O}(|E^{cand}| + n + 1 + 1) \leq \mathcal{O}(m)$.

- **Step 9** : Let us denote as k the maximal number of artificial intervals introduced with respect to one lower bound on a halfedge of the set E^{cand}. Then the theoretical complexity of Step 9 is of order $\mathcal{O}(k|E^{cand}|n)$. The number of lower bounds in the network with values less than or equal to R^{UB} (3.36) is no greater than the maximal number of solutions of MWAP and the vertex center problem, which order is $\mathcal{O}(n)$. Otherwise it contradicts the optimality of solutions using in (3.36). On each lower bound one artificial interval needs to be introduced if there are at least four unexplored points over it. Therefore, the number of iterations on one lower bound is of order $log_2 n$. Hence, the theoretical complexity of Step 9 is $\mathcal{O}(n^2 logn)$.

  This is justified by the observation [32], [20] that about 95% of arcs are usually eliminated by Halpern's bound comparing it with the vertex radius of a network. Our computational experiments on 1348 solved problems (see Table 3.5 column (6), Table 3.8 column (6) and Table 3.11 in Section 3.3.7) encouraged by Theorem 3.19 show that at least 99.7% of edges are eliminated after LB Elimination Criterion. As it can be seen from columns (2)–(7) of Table 3.6 and columns (2)–(4) of Table 3.9, the maximal number of halfedges in the set E^{cand} was equal to 6 and in 96.59% of problems the set E^{cand} was empty. The cardinality $|E^{cand}|$ does not depend on numbers of vertexes and edges of the network and was very small constant in the numerical experiments. Moreover, the maximal number k of artificial intervals introduced on halfedges of the set E^{cand} was equal to 1 (see Table 3.6 column (11) and Table 3.9 column (8)). So, practical complexity of Step 9 tends to $\mathcal{O}(n)$.

In summary, the worst case complexity of Algorithm 3.5 is $\mathcal{O}(mn + n^2 logn)$. It is similar to the best known result obtained by Dvir and Handler [20] who use presorting. However, we conjecture that the complexity of Step 9 can be further improved to $\mathcal{O}(n^2)$. This would lead to a worst case bound of $\mathcal{O}(mn)$, a bound which coincides with the empirical complexity of Algorithm 3.5.

We have assumed that the shortest distance matrix is given. When this matrix needs to be computed, Fredman and Tarjan's $\mathcal{O}(m+nlogn)$ algorithm for the shortest distances from a node to all nodes in V can be applied $n-1$ times [26]. Hence, the total complexity for solution of PLP would be $\mathcal{O}(mn + n^2 logn)$.

3.3.6 Applying the algorithm to an example network

We illustrate Algorithm 3.5 in a network with 19 nodes and 26 arcs (see Figure 3.14). The network is taken from the paper of Dvir and Handler [20]. We compare our solution approach with their algorithm operating in terms of diameter instead of radius. The example in Figure 3.14 shows that Algorithm 3.5 can be easily applied manually to a rather complex network.

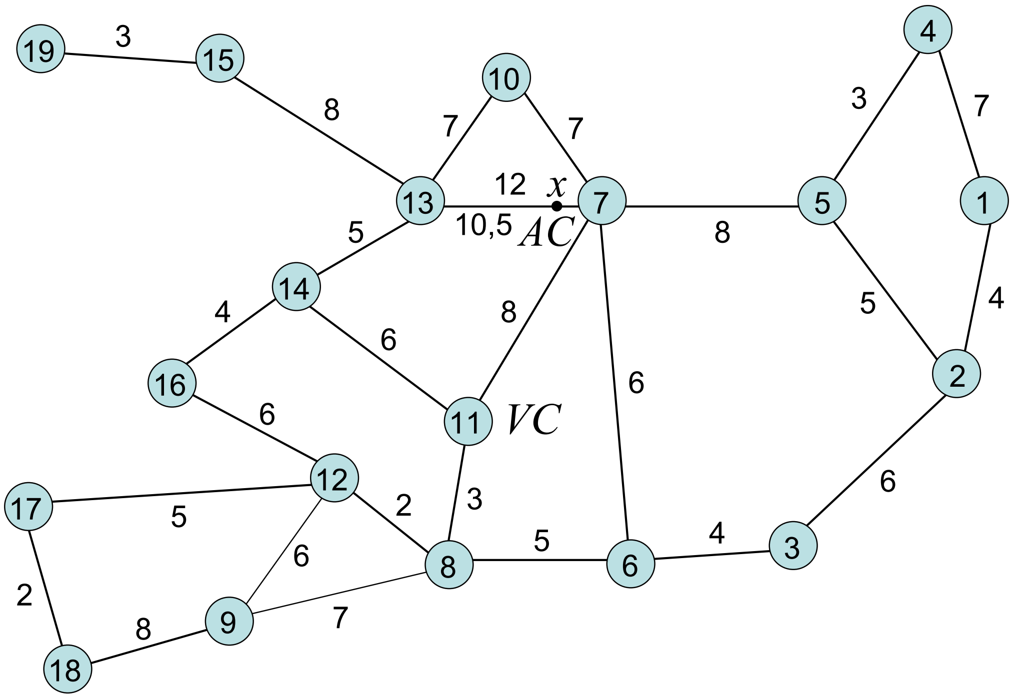


Figure 3.14: *Network for example.*

Initially, the Dvir and Handler's algorithm finds an upper bound $UB = 2R_{VC}$ and current candidate center point v_{11} corresponding to UB. Our algorithm performs a similar step. It finds the farthest point from each vertex of the network and R_{VC} (see Table 3.4). Then both algorithms use the elimination criterion described in Lemma 3.20 which eliminates 21 from 26 arcs in the network. The remaining five arcs, for which Halpern's bound in Dvir and Handler's algorithm and bounds (3.37), (3.38) and (3.36) must be computed, are $(7, 10)$, $(7, 13)$, $(10, 13)$, $(14, 16)$ and $(12, 16)$. The last arc was not considered in [20] despite of the fact that elimination criterion for this arc is not satisfied.

To calculate bounds (3.37), (3.38) and (3.36) with respect to the arc (i, j) we need the farthest points from the endpoints of the arc and first and second nondominated farthest vertexes from the middle point of the edge (i, j), which can be easily obtained from the distance matrix.

Let us show on example of arc $(7, 10)$ how computations of lower and upper bounds can be performed. Since the value of $\max_{v_k \in V} \min\{d(v_k, v_7), d(v_k, v_{10})\}$ is equal to $d(v_{18}, v_7) = 20$ (see Table 3.4) the first farthest vertex from the middle point of the edge $(7, 10)$ is the vertex v_{18}. Therefore, $b_{10} = d(v_{18}, v_{10}) = 27$.

Table 3.4: Shortest-distance matrix for example.

v_i/v_j	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
1	0	4	10	7	9	14	17	19	26	24	22	21	29	28	37	27	26	28	$\underline{40}$
2	4	0	6	8	5	10	13	15	22	20	18	17	25	24	33	23	22	24	$\underline{36}$
3	10	6	0	14	11	4	10	9	16	17	12	11	22	18	30	17	16	18	$\underline{33}$
4	7	8	14	0	3	17	11	22	29	18	19	24	23	25	31	29	29	31	$\underline{34}$
5	9	5	11	3	0	14	8	19	26	15	16	21	20	22	28	26	26	28	$\underline{31}$
6	14	10	4	17	14	0	6	5	12	13	8	7	18	14	26	13	12	14	$\underline{29}$
7	17	13	10	11	8	6	0	11	18	7	8	13	12	14	20	18	18	20	$\underline{23}$
8	19	15	9	22	19	5	11	0	7	18	3	2	14	9	22	8	7	9	$\underline{25}$
9	26	22	16	29	26	12	18	7	0	25	10	6	21	16	29	12	10	8	$\underline{32}$
10	24	20	17	18	15	13	7	18	25	0	15	20	7	12	15	16	25	$\underline{27}$	18
11	$\underline{22}$	18	12	19	16	8	8	3	10	15	0	5	11	6	19	10	10	12	$\underline{22}$
12	21	17	11	24	21	7	13	2	6	20	5	0	15	10	23	6	5	7	$\underline{26}$
13	$\underline{29}$	25	22	23	20	18	12	14	21	7	11	15	0	5	8	9	20	22	11
14	$\underline{28}$	24	18	25	22	14	14	9	16	12	6	10	5	0	13	4	15	17	16
15	$\underline{37}$	33	30	31	28	26	20	22	29	15	19	23	8	13	0	17	28	30	3
16	27	23	17	$\underline{29}$	26	13	18	8	12	16	10	6	9	4	17	0	11	13	20
17	26	22	16	29	26	12	18	7	10	25	10	5	20	15	28	11	0	2	$\underline{31}$
18	28	24	18	31	28	14	20	9	8	27	12	7	22	17	30	13	2	0	$\underline{33}$
19	$\underline{40}$	36	33	34	31	29	23	25	32	18	22	26	11	16	3	20	31	33	0

The distance $d(v_{18}, v_7)$ is strictly less than $d(v_{18}, v_{10})$. Hence, only one second nondominated vertex $v^{sf7}_{((7,10),\frac{1}{2})}$ farthest from the middle point of $(7, 10)$ must be computed. This vertex is the vertex v_{19}, because it satisfies $d(v_{19}, v_7) > d(v_{18}, v_7) = 20$ and the distance $d(v_{19}, v_{10}) = 18$ is maximal among vertexes $v_k \in V$ with $d(v_k, v_7) > d(v_{18}, v_7)$. Thus, $b_7 = d(v_{19}, v_7) = 23$ and $t^1_{(7,10)} = \frac{l_{7,10}-d(v_{18},v_7)+d(v_{19},v_{10})}{2l_{7,10}} = \frac{7-20+18}{14} = \frac{5}{14}$, $t^2_{(7,10)} = \frac{1}{2}$, $R^{(7,10)} = d(v_{18}, v_7) + t^1_{(7,10)} l_{7,10} = 22,5$. Since the farthest vertexes from v_7 and v_{10} are v_{19} and v_{18}, respectively, $B_7 = d(v_{19}, v_{10}) + l_{7,10} = 18 + 7 = 25$ and $B_{10} = d(v_{18}, v_7) + l_{7,10} = 20 + 7 = 27$. Accordingly, $LB^7_{7,10} = (23 + 25)/2 = 24$ and $LB^{10}_{7,10} = (27 + 27)/2 = 27$.

Doing similar calculations on remaining four arcs, the following bounds are calculated: $LB^7_{7,13} = 23$, $LB^{13}_{7,13} = 25,5$ and $R^{(7,13)} = 21,5$ for $(7, 13)$, $LB^{10}_{10,13} = 28$, $LB^{13}_{10,13} = 30$ and $R^{(10,13)} = 26,5$ for $(10, 13)$, $LB^{14}_{14,16} = 29,5$, $LB^{16}_{14,16} = 29$ and $R^{(14,16)} = 28$ for $(14, 16)$, $LB^{12}_{12,16} = 26$, $LB^{16}_{12,16} = 29,5$ and $R^{(12,16)} = 25$ for $(12, 16)$. Hence, $R^{ub} = 21,5$ at the point $x = ((7, 13), \frac{1,5}{12})$. As can be seen above all lower bounds are strictly greater than the upper bound R^{ub}. Therefore, Algorithm 3.5 stops on the Step 6. Hence, the central point x of local minimum width annulus is optimal solution of PLP.

3.3.7 Computational results

Algorithm 3.5 was implemented in C++ and compiled with g++ v.3.3.3. All computations were done at the University of Kaiserslautern on a server equipped with Dual Intel Xeon 3.2 GHz CPUs, 4 GB RAM running on Linux Kernel 2.6.5. We have constructed three types of randomly generated problems in total 1348 networks.

The first block containing 368 problems was constructed by a principle of reduction the density of a network. In the basis of this block lay 23 randomly generated problems with 100 vertexes used in the computational experiments for Algorithm 3.1 and Algorithm 3.4. Average node degree of this sequence of problems is decreased from 100 % to about 1 % for approximately 5 % from one network to the next. Each of the 23 problems was solved 16 times – 1 time for constructed network, which is connected and has the fixed lengths of edges, and 15 times for the network with the same edges connectivity but with random lengths of edges (Table 3.5, columns (1) – (3)). Columns (4) – (5) show range and average number of edges eliminated from the consideration by Interior Elimination Criterion. The average percent of deleted arcs is about 42.57 %. The same criterion in [20] – Coincidence criterion – eliminates about 60 % of edges in 400 randomly generated problems with 100 – 260 nodes, 146 – 1379 edges and average node degree 2 – 13. Therefore, we can assume more complex design of our networks compare to [20]. Columns (6) – (7) of Table 3.5 show range and average number of eliminated edges by LB Elimination Criterion after applying of Interior Elimination Criterion. The average percentage

Table 3.5: Summary of computational results for the first block (368 problems): cumulative percent of eliminated halfedges by LB Elimination Criterion in comparison with Halpern's criterion.

n= 100, m= (1)	av nod deg (2)	No. ne-ts (3)	Int. Elim. Cr. (%) range (4)	Int. Elim. Cr. (%) aver. (5)	Int. Elim. Cr. + LB Elim. Cr. (%) range (6)	Int. Elim. Cr. + LB Elim. Cr. (%) aver. (7)	Int. Elim. Cr. + Halpern's b-d (%) range (8)	Int. Elim. Cr. + Halpern's b-d (%) aver. (9)
4950	99	16	5.7 – 55.3	19.5	99.98 – 100	99.996	98.89 – 99.94	99.49
4783	95	16	6.5 – 56	16	99.96 – 100	99.995	98.95 – 99.88	99.4
4549	90	16	10 – 99.74	34.9	99.97 – 100	99.998	98.5 – 99.91	99.5
4224	84	16	5.6 – 51.2	19.4	99.99 – 100	99.998	98.7 – 99.93	99.5
4018	80	16	8.3 – 43.9	18.2	99.96 – 100	99.995	98.98 – 99.93	99.56
3730	74	16	10.1 – 93.35	26.36	99.92 – 100	99.99	97.53 – 99.92	99.26
3468	69	16	9.7 – 38.1	20.9	99.99 – 100	99.995	99 – 99.86	99.47
3218	64	16	7.5 – 99.6	35.5	99.98 – 100	99.997	98.26 – 99.9	99.3
2991	59	16	16.6 – 89.7	34.2	99.97 – 100	99.993	99 – 99.9	99.57
2784	55	16	15.7 – 79.7	32	99.96 – 100	99.994	98.1 – 99.93	99.45
2539	50	16	13.4 – 96.7	38.3	99.96 – 100	99.994	98.66 – 99.96	99.56
2279	45	16	17.9 – 57.2	36	100	100	99.3 – 99.96	99.67
2033	40	16	12 – 67.8	34.3	99.95 – 100	99.997	98.52 – 99.95	99.48
1757	35	16	11 – 71	44.4	99.97 – 100	99.993	98.7 – 99.94	99.48
1548	30	16	15.4 – 96.3	46.3	99.97 – 100	99.996	99.2 – 99.94	99.66
1282	25	16	12.9 – 83.2	52.9	100	100	99.1 – 99.92	99.61
1008	20	16	16.6 – 98.5	52.1	99.95 – 100	99.997	96.8 – 99.9	99.2
772	15	16	29 – 99.1	51.3	99.87 – 100	99.99	96.9 – 99.9	99.36
509	10	16	32 – 99.6	73	100	100	98.6 – 99.8	99.47
275	5	16	69.8 – 98.2	86.7	99.8 – 100	99.99	98.2 – 99.6	99.3
117	2	16	69.2 – 84.6	78.8	100	100	89.7 – 99.1	94.1
100	2	16	16 – 31	24.9	100	100	71 – 89	81.88
99	1	16	98.99	98.99	100	100	98.99	98.99
all		368	5.6 – 99.74	42.57	99.8 – 100	99.996	71 – 99.96	98.446

of deleted arcs is about 99.996 % and in range 99.8 – 100 %. As we have expected by Lemma 3.19, this is better than Halpern's bound calculated in columns (8) – (9) of Table 3.5. Concerning all 368 solved problems the average percent of elimination with Halpern's lower bound is 98.446 %. It is the same as for 400 solved problems in [20]. However, in some problems it is substantially less: 71 – 89 % of elimination in range and 81.88 % in average. Moreover, for the mentioned problems with „bad" percentage of deletion by Halpern's lower bound we have in average 100 % of elimination by LB Elimination Criterion (columns (6) – (9) of Table 3.5).

Table 3.6 illustrates effectiveness of Steps 5 – 9 in Algorithm 3.5 for the first block of problems. Here, columns (2) – (7) demonstrate how much halfedges in a network must be explored after checking LB Elimination Criterion (Lemma 3.21). In column (2) 309 from 368 solved problems have no edges for further consideration. For these problems, Algorithm 3.5 stops in Step 6. Only one halfedge satisfying $LB \leq UB$ appeared in 46 problems, two in 8, three in 3, four in one and six in one problem. By checking in Steps 6 – 8 of the algorithm these 59 nets whether minimal lower bound is attained or ineffective we have solved in addition 52 problems. Hence, Steps 1 – 8 of Algorithm 3.5 solve 361 or 98.1 % from the whole number of problems (column (8)). Column (9) shows 7 problems which must be further investigated with the help of artificial edges. Columns (10) – (11) demonstrate maximal number of halfedges where an artificial edge is introduced and maximal number of iteration on it, i. e. how many times artificial edges are introduced with respect to the halfedge. As we can see, only one artificial arc appears in those 7 networks and all problems are solved after performing of one iteration on it. That is Algorithm 3.5 goes in Step 9 to Step 1 only once and for one lower bound only. Dvir and Handler's algorithm uses working tables for non–eliminated edges and the major effort of their algorithm involves finding maximal values in and updating the working table. As summarised in [20], Dvir and Handler's algorithm performs up to 9 iterations on the working table with respect to the edge $e \in E$.

In Table 3.7 percentage of problems in the first block solved by local MWAP and by calculation of LB is estimated. So, a solution of local MWAP is optimal for PLP in 335 nets or in 91 % of the problems. In this case conditions of Theorem 3.18 are satisfied. Moreover, 364 or 98.9 % of problems are solved either by local MWAP and lower bound(s) or by lower bound(s) only. Finally, in remainded 1.1 % of nets we have solved PLP investigating one artificial edge.

Solving the first block of problems, we were convinced, that the number of the edges removed by LB Elimination Criterion and the number of introduced artificial edges is stable and does not depend on density of a network. On the other hand, a number of distance functions and, as follows, a number of intersection points of them with respect to each edge depends on the number of vertexes in the network. Therefore, in the second block of problems we fix density of a network and enable growth of number of vertexes. Here we solve 144 problems with average node degree

Table 3.6: Summary of computational results for the first block (368 problems): details of terminating of Algorithm 3.5 on different steps.

aver node degr	No. of halfedges with LB $\leq$ UB is equal to (in No. of nets)						LB attained or ineffective				not solv nets (%)
							solv	not solved			
	0	1	2	3	4	6	No. nets	No. nets	max No. edges	max No. iter.	
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)
99	12	2	2	0	0	0	16	0	0	0	0
95	11	4	0	0	1	0	15	1	1/2	1	0
90	15	0	0	1	0	0	15	1	1/2	1	0
84	13	3	0	0	0	0	16	0	0	0	0
80	11	4	0	1	0	0	16	0	0	0	0
74	8	5	1	1	0	1	16	0	0	0	0
69	11	5	0	0	0	0	16	0	0	0	0
64	13	3	0	0	0	0	15	1	1/2	1	0
59	10	5	1	0	0	0	14	2	1/2	1	0
55	12	3	1	0	0	0	16	0	0	0	0
50	12	3	1	0	0	0	16	0	0	0	0
45	16	0	0	0	0	0	16	0	0	0	0
40	15	0	1	0	0	0	15	1	1/2	1	0
35	12	4	0	0	0	0	15	1	1/2	1	0
30	14	2	0	0	0	0	16	0	0	0	0
25	16	0	0	0	0	0	16	0	0	0	0
20	15	1	0	0	0	0	16	0	0	0	0
15	14	1	1	0	0	0	16	0	0	0	0
10	16	0	0	0	0	0	16	0	0	0	0
5	15	1	0	0	0	0	16	0	0	0	0
2	16	0	0	0	0	0	16	0	0	0	0
2	16	0	0	0	0	0	16	0	0	0	0
1	16	0	0	0	0	0	16	0	0	0	0
368	309	46	8	3	1	1	361	7			0

Table 3.7: Percentage of PLP in the first block solved by LB – calculation.

	No. nets	%
Solved by local MWAP (Theorem 3.18)	335	91
Solved by local MWAP and(or) LB (Steps 1 – 5 of Alg. 3.5)	364	98.9
Solved by other point (on artificial interval)	4	1.1
All solved problems	368	

Table 3.8: Summary of computational results for the second block (144 problems): cumulative percent of eliminated halfedges by LB Elimination Criterion in comparison with Halpern's criterion.

n=	m=	No. nets	Int. Elim. Cr. (%)		Int. Elim. Cr. + LB Elim. Cr. (%)		Int. Elim. Cr. + Halpern's b-d (%)	
			range	aver.	range	aver.	range	aver.
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
100	196	16	30.6 – 71.9	41.3	99.74 – 100	99.95	89.31 – 98.53	95.4
200	377	16	35.8 – 66.6	56.2	99.73 – 100	99.98	98.7 – 99.73	99.35
300	450	16	59.1 – 68.2	63.5	100	100	93.56 – 98.67	96.3
400	698	16	53 – 77.8	65.5	99.93 – 100	99.99	98.28 – 99.71	99.21
500	853	16	28.8 – 55.5	44.8	99.94 – 100	99.98	98.59 – 99.77	99.33
600	1091	16	34.5 – 60.6	47.7	99.95 – 100	99.99	99.08 – 99.91	99.5
700	1300	16	34.2 – 56.2	46.1	99.96 – 100	99.99	98.92 – 99.92	99.48
800	1475	16	39 – 59.8	46	99.93 – 100	99.99	98.98 – 99.93	99.7
900	1747	16	30.6 – 65	39.8	99.97 – 100	99.996	98.57 – 99.94	99.56
all			30.6 – 77.8	50.1	99.73 – 100	99.99	89.31 – 99.94	98.65

Table 3.9: Summary of computational results for the second block (144 problems): details of terminating of Algorithm 3.5 on different steps.

n=	No. of halfedges with LB $\leq$ UB is equal to (in No. of nets)			LB attained or ineffective				not solved nets (%)
				solved	not solved			
	0	1	2	No. nets	No. nets	max No. edges	max No. iter.	
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
100	13	3	0	15	1	1/2	1	0
200	14	1	1	16	0	0	0	0
300	16	0	0	16	0	0	0	0
400	14	2	0	16	0	0	0	0
500	11	5	0	16	0	0	0	0
600	13	3	0	16	0	0	0	0
700	12	4	0	16	0	0	0	0
800	13	2	1	14	2	1/2	1	0
900	14	2	0	16	0	0	0	0
all	120	22	2	141	3			0

Table 3.10: Percentage of PLP in the second block solved by LB – calculation.

	No. nets	%
Solved by local MWAP (Theorem 3.18)	123	85.4
Solved by local MWAP and(or) LB (Steps 1 – 5 of Alg. 3.5)	141	97.9
Solved by other point (on artificial interval)	3	2.1
All solved problems	144	

3 and the number of vertexes growing from 100 up to 900 (Table 3.8). The problems base on 9 connected networks which were solved 16 times in the similar with the first block way. In this block range and average percent of elimination by LB Elimination Criterion is 99.73 – 100 % and 99.99 %, respectively. As it is additionally illustrated in Table 3.9, LB Elimination Criterion remains stable in relation to growth of number of vertexes. Table 3.10 demonstrated that percent of PLP solved by local MWAP and LB remains close to the values obtained in Table 3.7 for the first block of nets.

Comparing results of two elimination criteria for two blocks of problems, we can see there is a gap for Interior Elimination Criterion and almost equivalence for LB Elimination Criterion (Table 3.5 and Table 3.8 columns (4) – (7)). It is in average 42.57 % against 50.1 % of deleted arcs for Interior Elimination Criterion and 99.996 % against 99.99 % for LB Elimination Criterion.

Table 3.11: Percentage of PLP solved by LB – calculation for three blocks.

	No. nets	%
Total number of solved problems	1348	100
Cumulative percent of eliminated halfedges after LB Elimination Criterion: range/average		99.73 – 100/99.996
Solved by local MWAP (Theorem 3.18)	1141	84.64
Solved by local MWAP and(or) LB (Steps 1 – 5 of Algorithm 3.5)	1302	96.59
Solved by other point (on artificial interval)	46	3.41

The third block of independent from each other problems consists of 836 connected networks with randomly generated number of vertexes and edges. The networks have 100 – 900 nodes and 137 – 16347 edges with average node degrees 2 – 36. Here the range and average percent of deleted arcs by LB Elimination Criterion is similar to the percents in the previous blocks of problems. It is 99.8 – 100 % and 99.998 %, respectively. Finding of the shortest path matrix was done for all nets by Floyd – Warshall algorithm. The maximal time which we have spent for the solution of the problems with the calculation of the shortest path matrix was about 95 seconds. For the largest problems it takes maximal 2.73 seconds to compute an absolute center without the calculation of the shortest path matrix.

Table 3.11 cumulates most important results for total number of solved problems. The range and average percents of eliminated arcs show that our LB Elimination Criterion depends neither on the density of a network nor on the number of vertexes in it and is very powerful and stable. More than 96 % of PLP were solved by local MWAP and (or) received from it lower bound.

So, the implementation of Algorithm 3.5 shows its stability and rapid convergence of the lower and upper estimations of an absolute radius of a network to its optimal value.

Chapter 4

Conclusions, extensions and future research

In this chapter, the problems formulated and studied in Chapter 3 are considered in directed weighted and unweighted networks. Also the summary of this work and topics for future investigation are presented.

4.1 Modifications of MWAP, CLP and PLP in directed networks

4.1.1 MWAP, PLP and CLP in unweighted directed networks

A *strongly connected directed graph* is a graph in which a directed path exists from every node i to every node j. In other words, in strongly connected directed network each node i has at least one ingoing and at least one outgoing arc.

Let us denote as $\vec{G}(V, E)$ an unweighted strongly connected directed network, where $V = \{v_1, ..., v_n\}$ is the set of nodes and $E = \{e_1, ..., e_m\}$ is the set of edges with cardinality $|V| = n$ and $|E| = m$, respectively. $\mathbb{P}(\vec{G})$ denotes the set of all points in the network $\vec{G}(V, E)$, $\vec{e} = [i, j] \in E$ is a directed edge formed by two vertexes $v_i, v_j \in V$ with the length $l_{\vec{e}}$. A point $x \in \vec{e}$ can be written as $x = (\vec{e}, (1-t))$, where $0 \leq t \leq 1$. The distance between two points $x, y \in \mathbb{P}(\vec{G})$ of the network $\vec{G}(V, E)$, where $x = (\vec{e}_1, (1-t_1))$, $\vec{e}_1 = [i_1, j_1]$, $0 < t_1 \leq 1$, $y = (\vec{e}_2, (1-t_2))$, $\vec{e}_2 = [i_2, j_2]$, $0 < t_2 < 1$ is defined as

$$\vec{d}(x, y) = \vec{d}(x, i_2) + \vec{d}(i_2, y), \tag{4.1}$$

where $\vec{d}(x, i_2), \vec{d}(i_2, y)$ are the lengths of shortest directed paths between the point

x and the vertex i_2 and the vertex i_2 and the point y, respectively. Analogous, $\vec{d}(y,x) = \vec{d}(y,i_1) + \vec{d}(i_1,x)$, where $0 < t_1 < 1$ and $0 < t_2 \leq 1$.

Definition 4.1. MWAP in $\vec{G}(V,E)$

For a given network $\vec{G}(V,E)$ with cardinality $|V| = n$ and $|E| = m$ find a node covering annulus $\mathcal{A}(x,R,r)$, $x \in \mathbb{P}(\vec{G})$ of minimum width

$$width_{\mathcal{A}} = R - r. \tag{4.2}$$

The objective function (4.2) can be rewritten as follows

$$width_{\mathcal{A}}(x) = \max_{v_k \in V} \vec{d}(x,v_k) - \min_{v_k \in V} \vec{d}(x,v_k). \tag{4.3}$$

For any point $x \in \vec{e} = [i,j]$ the minimal distance $\min_{v_k \in V} \vec{d}(x,v_k)$ is equal to $\vec{d}(x,j)$. Using definition (4.1) the objective function (4.3) is equal to

$$\begin{aligned} width_{\mathcal{A}}(x) &= \max_{v_k \in V} \vec{d}(x,v_k) - \min_{v_k \in V} \vec{d}(x,v_k) \\ &= \max_{v_k \in V} \{\vec{d}(x,j) + \vec{d}(j,v_k)\} - \vec{d}(x,j) \\ &= \max_{v_k \in V} \vec{d}(j,v_k). \end{aligned} \tag{4.4}$$

This means that for all points $x \in \vec{e}$ the objective function $width_{\mathcal{A}}(x)$ has an identical value. Therefore, in order to find all optimal solutions of MWAP it is enough to consider as candidate solutions the points $v_k \in V \subset \mathbb{P}(\vec{G})$ only. If the point $v_k \in V$ solves MWAP then any interia point on each edge $\vec{e}$ ingoing in the vertex v_k solves it too. According to (4.1) the distance between the point $x \in \vec{e}$, $0 < t \leq 1$ and any vertex $v_k \in V$ is equal to

$$\vec{d}(x,v_k) = \vec{d}(x,j) + \vec{d}(j,v_k) = l_{\vec{e}}(1-t) + \vec{d}(j,v_k). \tag{4.5}$$

The sum is minimal when $t = 1$. It corresponds to the point $x = v_j$.

Now let us formulate the problem of finding an absolute center in an unweighted directed network:

Definition 4.2. PLP in $\vec{G}(V,E)$

For a given network $\vec{G}(V,E)$, $|V| = n$ and $|E| = m$ find a point $x \in \mathbb{P}(\vec{G})$ so that the maximal distance

$$\max_{v_k \in V} \vec{d}(x,v_k) \tag{4.6}$$

is minimal.

Term (4.5) shows that PLP in $\vec{G}(V, E)$ can have an optimal solution at points $v_k \in V$ only. Moreover, the equality (4.4) proves the following

Lemma 4.3. *Each vertex $v_k \in V$ solving MWAP in an unweighted strongly connected directed network $\vec{G}(V, E)$ solves PLP in this network to optimality and vice versa.*

Hence, the optimal solutions set of PLP is a subset of the optimal centers for MWAP in the unweighted strongly connected directed network $\vec{G}(V, E)$. As follows, the worst case complexity of finding an absolute center of a network and the center of an minimum width annulus is of order $\mathcal{O}(n^2)$.

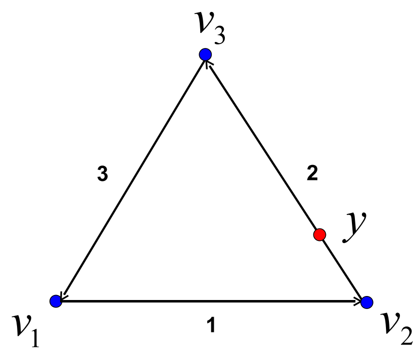


Figure 4.1: *Network for example.*

In addition to the differences of planar and network circles mentioned in Section 3.1.1 the distance between points $v_k \in V$ and a circle in $\vec{G}(V, E)$ is directed. Therefore, MWAP and CLP are not equivalent in $\vec{G}(V, E)$.

Definition 4.4. CLP in $\vec{G}(V, E)$

For a given network $\vec{G}(V, E)$ with cardinality $|V| = n$ and $|E| = m$ find a circle $\mathcal{C}(x, \rho)$, $x \in \mathbb{P}(\vec{G})$ so that the maximal distance

$$\max_{v_k \in V} \vec{d}(v_k, \mathcal{C}(x, \rho)) \tag{4.7}$$

is minimal.

For instance, in the directed network shown in Figure 4.1 an optimal MWA $\mathcal{A}(v_1, 3, 0)$ is centered at the vertex v_1 and has $width_{\mathcal{A}} = 3$. The circle $\mathcal{C}(v_1, \rho)$, where $\rho = (3+0)/2 = 1.5$ consists of one point y only. Hence, $\max_{v_1, v_2, v_3} \vec{d}(v_k, \mathcal{C}(v_1, 1.5)) = 4.5$. On the other hand, the maximal distance between vertexes and the circle $\mathcal{C}(v_1, 3) = \{v_3\}$ is equal to 3. It proves nonequivalence of CLP and MWAP in $\vec{G}(V, E)$.

4.1.2 PLP and MWAP in weighted directed networks

In the case of weighted strongly connected directed network $\vec{G}_w(V, E)$ we assume that each vertex $v_k \in V$ has its positive weight w_k. Then for any point $x = l_{\vec{e}}(1-t) \in \vec{e} = [i, j]$, $0 < t \leq 1$ the distance between this point x and any vertex $v_k \in V$ is equal to

$$\vec{d}_{w_k}(x, v_k) = w_k\vec{d}(x, j) + w_k\vec{d}(j, v_k) = w_k l_{\vec{e}}(1-t) + w_k\vec{d}(j, v_k).$$

This distance is minimal when $t = 1$. Hence, PLP in an weighted strictly connected directed network has its optimal solution at points $v_k \in V$ only.

Definition 4.5. PLP in $\vec{G}_w(V, E)$

For a given weighted network $\vec{G}_w(V, E)$, $|V| = n$ and $|E| = m$ find a point $x \in \mathbb{P}(\vec{G}_w)$ so that the maximal weighted distance

$$\max_{v_k \in V} \vec{d}_{w_k}(x, v_k) = \max_{v_k \in V} w_k\vec{d}(x, v_k) \tag{4.8}$$

is minimal.

Therefore, to solve PLP in weighted $\vec{G}_w(V, E)$ we should calculate the minimum of maximal weighted distance between vertexes of the network

$$\min_{x \in V} \max_{v_k \in V} w_k\vec{d}(x, v_k).$$

This procedure is of order $\mathcal{O}(n^2)$ assuming the shortest path matrix is given.

Now let us formulate MWAP in an weighted directed network.

Definition 4.6. MWAP in $\vec{G}_w(V, E)$

For a given network $\vec{G}_w(V, E)$ with cardinality $|V| = n$ and $|E| = m$ find a node covering annulus $\mathcal{A}(x, R, r)$, $x \in \mathbb{P}(\vec{G}_w)$ of minimum width

$$width_{\mathcal{A}} = R - r. \tag{4.9}$$

The linear function $\vec{d}_{w_k}(x, v_k)$ is decreasing function on interval $0 < t \leq 1$, where $x = l_{\vec{e}}(1-t) \in \vec{e} = [i, j]$. The objective function (4.9) and $x = l_{\vec{e}}(1-t) \in \vec{e} = [i, j]$, $0 < t \leq 1$ are functions of variable t on each edge $\vec{e}$:

$$width_{\mathcal{A}}(t) = R(t) - r(t).$$

The functions $R(t) = \max_{v_k \in V} \vec{d}_{w_k}(x, v_k)$ and $r(t) = \min_{v_k \in V} \vec{d}_{w_k}(x, v_k)$ are convex and concave piecewise linear functions, respectively (Figure 4.2). The pieces of the function $R(t)$ have strictly decreasing weights and the pieces of the function $r(t)$

have strictly increasing weights for $0 < t \leq 1$. Moreover, the edge $\vec{e} = tl_{\vec{e}}$ can be divided into intervals $(t_p, t_{p+1}]$, $0 < t_1 < ... < t_p < ... < t_P \leq 1$, $P \leq n+1$, where the functions $R(t)$ and $r(t)$ are presented by distance functions of two points, for instance v_{k_1} and v_{k_2}, from the set V (see Figure 4.2)

$$R(t) = \vec{d}_{w_{k_1}}(x, v_{k_1}), \quad r(t) = \vec{d}_{w_{k_2}}(x, v_{k_2}).$$

Then the difference $width_{\mathcal{A}}(t) = \vec{d}_{w_{k_1}}(x, v_{k_1}) - \vec{d}_{w_{k_2}}(x, v_{k_2})$ on the interval $(t_p, t_{p+1}]$ is equal to

$$\begin{aligned} width_{\mathcal{A}}(t) &= w_{k_1}\vec{d}(x, j) + w_{k_1}\vec{d}(j, v_{k_1}) - (w_{k_2}\vec{d}(x, j) + w_{k_2}\vec{d}(j, v_{k_2})) \\ &= (w_{k_1} - w_{k_2})l_{\vec{e}}(1-t) + (w_{k_1}\vec{d}(j, v_{k_1}) - w_{k_2}\vec{d}(j, v_{k_2})). \end{aligned} \quad (4.10)$$

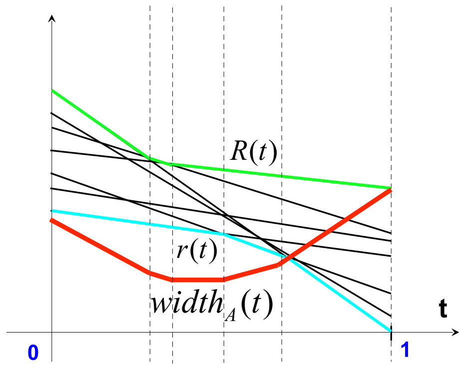


Figure 4.2: *Functions $R(t)$, $r(t)$ and $width_{\mathcal{A}}(t)$ on edge $\vec{e} = tl_{\vec{e}}$, $0 < t \leq 1$.*

In the difference (4.10), the second part on the right hand side is a constant for $t \in (t_p, t_{p+1}]$. Therefore, the function $width_{\mathcal{A}}(t)$ in the interval $(t_p, t_{p+1}]$ depends on the term $(w_{k_1} - w_{k_2})l_{\vec{e}}(1-t)$ only. If $w_{k_1} > w_{k_2}$ then $width_{\mathcal{A}}(t)$ decreases from t_p to t_{p+1}. Next, if $w_{k_1} < w_{k_2}$ then $width_{\mathcal{A}}(t)$ increases from t_p to t_{p+1}. And finally, if $w_{k_1} = w_{k_2}$ then $width_{\mathcal{A}}(t)$ remains constant from t_p to t_{p+1}. Hence, the function $width_{\mathcal{A}}(t)$ can attain its minimum or at point, where weight of the vertex defining the function $R(t)$ become smaller than weight of the vertex defining the function $r(t)$, or at point, where weight of the vertex defining the function $r(t)$ become large than weight of the vertex defining the function $R(t)$, or at any point between these two points.

First, we solve local MWAP on an directed edge $\vec{e} = [i, j]$. Let us sort all weighted distances $\vec{d}_{w_k}(j, v_k)$, $v_k \in V$ in non-increasing order:

$$w_{(1)}\vec{d}(j, v_{(1)}) \geq ... \geq w_{(k)}\vec{d}(j, v_{(k)}) \geq ... \geq w_{(n)}\vec{d}(j, v_{(n)}). \quad (4.11)$$

Based on the inequalities (4.11) two sequences for design of the functions $R(t)$ and $r(t)$ are constructed. For the function $R(t)$ we move through the sequence (4.11) from the left to the right eliminating all points $v_{(k+1)}$, $k = 1, ..., n-1$, for which $w_{(k)}\vec{d}(j, v_{(k)}) \geq w_{(k+1)}\vec{d}(j, v_{(k+1)})$ and $w_{(k)} \geq w_{(k+1)}$. For the function $r(t)$ we also start with the sequence (4.11) moving from the right to the left and eliminating the points $v_{(k)}$, $k = 1, ..., n-1$, for which $w_{(k)}\vec{d}(j, v_{(k)}) \geq w_{(k+1)}\vec{d}(j, v_{(k+1)})$ and $w_{(k)} \geq w_{(k+1)}$. After that we have two different decreasing sequences for the functions $R(t)$ and $r(t)$ with an equal number of elements $n_1 \leq n$:

$$w_{(1)}^{R(t)}\vec{d}(j, v_{(1)}^{R(t)}) > ... > w_{(k)}^{R(t)}\vec{d}(j, v_{(k)}^{R(t)}) > ... > w_{(n_1)}^{R(t)}\vec{d}(j, v_{(n_1)}^{R(t)}) \tag{4.12}$$

and

$$w_{(1)}^{r(t)}\vec{d}(j, v_{(1)}^{r(t)}) > ... > w_{(k)}^{r(t)}\vec{d}(j, v_{(k)}^{r(t)}) > ... > w_{(n_1)}^{r(t)}\vec{d}(j, v_{(n_1)}^{r(t)}), \tag{4.13}$$

where

$$w_{(1)}^{R(t)} < ... < w_{(k)}^{R(t)} < ... < w_{(n_1)}^{R(t)}$$

and

$$w_{(1)}^{r(t)} < ... < w_{(k)}^{r(t)} < ... < w_{(n_1)}^{r(t)}.$$

The points, which can form the functions $R(t)$ and $r(t)$ are $v_{(1)}^{R(t)}$, ..., $v_{(n_1)}^{R(t)} \in V$ and $v_{(1)}^{r(t)}$, ..., $v_{(n_1)}^{r(t)} \in V$, respectively.

Now, if $w_{(n_1)}^{r(t)} < w_{(1)}^{R(t)}$ then the vertex v_j is unique point of minimum of the function $width_{\mathcal{A}}(t)$ on the edge $\vec{e}$ (Figure 4.3 a)). Otherwise, $w_{(n_1)}^{r(t)} \geq w_{(1)}^{R(t)}$ and we should find a value of t, $0 < t < 1$, where weights of $r(t)$ become smaller than weights of the function $R(t)$ when we move from the vertex v_j to the vertex v_i on the edge e. We calculate values of intersection points and corresponding weights for the functions $R(t)$ and $r(t)$ with the help of the following two procedures:

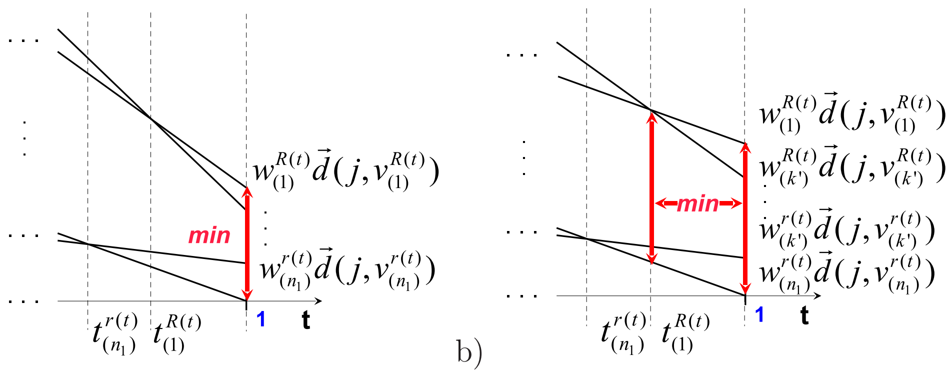


Figure 4.3: *Function* $width_{\mathcal{A}_{\vec{e}}}(t)$ *has its minimum: a) at the point* v_j, $t = 1$; *b) at all points* $l_{\vec{e}}(1-t)$, $t_{(1)}^{R(t)} \leq t \leq 1$.

Procedure 1 – *IP* $t(v_{(k)}^{R(t)})$: *Intersection points of* $R(t)$

Input : current point $v_{(k)}^{R(t)}$ defining $R(t)$

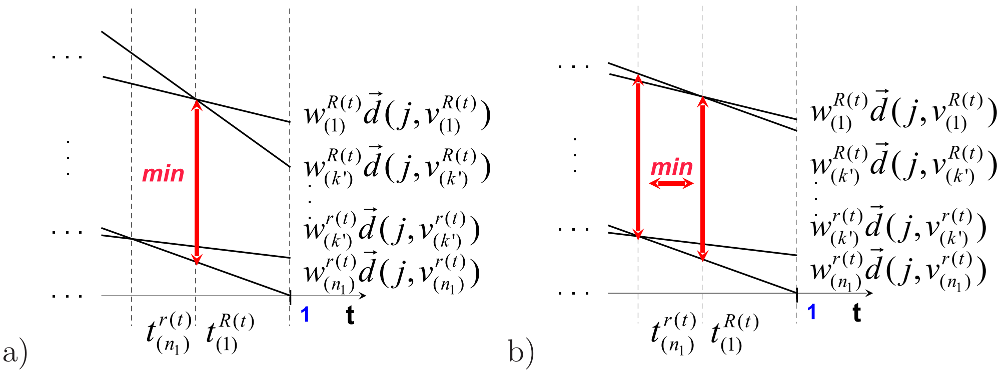


Figure 4.4: *Function* $width_{\mathcal{A}_{\vec{e}}}(t)$ *has its minimum: a) at the point* $l_{\vec{e}}(1 - t^{R(t)}_{(1)})$*; b) at all points* $l_{\vec{e}}(1 - t)$, $t^{r(t)}_{(n_1)} \leq t \leq t^{R(t)}_{(1)}$.

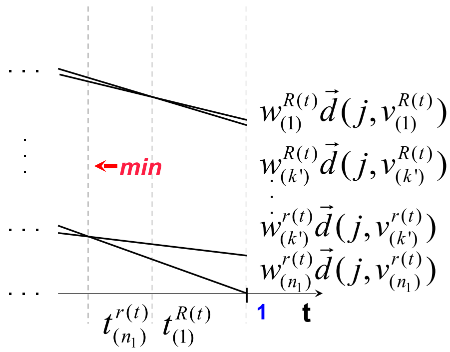


Figure 4.5: *Function* $width_{\mathcal{A}_{\vec{e}}}(t)$ *attains its minimum on a subset of the halfinterval* $0 < t \leq t^{r(t)}_{(n_1)}$.

- **Step 1** : $\forall v^{R(t)}_{(\bar{k})} : w^{R(t)}_{(k)}\vec{d}(j, v^{R(t)}_{(k)}) > w^{R(t)}_{(\bar{k})}\vec{d}(j, v^{R(t)}_{(\bar{k})})$ in (4.12) calculate $t^{R(t)}_{(\bar{k})} = 1 - \frac{w^{R(t)}_{(k)}\vec{d}(j,v^{R(t)}_{(k)}) - w^{R(t)}_{(\bar{k})}\vec{d}(j,v^{R(t)}_{(\bar{k})})}{(w^{R(t)}_{(\bar{k})} - w^{R(t)}_{(k)})l_{\vec{e}}}$,
- **Step 2** : find $t^{R(t)}_{(k)} = \max t^{R(t)}_{(\bar{k})}$ for some $v^{R(t)}_{(k')}$; if $\max t^{R(t)}_{(\bar{k})}$ is not unique, then $v^{R(t)}_{(k')}$ is the point with maximal weight,
- **Step 3** : the weight $w_{t^{R(t)}_{(k)}} = w^{R(t)}_{(k')}$,
- **Step 4** : the next current point $v^{R(t)}_{(k)} = v^{R(t)}_{(k')}$.

Output: intersection points $t^{R(t)}_{(k)}$ with associated weights $w_{t^{R(t)}_{(k)}}$ and the next current point $v^{R(t)}_{(k)}$.

Procedure 2 – *IP* $t(v^{r(t)}_{(k)})$: *Intersection points of* $r(t)$

Input : current point $v^{r(t)}_{(k)}$ defining $r(t)$

- **Step 1** : $\forall v^{r(t)}_{(\bar{k})}$: $w^{r(t)}_{(\bar{k})}\vec{d}(j, v^{r(t)}_{(\bar{k})}) > w^{r(t)}_{(k)}\vec{d}(j, v^{r(t)}_{(k)})$ in (4.13) calculate $t^{r(t)}_{(\bar{k})} = 1 - \frac{w^{r(t)}_{(k)}\vec{d}(j,v^{r(t)}_{(k)}) - w^{r(t)}_{(\bar{k})}\vec{d}(j,v^{r(t)}_{(\bar{k})})}{(w^{r(t)}_{(\bar{k})} - w^{r(t)}_{(k)})l_{\vec{e}}}$,
- **Step 2** : find $t^{r(t)}_{(k)} = \max t^{r(t)}_{(\bar{k})}$ for some $v^{r(t)}_{(k')}$; if $\max t^{r(t)}_{(\bar{k})}$ is not unique, then $v^{r(t)}_{(k')}$ is the point with minimal weight,
- **Step 3** : the weight $w_{t^{r(t)}_{(k)}} = w^{r(t)}_{(k')}$,
- **Step 4** : the next current point $v^{r(t)}_{(k)} = v^{r(t)}_{(k')}$.

Output: intersection points $t^{r(t)}_{(k)}$ with associated weights $w_{t^{r(t)}_{(k)}}$ and the next current point $v^{r(t)}_{(k)}$.

All steps of finding of minimum width covering annuli in an directed weighted network are summarized in Algorithm 4.1.

Let us illustrate a few iterations of the algorithm on some edge $\vec{e} \in E$ in the case $t^{r(t)}_{(n_1)} < t^{R(t)}_{(1)}$, where points follow directly one another. If $t^{r(t)}_{(n_1)} < t^{R(t)}_{(1)}$ and $w^{R(t)}_{(1)} = w^{r(t)}_{(n_1)}$ then all points $x = l_{\vec{e}}(1-t) \in \vec{e}$, $t^{R(t)}_{(1)} \leq t \leq 1$ are optimal for local MWAP with objective value $width_{\mathcal{A}_{\vec{e}}}(t) = w^{R(t)}_{(1)}\vec{d}(j, v^{R(t)}_{(1)}) - w^{r(t)}_{(n_1)}\vec{d}(j, v^{r(t)}_{(n_1)})$ (Figure 4.3 b)).

If $t^{r(t)}_{(n_1)} < t^{R(t)}_{(1)}$, $w^{r(t)}_{(n_1)} > w^{R(t)}_{(1)}$ and $w^{r(t)}_{(n_1)} < w^{R(t)}_{(k')}$ (see Figure 4.4 a)) then the point $x = l_{\vec{e}}(1 - t^{R(t)}_{(1)}) \in \vec{e}$ is optimal for local MWAP with objective value $width_{\mathcal{A}_{\vec{e}}}(t)$ equal to

$$w^{R(t)}_{(k')}l_{\vec{e}}(1 - t^{R(t)}_{(1)}) + w^{R(t)}_{(k')}\vec{d}(j, v^{R(t)}_{(k')}) - w^{r(t)}_{(n_1)}l_{\vec{e}}(1 - t^{R(t)}_{(1)}) - w^{r(t)}_{(n_1)}\vec{d}(j, v^{r(t)}_{(n_1)}). \quad (4.14)$$

If $t^{r(t)}_{(n_1)} < t^{R(t)}_{(1)}$, $w^{r(t)}_{(n_1)} > w^{R(t)}_{(1)}$ and $w^{r(t)}_{(n_1)} = w^{R(t)}_{(k')}$ (see Figure 4.4 b)) then the points $x = l_{\vec{e}}(1-t) \in \vec{e}$, $t^{r(t)}_{(n_1)} \leq t \leq t^{R(t)}_{(1)}$ are optimal for local MWAP with objective value (4.14).

If $t^{r(t)}_{(n_1)} < t^{R(t)}_{(1)}$, $w^{r(t)}_{(n_1)} > w^{R(t)}_{(1)}$ and $w^{r(t)}_{(n_1)} > w^{R(t)}_{(k')}$ (see Figure 4.5) then the center of local MWA lies left from the point $t^{r(t)}_{(n_1)}$.

Algorithm 4.1 *Optimal solution of MWAP in $\vec{G}_w(V,E)$*

Input: $\vec{G}_w(V,E)$, $|E|=m$, $|V|=n$, distance matrix;

Step 1. $\forall \vec{e} \in E$, $\vec{e}=[i,j]$ $t^{R(t)}_{(k)}=1$, $t^{r(t)}_{(k)}=1$, $w^{R(t)}_{(k)}=w^{R(t)}_{(1)}$, $w^{r(t)}_{(k)}=w^{r(t)}_{(n_1)}$;

if $w^{r(t)}_{(k)} < w^{R(t)}_{(k)}$ **then**
  STOP – point $t=1$ minimizes $width_{\mathcal{A}_{\vec{e}}}(t)$;
else
  apply $IP\ t(v^{R(t)}_{(k)})$, $IP\ t(v^{r(t)}_{(k)})$;
  while $t^{r(t)}_{(k)} = t^{R(t)}_{(k)} = 0$ **do**
    if $t^{r(t)}_{(k)} < t^{R(t)}_{(k)}$ **then**
      if $w^{r(t)}_{(k)} < w^{R(t)}_{(k)}$ **then**
        STOP – point $t^{R(t)}_{(k)}$ (or the interval right from $t^{R(t)}_{(k)}$ to the nearest calculated intersection point, when $w^{r(t)} = w^{R(t)}$ on the interval) minimizes $width_{\mathcal{A}_{\vec{e}}}(t)$;
      else
        apply $IP\ t(v^{R(t)}_{(k)})$;
      end if
    end if
    if $t^{r(t)}_{(k)} > t^{R(t)}_{(k)}$ **then**
      if $w^{r(t)}_{(k)} < w^{R(t)}_{(k)}$ **then**
        STOP – point $t^{r(t)}_{(k)}$ (or $[t^{r(t)}_{(k)}, .]$) minimizes $width_{\mathcal{A}_{\vec{e}}}(t)$;
      else
        apply $IP\ t(v^{r(t)}_{(k)})$;
      end if
    end if
    if $t^{r(t)}_{(k)} = t^{R(t)}_{(k)}$ **then**
      apply $IP\ t(v^{r(t)}_{(k)})$, $IP\ t(v^{R(t)}_{(k)})$;
    end if
  end while
end if
if $t^{r(t)}_{(k)} = t^{R(t)}_{(k)} = 0$ **then**
  point $t=0$ (or $[0,.]$) minimizes $width_{\mathcal{A}_{\vec{e}}}(t)$;
end if

Step 2. find the set T of $argmin_{\vec{e}\in E} \min width_{\mathcal{A}_{\vec{e}}}(t)$.

Output: minimum width annuli $\mathcal{A}(x,R,r)$, where $x = t_{\vec{e}} l_{\vec{e}}$, $t_{\vec{e}} \in T$.

Sorting of weighted distances on each edge $\vec{e} \in E$ takes $\mathcal{O}(nlogn)$ time. The maximal number of applying of Procedures 1 and 2 on each edge $\vec{e} \in E$ is the sum of the first n natural numbers, which is equal to $(n+1)n/2$. Therefore, the worse–case complexity of Algorithm 4.1 is of order $\mathcal{O}(mn^2)$.

4.2 Conclusions and future research

As a result of this work different location problems were investigated on Rectilinear and Chebyshev planes as well as in networks. We have considered three basic location problems – MWAP, CLP, PLP – and their relations between each other. These relations have served as a basis for finding of elegant solution, algorithms for both new and well–known problems.

So, MWAP was formulated and investigated in Rectilinear space. In contrast to Euclidean metric, MWAP and PLP have at least one common optimal point. Therefore, MWAP on Rectilinear plane was solved with the help of PLP. Hence, the solution sequence was PLP$\Rightarrow$MWAP. It was shown, that MWAP and CLP are equivalent. Thus, CLP can be also solved in linear time. The obtained results were analysed and transfered to Chebyshev metric.

After that, the notions of circle, sphere and annulus in networks were introduced. It should be noted that the notion of a circle in a network is different from the notion of a cycle. An $\mathcal{O}(mn)$ time algorithm for solution of MWAP was constructed and implemented. The algorithm is based on the fact that the middle point of an edge represents an optimal solution of a local minimum width annulus on this edge. The resulting complexity is better than the complexity $\mathcal{O}(mn + n^2logn)$ in unweighted case of the fastest known algorithm for minimizing of the range function, which is mathematically equivalent to MWAP. The obtained computational results for MWAP show that the number of solution edges for the problem does not depend on the density of a network, even on trees. MWAP in unweighted undirected networks was extended to the MWAP on subsets and to the restricted MWAP. Resulting problems were analysed and solved. Also the p–minimum width annulus problem was formulated and explored. This problem is NP–hard. However, the p–MWAP has been solved in polynomial $\mathcal{O}(m^2n^3p)$ time with a natural assumption, that each minimum width annulus covers all vertexes of a network having distances to the central point of annulus less than or equal to the radius of its outer circle.

In contrast to the planar case MWAP in undirected unweighted networks have appeared to be a root problem among considered problems. During investigation of properties of circles in networks it was shown that the difference between planar and network circles is significant. This leads to the nonequivalence of CLP and MWAP in the general case. However, MWAP was effectively used in solution procedures for CLP giving the sequence MWAP$\Rightarrow$CLP. The complexity of the developed and implemented algorithm is of order $\mathcal{O}(m^2n^2)$. It is important to mention that CLP in networks has been formulated for the first time in this work and differs from the well–studied location of cycles in networks.

We have constructed an $\mathcal{O}(mn+n^2logn)$ algorithm for well–known PLP. The complexity of this algorithm is not worse than the complexity of the currently best algorithms. But the concept of the solution procedure is new – we use MWAP in order to solve PLP building the opposite to the planar case solution sequence MWAP⇒PLP and this method has the following advantages: First, the lower bounds LB obtained in the solution procedure are proved to be in any case better than the strongest Halpern's lower bound. It is pointed out in the literature by Halpern [32], Dvir and Handler [20] that Halpern's bound eliminates, in average, more than 95% of arcs and in worst case at least 90% [20]. However, in our computational experiments the minimal percent of arc deletion by Halpern's criterion was 71% while in the whole range of solved problems at least 99.73% of edges were eliminated from future consideration by LB bounds. It was shown that percent of eliminated edges by LB criterion depends neither on the average node degree of a network nor on the number of vertexes in it. This proves stability of the elimination criterion deducted in the algorithm. Second, the developed algorithm is so simple that it can be easily applied to complex networks manually. Third, the empirical complexity of the algorithm is equal to $\mathcal{O}(mn)$. Based on our observation and a wide range of various practical experiments we expect that the theoretical complexity of the algorithm is indeed of order $\mathcal{O}(mn)$ assuming that the shortest path matrix is given.

MWAP was extended to and explored in directed unweighted and weighted networks. The complexity bound $\mathcal{O}(n^2)$ of the developed algorithm for finding of the center of a minimum width annulus in the unweighted case does not depend on the number of edges in a network, because the problems can be solved in the order PLP⇒MWAP. In the weighted case computational time is of order $\mathcal{O}(mn^2)$.

All results are summarized in Figure 4.6. The algorithms and relations defined in this work are contained in boxes. However, there are some interesting topics for further research including the following:

1. Proving or disproving of the assumption, that the computational time order of Step 9 in Algorithm 3.5 is at most $\mathcal{O}(n^2)$. Having this proof the overall complexity of finding an absolute center of an unweighted network would be equal to $\mathcal{O}(mn)$.

2. Developing of approximation techniques needed to reduce the computation time of p–MWAP in unweighted undirected networks – an NP–hard problem.

3. Exploring CLP in weighted undirected networks $G_w(V,E)$ and p–CLP in undirected networks.

4. Formulation and investigation of p–MWAP or p–CLP on Euclidean $\mathbb{R}^2_{l_2}$, Rectilinear $\mathbb{R}^2_{l_1}$ and Chebyshev $\mathbb{R}^2_{l_\infty}$ planes and finding of a minimum width annulus in higher dimensions with different metrics.

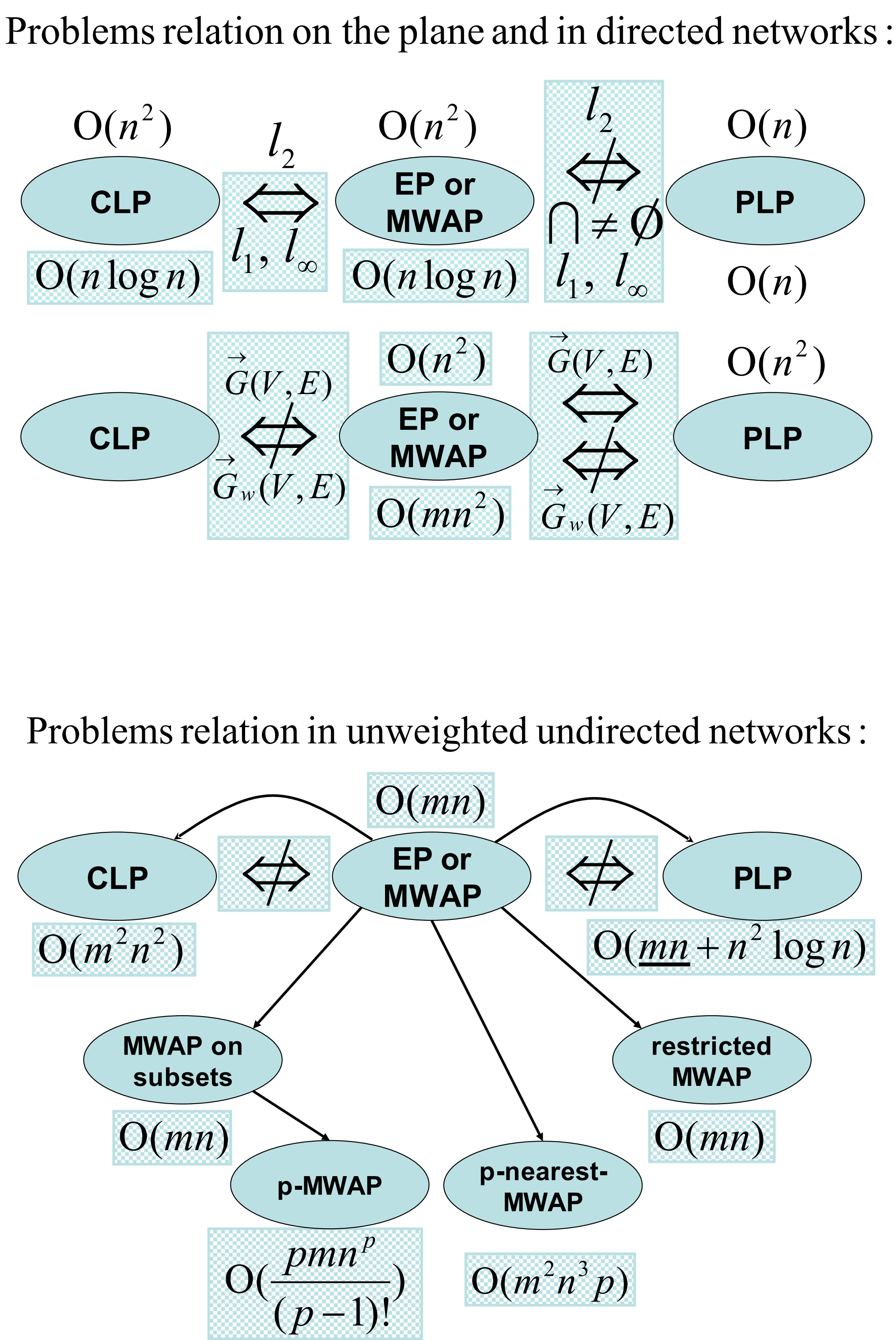


Figure 4.6: *Summary of major results of the work.*

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